

# NOTE ON VARIABLE ROLE-REVERSALS

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ABSTRACT. In [2], the author uses variable role-reversals only when  $\theta < \frac{45}{89} - \varepsilon$ . In this note, we shall discuss how to use variable role-reversals when  $\frac{45}{89} - \varepsilon \leq \theta < \frac{11}{20} - \varepsilon$ .

In a recent paper [2], the author studied the average distribution of majorants and minorants of  $\mathbb{1}_p(x)$  in arithmetic progressions to moduli larger than  $\sqrt{x}$ . During the constructing process of those majorants and minorants, the author applied the *variable role-reversal*, which is also a very powerful technique in many other applications of Harman's sieve. For the sake of simplicity, the author only uses role-reversals when  $\theta < \frac{45}{89} - \varepsilon$ . Here, we generalize the application of role-reversals in [2] to a wider range  $\theta < \frac{11}{20} - \varepsilon$ . We mention that further extension to  $\theta < \frac{4}{7} - \varepsilon$  may still be possible.

From here we suppose that  $\frac{1}{2} \leq \theta < \frac{11}{20} - \varepsilon$ . Almost all of the notations and definitions in this note are same as those in [2]. We only change the definitions of  $\kappa$  and  $\tau$  in order to state our Lemma 0.2 and Lemma 0.3 below simpler. Note that  $\kappa$  here equals to  $\kappa$  in [2] when  $\theta \leq \frac{7}{13} - \varepsilon$  and becomes  $\kappa'$  in [2] when  $\theta > \frac{7}{13} - \varepsilon$ . The definitions of some sets below are changed correspondingly.

$$\kappa = \kappa(\theta) = \begin{cases} \frac{5-8\theta}{6} - \varepsilon, & \theta \leq \frac{17}{32} - \varepsilon, \\ \frac{5-8\theta}{12} - 3\varepsilon, & \frac{17}{32} - \varepsilon < \theta \leq \frac{7}{13} - \varepsilon, \\ \frac{11-20\theta}{6} - 2\varepsilon, & \frac{7}{13} - \varepsilon < \theta \leq \frac{11}{20} - \varepsilon, \end{cases}$$

$$\tau = \tau(\theta) = \begin{cases} \frac{3(1-\theta)}{5} - \varepsilon, & \theta \leq \frac{11}{21}, \\ \frac{2}{7} - \varepsilon, & \frac{11}{21} < \theta \leq \frac{6}{11} - \varepsilon, \\ \frac{5-6\theta}{7} - \varepsilon, & \frac{6}{11} - \varepsilon < \theta. \end{cases}$$

$$\mathcal{S} = \mathcal{S}(\theta) = \{(s, t) : s \leq 1 - \theta - \varepsilon, s + 2t \leq 2 - 2\theta - \varepsilon, s + 4t \leq 2 - \theta - \varepsilon\},$$

$$\mathcal{S}_j = \mathcal{S}_j(\theta) = \{\alpha_j : \alpha_j \text{ partitions exactly into } \mathcal{S}\},$$

$$\mathcal{A}_j = \mathcal{A}_j(\theta) = \{\alpha_j : (\log \log x)^{-3} \leq \alpha_j \leq \dots \leq \alpha_1 < \tau, \alpha_1 + \dots + \alpha_j \leq 1\},$$

$$\mathcal{A}'_j = \mathcal{A}'_j(\theta) = \left\{ \alpha_j : (\log \log x)^{-3} \leq \alpha_j \leq \dots \leq \alpha_1 < \frac{3}{7} + \varepsilon, \alpha_1 + \dots + \alpha_j \leq 1 \right\},$$

$$\mathcal{U}_j = \mathcal{U}_j(\theta) = \{\alpha_j : \alpha_j \in \mathcal{A}_j, (\alpha_1, \dots, \alpha_j, 2\theta - 1 + \varepsilon) \in \mathcal{S}_{j+1}\},$$

$$\mathcal{U}'_j = \mathcal{U}'_j(\theta) = \{\alpha_j : \alpha_j \in \mathcal{A}'_j, (\alpha_1, \dots, \alpha_j, 2\theta - 1 + \varepsilon) \in \mathcal{S}_{j+1}\}.$$

The next three lemmas come directly from [2].

**Lemma 0.1.** *Let  $\frac{1}{2} \leq \theta \leq \frac{45}{89} - \varepsilon$ ,  $m_1 \sim M_1$  and  $m_2 \sim M_2$ . Then*

$$\sum_{\left(\frac{\log M_1}{\log x}, \frac{\log M_2}{\log x}\right) \in \mathcal{S}} a_{1,m_1} a_{2,m_2} S(\mathcal{A}_{m_1 m_2}^q, x^\kappa)$$

*has an asymptotic formula.*

*Proof.* This lemma is [[2], Lemma 2.14]. □

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**Lemma 0.2.** Let  $\frac{1}{2} \leq \theta \leq \frac{17}{32} - \varepsilon$ . Then

$$\sum_{\alpha_j \in U'_j} S(\mathcal{A}_{p_1 \dots p_j}^q, x^\kappa)$$

has an asymptotic formula.

*Proof.* This lemma can be proved in the same way of [[2], Lemma 2.13].  $\square$

**Lemma 0.3.** Let  $\frac{1}{2} \leq \theta \leq \frac{11}{20} - \varepsilon$ . Then

$$\sum_{\alpha_j \in U_j} S(\mathcal{A}_{p_1 \dots p_j}^q, x^\kappa)$$

has an asymptotic formula.

*Proof.* This lemma can be proved in the same way of [[2], Lemma 2.13].  $\square$

Next we shall briefly introduce the use of role-reversals in this problem. Suppose that we want to give a nontrivial estimate for the sum

$$\sum_{\alpha_j} S(\mathcal{A}_{p_1 \dots p_j}^q, p_j). \quad (1)$$

Assume that the sum (1) has a “right sign” (that is,  $-$  in the construction of a majorant or  $+$  in the construction of a minorant) and satisfies  $\alpha_j \in \mathbf{S}_j$  (or  $U'_j$  or  $U_j$ , depending on the value of  $\theta$ ). A direct application of Buchstab’s identity gives

$$\sum_{\alpha_j} S(\mathcal{A}_{p_1 \dots p_j}^q, p_j) = \sum_{\alpha_j} S(\mathcal{A}_{p_1 \dots p_j}^q, x^\kappa) - \sum_{\substack{\alpha_j \\ \kappa \leq \alpha_{j+1} < \min(\alpha_j, \frac{1}{2}(1-\alpha_1-\dots-\alpha_j))}} S(\mathcal{A}_{p_1 \dots p_j p_{j+1}}^q, p_{j+1}). \quad (2)$$

After applying Buchstab’s identity once, sometimes we cannot use it again since  $(\alpha_1, \dots, \alpha_j, \alpha_j) \notin \mathbf{S}_{j+1}$  (or  $U'_{j+1}$  or  $U_{j+1}$ ). In this case, we cannot give an asymptotic formula for the sum (with a “wrong sign”)

$$\sum_{\substack{\alpha_j \\ \kappa \leq \alpha_{j+1} < \min(\alpha_j, \frac{1}{2}(1-\alpha_1-\dots-\alpha_j))}} S(\mathcal{A}_{p_1 \dots p_j p_{j+1}}^q, x^\kappa) \quad (3)$$

after the second Buchstab iteration. The idea of a role-reversal is to apply the second Buchstab iteration on a different variable, usually the largest explicit prime variable (denoted by  $p_1$ ):

$$(p_1 p_2 \dots p_{j+1})\beta \Rightarrow (\beta p_2 \dots p_{j+1})p_1.$$

Reversing the roles of  $\beta$  and  $p_1$  gives

$$\begin{aligned} & \sum_{\substack{\alpha_j \\ \kappa \leq \alpha_{j+1} < \min(\alpha_j, \frac{1}{2}(1-\alpha_1-\dots-\alpha_j))}} S(\mathcal{A}_{p_1 \dots p_j p_{j+1}}^q, p_{j+1}) \\ &= \sum_{\substack{\alpha_j \\ \kappa \leq \alpha_{j+1} < \min(\alpha_j, \frac{1}{2}(1-\alpha_1-\dots-\alpha_j))}} S\left(\mathcal{A}_{\beta p_2 \dots p_j p_{j+1}}^q, \left(\frac{2x}{\beta p_2 \dots p_{j+1}}\right)^{\frac{1}{2}}\right). \end{aligned} \quad (4)$$

Then, using the second Buchstab’s identity (actually on the variable  $p_1$  in (3)), we have

$$\begin{aligned} & \sum_{\alpha_{j+1}} S\left(\mathcal{A}_{\beta p_2 \dots p_j p_{j+1}}^q, \left(\frac{2x}{\beta p_2 \dots p_{j+1}}\right)^{\frac{1}{2}}\right) \\ &= \sum_{\alpha_{j+1}} S(\mathcal{A}_{\beta p_2 \dots p_j p_{j+1}}^q, x^\kappa) - \sum_{\substack{\alpha_{j+1} \\ \kappa \leq \alpha_{j+2} < \frac{1}{2}\alpha_1}} S(\mathcal{A}_{\beta p_2 \dots p_j p_{j+1} p_{j+2}}^q, p_{j+2}). \end{aligned} \quad (5)$$

Note that  $\beta \asymp x^{1-\alpha_1-\dots-\alpha_{j+1}}$ . Now we only need the condition  $(1-\alpha_1-\dots-\alpha_j, \alpha_2, \dots, \alpha_j) \in \mathbf{S}_j$  (or  $U'_j$  or  $U_j$ ) instead of  $(\alpha_1, \dots, \alpha_j, \alpha_j) \in \mathbf{S}_{j+1}$  (or  $U'_{j+1}$  or  $U_{j+1}$ ). We can then discard parts of the last sum

on the right-hand side of (5) with  $(1 - \alpha_1 - \dots - \alpha_{j+1}, \alpha_2, \dots, \alpha_{j+2}) \notin \mathbf{G}_{j+2}$ . More role-reversals are also possible to apply on other variables (e.g.  $p_2, p_{j+2}, \dots$ ) of this sum.

When  $\theta \leq \frac{45}{89} - \varepsilon$ ,  $\alpha_j \in \mathbf{S}_j$  and  $(1 - \alpha_1 - \dots - \alpha_j, \alpha_2, \dots, \alpha_j) \in \mathbf{S}_j$ , by the definition of  $\mathbf{S}_j$ , we can give asymptotic formulas for the first sums on the right-hand side of (2) and (5) using Lemma 0.1 directly. In [2] the author indeed used Lemma 0.1 to give an asymptotic formula for the first sum on the right-hand side of (5). When  $\theta > \frac{45}{89} - \varepsilon$ , however, the available results (Lemmas 0.2–0.3) require a descending order between prime variables in the definitions of  $\mathbf{A}_j$  and  $\mathbf{A}'_j$ . Thus, at least a splitting argument on the (possibly non-prime) variable  $\beta$  is necessary.

Since we have  $(\beta, P(p_{j+1})) = 1$  and  $\alpha_1, \alpha_2, \dots, \alpha_{j+1} \geq \kappa$ , any prime factor of  $\beta$  is  $\geq x^\kappa$ , and  $\frac{\log \beta}{\log x} = 1 - \alpha_1 - \dots - \alpha_{j+1} \leq 1 - (j+1)\kappa$ . Hence, the number of prime factors of  $\beta$  is no more than

$$\left\lceil \frac{1 - (j+1)\kappa}{\kappa} \right\rceil < \infty. \quad (6)$$

Now, we can split  $\beta$  into a product of finitely many prime variables. Write  $\beta = p'_1 p'_2 \dots p'_l$ ,  $p'_i \asymp x^{\alpha'_i}$  and  $\alpha'_1 \geq \alpha'_2 \geq \dots \geq \alpha'_l \geq \alpha_{j+1} \geq \kappa$ . Note that we have  $1 - \alpha_1 - \dots - \alpha_{j+1} = \alpha'_1 + \dots + \alpha'_l$ .

**Case 1.**  $\frac{45}{89} - \varepsilon < \theta \leq \frac{17}{32} - \varepsilon$ . In order to give an asymptotic formula for the first sum on the right-hand side of (2), we need  $\alpha_j \in \mathbf{U}'_j$ , and thus  $\alpha_1 < \frac{3}{7} + \varepsilon$ . We also need to give an asymptotic formula for the first sum on the right-hand side of (5). Assume first that  $\alpha'_1 < \alpha_1$ . Since

$$\alpha_1 > \alpha_2 > \dots > \alpha_{j+1} \geq \kappa \quad \text{and} \quad \alpha_1 > \alpha'_1 \geq \alpha'_2 \geq \dots \geq \alpha'_l \geq \alpha_{j+1} \geq \kappa,$$

we know that the  $j+l$  variables  $\alpha_2, \dots, \alpha_{j+1}, \alpha'_1, \dots, \alpha'_l$  can be arranged into a descending order (denoted by  $\alpha'_{j+l}$ ) with the largest variable ( $= \max(\alpha_2, \alpha'_1)$ ) less than  $\alpha_1$ . Hence, we have  $\alpha'_{j+l} \in \mathbf{A}'_{j+l}$ . Now, we only need the condition  $(\alpha'_{j+l}, 2\theta - 1 + \varepsilon) \in \mathbf{S}_{j+l+1}$ , which can be satisfied if we have

$$(\alpha_2, \dots, \alpha_j, \alpha_{j+1}, 1 - \alpha_1 - \dots - \alpha_{j+1}, 2\theta - 1 + \varepsilon) \in \mathbf{S}_{j+2}.$$

Again, this can be satisfied if we have

$$(\alpha_2, \dots, \alpha_j, 1 - \alpha_1 - \dots - \alpha_j, 2\theta - 1 + \varepsilon) \in \mathbf{S}_{j+1}.$$

Assume that  $\alpha'_1 \geq \alpha_1$ . Then we have  $1 - \alpha_1 - \dots - \alpha_{j+1} = \alpha'_1 + \dots + \alpha'_l > \alpha'_1 \geq \alpha_1$ . Now, if we have

$$(\alpha_2, \dots, \alpha_j, \alpha_{j+1}, 1 - \alpha_1 - \dots - \alpha_{j+1}, 2\theta - 1 + \varepsilon) \in \mathbf{S}_{j+2},$$

we must have

$$(\alpha_1, \alpha_2, \dots, \alpha_j, \alpha_{j+1}, 2\theta - 1 + \varepsilon) \in \mathbf{S}_{j+2},$$

and we can give an asymptotic formula for the sum (3). Thus, a role-reversal is not necessary to apply in this case, and we can just perform another direct Buchstab iteration on the last sum on the right-hand side of (2) to get the sums (3) and

$$\sum_{\substack{\alpha_j \\ \kappa \leq \alpha_{j+1} < \min(\alpha_j, \frac{1}{2}(1 - \alpha_1 - \dots - \alpha_j)) \\ \kappa \leq \alpha_{j+2} < \min(\alpha_{j+1}, \frac{1}{2}(1 - \alpha_1 - \dots - \alpha_j - \alpha_{j+1}))}} S\left(\mathbf{A}_{p_1 \dots p_j p_{j+1} p_{j+2}}^q, p_{j+2}\right). \quad (7)$$

We can then discard parts of (7) with  $\alpha_{j+2} \notin \mathbf{G}_{j+2}$ .

**Case 2.**  $\frac{17}{32} - \varepsilon < \theta < \frac{11}{20} - \varepsilon$ . The discussions are very similar to those in **Case 1**. The only difference is that we need  $\alpha_j \in \mathbf{U}_j$  instead of  $\alpha_j \in \mathbf{U}'_j$  (Lemma 0.2 is not available here), so that  $\alpha_1 < \tau$ .

In other similar cases where  $2\theta - 1 + \varepsilon$  is replaced by  $2\theta_1 + \theta_2 - 1 + \varepsilon$  or  $\varepsilon$  (see [2]), and when performing further role-reversals on a sum that a role-reversal has already been performed on it (which means that more almost-prime variables become explicit, see [[1], (34)] for example), the required conditions are similar.

As an application, we get the following better bounds for  $C_0^*(\theta)$  using the **first method** in [2] by applying role-reversals in the extended range of  $\theta$ . Many other bounds in [2] can be improved using a similar approach.

| $\theta$ | $C_0^*(\theta)$ (Old) | $C_0^*(\theta)$ (New) | $\theta$ | $C_0^*(\theta)$ (Old) | $C_0^*(\theta)$ (New) |
|----------|-----------------------|-----------------------|----------|-----------------------|-----------------------|
| 0.501    | <b>0.8636</b>         | <b>0.8636</b>         | 0.517    | 0.4455                | <b>0.4936</b>         |
| 0.502    | <b>0.8455</b>         | <b>0.8455</b>         | 0.518    | 0.4086                | <b>0.4624</b>         |
| 0.503    | <b>0.8298</b>         | <b>0.8298</b>         | 0.519    | 0.3744                | <b>0.4261</b>         |
| 0.504    | <b>0.8135</b>         | <b>0.8135</b>         | 0.520    | 0.3363                | <b>0.3890</b>         |
| 0.505    | <b>0.7972</b>         | <b>0.7972</b>         | 0.521    | 0.2973                | <b>0.3510</b>         |
| 0.506    | 0.7299                | <b>0.7730</b>         | 0.522    | 0.2546                | <b>0.3113</b>         |
| 0.507    | 0.7082                | <b>0.7505</b>         | 0.523    | 0.2148                | <b>0.2714</b>         |
| 0.508    | 0.6894                | <b>0.7334</b>         | 0.524    | 0.1706                | <b>0.2277</b>         |
| 0.509    | 0.6606                | <b>0.7045</b>         | 0.525    | 0.1218                | <b>0.1785</b>         |
| 0.510    | 0.6369                | <b>0.6808</b>         | 0.526    | 0.0778                | <b>0.1324</b>         |
| 0.511    | 0.6128                | <b>0.6551</b>         | 0.527    | 0.0267                | <b>0.0779</b>         |
| 0.512    | 0.5907                | <b>0.6373</b>         | 0.528    | 0                     | <b>0.0256</b>         |
| 0.513    | 0.5567                | <b>0.6033</b>         | 0.529    | 0                     | 0                     |
| 0.514    | 0.5332                | <b>0.5796</b>         | 0.530    | 0                     | 0                     |
| 0.515    | 0.5040                | <b>0.5496</b>         | 0.531    | 0                     | 0                     |
| 0.516    | 0.4762                | <b>0.5245</b>         |          |                       |                       |

Table 1: Lower Bounds for  $C_0^*(\theta)$  (First Method,  $\frac{1}{2} < \theta \leq \frac{17}{32} - \varepsilon$ )

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