

A COUNTEREXAMPLE TO THE EISENBUD–MAZUR CONJECTURE

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ABSTRACT. We construct a height-five prime ideal P in the six-variable formal power series ring R over $\mathbf{Q}$ such that $P^{(2)} \not\subseteq \mathfrak{m}P$, thus providing a prime-ideal counterexample to the Eisenbud–Mazur conjecture (with $\mathbf{Q}$ being the coefficient field). The ideal is the kernel of an explicit quadratic parametrization over a number field of degree 28 over $\mathbf{Q}$. A witness has order three, with leading term the determinant of a symmetric 3×3 matrix. We construct the witness by eliminating critical values to all orders and then straightening the resulting formal curve. Interpolation at the 28 nodes of an arrangement of eight lines supplies the required lifting maps. A rational linear functional on third and fourth jets certifies nonmembership in $\mathfrak{m}P$. The construction also yields an unmixed radical ideal with 28 smooth branches over $\mathbf{C}$, and provides a counterexample to the more general form of the Eisenbud–Mazur conjecture stated by More (with $\mathbf{C}$ being the coefficient field). Finally, the prime ideal counterexample descends to the localized polynomial ring $\mathbf{Q}[x_0, \dots, x_5]_{(x_0, \dots, x_5)}$, where a polynomial witness with the same fourth jet exists. All finite identities used in the proof admit exact verification.

1. INTRODUCTION

The Eisenbud–Mazur conjecture asks whether

$$P^{(2)} \subseteq \mathfrak{m}P \tag{1.1}$$

for every prime ideal P in a regular local ring $(R, \mathfrak{m})$ containing a field of characteristic zero. Here $P^{(2)} = P^2 R_P \cap R$ is the symbolic square of P . Equivalently, it predicts that no element of the symbolic square can be a minimal generator of P . Eisenbud and Mazur related this containment to the existence of nontrivial evolutions of reduced local algebras [EM97]. The formulation for arbitrary regular local rings containing a characteristic-zero field, and more generally for unmixed radical ideals, appears in [Mor15, Conjecture 1.5].

We construct a counterexample to the original Eisenbud–Mazur conjecture in the formal power series ring in six variables over $\mathbf{Q}$. The quotient by P is a one-dimensional complete local domain, and its normalization has

Date: October 19, 2026.

2020 Mathematics Subject Classification. Primary 13A15; Secondary 13H05, 13N05.

Key words and phrases. Symbolic powers, Eisenbud–Mazur conjecture, formal curves, Veronese surface, critical values.

a residue-field extension of $\mathbf{Q}$ of degree 28. After extending coefficients to $\mathbf{C}$, the construction gives a counterexample to [Mor15, Conjecture 1.5] with 28 smooth branches. We also give a counterexample to the original Eisenbud–Mazur conjecture in a localized polynomial ring over $\mathbf{Q}$; this is established in Theorem 8.1.

There are substantial positive results for special classes of ideals. Eisenbud and Mazur prove the containment for generically complete intersection licci ideals, including perfect ideals of height two and Gorenstein ideals of height three, and for generically complete intersection almost complete intersections [EM97]. Huneke and Ribbe obtain further positive results using residual intersections, integral closure, and differentials [HR98]. More proves the conjecture for prime ideals obtained by contracting a parametrized smooth curve along the extension $k[[t^2, x_1, \dots, x_m]] \subset k[[t, x_1, \dots, x_m]]$ [Mor15, Theorem 2.1]. A related uniform bound is $P^{(h+1)} \subseteq \mathfrak{m}P$, where h is the height of P : Hochster and Huneke prove this for regular local rings containing a field [HH07, Theorems 3.5 and 4.2], and Murayama proves a generalization for arbitrary regular local rings [Mur23, Theorem C]. For the height-five prime ideal constructed below, this bound concerns the sixth symbolic power and is compatible with the failure of (1.1).

Two features of the problem account for both these positive results and the difficulty of finding a counterexample. First, in characteristic zero the symbolic square of a radical ideal has a differential description: $f \in P^{(2)}$ if and only if f and all its partial derivatives lie in P [EM97, Theorem 3]. So (1.1) is a statement about critical loci: a witness is a hypersurface $f = 0$ that is singular along $V(P)$ and is nevertheless a minimal generator of P . Second, if P is stable under a derivation $\sum w_i x_i \partial_i$ with positive integer weights, the Euler identity $\sum w_i x_i \partial_i f = (\deg f) f$ places every such f in $\mathfrak{m}P$ [EM97, Proposition 10]. A counterexample must therefore be non-quasi-homogeneous in every system of coordinates. The hypothesis on the characteristic is essential as well: in characteristic p the differential description breaks down, and [EM97] exhibits monomial curves in characteristics 2 and 3 that violate (1.1).

We set

$$R = \mathbf{Q}[[x_0, x_1, x_2, x_3, x_4, x_5]], \quad \mathfrak{m} = (x_0, x_1, x_2, x_3, x_4, x_5).$$

The first two nonzero homogeneous terms of our witness are the cubic

$$\begin{aligned} D &= \det \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_3 & x_4 \\ x_2 & x_4 & x_5 \end{pmatrix} \\ &= x_0 x_3 x_5 - x_0 x_4^2 - x_1^2 x_5 + 2x_1 x_2 x_4 - x_2^2 x_3 \end{aligned} \tag{1.2}$$

and the quartic

$$\begin{aligned} W = & x_0^4 + x_0^3x_1 - x_0^3x_2 - 2x_0^2x_3^2 + 16x_0x_1x_3x_5 - 7x_0x_2x_3^2 \\ & + 14x_0x_3x_4x_5 - 20x_0x_3x_5^2 + 8x_1x_5^3 - 7x_2x_3x_5^2 + x_3^4 + x_4x_5^3 - x_5^4. \end{aligned} \quad (1.3)$$

The significance of the quartic W is explained after Theorem 1.1; see also Sections 3 and 4.

Theorem 1.1. *There is an explicitly defined continuous $\mathbf{Q}$ -algebra homomorphism*

$$\varphi : R \longrightarrow L[[t]], \quad \varphi(x_0) = t, \quad \varphi(x_i) = t\ell_i + t^2w_i \quad (1 \leq i \leq 5),$$

where L is a number field of degree 28 over $\mathbf{Q}$, with $\ell_i, w_i \in L$ for $1 \leq i \leq 5$, such that $P = \ker \varphi$ is a height-five prime ideal and there exists $f \in R$ satisfying

$$f \equiv D + W \pmod{\mathfrak{m}^5}, \quad f \in P^{(2)}, \quad f \notin \mathfrak{m}P. \quad (1.4)$$

The normalization of R/P is $L[[t]]$, whereas the residue field of R/P is $\mathbf{Q}$.

The field L and all coefficients of φ are specified by rational polynomial formulas below. The witness f in Theorem 1.1 is a complete formal power series, given in equation (5.1).

The construction combines a geometric lifting argument with a finite obstruction to membership in $\mathfrak{m}P$. The projectivized singular locus of D is the quadratic Veronese surface: the partial derivatives of D are the 2×2 minors, which generate the prime ideal Q of the cone over this surface, so that $D \in Q^{(2)}$. The Laplace expansion, however, writes D as a combination of the minors with coefficients in $\mathfrak{m}$, so $D \in \mathfrak{m}Q$ and Q itself is no counterexample. We therefore replace Q by the ideal of a formal arc whose tangent line lies on the cone but which leaves the cone at second order. Such an ideal no longer contains all the minors, so the Laplace expansion no longer applies, and an element of order three with leading form D can be a minimal generator; what has to be arranged is that its gradient vanishes along the arc, and the quartic W supplies the critical point that makes this possible. On its affine chart

$$(x_0, x_1, x_2, x_3, x_4, x_5) = (1, U, V, U^2, UV, V^2),$$

the restriction of W is the product of the eight lines $U + \alpha V + \alpha^2 = 0$, where α runs through the roots of $z^8 - z - 1$. Their 28 nodes form a single Galois orbit. Evaluation at one node identifies the polynomials in U, V of degree at most six with L . Consequently, evaluation at the corresponding point of the Veronese surface maps homogeneous cubics in the six variables onto L , and the same holds in every higher degree.

The interpolation has two uses. First, the cubic determinant has a non-degenerate quadratic form in the directions normal to the Veronese surface, and the restricted quartic has a nondegenerate critical point at each node. Two successive applications of the formal implicit function theorem therefore

produce a critical arc. A quintic term removes the first obstruction to its critical value being zero; the surjectivity of evaluation then removes all subsequent obstructions, one homogeneous degree at a time. Second, the same interpolation gives $\varphi(\mathfrak{m}^3) = t^3 L[[t]]$. This permits a formal coordinate change with identity two-jet that replaces the critical arc by the prescribed quadratic parametrization while preserving $D+W$ modulo $\mathfrak{m}^5$. The two-term shape of the arc and of the witness is what defeats the Euler argument: no positive grading makes both homogeneous.

The obstruction to $f \in \mathfrak{m}P$ can be checked using only the fourth jet. We determine the image of P modulo $\mathfrak{m}^4$ and display a rational linear functional supported in degrees three and four after quotienting by x_0 . It annihilates the image of $\mathfrak{m}P$ and takes the value 1 on $D+W$. Note that the proof of symbolic-square membership uses the full formal series, whereas the nonmembership argument depends only on its fourth jet. The exact computations needed to verify the finite identities are recorded in the appendix.

Finally, Proposition 7.1 describes the extension of P to $\mathbf{C}[[x_0, \dots, x_5]]$ as the intersection of 28 distinct smooth branch primes. This also gives a counterexample to the extended form of the Eisenbud–Mazur conjecture considered in [Mor15, Conjecture 1.5].

Declaration of generative AI: The counterexample was discovered by a ChatGPT model. The authors reviewed and edited the content as needed and take full responsibility for the content of the published article.

2. THE COEFFICIENT FIELD L

Throughout the paper, orders and homogeneous components refer to ordinary total degree in $x_0, \dots, x_5$. We write $H_d = \mathbf{Q}[x_0, \dots, x_5]_d$ for the space of homogeneous forms of degree d . Let

$$h(z) = z^8 - z - 1,$$

choose two distinct roots β, γ in an algebraic closure of $\mathbf{Q}$, and put

$$u = \beta\gamma, \quad v = -(\beta + \gamma), \quad L = \mathbf{Q}(u, v). \quad (2.1)$$

Lemma 2.1. *The extension $L/\mathbf{Q}$ has degree 28. Its embeddings into a splitting field of h are indexed by the unordered pairs of distinct roots of h .*

Proof. The polynomial h is irreducible modulo 7. Modulo 19 it factors as

$$h(z) = (z - 4)(z^7 + 4z^6 - 3z^5 + 7z^4 + 9z^3 - 2z^2 - 8z + 5), \quad (2.2)$$

where the degree-seven factor is irreducible. Both reductions are squarefree; exact certificates are given in Appendix A. Consequently the Galois group of h is transitive on its eight roots and contains a 7-cycle. The stabilizer of the root fixed by this cycle acts transitively on the other seven roots. Transitivity of the whole group therefore implies double transitivity, and hence transitivity on the 28 unordered pairs of distinct roots.

The pair (u, v) determines $\{\beta, \gamma\}$ as the roots of $z^2 + vz + u$. Its orbit thus has size 28, which is $[L : \mathbf{Q}]$. $\square$

For each root α of h , consider the line

$$\Lambda_\alpha : U + \alpha V + \alpha^2 = 0.$$

The lines corresponding to α and α' meet at $(\alpha\alpha', -\alpha - \alpha')$. No three are concurrent, since a monic quadratic cannot have three distinct roots.

Lemma 2.2. *Evaluation at (u, v) gives an isomorphism of $\mathbf{Q}$ -vector spaces*

$$\mathbf{Q}[U, V]_{\leq 6} \xrightarrow{\sim} L$$

and with:

$$\ell = (1, u, v, u^2, uv, v^2) \in L^6, \quad (2.3)$$

the evaluation map $H_d \rightarrow L$, $E \mapsto E(\ell)$, is surjective for every $d \geq 3$.

Proof. Suppose that $g \in \mathbf{Q}[U, V]_{\leq 6}$ vanishes at (u, v) . Lemma 2.1 implies that it vanishes at all 28 intersections. Each line Λ_α contains seven distinct intersections, so the restriction of g to that line is zero. Over the splitting field, the product of the eight distinct line equations therefore divides g . Since $\deg g \leq 6$, we obtain $g = 0$. Both vector spaces have dimension 28, proving the first assertion.

Every monomial $U^i V^j$ with $i + j \leq 6$ is a product of at most three elements of $\{U, V, U^2, UV, V^2\}$. Padding with powers of x_0 gives a homogeneous cubic restricting to this monomial under

$$(x_0, \dots, x_5) = (1, U, V, U^2, UV, V^2).$$

Multiplication by x_0^{d-3} proves the assertion for all $d \geq 3$. $\square$

We use the ordered basis

$$\mathcal{B} = (U^i V^{a-i} : 0 \leq a \leq 6, 0 \leq i \leq a), \quad (2.4)$$

where a and then i increase, and where the entries are evaluated at (u, v) when regarded as elements of L . For exact computations, one can instead use the relations

$$\begin{aligned} r_0 &= U^4 - 6U^3V^2 + 5U^2V^4 - UV^6 - 1, \\ r_1 &= 4U^3V - 10U^2V^3 + 6UV^5 - V^7 - 1. \end{aligned} \quad (2.5)$$

Indeed, division of $h(z)$ by $z^2 + Vz + U$ leaves remainder $r_0 + zr_1$. The quotient $\mathbf{Q}[U, V]/(r_0, r_1)$ then identifies with L ; the verification is given in Appendix A, by giving a Gröbner basis and proving that this quotient has dimension 28, and thus that its natural surjection to L is an isomorphism.

3. THE EXPLICIT FORMAL CURVE

Retain the field $L = \mathbf{Q}(u, v)$ from Lemma 2.1 and its ordered basis of monomials of degree at most six. Define the resultant:

$$H(U, V) = \text{Res}_z(z^8 - z - 1, z^2 + Vz + U). \quad (3.1)$$

Explicitly:

$$\begin{aligned} H(U, V) &= 1 + U - V - 2U^4 + 16U^3V^2 - 7U^4V + 14U^3V^3 \\ &\quad - 20U^2V^4 + 8UV^6 - 7U^2V^5 + U^8 + UV^7 - V^8. \end{aligned}$$

If α ranges over the eight roots of $z^8 - z - 1$ in a splitting field, then

$$H(U, V) = \prod_{\alpha} (U + \alpha V + \alpha^2).$$

We use the affine coordinates

$$(x_0, x_1, x_2, x_3, x_4, x_5) = (1, U, V, U^2 + A, UV + B, V^2 + C). \quad (3.2)$$

For a homogeneous polynomial G in the variables $x_0, \dots, x_5$, write G_{ch} for its substitution under (3.2). Direct expansion gives

$$D_{\text{ch}} = AC - B^2, \quad W_{\text{ch}}(U, V, 0, 0, 0) = H(U, V). \quad (3.3)$$

The three normal derivatives a, b, c of W_{ch} on $A = B = C = 0$, namely $a = \partial_A W_{\text{ch}}|_{A=B=C=0}$, and similarly for b and c , are the following polynomials:

$$\begin{aligned} a(U, V) &= 4U^6 - 14U^2V - 4U^2 + 14UV^3 + 16UV^2 - 7V^5 - 20V^4, \\ b(U, V) &= 14U^2V^2 + V^6, \\ c(U, V) &= 14U^3V + 16U^3 - 14U^2V^3 - 40U^2V^2 + 3UV^5 + 24UV^4 - 4V^6. \end{aligned} \quad (3.4)$$

Set

$$\begin{aligned} K &= ac - \frac{b^2}{4}, \\ V_0 &= -231U^6 + \frac{1169}{2}U^2V + 215U^2 - \frac{833}{2}UV^3 - 988UV^2 + \frac{1}{4}V^5 + 411V^4. \end{aligned} \quad (3.5)$$

Lemma 3.1. *At (u, v) one has*

$$H = H_U = H_V = 0, \quad V_0 - K = 0,$$

and $\text{Hess}(H)(u, v)$ is invertible.

Proof. The point $(u, v) = (\beta\gamma, -\beta - \gamma)$ is the intersection of the two lines indexed by β and γ . No third line passes through it. Let c_* be the product of the other six line equations, evaluated at this point. Then $c_* \neq 0$, and differentiating the product of the eight line equations gives

$$\det \text{Hess}(H)(u, v) = -c_*^2(\beta - \gamma)^2 \neq 0.$$

The same product description gives the three stated vanishings of H . Finally, expansion of $K = ac - b^2/4$ using (3.4) and then reduction by the two relations (2.5) gives

$$ac - \frac{b^2}{4} \equiv V_0 \pmod{(r_0, r_1)}.$$

The displayed reduction basis in the finite-certificate appendix makes this a rational polynomial identity; the supplementary verifier checks it independently. This shows that $V_0 - K$ vanishes at (u, v) . $\square$

All derivatives in the next definition are taken as polynomial derivatives in U, V before evaluation at (u, v) . Define $u_1, v_1 \in L$ by

$$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = -\text{Hess}(H)(u, v)^{-1} \nabla(V_0 - K)(u, v). \quad (3.6)$$

Put

$$\begin{aligned} \ell &= (1, u, v, u^2, uv, v^2), & w_0 &= 0, & w_1 &= u_1, & w_2 &= v_1, \\ w_3 &= 2uu_1 - c(u, v), & w_4 &= vu_1 + uv_1 + \frac{1}{2}b(u, v), \\ w_5 &= 2vv_1 - a(u, v). \end{aligned} \quad (3.7)$$

These formulas specify a continuous $\mathbf{Q}$ -algebra homomorphism from R to $L[[t]]$, i.e. an arc:

$$\varphi : R = \mathbf{Q}[[x_0, \dots, x_5]] \longrightarrow L[[t]], \quad x_i \longmapsto t\ell_i + t^2w_i, \quad P = \ker \varphi. \quad (3.8)$$

(Here ℓ_i is the i -th coordinate of ℓ .) In particular, $\varphi(x_0) = t$. Geometrically φ corresponds to a formal curve in the formal spectrum of R .

Lemma 3.2. *For every $d \geq 3$,*

$$\varphi(\mathfrak{m}^d) = t^d L[[t]]. \quad (3.9)$$

The image $A = \varphi(R)$ is a one-dimensional complete local domain, finite over $\mathbf{Q}[[t]]$, with residue field $\mathbf{Q}$. Its fraction field is $L((t))$, and its normalization is $L[[t]]$. Consequently, P is a height-five prime ideal and $P \subseteq \mathfrak{m}^2$.

Proof. By Lemma 2.2, choose homogeneous cubics $b_1, \dots, b_{28}$ such that $b_j(\ell)$ form a $\mathbf{Q}$ -basis of L . Relative to this basis, the elements $t^{-3}\varphi(b_j)$ form the columns of a matrix in $\text{Mat}_{28}(\mathbf{Q}[[t]])$ whose reduction modulo t is invertible. They therefore form a $\mathbf{Q}[[t]]$ -basis of $L[[t]]$. Since $\varphi(x_0) = t$, every element of $t^3L[[t]]$ lifts to an element of $\mathfrak{m}^3$. Multiplication by x_0^{d-3} proves the reverse inclusion in (3.9); the forward inclusion follows from $\varphi(x_i) \in tL[[t]]$.

The image A contains $\mathbf{Q}[[t]]$ and is a $\mathbf{Q}[[t]]$ -submodule of the finite free module $L[[t]]$. It is therefore finite over $\mathbf{Q}[[t]]$. It is a domain of dimension one. As a quotient of the complete local ring R , it is complete and local. Reduction modulo t takes A onto $\mathbf{Q}$; its kernel is the image of $\mathfrak{m}$, so its residue field is $\mathbf{Q}$. The inclusion $t^3L[[t]] \subseteq A$ implies that $\text{Frac}(A) = L((t))$.

The ring $L[[t]]$ is finite over A and integrally closed in this field, hence is the normalization of A .

It follows that $\text{ht } P = 6 - 1 = 5$. The six coordinates of ℓ are linearly independent over $\mathbf{Q}$ by Lemma 2.2. The coefficient of t in $\varphi(p)$ therefore excludes a nonzero linear initial form for any $p \in P$. The constant coefficient also vanishes, so $P \subseteq \mathfrak{m}^2$. $\square$

4. ELIMINATING CRITICAL VALUES TO ALL ORDERS

Consider the homogeneous cubic

$$\begin{aligned} V_0^\sharp = & -231x_3^3 + \frac{1169}{2}x_0x_2x_3 + 215x_0^2x_3 - \frac{833}{2}x_0x_4x_5 \\ & - 988x_0x_1x_5 + \frac{1}{4}x_2x_5^2 + 411x_0x_5^2. \end{aligned} \quad (4.1)$$

Its restriction to $A = B = C = 0$ under (3.2) is exactly $V_0(U, V)$. Set $E_5 = x_0^2V_0^\sharp$.

The following proposition is the technical core of the construction.

Proposition 4.1. *There exist homogeneous forms $E_n \in H_n$ for all $n \geq 6$ and a continuous $\mathbf{Q}$ -algebra homomorphism $\psi : R \rightarrow L[[t]]$, i.e. an arc, such that with:*

$$F := D + W + E_5 + \sum_{n \geq 6} E_n, \quad (4.2)$$

one has:

$$\psi(x_0) = t, \quad \psi(x_i) = t\ell_i + t^2w_i + O(t^3) \quad (1 \leq i \leq 5), \quad (4.3)$$

the critical condition:

$$\psi(\partial_i F) = 0 \quad (0 \leq i \leq 5), \quad (4.4)$$

and

$$\psi(F) = 0.$$

Proof. We first associate to every (tentative) $F = D + W + E_5 + O(\mathfrak{m}^6)$ an arc $\psi_F : R \rightarrow L[[t]]$ that satisfies the condition:

$$\psi_F(\partial_i F) = 0 \quad (1 \leq i \leq 5).$$

First, make the following substitution (the affine cone version of (3.2)):

$$x_0 = t, \quad (x_1, \dots, x_5) = t(U, V, U^2 + A, UV + B, V^2 + C)$$

into F , and then divide by t^3 ; the resulting formal series G_F is then:

$$G_F = AC - B^2 + tW_{\text{ch}} + t^2(E_5)_{\text{ch}} + O(t^3). \quad (4.5)$$

We work over $L[[t, U - u, V - v, A, B, C]]$ (and thus $G_F \in L[[t, U - u, V - v, A, B, C]]$). The Hessian of $AC - B^2$ in the three variables A, B, C is

invertible. The formal implicit function theorem then gives unique critical solutions in $tL[[t, U - u, V - v]]$ satisfying:

$$A_F = -tc + O(t^2), \quad B_F = \frac{1}{2}tb + O(t^2), \quad C_F = -ta + O(t^2). \quad (4.6)$$

Here a, b, c are the polynomials in U, V as before; the corresponding critical equations are:

$$\begin{aligned} \partial_A G_F(t, U, V, A_F, B_F, C_F) &= 0, \\ \partial_B G_F(t, U, V, A_F, B_F, C_F) &= 0, \\ \partial_C G_F(t, U, V, A_F, B_F, C_F) &= 0. \end{aligned}$$

Let $\tilde{G}_F \in tL[[t, U - u, V - v]]$ be the result of substituting these three solutions into G_F . A calculation through degree two in t gives

$$\tilde{G}_F(t, U, V) = tH(U, V) + t^2(V_0 - K)(U, V) + O(t^3). \quad (4.7)$$

Put $\Psi_F := t^{-1}\tilde{G}_F \in L[[t, U - u, V - v]]$. Lemma 3.1 and the formal implicit function theorem give unique series $U_F, V_F \in L[[t]]$ with constant terms u, v satisfying the critical equations:

$$\partial_U \Psi_F(t, U_F, V_F) = \partial_V \Psi_F(t, U_F, V_F) = 0.$$

Their first terms are

$$U_F = u + tu_1 + O(t^2), \quad V_F = v + tv_1 + O(t^2), \quad (4.8)$$

by (3.6).

For abbreviations we write:

$$\begin{aligned} \tilde{A}_F &= A_F(t, U_F, V_F), \\ \tilde{B}_F &= B_F(t, U_F, V_F), \\ \tilde{C}_F &= C_F(t, U_F, V_F), \end{aligned}$$

for the elements of $L[[t]]$ obtained by substituting U_F, V_F into the variables U, V .

Substituting $U_F, V_F, \tilde{A}_F, \tilde{B}_F, \tilde{C}_F \in L[[t]]$ into the variables U, V, A, B, C , we then have the formulas

$$x_0 = t, \quad (x_1, \dots, x_5) = t(U_F, V_F, U_F^2 + \tilde{A}_F, U_F V_F + \tilde{B}_F, V_F^2 + \tilde{C}_F)$$

which then defines an arc $\psi_F : R \rightarrow L[[t]]$.

Now (in the affine chart) the change of coordinate map

$$(U, V, A, B, C) \mapsto (U, V, U^2 + A, UV + B, V^2 + C)$$

has determinant-one Jacobian. The five critical equations that the five critical solutions A_F, B_F, C_F, U_F, V_F satisfy, thus give:

$$\psi_F(\partial_i F) = 0 \quad (1 \leq i \leq 5). \quad (4.9)$$

Note also that equations (4.6) and (4.8) give precisely the coefficients w_i in (3.7). Thus every such ψ_F satisfies (4.3).

Now define the critical value $\mathcal{V}(F)$ of ψ_F by:

$$\mathcal{V}(F) = \psi_F(F) \in L[[t]].$$

We thus have by definition:

$$\mathcal{V}(F) = t^3 \tilde{G}_F(t, U_F, V_F).$$

The vanishings in Lemma 3.1 give:

$$H(U_F, V_F) = O(t^2), \quad (V_0 - K)(U_F, V_F) = O(t)$$

and hence (4.7) then implies

$$\mathcal{V}(F) \in t^6 L[[t]]. \quad (4.10)$$

Next we claim, for a homogeneous $E \in H_n$, $n \geq 6$, that

$$\mathcal{V}(F + E) - \mathcal{V}(F) \equiv t^n E(\ell) \pmod{t^{n+1}}. \quad (4.11)$$

For this, introduce a parameter s and carry out the two recursive implicit constructions for $F + sE$. Each coefficient of the resulting arc lies in $L[s]$: the constant matrices to be inverted are unchanged, and all subsequent recursion uses polynomial operations and these fixed inverse matrices. Thus the chain rule applies coefficientwise in $L[s][[t]]$. Since $x_0 = t$ is fixed and the five spatial derivatives vanish, it gives

$$\frac{d}{ds} \mathcal{V}(F + sE) = \psi_{F+sE}(E) \equiv t^n E(\ell) \pmod{t^{n+1}}.$$

Integrating from $s = 0$ to $s = 1$ proves (4.11). Note that this is also a crucial point that makes use of the fact that the coefficient field has characteristic zero.

Now start with $F^{[5]} = D + W + E_5$. Inductively suppose that $F^{[n-1]}$ has been chosen and that its critical value has no term of degree less than n . Let $c_n \in L$ be the coefficient of t^n . By Lemma 2.2, choose $E_n \in H_n$ with $E_n(\ell) = -c_n$. Equation (4.11) shows that $F^{[n]} = F^{[n-1]} + E_n$ kills the coefficient of t^n while preserving all previous vanishings.

Now adding a form of degree n changes G_F and its normal equations by $O(t^{n-3})$. The fixed invertible normal Hessian then shows that the normal critical solutions, as functions of t, U, V , change by $O(t^{n-3})$. After reduction and division by t , the tangential equations change by $O(t^{n-4})$. Their fixed invertible constant Hessian shows that U_F, V_F change by $O(t^{n-4})$. Since the normal solutions and their U, V derivatives are divisible by t , their evaluations on the final tangential arc change by $O(t^{n-3})$. The original coordinates $x_1, \dots, x_5$ therefore change by $O(t^{n-3})$. These orders tend to infinity with n .

It follows that $F^{[n]}$ converges $\mathfrak{m}$ -adically to the series F with the shape in (4.2), and the arcs converge coefficientwise to an arc $\psi = \psi_F$ satisfying (4.3). Substitution and formal differentiation are continuous for these filtrations, so the induction and (4.9) give

$$\psi(F) = 0, \quad \psi(\partial_i F) = 0 \quad (1 \leq i \leq 5).$$

Finally to complete the proof of (4.4), it remains to show that $\psi(\partial_0 F) = 0$. Differentiate $\psi(F) = 0$ with respect to t . The chain rule and $\psi(x_0) = t$ give

$$0 = \psi(\partial_0 F) + \sum_{i=1}^5 \psi(\partial_i F) \frac{d\psi(x_i)}{dt} = \psi(\partial_0 F),$$

which thus completes the proof of (4.4). $\square$

5. STRAIGHTENING AND SYMBOLIC-SQUARE MEMBERSHIP

Lemma 5.1. *Let $\psi : R \rightarrow L[[t]]$ be any arc that satisfies (4.3). There is a continuous $\mathbf{Q}$ -algebra automorphism θ of R such that*

$$\theta(x_0) = x_0, \quad \theta(x_i) - x_i \in \mathfrak{m}^3 \quad (1 \leq i \leq 5), \quad \varphi \circ \theta = \psi.$$

Proof. Each difference $\psi(x_i) - \varphi(x_i)$ belongs to $t^3 L[[t]]$. By (3.9), choose $g_i \in \mathfrak{m}^3$ whose image under φ is this difference. Define $\theta(x_i) = x_i + g_i$ for $1 \leq i \leq 5$ and $\theta(x_0) = x_0$. This substitution has identity linear part, so recursive solution of its homogeneous terms gives a continuous inverse. Its composite with φ agrees with ψ on the six variables and hence on R by continuity. $\square$

Apply this lemma to the arc of Proposition 4.1, and put

$$f = \theta(F). \tag{5.1}$$

Changing variables by terms of order at least three changes a cubic only in degrees at least five. Therefore

$$f \equiv D + W \pmod{\mathfrak{m}^5}. \tag{5.2}$$

The chain rule and $\varphi \circ \theta = \psi$ imply

$$\varphi(f) = 0, \quad \varphi(\partial_i f) = 0 \quad (0 \leq i \leq 5). \tag{5.3}$$

Proposition 5.2. *The series f in (5.1) belongs to $P^{(2)}$. More generally, if $g \in R$ satisfies $\varphi(g) = \varphi(\partial_i g) = 0$ for $1 \leq i \leq 5$, then $g \in P^{(2)}$.*

Proof. Let $S = L[[x_0, \dots, x_5]]$ and set

$$Q_{\text{arc}} = (x_i - x_0 \ell_i - x_0^2 w_i : 1 \leq i \leq 5)S.$$

The same formulas that define $\varphi : R \rightarrow L[[t]]$ as before then give a surjective homomorphism $S \rightarrow L[[t]]$ with kernel Q_{arc} , and $Q_{\text{arc}} \cap R = P$. Use x_0 and $y_i := x_i - x_0 \ell_i - x_0^2 w_i$ ($1 \leq i \leq 5$) as formal coordinates on S . In terms of these new coordinates, we have $S = L[[x_0]][[y_1, \dots, y_5]]$, and then Q_{arc} is the ideal of S generated by y_i for $1 \leq i \leq 5$, and hence Q_{arc}^2 is the ideal of S generated by $y_i y_j$ for $1 \leq i, j \leq 5$. One then sees immediately, by using Taylor expansion for example (with respect to the original coordinates $x_1, \dots, x_5$, with x_0 being regarded as fixed), that if $g \in R$ satisfies $\varphi(g) = \varphi(\partial_i g) = 0$ for $1 \leq i \leq 5$, then $g \in Q_{\text{arc}}^2$.

Since $L/\mathbf{Q}$ is finite and separable, $S = R \otimes_{\mathbf{Q}} L$ is finite *étale* over R . The local map $R_P \rightarrow S_{Q_{\text{arc}}}$ is faithfully flat. Its closed fiber is a localization of a finite *étale* algebra over $\kappa(P)$, hence a field. Consequently,

$$PS_{Q_{\text{arc}}} = Q_{\text{arc}}S_{Q_{\text{arc}}}.$$

Faithful flatness therefore gives

$$g \in P^2S_{Q_{\text{arc}}} \cap R_P = P^2R_P.$$

Intersecting with R proves $g \in P^{(2)}$. Finally the particular series f satisfies the hypotheses by (5.3). $\square$

6. A FINITE CERTIFICATE FOR NONMEMBERSHIP

Put

$$T = \mathbf{Q}[[x_1, \dots, x_5]], \quad \mathfrak{q} = (x_1, \dots, x_5)T, \quad J = (P + (x_0))/(x_0) \subset T.$$

A bar denotes the image under the substitution $x_0 = 0$ (here we identify $R/(x_0)$ with T). We shall construct a $\mathbf{Q}$ -linear functional $\lambda : T \rightarrow \mathbf{Q}$ such that

$$\lambda(\mathfrak{q}J) = 0, \quad \lambda(\mathfrak{q}^5) = 0, \quad \lambda(\overline{D} + \overline{W}) = 1. \quad (6.1)$$

Together with (5.2), these identities prove the required nonmembership in $\mathfrak{m}P$.

Consider the six quadrics:

$$\begin{aligned} q_1 &= x_0x_3 - x_1^2, & q_2 &= x_0x_4 - x_1x_2, & q_3 &= x_0x_5 - x_2^2, \\ q_4 &= x_1x_4 - x_2x_3, & q_5 &= x_1x_5 - x_2x_4, & q_6 &= x_3x_5 - x_4^2 \end{aligned} \quad (6.2)$$

with their accompanying homogeneous cubic terms given by:

$$\begin{aligned} C_1 &= 16x_0x_1x_3 + 14x_0x_3x_4 - 40x_0x_3x_5 + 24x_1x_5^2 \\ &\quad - 14x_2x_3x_5 + 3x_4x_5^2 - 4x_5^3, \\ C_2 &= -7x_0x_3x_5 - \frac{1}{2}x_5^3, \\ C_3 &= -4x_0^2x_3 + 16x_0x_1x_5 - 14x_0x_2x_3 + 14x_0x_4x_5 \\ &\quad - 20x_0x_5^2 - 7x_2x_5^2 + 4x_3^3, \\ C_4 &= -\frac{1}{2}x_0^3 - \frac{7}{2}x_0x_3^2 - \frac{7}{2}x_3x_5^2, \\ C_5 &= -\frac{1}{2}x_0^3 - 8x_0x_3x_5 - 4x_5^3, \\ C_6 &= 4x_0^3 + 3x_0^2x_1 - 3x_0^2x_2 - 4x_0x_3^2 + 16x_1x_3x_5 \\ &\quad - 7x_2x_3^2 + 14x_3x_4x_5 - 20x_3x_5^2. \end{aligned} \quad (6.3)$$

Lemma 6.1. *The image of P in $R/\mathfrak{m}^4$ is the rational span of the classes of*

$$q_i + C_i \quad (1 \leq i \leq 6), \quad x_jq_i \quad (0 \leq j \leq 5, 1 \leq i \leq 6). \quad (6.4)$$

Proof. Let ρ denote substitution of $(1, U, V, U^2, UV, V^2)$ into homogeneous forms. The degree-two kernel of ρ is spanned by the six q_i , and its degree-three kernel is spanned by their linear multiples. These statements can be seen by homogenizing the substitution to the map

$$(x_0, x_1, x_2, x_3, x_4, x_5) \mapsto (s^2, sr, ss', r^2, rs', s'^2).$$

The six quadrics in (6.2) generate all relations among these degree-two monomials: any two products representing the same monomial in s, r, s' are related by changing the pairing of four factors. The quotient dimensions in degrees two and three are therefore 15 and 28, respectively. Since $\rho(H_d)$ consists of polynomials of degree at most $2d$, Lemma 2.2 shows that evaluation at ℓ has these same kernels for $d = 2, 3$.

By the definition of φ as given in (3.8), the coefficient of t^3 in $\varphi(q_i)$ is $\sum_{j=0}^5 \partial_j q_i(\ell) w_j$. The negatives of these six coefficients, in order, are

$$c, \quad -\frac{1}{2}b, \quad a, \quad -(\frac{1}{2}ub + vc), \quad ua + \frac{1}{2}vb, \quad u^2a + uvb + v^2c, \quad (6.5)$$

where a, b, c are evaluated at (u, v) . The terms containing u_1, v_1 cancel. Substitution into the displayed C_i , followed by reduction using (2.5), gives exactly the values in (6.5). Consequently,

$$\varphi(q_i + C_i) \in t^4 L[[t]] \quad (1 \leq i \leq 6). \quad (6.6)$$

For $p \in P$, Lemma 3.2 gives $p = p_2 + p_3 + O(\mathfrak{m}^4)$ with $p_d \in H_d$. The coefficient of t^2 in $\varphi(p)$ gives $p_2 = \sum_i \alpha_i q_i$ for some $\alpha_i \in \mathbf{Q}$. By (6.6), the cubic $p_3 - \sum_i \alpha_i C_i$ evaluates to zero at ℓ . It is therefore a linear combination of the $x_j q_i$, proving one inclusion.

Conversely, every expression in (6.4) maps into $t^4 L[[t]]$. Lemma 3.2 gives $\varphi(\mathfrak{m}^4) = t^4 L[[t]]$, so its image can be canceled by an element of $\mathfrak{m}^4$. Its class modulo $\mathfrak{m}^4$ thus belongs to the image of P . $\square$

Now define the functional λ on monomials by Table 1, and set it to zero on every monomial not listed. It extends to all of T by taking this finite linear combination of coefficients; in particular, $\lambda(\mathfrak{q}^5) = 0$.

Proposition 6.2. *The functional λ satisfies (6.1). Every series $f \equiv D + W \pmod{\mathfrak{m}^5}$ consequently lies outside $\mathfrak{m}P$.*

Proof. The displayed coefficients give the rational identities

$$\lambda(\overline{x_j(q_i + C_i)}) = 0 \quad (1 \leq j \leq 5, 1 \leq i \leq 6), \quad (6.7)$$

$$\lambda(\overline{x_j x_k q_i}) = 0 \quad (1 \leq j \leq 5, 0 \leq k \leq 5, 1 \leq i \leq 6). \quad (6.8)$$

These are $30 + 180$ finite identities; all are checked by the exact-arithmetic verifier included with the supplementary material. Note that of course for (6.8) it is trivial when $k = 0$, so that the number of non-trivial identities in (6.8) is 150, and the total number of non-trivial identities that have to be checked in (6.7)–(6.8) is 180.

Note also how the dimension count works here: the dimension of the space of cubics in five variables is 56, and the dimension of the space of quartics in

TABLE 1. All nonzero values of λ . An exponent tuple $e = (e_1, \dots, e_5)$ denotes $x_1^{e_1} \cdots x_5^{e_5}$.

Degree-three tuple	Value	Degree-four tuple	Value
(0, 0, 0, 2, 1)	$-5632/49$	(0, 0, 0, 0, 4)	3
(0, 0, 0, 3, 0)	$-10240/343$	(0, 0, 0, 1, 3)	8
(0, 1, 0, 1, 1)	-12	(0, 0, 0, 2, 2)	$384/49$
(0, 1, 0, 2, 0)	-32	(0, 0, 0, 3, 1)	$512/343$
(0, 1, 1, 0, 1)	$-192/7$	(0, 0, 1, 0, 3)	$384/49$
(0, 1, 1, 1, 0)	$-1536/49$	(0, 0, 1, 1, 2)	$512/343$
(0, 2, 0, 1, 0)	$-3/2$	(0, 0, 3, 1, 0)	$-3/8$
(0, 2, 1, 0, 0)	-8	(0, 1, 0, 2, 1)	$8/7$
(1, 0, 0, 2, 0)	$-1280/49$	(0, 1, 0, 3, 0)	$64/49$
(1, 1, 0, 0, 1)	$-3/2$	(0, 1, 1, 0, 2)	$8/7$
(1, 1, 0, 1, 0)	-4	(0, 1, 1, 1, 1)	$64/49$
(1, 1, 1, 0, 0)	$-192/49$	(1, 0, 0, 1, 2)	$8/7$
(2, 0, 0, 0, 1)	-4	(1, 0, 0, 2, 1)	$64/49$
(2, 0, 0, 1, 0)	$32/49$	(1, 0, 1, 0, 2)	$64/49$
(2, 0, 1, 0, 0)	$1536/343$		
(3, 0, 0, 0, 0)	$24/7$		

five variables is 126; as $(56 + 126) - 180 = 2$, so this leaves two degrees of freedom to define the functional λ satisfying (6.7)–(6.8) (these two remaining degrees of freedom are used to make $\lambda(\overline{D}) = 4$ and $\lambda(\overline{W}) = -3$ as shown below; incidentally this also explains why λ is defined this way).

For $g \in \mathfrak{q}J$, choose $p_j \in P$ such that $g = \sum_{j=1}^5 x_j \overline{p}_j$. The terms of g through degree four depend only on the classes of the p_j modulo $\mathfrak{m}^4$. Lemma 6.1 and (6.7)–(6.8) therefore give $\lambda(g) = 0$. This proves vanishing of λ on the whole ideal $\mathfrak{q}J$.

Finally, (1.2) and (1.3) give

$$\overline{D} = 2x_1x_2x_4 - x_1^2x_5 - x_2^2x_3$$

and

$$\overline{W} = 8x_1x_5^3 - 7x_2x_3x_5^2 + x_3^4 + x_4x_5^3 - x_5^4.$$

Table 1 gives $\lambda(\overline{D}) = 4$ and $\lambda(\overline{W}) = -3$. Thus $\lambda(\overline{f}) = 1$ whenever $f \equiv D + W \pmod{\mathfrak{m}^5}$. If $f \in \mathfrak{m}P$, then $\overline{f} \in \mathfrak{q}J$, a contradiction. $\square$

Proof of Theorem 1.1. The field L and $\varphi : R \rightarrow L[[t]]$ are specified in (2.1) and (3.6)–(3.8). Lemma 3.2 proves the assertions about the height, residue field, and normalization. Proposition 4.1 and Lemma 5.1 construct a series f given in equation (5.1), with $f \equiv D + W \pmod{\mathfrak{m}^5}$. Proposition 5.2 gives $f \in P^{(2)}$, and Proposition 6.2 gives $f \notin \mathfrak{m}P$. $\square$

7. EXTENSION OF THE COEFFICIENT FIELD

For an ideal I in a Noetherian ring, recall the symbolic square:

$$I^{(2)} = \bigcap_{\mathfrak{q} \in \text{Min}(I)} (I^2 R_{\mathfrak{q}} \cap R).$$

The prime ideal P of R in the previous constructions acquires a particularly simple branch description after extension of the coefficient field.

Proposition 7.1. *Let $R_{\mathbf{C}} = \mathbf{C}[[x_0, \dots, x_5]]$, let $\mathfrak{m}_{\mathbf{C}} = (x_0, \dots, x_5)R_{\mathbf{C}}$, and put $I = PR_{\mathbf{C}}$. Then I is an unmixed radical ideal, and we have the primary decomposition:*

$$I = \bigcap_{\sigma: L \hookrightarrow \mathbf{C}} (x_i - x_0 \sigma(\ell_i) - x_0^2 \sigma(w_i) : 1 \leq i \leq 5) R_{\mathbf{C}}. \quad (7.1)$$

These 28 ideals are the distinct smooth branch primes of I , and the series f of equation (5.1) satisfies

$$f \in I^{(2)} \setminus \mathfrak{m}_{\mathbf{C}} I.$$

Proof. By Lemma 3.2, the inclusion $A \subset L[[t]]$ is an inclusion of finite $\mathbf{Q}[[t]]$ -modules. The maximal-ideal topology on A agrees with its t -adic topology: the quotient A/tA is a finite-dimensional local $\mathbf{Q}$ -algebra. Completed extension of the coefficient field consequently gives

$$R_{\mathbf{C}}/PR_{\mathbf{C}} \cong A \otimes_{\mathbf{Q}[[t]]} \mathbf{C}[[t]]. \quad (7.2)$$

For completeness, this identification can be seen by reducing the finite $\mathbf{Q}[[t]]$ -module A modulo t^n , extending coefficients to $\mathbf{C}$, and taking the inverse limit. Finiteness identifies that limit with the tensor product in (7.2); the equivalence of topologies identifies it with the completed presentation $\mathbf{C}[[x]]/PC[[x]]$.

The extension $\mathbf{Q}[[t]] \rightarrow \mathbf{C}[[t]]$ is flat, since its target is torsion-free over the discrete valuation ring $\mathbf{Q}[[t]]$. It follows that

$$R_{\mathbf{C}}/I \hookrightarrow \mathbf{C}[[t]] \otimes_{\mathbf{Q}[[t]]} L[[t]] \cong \prod_{\sigma: L \hookrightarrow \mathbf{C}} \mathbf{C}[[t]].$$

The image contains t^3 times the product on the right. The coordinate maps have the kernels displayed in (7.1); their intersection is therefore I . Each quotient is $\mathbf{C}[[t]]$, and distinct embeddings give distinct pairs $(\sigma(u), \sigma(v))$, hence distinct tangent directions. Thus these are precisely the minimal primes of I .

The identities $\varphi(f) = \varphi(\partial_i f) = 0$ persist on every branch. Taylor expansion in the five transverse branch coordinates therefore places f in the square of each branch ideal, and hence in $I^{(2)}$. Extend the functional of Table 1 $\mathbf{C}$ -linearly on fourth jets. The identities in the proof of Proposition 6.2 remain valid under this extension. The extended functional annihilates the image of $\mathfrak{m}_{\mathbf{C}} I$ modulo x_0 and takes value 1 on f , proving the nonmembership. $\square$

Corollary 7.2. *There is a formal power series $f \in \mathbf{C}[[x_0, \dots, x_5]]$ without constant term such that*

$$f \notin \mathfrak{m}_{\mathbf{C}} \sqrt{(f, \partial_0 f, \dots, \partial_5 f)}.$$

Proof. Take the series given by equation (5.1). The radical ideal I of Proposition 7.1 contains f and all its partial derivatives, so the displayed radical is contained in I . The assertion follows from $f \notin \mathfrak{m}_{\mathbf{C}} I$. $\square$

This is the formal-power-series problem in [EM97, Introduction and Corollary 2]. Proposition 7.1 then also gives a counterexample to the general form of the Eisenbud–Mazur conjecture stated in [Mor15, Conjecture 1.5] (with $\mathbf{C}$ being the coefficient field).

8. A COUNTEREXAMPLE IN A LOCALIZED POLYNOMIAL RING

The formulas that define $\varphi : R \rightarrow L[[t]]$ also define an algebraic curve. The construction of f uses a complete formal power series, but by using flatness and a differential characterization of the symbolic square, we are able to construct a counterexample to the original Eisenbud–Mazur conjecture in a localized polynomial ring.

Let

$$S_0 = \mathbf{Q}[x_0, \dots, x_5]_{\mathfrak{n}}, \quad \mathfrak{n} = (x_0, \dots, x_5),$$

where the same notation $\mathfrak{n}$ denotes the maximal ideal after localization. Let $\mathfrak{p}_0$ be the kernel of the polynomial homomorphism

$$\mathbf{Q}[x_0, \dots, x_5] \longrightarrow L[t], \quad x_i \longmapsto t\ell_i + t^2 w_i,$$

and put $\mathfrak{p} = \mathfrak{p}_0 S_0$.

Theorem 8.1. *The ideal $\mathfrak{p}$ is a height-five prime of S_0 . Its extension to the completion $R = \widehat{S_0}$ is P , and*

$$\mathfrak{p}^{(2)} R = P^{(2)}. \tag{8.1}$$

There exists a polynomial $g \in \mathbf{Q}[x_0, \dots, x_5]$ such that

$$g \equiv D + W \pmod{(x_0, \dots, x_5)^5}, \quad g \in \mathfrak{p}^{(2)} \setminus \mathfrak{np}. \tag{8.2}$$

In particular, the counterexample exists in a regular local ring essentially of finite type over $\mathbf{Q}$.

Proof. We first identify the completed prime. Write

$$B = \mathbf{Q}[t, t\ell_1 + t^2 w_1, \dots, t\ell_5 + t^2 w_5] \subseteq L[t].$$

The ring B contains $\mathbf{Q}[t]$ and is a $\mathbf{Q}[t]$ -submodule of the finite free module $L[t]$. It is therefore finite over $\mathbf{Q}[t]$. The fiber B/tB has only one prime ideal, and its residue field is $\mathbf{Q}$. Indeed, each generator $b_i = t\ell_i + t^2 w_i = ta_i(t)$, with $a_i(t) \in L[t]$, satisfies a monic equation

$$b_i^{28} + tc_1(t)b_i^{27} + \dots + t^{28}c_{28}(t) = 0, \quad c_j(t) \in \mathbf{Q}[t],$$

obtained from the characteristic polynomial of multiplication by $a_i(t)$ on $L[t]$. Thus each b_i is nilpotent modulo tB . The quotient by these generators is $\mathbf{Q}$, proving the assertion about the fiber.

Let $\mathfrak{b} = (t, b_1, \dots, b_5)B$. It follows that $B \otimes_{\mathbf{Q}[t]} \mathbf{Q}[t]_{(t)}$ is local and equals $B_{\mathfrak{b}}$. The map from S_0 has image $B_{\mathfrak{b}}$ and kernel $\mathfrak{p}$. In particular, $S_0/\mathfrak{p}$ is a one-dimensional domain, and $\text{ht } \mathfrak{p} = 5$. The $\mathfrak{b}$ -adic and t -adic topologies on this finite local algebra are equivalent. Hence

$$\widehat{S_0/\mathfrak{p}} = B_{\mathfrak{b}} \otimes_{\mathbf{Q}[t]_{(t)}} \mathbf{Q}[[t]].$$

Flatness of $\mathbf{Q}[[t]]$ over $\mathbf{Q}[t]_{(t)}$ embeds this ring in $L[[t]]$. On the other hand, completion of the quotient presentation identifies it with $R/\mathfrak{p}R$. The resulting map $R \rightarrow L[[t]]$ is precisely φ , since it has the same values on all six variables. Thus

$$\mathfrak{p}R = P. \quad (8.3)$$

We next justify (8.1) by an explicit finite-module kernel. The map

$$\delta : \mathfrak{p} \longrightarrow (S_0/\mathfrak{p})^6, \quad h \longmapsto (\partial_0 h, \dots, \partial_5 h) \bmod \mathfrak{p}$$

is S_0 -linear by the product rule, and

$$\ker \delta = \mathfrak{p}^{(2)}. \quad (8.4)$$

This is the differential description of the symbolic square in characteristic zero; in the present setting it also has the following direct proof. If $h \in \mathfrak{p}^{(2)}$, choose $s \in S_0 \setminus \mathfrak{p}$ with $sh \in \mathfrak{p}^2$. Differentiation and reduction modulo $\mathfrak{p}$ show $s\partial_i h \in \mathfrak{p}$, hence $\partial_i h \in \mathfrak{p}$ for every i .

Conversely, suppose that $h \in \mathfrak{p}$ and all its partial derivatives belong to $\mathfrak{p}$. Write $h = a/b$ with $a, b \in \mathbf{Q}[x_0, \dots, x_5]$ and $b \notin \mathfrak{n}$. Then a and all its partial derivatives vanish under the polynomial parametrization into $L[t]$. The polynomial change of variables

$$y_i = x_i - x_0 \ell_i - x_0^2 w_i \quad (1 \leq i \leq 5)$$

therefore shows that a belongs to $(y_1, \dots, y_5)^2$ in $L[x_0, \dots, x_5]$: both its constant term and all its linear coefficients in the y_i vanish. The extension $S_0 \rightarrow S_0 \otimes_{\mathbf{Q}} L$ is finite *étale*. The ideal generated by the y_i is a prime contracting to $\mathfrak{p}$. Localizing at that prime gives a faithfully flat local extension of $(S_0)_{\mathfrak{p}}$ with closed fiber a field. As in Proposition 5.2, contraction of its squared maximal ideal gives $h \in \mathfrak{p}^2(S_0)_{\mathfrak{p}}$. This proves (8.4).

Since completion is flat, tensoring the kernel description with R and using (8.3) gives

$$\mathfrak{p}^{(2)}R = \ker(P \longrightarrow (R/P)^6, \quad h \longmapsto (\partial_i h \bmod P)_{i=0}^5). \quad (8.5)$$

Here the tensor-extended map is indeed formal differentiation modulo P : if $h = \sum_j a_j p_j$ with $p_j \in \mathfrak{p}$ and $a_j \in R$, the terms $p_j \partial_i a_j$ vanish modulo P . Proposition 5.2 identifies the kernel on the right with a submodule of $P^{(2)}$. The reverse inclusion follows from the same product-rule argument applied to $sh \in P^2$ with $s \notin P$. This proves (8.1).

Choose polynomials $g_1, \dots, g_r$ in $\mathbf{Q}[x_0, \dots, x_5]$ that generate $\mathfrak{p}^{(2)}$ as an S_0 -ideal. Such generators are obtained from any finite generating set by clearing denominators that are units in S_0 . By (8.1), write the formal witness as

$$f = \sum_{j=1}^r a_j g_j, \quad a_j \in R.$$

Choose polynomials a_j^b with $a_j - a_j^b \in \mathfrak{m}^5$, and set $g = \sum_j a_j^b g_j$. Then $g \in \mathfrak{p}^{(2)}$ and $g - f \in \mathfrak{m}^5$. In particular, $g \equiv D + W \pmod{\mathfrak{m}^5}$. Proposition 6.2 gives $g \notin \mathfrak{m}P$. If g belonged to $\mathfrak{np}$, its extension to R would belong to $\mathfrak{m}P$, a contradiction. This proves (8.2). $\square$

Remark 8.2. Theorem 8.1 proves the existence of a polynomial witness g with the displayed fourth jet. Its proof approximates the coefficients of a representation of f in generators of the algebraic symbolic square. It does not identify a degree bound, display the higher coefficients of g , or assert that a finite truncation of the particular formal series f is a witness.

APPENDIX A. ELEMENTARY FINITE CERTIFICATES

A.1. The two finite-field irreducibility checks. Let $\bar{h} = z^8 - z - 1 \in \mathbf{F}_7[z]$. Repeated modular powering gives

$$z^{7^8} \equiv z \pmod{\bar{h}}, \quad z^{7^4} - z \equiv a_7 \pmod{\bar{h}},$$

where

$$a_7 = z^7 - 3z^6 + 2z^5 + z^4 - 3z^3 + 2z^2.$$

A Bézout certificate is

$$A_7 a_7 + B_7 \bar{h} = 1 \quad \text{in } \mathbf{F}_7[z],$$

with

$$\begin{aligned} A_7 &= 3z^7 + 3z^5 + 3z^4 - z^3 + 2z^2 + z, \\ B_7 &= -3z^6 + 2z^5 - 2z^4 + 3z^3 - z^2 + z - 1. \end{aligned}$$

The finite-field irreducibility criterion proves that $\bar{h}$ is irreducible: the only prime divisor of 8 is 2.

In $\mathbf{F}_{19}[z]$, put

$$q_{19} = z^7 + 4z^6 - 3z^5 + 7z^4 + 9z^3 - 2z^2 - 8z + 5.$$

Then

$$\bar{h} = (z - 4)q_{19}, \quad z^{19^7} \equiv z \pmod{q_{19}},$$

and

$$z^{19} - z \equiv a_{19} = z^5 + 2z^4 + z^3 - z \pmod{q_{19}}.$$

Here $A_{19}a_{19} + B_{19}q_{19} = 1$, where

$$\begin{aligned} A_{19} &= 4z^6 + 2z^5 - 7z^4 + 4z^3 + 5z^2 - 4z + 9, \\ B_{19} &= -4z^4 + 6z^3 + z^2 - 7z + 4. \end{aligned}$$

Since 7 is prime, these identities prove that q_{19} is irreducible. Moreover, $q_{19}(4) = 9$ in $\mathbf{F}_{19}$, so the displayed factorization of h is squarefree. All powers in these checks are computed by repeated powering with reduction after each step.

A.2. Rational reduction in the coefficient field. For graded reverse lexicographic order with $U > V$, a Gröbner basis of the ideal (r_0, r_1) from (2.5) is

$$\begin{aligned}
g_0 &= U^7 - 3U^3V - U^3 + 4U^2V^3 + 6U^2V^2 - UV^5 - 5UV^4 + V^6, \\
g_1 &= U^6V - U^3 + 3U^2V^2 + 3U^2V - UV^4 - 4UV^3 + V^5, \\
g_2 &= U^5V^2 - U^6 + 2U^2V + U^2 - UV^3 - 3UV^2 + V^4, \\
g_3 &= U^4V^3 - 2U^5V + U^2 - UV^2 - 2UV + V^3, \\
g_4 &= U^3V^4 + U^5 - 3U^4V^2 - UV - U + V^2, \\
g_5 &= U^2V^5 + 3U^4V - 4U^3V^3 - U + V, \\
g_6 &= UV^6 - U^4 + 6U^3V^2 - 5U^2V^4 + 1, \\
g_7 &= V^7 - 4U^3V + 10U^2V^3 - 6UV^5 + 1.
\end{aligned} \tag{A.1}$$

Their leading monomials are precisely the monomials of total degree seven. Thus $\mathbf{Q}[U, V]/(r_0, r_1)$ has dimension 28, and its natural surjection onto the degree-28 field L is an isomorphism. This also gives the normal form used for the identities involving $V_0 - K$ and the C_i .

A.3. Reproduction of the finite checks. The supplementary Python program uses exact rational arithmetic and finite-field arithmetic to check the displayed certificates. It checks all 28 S-polynomials of (A.1), verifies equality of the ideal they generate with (r_0, r_1) , checks (6.5) and all 210 identities (6.7)–(6.8), and verifies the quadratic and cubic Veronese relation-space dimensions. It also checks the normal derivative formulas, the congruence $V_0 \equiv K$, and the stated homogeneous lift of V_0 .

A separate Singular input in the supplementary material reconstructs the resultant and checks its equality with the restriction of W , together with the normal derivative identities and the nonvanishing of the tangential Hessian determinant in L . These finite calculations use no floating-point arithmetic and do not replace the all-orders argument in the body of the paper.

APPENDIX B. GRADIENT INTERPOLATION IN HIGHER DEGREES

The proof of Theorem 1.1 does not require the calculations in this appendix. The purpose here is to describe a direct continuation procedure on the fixed quadratic parametrization and provide additional finite checks of the interpolation mechanism.

For $d \geq 1$, define the $\mathbf{Q}$ -linear map

$$J_d : H_d \longrightarrow L^6, \quad E \longmapsto (\partial_0 E(\ell), \dots, \partial_5 E(\ell)).$$

Using the ordered basis (2.4) in each of the six coordinate blocks, the target has dimension 168 over $\mathbf{Q}$. Exact rational row reduction gives

d	5	6	7
$\dim_{\mathbf{Q}} H_d$	252	462	792
$\text{rank}_{\mathbf{Q}} J_d$	144	160	168

Here monomial columns are ordered lexicographically by their six-tuples of exponents, starting with the smallest tuple. Each matrix entry is obtained by differentiating a monomial, substituting ℓ , and reducing by the Gröbner basis in Appendix A.

Lemma B.1. *If J_7 is surjective, then J_d is surjective for every $d \geq 7$.*

Proof. For $G \in H_7$, Euler's identity and $\ell_0 = 1$ give

$$J_d(x_0^{d-7}G) = T_d J_7(G), \quad T_d(a) = a + \frac{d-7}{7} \left(\sum_{j=0}^5 \ell_j a_j \right) e_0.$$

The L -linear map T_d has determinant $d/7 \neq 0$, so it is bijective. Surjectivity of J_7 over $\mathbf{Q}$ therefore implies that of J_d . $\square$

Proposition B.2. *Suppose J_7 is surjective and that forms $G_5 \in H_5$ and $G_6 \in H_6$ satisfy*

$$\varphi(\partial_i(D + W + G_5 + G_6)) \equiv 0 \pmod{t^6} \quad (0 \leq i \leq 5). \quad (\text{B.1})$$

Then there are forms $G_n \in H_n$ for all $n \geq 7$ such that

$$f = D + W + \sum_{n \geq 5} G_n$$

satisfies $\varphi(f) = \varphi(\partial_i f) = 0$ for every i . Consequently $f \in P^{(2)} \setminus \mathfrak{m}P$.

Proof. Suppose that the gradient of the sum through degree $n-1$ vanishes modulo t^{n-1} , where $n \geq 7$. By Lemma B.1, choose $G_n \in H_n$ whose image under J_n is the negative of the remaining coefficient of t^{n-1} . Since

$$\nabla G_n(t\ell + t^2w) = t^{n-1}J_n(G_n) + O(t^n),$$

this kills the next coefficient without changing the preceding ones. The resulting sum converges in the $\mathfrak{m}$ -adic topology, and all six derivatives belong to P . The chain rule gives $\frac{d}{dt}\varphi(f) = 0$; the constant term is zero, so $\varphi(f) = 0$ as we are in characteristic zero. Propositions 5.2 and 6.2, applied through their proofs, give the last assertion. $\square$

The initial forms in (B.1) exist by taking the degree-five and degree-six parts of the series already constructed in Sections 4 and 5. Thus the continuation statement is consistent with the main proof. The rank calculations alone do not prove the initial lifting condition, and no separate explicit coefficient lists for G_5, G_6 are needed or asserted here. This appendix is not used as an independent proof of the initial lifting step.

Supplementary material. The accompanying supplementary material consists of a Python program and a Singular input, which carry out the finite checks described in Appendix A. The Python program requires only Python 3 with its standard library; the Singular input performs the resultant check in Section 3. The recorded runs used Python 3.14.7 and Singular 4.4.1. The checks concern the finite identities; the existence of the full witness is proved by the formal arguments in the text.

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