
Octonionic Heisenberg-Robertson-Schrodinger and Maccone-Pati Uncertainty Principles

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Abstract: Let $\mathcal{O}$ be an octonionic Hilbert space. Let $A : \mathcal{D}(A) \subseteq \mathcal{O} \rightarrow \mathcal{O}$ and $B : \mathcal{D}(B) \subseteq \mathcal{O} \rightarrow \mathcal{O}$ be possibly unbounded self-adjoint linear operators. Then for all $h \in \mathcal{D}(AB) \cap \mathcal{D}(BA)$ with $\|h\| = 1$, we show that

$$(1) \quad \Delta_h(A)\Delta_h(B) \geq \frac{\sqrt{|\langle \{A, B\}h, h \rangle - 2\langle Ah, h \rangle \langle Bh, h \rangle|^2 + |\langle [A, B]h, h \rangle|^2}}{2}$$

and

$$(2) \quad \Delta_h(A)^2 + \Delta_h(B)^2 \geq \frac{|\langle (A+B)h, y \rangle|^2 + |\langle (A-B)h, y \rangle|^2}{2}, \quad \forall y \in \mathcal{O} \text{ satisfying } \|y\| \leq 1 \text{ and } \langle h, y \rangle = 0,$$

where $\Delta_h(A) := \|Ah - \langle Ah, h \rangle h\|$. We call Inequality (1) as octonionic Heisenberg-Robertson-Schrodinger uncertainty principle and Inequality (2) as octonionic Maccone-Pati uncertainty principle.

Keywords: Uncertainty principle, Octonionic Hilbert space.

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1. INTRODUCTION

Let $\mathcal{H}$ be a complex Hilbert space and A be a possibly unbounded self-adjoint linear operator defined on the domain $\mathcal{D}(A) \subseteq \mathcal{H}$. For $h \in \mathcal{D}(A)$ with $\|h\| = 1$, define the **uncertainty** or **variance** of A at the point h as

$$\Delta_h(A) := \|Ah - \langle Ah, h \rangle h\| = \sqrt{\|Ah\|^2 - \langle Ah, h \rangle^2}.$$

In 1929, Robertson [7] derived the following operator form of the uncertainty principle (also called as uncertainty relation or uncertainty inequality) by Heisenberg in 1927 [5]. Recall that for two linear operators $A : \mathcal{D}(A) \subseteq \mathcal{H} \rightarrow \mathcal{H}$ and $B : \mathcal{D}(B) \subseteq \mathcal{H} \rightarrow \mathcal{H}$, we define the commutator $[A, B] := AB - BA$ and the anti-commutator $\{A, B\} := AB + BA$.

Theorem 1.1. [3, 5, 7, 9] (***Heisenberg-Robertson Uncertainty Principle***) Let $A : \mathcal{D}(A) \subseteq \mathcal{H} \rightarrow \mathcal{H}$ and $B : \mathcal{D}(B) \subseteq \mathcal{H} \rightarrow \mathcal{H}$ be possibly unbounded self-adjoint linear operators. Then for all $h \in \mathcal{D}(AB) \cap \mathcal{D}(BA)$ with $\|h\| = 1$, we have

$$(1) \quad \frac{1}{2} (\Delta_h(A)^2 + \Delta_h(B)^2) \geq \frac{1}{4} (\Delta_h(A) + \Delta_h(B))^2 \geq \Delta_h(A)\Delta_h(B) \geq \frac{1}{2} |\langle [A, B]h, h \rangle|.$$

In 1930, Schrodinger improved Inequality (1) [8].

Theorem 1.2. [2, 8] (**Heisenberg-Robertson-Schrodinger Uncertainty Principle**) Let $A : \mathcal{D}(A) \subseteq \mathcal{H} \rightarrow \mathcal{H}$ and $B : \mathcal{D}(B) \subseteq \mathcal{H} \rightarrow \mathcal{H}$ be possibly unbounded self-adjoint linear operators. Then for all $h \in \mathcal{D}(AB) \cap \mathcal{D}(BA)$ with $\|h\| = 1$, we have

$$\begin{aligned} \Delta_h(A)\Delta_h(B) &\geq |\langle Ah, Bh \rangle - \langle Ah, h \rangle \langle Bh, h \rangle| \\ &= \frac{\sqrt{|\langle \{A, B\}h, h \rangle - 2\langle Ah, h \rangle \langle Bh, h \rangle|^2 + |\langle [A, B]h, h \rangle|^2}}{2} \\ &= \frac{\sqrt{(\langle \{A, B\}h, h \rangle - 2\langle Ah, h \rangle \langle Bh, h \rangle)^2 - \langle [A, B]h, h \rangle^2}}{2}. \end{aligned}$$

In 2014, Maccone and Pati derived the following uncertainty principles which work for orthogonal vectors [6].

Theorem 1.3. [6] (**Maccone-Pati Uncertainty Principles**) Let $A : \mathcal{D}(A) \subseteq \mathcal{H} \rightarrow \mathcal{H}$ and $B : \mathcal{D}(B) \subseteq \mathcal{H} \rightarrow \mathcal{H}$ be possibly unbounded self-adjoint linear operators. Then for all $h \in \mathcal{D}(AB) \cap \mathcal{D}(BA)$ with $\|h\| = 1$, we have

(i)

$$\begin{aligned} \Delta_h(A)^2 + \Delta_h(B)^2 &\geq |\langle (A+B)h, k \rangle|^2 - \langle \{A, B\}h, h \rangle + 2\langle Ah, h \rangle \langle Bh, h \rangle, \\ &\quad \forall k \in \mathcal{H} \text{ satisfying } \|k\| = 1 \text{ and } \langle h, k \rangle = 0. \end{aligned}$$

(ii)

$$\begin{aligned} \Delta_h(A)^2 + \Delta_h(B)^2 &\geq |\langle (A-B)h, k \rangle|^2 + \langle \{A, B\}h, h \rangle - 2\langle Ah, h \rangle \langle Bh, h \rangle, \\ &\quad \forall k \in \mathcal{H} \text{ satisfying } \|k\| = 1 \text{ and } \langle h, k \rangle = 0. \end{aligned}$$

(iii)

$$\Delta_h(A)^2 + \Delta_h(B)^2 \geq \frac{|\langle (A+B)h, k \rangle|^2 + |\langle (A-B)h, k \rangle|^2}{2}, \quad \forall k \in \mathcal{H} \text{ satisfying } \|k\| = 1 \text{ and } \langle h, k \rangle = 0.$$

(iv)

$$\Delta_h(A)\Delta_h(B) \geq \frac{2\langle Ah, h \rangle \langle Bh, h \rangle - \langle \{A, B\}h, h \rangle}{2 - \left| \left\langle \left(\frac{A}{\Delta_h(A)} + \frac{B}{\Delta_h(B)} \right) h, k \right\rangle \right|^2}, \quad \forall k \in \mathcal{H} \text{ satisfying } \|k\| = 1 \text{ and } \langle h, k \rangle = 0.$$

(v)

$$\Delta_h(A)\Delta_h(B) \geq \frac{-2\langle Ah, h \rangle \langle Bh, h \rangle + \langle \{A, B\}h, h \rangle}{2 - \left| \left\langle \left(\frac{A}{\Delta_h(A)} - \frac{B}{\Delta_h(B)} \right) h, k \right\rangle \right|^2}, \quad \forall k \in \mathcal{H} \text{ satisfying } \|k\| = 1 \text{ and } \langle h, k \rangle = 0.$$

We derive octonionic analogues of Theorem 1.2 in Theorem 2.1 and Theorem 1.3 in Theorem 3.1. We end the introduction by recalling the definition of octonionic Hilbert spaces.

Let $\{e_j\}_{j=1}^7$ be the free symbols with multiplicative relations governed by the following table [1].

Let

$$\mathbb{O} := \left\{ a_0 + \sum_{j=1}^7 a_j e_j : a_0, \dots, a_7 \in \mathbb{R} \right\}$$

$\times$	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	-1	e_4	e_7	$-e_2$	$-e_6$	e_5	$-e_3$
e_2	$-e_4$	-1	e_5	e_1	$-e_3$	e_7	$-e_6$
e_3	$-e_7$	$-e_5$	-1	e_6	e_2	$-e_4$	e_1
e_4	e_2	$-e_1$	$-e_6$	-1	e_7	e_3	$-e_5$
e_5	e_6	e_3	$-e_2$	$-e_7$	-1	$-e_1$	e_4
e_6	$-e_5$	$-e_7$	e_4	$-e_3$	e_1	-1	e_2
e_7	e_3	e_6	$-e_1$	e_5	$-e_4$	$-e_2$	-1

be the non-associative division algebra of octonion numbers with the conjugation

$$\overline{a_0 + \sum_{j=1}^7 a_j e_j} := a_0 - \sum_{j=1}^7 a_j e_j, \quad \forall a_0 + \sum_{j=1}^7 a_j e_j \in \mathbb{O}$$

and the modulus

$$\left| a_0 + \sum_{j=1}^7 a_j e_j \right| := \sqrt{\sum_{j=0}^7 a_j^2}, \quad \forall a_0 + \sum_{j=1}^7 a_j e_j \in \mathbb{O}.$$

Definition 1.4. [4] A right octonionic vector space $\mathcal{O}$ over $\mathbb{O}$ is said to be a (right) **octonionic Hilbert space** if there exists a map (called as octonionic inner product) $\langle \cdot, \cdot \rangle : \mathcal{O} \times \mathcal{O} \rightarrow \mathbb{O}$ such that the following hold.

- (i) $\langle x, x \rangle \geq 0, \forall x \in \mathcal{O}$. If $x \in \mathcal{O}$ satisfies $\langle x, x \rangle = 0$, then $x = 0$.
- (ii) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle, \forall x, y, z \in \mathcal{O}$.
- (iii) $\text{Re}(\langle x, yq \rangle) = \text{Re}(\langle x, y \rangle q), \forall x, y \in \mathcal{O}, \forall q \in \mathbb{O}$.
- (iv) $\langle x, yr \rangle = \langle x, y \rangle r, \forall x, y \in \mathcal{O}, \forall r \in \mathbb{R}$.
- (v) $\langle x, xq \rangle = \langle x, x \rangle q, \forall x \in \mathcal{O}, \forall q \in \mathbb{O}$.
- (vi) $\langle x, y \rangle = \overline{\langle y, x \rangle}, \forall x, y \in \mathcal{O}$.
- (vii) $\mathcal{O}$ is complete w.r.t. the norm $\|x\| := \sqrt{\langle x, x \rangle}, \forall x \in \mathcal{O}$.

2. OCTONIONIC HEISENBERG-ROBERTSON-SCHRODINGER UNCERTAINTY PRINCIPLE

In this article, $\mathcal{O}$ denotes a right octonionic Hilbert space. Throughout the entire paper, given $x \in \mathcal{O}$ and a $r \in \mathbb{R} \setminus \{0\}$, the expression x/r denotes xr^{-1} . Note that this is meaningful only for division by non-zero real numbers and not for arbitrary octonions, since the octonions are not commutative. Let $\mathcal{O}$ be an octonionic Hilbert space and A be a possibly unbounded self-adjoint linear operator defined on domain $\mathcal{D}(A) \subseteq \mathcal{O}$. For $h \in \mathcal{D}(A)$ with $\|h\| = 1$, we define the **uncertainty** or **variance** of A at the point h as

$$\Delta_h(A) := \|Ah - \langle Ah, h \rangle h\| = \sqrt{\|Ah\|^2 - \langle Ah, h \rangle^2}.$$

Note that $\langle Ah, h \rangle \in \mathbb{R}$. For two maps $A : \mathcal{D}(A) \subseteq \mathcal{O} \rightarrow \mathcal{O}$ and $B : \mathcal{D}(B) \subseteq \mathcal{O} \rightarrow \mathcal{O}$, we define the commutator $[A, B] := AB - BA$ and the anti-commutator $\{A, B\} := AB + BA$.

Theorem 2.1. (Octonionic Heisenberg-Robertson-Schrodinger Uncertainty Principles) Let $\mathcal{O}$ be an octonionic Hilbert space. Let $A : \mathcal{D}(A) \subseteq \mathcal{O} \rightarrow \mathcal{O}$ and $B : \mathcal{D}(B) \subseteq \mathcal{O} \rightarrow \mathcal{O}$ be possibly unbounded self-adjoint linear operators. Then for all $h \in \mathcal{D}(AB) \cap \mathcal{D}(BA)$ with $\|h\| = 1$, we have

(i)

$$\Delta_h(A)\Delta_h(B) \geq \frac{|\langle [A, B]h, h \rangle|}{2}.$$

(ii)

$$\Delta_h(A)\Delta_h(B) \geq \frac{|\langle \{A, B\}h, h \rangle - 2\langle Ah, h \rangle \langle Bh, h \rangle|}{2}.$$

(iii)

$$\begin{aligned} \Delta_h(A)\Delta_h(B) &\geq \frac{\sqrt{(|\langle \{A, B\}h, h \rangle - 2\langle Ah, h \rangle \langle Bh, h \rangle|^2 - \langle [A, B]h, h \rangle^2)}}{2} \\ &= \frac{\sqrt{|\langle \{A, B\}h, h \rangle - 2\langle Ah, h \rangle \langle Bh, h \rangle|^2 + |\langle [A, B]h, h \rangle|^2}}{2}. \end{aligned}$$

Proof. Let $h \in \mathcal{D}(AB) \cap \mathcal{D}(BA)$ be such that $\|h\| = 1$. Since A and B are self-adjoint, we first make an important observation: $\langle Ah, h \rangle, \langle Bh, h \rangle \in \mathbb{R}$.

(i)

$$\begin{aligned} \Delta_h(A)\Delta_h(B) &\geq \langle Ah - \langle Ah, h \rangle h, Bh - \langle Bh, h \rangle h \rangle \\ &= |\langle Ah, Bh \rangle - \langle Ah, h \rangle \langle Bh, h \rangle| \\ &\geq \frac{|\langle Ah, Bh \rangle - \langle Ah, h \rangle \langle Bh, h \rangle - \overline{\langle Ah, Bh \rangle - \langle Ah, h \rangle \langle Bh, h \rangle}|}{2} \\ &= \frac{|\langle Ah, Bh \rangle - \langle Ah, h \rangle \langle Bh, h \rangle - \langle Bh, Ah \rangle + \langle Bh, h \rangle \langle Ah, h \rangle|}{2} \\ &= \frac{|\langle BA h, h \rangle - \langle AB h, h \rangle|}{2} \\ &= \frac{|-\langle (AB - BA)h, h \rangle|}{2} \\ &= \frac{|\langle [A, B]h, h \rangle|}{2}. \end{aligned}$$

(ii)

$$\begin{aligned} \Delta_h(A)\Delta_h(B) &\geq \langle Ah - \langle Ah, h \rangle h, Bh - \langle Bh, h \rangle h \rangle \\ &= |\langle Ah, Bh \rangle - \langle Ah, h \rangle \langle Bh, h \rangle| \\ &\geq \frac{|\langle Ah, Bh \rangle - \langle Ah, h \rangle \langle Bh, h \rangle + \overline{\langle Ah, Bh \rangle - \langle Ah, h \rangle \langle Bh, h \rangle}|}{2} \\ &= \frac{|\langle Ah, Bh \rangle - \langle Ah, h \rangle \langle Bh, h \rangle + \langle Bh, Ah \rangle - \langle Bh, h \rangle \langle Ah, h \rangle|}{2} \\ &= \frac{|\langle BA h, h \rangle - \langle Ah, h \rangle \langle Bh, h \rangle + \langle AB h, h \rangle - \langle Bh, h \rangle \langle Ah, h \rangle|}{2} \\ &= \frac{|\langle (AB + BA)h, h \rangle - 2\langle Ah, h \rangle \langle Bh, h \rangle|}{2} \\ &= \frac{|\langle \{A, B\}h, h \rangle - 2\langle Ah, h \rangle \langle Bh, h \rangle|}{2}. \end{aligned}$$

(iii) Define

$$z := Ah - \langle Ah, h \rangle h, \quad w := Bh - \langle Bh, h \rangle h.$$

Then

$$\begin{aligned}
 \langle z, w \rangle + \langle w, z \rangle &= \langle Ah - \langle Ah, h \rangle h, Bh - \langle Bh, h \rangle h \rangle + \langle Bh - \langle Bh, h \rangle h, Ah - \langle Ah, h \rangle h \rangle \\
 &= \langle BAh, h \rangle - \langle Ah, h \rangle \langle Bh, h \rangle + \langle ABh, h \rangle - \langle Bh, h \rangle \langle Ah, h \rangle \\
 &= \langle \{A, B\}h, h \rangle - 2\langle Ah, h \rangle \langle Bh, h \rangle
 \end{aligned}$$

and

$$\begin{aligned}
 \langle z, w \rangle - \langle w, z \rangle &= \langle Ah - \langle Ah, h \rangle h, Bh - \langle Bh, h \rangle h \rangle - \langle Bh - \langle Bh, h \rangle h, Ah - \langle Ah, h \rangle h \rangle \\
 &= \langle BAh, h \rangle - \langle Ah, h \rangle \langle Bh, h \rangle - \langle ABh, h \rangle + \langle Bh, h \rangle \langle Ah, h \rangle \\
 &= -\langle [A, B]h, h \rangle.
 \end{aligned}$$

Therefore

$$\begin{aligned}
 \Delta_h(A)^2 \Delta_h(B)^2 + \Delta_h(B)^2 \Delta_h(A)^2 &\geq |\langle z, w \rangle|^2 + |\langle w, z \rangle|^2 \\
 &= \frac{(\langle z, w \rangle + \langle w, z \rangle)^2 - (\langle z, w \rangle - \langle w, z \rangle)^2}{2} \\
 &= \frac{(\langle \{A, B\}h, h \rangle - 2\langle Ah, h \rangle \langle Bh, h \rangle)^2 - \langle [A, B]h, h \rangle^2}{2}.
 \end{aligned}$$

Note that

$$\overline{\langle [A, B]h, h \rangle} = -\langle [A, B]h, h \rangle$$

and

$$\overline{\langle \{A, B\}h, h \rangle - 2\langle Ah, h \rangle \langle Bh, h \rangle} = \langle \{A, B\}h, h \rangle - 2\langle Ah, h \rangle \langle Bh, h \rangle.$$

Therefore

$$-\langle [A, B]h, h \rangle^2 = |\langle [A, B]h, h \rangle|^2$$

and

$$(\langle \{A, B\}h, h \rangle - 2\langle Ah, h \rangle \langle Bh, h \rangle)^2 = |\langle \{A, B\}h, h \rangle - 2\langle Ah, h \rangle \langle Bh, h \rangle|^2.$$

□

3. OCTONIONIC MACCONE-PATI UNCERTAINTY PRINCIPLES

Here we derive octonionic Maccone-Pati uncertainty principles.

Theorem 3.1. (Octonionic Maccone-Pati Uncertainty Principles) *Let $\mathcal{O}$ be an octonionic Hilbert space. Let $A : \mathcal{D}(A) \subseteq \mathcal{O} \rightarrow \mathcal{O}$ and $B : \mathcal{D}(B) \subseteq \mathcal{O} \rightarrow \mathcal{O}$ be possibly unbounded self-adjoint linear operators. Then for all $h \in \mathcal{D}(AB) \cap \mathcal{D}(BA)$ with $\|h\| = 1$, we have*

(i)

$$\begin{aligned}
 \Delta_h(A)^2 + \Delta_h(B)^2 &\geq |\langle (A + B)h, y \rangle|^2 - \langle \{A, B\}h, h \rangle + 2\langle Ah, h \rangle \langle Bh, h \rangle, \\
 &\quad \forall y \in \mathcal{O} \text{ satisfying } \|y\| \leq 1 \text{ and } \langle h, y \rangle = 0.
 \end{aligned}$$

(ii)

$$\Delta_h(A)^2 + \Delta_h(B)^2 \geq |\langle (A - B)h, y \rangle|^2 + \langle \{A, B\}h, h \rangle - 2\langle Ah, h \rangle \langle Bh, h \rangle,$$

$$\forall y \in \mathcal{O} \text{ satisfying } \|y\| \leq 1 \text{ and } \langle h, y \rangle = 0.$$

(iii)

$$\Delta_h(A)^2 + \Delta_h(B)^2 \geq \frac{|\langle (A + B)h, y \rangle|^2 + |\langle (A - B)h, y \rangle|^2}{2}, \quad \forall y \in \mathcal{O} \text{ satisfying } \|y\| \leq 1 \text{ and } \langle h, y \rangle = 0.$$

(iv)

$$\Delta_h(A)\Delta_h(B) \geq \frac{2\langle Ah, h \rangle \langle Bh, h \rangle - \langle \{A, B\}h, h \rangle}{2 - \left| \left\langle \left(\frac{A}{\Delta_h(A)} + \frac{B}{\Delta_h(B)} \right) h, y \right\rangle \right|^2}, \quad \forall y \in \mathcal{O} \text{ satisfying } \|y\| \leq 1 \text{ and } \langle h, y \rangle = 0.$$

(v)

$$\Delta_h(A)\Delta_h(B) \geq \frac{-2\langle Ah, h \rangle \langle Bh, h \rangle + \langle \{A, B\}h, h \rangle}{2 - \left| \left\langle \left(\frac{A}{\Delta_h(A)} - \frac{B}{\Delta_h(B)} \right) h, y \right\rangle \right|^2}, \quad \forall y \in \mathcal{O} \text{ satisfying } \|y\| \leq 1 \text{ and } \langle h, y \rangle = 0.$$

Proof. Let $h \in \mathcal{D}(AB) \cap \mathcal{D}(BA)$ with $\|h\| = 1$. Let $y \in \mathcal{O}$ be such that $\|y\| \leq 1$ and $\langle h, y \rangle = 0$.

(i)

$$\begin{aligned} |\langle (A + B)h, y \rangle|^2 &= |\langle Ah - \langle Ah, h \rangle h + Bh - \langle Bh, h \rangle h, y \rangle|^2 \\ &\leq \|y\|^2 \langle (Ah - \langle Ah, h \rangle h) + (Bh - \langle Bh, h \rangle h), (Ah - \langle Ah, h \rangle h) + (Bh - \langle Bh, h \rangle h) \rangle \\ &\leq \langle Ah - \langle Ah, h \rangle h, Ah - \langle Ah, h \rangle h \rangle + \langle Bh - \langle Bh, h \rangle h, Bh - \langle Bh, h \rangle h \rangle \\ &\quad + \langle Bh - \langle Bh, h \rangle h, Ah - \langle Ah, h \rangle h \rangle + \langle Ah - \langle Ah, h \rangle h, Bh - \langle Bh, h \rangle h \rangle \\ &= \Delta_h(A)^2 + \Delta_h(B)^2 + \langle Bh, Ah \rangle - \langle Bh, h \rangle \langle Ah, h \rangle + \langle Ah, Bh \rangle - \langle Ah, h \rangle \langle Bh, h \rangle \\ &= \Delta_h(A)^2 + \Delta_h(B)^2 + \langle \{A, B\}h, h \rangle - 2\langle Ah, h \rangle \langle Bh, h \rangle. \end{aligned}$$

(ii) Similar to (i).

(iii) We get by adding (i) and (ii).

(iv)

$$\begin{aligned} \left| \left\langle \left(\frac{A}{\Delta_h(A)} + \frac{B}{\Delta_h(B)} \right) h, y \right\rangle \right|^2 &= \left| \left\langle \frac{Ah - \langle Ah, h \rangle h}{\Delta_h(A)} + \frac{Bh - \langle Bh, h \rangle h}{\Delta_h(B)}, y \right\rangle \right|^2 \\ &\leq \|y\|^2 \left\langle \frac{Ah - \langle Ah, h \rangle h}{\Delta_h(A)} + \frac{Bh - \langle Bh, h \rangle h}{\Delta_h(B)}, \frac{Ah - \langle Ah, h \rangle h}{\Delta_h(A)} + \frac{Bh - \langle Bh, h \rangle h}{\Delta_h(B)} \right\rangle \\ &\leq \left\langle \frac{Ah - \langle Ah, h \rangle h}{\Delta_h(A)}, \frac{Ah - \langle Ah, h \rangle h}{\Delta_h(A)} \right\rangle + \left\langle \frac{Bh - \langle Bh, h \rangle h}{\Delta_h(B)}, \frac{Bh - \langle Bh, h \rangle h}{\Delta_h(B)} \right\rangle \\ &\quad + \left\langle \frac{Bh - \langle Bh, h \rangle h}{\Delta_h(B)}, \frac{Ah - \langle Ah, h \rangle h}{\Delta_h(A)} \right\rangle + \left\langle \frac{Ah - \langle Ah, h \rangle h}{\Delta_h(A)}, \frac{Bh - \langle Bh, h \rangle h}{\Delta_h(B)} \right\rangle \\ &= 1 + 1 + \frac{\langle Bh, Ah \rangle - \langle Bh, h \rangle \langle Ah, h \rangle}{\Delta_h(A)\Delta_h(B)} + \frac{\langle Ah, Bh \rangle - \langle Ah, h \rangle \langle Bh, h \rangle}{\Delta_h(A)\Delta_h(B)} \\ &= 2 + \frac{-2\langle Ah, h \rangle \langle Bh, h \rangle + \langle \{A, B\}h, h \rangle}{\Delta_h(A)\Delta_h(B)}. \end{aligned}$$

Therefore

$$\begin{aligned}
 & -\frac{2\langle Ah, h \rangle \langle Bh, h \rangle + \langle \{A, B\}h, h \rangle}{\Delta_h(A)\Delta_h(B)} \leq 2 - \left| \left\langle \left(\frac{A}{\Delta_h(A)} + \frac{B}{\Delta_h(B)} \right) h, y \right\rangle \right|^2 \\
 & \implies \frac{2\langle Ah, h \rangle \langle Bh, h \rangle - \langle \{A, B\}h, h \rangle}{2 - \left| \left\langle \left(\frac{A}{\Delta_h(A)} + \frac{B}{\Delta_h(B)} \right) h, y \right\rangle \right|^2} \leq \Delta_h(A)\Delta_h(B).
 \end{aligned}$$

(v) Similar to (iv).

□

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