

A PROOF OF THE SECOND PINCHING THEOREM IN DIMENSION FOUR

JINXI ZHANG

ABSTRACT. Let M^4 be a closed connected manifold minimally immersed in $\mathbb{S}^5(1)$, and let h be its second fundamental form. We prove that constant $S = |h|^2 > 4$ implies $S > 8$. No constancy of $\text{tr}(h^3)$ is assumed. The proof combines integral curvature identities with a joint estimate for two contractions of the symmetrized Hessian. The remaining algebraic inequality is verified in Mathematica using exact rational arithmetic. The code and data are included in the supplementary material.

1. INTRODUCTION

Let $F : M^n \rightarrow \mathbb{S}^{n+1}(1)$ be a smooth minimal immersion of a closed connected manifold, with second fundamental form h and $S = |h|^2$. Here closed means compact without boundary. The Gauss equation

$$\text{Scal}_M = n(n-1) - S$$

shows that the scalar curvature Scal_M of the induced metric is constant if and only if S is constant. Let $\mathcal{C}_n$ denote the set of constant values of S realized by such immersions.

Conjecture 1.1 (Chern; see [2, 8]). *For each $n \geq 2$, the set $\mathcal{C}_n$ is discrete.*

The *strong Chern conjecture* asserts that every such immersion with constant S is isoparametric: its principal curvatures and their multiplicities are locally constant; see [8, Conjecture 1.4]. In dimension four, the minimal isoparametric examples have $S = 0, 4, 12$ [1, 11, 12]. The *second pinching problem* asks whether $S > n$ implies $S \geq 2n$ when S is constant. We prove a strict bound in dimension four.

Theorem 1.2. *Let $F : M^4 \rightarrow \mathbb{S}^5(1)$ be a smooth minimal immersion of a nonempty closed connected manifold, and let h be its second fundamental form. If $S = |h|^2$ is constant and $S > 4$, then*

$$S > 8.$$

Recall the classical first pinching theorem.

Theorem 1.3 ([15, 9, 2]). *If M^n is closed and connected, $F : M^n \rightarrow \mathbb{S}^{n+1}(1)$ is minimal, and $S = |h|^2 \leq n$, then either $S \equiv 0$ or $S \equiv n$. The image is, respectively, an equatorial sphere or a Clifford minimal hypersurface*

$$\mathbb{S}^k\left(\sqrt{k/n}\right) \times \mathbb{S}^{n-k}\left(\sqrt{(n-k)/n}\right), \quad 1 \leq k \leq n-1.$$

Together, these theorems give the following classification for $S \leq 8$.

Corollary 1.4. *Let $F : M^4 \rightarrow \mathbb{S}^5(1)$ be a smooth minimal immersion of a nonempty closed connected manifold. Let h be its second fundamental form. If $S = |h|^2$ is constant and $S \leq 8$, then $S = 0$ or $S = 4$. Up to an ambient isometry, the image is an equatorial $\mathbb{S}^4$, or one of*

$$\mathbb{S}^1(1/2) \times \mathbb{S}^3(\sqrt{3}/2), \quad \mathbb{S}^2(1/\sqrt{2}) \times \mathbb{S}^2(1/\sqrt{2}).$$

In particular, $\mathcal{C}_4 \cap [0, 8] = \{0, 4\}$.

Date: 18 October 2026.

2020 Mathematics Subject Classification. 53C42, 53C24.

Key words and phrases. Minimal hypersurface, constant scalar curvature, Chern conjecture, second pinching problem.

Proof. If $S > 4$, Theorem 1.2 contradicts $S \leq 8$. Apply Theorem 1.3 to the remaining interval. $\square$

The region $S > 8$ remains unclassified here.

For earlier bounds on the second gap, see [13, 14, 16]. For closed connected minimal immersions with constant $S > n$, Chen [3, Theorem 1.1] obtains $S > 85n/59$ when $n \geq 4$. For complete minimal immersions in dimension four with constant $S > 4$, Cheng–Li–Wei [4, Theorem 1.2] obtain $S > 200000/29609$. Neither result assumes constant $f_3 = \text{tr}(h^3)$.

For closed connected minimal four-dimensional immersions with constant S , isoparametricity follows from constant third mean curvature $H_3 = f_3/12$ [8, Theorem 1.8] or constant Gauss–Kronecker curvature [5, Theorem 1.1]. With constant S, f_3 , the implication $S > n \Rightarrow S \geq 2n$ holds for closed connected minimal immersions with $n \geq 5$ [7, Theorem 1.1], and for closed oriented minimal immersions with $n \geq 3$ [6, Theorem 1.1].

Proof strategy. The argument combines pointwise estimates with integral curvature identities, as in [10, Sections 2 and 4] and [3, Section 4]. We use the curvature defect, Hessian symmetrization and two trace constraints in the form given in [3, Sections 2.2, 3.1 and 3.3].

The two contraction constraints act on the same reduced Hessian. A single square-completion estimate retains the mixed term between the two contractions. The second constraint involves ∇f_3 , represented after normalization by the vector d in (2.15).

Assume $4 < S \leq 8$ and put $t = (S - 4)/S$. A joint estimate for two contractions of the symmetrized Hessian, combined with three zero-average identities, gives a quadratic polynomial W_t with nonnegative average. For each fixed admissible pair of normalized tensors (Definition 2.2), $W_0 \leq 0$ and W_t is convex in t . Appendix A proves $W_{1/2} \leq -10^{-4}$ by a finite calculation using exact rational arithmetic. Convexity gives $W_t < 0$ throughout $0 < t \leq 1/2$, contradicting its nonnegative average.

2. CURVATURE IDENTITIES AND NORMALIZATION

We first collect the curvature identities that determine the Hessian norm, then normalize the second fundamental form and its first derivative. The normalization separates the constant S from the pointwise algebraic constraints.

Let $F : M^4 \rightarrow \mathbb{S}^5(1)$ be a smooth minimal immersion. The metric and covariant derivative ∇ are induced by F . Choose a local unit normal and an orthonormal tangent frame $e_1, \dots, e_4$. In this frame, let $h = (h_{ij})$ be the second fundamental form, $S = |h|^2$, and $f_j = \text{tr}(h^j)$. Powers of h refer to the associated self-adjoint endomorphism.

Indices range from 1 to 4. For covariant tensors of order r ,

$$\langle U, V \rangle = \sum_{i_1, \dots, i_r=1}^4 U_{i_1 \dots i_r} V_{i_1 \dots i_r}, \quad |U|^2 = \langle U, U \rangle.$$

Thus $\langle X, Y \rangle = \text{tr}(XY)$ for symmetric matrices. Write I for the 4×4 identity matrix and $\text{Sym}^r \mathbb{R}^4$ for the symmetric covariant r -tensors. For symmetric matrices, $X \preceq Y$ means $v^T(Y - X)v \geq 0$ for every $v \in \mathbb{R}^4$; $X \prec Y$ means strict inequality for every $v \neq 0$. The derivative and Laplacian conventions are

$$h_{ijk} = (\nabla_k h)_{ij}, \quad h_{ijkl} = (\nabla_l \nabla h)_{ijk}, \quad \Delta = \text{div } \nabla.$$

For a scalar function f , write $f_{;k} = (\nabla f)_k$. On tensors, Δ is the rough Laplacian with the same sign. Pointwise formulas use a local unit normal. If M is closed and has no global unit normal, the integral formulas are applied on its compact unit normal double cover.

The scalar integrands are unchanged by normal reversal, so their normalized integrals agree with those on M .

Let δ_{ij} be the Kronecker symbol and Rm_{ijkl} the curvature tensor. The Gauss, Codazzi and Ricci equations are

$$(2.1) \quad \text{Rm}_{ijkl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} + h_{ik}h_{jl} - h_{il}h_{jk},$$

$$(2.2) \quad h_{ijk} = h_{ikj},$$

$$(2.3) \quad h_{ijkl} - h_{ijlk} = \sum_r h_{rj}\text{Rm}_{rikl} + \sum_r h_{ir}\text{Rm}_{rjkl}.$$

Thus ∇h is fully symmetric and $\nabla^2 h$ is symmetric in its first three indices. At a fixed point choose a principal frame $h_{ij} = \lambda_i \delta_{ij}$, where λ_i are the principal curvatures. Covariant differentiations are performed before this choice. In particular,

$$(2.4) \quad h_{iijj} - h_{jjii} = (\lambda_i - \lambda_j)(1 + \lambda_i \lambda_j).$$

Define the three scalar contractions

$$(2.5) \quad \mathcal{A} = \sum_{ijk} \lambda_i^2 h_{ijk}^2, \quad \mathcal{B} = \sum_{ijk} \lambda_i \lambda_j h_{ijk}^2, \quad \mathcal{C} = \sum_{ijk} \lambda_i h_{ijk}^2.$$

Lemma 2.1 (Curvature identities). *Let $F : M^4 \rightarrow \mathbb{S}^5(1)$ be a smooth minimal immersion. Let h be its second fundamental form, $S = |h|^2$, $f_j = \text{tr}(h^j)$, and let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be as in (2.5). Then*

$$(2.6) \quad \Delta h = (4 - S)h, \quad \frac{1}{2}\Delta S = |\nabla h|^2 + S(4 - S),$$

$$(2.7) \quad \frac{1}{2}\Delta|\nabla h|^2 = |\nabla^2 h|^2 + (11 - S)|\nabla h|^2 - 3(\mathcal{A} - 2\mathcal{B}) - \frac{3}{2}|\nabla S|^2.$$

Moreover,

$$(2.8) \quad \Delta f_3 = 3(4 - S)f_3 + 6\mathcal{C}, \quad (f_3)_{;k} = 3 \sum_i \lambda_i^2 h_{iik},$$

$$(2.9) \quad \Delta f_4 = 4(4 - S)f_4 + 8\mathcal{A} + 4\mathcal{B}.$$

Proof. The tensor equation and the first scalar equation in (2.6) are Simons' identities [15]. Equation (2.7) is the Peng-Terng identity [13, 14]; the scalar formulas, with these signs, are recorded in [10, Section 2, equations (2.1)–(2.2)]. The product rule applied to $\text{tr}(h^3)$ and $\text{tr}(h^4)$ gives (2.8) and (2.9). $\square$

2.1. Normalized tensors. From this point on, the geometric applications have constant $S > 4$. The following normalization gives unit tensors and puts the interval $4 < S \leq 8$ in correspondence with $0 < t \leq 1/2$. Define

$$(2.10) \quad t = \frac{S - 4}{S}, \quad A = \frac{h}{\sqrt{S}}, \quad \mu_i = \frac{\lambda_i}{\sqrt{S}}, \quad \tau = \frac{\nabla h}{\sqrt{S(S - 4)}}.$$

Simons' identity gives $|\nabla h|^2 = tS^2$. Thus A and τ have norm one.

Definition 2.2 (Admissible pair). An *admissible pair* consists of a symmetric matrix $A \in \mathbb{R}^{4 \times 4}$ and a tensor $\tau \in \text{Sym}^3 \mathbb{R}^4$ such that

$$(2.11) \quad \text{tr } A = 0, \quad |A| = |\tau| = 1, \quad \sum_i \tau_{iik} = 0, \quad \sum_{ij} A_{ij} \tau_{ijk} = 0 \quad (1 \leq k \leq 4).$$

In a basis with $A = \text{diag}(\mu_1, \dots, \mu_4)$, the last condition is $\sum_i \mu_i \tau_{iik} = 0$. All component formulas below use such a basis unless a different basis is specified.

The pair (2.10) is admissible: differentiating $\text{tr } h = 0$ gives its ordinary trace condition, and differentiating S gives its A -weighted trace condition.

For each k , the symmetric matrix $(\tau_{ijk})_{ij}$ is orthogonal to I and A by the two trace constraints. Its contraction with A^2 therefore depends only on the component of A^2 orthogonal

to I and A . For an admissible pair (A, τ) , define the scalars m, u, p and the symmetric matrix Φ by

$$(2.12) \quad m = \operatorname{tr} A^3, \quad u = \operatorname{tr} A^4,$$

$$(2.13) \quad \Phi = A^2 - mA - \frac{1}{4}I, \quad p = |\Phi|^2 = u - m^2 - \frac{1}{4}.$$

The scalars a, b, c below are the normalized versions of $\mathcal{A}, \mathcal{B}, \mathcal{C}$. We also define the contraction d of A^2 with τ and the Gram matrix R of the four matrices $(\tau_{ijk})_{ij}$, indexed by k :

$$(2.14) \quad a = \sum_{ijk} \mu_i^2 \tau_{ijk}^2, \quad b = \sum_{ijk} \mu_i \mu_j \tau_{ijk}^2, \quad c = \sum_{ijk} \mu_i \tau_{ijk}^2,$$

$$(2.15) \quad d_k = \sum_i \mu_i^2 \tau_{iik},$$

$$(2.16) \quad R_{kl} = \sum_{ij} \tau_{ijk} \tau_{ijl}.$$

Thus $d \in \mathbb{R}^4$ and $R \in \operatorname{Sym}^2 \mathbb{R}^4$, with $R \succeq 0$ and $\operatorname{tr} R = 1$. For the geometric pair (2.10), these contractions recover the unnormalized quantities:

$$\mathcal{A} = tS^3 a, \quad \mathcal{B} = tS^3 b, \quad \mathcal{C} = tS^{5/2} c.$$

The vector d records the variation of the cubic trace:

$$\nabla f_3 = 3S^2 \sqrt{t} d.$$

The two trace constraints in (2.11) do not imply $d = 0$. The identities $\operatorname{tr} A = 0$ and $|A| = 1$ imply

$$(2.17) \quad \operatorname{tr} \Phi = \operatorname{tr}(A\Phi) = 0, \quad p \geq 0, \quad \mu_i^2 \leq \frac{3}{4}.$$

The last inequality follows from $\mu_i^2 = (\sum_{j \neq i} \mu_j)^2 \leq 3(1 - \mu_i^2)$. Since S and $|\nabla h|^2$ are constant, (2.7) now reads

$$(2.18) \quad \frac{|\nabla^2 h|^2}{S^3} = 3t(a - 2b) + \frac{t(11t - 7)}{4}.$$

3. THE REDUCED HESSIAN

Equation (2.18) gives the normalized squared norm of the Hessian. Differentiating the constant quantities $\operatorname{tr} h$, S and $|\nabla h|^2$ gives three of its contractions. To use this information in a norm estimate, we pass to a fully symmetric tensor and remove its ordinary trace. The tensor that remains will have two prescribed contractions and a known squared norm.

3.1. The reduction statement. The three contractions are the ordinary trace, the A -weighted trace, and contraction against the cubic tensor τ .

Definition 3.1 (Contraction maps). Fix an admissible pair (A, τ) . Define

$$\mathsf{T}_0, \mathsf{T}_A : \operatorname{Sym}^4 \mathbb{R}^4 \longrightarrow \operatorname{Sym}^2 \mathbb{R}^4, \quad \mathsf{N}_\tau : \operatorname{Sym}^4 \mathbb{R}^4 \longrightarrow \mathbb{R}^4$$

by

$$(3.1) \quad (\mathsf{T}_0 U)_{kl} = \sum_i U_{iikl}, \quad (\mathsf{T}_A U)_{kl} = \sum_i \mu_i U_{iikl}, \quad (\mathsf{N}_\tau U)_l = \sum_{ijk} \tau_{ijk} U_{ijkl}.$$

Let

$$\mathcal{H}_4 = \{U \in \operatorname{Sym}^4 \mathbb{R}^4 : \mathsf{T}_0 U = 0\}, \quad \mathsf{P}_0 : \operatorname{Sym}^4 \mathbb{R}^4 \longrightarrow \mathcal{H}_4$$

be, respectively, the trace-free subspace and its orthogonal projection. For four-tensors symmetric only in the first three indices, $\mathsf{T}_0, \mathsf{T}_A, \mathsf{N}_\tau$ still denote the component contractions in (3.1).

The two contractions retained after trace removal will be denoted by E_t and b_0 . Write

$$\text{Sym}_0^2 \mathbb{R}^4 = \{X \in \mathbb{R}^{4 \times 4} : X^T = X, \text{tr } X = 0\}.$$

For an admissible pair (A, τ) and $t \in \mathbb{R}$, define

$$(3.2) \quad \Psi = \frac{1}{4}A^2 + \frac{3}{16}I - R, \quad E_t = \frac{1}{2}\Phi + t\Psi, \quad b_0 = -\frac{3}{2}Ad.$$

Here $\Psi, E_t \in \text{Sym}_0^2 \mathbb{R}^4$ and $b_0 \in \mathbb{R}^4$. The term $\Phi/2$ is independent of t , while Ψ is the coefficient of t in E_t .

Proposition 3.2 (Hessian contractions). *Let $F : M^4 \rightarrow \mathbb{S}^5(1)$ be a smooth minimal immersion with constant $S = |h|^2 > 4$. At each point, define t, A, τ by (2.10) and E_t, b_0 by (3.2). There exists $V \in \mathcal{H}_4$ such that*

$$(3.3) \quad \mathsf{T}_A V = E_t,$$

$$(3.4) \quad \mathsf{N}_\tau V = b_0,$$

$$(3.5) \quad |V|^2 = 3t(a - 2b) + \frac{t(35t - 28)}{16} - \frac{3}{2}p.$$

3.2. Symmetrization and trace removal. We prove the proposition by subtracting two orthogonal corrections: the part fixed by the Ricci equation, and the least-norm tensor with the remaining ordinary trace. The latter correction is supplied by Lemma 3.3.

For a covariant r -tensor T , define

$$(\text{Sym } T)_{i_1 \dots i_r} = \frac{1}{r!} \sum_{\sigma \in \mathfrak{S}_r} T_{i_{\sigma(1)} \dots i_{\sigma(r)}},$$

where $\mathfrak{S}_r$ is the permutation group. Adjoints L^* are taken with respect to the ordered-index inner product of Section 2. The product $F \otimes G$ denotes the tensor product without symmetrization.

Lemma 3.3 (Ordinary trace and its adjoint). *For every $X \in \text{Sym}^2 \mathbb{R}^4$,*

$$(3.6) \quad \mathsf{T}_0^* X = \text{Sym}(I \otimes X), \quad \mathsf{T}_0 \mathsf{T}_0^* X = \frac{4}{3}X + \frac{1}{6} \text{tr}(X)I,$$

$$(3.7) \quad (\mathsf{T}_0 \mathsf{T}_0^*)^{-1} X = \frac{3}{4}X - \frac{1}{16} \text{tr}(X)I, \quad \mathsf{P}_0 = \text{Id}_{\text{Sym}^4 \mathbb{R}^4} - \mathsf{T}_0^* (\mathsf{T}_0 \mathsf{T}_0^*)^{-1} \mathsf{T}_0.$$

Proof. The adjoint formula follows by symmetry. In components,

$$(\mathsf{T}_0^* X)_{ijkl} = \frac{1}{6}(\delta_{ij}X_{kl} + \delta_{ik}X_{jl} + \delta_{il}X_{jk} + \delta_{jk}X_{il} + \delta_{jl}X_{ik} + \delta_{kl}X_{ij}).$$

Tracing the first two indices proves (3.6). This operator acts by $4/3$ on trace-free matrices and by 2 on multiples of I , giving the stated inverse. In (3.7), the subtracted tensor has the same ordinary trace as the input and lies in $\text{im } \mathsf{T}_0^* = (\ker \mathsf{T}_0)^\perp$. The remainder is therefore the orthogonal projection onto $\mathcal{H}_4$. $\square$

Proof of Proposition 3.2. Set $\mathcal{D} = S^{-3/2} \nabla^2 h$. Since S is constant, (2.6) makes $|\nabla h|^2$ constant as well. Differentiating $\text{tr } h = 0$ and S twice, and $|\nabla h|^2$ once, gives

$$(3.8) \quad \mathsf{T}_0 \mathcal{D} = 0, \quad \mathsf{T}_A \mathcal{D} = -tR, \quad \mathsf{N}_\tau \mathcal{D} = 0.$$

First remove the failure of full symmetry. Put $\hat{U} = \text{Sym } \mathcal{D}$, $C = \mathcal{D} - \hat{U}$ and $\kappa = S^{-1} = (1-t)/4$. Since $\mathcal{D}$ is symmetric in its first three indices,

$$\hat{U}_{ijkl} = \frac{1}{4}(\mathcal{D}_{ijkl} + \mathcal{D}_{ijlk} + \mathcal{D}_{iklj} + \mathcal{D}_{jkl i}).$$

For $i \neq j$, symmetry in the first three indices gives $\widehat{U}_{ijj} = (\mathcal{D}_{ijj} + \mathcal{D}_{jji})/2$. Thus $C_{ijj} = (\mathcal{D}_{ijj} - \mathcal{D}_{jji})/2$; (2.4), divided by $S^{3/2}$, determines its sign and the factor $1/2$. In a principal frame, (2.1) and (2.3) show that C vanishes outside the index patterns $\{i, i, j, j\}$. The possibly nonzero components are therefore

$$\begin{aligned} C_{ijj} &= \frac{1}{2}(\mu_i - \mu_j)(\kappa + \mu_i \mu_j), & C_{ijj} &= C_{jii} = C_{ijj}, \\ C_{jji} &= C_{jji} = C_{ijj} = -C_{ijj}. \end{aligned}$$

Summing these components, using $\sum_i \mu_i = 0$, $\sum_i \mu_i^2 = 1$ and $\sum_i \mu_i^3 = m$, gives the first two contractions below. For the third, each pattern (i, i, l) with $i \neq l$ occurs three times in the ordered contraction, giving

$$(3.9) \quad \mathsf{T}_0 C = \frac{t}{2} A, \quad \mathsf{T}_A C = \frac{1}{2}(\kappa I + mA - A^2),$$

$$(3.10) \quad (\mathsf{N}_\tau C)_l = \frac{3}{2} \sum_i (\mu_i - \mu_l)(\kappa + \mu_i \mu_l) \tau_{iil} = \frac{3}{2} \mu_l d_l.$$

Indeed, the weight in this sum is $(\kappa - \mu_l^2)\mu_i - \kappa \mu_l + \mu_l \mu_i^2$. The first two terms vanish after summation by the two trace constraints; the last gives $\mu_l d_l$ by (2.15). Subtracting these contractions from (3.8) gives

$$(3.11) \quad \mathsf{T}_0 \widehat{U} = -\frac{t}{2} A, \quad \mathsf{T}_A \widehat{U} = -tR - \frac{1}{2}(\kappa I + mA - A^2), \quad \mathsf{N}_\tau \widehat{U} = -\frac{3}{2} Ad.$$

Next remove the ordinary trace of $\widehat{U}$. By Lemma 3.3, the least-norm tensor with trace $-tA/2$ is

$$U_0 = \mathsf{T}_0^*(\mathsf{T}_0 \mathsf{T}_0^*)^{-1}(-tA/2) = -\frac{3t}{8} \text{Sym}(I \otimes A).$$

Its squared norm follows from $|U_0|^2 = \langle -tA/2, (\mathsf{T}_0 \mathsf{T}_0^*)^{-1}(-tA/2) \rangle$; its contractions follow from the six-term adjoint formula in Lemma 3.3. Thus

$$(3.12) \quad |U_0|^2 = \frac{3}{16} t^2, \quad \mathsf{T}_A U_0 = -\frac{t}{16}(I + 4A^2), \quad \mathsf{N}_\tau U_0 = 0.$$

Set $V = \widehat{U} - U_0$. Then $V \in \mathcal{H}_4$ and (3.11)–(3.12) give (3.3) and (3.4).

It remains to compute $|V|^2$. Symmetrization is an orthogonal projection, so $C \perp \text{Sym}^4 \mathbb{R}^4$; also $U_0 \in (\ker \mathsf{T}_0)^\perp$ and $V \in \ker \mathsf{T}_0$. Thus $\mathcal{D} = C + U_0 + V$ is an orthogonal decomposition.

The components of C give

$$(3.13) \quad |C|^2 = \frac{3}{4} \sum_{ij} (\mu_i - \mu_j)^2 (\kappa + \mu_i \mu_j)^2 = \frac{3}{2} p + \frac{3}{8} t^2.$$

For each unordered pair $\{i, j\}$, the six ordered components displayed above have squared sum $6C_{ijj}^2$; this gives the factor $3/4$ in the sum over ordered pairs. Expanding the sum before multiplication by $3/4$, and using $\sum_i \mu_i = 0$, $\sum_i \mu_i^2 = 1$ gives $8\kappa^2 - 4\kappa + 2u - 2m^2 = 2p + t^2/2$. Together with $|U_0|^2$ in (3.12), this gives

$$(3.14) \quad \frac{|\nabla^2 h|^2}{S^3} = |V|^2 + \frac{3}{2} p + \frac{9}{16} t^2.$$

Equation (2.18) now gives (3.5). □

4. THE JOINT CONTRACTION ESTIMATE

We now bound the squared norm of a tensor in $\mathcal{H}_4$ from its two prescribed contractions. The estimate is purely algebraic. For the geometric application in Proposition 3.2, take $E = E_t$ and $b_0 = -\frac{3}{2} Ad$. Fix an admissible pair (A, τ) .

4.1. The estimate. Suppose $V \in \mathcal{H}_4$ satisfies $\mathsf{T}_A V = E$ and $\mathsf{N}_\tau V = b_0$, where $E \in \text{Sym}_0^2 \mathbb{R}^4$ and $b_0 \in \mathbb{R}^4$. For any $Y \in \text{Sym}_0^2 \mathbb{R}^4$ and $v \in \mathbb{R}^4$, square completion gives

$$|V|^2 \geq 2\langle E, Y \rangle + 2b_0 \cdot v - |\mathsf{P}_0(\mathsf{T}_A^* Y + \mathsf{N}_\tau^* v)|^2,$$

where $b_0 \cdot v = b_0^T v$. The two terms inside the projection contribute a cross term. The squared norm has a matrix term, a mixed term and a vector term, represented by $\mathcal{K}_A$, Z and H_τ , respectively.

For $Y \in \text{Sym}_0^2 \mathbb{R}^4$, set

$$\begin{aligned} \mathcal{K}_A Y &= \frac{1}{6}Y + \frac{1}{4}(A^2 Y + Y A^2) - \frac{1}{6}A Y A \\ &\quad + \frac{7}{36} \text{tr}(A Y) A - \frac{1}{12} \text{tr}(A^2 Y) I, \end{aligned} \quad (4.1)$$

$$(ZY)_l = \frac{1}{8} \sum_{ij} (4\mu_l - \mu_i - \mu_j) \tau_{ijl} Y_{ij}, \quad H_\tau = \frac{1}{4}I + \frac{9}{16}R. \quad (4.2)$$

The map Z depends on both A and τ . The domains and codomains are

$$\mathcal{K}_A : \text{Sym}_0^2 \mathbb{R}^4 \rightarrow \text{Sym}_0^2 \mathbb{R}^4, \quad Z : \text{Sym}_0^2 \mathbb{R}^4 \rightarrow \mathbb{R}^4, \quad H_\tau : \mathbb{R}^4 \rightarrow \mathbb{R}^4.$$

Proposition 4.1 (Two prescribed contractions). *Let (A, τ) be admissible. Let $E \in \text{Sym}_0^2 \mathbb{R}^4$, $b_0 \in \mathbb{R}^4$, and $V \in \mathcal{H}_4$ satisfy*

$$\mathsf{T}_A V = E, \quad \mathsf{N}_\tau V = b_0.$$

For every $Y \in \text{Sym}_0^2 \mathbb{R}^4$ and $v \in \mathbb{R}^4$,

$$|V|^2 \geq 2\langle E, Y \rangle + 2b_0 \cdot v - \langle Y, \mathcal{K}_A Y \rangle - 2v \cdot ZY - v^T H_\tau v. \quad (4.3)$$

4.2. The Gram identities. To prove Proposition 4.1, we identify the three terms in the projected squared norm. The same calculation gives a lower bound for $\mathcal{K}_A$ that will control the quadratic coefficient in Section 5.

Lemma 4.2 (Contraction identities). *For an admissible pair (A, τ) ,*

$$\mathsf{T}_A \mathsf{P}_0 \mathsf{T}_A^* = \mathcal{K}_A, \quad \mathsf{N}_\tau \mathsf{P}_0 \mathsf{T}_A^* = Z, \quad \mathsf{N}_\tau \mathsf{P}_0 \mathsf{N}_\tau^* = H_\tau. \quad (4.4)$$

The first two identities act on $\text{Sym}_0^2 \mathbb{R}^4$, and the third on $\mathbb{R}^4$. For every $Y \in \text{Sym}_0^2 \mathbb{R}^4$,

$$\langle Y, \mathcal{K}_A Y \rangle \geq \frac{1}{6}|Y|^2 + \frac{1}{3} \text{tr}(A^2 Y^2). \quad (4.5)$$

Proof. The adjoints are $\mathsf{T}_A^* Y = \text{Sym}(A \otimes Y)$ and $\mathsf{N}_\tau^* v = \text{Sym}(\tau \otimes v)$. For the matrix term, compute the unprojected contraction and then subtract the ordinary trace correction from (3.7):

$$\begin{aligned} \mathsf{T}_A \mathsf{T}_A^* Y &= \frac{1}{6}Y + \frac{1}{3}(A^2 Y + Y A^2) + \frac{1}{6} \text{tr}(A Y) A, \\ \mathsf{T}_0 \mathsf{T}_A^* Y &= \frac{1}{3}(A Y + Y A), \\ \mathsf{T}_A \mathsf{T}_0^* (\mathsf{T}_0 \mathsf{T}_0^*)^{-1} \mathsf{T}_0 \mathsf{T}_A^* Y &= \frac{1}{12}(A^2 Y + 2A Y A + Y A^2) - \frac{1}{36} \text{tr}(A Y) A + \frac{1}{12} \text{tr}(A^2 Y) I. \end{aligned}$$

The first expression minus the third is $\mathcal{K}_A Y$, proving the matrix identity.

For the mixed term, the vanishing A -weighted trace of τ gives

$$(\mathsf{N}_\tau \mathsf{T}_A^* Y)_l = \frac{1}{2} \mu_l \sum_{ij} \tau_{ijl} Y_{ij}.$$

Substitution in (3.7) gives the trace correction $\frac{1}{8} \sum_{ij} (\mu_i + \mu_j) \tau_{ijl} Y_{ij}$; the scalar part of $(\mathsf{T}_0 \mathsf{T}_0^*)^{-1}$ contributes nothing because the ordinary trace of τ vanishes. Subtraction gives Z .

For the vector term,

$$\mathsf{N}_\tau \mathsf{N}_\tau^* = \frac{1}{4}I + \frac{3}{4}R, \quad (\mathsf{T}_0 \mathsf{N}_\tau^* v)_{ij} = \frac{1}{2} \sum_k \tau_{ijk} v_k.$$

The latter matrix is trace-free and has squared norm $v^T R v / 4$. On trace-free matrices, $(\mathsf{T}_0 \mathsf{T}_0^*)^{-1}$ acts by $3/4$. The projection therefore subtracts $3R/16$, giving H_τ .

To obtain (4.5), expand

$$\langle Y, \mathcal{K}_A Y \rangle = \frac{1}{6} |Y|^2 + \frac{1}{2} \operatorname{tr}(A^2 Y^2) - \frac{1}{6} \operatorname{tr}(A Y A Y) + \frac{7}{36} \operatorname{tr}(A Y)^2$$

The squared trace term is nonnegative, and

$$\operatorname{tr}(A^2 Y^2) - \operatorname{tr}(A Y A Y) = \frac{1}{2} \sum_{ij} (\mu_i - \mu_j)^2 Y_{ij}^2 \geq 0. \quad \square$$

Proof of Proposition 4.1. Set $U = \mathbf{P}_0(\mathbf{T}_A^* Y + \mathbf{N}_\tau^* v)$. Since $V = \mathbf{P}_0 V$,

$$\langle V, U \rangle = \langle E, Y \rangle + b_0 \cdot v.$$

Lemma 4.2 gives

$$|U|^2 = \langle Y, \mathcal{K}_A Y \rangle + 2v \cdot ZY + v^T H_\tau v.$$

Substitution in $0 \leq |V - U|^2 = |V|^2 - 2\langle V, U \rangle + |U|^2$ proves (4.3). $\square$

5. THE AVERAGED INEQUALITY AND THE CONTRADICTION

For the geometric tensor in Proposition 3.2, the difference between its squared norm and the lower bound in Proposition 4.1 is itself a squared norm. We will add three divergence terms to this difference, preserving its average. The resulting polynomial W_t will be quadratic in t ; convexity will reduce its sign on $0 < t \leq 1/2$ to the two endpoints.

For a nonempty closed Riemannian manifold M and an integrable scalar function g , write

$$\langle g \rangle_M = \frac{1}{\operatorname{Vol}(M)} \int_M g \, d\operatorname{Vol}.$$

5.1. The polynomial W_t . We use (4.3) with $E = E_t$. Its terms linear in Y are $2\langle E_t - Z^* v, Y \rangle$. We take v proportional to b_0 and Y_t linear in t , with coefficient proportional to the matrix $E_{1/2} - Z^* v$.

For an admissible pair (A, τ) and real t , define

$$(5.1) \quad L_4 = t(2a + b - u), \quad L_3 = t(-m^2 + 2mc + 3|d|^2), \quad J = t(a - 2b) - p - \frac{t}{4}.$$

On the geometric pair these are divergences; their zero averages will be proved in Lemma 5.3.

Definition 5.1 (The polynomial W_t). Fix an admissible pair (A, τ) . For a real parameter t , set

$$(5.2) \quad v = -7Ad, \quad B_t = E_t - Z^* v, \quad B = B_{1/2}, \quad Y_t = 8tB.$$

Thus $v \in \mathbb{R}^4$, $B_t, B, Y_t \in \operatorname{Sym}_0^2 \mathbb{R}^4$, and $Y_{1/2} = 4B$. The pair (A, τ) and the matrix B are held fixed as t varies. Let Q_t be the right-hand side of (4.3) for these choices. Using the norm expression (3.5) and L_4, L_3, J from (5.1), set

$$(5.3) \quad Q_t = 16t\langle B_t, B \rangle - 64t^2\langle B, \mathcal{K}_A B \rangle + 2b_0 \cdot v - v^T H_\tau v,$$

$$(5.4) \quad W_t = 3t(a - 2b) + \frac{t(35t - 28)}{16} - \frac{3}{2}p - Q_t - \frac{89}{8}L_4 + 16L_3 + \frac{13}{4}J.$$

The adjoint $Z^* : \mathbb{R}^4 \rightarrow \operatorname{Sym}_0^2 \mathbb{R}^4$ and B have components

$$(5.5) \quad \begin{aligned} (Z^* v)_{ij} &= \frac{1}{8} \sum_l (4\mu_l - \mu_i - \mu_j) \tau_{ijl} v_l, \\ B &= \frac{5}{8} A^2 - \frac{m}{2} A - \frac{1}{32} I - \frac{1}{2} R + 7Z^* Ad. \end{aligned}$$

The matrix trial vanishes at $t = 0$ and is proportional to $B_{1/2}$ at $t = 1/2$. Keeping $B = B_{1/2}$ fixed makes W_t quadratic in t ; the divergence terms change only its constant and linear coefficients. The constants in (5.2) and (5.4) must give nonpositivity at $t = 0$, convexity in t , and strict negativity at $t = 1/2$. Proposition 5.4 proves the first two; Appendix A proves the third using positive-definite Gram representations checked in exact rational arithmetic.

5.2. The nonnegative average. The formulas for a, b, c, d, R are tensor contractions of A and τ , and (4.4) expresses $\mathcal{K}_A, Z, H_\tau$ in terms of contractions and orthogonal projection. Thus W_t is independent of the orthonormal frame. Reversing the unit normal sends (A, τ) to $(-A, -\tau)$ and also leaves W_t unchanged. For the geometric pair, W_t therefore defines a smooth scalar function on M , including at repeated principal curvatures.

The joint estimate gives a nonnegative average for W_t once L_4, L_3, J are shown to have zero average.

Proposition 5.2 (The integral inequality). *Let $F : M^4 \rightarrow \mathbb{S}^5(1)$ be a smooth minimal immersion of a nonempty closed manifold, with constant $S = |h|^2 > 4$. Define t, A, τ by (2.10) and W_t by Definition 5.1. Then*

$$(5.6) \quad \langle W_t \rangle_M \geq 0.$$

The following divergence identities supply these zero averages.

Lemma 5.3 (Three zero averages). *Let $F : M^4 \rightarrow \mathbb{S}^5(1)$ be a smooth minimal immersion of a nonempty closed manifold, with constant $S = |h|^2 > 4$. Define t, A, τ by (2.10), and L_4, L_3, J by (5.1). Then*

$$(5.7) \quad L_4 = \frac{\Delta(Sf_4/4)}{S^4}, \quad L_3 = \frac{\Delta(f_3^2/6)}{S^4}, \quad J = -\frac{1}{S^3} \operatorname{div}(h\nabla f_3/3).$$

Consequently,

$$\langle L_4 \rangle_M = \langle L_3 \rangle_M = \langle J \rangle_M = 0.$$

Proof. Equation (2.9) gives the first formula. For the second, use

$$\Delta(f_3^2/6) = \frac{1}{3}f_3\Delta f_3 + \frac{1}{3}|\nabla f_3|^2, \quad |\nabla f_3|^2 = 9tS^4|d|^2.$$

For the third identity, put $D = \nabla f_3/3$ and compute $\operatorname{div}(hD)$. In a principal frame, $D_k = \sum_i \lambda_i^2 h_{iik} = (f_3)_{;k}/3$. Codazzi and $\operatorname{tr} h = 0$ give $\operatorname{div} h = 0$, so

$$\begin{aligned} \operatorname{div}(hD) &= 2\mathcal{B} + \sum_{ik} \lambda_i^2 \lambda_k h_{iik} \\ &= 2\mathcal{B} - \mathcal{A} + \sum_{ik} \lambda_i^2 \lambda_k (\lambda_i - \lambda_k)(1 + \lambda_i \lambda_k) \\ &= 2\mathcal{B} - \mathcal{A} + Sf_4 - f_3^2 - S^2. \end{aligned}$$

In the second line we used (2.4) and $\sum_k \lambda_k h_{kkii} = -\sum_{jk} h_{jki}^2$, obtained by differentiating S twice. Division by $-S^3$ gives the stated J . The integral identities follow from the divergence theorem. $\square$

Proof of Proposition 5.2. Take V from Proposition 3.2 and put $U_t = P_0(T_A^* Y_t + N_\tau^* v)$. The adjoint relations and Lemma 4.2 give

$$|V|^2 - Q_t = |V - U_t|^2.$$

By (3.5), W_t differs from $|V|^2 - Q_t$ by the linear combination of L_4, L_3, J in (5.4). This combination has zero average by Lemma 5.3, so

$$\langle W_t \rangle_M = \langle |V - U_t|^2 \rangle_M \geq 0. \quad \square$$

5.3. The value at zero and convexity. It remains to make W_t pointwise negative on $0 < t \leq 1/2$. We first control its value at zero and its quadratic coefficient. The evaluation at $t = 0$ is algebraic: the geometric normalization of τ requires $S > 4$.

Proposition 5.4 (Convexity in t). *Fix an admissible pair (A, τ) , and let W_t be the polynomial in Definition 5.1. With $p = |A^2 - mA - I/4|^2$ and $m = \operatorname{tr} A^3$, one has*

$$(5.8) \quad W_0 \leq -\frac{75}{32}p \leq 0, \quad \frac{d^2}{dt^2} W_t \geq \frac{1}{3} \quad (t \in \mathbb{R}).$$

The value at zero is controlled by H_τ and Ad . The quadratic coefficient is controlled by Ψ and $\mathcal{K}_A$. The next two lemmas give these bounds.

Lemma 5.5 (Bounds for R and Ad). *For an admissible pair (A, τ) ,*

$$(5.9) \quad R \preceq \frac{2}{3}I, \quad H_\tau \preceq \frac{5}{8}I, \quad |Ad|^2 \leq \frac{p}{4}.$$

Proof. To estimate R in a unit direction, choose that direction as e_1 and split τ into

$$a_0 = \tau_{111}, \quad \xi_\alpha = \tau_{11\alpha}, \quad \Gamma_{\alpha\beta} = \tau_{1\alpha\beta}, \quad \Xi_{\alpha\beta\gamma} = \tau_{\alpha\beta\gamma}, \quad \alpha, \beta, \gamma > 1.$$

Then $\text{tr } \Gamma = -a_0$ and

$$|\tau|^2 = a_0^2 + 3|\xi|^2 + 3|\Gamma|^2 + |\Xi|^2, \quad R_{11} = a_0^2 + 2|\xi|^2 + |\Gamma|^2.$$

Cauchy–Schwarz gives $a_0^2 = (\text{tr } \Gamma)^2 \leq 3|\Gamma|^2$. Consequently

$$2|\tau|^2 - 3R_{11} = -a_0^2 + 3|\Gamma|^2 + 2|\Xi|^2 \geq 0.$$

This holds in every direction and proves the first two bounds.

For the bound on Ad , choose a basis with A diagonal. Fix k and set $z^{(k)} = (\tau_{iik})_i$ and $\theta_k = 3|z^{(k)}|^2 - 2\tau_{kkk}^2$. Here θ_k is the contribution of the index patterns $\{i, i, k\}$ to the ordered norm: each pattern with $i \neq k$ occurs three times, while (k, k, k) occurs once. Write $\mathbf{1} = (1, 1, 1, 1)^T$ and $\mu = (\mu_1, \dots, \mu_4)^T$. The two trace conditions give $z^{(k)} \perp \mathbf{1}, \mu$.

The projection of e_k onto this common orthogonal complement has squared norm $3/4 - \mu_k^2$. Hence $\tau_{kkk}^2 \leq (3/4 - \mu_k^2)|z^{(k)}|^2$. Using $d_k = \sum_i \Phi_{ii} z_i^{(k)}$ and Cauchy–Schwarz gives

$$\theta_k \geq \left(\frac{3}{2} + 2\mu_k^2\right)|z^{(k)}|^2, \quad d_k^2 = \left(\sum_i \Phi_{ii} z_i^{(k)}\right)^2 \leq \frac{2p\theta_k}{3 + 4\mu_k^2}.$$

The repeated-index groups have disjoint ordered supports, so $\sum_k \theta_k \leq 1$. Since $\mu_k^2 \leq 3/4$,

$$|Ad|^2 \leq p \sum_k \frac{2\mu_k^2 \theta_k}{3 + 4\mu_k^2} \leq \frac{p}{4}. \quad \square$$

The next estimate will be applied with $X = 8B$ in the quadratic coefficient.

Lemma 5.6 (A quadratic bound). *For an admissible pair (A, τ) and every $X \in \text{Sym}_0^2 \mathbb{R}^4$,*

$$(5.10) \quad 2\langle \Psi, X \rangle - \langle X, \mathcal{K}_A X \rangle \leq \frac{97}{48}.$$

Proof. By Lemma 5.5, the eigenvalues of R lie in $[0, 2/3]$ and sum to one. The vertices of this constraint set are the permutations of $(2/3, 1/3, 0, 0)$, so convexity of the sum of squares gives $\text{tr } R^2 \leq 5/9$ and $|R - I/4|^2 \leq 11/36$.

The contribution involving A^2 is controlled by

$$X/2 - X^2/3 = 3I/16 - \frac{1}{3}(X - 3I/4)^2 \preceq 3I/16.$$

Taking the trace against the positive semidefinite matrix A^2 preserves this inequality; $\text{tr } A^2 = 1$ bounds that contribution by $3/16$. Combining this with (4.5) gives

$$\begin{aligned} 2\langle \Psi, X \rangle - \langle X, \mathcal{K}_A X \rangle &\leq -2\langle R - I/4, X \rangle - \frac{1}{6}|X|^2 + \text{tr}(A^2(X/2 - X^2/3)) \\ &\leq -2\langle R - I/4, X \rangle - \frac{1}{6}|X|^2 + \frac{3}{16} \\ &\leq 6|R - I/4|^2 + \frac{3}{16} \leq \frac{97}{48}. \end{aligned}$$

The last line completes the square in X and uses the bound on $|R - I/4|^2$. $\square$

Proof of Proposition 5.4. For the constant term, put $w = Ad$ and use Lemma 5.5:

$$W_0 = -\frac{19}{4}p - 21|w|^2 + 49w^T H_\tau w \leq -\frac{19}{4}p + \frac{77}{8}|w|^2 \leq -\frac{75}{32}p.$$

For the quadratic term, B is fixed, B_t is affine in t , and so are L_4, L_3, J . Equations (5.3)–(5.4) therefore give the coefficient of t^2 as

$$\frac{35}{16} + 64\langle B, \mathcal{K}_A B \rangle - 16\langle \Psi, B \rangle.$$

Apply (5.10) with $X = 8B$ to bound this below by $35/16 - 97/48 = 1/6$. Thus $W_t'' \geq 1/3$. $\square$

5.4. Completing the proof. Appendix A proves the following bound for every admissible pair.

Proposition 5.7 (The bound at $t = 1/2$). *Let (A, τ) be an admissible pair as in Definition 2.2, and let W_t be defined by Definition 5.1. Then*

$$(5.11) \quad W_{1/2}(A, \tau) \leq -\frac{1}{10000}.$$

We finish by applying convexity on $[0, 1/2]$.

Proof of Theorem 1.2. Suppose $4 < S \leq 8$. Then $0 < t \leq 1/2$, and the tensors defined by (2.10) satisfy (2.11) at every point. Propositions 5.4 and 5.7 give

$$W_t \leq (1 - 2t)W_0 + 2tW_{1/2} \leq -\frac{t}{5000} < 0.$$

Since t is constant, integration contradicts (5.6). $\square$

APPENDIX A. THE BOUND AT $t = 1/2$

We prove Proposition 5.7 by parametrizing the spectrum of A and the components of τ , then reducing the inequality to positivity of a quartic. Bernstein expansion on a rectangular cover of $[0, 1]^2$ reduces this to Gram identities and integer matrix inequalities. Setting $t = 1/2$ in (5.4) gives

$$(A.1) \quad \begin{aligned} W_{1/2} = & \frac{21}{32} + \frac{13}{16}p - \frac{39}{16}m^2 - 8a - \frac{189}{16}b + 16mc + 24|d|^2 \\ & - 8|B|^2 + 16\langle B, \mathcal{K}_A B \rangle - 21|Ad|^2 + 49(Ad)^T H_\tau(Ad). \end{aligned}$$

Parametrization. Since $W_{1/2}$ is invariant under orthogonal changes of basis, we may diagonalize A . Its eigenvalue vector $\mu = (\mu_1, \dots, \mu_4)^T$ is a unit vector in $\mathbf{1}^\perp$, where $\mathbf{1} = (1, 1, 1, 1)^T$. The separate transformations $A \mapsto -A$ and $\tau \mapsto -\tau$ preserve (A.1). Define

$$U = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}.$$

Its columns form an orthonormal basis of $\mathbf{1}^\perp$. Permuting the four eigenvalues acts on $U^T \mu$ by coordinate permutations and even sign changes. Together with $\mu \mapsto -\mu$, these allow us to arrange the three coordinates of $U^T \mu$ in nonincreasing order, all nonnegative. The first coordinate is positive; dividing by it gives $(1, r, ry)^T$ with $0 \leq r, y \leq 1$. If the second coordinate of $U^T \mu$ is zero, take $r = y = 0$. Thus every spectrum is represented, up to these symmetries, by

$$(A.2) \quad \begin{aligned} \ell &= (1 + r + ry, 1 - r - ry, -1 + r - ry, -1 - r + ry), \\ L &= |\ell|^2 = 4(1 + r^2 + r^2 y^2), \quad \mu = \ell / \sqrt{L}. \end{aligned}$$

For each repeated-index group, the two trace constraints leave the two-dimensional space $\{\mathbf{1}, \ell\}^\perp$. A basis valid throughout the closed square is given by the columns of

$$(A.3) \quad C(r, y) = \begin{pmatrix} -1 + ry & -r + ry \\ 1 + r & 0 \\ 0 & 1 + r \\ -r - ry & -1 - ry \end{pmatrix}.$$

The columns satisfy both trace constraints; the two rows containing $1 + r$ prove independence, including at the boundary. The four repeated-index groups and four distinct-index components give twelve coordinates. For $x = (x_1, \dots, x_{12}) \in \mathbb{R}^{12}$, define the symmetric cubic tensor $T = T(r, y, x)$ by

$$(T_{234}, T_{134}, T_{124}, T_{123}) = (x_1, x_2, x_3, x_4)$$

and, for $k = 1, \dots, 4$,

$$(A.4) \quad (T_{iik})_{i=1}^4 = C(r, y)(x_{2k+3}, x_{2k+4})^T.$$

Since these groups of components are disjoint, the formulas parametrize the whole constrained cubic space, with

$$N(r, y, x) := |T(r, y, x)|^2 > 0 \quad (x \neq 0).$$

Set

$$(A.5) \quad A(r, y) = \text{diag}(\ell / \sqrt{L}), \quad \tau(r, y, x) = T(r, y, x) / \sqrt{N(r, y, x)}.$$

These formulas also apply at repeated or zero principal curvatures.

Polynomial identity. Set $\mathcal{D}(r, y) = 5625L^5/32 > 0$. For $x \neq 0$, define

$$(A.6) \quad P(r, y, x) = \mathcal{D}(r, y)N(r, y, x)^2 \left[-W_{1/2}(A(r, y), \tau(r, y, x)) - \frac{1}{10000} \right].$$

Substitution of (A.1) cancels the denominators and extends P to an integer polynomial, homogeneous of degree four in x and of degree at most 14 in each of r, y . It has 196 quartic monomials and 19 020 nonzero bivariate coefficient terms.

Bernstein coefficients and positivity. Fix $r_0 < r_1$ and $y_0 < y_1$. On the rectangle $[r_0, r_1] \times [y_0, y_1]$, set

$$s_1 = \frac{r - r_0}{r_1 - r_0}, \quad s_2 = \frac{y - y_0}{y_1 - y_0}.$$

For $0 \leq i, j \leq 14$, define

$$(A.7) \quad \beta_{ij}(r, y) = \binom{14}{i} s_1^i (1 - s_1)^{14-i} \binom{14}{j} s_2^j (1 - s_2)^{14-j}.$$

These nonnegative polynomials satisfy $\sum_{i,j=0}^{14} \beta_{ij} = 1$. Define the quartics P_{ij} and scalars d_{ij} by

$$(A.8) \quad P = \sum_{i,j=0}^{14} \beta_{ij} P_{ij}(x), \quad \mathcal{D} = \sum_{i,j=0}^{14} \beta_{ij} d_{ij}.$$

Label each quadratic monomial $x_a x_b$ by (a, b) , with $1 \leq a \leq b \leq 12$, and let $z(x) \in \mathbb{R}^{78}$ list these monomials in lexicographic order. It contains every x_a^2 , so $z(x) \neq 0$ whenever $x \neq 0$.

Lemma A.1 (Bernstein positivity). *Fix a rectangle and the expansions (A.8). Suppose that $d_{ij} > 0$ for every $0 \leq i, j \leq 14$, and that for each such pair there is a positive definite symmetric matrix $G_{ij} \in \mathbb{R}^{78 \times 78}$ satisfying*

$$P_{ij}(x) = d_{ij} z(x)^T G_{ij} z(x) \quad (x \in \mathbb{R}^{12}).$$

Then $P(r, y, x) > 0$ for every point (r, y) of the rectangle and every $x \neq 0$.

Proof. For $x \neq 0$, each $P_{ij}(x)$ is positive. Equation (A.8) is a weighted sum of these values with nonnegative weights of sum one. $\square$

Lemma A.2 (Integer positivity test). *Let $s \geq 1$, let $Q \in \mathbb{Z}^{s \times s}$ be symmetric, and let $V \in \mathbb{Z}^{s \times s}$. If*

$$(A.9) \quad (VQV^T)_{ii} > \sum_{j \neq i} |(VQV^T)_{ij}| \quad (1 \leq i \leq s),$$

then Q is positive definite.

Proof. Put $M = VQV^T$. For $w \in \mathbb{R}^s$, the inequality $2|w_i w_j| \leq w_i^2 + w_j^2$ gives

$$w^T M w \geq \sum_i \left(M_{ii} - \sum_{j \neq i} |M_{ij}| \right) w_i^2.$$

Hence M is positive definite. If V were singular, a nonzero $w \in \ker V^T$ would satisfy $w^T M w = 0$. Thus V is invertible and Q is positive definite by congruence. $\square$

The rational certificate. A binary word selects closed halves of $[0, 1]^2$: bisect in r at even depth and y at odd depth, starting at depth zero, and retain the lower half for 0 and the upper half for 1. The 24 leaves listed in `DATA_SPEC.md` form a complete prefix tree: every non-leaf prefix has both children, and no leaf has a listed descendant. Their rectangles therefore cover the closed square with disjoint interiors.

To prove $d_{ij} > 0$, use $\mathcal{D} = 180000(1+r^2+r^2y^2)^5$. On any listed rectangle, write $r = r_0 + hs_1$, $y = y_0 + ks_2$, where $r_0, y_0 \geq 0$ and $h, k > 0$. If $\mathcal{D} = \sum_{a,b} e_{ab} s_1^a s_2^b$, then $e_{ab} \geq 0$ and $e_{00} > 0$. The degree is at most ten in each variable. Conversion to degree (14, 14) gives

$$d_{ij} = \sum_{a \leq i, b \leq j} e_{ab} \frac{\binom{i}{a}}{\binom{14}{a}} \frac{\binom{j}{b}}{\binom{14}{b}} > 0.$$

The positive term e_{00} contributes to every coefficient.

Reversing a basis vector e_k changes the sign of a tensor component once for each occurrence of k among its indices. Group the quadratic monomials according to their signs under these four reversals. This gives eight groups of sizes 16, 9, 9, 9, 9, 9, 9, 8. We take G_{ij} block diagonal with respect to these groups, each block indexed by the monomials in that group. The encoding and ordering are specified in `DATA_SPEC.md`. Index the 442 upper-triangular positions by e , with weight $w_e = 1$ on the diagonal and $w_e = 2$ otherwise. The sorted concatenation of the two quadratic labels gives a quartic label $\gamma(e)$; there are 196 such labels. Set

$$c_\gamma = \sum_{\gamma(e)=\gamma} w_e.$$

These labels give precisely the degree-four monomials unchanged by all four sign reversals. The other 169 of the $\binom{15}{4} = 1365$ quartic types have zero coefficient; all are included in the coefficient checks.

At each rectangle and control index (i, j) , choose a common positive integer scale σ clearing the denominators of P_{ij} and d_{ij} , as specified in `DATA_SPEC.md`. Let p_γ be the coefficient of x^γ in σP_{ij} , and let d be the positive integer σd_{ij} . Here $x^\gamma = x_{a_1} x_{a_2} x_{a_3} x_{a_4}$ if $\gamma = (a_1, a_2, a_3, a_4)$. The canonical Gram entry is $p_{\gamma(e)} / (c_{\gamma(e)} d)$. Each rectangle supplies $n^{00}, n^{10}, n^{01}, n^{11} \in \mathbb{Z}^{442}$ and the positive denominator $n_* = 12884901888$, with

$$\sum_{\gamma(e)=\gamma} w_e n_e^{ab} = 0 \quad \text{for every label } \gamma \text{ and } ab \in \{00, 10, 01, 11\}.$$

Define the integer vector

$$\nu(i, j) = (14 - i)(14 - j)n^{00} + i(14 - j)n^{10} + (14 - i)jn^{01} + ijn^{11}.$$

The Gram entry at position e is

$$(A.10) \quad (G_{ij})_e = \frac{p_{\gamma(e)}}{c_{\gamma(e)} d} + \frac{\nu_e(i, j)}{196 n_*}.$$

The first term supplies the coefficients of P_{ij}/d_{ij} ; the second represents the zero quartic. Thus

$$(A.11) \quad P_{ij}(x)/d_{ij} = z(x)^T G_{ij} z(x).$$

Positive definiteness is checked at every control index, not inferred from the four corners.

The integer numerator $Q_{ij} = (196 n_* d) G_{ij}$ has entries

$$(Q_{ij})_e = \frac{196 n_*}{c_{\gamma(e)}} p_{\gamma(e)} + d \nu_e(i, j).$$

The possible multiplicities c_γ are 1, 2, 3, 4, 6, according to the repeated indices in γ . Each divides $196 n_*$, so these entries are integers. The positive factor $196 n_* d$ preserves definiteness. For each block, the data supply an integer matrix V satisfying (A.9). The Bernstein coefficients are computed both by affine substitution and by midpoint restriction. For each computation, the verification includes $24 \cdot 15^2 = 5400$ full Gram identities, 43 200 block tests and $5400 \cdot 78 = 421\,200$ strict row inequalities.

The verification consists of checking:

- (1) the full coefficient identity (A.6), reconstructed both from (4.1)–(4.2) and from contractions on $\text{Sym}^4 \mathbb{R}^4$ with the ordered inner product;
- (2) the 24-rectangle cover, the Bernstein expansions and (A.11) coefficient by coefficient, including positivity of all denominators and all d_{ij} ;
- (3) symmetry and the strict inequalities (A.9) after the supplied congruence, for all 43 200 integer blocks.

These checks have been completed using exact rational arithmetic. Lemma A.1 and (A.6) then prove Proposition 5.7.

Electronic supplement. The TeX source, Mathematica notebook and PDF export, and verification code and data are at <https://github.com/Jackyluckin/proof-of-second-pinchng-prob-dim4>. See `README.md` for instructions and `DATA_SPEC.md` for the data format.

The Lean version of Theorem 1.2 is conditional on the finite coefficient identities and integer positivity checks in `lean/README.md`. Mathematica verifies them; their Lean proof is not included. The classical classification results and Corollary 1.4 are not formalised.

Use of AI. AI tools, including GPT-6 Astra, were used to explore candidate estimates, develop the verification code, review the arguments, and revise the manuscript. The author is responsible for the mathematical content and its presentation.

REFERENCES

- [1] É. Cartan, *Sur des familles remarquables d'hypersurfaces isoparamétriques dans les espaces sphériques*, Math. Z. **45** (1939), 335–367, doi:10.1007/BF01580289.
- [2] S.-S. Chern, M. do Carmo and S. Kobayashi, *Minimal submanifolds of a sphere with second fundamental form of constant length*, in *Functional Analysis and Related Fields*, F. E. Browder (ed.), Springer, Berlin, Heidelberg, 1970, pp. 59–75, doi:10.1007/978-3-642-48272-4_2.
- [3] R. Chen, *An improved second pinching theorem for minimal hypersurfaces with constant scalar curvature in spheres*, preprint (2026), arXiv:2609.17586v1.
- [4] Q.-M. Cheng, F. Li and G. Wei, *An improved result on Chern conjecture*, preprint (2026), arXiv:2610.05276v1.
- [5] J. Ge, T. Liu, K. Luo and W. Yan, *Rigidity of closed minimal hypersurfaces in $\mathbb{S}^5$* , preprint (2026), arXiv:2606.29246v1.
- [6] S. Guan, *The second gap for minimal hypersurfaces in the unit sphere with constant cubic trace*, preprint (2026), arXiv:2609.26852v1.
- [7] J. Ge, H. Tan, W. Yan and Y. Zhang, *The second gap and rigidity in Chern's conjecture with constant cubic trace*, preprint (2026), arXiv:2609.27661v1.
- [8] C. He, H. Xu and E. Zhao, *Classification of closed minimal hypersurfaces with constant scalar curvature in $\mathbb{S}^5$* , preprint (2026), arXiv:2603.01181v1.
- [9] H. B. Lawson, Jr., *Local rigidity theorems for minimal hypersurfaces*, Ann. of Math. (2) **89** (1969), 187–197, doi:10.2307/1970816.
- [10] L. Lei, H. Xu and Z. Xu, *On Chern's conjecture for minimal hypersurfaces in spheres*, preprint (2017), arXiv:1712.01175v1.
- [11] H. F. Münzner, *Isoparametrische Hyperflächen in Sphären*, Math. Ann. **251** (1980), 57–71, doi:10.1007/BF01420281.
- [12] H. F. Münzner, *Isoparametrische Hyperflächen in Sphären. II. Über die Zerlegung der Sphäre in Ballbündel*, Math. Ann. **256** (1981), 215–232, doi:10.1007/BF01450799.
- [13] C. K. Peng and C. L. Terng, *Minimal hypersurfaces of spheres with constant scalar curvature*, in *Seminar on Minimal Submanifolds*, E. Bombieri (ed.), Ann. of Math. Stud. **103**, Princeton University Press, Princeton, NJ, 1983, pp. 177–198.
- [14] C. K. Peng and C. L. Terng, *The scalar curvature of minimal hypersurfaces in spheres*, Math. Ann. **266** (1983), 105–113, doi:10.1007/BF01458707.
- [15] J. Simons, *Minimal varieties in Riemannian manifolds*, Ann. of Math. (2) **88** (1968), 62–105, doi:10.2307/1970556.
- [16] Y. J. Suh and H. Y. Yang, *The scalar curvature of minimal hypersurfaces in a unit sphere*, Commun. Contemp. Math. **9** (2007), no. 2, 183–200, doi:10.1142/S021919970700237X.

UNIVERSITY OF CAMBRIDGE, CAMBRIDGE, UNITED KINGDOM