

Non-flat toroidal compactifications of the universal abelian variety in genus five and above

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Attribution

The computations were carried out, and both counterexamples were produced, by AI models (Claude Opus 4.8 and GPT-5.5 for the initial code, GPT-5.6-Sol for the search and certificates, later Claude models for the exposition and citations). This work was directed by David Holmes, who also posed the original problem, and assisted the models in understanding the mathematics at various points.

Abstract

We prove that two mixed Delaunay–Voronoi models of the universal abelian variety are not flat in genus five: the classical model over the second-Voronoi compactification and the model over the finite cover that records a symmetric theta divisor. Both counterexamples arise from the same monoid-theoretic obstruction: two inequivalent minimal elements in one fibre-degree slice acquire a common positive translate. The ordinary Delaunay cells and cones are certified by exact all-lattice-point computations; after this geometric input, the obstruction is verified by the integer calculations displayed below. Combining the genus-five example in the fixed-theta model with a theorem of Nakamura–Sugawara gives the sharp threshold: the morphism is flat exactly in genera at most four.

1 Introduction

Let $\mathcal{A}_g$ denote the moduli stack over $\mathbb{Z}$ of principally polarized abelian varieties of dimension g , and let $\mathcal{A}_{g,1} \rightarrow \mathcal{A}_g$ denote the universal abelian variety. We study two toroidal models of this universal abelian variety. The first is the classical mixed Delaunay–Voronoi model over the second-Voronoi compactification of $\mathcal{A}_g$. The fan used there is the one produced by Voronoi’s second reduction theory [23, 24]; the toroidal compactification it defines, and the model of the universal abelian variety over it, go back to Mumford’s analytic construction [15] and to Namikawa [16, 17, 18], Ash–Mumford–Rapoport–Tai [3] and Faltings–Chai [9, Chapter VI, Section 1]; see Alexeev–Nakamura [2] and Alexeev [1] for the moduli-theoretic form of the family, and [7] for a recent survey of the combinatorics. The second lies over $\mathcal{N}_g$, the stack parametrizing a principally polarized abelian variety together with a symmetric theta divisor. The morphism $\mathcal{N}_g \rightarrow \mathcal{A}_g$ is a torsor under the finite group scheme $\mathcal{A}_{g,1}[2]$; this is the theta level structure of Faltings–Chai [9, Chapter IV, Section 7, p. 132], and we use it in the form of [4, Section 1.1]. We call the latter model the *fixed-theta model*.

Both toroidal morphisms are defined over $\mathrm{Spec} \mathbb{Z}$, and all flatness assertions below concern these integral morphisms. In particular, the positive fixed-theta result includes residue characteristic two.

Our conclusions are the following.

Theorem 1 (Main results).

- (1) *The classical mixed Delaunay–Voronoi morphism is not flat in genus five. The same rank-five chart occurs on the boundary in every genus $g \geq 5$, so the classical morphism is non-flat in all such genera.*
- (2) *The fixed-theta mixed Delaunay–Voronoi morphism is non-flat in every genus $g \geq 5$ and flat in every genus $g \leq 4$.*

Context. Flatness of these morphisms in genus at least five has been an open question. Alexeev, discussing the normalized Namikawa family, writes that “the flatness of the second is somewhat unclear” [1, Section 2.15, p. 661]; the survey [7, Proposition 6.12, p. 74] describes the second-Voronoi family as proper and equidimensional, not as flat; and [4, Section 1.4] records that the fixed-theta morphism “is not known to be flat for $g \geq 5$ ”. The failures for $g \geq 5$ already in the literature concern integrality of tilings, whose stated consequence is non-reduced fibres, not non-flatness [7, Proposition 6.10].

What is new. Both genus-five examples come from our own computer search. The classical form A_{cl} of Section 4 is, up to a factor of four, a Gram matrix of the classical lattice D_5^* ; the fixed-theta form B_θ of Section 5 is, as far as we know, new. The non-flat slices, the minimality certificates and the parity obstruction built on them are new. The geometric phenomenon behind the fixed-theta example is not new: Delaunay simplices of determinant two exist in dimension five [8, p. 796]; see Section 5.2 for exactly what is known. The low-genus flatness assertion combines our slice lemma with a theorem of Nakamura–Sugawara [19].

The proof of the fixed-theta flatness assertion for $g \leq 4$ is independent of the geometric certificate. The argument does not settle the classical problem in genera at most four.

The mechanism. The coordinate algebra of a mixed toric chart decomposes according to fibre degree. Flatness of the whole algebra over the base would force every degree summand to be flat. In each genus-five example, we isolate one such summand and find two elements with the following properties:

- neither element can be reduced by a nonunit of the base monoid;
- the two elements are inequivalent modulo the units of the base; and
- suitable positive translates of the two elements coincide.

The last equality gives a relation between two elements that would have to belong to an independent local basis if the summand were flat. Lemma 2 packages this obstruction.

The two examples establish minimality differently. In the classical chart, five active inequalities at each candidate form a unimodular matrix. In the fixed-theta chart, a parity relation among four rays forces the only possible slack to vanish.

Route through the proof. Section 2 derives a formula for a fibre-degree slice from an ordinary Delaunay cell and proves the algebraic obstruction. Section 3 explains the exact finite certificate required to identify the two ordinary Delaunay cones. Sections 4 and 5 apply the obstruction to the classical and fixed-theta charts, respectively. Section 6 propagates the examples to higher genus and treats the low-genus fixed-theta case.

Trust boundary. The proof has three kinds of input.

- We use the standard toroidal and toric descriptions of the mixed Delaunay–Voronoi charts. These go back to [15, 16, 17, 18, 3, 9, 2]; we follow the integral structures of Faltings–Chai [9, Notation IV.2.1, Chapter IV, Section 7, and Chapter VI, Section 1], in the form used in [4, Sections 4.3 and 6.2].
- Exact finite computations certify the two ordinary Delaunay cones and their cell decompositions. Section 3 states exactly what is checked.
- For genera at most four, we use the theorem of Nakamura–Sugawara [19, Theorem 1.3] that the relevant Delaunay decompositions are simplicially generating. They attribute the statement itself to Voronoi [24], recording that it is only implicit there; compare [2, 1.14].

Once the two ordinary cone presentations have been certified, the non-flatness arguments consist entirely of the displayed integer calculations. They may be checked by hand or by the pure-Python scripts listed in Section 7. A separate Sage script verifies directly, over all lattice points, that the displayed ordinary cones are the required Delaunay cones. No floating-point computation and no stabilization of a bounded search is used as a proof.

The certificate data and all three checking scripts accompany this note as ancillary files. They are named, with the commands that run them, in Section 7.

2 Monoid slices and the flatness obstruction

This section separates the common argument from the two genus-five certificates. We first fix notation, then recover one fibre-degree slice from an ordinary Delaunay cell, and finally prove the algebraic obstruction used in both examples.

2.1 Notation

All monoids are written additively. For a lattice L and a commutative ring A , write

$$L_A = L \otimes_{\mathbb{Z}} A.$$

Throughout the non-flatness arguments, k denotes an arbitrary field. The underlying toric charts are defined over $\mathbb{Z}$: failure after base change to k detects non-flatness of the integral morphism, whereas the low-genus positive result will be proved directly for $\mathbb{Z}$ -monoid algebras.

General notation

W	The rank- g fibre lattice.
$N, M = \text{Hom}(N, \mathbb{Z})$	The lattice of quadratic data on the base and its dual.
σ	A closed ordinary Delaunay cone in $N_{\mathbb{R}}$.
$\sigma^{\vee}, \sigma^{\perp}$	The dual cone in $M_{\mathbb{R}}$ and the annihilator of σ .
a, ℓ	A quadratic potential in σ and a real linear form on W .
$P = \sigma^{\vee} \cap M$	The base monoid.
$U = P^{\times} = \sigma^{\perp} \cap M$	The group of units of P .
$\tilde{\sigma}_0, Q$	The mixed cone at the cell $\{0\}$ and its integral dual monoid.
deg, Q_c	The fibre-degree map $Q \rightarrow W$ and its degree- c slice.
ϵ_x	Evaluation at $x \in W$, viewed as an element of $M_{\mathbb{Q}}$.
u_D	The linear Delaunay interpolant attached to a cell D .
$\text{Cone}(D)$	The nonnegative real span of a cell D containing 0.
$k[-]$	A monoid algebra, or the corresponding monomial module for a monoid-set.

Classical example

$A_{\text{cl}}, D_{\text{cl}}, c_{\text{cl}}$	The form, selected Delaunay cell, and selected fibre degree.
C_{cl}, C_i	The 10×5 inequality matrix and its rows.
π, j	A quotient map from the dual base lattice and a chosen right inverse.
H_i	The ten primitive ray forms.
b_{cl}	The vector of integral lower bounds for the selected slice.
$P_{\text{cl}}^{\sharp}, S_{\text{cl}}$	The sharpened classical base monoid and its selected slice.

Fixed-theta example

ρ, L_0, K_0	The theta characteristic, the $\rho = 0$ base lattice, and an intrinsic basis.
$B_{\vartheta}, D_{\vartheta}, c_{\vartheta}$	The form, selected simplex, and selected fibre degree.
r_i, R_{ϑ}	The eight primitive rays and the matrix having them as rows.
b_{ϑ}	The vector of integral lower bounds for the selected slice.
$P_{\vartheta}, S_{\vartheta}$	The fixed-theta base monoid and its selected slice.

2.2 The mixed cone and its grading

Let W be a free abelian group of rank g . In the classical model, the base lattice is

$$N_{\text{cl}} = \text{SymBil}(W, \mathbb{Z}),$$

the lattice of integral symmetric bilinear forms on W ; this is the lattice $B(X)$ of [9, Notation IV.2.1, p. 96]. On the fixed-theta chart with theta characteristic $\rho = 0$, it is the lattice $N_{\vartheta} = Q_0(W)$ of integer-valued quadratic functions a on W with $a(-w) = a(w)$ and $a(0) = 0$, which is the lattice $Q_0(X)$ of [9, Chapter IV, Section 7, p. 132], described concretely as the even-diagonal lattice L_0 in Section 5.1. In either of the two genus-five charts, write N for the relevant base lattice and put

$$M = \text{Hom}(N, \mathbb{Z}).$$

For $a \in N_{\mathbb{R}}$, let B_a be its associated symmetric bilinear form. On the classical chart and the fixed-theta chart with $\rho = 0$, the corresponding quadratic potential is

$$a(x) = \frac{1}{2} B_a(x, x). \tag{1}$$

Thus, if A is the symmetric matrix of B_a , then $a(x) = \frac{1}{2}x^tAx$.

The factor $\frac{1}{2}$ in (1) is the integral structure of Faltings–Chai. In their compactification of the universal abelian variety, a point (b, l) of the lattice $B(X) \times X^*$ is read as the quadratic function $\mu \mapsto b(\mu, \mu) + 2l(\mu)$ [9, Chapter VI, Section 1, p. 195]; dividing by two changes neither the lattice nor any of the inequalities below, and [4, Section 4.3] writes the potential in the divided form (1) that we use. It is not a harmless choice of scale. The classical literature on Delaunay decompositions [6, 23, 24, 8, 21] works with B_a itself. Rescaling B_a by a positive constant changes no Delaunay cell and no Delaunay cone, so those sources may be, and below are, cited for the ordinary Delaunay geometry; but it does move the evaluation characters ϵ_x of Section 2.3 inside $M_{\mathbb{Q}}$, and hence changes the slices Q_c . Citations below are therefore split accordingly.

Let $\sigma \subset N_{\mathbb{R}}$ be a closed ordinary Delaunay cone. Its dual cone and annihilator are

$$\sigma^{\vee} = \{m \in M_{\mathbb{R}} : \langle m, r \rangle \geq 0 \text{ for every } r \in \sigma\}$$

and

$$\sigma^{\perp} = \{m \in M_{\mathbb{R}} : \langle m, r \rangle = 0 \text{ for every } r \in \sigma\}.$$

It is enough to test these conditions on the rays of σ . The integral dual monoid is

$$P = \sigma^{\vee} \cap M.$$

Its unit group is

$$P^{\times} = \sigma^{\perp} \cap M,$$

because both p and $-p$ lie in $\sigma^{\vee}$ exactly when p annihilates σ . We abbreviate this group by U . A monoid is *sharp* if its only unit is 0.

For an ordinary Delaunay cell D with vertex set $\text{Vert}(D)$, the mixed cone is

$$\tilde{\sigma}_D = \left\{ (a, \ell) \in \sigma \times \text{Hom}(W, \mathbb{R}) : \begin{array}{l} a(x) + \ell(x) \geq a(v) + \ell(v) \text{ for every } x \in W \\ \text{and } v \in \text{Vert}(D) \end{array} \right\}. \quad (2)$$

For the zero-dimensional cell $\{0\}$, this specializes to

$$\tilde{\sigma}_0 = \{(a, \ell) \in \sigma \times \text{Hom}(W, \mathbb{R}) : a(x) + \ell(x) \geq 0 \text{ for every } x \in W\}. \quad (3)$$

The cone (2) over a Delaunay cell is the one appearing in Mumford’s construction [15], in [16, 17, 18, 2], and in [9, Remark VI.1.6(b), p. 197], where the mixed decomposition is cut out by the polarization function $(a_1, \dots, a_s) \mapsto -\sum_j \min_{\mu} a_j(\mu)$ and called a mild generalization of the Voronoi–Delaunay decomposition; the passage from a rational polyhedral cone to the affine toric chart $\text{Spec } k[\sigma^{\vee} \cap M]$ is the standard one of [14, 5]. In the integral structure of [9, Chapter VI, Section 1] and [4, Sections 4.3.3, 4.3.4, and 6.2], projection onto the first factor induces an injection of dual monoids

$$P = \sigma^{\vee} \cap M \longrightarrow Q = \tilde{\sigma}_0^{\vee} \cap (M \oplus W). \quad (4)$$

The second coordinate defines the fibre-degree map

$$\text{deg}: Q \longrightarrow W.$$

For $c \in W$, set

$$Q_c = \{m \in M : (m, c) \in Q\}.$$

For any commutative monoid T , the notation $k[T]$ means the k -vector space with basis $\{X^t : t \in T\}$ and multiplication $X^t X^{t'} = X^{t+t'}$. Likewise, $k[Q_c]$ denotes the $k[P]$ -module with basis $\{X^m : m \in Q_c\}$. Since the action of P preserves fibre degree,

$$k[Q] = \bigoplus_{c \in W} k[Q_c] \quad (5)$$

as $k[P]$ -modules. The same decomposition holds over $\mathbb{Z}$.

2.3 Recovering a fibre-degree slice

The next formula is the bridge from ordinary Delaunay geometry to the mixed chart. It recovers the required slice Q_c without enumerating the full mixed cone.

Let D be a full-dimensional Delaunay cell that contains 0. Define

$$\text{Cone}(D) = \left\{ \sum_{v \in D} \lambda_v v : \lambda_v \in \mathbb{R}_{\geq 0} \right\}.$$

This is the real cone spanned by D , not the semigroup generated by the vertices of D . The distinction will be essential in (25).

For every $x \in W$, define the evaluation character

$$\epsilon_x \in M_{\mathbb{Q}}, \quad \epsilon_x(a) = a(x).$$

The co-sphericity relations for D identify the classes of the vertex evaluations with the values of a unique linear map

$$u_D : W_{\mathbb{R}} \longrightarrow M_{\mathbb{R}} / \sigma^{\perp}, \quad u_D(v) = \epsilon_v \bmod \sigma^{\perp}$$

for each vertex v of D .

For $c \in \text{Cone}(D) \cap W$, polyhedral duality gives

$$Q_c = (u_D(c) + \sigma^{\vee}) \cap M. \quad (6)$$

Only the class of $u_D(c)$ modulo $\sigma^{\perp}$ matters in this formula.

We include the derivation because it explains exactly which part of the ordinary geometry controls the mixed slice. The real dual of (3) is

$$(\sigma^{\vee} \times \{0\}) + \text{Cone}\{(\epsilon_x, x) : x \in W\}. \quad (7)$$

Although (3) is written using all lattice points, it is rational polyhedral; a finite subcollection of its half-spaces suffices, and (7) is the corresponding finite Farkas-duality statement [10]; see also [5, Chapter 1].

For every $a \in \sigma$, the empty-sphere inequality of Delaunay [6] for D says

$$\epsilon_x(a) - u_D(x)(a) \geq 0 \quad \text{for every } x \in W. \quad (8)$$

Equivalently, if a is represented by a matrix H and h_D is the affine interpolant of $q_H(x) = \frac{1}{2}x^t H x$ through the vertices of D , then (8) is the inequality $q_H(x) \geq h_D(x)$ used in Section 3.1. Consequently, $\epsilon_x - u_D(x)$ belongs to $\sigma^{\vee}$.

Suppose that (m, c) is represented by the right-hand side of (7). Linearity of u_D and (8) then imply

$$m - u_D(c) \in \sigma^{\vee}.$$

This proves one inclusion in (6).

Conversely, write

$$c = \sum_{v \in \text{Vert}(D)} \lambda_v v, \quad \lambda_v \geq 0.$$

Choose a lift $\tilde{u}_D(c) \in M_{\mathbb{R}}$ of the class $u_D(c)$, and put

$$\delta = \sum_{v \in \text{Vert}(D)} \lambda_v \epsilon_v - \tilde{u}_D(c).$$

The interpolation property gives $\delta \in \sigma^\perp$, so $-\delta \in \sigma^\vee$. Hence

$$(\tilde{u}_D(c), c) = \sum_{v \in \text{Vert}(D)} \lambda_v (\epsilon_v, v) + (-\delta, 0)$$

belongs to (7). Adding $\sigma^\vee \times \{0\}$ proves the reverse inclusion.

Let $r_1, \dots, r_s$ be the primitive extreme rays of σ , where a lattice vector is *primitive* if the greatest common divisor of its coordinates is one. Pairing (6) with these rays gives

$$Q_c = \{m \in M : \langle m, r_i \rangle \geq \lceil \langle u_D(c), r_i \rangle \rceil \text{ for } 1 \leq i \leq s\}. \quad (9)$$

The pairing with $u_D(c)$ is well-defined because every r_i annihilates $\sigma^\perp$.

The ceilings in (9) give valid lower bounds, but an individual bound need not be attained by a lattice point. In the fixed-theta example, two explicit points together attain every displayed bound; the classical argument does not require such an attainment statement.

2.4 Overlapping minimal elements

We now isolate the algebraic obstruction. A minimal element of a slice would have to begin a generator in a flat local module. Two inequivalent minimal elements would therefore begin independent generators. If positive translates of those elements coincide, the meeting point supplies an impossible relation.

A commutative monoid is *fine* if it is finitely generated and integral. It is *saturated* if, whenever x belongs to its groupification and nx belongs to the monoid for some integer $n > 0$, the element x already belongs to the monoid. It is *torsion-free* if its groupification is torsion-free. These are the standard monoid-theoretic notions; see [13, (1.1)] and [20, Section I.1.3]. Flatness of a monomial module over a monoid algebra is governed by a general criterion, which we record because Lemma 2 is a special case of it. For a homomorphism $h: P \rightarrow Q$ of integral monoids, Kato proved that $k[Q]$ is flat over $k[P]$ for every field k if and only if h is injective and satisfies the condition I1 below, and that I1 is equivalent to h being *integral*, that is, to every push-out of h along a homomorphism of integral monoids being integral [12, Proposition 4.1(1)]. For an integral monoid P acting on a set S , the module $k[S]$ is flat over $k[P]$ for every field k exactly when S is a filtered direct limit of free P -sets, equivalently when S satisfies the two conditions I1 and I2 of [20, Proposition I.4.5.6]; see [20, Theorem I.4.5.7 and Proposition I.4.5.12], which extends Kato's criterion from homomorphisms to monoid actions; integrality is [20, Definition I.4.6.2]. The hypotheses of Lemma 2 violate condition I1, and we give the short direct argument rather than invoking the general statement.

For a P -set S , let $k[S]$ be the $k[P]$ -module with basis $\{X^s : s \in S\}$ and action

$$X^p X^s = X^{p+s}.$$

Lemma 2 (Overlapping minimal elements). *Let P be a fine, saturated, torsion-free monoid, and let $U = P^\times$. Let S be a P -stable subset of a lattice that is finitely generated as a P -set. An element $x \in S$ is minimal modulo units if*

$$x - p \in S, \quad p \in P \quad \implies \quad p \in U.$$

Suppose that $x, y \in S$ are minimal modulo units, are not translates of one another by U , and satisfy

$$x + p = y + q \tag{10}$$

for some nonunits $p, q \in P$. Then $k[S]$ is not flat over $k[P]$ for any field k . Consequently, a toric map with $k[S]$ as one of its fibre-degree summands is not flat.

Proof. Put

$$A = k[U], \quad K = \text{Frac}(A),$$

and base-change from A to K :

$$R_K = K \otimes_A k[P], \quad F_K = K \otimes_A k[S].$$

Let $\mathfrak{m} \subset R_K$ be the ideal generated by the monomials associated with nonunits of P . Then $\mathfrak{m}$ is maximal and

$$R_K/\mathfrak{m} = K.$$

Minimality modulo units implies that the classes of X^x and X^y survive in $F_K/\mathfrak{m}F_K$. Because x and y are not unit translates, these two classes are linearly independent over K .

Suppose, for contradiction, that $(F_K)_{\mathfrak{m}}$ is flat over $(R_K)_{\mathfrak{m}}$. The module is finite because S is finitely generated as a P -set. Since $(R_K)_{\mathfrak{m}}$ is a noetherian local ring, the finite flat module $(F_K)_{\mathfrak{m}}$ is free [22, Tag 00NZ]. Extend the residue classes of X^x and X^y to a K -basis of

$$(F_K)_{\mathfrak{m}}/\mathfrak{m}(F_K)_{\mathfrak{m}},$$

using X^x and X^y themselves as the first two lifts. Choose arbitrary lifts of the remaining basis elements. Nakayama's lemma makes these lifts an $(R_K)_{\mathfrak{m}}$ -basis.

Equation (10) gives the relation

$$X^p X^x - X^q X^y = 0.$$

Both coefficients are nonzero in the domain $(R_K)_{\mathfrak{m}}$, which contradicts the linear independence of a basis. Therefore $(F_K)_{\mathfrak{m}}$ is not flat.

Flatness is preserved by base change and localization, so $k[S]$ was not flat over $k[P]$. Finally, (5) makes $k[S]$ a direct summand of the full mixed monoid algebra in the applications below. Flatness of that algebra would therefore imply flatness of $k[S]$. $\square$

If P is sharp, then $P^\times = \{0\}$, so minimality modulo units is ordinary minimality.

3 The geometric certificate

Both counterexamples require the same geometric input: a complete list of Delaunay-cell orbits and an exact presentation of the corresponding closed Delaunay cone. The cells and their decomposition are those of Delaunay [6]; the corresponding cone in the space of quadratic forms is a cell of Voronoi's second reduction theory [23, 24], studied in this form in [8], with [21] as a modern computational

reference. As noted after (1), those sources omit the factor $\frac{1}{2}$; this rescaling leaves every cell and every cone unchanged, so it is harmless here. This section describes how those facts are certified from finite static data. The later sections use only the resulting cone presentations and one selected cell from each list.

A static data file, `counterexample_cells.py`, records 30 classical and 58 fixed-theta full-dimensional lattice polytopes. It records the output of the search, not the search procedure itself. The geometric checker described in Section 7 imports no part of the project's search code.

3.1 Certifying and completing the cell lists

Let H be a positive-definite symmetric form, and let D be a proposed five-dimensional Delaunay cell. Choose six affinely independent vertices of D , and solve for the affine function

$$h_D(x) = \ell_D \cdot x + d_D$$

that agrees at those vertices with

$$q_H(x) = \frac{1}{2}x^t H x.$$

For every remaining listed vertex v , the equality

$$q_H(v) - h_D(v) = 0$$

is a linear condition on H . We call these the *co-sphericity equations* of D ; a simplex contributes no additional equation.

The checker first verifies the co-sphericity equations. Completing the square then gives

$$q_H(x) - h_D(x) = \frac{(x - H^{-1}\ell_D)^t H (x - H^{-1}\ell_D)}{2} - \left(d_D + \frac{\ell_D \cdot H^{-1}\ell_D}{2}\right). \quad (11)$$

If the left-hand side is nonpositive, the Cauchy–Schwarz inequality bounds each coordinate of $x - H^{-1}\ell_D$ by an explicit rational number involving the corresponding diagonal entry of H^{-1} . The checker exhausts this rigorously bounded integer box and verifies that the nonpositive locus consists exactly of the listed vertices. These checks prove that every polytope in the data is an exact Delaunay cell; no arbitrary search window enters the proof.

The checker verifies that the cells are pairwise inequivalent under lattice translation and computes their normalized volumes exactly. Volumes are normalized so that a fundamental domain of $\mathbb{Z}^5$ has volume one. Distinct Delaunay-cell orbits have disjoint interiors, so a total normalized volume of one proves that the list is complete.

The classical list contains 30 cells, each with eight vertices and normalized volume $1/30$. Thus the checker verifies

$$30 \left(\frac{1}{30} \right) = 1.$$

The fixed-theta list contains 20 six-vertex cells of normalized volume $1/120$, two six-vertex cells of normalized volume $1/60$, 32 seven-vertex cells of normalized volume $1/60$, and four nine-vertex cells of normalized volume $1/15$. Thus the checker verifies

$$20 \left(\frac{1}{120} \right) + 2 \left(\frac{1}{60} \right) + 32 \left(\frac{1}{60} \right) + 4 \left(\frac{1}{15} \right) = 1.$$

3.2 Identifying the Delaunay cones

For every displayed ray form H and every listed cell D , the checker proves the global inequality

$$q_H(x) \geq h_D(x) \quad \text{for every } x \in \mathbb{Z}^5. \quad (12)$$

For a positive-definite ray, this follows from the bounded enumeration derived from (11).

Five classical rays are singular. Direct calculation gives

$$\text{rank}(H_i) = 4 \quad (1 \leq i \leq 5), \quad \text{rank}(H_i) = 5 \quad (6 \leq i \leq 10).$$

For each of the five singular rays, the checker first passes to the quotient by the kernel of H_i , taken as a saturated sublattice of $\mathbb{Z}^5$. Here *saturated* means that the quotient lattice is torsion-free. For each $1 \leq i \leq 5$, the certificate supplies a kernel-plus-complement matrix of determinant ± 1 , so the resulting change of basis is unimodular. For every listed cell D , the checker also verifies explicitly that the linear part ℓ_D of its affine interpolant annihilates $\ker H_i$. Since q_{H_i} is constant on cosets of $\ker H_i$, this additional condition ensures that $q_{H_i} - h_D$ descends to the quotient. Formula (11) can then be applied to the induced positive-definite form on that quotient.

Six of the eight fixed-theta rays have an even simpler certificate. For a primitive linear form $w \in (\mathbb{Z}^5)^\vee$ and an integer k , they satisfy

$$H = 2ww^t, \quad \{w \cdot v : v \in D\} = \{k, k+1\}, \quad (13)$$

and therefore

$$q_H(x) - h_D(x) = (w \cdot x - k)(w \cdot x - k - 1) \geq 0 \quad \text{for every } x \in \mathbb{Z}^5.$$

Once verified, these inequalities show that the cone generated by the displayed rays is contained in the closed Delaunay cone. For the reverse containment, the checker first verifies that the co-sphericity equations have the codimension of the displayed ray span: ten in the classical example and seven in the fixed-theta example. It then exhibits a genuine cell–lattice-point inequality for every facet of the ray cone, producing ten facets in the classical case and eight in the fixed-theta case.

Finally, the checker verifies the ray structure directly. In a d -dimensional span, an extreme ray is obtained by imposing $d - 1$ independent active facet inequalities. The checker exhausts all

$$\binom{10}{4} = 210$$

choices in the five-dimensional classical span and all eight choices in the eight-dimensional fixed-theta span, and it recovers exactly the displayed rays. These checks establish that every form preserving the listed subdivision lies in the displayed ray cone and hence prove equality with the closed Delaunay cone, not merely containment in a refinement.

3.3 Why the ordinary certificate suffices

Once an ordinary Delaunay cone and a cell containing 0 have been certified, formula (6) identifies the required fibre-degree slice of the actual mixed chart. Lemma 2 uses only that slice, so there is no need to construct or enumerate the full mixed cone.

4 The classical counterexample

We now construct the arithmetic part of the classical genus-five obstruction. The identification of this arithmetic data with a mixed toric chart rests on the geometric certificate described in Section 3, which supplies a five-dimensional sharpened base cone and one fibre-degree slice. Five active inequalities prove the minimality of each of two lattice points, and a final equality shows that their positive translates meet.

4.1 The form and its Delaunay cone

Take $W = \mathbb{Z}^5$ and consider the symmetric form below. It arose in our computer search, but the lattice is classical: A_{cl} is four times a Gram matrix of the dual root lattice D_5^* . What is new is the non-flat slice over its Delaunay cone constructed below.

$$A_{\text{cl}} = \begin{pmatrix} 4 & 0 & 0 & 0 & -2 \\ 0 & 4 & 0 & 0 & -2 \\ 0 & 0 & 4 & 0 & -2 \\ 0 & 0 & 0 & 4 & -2 \\ -2 & -2 & -2 & -2 & 5 \end{pmatrix}.$$

Writing $x = (x_1, x_2, x_3, x_4, z)$, we have

$$2q_{A_{\text{cl}}}(x) = x^t A_{\text{cl}} x = \sum_{i=1}^4 (2x_i - z)^2 + z^2. \quad (14)$$

Thus A_{cl} is positive definite. Its determinant is 256.

Use the upper-triangular coordinate order

$$(00, 01, 02, 03, 04, 11, 12, 13, 14, 22, 23, 24, 33, 34, 44)$$

on the lattice of symmetric forms, and use the dual coordinate order on $M_{\text{cl}} = \text{Hom}(N_{\text{cl}}, \mathbb{Z}) \cong \mathbb{Z}^{15}$. Define a surjection

$$\pi: M_{\text{cl}} \longrightarrow \mathbb{Z}^5$$

by

$$\pi(m) = (-2m_{33} + m_{34}, -2m_{22} + m_{24}, -2m_{11} + m_{14}, -2m_{00} + m_{04}, m_{44}). \quad (15)$$

Let

$$j: \mathbb{Z}^5 \longrightarrow M_{\text{cl}}$$

place the five input coordinates, in order, in $m_{34}, m_{24}, m_{14}, m_{04}, m_{44}$ and set all remaining coordinates to zero. Then

$$\pi \circ j = \text{id}_{\mathbb{Z}^5}.$$

Let C_{cl} be the 10×5 matrix whose rows are

$$\begin{aligned} C_1 &= (-1, -1, -1, 0, 2), & C_6 &= (-1, -1, -1, -1, 3), \\ C_2 &= (-1, -1, 0, -1, 2), & C_7 &= (-2, -1, -1, -1, 3), \\ C_3 &= (-1, 0, -1, -1, 2), & C_8 &= (-1, -2, -1, -1, 3), \\ C_4 &= (0, -1, -1, -1, 2), & C_9 &= (-1, -1, -2, -1, 3), \\ C_5 &= (-1, -1, -1, -1, 2), & C_{10} &= (-1, -1, -1, -2, 3). \end{aligned}$$

For $1 \leq i \leq 10$, let H_i be the symmetric form whose coordinate row is $C_i \pi$.

The geometric checker proves directly from the cell data that $H_1, \dots, H_{10}$ are precisely the ten primitive rays of the closed Delaunay cone σ_{cl} . Their span has rank five, C_{cl} has rank five, and

$$H_1 + \dots + H_{10} = 5A_{\text{cl}}. \quad (16)$$

This shows that A_{cl} lies in the relative interior of σ_{cl} .

Since π is surjective and the H_i span the row space of π , its kernel is the integral annihilator of the cone:

$$\ker \pi = \sigma_{\text{cl}}^\perp \cap M_{\text{cl}} = P_{\text{cl}}^\times.$$

Consequently, π induces an isomorphism

$$M_{\text{cl}}/(\sigma_{\text{cl}}^\perp \cap M_{\text{cl}}) \xrightarrow{\sim} \mathbb{Z}^5. \quad (17)$$

One of the certified Delaunay cells, translated to contain the origin, is

$$\begin{aligned} D_{\text{cl}} = \text{conv} &((-1, -1, -1, -1, -2), (-1, 0, -1, -1, -2), \\ &(-1, 0, 0, -1, -1), (-1, 0, 0, 0, -1), \\ &(0, 0, 0, -1, -1), (0, 0, 0, 0, -1), \\ &(0, 0, 0, 0, 0), (0, 1, 0, 0, 0)). \end{aligned} \quad (18)$$

It contains the fibre degree

$$c_{\text{cl}} = (0, 0, 0, 0, -1).$$

Because c_{cl} is a vertex of D_{cl} , its Delaunay interpolant is evaluation at that vertex. Direct evaluation on the ten rays gives

$$(\langle u_{D_{\text{cl}}}(c_{\text{cl}}), H_i \rangle)_{i=1}^{10} = (1, 1, 1, 1, 1, 3/2, 3/2, 3/2, 3/2, 3/2).$$

After taking ceilings, define

$$b_{\text{cl}} = (1, 1, 1, 1, 1, 2, 2, 2, 2, 2).$$

Under the quotient (17), the sharpened base monoid and the image of the selected slice are

$$P_{\text{cl}}^\sharp = \{p \in \mathbb{Z}^5 : C_{\text{cl}}p \geq 0\}, \quad S_{\text{cl}} = \{y \in \mathbb{Z}^5 : C_{\text{cl}}y \geq b_{\text{cl}}\}. \quad (19)$$

Since C_{cl} has rank five, $P_{\text{cl}}^\sharp$ is sharp.

The rational polyhedron defining S_{cl} has recession cone

$$\{p \in \mathbb{R}^5 : C_{\text{cl}}p \geq 0\},$$

which is the real cone underlying $P_{\text{cl}}^\sharp$. The polyhedral form of Gordan's lemma [11, 5] therefore shows that S_{cl} is finitely generated as a $P_{\text{cl}}^\sharp$ -set. All geometric input to (19) comes from the ordinary certificate and the slice formula; no presentation of the full mixed cone is used.

4.2 Two minimal elements whose translates meet

The remaining proof is an integer calculation. Define

$$y_a = (0, 0, -1, -1, 0), \quad y_b = (0, 1, 1, 1, 2).$$

Their slack vectors are

$$\begin{aligned} C_{\text{cl}}y_a - b_{\text{cl}} &= (0, 0, 1, 1, 1, 0, 0, 0, 1, 1), \\ C_{\text{cl}}y_b - b_{\text{cl}} &= (1, 1, 1, 0, 0, 1, 1, 0, 0, 0). \end{aligned} \tag{20}$$

Both are nonnegative, so $y_a, y_b \in S_{\text{cl}}$.

To prove that y_a is minimal, suppose that $p \in P_{\text{cl}}^\sharp$ and $y_a - p \in S_{\text{cl}}$. On each row where the first slack vector in (20) vanishes, the inequalities for p and $y_a - p$ give

$$C_i p \geq 0 \quad \text{and} \quad C_i p \leq 0.$$

Thus

$$C_1 p = C_2 p = C_6 p = C_7 p = C_8 p = 0.$$

The matrix with rows C_1, C_2, C_6, C_7, C_8 has determinant -1 , so $p = 0$. Hence y_a is minimal.

The same argument applies to y_b . If $p \in P_{\text{cl}}^\sharp$ and $y_b - p \in S_{\text{cl}}$, then

$$C_4 p = C_5 p = C_8 p = C_9 p = C_{10} p = 0.$$

The matrix with these five rows also has determinant -1 , so again $p = 0$. Thus y_b is minimal.

Now define

$$p_a = (0, 0, 1, 1, 1), \quad p_b = (0, -1, -1, -1, -1).$$

Their base-cone values are

$$\begin{aligned} C_{\text{cl}}p_a &= (1, 1, 0, 0, 0, 1, 1, 1, 0, 0), \\ C_{\text{cl}}p_b &= (0, 0, 0, 1, 1, 0, 0, 1, 1, 1). \end{aligned}$$

Thus p_a and p_b are nonzero elements of the sharp monoid $P_{\text{cl}}^\sharp$, hence nonunits. Their translates meet:

$$y_a + p_a = (0, 0, 0, 0, 1) = y_b + p_b. \tag{21}$$

We must transfer this obstruction from the sharpened lattice back to the original slice. Lift y_a, y_b, p_a, p_b along the right inverse j from (15). The inequalities in (19) show that j carries $P_{\text{cl}}^\sharp$ into the original base monoid P_{cl} and S_{cl} into the original c_{cl} -slice. It also preserves (21).

If an element of P_{cl} could be subtracted from either lifted minimal element, its image under π would be zero by the minimality just proved. Such an element lies in $\ker \pi = P_{\text{cl}}^\times$ and is therefore a unit. The two lifts are not unit translates because their images under π are distinct. More precisely, the right inverse gives

$$M_{\text{cl}} = \ker \pi \oplus j(\mathbb{Z}^5),$$

together with the decompositions

$$P_{\text{cl}} = \ker \pi \oplus j(P_{\text{cl}}^\sharp), \quad Q_{c_{\text{cl}}} = \ker \pi \oplus j(S_{\text{cl}}),$$

where $\ker \pi$ is a group. These splittings also lift finite generation from S_{cl} to $Q_{c_{\text{cl}}}$.

With the geometric cone and cell identifications certified in Section 3, all hypotheses of Lemma 2 now hold for the original fibre-degree summand. For every field k , that summand is not flat over the base monoid algebra, so the corresponding base change of the genus-five toric chart is not flat. Since flatness is preserved by base change, the integral classical chart cannot be flat either. Hence the classical mixed Delaunay–Voronoi morphism over $\mathbb{Z}$ is not flat in genus five.

The ancillary script `verify_classical_certificate.py` independently checks the arithmetic from (15) through (21), using only integer arithmetic.

5 The fixed-theta counterexample

The fixed-theta example uses the same algebraic obstruction but a different lattice and a different minimality argument. A non-unimodular Delaunay simplex produces a lattice degree not generated by its vertices. A parity relation among four rays then forces two elements in that degree to be minimal modulo units.

5.1 The $\rho = 0$ lattice

The local tropical charts of $\mathcal{N}_g$ are indexed by a theta characteristic

$$\rho \in W/2W.$$

This use of “characteristic” refers to the theta-characteristic label and not to the residue characteristic of the ground field. For the genus-five counterexample, we work only on the chart $\rho = 0$.

Let $\text{SymBil}(W, \mathbb{Z})$ denote the group of integral symmetric bilinear forms on W , and let $\text{diag}(B)$ be the vector of diagonal entries of B . The $\rho = 0$ fixed-theta lattice is

$$L_0 = \{B \in \text{SymBil}(W, \mathbb{Z}) : \text{diag}(B) \text{ is coordinatewise even}\}. \quad (22)$$

It is identified with $Q_0(W)$ by sending B to the integral quadratic function

$$a_B(x) = \frac{1}{2}B(x, x).$$

In the notation of Faltings–Chai [9, Chapter IV, Section 7, pp. 132–133], L_0 is the image of $Q_0(X)$ in $B(X)$, which they record has index 2^g in $B(X)$; here $2^5 = 32$ is the determinant of (23).

In the symmetric coordinate order

$$(00, 01, 02, 03, 04, 11, 12, 13, 14, 22, 23, 24, 33, 34, 44),$$

the columns of

$$K_0 = \text{diag}(2, 1, 1, 1, 1, 2, 1, 1, 1, 2, 1, 1, 2, 1, 2) \quad (23)$$

form a $\mathbb{Z}$ -basis of L_0 . Thus $z \in \mathbb{Z}^{15}$ is the intrinsic coordinate vector of the form whose ambient coordinate vector is $K_0 z$.

Let

$$M_\vartheta = \text{Hom}(L_0, \mathbb{Z}).$$

The basis (23) and its dual identify both L_0 and M_ϑ with copies of $\mathbb{Z}^{15}$. We distinguish them by role: ray vectors lie in the primal copy, slice variables lie in the dual copy, and their pairing is the standard dot product.

5.2 A Delaunay simplex and a missing degree

Take $W = \mathbb{Z}^5$ and consider the following form, which we found by computer search; we know of no earlier appearance of it:

$$B_\vartheta = \begin{pmatrix} 14 & 9 & 7 & 7 & 14 \\ 9 & 16 & 7 & 7 & 14 \\ 7 & 7 & 16 & 7 & 14 \\ 7 & 7 & 7 & 16 & 14 \\ 14 & 14 & 14 & 14 & 24 \end{pmatrix}.$$

Its leading principal minors are

$$14, \quad 143, \quad 1700, \quad 19017, \quad 57744.$$

They are positive, so Sylvester's criterion shows that B_ϑ is positive definite. Its diagonal entries are even, and therefore $B_\vartheta \in L_0$.

The geometric certificate lists 58 full-dimensional Delaunay-cell orbits: 22 have six vertices, 32 have seven vertices, and four have nine vertices. The checker verifies that their exact normalized volumes sum to one.

The checker also verifies that the co-sphericity equations have rank seven and that eight explicit cell-lattice-point inequalities witness all eight facets of the ray cone displayed in Section 5.3. These checks establish that this eight-dimensional cone is exactly the closed Delaunay cone. The checker further verifies that it contains B_ϑ in its relative interior and contains the full-dimensional Delaunay simplex

$$D_\vartheta = \text{conv}(0, e_1, e_2, e_3, e_4, v), \quad v = (1, 1, 1, 1, -2), \quad (24)$$

where $e_1, \dots, e_4$ are the first four standard basis vectors.

The determinant of the five nonzero vertex columns is -2 . The lattice point

$$c_\vartheta = (1, 1, 1, 1, -1) = \frac{e_1 + e_2 + e_3 + e_4 + v}{2} \quad (25)$$

therefore lies in $\text{Cone}(D_\vartheta)$. The five nonzero vertices are linearly independent, so the coefficients in (25) are unique. Since those coefficients are not integers, c_ϑ does not lie in the semigroup generated by the vertices of D_ϑ .

This is the known phenomenon of Delaunay cells that fail to be generating in the sense of [2, Definition 1.13]. Such cells do not occur for $g \leq 4$ and first occur in dimension five. Erdahl and Ryshkov state, without an explicit example, that in dimension five there are exactly two types of Delaunay simplex: the edge vectors of the first generate the lattice, and those of the second generate a sublattice of index two [8, p. 796]; [2, 1.14] cites this for the existence of non-generating cells in dimension five, and [2, Remark 1.17] records that such cells are responsible for the combinatorial complications of genus five. The simplex D_ϑ is therefore of the second type. What is ours is the form B_ϑ realizing it in the even-diagonal lattice L_0 , and the parity obstruction of Section 5.4 that it supports.

5.3 The eight-ray slice

The primitive rays of the closed Delaunay cone, expressed in the intrinsic primal coordinates of (23), are

$$\begin{aligned} r_0 &= (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0), \\ r_1 &= (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0), \\ r_2 &= (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 2, 2, 1, 2, 1), \\ r_3 &= (0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0), \\ r_4 &= (1, 1, 1, 1, 2, 1, 1, 1, 2, 1, 1, 2, 1, 2, 2), \\ r_5 &= (1, 2, 0, 2, 2, 1, 0, 2, 2, 0, 0, 0, 1, 2, 1), \\ r_6 &= (1, 2, 2, 0, 2, 1, 2, 0, 2, 1, 0, 2, 0, 0, 1), \\ r_7 &= (4, 4, 4, 4, 8, 4, 4, 4, 8, 4, 4, 8, 4, 8, 7). \end{aligned} \quad (26)$$

They have rank eight and satisfy

$$K_0(r_0 + \dots + r_7) = \text{vec}(B_\vartheta). \quad (27)$$

This positive decomposition shows directly that B_ϑ lies in the relative interior.

Equation (25) determines the Delaunay interpolant:

$$\begin{aligned} u_{D_\vartheta}(c_\vartheta) &= \frac{\epsilon_{e_1} + \epsilon_{e_2} + \epsilon_{e_3} + \epsilon_{e_4} + \epsilon_v}{2} \\ &= (1, 1/2, 1/2, 1/2, -1, 1, 1/2, 1/2, -1, \\ &\quad 1, 1/2, -1, 1, -1, 2) \in (M_\vartheta)_\mathbb{Q}. \end{aligned} \tag{28}$$

Pairing this vector with the rays in (26) gives the integral lower-bound vector

$$b_\vartheta = (1, 1, 1, 1, 3, 2, 2, 10). \tag{29}$$

Let R_ϑ be the 8×15 matrix whose rows are $r_0, \dots, r_7$. For $m \in M_\vartheta \cong \mathbb{Z}^{15}$, the entries of $R_\vartheta m$ are the pairings $\langle m, r_i \rangle$. The base monoid and the selected slice are therefore

$$P_\vartheta = \{p \in \mathbb{Z}^{15} : R_\vartheta p \geq 0\}, \quad S_\vartheta = \{m \in \mathbb{Z}^{15} : R_\vartheta m \geq b_\vartheta\}. \tag{30}$$

The unit group is

$$P_\vartheta^\times = \ker(R_\vartheta) \cap \mathbb{Z}^{15}.$$

The polyhedron defining S_ϑ has recession cone

$$\{p \in \mathbb{R}^{15} : R_\vartheta p \geq 0\},$$

the real cone underlying P_ϑ . Gordan's lemma [11, 5] therefore shows that S_ϑ is finitely generated as a P_ϑ -set.

5.4 The parity obstruction

We now exhibit two points whose slack is concentrated in different coordinates. A parity relation among four rays prevents either slack from being reduced by a nonunit.

Define

$$\begin{aligned} f_a &= (-4, 1, 0, 0, 0, 1, 0, 0, 0, 1, -1, 0, 1, 0, 2), \\ f_b &= (-1, 0, 0, 0, 0, 1, 0, 0, 0, 1, -1, 0, 1, 0, 1). \end{aligned}$$

Direct multiplication gives

$$\begin{aligned} R_\vartheta f_a - b_\vartheta &= (0, 0, 1, 0, 0, 0, 0, 0), \\ R_\vartheta f_b - b_\vartheta &= (0, 0, 0, 0, 0, 0, 0, 1). \end{aligned} \tag{31}$$

Thus $f_a, f_b \in S_\vartheta$. Together, the two slack vectors also show that each coordinate of b_ϑ is attained as a minimum on S_ϑ .

The required parity relation is

$$\begin{aligned} r_0 + r_1 + r_2 + r_7 &= (4, 4, 4, 4, 8, 4, 4, 4, 8, 6, 6, 10, 6, 10, 8) \\ &= 2(2, 2, 2, 2, 4, 2, 2, 2, 4, 3, 3, 5, 3, 5, 4), \end{aligned} \tag{32}$$

whereas the corresponding lower bounds have odd sum:

$$(b_\vartheta)_0 + (b_\vartheta)_1 + (b_\vartheta)_2 + (b_\vartheta)_7 = 1 + 1 + 1 + 10 = 13. \tag{33}$$

Consequently, no integral point can attain all eight lower bounds simultaneously: pairing with the even vector in (32) gives an even integer, while equality in the four indicated coordinates would give the odd integer in (33).

Suppose that $p \in P_{\vartheta}$ and $f_a - p \in S_{\vartheta}$. From (31),

$$0 \leq R_{\vartheta}p \leq (0, 0, 1, 0, 0, 0, 0, 0)$$

componentwise. Pairing p with (32) shows that

$$(R_{\vartheta}p)_0 + (R_{\vartheta}p)_1 + (R_{\vartheta}p)_2 + (R_{\vartheta}p)_7$$

is even. Only $(R_{\vartheta}p)_2$ can be nonzero, and it lies between zero and one. Hence $(R_{\vartheta}p)_2 = 0$, so $R_{\vartheta}p = 0$. Thus p is a unit, and f_a is minimal modulo units.

For f_b , the same argument leaves only $(R_{\vartheta}p)_7$ potentially nonzero. Parity forces this coordinate to be even, while (31) bounds it between zero and one. Therefore $R_{\vartheta}p = 0$, and f_b is also minimal modulo units.

The two elements are not unit translates because

$$R_{\vartheta}f_a \neq R_{\vartheta}f_b.$$

Indeed, unit translates have the same image under R_{ϑ} .

To make their positive translates meet, define

$$\begin{aligned} p_a &= (4, -1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, -2), \\ p_b &= (1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, -1). \end{aligned}$$

Their ray values are

$$R_{\vartheta}p_a = (0, 0, 0, 0, 0, 0, 0, 2), \quad R_{\vartheta}p_b = (0, 0, 1, 0, 0, 0, 0, 1).$$

Both are nonzero and nonnegative, so $p_a, p_b \in P_{\vartheta}$ are nonunits. Finally,

$$f_a + p_a = f_b + p_b = (0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 1, 0, 0). \quad (34)$$

Within the slice, we have found two elements that are minimal modulo units, are inequivalent modulo units, and have coinciding translates by nonunits. With the geometric cone and cell identifications certified in Section 3, the finite-generation statement following (30) and Lemma 2 prove that, for every field k , this fibre-degree summand is not flat. Hence the corresponding base change of the fixed-theta genus-five chart is not flat. Since flatness is preserved by base change, the integral fixed-theta chart, and therefore the fixed-theta mixed Delaunay–Voronoi morphism over $\mathbb{Z}$, is non-flat in genus five.

The ancillary script `verify_theta_certificate.py` independently checks the arithmetic from Section 5.2 through (34), using only integer and rational arithmetic.

6 Consequences in other genera

We now show that the genus-five examples propagate to higher genus, and we prove that the fixed-theta model is flat in genera at most four.

The genus- g cone stack for $\mathcal{N}_g$ contains the charts attached to every free lattice of rank at most g [4, Definition 4.14 and Proposition 6.18]. Consequently, the rank-five fixed-theta chart constructed above occurs on the boundary in every genus $g \geq 5$. If the global morphism were flat, its base change to this chart would be flat, contradicting the conclusion of Section 5. The fixed-theta morphism is therefore non-flat for every $g \geq 5$.

For $g \leq 4$, Nakamura–Sugawara [19, Theorem 1.3] prove that every Delaunay decomposition is simplicially generating; they attribute the statement to Voronoi [24], where they record it as only implicit, and [2, 1.14] attributes the corresponding total generation to Voronoi as well. In the form used here, the cone on each Delaunay cell is covered by generating subcones such that every lattice degree in one of those subcones is a nonnegative integral sum of lattice points of the cell; these lattice points are its vertices.

It remains to check that this result applies on every theta-characteristic chart. Let $\rho \in W/2W$, and choose a representative in W , also denoted ρ . Let $Q_\rho(W)$ be the corresponding lattice of integer-valued quadratic functions a on W satisfying $a(\rho - w) = a(w)$ and $a(0) = 0$. Modulo constants, this is the lattice $Q_\rho(X)$ of Faltings–Chai [9, Chapter IV, Section 7, pp. 132–133]; we use it in the form of [4, Section 4.3.2]. If $a \in Q_\rho(W)$ has associated symmetric bilinear form B , then

$$a(w) = \frac{1}{2}(B(w, w) - B(\rho, w)). \quad (35)$$

Thus a differs from the ordinary quadratic form $w \mapsto \frac{1}{2}B(w, w)$ by a linear function. Since the linear term ℓ in (2) varies freely, this difference is absorbed into ℓ . The resulting cells are therefore the ordinary Delaunay cells to which the theorem of Nakamura–Sugawara applies.

Moreover, the definition of $Q_\rho(W)$ gives

$$\epsilon_v(a) = a(v) \in \mathbb{Z} \quad \text{for every } v \in W.$$

Although (35) contains a factor of $1/2$, its value is integral by the defining condition on $Q_\rho(W)$. This construction is over $\mathbb{Z}$ and does not invert 2, so the same lattice and monoid calculation applies after base change to residue characteristic two.

Now fix any degree $c \in W$. The cones on the full-dimensional Delaunay cells in the star of 0 cover $W_{\mathbb{R}}$. Hence there is a full-dimensional cell D containing 0 such that $c \in \text{Cone}(D)$. By simplicial generation, c belongs to a generating subcone and therefore admits an expression

$$c = \sum_v n_v v \quad \text{with } n_v \in \mathbb{Z}_{\geq 0},$$

where v ranges over vertices of D . Consequently,

$$u_D(c) = \sum_v n_v \epsilon_v \pmod{\sigma^\perp}$$

has an integral representative, say $m_c \in M$. Formula (6) then gives

$$Q_c = m_c + P$$

and therefore

$$\mathbb{Z}[Q_c] = X^{m_c} \mathbb{Z}[P] \cong \mathbb{Z}[P]$$

as $\mathbb{Z}[P]$ -modules. Since this holds for every $c \in W$, the integral degree decomposition

$$\mathbb{Z}[Q] = \bigoplus_{c \in W} \mathbb{Z}[Q_c]$$

expresses $\mathbb{Z}[Q]$ as a free, and hence flat, $\mathbb{Z}[P]$ -module.

It remains to pass from the zero-cell charts to all mixed cones. Every Delaunay cell can first be translated so that one of its vertices is 0, and the mixed fan is invariant under these lattice

translations. A mixed cone attached to a nonzero-dimensional cell is then a face of the corresponding zero-cell cone; on affine toric charts, passage to such a face is localization [14, 5]. These translation and face compatibilities are part of the mixed-fan construction [4, Definition 4.28 and Definition 6.16]. Thus all mixed toric charts are flat over $\mathbb{Z}$.

Finally, the standard toroidal chart construction of [18, 3] and [9, Chapter IV, Section 7, and Chapter VI, Section 1], in the form of [4, Sections 2.5, 4.3.2, 4.3.4, and 6.2], supplies strict flat covers by these local toric models. Flatness is fpqc-local on both source and target [22, Tags 02L2 and 036K], so it descends from the toric charts to the morphism of algebraic stacks. This proves that the fixed-theta mixed Delaunay–Voronoi morphism is flat over $\mathbb{Z}$ for $g \leq 4$, including in residue characteristic two.

Combining this with the genus-five obstruction gives the sharp threshold

$$\text{the fixed-theta mixed Delaunay–Voronoi morphism is flat} \iff g \leq 4. \quad (36)$$

The classical rank-five chart likewise occurs on the boundary in every genus $g \geq 5$, so the classical morphism is non-flat in all those genera. The low-genus fixed-theta proof does not settle the corresponding classical question: the classical model uses $N_{\text{cl}} = \text{SymBil}(W, \mathbb{Z})$, whereas the fixed-theta model on the $\rho = 0$ chart uses the even-diagonal sublattice L_0 from (22); see [4, Sections 4.3.1–4.3.4 and 6.2] for the two integral chart constructions.

7 Reproduction and trust boundary

The certificate consists of four ancillary files accompanying this note: the static data file `counterexample_cells.py` and the three checking scripts named below. The computational material is divided according to its mathematical role.

The two arithmetic checkers verify the displayed lattice calculations and require only Python 3, with no further dependencies:

```
python3 verify_classical_certificate.py
python3 verify_theta_certificate.py
```

The geometric certificate requires SageMath. It is run from the directory containing the ancillary files:

```
sage -python verify_geometric_link.py both
```

It imports no search code; its only input is the explicit data in `counterexample_cells.py`. With SageMath 10.7 it runs in about ten seconds. Each script prints its checks and stops with an error if any assertion fails.

The SHA-256 checksums of the four files are:

```
counterexample_cells.py
f44b5c36133f66b798fdf76529097a39bd0c56da2b158adae9346d3f8ecf80e9
verify_classical_certificate.py
186a78c952c23be654ada9f0c290f5a0de41271286404465092f65405f6f768c
verify_geometric_link.py
4f9854d8c1304563ed0dc025d76f6360c286cdf07ac56e44ae8a42cd0e3c408e
verify_theta_certificate.py
4e880b9d8ad0c86a0132f43b25d8540d2c750db3f5433412e7c44c2fe51dee5d
```

The scripts certify the finite geometric presentations and the subsequent integer and rational arithmetic: the completeness of the 30- and 58-cell lists, the global empty-sphere inequalities, the

facet witnesses, and the equality of the displayed ray cones with the closed Delaunay cones. That is the machine-checked part of the argument, and it is the whole of it. Everything outside that boundary is ordinary mathematics, to be checked by reading: the standard toroidal-chart description of the mixed Delaunay–Voronoi models, the polyhedral duality behind the slice formula (6), the moduli-theoretic identification of the $\mathcal{N}_g$ cover, the local-algebra argument of Lemma 2, and the theorem of Nakamura–Sugawara.

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