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# SMOOTH LOCAL ISOMETRIC AND CONFORMAL EMBEDDINGS IN OPTIMAL DIMENSION

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ABSTRACT. Let $s_n = n(n+1)/2$. For every $n \geq 5$, the smallest Euclidean dimension in which every smooth $n$-dimensional Riemannian metric admits a local isometric embedding is $s_n$. The corresponding dimension for smooth local conformal embedding is $s_n - 1$. We construct two flat reference embeddings with the same reduced principal symbol. In the conformal reference, the metric line takes the place of one normal direction. The proof uses quadratic products of characteristic waves to correct the metric and gives a sequence converging in every derivative order on a fixed neighborhood. Polynomial metric germs give both lower bounds.

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## 1. Introduction

Put $s_n = n(n+1)/2$ and let $\delta_m$ denote the Euclidean metric on $\mathbb{R}^m$. The universal target dimension is the smallest integer $m$ that works for every metric germ in the stated class. The two equations in this paper are

$$\gamma(F) := (dF)^T dF = g \qquad \text{and} \qquad \gamma(F) = \rho g, \quad \rho > 0. \tag{1.1}$$

The conformal factor $\rho$ is a smooth function on the neighborhood where the map is defined.

**Theorem 1.1** (Isometric embedding)**.** *For every $n \geq 5$, every smooth Riemannian metric $g$ near a point $p$ of an $n$-manifold admits a smooth embedding $F : U \to \mathbb{R}^{s_n}$ on a neighborhood $U$ of $p$, with $F^*\delta_{s_n} = g$. The target dimension $s_n$ is optimal.*

**Theorem 1.2** (Conformal embedding)**.** *For every $n \geq 5$, every smooth Riemannian metric $g$ near a point $p$ of an $n$-manifold admits a smooth embedding $F : U \to \mathbb{R}^{s_n-1}$ and a positive smooth function $\rho$ on a neighborhood $U$ of $p$, with $F^*\delta_{s_n-1} = \rho g$. The target dimension $s_n - 1$ is optimal.*

At $n = 5$ the two target dimensions are 15 and 14. For $n = 6$ they are 21 and 20. The lower bounds concern a dimension that works for all metrics. Individual metrics may admit embeddings in smaller dimensions.

For analytic metrics, the local isometric theorem in dimension $s_n$ is due to Janet and Cartan [J26, C27]. Jacobowitz and Moore proved the analytic conformal theorem in dimension $s_n - 1$ [JM73]. The classical smooth constructions of Nash and Greene give existence in higher dimensions [N56, G70]. At the isometric dimension $s_n$, eliminating the normal variation leaves a determined first-order system. Its characteristics were studied in [BGY83, HK12]. The construction below works with a reference whose symbol has singular characteristics. It combines the linear image of the symbol with quadratic terms generated on regular characteristic patches.

### 1.1. The dimension obstructions.

*Proof of the two lower bounds.* Let $b_k = \binom{n+k}{n}$, the dimension of scalar $k$-jets at the origin. A symmetric metric jet has $s_n b_k$ coefficients. For a target of dimension $m$, the polynomial maps relevant to the two equations are

$$j_0^{k+1}F \longmapsto j_0^k\Big((dF)^T dF\Big), \qquad (j_0^{k+1}F, j_0^k a) \longmapsto j_0^k\Big(a(dF)^T dF\Big).$$

In the conformal case an actual solution has $a = \rho^{-1}$. Allowing all scalar $k$-jets can only enlarge the image. The domain dimensions are $mb_{k+1}$ and $mb_{k+1} + b_k$, respectively. Since $b_{k+1}/b_k = 1 + n/(k+1)$, these dimensions are strictly less than $s_n b_k$ for all sufficiently large $k$ if $m < s_n$ in the first case or $m < s_n - 1$ in the second.

By Sard's theorem each such image has measure zero. Choose a jet outside the image in the nonempty open set of jets with positive definite constant term. Its Taylor polynomial is a Riemannian metric near zero. Any smooth immersion solving the corresponding metric equation would produce the excluded jet. This proves optimality once the upper bounds are established. □

## 2. Structure of the proof

Write $q = s_n - n$. The isometric reference has $q$ independent normal forms. The conformal reference retains $q - 1$ of these forms and changes the constant metric so that adjoining the metric line recovers their full span. Section 4 gives the linearization and its reduced symbol. Section 5 constructs the second reference by circular bendings. Both references have the same frozen principal matrix

$$M_0(\xi) = \operatorname{diag}(\xi) - (n-4)^{-2}\Big(\sum_i \xi_i\Big)\mathbf{1}\mathbf{1}^T.$$

The normal reconstruction leaves $n$ differential equations for a tangential covector field. In the conformal case the projection along the metric line removes the auxiliary scalar from the residual. The iteration updates only the ambient map.

Section 6 finds finitely many three-parameter patches of simple characteristics. For each output covector the linear image and the quadratic tensors from the incident patches span the reduced output space, with a uniform lower bound. The same patches separate the frequencies of signed products from the singular characteristic directions. These two properties govern the mean and the oscillatory parts of the correction, respectively.

The regular linear parametrices are constructed in Sections 9 and 10. They give corrected characteristic waves in Section 11. The wave differentials have a nonzero leading physical normal polarization, also in the conformal case. Their squared lengths enter a continuous normal operator. Section 13 proves its coercivity and constructs a positive background density. The separated sample of Section 12 is used in Section 14 to correct the mean. Section 15 uses regular parametrices to remove the oscillatory products.

The corrections use the exact expansion

$$\gamma(F+Y) = \gamma(F) + D\gamma_F Y + (dY)^T dY. \tag{2.1}$$

Section 16 checks this calculation with the fixed target projection for the conformal equation. Section 17 bounds one increment $U$. For a defect of size at most $\delta^2$, it gives a new defect of size at most $C_Q(\delta R^{-Q} + \delta^3 R^{B_3})$. It also gives a global $L^2$ bound and exact Fourier support below $CR^{b_*}$, with $b_* < 3/2$. The exponents and domains are independent of the finite accuracy $Q$.

The Fourier history controls all coefficient norms during iteration. Sections 20 and 21 fix the reference, choose an initial target dilation and let the accuracy increase along one sequence. The same neighborhood works at every derivative order.

## 3. Notation, conventions and fixed domains

The metric and normal-map conventions are introduced in Section 4. Every reduced operator is used with the normalization displayed there, including the factor $1/2$ in (4.6). In this section we collect the conventions for norms, Fourier transforms and constants, the list of parameters, and the fixed domains and cutoffs used in all later constructions.

3.1. **Norms, adjoints and Fourier transforms.**

**Notation 3.1** (Norms and Fourier transforms)**.** For a function $f$ smooth up to the boundary of a closed set $D$ we put

$$\|f\|_{C^r(D)} = \max_{|\alpha|\le r} \sup_{x\in D} |\partial^\alpha f(x)|,$$

and similarly on other domains. Norms of vectors, matrices and tensors are taken componentwise in fixed coordinates; a coefficient norm includes every matrix entry. Fields with a fixed compact support inside a chart are extended by zero to $\mathbb{R}^n$. Unless another domain is specified, Sobolev and $L^2$ norms are taken on $\mathbb{R}^n$, whereas physical $C^r$ norms always carry their domain. Integer Sobolev norms are the $L^2$ norms of all Cartesian derivatives through the stated order; on $\mathbb{R}^n$ this is equivalent to $\|f\|_{H^s} = \|\langle D\rangle^s f\|_2$, where $D = -\mathrm{i}\partial$ and $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$. We also write $\|u\|_s = \|u\|_{H^s}$ and $H^\infty = \bigcap_{s\ge 0} H^s$.

In $d$ variables we use the Fourier normalization

$$\widehat{f}(\xi) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-\mathrm{i}x\cdot\xi} f(x)\,\mathrm{d}x, \qquad f(x) = \int_{\mathbb{R}^d} e^{\mathrm{i}x\cdot\xi} \widehat{f}(\xi)\,\mathrm{d}\xi,$$

so that $\|f\|_2^2 = (2\pi)^d \|\widehat{f}\|_2^2$. For the initial variables of a wave $d = n-1$. A finite complex measure $m$ on $\mathbb{R}^d$ is regarded as Fourier data: writing $dm(\xi) = \widehat{f}(\xi)\,\mathrm{d}\xi$ gives $f(y) = \int e^{\mathrm{i}y\cdot\xi}\,\mathrm{d}m(\xi)$. Its size is the total variation $\|m\|_{\mathrm{TV}}$. The Dirac mass at the origin is denoted by $\delta_0$; it represents the constant function 1.

For complex matrices, $X^T$ denotes the transpose and $X^*$ the conjugate transpose; real parts are taken entrywise. Thus $\mathrm{Re}(X^*X)$ is a real symmetric tensor, whereas a cross term $\mathrm{Re}(X^*Y)$ has to be symmetrized when it occurs alone. Complex Hilbert inner products are linear in the first argument, and adjoints are complex Hilbert adjoints, restricted to real fields when the operator is real. Operator norms refer to the entire stated spaces. A normal operator $AA^*$ acts on the output space of $A$; it should not be confused with the normal map $B_F$ of the immersion.

3.2. **Amplitudes and constants.** For a complex wave $Z = e^{\mathrm{i}\lambda\phi} z$ we call $z$ its amplitude. Estimates for $z$ and for $e^{-\mathrm{i}\lambda\phi} dZ$ are stated separately. A bound for the first does not by itself give the second at the same scale, because

$$e^{-\mathrm{i}\lambda\phi} dZ = \mathrm{i}\lambda z\, d\phi + dz.$$

This is why the estimates below distinguish the scale $K$ on which the amplitude varies from the frequency $\lambda$. The leading amplitude of a characteristic wave is a scalar transported along the Hamilton flow times a null vector; its value at the initial time is called the *initial amplitude* and is denoted by $r$, and $r^2$ is called the *initial density*. The mean of a wave is linear in its initial density.

The letter $M$ denotes the number of samples, while $M_{\mathrm{an}} = 20(n+1)$ is a fixed order of differentiability used in the oscillatory integral estimates. Three normal operators are distinguished in Sections 13–14: the continuous operator $E_F$, the continuous operator $E_{F,K}$ with input frequencies cut at $K$, and the sampled operator $\widetilde{E}_K$.

Constants may depend on the dimension, the fixed charts and the fixed geometric neighborhood. When a bound holds at every order $r$, the constant $C_r$ may depend on $r$. In estimates of the form $C_r L^{r+B}$ we state explicitly when the exponent $B$ is independent of $r$ and of the accuracy. At each finite order and accuracy, the constants involve only finitely many derivative bounds of the data; they are uniform for immersions in a fixed $C^{r_*}$ neighborhood of the reference whose derivative bounds (see (11.23)) have fixed constants. A requested accuracy determines finite iteration depths before the waves and the sample are formed, and constants may depend on these finite choices. Domains, the order $r_*$, Fourier supports and exponents do not. In the single-step estimate this dependence is displayed as $C_Q$, and the convergence argument absorbs it in one threshold. Letters $B$ with subscripts denote fixed loss exponents.

We often use derivative bounds of the form

$$\|f\|_{C^r} \le C_r T^{(r-R)_+}, \qquad r \ge 0, \tag{3.1}$$

with $T \ge 1$, $R \ge 0$ and $x_+ = \max(x, 0)$: the first $R$ derivatives are bounded uniformly, and each further derivative costs a factor $T$. They are preserved by products, compositions, inverses and flows, with a fixed loss in $R$ (Section 7).

3.3. **Parameters.** The main parameters and fields have the following roles.

| | |
|---|---|
| $F$ | the immersion at the current step; all waves and correction operators are constructed at $F$ |
| $R$ | the frequency parameter of the step; also the derivative scale of the final modulated amplitudes |
| $\lambda = R^a$ | the frequency of the characteristic waves |
| $h = R^{-\beta}$ | the separation of the samples |
| $K$, $K_{\max} = R^{s_f}$ | the amplitude scale, and the fixed bound for the initial frequencies of the modulations |
| $T_0 = R^t$ | the derivative scale of the immersion $F$ |
| $\delta$, $b = \delta R^A$ | $\delta^2$ bounds the metric defect; $b$ is the amplitude of the primary waves |
| $P$, $V$, $S$ | the primary wave sum, the mean correction field, and the correction of the oscillatory products |
| $Q$, $r$ | the requested finite accuracy and a derivative order |

The exponents are fixed in Proposition 19.1. In later sections we write $T = T_0$, and $B_{\mathrm{cubic}} = B_3$.

**Notation 3.2** (Scales at one step)**.** We use the scales of Section 19,

$$\lambda = R^a, \quad h = R^{-\beta}, \quad L_F = T_0 = R^t, \quad K_{\max} = R^{s_f}.$$

The unmodulated wave, with initial data $\delta_0$, has amplitudes varying on the scale $C_{\mathrm{env}} T_0$. Waves modulated by measures with Fourier support in $\{|\xi| \le C K_{\max}\}$ have amplitudes varying on the scale $C_* K_{\max}$, and the waves obtained after the correction of the mean have amplitudes varying on the scale $C_* R$. The two inequalities in (19.5) place these

scales below the wave frequency and below the smallest product frequency, respectively. All exponents and support constants are independent of the accuracy and of the derivative order.

3.4. **Fixed domains and cutoffs.** Fix a closed ball $D$ of positive radius centered at 0 and a larger bounded open chart $W \subset \mathbb{R}^n$. The limit metric equation is imposed on $D$. All measurements, sources of the product corrections and physical output cutoffs are supported in a fixed compact set $D_{\mathrm{lag}} \Subset W$. For the sources of the primary waves we choose cutoffs with

$$D_{\mathrm{lag}} \Subset \operatorname{int}\{\chi_{\mathrm{src}} = 1\}, \qquad \operatorname{supp} \chi_{\mathrm{src}} \Subset \operatorname{int}\{\chi_{\mathrm{core}} = 1\} \Subset W. \tag{3.2}$$

The identities for the linearized metric hold near every measurement support and every source of the product corrections. We write $\chi_{\mathrm{pair}}$ for a fixed cutoff supported in $D_{\mathrm{lag}}$ and equal to one near all product sources, in particular near $D$. The mean and the product corrections may use different cutoffs, but all their supports lie, with positive margins, inside the regions where the corresponding constructions are valid. The larger cutoffs contain the corrections, which may extend beyond the supports of their sources. All these domains are fixed before the frequency, the accuracy and the sample are chosen; this keeps the limit neighborhood independent of the accuracy. The precise arrangement is as follows.

**Setup 3.3** (Fixed domains)**.** Let $W \subset \mathbb{R}^n$ be a bounded open neighborhood of 0. All inclusions below have positive margins, and all choices are made before the frequency, the sample, the accuracy and the final dilation parameter.

(i) *Primary waves.* First fix the solution operators for the primary waves on the whole characteristic patches (Setup 11.6), with a fixed cutoff $\chi_{2,\mathrm{LAG}}$. The finitely many regions on which their identities hold contain a common open set $U_{\mathrm{LAG}}$ on which $\chi_{2,\mathrm{LAG}} = 1$.

(ii) *Measurement region and the ball $D$.* Inside these regions choose a measurement region $V_{\mathrm{meas}}$, small enough for Proposition 13.5, then an open set $U_{\mathrm{BG}}$ and the closed ball $D$, so that

$$D \Subset U_{\mathrm{BG}} \Subset V_{\mathrm{meas}} \Subset \operatorname{int} D_{\mathrm{lag}}, \qquad \operatorname{supp} \chi_{\mathrm{meas}} \Subset V_{\mathrm{meas}}, \tag{3.3}$$

where $\chi_{\mathrm{meas}}$ equals one near $\overline{U_{\mathrm{BG}}}$.

(iii) *Source cutoffs of the primary waves.* Choose

$$\begin{aligned} D_{\mathrm{lag}} \Subset \operatorname{int}\{\chi_{\mathrm{lag,src}} = 1\}, \qquad \operatorname{supp} \chi_{\mathrm{lag,src}} \Subset \operatorname{int}\{\chi_{\mathrm{core,LAG}} = 1\}, \\ \operatorname{supp} \chi_{\mathrm{core,LAG}} \Subset U_{\mathrm{LAG}} \Subset W, \qquad \operatorname{supp} \chi_{\mathrm{lag,src}} \Subset \{\chi_{2,\mathrm{LAG}} = 1\}. \end{aligned} \tag{3.4}$$

A larger chart in $W$ on which the primary waves are defined contains $\overline{U_{\mathrm{LAG}}}$ and the supports of all these cutoffs.

(iv) *Product corrections.* Every measurement support, every source of the product corrections and every physical output support lies in $\operatorname{int} D_{\mathrm{lag}}$. The physical cutoff $\chi_{\mathrm{phys}}$ and the source cutoff $\chi_{\mathrm{pair,src}}$ of the product corrections equal one near $D$;

the latter is supported in the region $X_{\mathrm{pair}}$ where the cutoff $\chi_{2,\mathrm{pair}}$ of the regular parametrices equals one. The amplitude cutoff $\chi_{\mathrm{pair}}$ satisfies $\operatorname{supp}\chi_{\mathrm{pair,src}} \Subset \operatorname{int}\{\chi_{\mathrm{pair}} = 1\}$ and $\operatorname{supp}\chi_{\mathrm{pair}} \Subset \operatorname{int} D_{\mathrm{lag}}$.

(v) *Final cutoffs.* The cutoffs of the mean correction and the common physical cutoff of the final increment (the sum of the primary waves, the mean correction and the product correction) are supported in an open set $U_{\mathrm{fin}}$ with $D \Subset U_{\mathrm{fin}} \Subset U_{\mathrm{BG}}$.

(vi) *Cutoff for the sampling estimate.* The estimate for the sampled operators at fixed patch parameter uses a separate cutoff $\chi_{\mathrm{slice}}$, equal to one near $\operatorname{supp}\chi_{\mathrm{meas}}$ and supported in $\operatorname{int} D_{\mathrm{lag}}$. It enters only this estimate; the fields themselves carry their own physical cutoffs.

In Section 13, the sets $U_{\mathrm{core}}$ and $U_{\mathrm{meas}}$ stand for $U_{\mathrm{BG}}$ and $V_{\mathrm{meas}}$. In Section 11, $U_{\mathrm{core}}$ and the corresponding cutoff stand for $U_{\mathrm{LAG}}$ and $\chi_{\mathrm{core,LAG}}$. The other solution operators have their own fixed domains of validity.

The linearized metric of the primary waves, the errors of their corrections and the identities for the linearized metric at $F$ are estimated on $D_{\mathrm{lag}}$, which contains every measurement support and every source of the product corrections. The smallness of the metric defect and the smoothing of the target are imposed only on $D$.

*Remark* 3.4. The primary waves and the product corrections use different solution operators, each on its own regular patches. The continuous operator and its image component are constructed in Section 13. All samples in one characteristic patch use the same reduced solution operator and the same reconstruction map $\mathcal{V}_F$, for every initial datum, and the finite primary wave and its correction have the same phase. For each construction the solution operator and its residual are fixed; a requested precision only determines the number of compositions of the residual, and the resulting polynomial in the residual is then applied to every datum in that patch. Over the finite list of patches we take the minimum of the margins and the maximum of the derivative losses. Constants may depend on the finite depths, but the depths do not change the losses.

## 4. The reduced metric equation and the reference models

4.1. **The reduced metric equation.** Let $F$ be an immersion, $h_F = \gamma(F)$ its induced metric, $\mathcal{N}_F$ its normal bundle and $\mathrm{II}_F$ its second form. The target metric is denoted by $g$. On symmetric coordinate tensors use the Frobenius pairing and put

$$B_F H = \sum_{i,j} H^{ij} \mathrm{II}_{F,ij}.$$

For isometry set $\mathsf{P} = I$ and $\Xi_F(w) = B_F^* w$. For conformality set

$$\mathsf{P}H = H - \frac{\operatorname{tr}(g^{-1}H)}{n} g, \qquad \Xi_F(w,t) = B_F^* w + tg.$$

Assume $\Xi_F$ is injective and its image has codimension $n$. Let $K_F = (\operatorname{im}\Xi_F)^{\perp}$, choose a frame $Q_1, \dots, Q_n$ of $K_F$, and let $\mathsf{G}_F = (Q_i : Q_j)_{i,j}$. All orthogonal complements here are for the coordinate Frobenius pairing. In the conformal case $Q_i : g = 0$; this need not mean $\operatorname{tr}_g Q_i = 0$.

**Definition 4.1** (Reduced equation)**.** Define

$$\begin{aligned} q_F(H) &= (Q_i : H)_i, \qquad P_F u = (Q_i : \operatorname{sym} \nabla^{h_F} u)_i, \\ M_F(x,\xi)_{i\cdot} &= (Q_i \xi)^T, \qquad p_F = \det M_F, \qquad q_F(x,\xi) = q_F(\xi \otimes \xi). \end{aligned} \tag{4.1}$$

Thus $q_F \mathsf{P} = q_F$. Let

$$N_F = \operatorname{pr}_{\mathcal{N}_F} \Xi_F^{-1} \Pi_{K_F^\perp}, \qquad Y_F(u,H) = dF h_F^{-1} u + N_F(\operatorname{sym} \nabla^{h_F} u - H/2). \tag{4.2}$$

The scalar component of the inverse is omitted for isometry.

The quadratic metric identity is exact:

$$\gamma(F+Y) = \gamma(F) + D\gamma_F Y + (dY)^T dY, \qquad D\gamma_F Y = (dF)^T dY + (dY)^T dF. \tag{4.3}$$

**Lemma 4.2** (Normal elimination)**.** *For $Y = dF h_F^{-1} u + w$, with $w$ normal,*

$$D\gamma_F Y = 2 \operatorname{sym} \nabla^{h_F} u - 2 B_F^* w. \tag{4.4}$$

*For $Y = Y_F(u,H)$,*

$$\mathsf{P}(D\gamma_F Y - H) = 2\mathsf{P} \sum_{i,j} Q_i (\mathsf{G}_F^{-1})_{ij} (P_F u - q_F(H)/2)_j. \tag{4.5}$$

*Consequently the reduced equation is*

$$P_F u = q_F(H)/2. \tag{4.6}$$

*In the conformal case $Y_F(u, H + fg) = Y_F(u,H)$ for every smooth scalar $f$, even for an approximate solution of the reduced equation.*

*Proof.* Differentiate $u_j = \langle F_j, Y \rangle$ and $\langle F_j, w \rangle = 0$. The tangential part of $F_{ij}$ is $\Gamma(h_F)_{ij}^k F_k$; its normal part is $\mathrm{II}_{F,ij}$. Symmetrizing gives (4.4). Put $T = \operatorname{sym} \nabla^{h_F} u - H/2$. Then $B_F^* N_F T = \Pi_{K_F^\perp} T - tg$, with the scalar term absent for isometry. Apply $\mathsf{P}$ and use

$$\Pi_{K_F} T = \sum_{i,j} Q_i (\mathsf{G}_F^{-1})_{ij} (Q_j : T).$$

This proves (4.5). For conformality $\Pi_{K_F} g = 0$ and $\Xi_F(0,1) = g$, so replacing $H$ by $H + fg$ changes only the scalar component, by $-f/2$. □

The connection in these equations is the connection of $h_F$, not of the target metric. The differential lift is

$$\mathcal{V}_F(u) = Y_F(u,0), \qquad \mathsf{P} D\gamma_F \mathcal{V}_F(u) = 2\mathsf{P} \sum_{i,j} Q_i (\mathsf{G}_F^{-1})_{ij} (P_F u)_j. \tag{4.7}$$

One must keep the first projection on the right in the conformal case: $\mathsf{P}\Pi_{K_F}$ is generally different from $\Pi_{K_F}$.

For a nonzero characteristic polarization $b \in \ker M_F(x,\xi)$, the physical normal $\nu = N_F(\xi \odot b)$ is nonzero. Indeed $\xi \odot b = (\xi b^T + b\xi^T)/2$ is a nonzero tensor of rank at most two. If its normal inverse were zero in the conformal problem, it would equal $tg$,

impossible for $n \geq 3$. For isometry the inverse is injective. On compact normalized families this gives a positive normal gap. The exact oscillatory formula is

$$\begin{aligned} \mathcal{V}_F(\tau^2 e^{\mathrm{i}S/\tau} b) &= e^{\mathrm{i}S/\tau}(\mathrm{i}\tau N_F(dS \odot b) + \tau^2 \mathcal{V}_F(b)), \\ A_0 = -\nu(dS)^T, &\qquad A_0^* A_0 = |\nu|^2 dS(dS)^T. \end{aligned} \tag{4.8}$$

Only the physical normal length occurs in this quadratic tensor.

The isometric reference is constructed next. The conformal reference is constructed in Section 5. The technical estimates are written first with $\mathsf{P} = I$; their projected versions, including the altered inverse formulas, are proved in Section 16. The reduced system and the regular oscillatory integral arguments are common to both.

## 4.2. **The constant-coefficient model.**

**Definition 4.3** (The model tensors)**.** Let $n \geq 5$, $N = s_n = n(n+1)/2$, $q = n(n-1)/2$, $d = n-4$ and $c_n = d^{-2}$. Let $E = \operatorname{Sym}^2 \mathbb{R}^n$ carry the Frobenius pairing, and write $S_z = \sum_i z_i$ for $z \in \mathbb{R}^n$, $\mathbf{1} = (1, \dots, 1)$ and $J = \mathbf{1}\mathbf{1}^T$. Set

$$Q_i^0 = e_i e_i^T - c_n J, \qquad K_0 = \operatorname{span}\{Q_1^0, \dots, Q_n^0\}, \qquad \ell_{ab} = \mathbf{1} - 2e_a - 2e_b \quad (a < b). \tag{4.9}$$

With the normalization of (4.1), the corresponding pencil, its determinant and its quadratic map are

$$\begin{aligned} M_0(\xi) = \operatorname{diag}(\xi_1, \dots, \xi_n) - c_n S_\xi J, &\qquad p_0(\xi) = \det M_0(\xi), \\ q_0(\xi) = (\xi^T Q_i^0 \xi)_i = M_0(\xi)\xi. & \end{aligned} \tag{4.10}$$

Explicitly, $q_0(\xi)_i = \xi_i^2 - c_n S_\xi^2$. Note that $M_0(\xi)$ is symmetric and linear in $\xi$, and that $r^T M_0(k) r = k \cdot q_0(r)$.

By the matrix determinant lemma, $\det(\operatorname{diag}(\xi) - c_n S_\xi \mathbf{1}\mathbf{1}^T) = \det \operatorname{diag}(\xi)\,(1 - c_n S_\xi \mathbf{1}^T \operatorname{diag}(\xi)^{-1}\mathbf{1})$ when all $\xi_i \neq 0$, and by continuity

$$p_0(\xi) = \prod_i \xi_i - c_n S_\xi \sum_i \prod_{j \neq i} \xi_j. \tag{4.11}$$

**Proposition 4.4** (A flat form of full rank)**.** *The $\mathbb{R}^q$-valued symmetric form*

$$B_0(u, v) = \Big((\ell_{ab} \cdot u)(\ell_{ab} \cdot v)\Big)_{a<b}$$

*is onto, its kernel, as a map $E \to \mathbb{R}^q$, is $K_0$, and it satisfies the Gauss equations of a flat metric: $\langle B_0(X,Z), B_0(Y,W)\rangle - \langle B_0(X,W), B_0(Y,Z)\rangle = 0$.*

*Proof.* As a map on $E$, $B_0(H) = (\ell_{ab}^T H \ell_{ab})_{a<b}$, so $B_0^*$ maps the basis vector indexed by $(a,b)$ to $\ell_{ab}\ell_{ab}^T$. The $q$ matrices $(e_a + e_b)(e_a + e_b)^T$, $a < b$, are linearly independent, since only the one indexed by $(a,b)$ has a nonzero $(a,b)$ entry. Moreover

$$\ell_{ab} = -2\Big(I - \frac{J}{4}\Big)(e_a + e_b), \qquad \det\Big(I - \frac{J}{4}\Big) = 1 - \frac{n}{4} \neq 0, \tag{4.12}$$

so the matrices $\ell_{ab}\ell_{ab}^T = 4L(e_a+e_b)(e_a+e_b)^T L^T$, with $L = I - J/4$ invertible, are independent as well. Hence $B_0^*$ is injective and $B_0$ is onto. Each $Q_i^0$ annihilates these matrices: every coordinate of $\ell_{ab}$ is $\pm 1$ and $S_{\ell_{ab}} = n-4 = d$, so

$$Q_i^0 : \ell_{ab}\ell_{ab}^T = (\ell_{ab})_i^2 - c_n S_{\ell_{ab}}^2 = 1 - c_n d^2 = 0.$$

The $Q_i^0$ are independent: if $\sum_i \alpha_i Q_i^0 = \operatorname{diag}(\alpha) - c_n(S_\alpha)J = 0$, an off-diagonal entry gives $S_\alpha = 0$ and then $\alpha = 0$. Since $\dim\ker B_0 = s_n - q = n$ and $K_0 \subset \ker B_0$, we get $\ker B_0 = K_0$. Finally, every component of $B_0$ is a product of two linear forms, and for such a component the Gauss expression is $(\ell\cdot X)(\ell\cdot Z)(\ell\cdot Y)(\ell\cdot W) - (\ell\cdot X)(\ell\cdot W)(\ell\cdot Y)(\ell\cdot Z) = 0$. □

### 4.3. **The singular characteristic set.**

**Proposition 4.5** (Singular characteristics of the model)**.** *The singular characteristic set $\mathfrak{M} = \{p_0 = 0,\ dp_0 = 0\}\setminus\{0\}$ is the union, with the origin removed, of*

$$L_{ijl} = \{\xi_i = \xi_j = \xi_l = 0\}, \qquad L_{ijS} = \{\xi_i = \xi_j = S_\xi = 0\}, \qquad \mathbb{R}\ell_{ab}, \tag{4.13}$$

*for distinct indices. The first two families consist of the covectors at which $M_0$ has corank at least two; they have codimension three. Each line $\mathbb{R}\ell_{ab}$ consists of covectors of corank one with $dp_0 = 0$; every other zero of $p_0$ of corank one is a simple characteristic. The lines $\mathbb{R}\ell_{ab}$ are also exactly the nonzero common zeros of the components of $q_0$. We write $\mathcal{L}$ for the finite family of the subspaces in (4.13), so that $\mathfrak{M} = (\bigcup_{L\in\mathcal{L}} L)\setminus\{0\}$.*

*Proof. The kernel of $M_0(\xi)$.* Put $\alpha = c_n S_\xi$; the kernel equation reads

$$\xi_i v_i = \alpha S_v \qquad (1\le i\le n). \tag{4.14}$$

Let $Z = \{i : \xi_i = 0\}$. If $Z = \emptyset$, then $v_i = \alpha S_v/\xi_i$ is determined by the scalar $S_v$, so the kernel has dimension at most one. If $Z\neq\emptyset$ and $\alpha\neq 0$, an index in $Z$ gives $S_v = 0$, so $v$ is supported in $Z$ with $S_v = 0$, and the kernel has dimension $|Z|-1$. If $\alpha = 0$, the kernel consists of the vectors supported in $Z$ and has dimension $|Z|$. Thus the corank is at least two exactly on $\{|Z|\ge 3\}\cup\{|Z|\ge 2,\ S_\xi = 0\}$, which is the union of the spaces $L_{ijl}$ and $L_{ijS}$; each is defined by three independent linear equations. At such points the adjugate of $M_0$ vanishes, and since $dp_0(\xi)[\eta] = \operatorname{tr}(\operatorname{adj} M_0(\xi)\, M_0(\eta))$, also $dp_0(\xi) = 0$.

*Corank one.* If $M_0(\xi)$ has corank one and $v$ spans its kernel, then, $M_0$ being symmetric, $\operatorname{adj} M_0(\xi) = a\, vv^T$ with $a\neq 0$. Hence

$$dp_0(\xi)[\eta] = a\, v^T M_0(\eta) v = a\,\eta\cdot q_0(v), \qquad \text{that is} \qquad dp_0(\xi) = a\, q_0(v). \tag{4.15}$$

So $dp_0(\xi) = 0$ if and only if $q_0(v) = 0$.

*Common zeros of $q_0$.* Let $v\neq 0$ with $q_0(v) = 0$, that is, $v_i^2 = c_n S_v^2$ for all $i$. If $S_v = 0$ then $v = 0$; hence $S_v\neq 0$ and $v_i = \varepsilon_i S_v/d$ with $\varepsilon_i = \pm 1$. Summing over $i$ gives $\sum_i \varepsilon_i = d = n-4$, so exactly two signs are negative, and $v\in\mathbb{R}\ell_{ab}$. Conversely $q_0(\ell_{ab}) = 0$, as computed in the proof of Proposition 4.4.

*Conclusion.* Let $\xi\neq 0$ have corank one and $dp_0(\xi) = 0$. Then its kernel is spanned by some $\ell = \ell_{ab}$, and (4.14) gives $\xi_i\ell_i = c_n S_\xi d = S_\xi/d$, that is, $\xi_i = (S_\xi/d)\ell_i$, since $\ell_i = \pm 1$. Thus $\xi\in\mathbb{R}\ell_{ab}$. Conversely, for $\xi = \ell_{ab}$ all coordinates are nonzero, so the corank is at most one; $M_0(\ell)\ell = q_0(\ell) = 0$, so it equals one, and (4.15) gives $dp_0(\ell) = 0$. □

### 4.4. **Circular bendings and the isometric reference.**

**Lemma 4.6** (Prescribing rank-one normal forms)**.** *Let $H$ be a constant positive definite $n$-by-$n$ matrix and let $\ell_1, \dots, \ell_r$ be nonzero covectors, represented as columns in the chosen coordinates. There is an analytic local embedding $F$ into $\mathbb{R}^{n+r}$ with*

$$F(0) = 0, \qquad \gamma(F) = H, \qquad dF(0) = (H^{1/2}, 0).$$

*At the origin the $r$ normal components of $d^2F$ are the forms $\ell_j \ell_j^T$, in the order prescribed.*

*Proof.* Write $A = H^{1/2}$ and start with $x \mapsto Ax$. To add one ambient coordinate, decompose $Y = Y_\perp + tv$ with $v$ a unit vector and $Y_\perp \perp v$, and use the map

$$C_{v,\kappa}(Y) = \left( Y_\perp + \frac{\sin(\kappa t)}{\kappa} v,\ \frac{1 - \cos(\kappa t)}{\kappa} \right), \qquad \kappa > 0.$$

It fixes the lengths of vectors perpendicular to $v$ and sends the derivative in the $v$ direction to a perpendicular unit vector. Thus its differential preserves the Euclidean metric. It fixes the origin, its differential there is the standard inclusion and its Hessian there has only the new component $\kappa(v \cdot X)(v \cdot Z)$.

At step $j$ take

$$a_j = A^{-T} \ell_j, \qquad v_j = (a_j, 0)/|a_j|, \qquad \kappa_j = |a_j|^2.$$

The zeros extend $a_j$ into the already added coordinates. The chain rule shows that the old Hessian components are preserved and the new one is $(\ell_j \cdot X)(\ell_j \cdot Z)$. All the steps preserve the differential $(A, 0)$ at zero and the metric $H$. The projection onto the first $n$ coordinates has invertible differential at zero, giving an embedding after restricting the domain. □

**Proposition 4.7** (Flat isometric reference)**.** *There is an analytic local embedding $F_0$ into $\mathbb{R}^{s_n}$ with $\gamma(F_0) = I$, $F_0(0) = 0$ and $B_{F_0}(0) = B_0$ in the standard normal coordinates. In particular, $K_{F_0}(0) = K_0$, and the normal map has full rank near the origin.*

*Proof.* Apply Lemma 4.6 to $H = I$ and all $q$ covectors $\ell_{ab}$ of (4.9). The Hessian at zero is normal because the metric is constant. The identification with $B_0$, and the rank assertion, follow from Proposition 4.4 and continuity. □

**Corollary 4.8** (Rescaled flat references)**.** *For all sufficiently small $\sigma > 0$ define, on a fixed ball,*

$$G_\sigma(x) = \sigma^{-1}\Big(F_0(\sigma x) - F_0(0)\Big).$$

*Then $\gamma(G_\sigma) = I$, the smallest singular value of $B_{G_\sigma}$ is at least $c\sigma$, and the projected frame $Q_i(G_\sigma) = \Pi_{K_{G_\sigma}} Q_i^0$ converges to $(Q_i^0)$ in $C^m$ for every fixed $m \geq 0$ as $\sigma \to 0$. Consequently, one fixed sufficiently small $\rho > 0$ gives a reference immersion $G_{\mathrm{ref}} = G_\rho$ whose projected frame lies in any prescribed finite-order neighborhood of the constant frame. The metric, normal-rank and frame bounds persist on a fixed finite-order neighborhood of $G_{\mathrm{ref}}$. All geometric seminorms of $G_{\mathrm{ref}}$ are finite; later constants may depend on $\rho$.*

*Proof.* Let $Q_i(y) = \Pi_{\ker B_{F_0}(y)} Q_i^0$, which is smooth near 0 with $Q_i(0) = Q_i^0$. Since $dG_\sigma(x) = dF_0(\sigma x)$ and $d^2G_\sigma(x) = \sigma\, d^2F_0(\sigma x)$, the tangent and normal spaces of $G_\sigma$ at $x$ coincide with those of $F_0$ at $\sigma x$, and

$$B_{G_\sigma}(x) = \sigma B_{F_0}(\sigma x), \qquad Q_i(G_\sigma)(x) = Q_i(\sigma x).$$

The second identity gives $\partial^\alpha[Q_i(G_\sigma)](x) = \sigma^{|\alpha|}(\partial^\alpha Q_i)(\sigma x)$, which tends to 0 for $|\alpha| \geq 1$, while $Q_i(\sigma x) \to Q_i^0$; this is the asserted convergence. The first identity and the full normal rank of $F_0$ near 0 give the singular-value bound, and flatness gives $\gamma(G_\sigma) = I$. The frame depends smoothly on the second jet of the immersion, so all these bounds persist on a neighborhood in a topology with two more derivatives than the prescribed frame topology. □

After this fixed dilation, the projected frames of all immersions near $G_{\mathrm{ref}}$ are uniformly close to (4.9), and all later operators use these variable coefficients.

## 5. A flat reference for the conformal equation

For $a < b$ write $L_{ab} = \ell_{ab}\ell_{ab}^T$ and set

$$H_0 = \sum_{a<b} L_{ab}.$$

This matrix is positive definite. Indeed, (4.12) expresses the $\ell_{ab}$ as the images of the spanning family $e_a + e_b$ under an invertible linear map. For example, if $i, j, k$ are distinct then $2e_i = (e_i + e_j) + (e_i + e_k) - (e_j + e_k)$. Since each $L_{ab}$ lies in $K_0^\perp$, so does $H_0$.

**Proposition 5.1** (Flat conformal reference)**.** *Fix one pair $a_* < b_*$. There is an analytic local embedding $F_*$ into $\mathbb{R}^{s_n-1}$ with $F_*(0) = 0$ and $\gamma(F_*) = H_0$. At the origin its normal forms are the $q-1$ matrices $L_{ab}$ with $(a,b) \neq (a_*, b_*)$. If $C_0 = B_{F_*}(0)$, the map*

$$(w, t) \longmapsto C_0^* w + tH_0$$

*is injective with image $K_0^\perp$. Thus the conformal and isometric references give the same frozen pencil* (4.10)*.*

*Proof.* Apply Lemma 4.6 with $H = H_0$ and the retained covectors. The metric is constant and the displayed Hessian components are normal at zero. By Proposition 4.4, the $q$ matrices $L_{ab}$ are a basis of $K_0^\perp$. Replacing $L_{a_*b_*}$ in this basis by their sum $H_0$ leaves a basis, because its coefficient in the deleted direction is one. This proves the claim about the augmented map. Since the reduced symbol uses the Frobenius annihilator of its image and acts on tangential covector coefficients, the constant metric $H_0$ leaves the principal matrix unchanged. □

In the conformal construction choose coordinates with $g(0) = H_0$. For $F_\sigma(x) = \sigma^{-1}F_*(\sigma x)$ the metric is still $H_0$ and the second fundamental form equals $\sigma B_{F_*}(\sigma x)$. In a smooth local normal frame the normalized map

$$(w, t) \longmapsto \sigma^{-1} B_{F_\sigma}^* w + tg(s\sigma x)$$

therefore tends smoothly on the fixed coordinate domain to $(w, t) \mapsto C_0^* w + tH_0$, uniformly for $s$ in a compact interval. Its rank gap persists down to the normalized

endpoint. The projected stress frame tends to $Q_i^0$, and the metric projection makes the initial defect tend to zero as $s \to 0$ for any one fixed positive $\sigma$.

At $n = 5$, the matrix $H_0 = 12I - 2J$ has eigenvalue 2 on $\mathbb{R}\mathbf{1}$ and eigenvalue 12 on its orthogonal complement. The construction starts from ten independent forms and retains nine, giving $5 + 9 = 14$ ambient coordinates. Here $c_n = 1$ in (4.10), so the characteristic patch construction uses the same pencil at this endpoint.

## 6. Characteristic patches and phases

We now construct the finite family of characteristic patches which provides the oscillations, first for the constant model and then for immersions near the reference, together with their phases. The main results are the Gram inequality (Proposition 6.1), the classification of all quadratic interactions as elliptic or of simple real principal type (Propositions 6.4 and 6.9), and the properties of the phase differences needed for the oscillatory integrals (Proposition 6.5). The construction of phases on common domains uses the elementary estimates for flows and inverse charts of Section 7, which are independent of the present section.

6.1. **Heuristics.** At a simple characteristic covector $\xi$ the matrix $M_F(x, \xi)$ has a one-dimensional kernel, which determines the leading tangential polarization of a wave with phase gradient $\xi$. After normal reconstruction by (4.7), the leading part of the differential of such a wave is a rank-one matrix $\nu \otimes \xi$, and its Hermitian square is $|\nu|^2 \xi \otimes \xi$. Applying $q_F$ gives the vector $|\nu|^2 q_F(\xi \otimes \xi)$. The waves are also slowly modulated; the quadratic interaction between a wave with base covector $\xi_a$ and its modulation has, to leading order, an output frequency $k$ tangent to the characteristic variety at $\xi_a$. For the constant model these considerations lead to the following Gram inequality, which states that the quadratic vectors complete the image of $M_0$ at every output direction: for each unit $\widehat{k}$ there are selected base covectors $\xi_a$ with

$$\begin{aligned} \widehat{k} \cdot \partial_\xi p_0(\xi_a) &= 0, \\ |M_0(\widehat{k})^T w|^2 + \sum_a |w \cdot q_0(\xi_a)|^2 &\geq c|w|^2 \qquad (w \in \mathbb{R}^n). \end{aligned} \tag{6.1}$$

This holds even at singular output covectors. It is proved in Proposition 6.1, and the corresponding lower bound for the continuous normal operator is proved in Section 13. The quadratic products of distinct waves leave the characteristic patch; their corrections require that their frequencies lie in regions where the reduced system is elliptic or of simple real principal type. This is the second main property of the construction.

### 6.2. Characteristic patches for the constant model.

**Proposition 6.1** (A finite family of characteristic patches)**.** *Let $n \geq 5$ and use the common pencil (4.10). Let $\mathcal{L}$ denote the codimension-three spaces and lines in the description (4.13) of its singular set. There are finitely many smooth maps*

$$\xi_a \colon Z_a \longrightarrow \mathbb{R}^n,$$

*where each $Z_a$ is a closed convex cube in $\mathbb{R}^3$, and a finite open cover $(V_b)$ of the unit sphere $\mathbb{S}^{n-1}$, with the following properties.*

(i) *The norm of $\xi_a(z)$ is bounded above and away from zero. Its coordinates are nonzero, and it satisfies $p_0(\xi_a(z)) = 0$ and $\operatorname{rank} M_0(\xi_a(z)) = n-1$. With $r_{a,i} = 1/\xi_{a,i}$, the vector $v_a = q_0(r_a)$ is a nonzero multiple of $dp_0(\xi_a)$. For $L \in \mathcal{L}$ and $z, z' \in Z_a$,*

$$|\pi_{L^\perp}(\xi_a(z) - \xi_a(z'))| \geq c|z - z'|, \qquad |\xi_a(z) - \xi_a(z')| \leq C|z - z'|. \tag{6.2}$$

*The normalized directions of all nonzero differences and of all sums of two covectors from the same patch lie at a uniform positive distance from $\mathfrak{M}$.*

(ii) *For each $V_b$ there are a finite subcollection of patches and smooth incidence points $z_a(k)$ in their interiors, $k \in V_b$, such that, for one chosen first parameter $z_1$ on each patch,*

$$k \cdot v_a(z_a(k)) = 0, \qquad |\partial_{z_1}(k \cdot v_a(z))| \geq c$$

*near the incidence points, and*

$$|M_0(k)^T s|^2 + \sum_a |s \cdot q_0(\xi_a(z_a(k)))|^2 \geq c|s|^2, \qquad s \in \mathbb{R}^n, \quad k \in V_b. \tag{6.3}$$

(iii) *For any fixed finite grouping of the patches, positive radial factors $R_g$ can be fixed so that every sum or difference $R_g\xi_a(z) \pm R_{g'}\xi_{a'}(z')$ of covectors from different groups has normalized direction at a uniform positive distance from $\mathfrak{M}$. The factors do not depend on the later sample spacing or accuracy.*

Since $dp_0(\xi_a)$ is proportional to $v_a$, the incidence condition in (ii) is the first line of (6.1), and (6.3) is its second line.

*Proof. Step 1 (base covectors satisfying the Gram condition).* Fix $k \neq 0$ with $M_0(k)$ singular and a nonzero $s \in \ker M_0(k)$. Put $\alpha = c_n S_k$ and

$$Q(r) = r^T M_0(k) r = k \cdot q_0(r) = \sum_l k_l r_l^2 - \alpha S_r^2.$$

We first find two indices $i \neq j$ with $s_i + s_j \neq 0$ and a smooth zero of $Q$ in the subspace $\{r_i = r_j = 0\}$ with $S_r \neq 0$ and all other coordinates nonzero.

*Case A: some coordinate of $k$ vanishes.* The kernel equation $k_l s_l = \alpha S_s$ shows that $s$ is supported on the zero coordinates of $k$, and if $\alpha \neq 0$ also $S_s = 0$. If $\alpha = 0$, choose $p, q$ with $k_p > 0 > k_q$ (possible since $S_k = 0$ and $k \neq 0$). If $\alpha \neq 0$, choose $p$ with $\alpha k_p > 0$ and a zero coordinate $q$ of $k$. In both cases $s$ does not vanish identically on the remaining $n - 2 \geq 3$ indices: if $\alpha = 0$, $s_p = s_q = 0$; if $\alpha \neq 0$, $s_p = 0$ and $s$ has at least two nonzero entries because $S_s = 0$. Among three indices on which $s$ is not identically zero, two satisfy $s_i + s_j \neq 0$: if $s_i \neq 0$ and $s_i + s_j = s_i + s_{j'} = 0$, then $s_j + s_{j'} = -2s_i \neq 0$. Fix such $i, j$. If $\alpha = 0$, put $r_l = 1$ for $l \notin \{i, j, p, q\}$ and choose $r_p, r_q > 0$ with $k_p r_p^2 + k_q r_q^2 = -\sum_{l \notin \{i,j,p,q\}} k_l$, which is possible since $k_p > 0 > k_q$. Then $Q(r) = 0$, $S_r > 0$, and $\partial_{r_p} Q = 2k_p r_p \neq 0$. If $\alpha \neq 0$, choose nonzero $r_l$ for $l \notin \{i, j, q\}$ such that $\sum_l k_l r_l^2$ has the sign of $\alpha$ (take $r_p$ large), and then choose $r_q$ with $S_r = \pm(\sum_l k_l r_l^2/\alpha)^{1/2}$; of the two signs at least one gives $r_q \neq 0$. Since $k_q = 0$, $Q(r) = 0$, $S_r \neq 0$, and $\partial_{r_q} Q = -2\alpha S_r \neq 0$.

*Case B: all $k_i \neq 0$.* The kernel equation gives $s_i = \alpha S_s/k_i$ with $\alpha S_s \neq 0$, and summing, $\alpha \sum_i k_i^{-1} = 1$. Replacing $k$ by $-k$ (which changes $M_0(k)$ to $-M_0(k)$) we may assume $\alpha > 0$. Let $p$ and $q$ be the numbers of positive and negative coordinates of $k$. The matrix $H_0$ from Section 5 is positive definite and annihilates $K_0$ under the Frobenius pairing. Since $M_0(k) = \sum_i k_i Q_i^0$, its pairing with $H_0$ vanishes. Independence of the $Q_i^0$ makes $M_0(k) \neq 0$ when $k \neq 0$. Every nonzero semidefinite matrix has strictly signed pairing with $H_0$. It follows that $Q$ is indefinite. To compute the inertia, consider the bordered matrix

$$\mathcal{B} = \begin{pmatrix} \operatorname{diag}(k) & \mathbf{1} \\ \mathbf{1}^T & \alpha^{-1} \end{pmatrix}.$$

Its Schur complement with respect to the entry $\alpha^{-1}$ is $\operatorname{diag}(k) - \alpha \mathbf{1}\mathbf{1}^T = M_0(k)$, and with respect to $\operatorname{diag}(k)$ it is $\alpha^{-1} - \sum_l k_l^{-1} = 0$. By the Haynsworth inertia additivity, $\operatorname{In}(\mathcal{B}) = (p, q, 0) + (0, 0, 1) = (1, 0, 0) + \operatorname{In} M_0(k)$, so $\operatorname{In} M_0(k) = (p-1, q, 1)$. Indefiniteness gives $p \geq 2$ and $q \geq 1$. If $p \geq 3$, let $i, j$ be two positive coordinates; if $p = 2$, then $q = n - 2 \geq 3$, and let $i, j$ be two negative coordinates. In both cases $s_i + s_j = \alpha S_s(k_i^{-1} + k_j^{-1}) \neq 0$. The restriction of $Q$ to $\{r_i = r_j = 0\}$ is $\operatorname{diag}(k') - \alpha \mathbf{1}\mathbf{1}^T$, with $k'$ the remaining coordinates, and the same bordered matrix argument, now with Schur complement $\alpha^{-1} - \sum_{l \neq i,j} k_l^{-1} = k_i^{-1} + k_j^{-1} \neq 0$, shows that it is nonsingular with inertia $(p-2, q, 0)$ or $(1, q-1, 0)$, respectively. It is therefore a nonsingular indefinite form in $n - 2 \geq 3$ variables, and its null cone is a smooth hypersurface of dimension $n - 3$ away from the origin. No open subset of this cone lies in a hyperplane $\{\lambda = 0\}$: otherwise $Q$ would vanish on an open subset of the hyperplane, hence on the hyperplane, and $\lambda$ would divide $Q$, forcing $\operatorname{rank} Q \leq 2$. Hence we can choose a point of the cone with $S_r \neq 0$ and all remaining coordinates nonzero.

*Excluding further degeneracies.* In both cases the zero set of $Q$ in $\{r_i = r_j = 0\}$ is, near the chosen point, a smooth hypersurface of dimension $n - 3$, and $q_0(r)_i = q_0(r)_j = -c_n S_r^2 \neq 0$. The conditions that two other components $q_0(r)_l, q_0(r)_m$ vanish, or that three components $q_0(r)_l, q_0(r)_m, q_0(r)_t$ are equal, read

$$r_l = \varepsilon_l S_r/d, \quad r_m = \varepsilon_m S_r/d, \qquad \text{or} \qquad r_l = \varepsilon_l r_t, \quad r_m = \varepsilon_m r_t,$$

with indices distinct and outside $\{i, j\}$ and signs $\pm 1$; each is a pair of independent linear equations, hence defines a subspace of codimension two. (A coincidence involving $q_0(r)_i$ would force $r_l = 0$ for some $l \notin \{i, j\}$, and $q_0(r)_i$ never vanishes.) A hypersurface of dimension $n - 3$ is not contained in finitely many subspaces of dimension $n - 4$, so we may avoid all of them while keeping the strict conditions above. Now a vector in $L_{ijl}^{\perp} = \operatorname{span}\{e_i, e_j, e_l\}$ has at least $n - 3 \geq 2$ zero components, and a vector in $L_{ijS}^{\perp} = \operatorname{span}\{e_i, e_j, \mathbf{1}\}$ has at least $n - 2 \geq 3$ equal components. Hence $q_0(r) \notin L^{\perp}$ for every codimension-three $L \in \mathcal{L}$.

*Passing to the base covector.* At this point $r$ has exactly two zero coordinates and $S_r \neq 0$, so by the proof of Proposition 4.5 the kernel of $M_0(r)$ is $\mathbb{R}(e_i - e_j)$. By (4.15), $dp_0(r)$ is a nonzero multiple of $q_0(e_i - e_j) = e_i + e_j$, so $dp_0(r) \cdot s \neq 0$. On the other hand $dQ(r) = 2M_0(k)r$ is nonzero and annihilates $s$, because $r^T M_0(k) s = 0$. Hence $dp_0(r)$ and $dQ(r)$ are independent, and $\{p_0 = 0,\ Q = 0\}$ is a smooth submanifold

of codimension two near $r$ (note $p_0(r) = 0$ since $M_0(r)$ is singular). Its tangent space contains $e_i - e_j$: indeed $dp_0(r)(e_i - e_j) = 0$, and $dQ(r)(e_i - e_j) = 2(k_i r_i - \alpha S_r) - 2(k_j r_j - \alpha S_r) = 0$. Moving along a curve in this submanifold with tangent $e_i - e_j$ makes $r_i$ and $r_j$ nonzero while preserving all the open conditions obtained so far. For such a point put $\xi_l = r_l^{-1}$. Since all $r_l \neq 0$, (4.11) gives $p_0(r) = \prod_l r_l\,(1 - c_n S_r S_\xi) = 0$, so $S_r S_\xi = d^2$, and by symmetry $p_0(\xi) = 0$. The kernel of $M_0(r)$ is spanned by $(1/r_l)_l = \xi$ and the kernel of $M_0(\xi)$ by $r$, by (4.14); hence, by (4.15), $dp_0(r) \parallel q_0(\xi)$ and $dp_0(\xi) \parallel q_0(r)$. By continuity $s \cdot q_0(\xi) \neq 0$, and $q_0(r) \neq 0$; thus both covectors are simple characteristics, and

$$k \cdot q_0(r) = Q(r) = 0, \qquad s \cdot q_0(\xi) \neq 0.$$

The first identity is the incidence condition $k \cdot dp_0(\xi) = 0$. The map $\xi \mapsto (1/\xi_l)_l$ is a diffeomorphism of the set of characteristic covectors with nonzero coordinates onto itself, so it also preserves the transversality of the incidence condition, which at the level of $r$ is $dQ(r) \neq 0$ along $\{p_0 = 0\}$.

*Step 2 (three-parameter patches).* Let $c$ be a base covector from Step 1 for the output covector $k_0$, put $v = q_0(c^{-1})$, which is normal to the characteristic variety at $c$, and let $T = v^\perp$ be its tangent space. For a codimension-three $L \in \mathcal{L}$ the projection $T \to L^\perp$ is onto: its kernel $T \cap L$ has dimension $(n-1)+(n-3)-n = n-4$, because $v \notin L^\perp$ means $L \not\subset T$. For a line $L = \mathbb{R}\ell_{ab}$ the projection has rank at least $n-2 \geq 3$. We choose a linear map $E_3 \colon \mathbb{R}^3 \to T$ such that every $\pi_{L^\perp} E_3$ has rank three and $d(k_0 \cdot v) E_3 e_1 \neq 0$, where $d(k_0 \cdot v)$ denotes the differential along the characteristic variety of $\xi \mapsto k_0 \cdot q_0(\xi^{-1})$, which is nonzero by Step 1. Each of these conditions is the nonvanishing of a polynomial in the entries of $E_3$ (a sum of squares of $3 \times 3$ minors, or a linear form), and each polynomial is not identically zero by the rank statements just proved; hence their product is not identically zero, and a suitable $E_3$ exists.

Choose $e$ with $dp_0(c)e \neq 0$. By the implicit function theorem the characteristic variety near $c$ is parameterized as

$$\xi(z) = c + E_3 z + u(z)e, \qquad u(0) = 0, \quad Du(0) = 0, \quad D\xi(0) = E_3,$$

for $z$ near 0 in $\mathbb{R}^3$ (fixing the remaining $n-4$ tangent coordinates). Shrink to a convex cube on which $\|\pi_{L^\perp}(D\xi(z) - E_3)\|$ is at most half the smallest singular value $\sigma_L$ of $\pi_{L^\perp} E_3$, for every $L \in \mathcal{L}$. For $z, z'$ in the cube,

$$\pi_{L^\perp}(\xi(z) - \xi(z')) = \pi_{L^\perp} E_3 (z - z') + \int_0^1 \pi_{L^\perp}\Big(D\xi(z' + \theta(z - z')) - E_3\Big)(z - z')\,\mathrm{d}\theta,$$

whose norm is at least $\sigma_L |z - z'|/2$. This proves (6.2); the upper bound is the Lipschitz bound of $\xi$. Since the distance of a unit vector $w$ from a subspace $L$ equals $|\pi_{L^\perp} w|$, the two bounds give a uniform positive distance between the normalized differences and every $L \in \mathcal{L}$. On a smaller cube all sums $\xi(z) + \xi(z')$ have directions close to that of $c$, which is a simple characteristic, hence lie at a positive distance from $\mathfrak{M}$. Shrinking once more keeps $|\partial_{z_1}(k_0 \cdot v(z))|$ bounded below.

*Step 3 (finite covers and radial factors).* Let $k_0$ be a unit covector with $M_0(k_0)$ singular. By Step 1, for every unit $s \in \ker M_0(k_0)$ there is a base covector with $s \cdot q_0(\xi) \neq 0$. By compactness of the unit sphere of $\ker M_0(k_0)$ we select finitely many base covectors

with $\sum_a |s \cdot q_0(\xi_a)|^2 \geq c|s|^2$ on $\ker M_0(k_0)$. The quadratic form $G(w) = |M_0(k_0)^T w|^2 + \sum_a |w \cdot q_0(\xi_a)|^2$ is then positive definite on $\mathbb{R}^n$: if $G(w) = 0$ then $M_0(k_0)w = 0$ (recall $M_0$ is symmetric), so $w \in \ker M_0(k_0)$, and then $w = 0$. If $M_0(k_0)$ is invertible, no base covector is needed. The incidence points are continued by the implicit function theorem as $z_1 = z_1(k)$, with $z_2 = z_3 = 0$, for $k$ near $k_0$, and by continuity $G$ stays positive definite on a smaller neighborhood. Compactness of $\mathbb{S}^{n-1}$ gives a finite subcover, and we take the minimum of the finitely many positive constants.

All base covectors lie in a compact set of nonzero simple characteristics. We choose the radial factors successively, with ratios so large that every sum or difference of covectors from different groups has normalized direction as close as desired to $\pm$ the direction of the covector from the group with the larger factor; a finite ratio suffices because the norms of base covectors are bounded above and below. Each base covector and its negative lie at a positive distance from $\mathfrak{M}$, so the same holds for these directions. The factors are fixed independently of everything chosen later. The assertions for a single patch were proved in Step 2. □

6.3. **Characteristic phases near the reference immersion.** We now continue the patches to immersions near the reference and construct their phases on common domains.

**Lemma 6.2** (A homogeneous characteristic phase)**.** *Let $H(t, y, \zeta)$ be real and homogeneous of degree one in $\zeta$ on fixed physical boxes and nonzero conic boxes, with a fixed bound for its normalized $C^3$ norm. There are common smaller boxes and a phase $S(t, y, \zeta)$, homogeneous of degree one in $\zeta$, satisfying*

$$S_t = H(t, y, S_y), \qquad S(0, y, \zeta) = y \cdot \zeta. \tag{6.4}$$

*The point $S_\zeta(t, y, \zeta)$ lies in one larger box of initial points, and $S_{y\zeta}$ is uniformly close to $I$. If, for $T \geq 1$ and $R_* \geq 2$, the normalized derivatives of $H$ satisfy $\|H\|_{C^r} \leq C_r T^{(r-R_*)_+}$, then the phase, the flow and the inverse of the map from initial points to points satisfy bounds $C_r T^{(r-R_*+1)_+}$. Normalized $C^{m+1}$ convergence of $H$ implies $C^m$ convergence of these phases on the same domains. The initial time may be any $s$ in the same short interval, and all estimates are joint in $s$.*

*Proof.* Solve the Hamilton equations of $\tau - H(t, y, \zeta)$, on a short interval of both signs,

$$\dot{Y} = -H_\zeta(t, Y, Z), \qquad \dot{Z} = H_y(t, Y, Z), \qquad (Y, Z)|_{t=0} = (u, \zeta).$$

By the low-order bounds and homogeneity, the base point and the normalized covector stay inside their boxes, with $|Z| \asymp |\zeta|$, and on a common shorter interval $Y_u$ is uniformly close to $I$. Lemma 7.4 gives the inverse $u = U(t, y, \zeta)$ of $u \mapsto Y(t, u, \zeta)$ on common domains.

By Euler's identity $Z \cdot H_\zeta = H$; differentiating in $y$ and $\zeta$ gives $Z \cdot H_{\zeta y} = H_y$ and $Z \cdot H_{\zeta\zeta} = 0$. Hence, for any variation $\delta$ of the initial data $(u, \zeta)$,

$$\frac{d}{dt}(Z \cdot \delta Y) = H_y\, \delta Y - Z \cdot H_{\zeta y}\, \delta Y - (Z \cdot H_{\zeta\zeta})\, \delta Z = 0,$$

so $Z\cdot\delta Y=\zeta\cdot\delta u$, its value at $t=0$. Allowing $t$ to vary as well, $Z\cdot dY=Z\cdot\dot{Y}\,\mathrm{d}t+\zeta\cdot du=-H\,\mathrm{d}t+\zeta\cdot du$, that is,

$$Z\cdot dY+H\,\mathrm{d}t=\zeta\cdot du.$$

Put $S(t,y,\zeta)=U(t,y,\zeta)\cdot\zeta$. Substituting $u=U(t,y,\zeta)$, so that $Y=y$, we get $\zeta\cdot dU=H\,\mathrm{d}t+Z\cdot dy$, and therefore

$$dS=\zeta\cdot dU+U\cdot d\zeta=H\,\mathrm{d}t+Z\cdot dy+U\cdot d\zeta.$$

Thus $S_y=Z$, $S_t=H(t,y,S_y)$, $S_\zeta=U$ and $S(0,y,\zeta)=y\cdot\zeta$. Scaling $\zeta$ by $c>0$ scales $Z$ and leaves $Y$ unchanged (since $H_\zeta$ is homogeneous of degree zero and $H_y$ of degree one), so $U$ is homogeneous of degree zero and $S$ of degree one. Finally $S_{\zeta y}=U_y$, which is close to $I$.

The Hamilton vector field involves one derivative of $H$. The joint estimates for flows and inverse charts in Lemma 7.5 therefore give the bound $C_rT^{(r-R_*+1)_+}$, which is preserved by multiplication by the normalized $\zeta$. The initial time enters as one more parameter of the same equations. Subtracting the differentiated equations for two Hamiltonians gives the finite-order continuity. □

**Setup 6.3** (Immersions near the reference)**.** Fix the patches, cover, incidence points and radial factors of Proposition 6.1, and put $M_{\mathrm{an}}=20(n+1)$. Let $F\colon B_1\to\mathbb{R}^N$ be a smooth immersion whose second fundamental form $B_F\colon E\to\mathcal{N}_F$ is onto. Extend $B_F^*$ by zero on tangent vectors and put

$$\begin{aligned} P_{\tan,F}&=dF\,g^{-1}(dF)^T, \qquad \mathcal{A}_F=B_FB_F^*+P_{\tan,F},\\ N_F&=\mathcal{A}_F^{-1}B_F, \qquad \Pi_{K_F}=I-B_F^*N_F, \qquad Q_i=\Pi_{K_F}Q_i^0. \end{aligned} \tag{6.5}$$

We impose two kinds of conditions on $F$. They involve a normalization parameter $0<\sigma\le 1$, which measures the size of the second fundamental form; the constants $C_0$, $\kappa_{\mathrm{met}}$, $\kappa$, $\epsilon_0$ do not depend on $\sigma$.

- ($\mathrm{L}_\sigma$) *Low-order conditions:* $\|F\|_{C^4(B_1)}\le C_0$, the induced metric satisfies $g=(dF)^TdF\ge\kappa_{\mathrm{met}}I$, the smallest singular value of the normalized form $\sigma^{-1}B_F$ is at least $\kappa>0$, and $\max_i\|Q_i-Q_i^0\|_{C^6(B_1)}<\epsilon_0$.
- (H) *A high-order bound:* $\max_i\|Q_i\|_{C^{M_{\mathrm{an}}+9}(B_1)}\le C_{\mathrm{hi}}$, where $C_{\mathrm{hi}}$ is a constant fixed once and for all in Remark 6.6.

On the orthogonal splitting of $\mathbb{R}^N$ into tangent and normal spaces, $\mathcal{A}_F$ is the identity on tangent vectors and $B_FB_F^*\ge\kappa^2\sigma^2$ on normal vectors; hence it is invertible with $\|\mathcal{A}_F^{-1}\|\le(\kappa\sigma)^{-2}$ (for $\kappa\sigma\le 1$). The kernel $K_F$, the projection $\Pi_{K_F}$ and the frame are unchanged if $B_F$ is replaced by $\sigma^{-1}B_F$; they are therefore smooth functions of $(dF,\sigma^{-1}d^2F)$ with bounds depending only on the constants of ($\mathrm{L}_\sigma$), and not on $\sigma$. Consequently $N_F$ takes normal values and vanishes on $K_F$, $B_F^*N_F=B_F^*(B_FB_F^*)^{-1}B_F$ is the orthogonal projection onto $\operatorname{im}B_F^*=K_F^\perp$, and $\Pi_{K_F}$ is the orthogonal projection onto $K_F$; on $K_F^\perp$ the map $N_F$ inverts $B_F^*$, in agreement with the definition before (4.2). For small $\epsilon_0$ the $Q_i$ form a frame of $K_F$. All these are smooth functions of the second jet of $F$, with the stated bounds for the inverses. Define $M_F$, $p_F$ and $q_F$ by (4.1) with this frame.

We say that $F$ satisfies (L) if it satisfies ($\mathrm{L}_\sigma$) for some $0 < \sigma \le 1$. The rescaled references $G_\sigma$ of Corollary 4.8, whose second fundamental forms have size $\sigma$, satisfy ($\mathrm{L}_\sigma$) with constants independent of $\sigma$; in the proof of Theorem 1.1 the parameter is the fixed value $\sigma = \rho$ of the reference. The conditions (L) determine every domain, box, margin and lower bound below. Condition (H) enters only through upper bounds for finitely many derivatives, and it is not a consequence of (L): a $C^6$ neighborhood of the reference contains immersions whose frames have arbitrarily large $C^{M_{\mathrm{an}}+9}$ norm, and no estimate below is claimed uniformly on such a neighborhood without (H). The phases constructed below depend on $F$ only through the frame $Q_i$.

**Proposition 6.4** (Characteristic phases and quadratic interactions)**.** *There are fixed balls $B_{\mathrm{in}} \Subset B_{\mathrm{out}} \Subset B_1$ and a fixed $\epsilon_0 > 0$ such that, for every $F$ satisfying the low-order conditions (L) of Setup 6.3, each model patch has a jointly smooth real family of phases $\Theta_a(x, z)$ on $B_{\mathrm{out}}$ and its fixed parameter cube, with*

$$p_F(x, d_x\Theta_a) = 0, \qquad \operatorname{rank} M_F(x, d_x\Theta_a) = n - 1.$$

*The gradient $d_x\Theta_a$ is uniformly close to $\xi_a(z)$ in $C^1_z C^0_x$, and is uniformly Lipschitz in $z$. For all $L \in \mathcal{L}$, all $x$ and all $z, z'$ in one patch,*

$$|\pi_{L^\perp}(d_x\Theta_a(x, z) - d_x\Theta_a(x, z'))| \ge c|z - z'|. \tag{6.6}$$

*The normalized directions of all nonzero differences and all sums within one patch, and of all signed sums of covectors from different groups multiplied by the fixed radial factors, lie in one compact set $\Gamma$ at a positive distance from $\mathfrak{M}$. On $\Gamma$ the $(n-1)$st singular value of $M_F$ is bounded below, and at every zero of $p_F$ in $\Gamma$ one has $|d_\xi p_F| \ge c$. Thus every such direction is elliptic or a simple characteristic. The domains and constants depend only on the model data, $\kappa$, $\kappa_{\mathrm{met}}$, $C_0$ and the $C^6$ neighborhood, not on higher norms or on the accuracy. If, in addition, $\|F\|_{C^r} \le C_r T^{(r-R_*)_+}$ on the fixed construction neighborhood, with $T \ge 1$ and $R_* \ge 7$, then*

$$\|\mathscr{G}_F\|_{C^r} \le C_r T^{(r-R_*+6)_+}, \qquad \mathscr{G}_F = (S_F, d\Theta_a), \quad r \ge 0, \tag{6.7}$$

*where the norm is the maximum over the finite family, taken jointly in the physical variables, the normalized covectors and the parameters, on fixed domains.*

*Proof. Step 1 (phases).* On each slightly enlarged model patch choose adapted coordinates $x = (t, y)$, $\xi = (\tau, \zeta)$ with $\partial_\tau p_0 \ge c > 0$. By the implicit function theorem there is a unique homogeneous function $H_F$ with

$$p_F\Big(x, (H_F(x, \zeta), \zeta)\Big) = 0$$

near the model root, depending smoothly on the coefficients. Euler's identity $\xi \cdot \partial_\xi p_F = n\, p_F$ (recall $p_F$ is homogeneous of degree $n$) shows that $\zeta \ne 0$ on this set of zeros, since $\partial_\tau p_F \ne 0$. Lemma 6.2 gives a phase $S_F$ on common domains, with $S_0 = tH_0(\zeta) + y \cdot \zeta$ for the constant model, and its continuity statement gives $C^5$ closeness of the phases from $C^6$ closeness of the coefficients.

Under the derivative bounds on $F$, the projected frame is a smooth function of the second jet of $F$, so Lemma 7.2 gives

$$\max_i \|Q_i\|_{C^r} \le C_r T^{(r-R_*+2)_+}. \tag{6.8}$$

Implicit differentiation preserves this for $H_F$, and the phase lemma uses one more derivative, giving $\|S_F\|_{C^r} \le C_r T^{(r-R_*+3)_+}$. None of this changes the domains or the lower bounds.

*Step 2 (sums and differences).* Let $\zeta_a(z)$ be the spatial part of the model patch and put

$$\Theta_a(x,z) = S_F(x,\zeta_a(z)).$$

For the model its gradient is $\xi_a(z)$, and the $C^5$ comparison gives $C^1_z C^0_x$ closeness of the gradients for $F$. The characteristic equation holds exactly by construction, and a nonzero minor of order $n-1$ of the model persists. For every $x$ and $L$, the projected map $z \mapsto \pi_{L^\perp} d_x \Theta_a(x,z)$ is $C^1$ close to the model one on the convex cube, so the segment argument of Proposition 6.1 gives (6.6). Together with the upper Lipschitz bound it gives a fixed angular distance from every $L$, including for arbitrarily short nonzero differences. The distances for sums and for signed sums between groups persist by continuity.

Let $\Gamma_0$ be the closure of all normalized directions of these sums and differences, including the limits of normalized difference quotients. It is compact and disjoint from $\mathfrak{M}$. On $\Gamma_0$, the $(n-1)$st singular value of $M_0$ has a positive minimum (corank at least two occurs only on $\mathfrak{M}$), and by Proposition 4.5 $dp_0 \neq 0$ on $\Gamma_0 \cap \{p_0 = 0\}$, with a positive minimum by compactness. On a slightly larger compact neighborhood $\Gamma$ of $\Gamma_0$, $C^1$ closeness of $M_F$ and $p_F$ to $M_0$ and $p_0$ preserves the singular value bound and ensures $|d_\xi p_F| \ge c$ at every zero of $p_F$. This proves the classification.

The phase estimate of Step 1 and one further derivative prove (6.7). □

For the oscillatory integrals of Sections 8 and 12 we need the differences of nearby phases in a patch, divided by the shift, and their averaged gradients. Fix a patch, an incidence point and the corresponding base output direction $k_b = (\omega_b, \eta_b)$ in the adapted coordinates $x = (t,y)$, $\xi = (\tau,\eta)$ of the patch. Write $z = (s,v)$ with $s = z_1$ and $v = (z_2, z_3)$, and let $v_b$ be the value of $v$ at the incidence point. For $0 \le \epsilon \le \epsilon_*$ put

$$\phi_{F,\epsilon}(x,z,\eta) = \frac{S_F(x,\zeta_a(z)+\epsilon\eta) - S_F(x,\zeta_a(z))}{\epsilon} = \int_0^1 \partial_\zeta S_F(x,\zeta_a(z)+r\epsilon\eta)\cdot\eta\,\mathrm{d}r, \tag{6.9}$$

the integral giving the value at $\epsilon = 0$, and let

$$\Psi_{F,\epsilon}(x,x',s,v,\theta) = \int_0^1 d_x\phi_{F,\epsilon}\Big(x' + r(x-x'), s, v, \theta\Big)\,\mathrm{d}r$$

be the averaged $x$-gradient. For the model we put

$$G(s,\theta) = \nabla H_0(\zeta_a(s,v_b))\cdot\theta, \qquad \Psi_0(s,\theta) = (G(s,\theta),\theta).$$

**Proposition 6.5** (Phase differences and their averaged gradients)**.** *Let $\delta_1, \delta > 0$ be given. There are convex physical boxes inside $B_{\mathrm{out}}$, convex product boxes $D_0 \Subset D_1 \Subset D$ in the variables $(s,\theta)$, centered at $(s_b,\eta_b)$ and with $\theta$-factors in a fixed annulus, a domain*

*of $v$ around $v_b$, a number $\epsilon_* > 0$ and a bound $\epsilon_0 > 0$ for condition (L) of Setup 6.3, all depending only on the model data and on $\delta_1, \delta$, with the following properties.*

(i) *$|\partial_s G| \geq c > 0$ on $D$, and $D\Psi_0$ differs on $D$ from one constant invertible matrix $G_0$ by at most $\|G_0^{-1}\|^{-1}/4$.*
(ii) *For every $F$ satisfying (L) with this $\epsilon_0$, uniformly in $(x, x', v)$ and $0 \leq \epsilon \leq \epsilon_*$,*

$$\|\Psi_{F,\epsilon} - \Psi_0\|_{C^1_{s,\theta}(D)} \leq \delta_1, \qquad |\Psi_{F,\epsilon,y} - \theta| \leq \delta|\theta|, \qquad c \leq |\Psi_{F,\epsilon}| \leq C. \tag{6.10}$$

(iii) *If $F$ satisfies (L) and (H), then $\Psi_{F,\epsilon}$ has joint $C^{M_{\mathrm{an}}+2}$ bounds in $(x, x', s, v, \theta, \epsilon)$, and so has $\partial_\epsilon \Psi_{F,\epsilon}$, with constants depending only on the data of (L) and on $C_{\mathrm{hi}}$.*
(iv) *Under the derivative bounds of Proposition 6.4, the maps $\Psi_{F,\epsilon}$, their inverse charts in $(s, \theta)$ and in $\theta$ at fixed time, and their first $\epsilon$-derivatives satisfy (6.7) jointly in the physical, data and $v$ variables and in $\epsilon$; we include them in $\mathscr{G}_F$.*

Parts (i)–(iii) are exactly the hypotheses on the phase in Setup 8.3 below, with $M = M_{\mathrm{an}}$. The constants $\delta_1, \delta$ are those required there and in Proposition 8.4; they depend only on the model, so the boxes, $\epsilon_*$ and $\epsilon_0$ are fixed by low-order data alone, before any other choice.

*Proof. (i)* For the model, $p_0(\tau, \zeta) = 0$ near the patch is equivalent to $\tau = H_0(\zeta)$, so $d_\xi p_0 = \partial_\tau p_0\,(1, -\nabla H_0)$. The incidence condition $k_b \cdot d_\xi p_0 = 0$ gives $\omega_b = \eta_b \cdot \nabla H_0(\zeta_a(z_b))$, so $\eta_b \neq 0$, and its transversality (Proposition 6.1(ii)) gives $\partial_s G(s_b, \eta_b) \neq 0$. For the model at $\epsilon = 0$ and $v = v_b$ we have $S_0 = tH_0(\zeta) + y \cdot \zeta$, so the phase is $tG(s, \theta) + y \cdot \theta$ and its averaged gradient is the triangular map $\Psi_0$. Its derivative at $(s_b, \eta_b)$ is invertible, and on a small convex box it stays within the stated distance of its value there.

*(ii)* Let $\delta_1, \delta$ be given. By (6.9), the map $\Psi_{F,\epsilon}$ and its first $(s, \theta)$-derivatives involve the derivatives of $S_F$ of order at most three on a compact set of shifted covectors. We choose, in this order: first a domain of $v$ around $v_b$ so small that the model map at $\epsilon = 0$ is within $\delta_1/3$ of $\Psi_0$ in $C^1_{s,\theta}(D)$ for all such $v$; then $\epsilon_*$ so small that the model maps for $0 \leq \epsilon \leq \epsilon_*$ are within $\delta_1/3$ of the map at $\epsilon = 0$, and the shifted covectors stay in the domain of the phase; finally $\epsilon_0$ so small that, by the $C^5$ comparison of the phases in Proposition 6.4, which uses only (L), the maps for $F$ are within $\delta_1/3$ of the model maps. This gives the first inequality. The $y$-component of $\Psi_0$ is $\theta$, so the same choices, with $\delta|\theta|/3$ in place of $\delta_1/3$, give the second inequality, and the third follows since $|\theta|$ is bounded above and below on the annulus.

*(iii)* The joint derivatives of order $M_{\mathrm{an}} + 2$ of $\Psi_{F,\epsilon}$, and of $\partial_\epsilon \Psi_{F,\epsilon}$, are bounded by the derivatives of $S_F$ of order at most $M_{\mathrm{an}} + 5$ in $(x, \zeta)$ on the fixed compact set, multiplied by derivatives of the fixed smooth map $\zeta_a$. The phase is obtained from the Hamiltonian $H_F$ by Lemma 6.2, whose proof (the induction of Lemma 7.5, stopped at a finite order) bounds the $C^m$ norm of $S_F$ on the fixed domains by the $C^{m+1}$ norm of $H_F$. In turn $H_F$ is obtained from $p_F = \det M_F$ by the implicit function theorem with the lower bound of (L) for $\partial_\tau p_F$, so its $C^{m+1}$ norm is bounded by the $C^{m+1}$ norm of the frame. With $m = M_{\mathrm{an}} + 5$ these orders are covered by (H), since $M_{\mathrm{an}} + 6 \leq M_{\mathrm{an}} + 9$.

*(iv)* By (6.7) for $S_F$, the two phase derivatives in $\Psi_{F,\epsilon}$ give the bound $C_r T^{(r-R_*+5)_+}$. Joint inversion preserves it by Lemma 7.5, and one $\epsilon$-derivative changes the exponent to $(r-R_*+6)_+$. The domains were fixed in (i) and (ii) using low-order data only. □

*Remark* 6.6 (Finite-order assumptions on the immersion). The frame $Q_i$ depends smoothly on the second jet of $F$, so a fixed $C^8$ neighborhood of the reference suffices for the $C^6$ frame condition in (L). For the reference of Corollary 4.8 we have $\gamma(G_{\mathrm{ref}}) = I$, so $\|dF - dG_{\mathrm{ref}}\|_{C^0} < 1/2$ implies $|dF(x)v| \geq |v|/2$ and $\gamma(F) \geq I/4$. This neighborhood is chosen together with the neighborhoods for the normal rank and the kernel frame.

We now fix the constant $C_{\mathrm{hi}}$ of (H). Let $\sigma_1 \leq 1$ be so small that the rescaled references $G_\sigma$ are defined on $B_1$ for $0 < \sigma \leq \sigma_1$. They satisfy $Q_i(G_\sigma)(x) = Q_i(\sigma x)$ (proof of Corollary 4.8), where $Q_i(y)$ is the projected frame of $F_0$; hence $\|Q_i(G_\sigma)\|_{C^k(B_1)} \leq \|Q_i\|_{C^k(B_{\sigma_1})}$ for every $k$, uniformly in $\sigma$, and similarly $\|dG_\sigma\|_{C^k(B_1)} + \|\sigma^{-1}d^2G_\sigma\|_{C^k(B_1)} \leq C_k(F_0)$, since $\sigma^{-1}d^2G_\sigma(x) = (d^2F_0)(\sigma x)$.

The frame is a smooth function of $(dF, \sigma^{-1}d^2F)$ on the set defined by the lower bounds of $(\mathrm{L}_\sigma)$ (Setup 6.3), but not of the unnormalized second jet uniformly in $\sigma$: a perturbation $Y$ of $G_\sigma$ enters the projection through $\sigma^{-1}d^2Y$. Therefore a perturbation of fixed size in $C^{M_{\mathrm{an}}+11}$ does not give a bound independent of $\sigma$. We use instead the normalized neighborhood

$$\|d(F-G_\sigma)\|_{C^{M_{\mathrm{an}}+9}(B_1)} + \sigma^{-1}\|d^2(F-G_\sigma)\|_{C^{M_{\mathrm{an}}+9}(B_1)} \leq 1, \tag{6.11}$$

which holds in particular if $\|F - G_\sigma\|_{C^{M_{\mathrm{an}}+11}(B_1)} \leq \sigma$. On this set the arguments $(dF, \sigma^{-1}d^2F)$ are bounded in $C^{M_{\mathrm{an}}+9}$ by $C(F_0)+1$, and the Faà di Bruno formula gives a constant $C_{\mathrm{hi}}$, depending only on $F_0$, $n$ and the constants of (L), such that (H) holds for every $F$ satisfying $(\mathrm{L}_\sigma)$ and (6.11) for some $0 < \sigma \leq \sigma_1$. We take $B_1$ inside the chart $W$ of Setup 3.3, so that bounds on $W$ imply bounds on $B_1$.

In the proof of Theorem 1.1 the parameter is fixed, $\sigma = \rho \leq \sigma_1$, before the iteration. All iterates lie in a $C^{r_*}$ ball of radius $\varepsilon_* \leq \rho$ around $G_{\mathrm{ref}} = G_\rho$, with $r_* \geq M_{\mathrm{an}} + 11$ (Section 21), so (6.11) and hence (H) hold for all of them with this constant. Uniformity as $\sigma \to 0$ is needed only for the rescaled references $G_\sigma$ themselves (Section 13), for which it was shown above.

6.4. **Indexing the quadratic interactions.** The characteristic patches, and therefore the sums and differences of their covectors, are fixed before the frequencies and the sample.

**Definition 6.7** (Groups of patches)**.** Index the finite list of characteristic patches before the frequencies and the sample are chosen. If a patch is used more than once, each occurrence receives its own index and forms a separate group. Apply part (iii) of Proposition 6.1 to this list, so that different groups have radially separated factors. Within one group the deletion rule of Theorem 12.4 separates distinct samples by at least $h/4$. All modulations attached to one sample are combined into a single Fourier measure and a single wave.

*Remark* 6.8 (Sums and differences of phases). For two distinct samples in the same group, the chord estimate (6.6) keeps the direction of the phase difference away from

$\mathfrak{M}$ and gives it the frequency $\lambda|z - z'| \geq c\lambda h$. The sum of two phases has frequency comparable to $\lambda$ and the same angular separation. For different groups the radial factors give both properties for both signs, even if the parameters or the unscaled directions coincide. For a repeated index, the doubled phase is separated from $\mathfrak{M}$, while the Hermitian square is treated as part of the mean. These cases exhaust the real quadratic expansion of a sum of waves. Fourier modulations are included in the amplitudes (Section 15). All group factors are fixed before the sample is drawn.

We summarize the properties of the patches used in the rest of the paper.

**Proposition 6.9** (Characteristic patches)**.** *For $n \geq 5$ there is a finite family of three-parameter patches of simple characteristic covectors of the constant model with the following properties.*

(i) *For every unit output direction, the selected patches have transversal incidence points in their interiors and satisfy the uniform Gram inequality* (6.1).
(ii) *For immersions in a fixed neighborhood of the reference, on a fixed chart and after fixed radial factors are assigned to the patches, every doubled phase and every sum or difference of distinct sampled phases has its gradient in a fixed compact conic region where the reduced system is elliptic or of simple real principal type.*
(iii) *For samples separated by $h$, the frequency $\varrho$ of each such product satisfies $c\lambda h \leq \varrho \leq C\lambda$.*

*The Hermitian squares are treated by the equation for the mean and are excluded from (ii).*

*Proof.* Part (i) is part (ii) of Proposition 6.1. Part (ii) is Proposition 6.4, with the grouping of Definition 6.7. Part (iii) follows from Remark 6.8: the frequency of a difference is $\lambda$ times the length of a difference of gradients, which is between $c|z-z'|$ and $C|z - z'|$ by (6.6), and sums and products between groups have frequency comparable to $\lambda$. □

A convex three-parameter patch controls the limits of normalized differences as well as the differences of separated samples; this is why three parameters are used. The parametrices on the regions of (ii) also require control of their Hamilton trajectories. This is proved in Theorem 10.3: each chart is inverted separately, the successive retarded parts of the parametrix use the same time coordinate and together occupy one fixed time interval, and the charts are chosen so that the forward flowout of each source chart over this interval stays in a slightly larger chart of the same regular region.

## 7. Smoothing and derivative estimates

This section contains elementary estimates which do not depend on any phase, solution operator or frequency. Lemma 7.1 is used to smooth the metric defect (Lemma 17.1). The remaining results show that derivative bounds of the form (3.1) are preserved, with a fixed loss, by the algebraic operations, inverse charts and Hamilton flows used in Section 6 and in Sections 9–13.

### 7.1. **Smoothing on a ball.**

**Lemma 7.1** (Smoothing on a ball)**.** *Let $\Omega = B_a(c) \subset \mathbb{R}^n$ be a ball with $\overline{\Omega} \subset V$, where $V$ is open. For each integer $m \geq 1$ and all sufficiently large $R$ there is a real linear operator $S_{R,m} : C^\infty(\overline{\Omega}) \to C_c^\infty(V)$, with outputs supported in one fixed compact subset of $V$, such that for all integers $r \geq 0$*

$$\|S_{R,m} f\|_{C^r} + \|S_{R,m} f\|_{H^r} \leq C_{r,m} R^r \|f\|_{C^0(\overline{\Omega})}, \tag{7.1}$$

$$\|f - S_{R,m} f\|_{C^0(\overline{\Omega})} \leq C_m R^{-m} \|f\|_{C^m(\overline{\Omega})}. \tag{7.2}$$

*The operators act componentwise on vector- and tensor-valued functions and preserve real fields.*

*Proof.* After an affine change of variables we may assume $\Omega = B_1$; this changes the constants and the size of $R$ only by fixed factors. Fix $\rho \in C_c^\infty(B_{1/4})$ with $\rho \geq 0$ and $\int \rho = 1$. For $R \geq 2m$ choose $\chi_R \in C_c^\infty(B_{1+1/(4R)})$ with $\chi_R = 1$ on $B_1$ and $\|D^r \chi_R\|_\infty \leq C_r R^r$. Put

$$a_j = (-1)^{j-1} \binom{m}{j} \quad (1 \leq j \leq m), \qquad \sum_{j=1}^{m} a_j j^k = \delta_{k0} \quad (0 \leq k < m). \tag{7.3}$$

The identity holds because the $m$th finite difference annihilates polynomials of degree less than $m$: $\sum_{j=0}^{m} (-1)^j \binom{m}{j} j^k = 0$ for $0 \leq k < m$, and moving the term $j = 0$ (which equals $\delta_{k0}$) to the other side gives (7.3). Define

$$S_{R,m} f(x) = \chi_R(x) \sum_{j=1}^{m} a_j \int \rho(y) f\Big((1 - j/R)x + (j/R)y\Big)\, \mathrm{d}y \tag{7.4}$$

for $|x| < 1 + 1/(2R)$, and $S_{R,m} f = 0$ elsewhere.

*The formula is well defined.* Let $|x| \leq 1 + 1/(2R)$, $|y| \leq 1/4$ and $1 \leq j \leq m \leq R/2$. Then $1 - j/R > 0$ and

$$\Big|(1 - j/R)x + (j/R)y\Big| \leq \Big(1 - \frac{j}{R}\Big)\Big(1 + \frac{1}{2R}\Big) + \frac{j}{4R} \leq 1 + \frac{1}{2R} - \frac{3j}{4R} < 1,$$

so every argument of $f$ lies in $B_1$. Since $\chi_R$ vanishes outside $B_{1+1/(4R)}$, the extension by zero is smooth, and for large $R$ all outputs are supported in a fixed compact subset of $V$.

*Derivative bounds.* In the $j$th term substitute $w = (1 - j/R)x + (j/R)y$:

$$\int \rho(y) f\Big((1 - j/R)x + (j/R)y\Big)\, \mathrm{d}y = \Big(\frac{R}{j}\Big)^n \int \rho\Big(\frac{R}{j}\Big(w - (1 - j/R)x\Big)\Big) f(w)\, \mathrm{d}w.$$

All $x$-derivatives now fall on $\rho$, and each produces the factor $-(R/j)(1 - j/R) = -(R/j - 1)$, of absolute value at most $R$. Undoing the substitution,

$$\Big|\partial_x^\alpha \int \rho(y) f\Big((1 - j/R)x + (j/R)y\Big)\, \mathrm{d}y\Big| \leq R^{|\alpha|} \|\partial^\alpha \rho\|_{L^1} \|f\|_{C^0(B_1)}.$$

By the Leibniz rule and $\|D^r \chi_R\|_\infty \leq C_r R^r$ we obtain $\|S_{R,m} f\|_{C^r} \leq C_{r,m} R^r \|f\|_{C^0(B_1)}$. Since the supports lie in a fixed bounded set, the same bound holds for the $H^r$ norm.

*Approximation.* Let $x \in B_1$, so that $\chi_R(x) = 1$. For $y \in B_{1/4}$ the segment from $x$ to $y$ lies in $B_1$, and Taylor's formula at $x$ gives

$$f\Big(x + (j/R)(y-x)\Big) = \sum_{k=0}^{m-1} \frac{(j/R)^k}{k!} D^k f(x)[(y-x)^k] + E_j(x,y),$$

where $|E_j(x,y)| \le C_m (j/R)^m \|f\|_{C^m(B_1)}$. Multiplying by $a_j \rho(y)$, integrating in $y$ and summing over $j$, the term of order $k$ is multiplied by $\sum_j a_j j^k = \delta_{k0}$. Hence

$$S_{R,m} f(x) - f(x) = \sum_{j=1}^{m} a_j \int \rho(y) E_j(x,y)\, \mathrm{d}y,$$

whose absolute value is at most $C_m R^{-m} \|f\|_{C^m(B_1)}$. This proves (7.2). The operator is real and acts componentwise by construction. □

7.2. **Derivative bounds.** The following elementary inequality is used repeatedly: if $R \ge 0$ and $j_1, \dots, j_k \ge 0$, then

$$\sum_{i=1}^{k} (j_i - R)_+ \le \Big(\sum_{i=1}^{k} j_i - R\Big)_+. \tag{7.5}$$

Indeed, if all $j_i \le R$ the left side vanishes; otherwise let $I = \{i : j_i > R\} \neq \emptyset$. Then the left side equals $\sum_{i \in I} j_i - |I| R \le \sum_i j_i - R$.

**Lemma 7.2** (Derivative products)**.** *Let $T \ge 1$ and $R \ge 0$, and suppose finitely many arrays $f_i$ on fixed charts satisfy $\|f_i\|_{C^r} \le C_r T^{(r-R)_+}$ for all $r \ge 0$. Then*

$$\prod_i \|D^{j_i} f_i\|_\infty \le C_{j_1,\dots,j_k} T^{(\sum_i j_i - R)_+}. \tag{7.6}$$

*Let $\Phi$ be a smooth function, all of whose derivatives are bounded on a fixed bounded set containing the ranges of the $f_i$. Then $\Phi(f_1, \dots, f_k)$ satisfies the same bounds $\|\Phi(f)\|_{C^r} \le C_r T^{(r-R)_+}$. This applies in particular to matrix inversion on a set of matrices with a fixed lower bound for their smallest singular value. All coordinate and parameter derivatives are counted jointly.*

*Proof.* The product estimate follows from the hypothesis and (7.5). For the second assertion, the Faà di Bruno formula expresses $D^r[\Phi(f)]$, $r \ge 1$, as a finite sum of terms

$$(D^q \Phi)(f)\Big[D^{\alpha_1} f, \dots, D^{\alpha_q} f\Big], \qquad 1 \le q \le r, \quad \alpha_i \ge 1, \quad \alpha_1 + \cdots + \alpha_q = r,$$

with universal coefficients. The factor $(D^q\Phi)(f)$ is bounded by assumption, and the product of the remaining factors is bounded by $C T^{(r-R)_+}$ by (7.6). □

**Corollary 7.3** (Derivative estimates for finite jets)**.** *Let $T \ge 1$ and suppose $\|F\|_{C^j} \le A_j T^{(j-R)_+}$ for all $j \ge 0$. Let $\mathcal{F}_i(J^{c_i}F)$ be a fixed finite family of smooth functions of the $c_i$-jets of $F$, and assume that the values of these jets lie in fixed bounded sets on which the $\mathcal{F}_i$ have bounded derivatives of all orders (this includes fixed lower bounds for determinants to be inverted). If $\nu \ge 0$ and $R \ge \max_i c_i + \nu$, then*

$$\|\mathcal{F}_i(J^{c_i}F)\|_{C^r} \le C_r T^{(r-\nu)_+}, \qquad r \ge 0. \tag{7.7}$$

*Additional parameter derivatives of fixed order are included in $c_i$.*

*Proof.* A derivative of order $r$ of $J^{c_i}F$ is a derivative of $F$ of order at most $r + c_i$, so $\|J^{c_i}F\|_{C^r} \le CT^{(r+c_i-R)_+} \le CT^{(r-\nu)_+}$, since $R - c_i \ge \nu$. Lemma 7.2, applied with $\nu$ in place of $R$, gives the claim. □

**Lemma 7.4** (Common inverse domains)**.** *Let $D_0 \Subset D_1 \Subset D$ be bounded convex boxes in $\mathbb{R}^d$ and let $F_0$ be smooth near $\overline{D}$. Suppose a constant invertible matrix $G$ satisfies $\sup_D \|G^{-1}(DF_0 - G)\| \le 1/4$. Then for all $F$ in a sufficiently small $C^1$ neighborhood of $F_0$, the map $F$ is injective on $D$, with $|G^{-1}(F(u) - F(v))| \ge |u - v|/2$, and there is one open set $V$ with $F(D_0) \Subset V \Subset F(D_1)$, with positive distances between these sets that do not depend on $F$. The same holds uniformly for parameter families of maps in this neighborhood.*

*Proof.* Choose the $C^1$ neighborhood so small that $\sup_D \|I - G^{-1}DF\| \le 1/2$. For $u, v \in D$, convexity gives

$$G^{-1}(F(u) - F(v)) = (u - v) - \int_0^1 \Big(I - G^{-1}DF(v + \theta(u - v))\Big)(u - v)\,\mathrm{d}\theta,$$

and the integral has norm at most $|u - v|/2$; this proves the lower Lipschitz bound and injectivity.

Choose $r > 0$ such that $\overline{B}_r(u_0) \subset D_1$ for every $u_0 \in \overline{D}_0$, and let $V$ be the open $(r/(4\|G^{-1}\|))$-neighborhood of $F_0(\overline{D}_0)$. If $\|F - F_0\|_{C^0} < r/(4\|G^{-1}\|)$, then every $v \in \overline{V}$ has a point $u_0 \in \overline{D}_0$ with $|G^{-1}(v - F(u_0))| < r/2$. Consider

$$\Phi(u) = u - G^{-1}(F(u) - v), \qquad u \in \overline{B}_r(u_0).$$

Since $D\Phi = I - G^{-1}DF$ has norm at most $1/2$, $\Phi$ is a $1/2$-contraction, and

$$|\Phi(u) - u_0| \le |\Phi(u) - \Phi(u_0)| + |\Phi(u_0) - u_0| < \tfrac{1}{2}|u - u_0| + \tfrac{r}{2} \le r.$$

Thus $\Phi$ maps $\overline{B}_r(u_0)$ into itself, and its fixed point $u$ satisfies $F(u) = v$. Hence $\overline{V} \subset F(D_1)$. Finally, $F(\overline{D}_0)$ lies within distance $r/(4\|G^{-1}\|)$ of $F_0(\overline{D}_0)$; after halving the $C^0$ neighborhood once more we obtain $F(\overline{D}_0) \Subset V$ with a fixed margin. All constants depend only on $r$, $G$ and the size of the neighborhood. □

**Lemma 7.5** (Composition, inverse charts and flows)**.** *Work on fixed coordinate domains, which may include physical variables and external parameters, and let $T \ge 1$ and $R \ge 1$. Bounds of the form $\|f\|_{C^r} \le C_r T^{(r-R)_+}$ are preserved, with the same index $R$:*

(i) *by composition of maps satisfying such bounds;*
(ii) *by inverse and implicit function charts, whenever the charts exist on the given domains and the relevant first derivative has a uniformly bounded inverse;*
(iii) *by the flow of a vector field $V(t, x)$ satisfying such bounds, on a fixed time interval on which the trajectories stay in the working domains; the initial section may satisfy the same bounds;*
(iv) *by the solutions $U$ and $U^{-1}$ of $U' = -CU$, $U(0) = I$, when the matrix $C$ satisfies such bounds.*

*The constants depend only on the constants in the hypotheses and on the fixed lower bounds.*

*Proof. The basic inequality.* Let $q \geq 1$ and $\alpha_1, \dots, \alpha_q \geq 1$ with $\sum_i \alpha_i = r$. We claim

$$(q-R)_+ + \sum_{i=1}^{q} (\alpha_i - R)_+ \leq (r-R)_+. \tag{7.8}$$

If $q > R$, then $(\alpha_i - R)_+ \leq \alpha_i - 1$ for every $i$ (because $\alpha_i \geq 1$ and $R \geq 1$), so the left side is at most $(q-R) + \sum_i (\alpha_i - 1) = r - R$. If $q \leq R$, the first term vanishes and the claim is (7.5).

*Composition.* By the Faà di Bruno formula, $D^r(f \circ g)$ is a sum of terms $(D^q f)(g)[D^{\alpha_1} g, \dots, D^{\alpha_q} g]$ with $\sum \alpha_i = r$, $\alpha_i \geq 1$. The first factor is bounded by $CT^{(q-R)_+}$ and the others by $CT^{(\alpha_i - R)_+}$, so (7.8) gives the bound $CT^{(r-R)_+}$.

*Inverse charts.* Let $X = F^{-1}$. The first derivative $DX = DF(X)^{-1}$ is bounded by the hypothesis on the inverse. For $r \geq 2$, differentiating $F \circ X = \mathrm{Id}$ $r$ times gives

$$DF(X)\, D^r X = -\sum (D^q F)(X) \Big[ D^{\alpha_1} X, \dots, D^{\alpha_q} X \Big],$$

where the sum runs over $q \geq 2$ and $\alpha_i \geq 1$ with $\sum \alpha_i = r$; in particular every $\alpha_i < r$. Assume inductively that $\|D^\alpha X\|_\infty \leq C_\alpha T^{(\alpha - R)_+}$ for $\alpha < r$. Since $(D^q F)(X)$ is bounded by $CT^{(q-R)_+}$, (7.8) bounds the right side by $CT^{(r-R)_+}$, and multiplication by $DF(X)^{-1}$ completes the induction. Implicit functions are obtained by inverting $(z, w) \mapsto (z, F(z, w))$, with the parameters included in $z$. Existence of the charts on common domains is a separate low-order matter, treated by Lemma 7.4.

*Flows.* Let $\partial_t \Psi = V(t, \Psi)$. Differentiating $r$ times in the initial point gives

$$\partial_t D^r \Psi = D_x V(t, \Psi)\, D^r \Psi + \sum_{q \geq 2} (D_x^q V)(t, \Psi) \Big[ D^{\alpha_1} \Psi, \dots, D^{\alpha_q} \Psi \Big],$$

where again $\alpha_i < r$. By induction the sum is bounded by $CT^{(r-R)_+}$, and the coefficient $D_x V$ is bounded; Gronwall's inequality on the fixed time interval gives $\|D^r \Psi\| \leq C_r T^{(r-R)_+}$. Joint estimates in time and parameters follow by adjoining $t$ and the parameters as coordinates with $\dot{t} = 1$ and vanishing derivative. A derivative in the elapsed time is replaced by the equation, which costs no further derivative of $\Psi$; a variable initial section is handled by composition.

*Linear matrix equations.* Gronwall's inequality bounds $U$ and $U^{-1}$, which solves $(U^{-1})' = U^{-1} C$. The flow argument applied to these linear equations on the resulting bounded ranges gives the derivative bounds. □

*Remark* 7.6 (Homogeneous symbols). Symbols and phases which are homogeneous in a covector variable are estimated through their restrictions to a fixed annulus. For a symbol $a$ of order $p$ and a phase $S$ of degree one put $a_\mu(x, \omega) = \mu^{-p} a(x, \mu\omega)$ and $S_\mu(x, \omega) = \mu^{-1} S(x, \mu\omega)$ for $\omega$ in a fixed annulus. Then

$$\partial_x^\alpha \partial_\omega^\gamma a_\mu = \mu^{|\gamma| - p} (\partial_x^\alpha \partial_\eta^\gamma a)(x, \mu\omega),$$

and similarly for $S$ with $p = 1$. Thus the usual symbol estimates for $a$ are equivalent to uniform bounds for the normalized functions $a_\mu$, to which Lemmas 7.2 and 7.5 apply.

Amplitudes depending on two positions are treated in the same way, and different frequency shells are normalized separately; their radial powers and volume factors are kept outside the derivative bounds.

## 8. Oscillatory integral estimates

This section contains $L^2$ bounds for oscillatory integral operators whose amplitudes have finitely many derivatives, an exact formula for their normal operators, a comparison estimate, and an estimate for the Fourier tails of oscillatory functions. The phase hypotheses below are those established for the phases of our construction in Proposition 6.5; otherwise each result uses only the displayed bounds. The results are applied to the continuous normal operator (Section 13), the sampled operators (Section 12), and the parametrices (Sections 9 and 10).

### 8.1. **Normal operators.**

**Notation 8.1.** We use the Fourier convention of Notation 3.1 and the norm $\|f\|_{H^s} = \|\langle D\rangle^s f\|_2$ on $\mathbb{R}^n$. Outputs with compact support in a chart are extended by zero. Derivative orders are total orders of multi-indices, and matrix sizes are fixed.

**Lemma 8.2** (Multiplication of kernels by smooth amplitudes)**.** *Fix compact sets in $\mathbb{R}^d$, $d \geq 1$, and put $\ell = \lfloor d/2 \rfloor + 1$.*

(i) *Let $U$ be a measure space and let $T\colon L^2(U) \to L^2(\mathbb{R}^d)$ be a bounded integral operator with kernel $K(x,u)$. For a function $b(x,u)$ supported, in $x$, in the fixed compact set, let $T[b]$ be the operator with kernel $b(x,u)K(x,u)$. Then*

$$\|T[b]\| \leq C\|T\| \sup_u \|b(\cdot,u)\|_{H^\ell}. \tag{8.1}$$

(ii) *Let $b(x,y,\xi)$ be supported, in $(x,y)$, in a fixed compact set, and assume $\sup_\xi \|b(\cdot,\cdot,\xi)\|_{C^{2\ell}} \leq A$. Then the operator*

$$Bf(x) = (2\pi)^{-d} \iint e^{\mathrm{i}(x-y)\cdot\xi} b(x,y,\xi) f(y)\,\mathrm{d}y\,\mathrm{d}\xi$$

*satisfies $\|B\|_{L^2\to L^2} \leq CA$.*

*No derivatives in $u$ or $\xi$ are needed, and the frequency support may be unbounded. The constants depend only on $d$ and the compact sets; the same bounds hold for matrix-valued amplitudes of fixed size.*

*Proof.* (i) For $a \in \mathbb{R}^d$ define $G(a,x) = T\big(b(a,\cdot)f\big)(x)$, so that $T[b]f(x) = G(x,x)$. Since $\ell > d/2$, the Sobolev embedding in the variable $a$ gives, for every fixed $x$,

$$|G(x,x)|^2 \leq \sup_a |G(a,x)|^2 \leq C \sum_{|\alpha|\leq \ell} \int |\partial_a^\alpha G(a,x)|^2\,\mathrm{d}a,$$

where we use the continuous representative in $a$; the support of $b$ in its first variable is a fixed compact set, so the integral is over a fixed bounded region. Integrating in $x$

and using Fubini's theorem,

$$\|T[b]f\|_2^2 \le C\sum_{|\alpha|\le\ell}\int \big\|T\big(\partial_a^\alpha b(a,\cdot)f\big)\big\|_2^2\,\mathrm{d}a \le C\|T\|^2\sum_{|\alpha|\le\ell}\iint_U |\partial_a^\alpha b(a,u)|^2|f(u)|^2\,\mathrm{d}u\,\mathrm{d}a$$
$$= C\|T\|^2\int_U \|b(\cdot,u)\|_{H^\ell}^2|f(u)|^2\,\mathrm{d}u \le C\|T\|^2\sup_u\|b(\cdot,u)\|_{H^\ell}^2\|f\|_2^2.$$

(ii) For $a\in\mathbb{R}^d$ let $T_a$ be the operator $T_ag(y)=(2\pi)^{-d}\int e^{\mathrm{i}y\cdot\xi}b(a,y,\xi)^*g(\xi)\,\mathrm{d}\xi$ from $L^2(\mathbb{R}^d_\xi)$ to $L^2(\mathbb{R}^d_y)$, and put $G(a,x)=(\mathcal{F}^{-1}T_a^*f)(x)$, where $\mathcal{F}^{-1}$ is the inverse Fourier transform $g\mapsto\int e^{\mathrm{i}x\cdot\xi}g(\xi)\,\mathrm{d}\xi$. Writing out $T_a^*f(\xi)=(2\pi)^{-d}\int e^{-\mathrm{i}y\cdot\xi}b(a,y,\xi)f(y)\,\mathrm{d}y$, we get $G(a,x)=(2\pi)^{-d}\iint e^{\mathrm{i}(x-y)\cdot\xi}b(a,y,\xi)f(y)\,\mathrm{d}y\,\mathrm{d}\xi$, and hence $Bf(x)=G(x,x)$. As in (i),

$$\|Bf\|_2^2 \le C\sum_{|\alpha|\le\ell}\int\|\mathcal{F}^{-1}\partial_a^\alpha T_a^*f\|_2^2\,\mathrm{d}a \le C\sum_{|\alpha|\le\ell}\int\|\partial_a^\alpha T_a\|^2\,\mathrm{d}a\;\|f\|_2^2.$$

For each fixed $a$ and $\alpha$, the operator $\partial_a^\alpha T_a$ is, up to the constant $(2\pi)^{-d}$ and the unitary Fourier transform, the multiplication of the kernel of the inverse Fourier transform by the amplitude $\partial_a^\alpha b(a,y,\xi)^*$. Part (i), applied in the variable $y$, bounds its norm by $C\sup_\xi\|\partial_a^\alpha b(a,\cdot,\xi)\|_{H^\ell}\le CA$, because $|\alpha|\le\ell$ derivatives in $a$ and at most $\ell$ in $y$ use at most $2\ell$ derivatives. Since $b$ has compact support in $a$, this proves $\|B\|\le CA$. For unbounded frequency supports one first inserts a frequency cutoff, which multiplies $T_a^*f$, and passes to the limit by dominated convergence in $H^\ell_a(L^2_\xi)$; the bound is unchanged. The same argument works if $\mathcal{F}^{-1}$ is replaced by any bounded operator $L^2_\xi\to L^2_x$, at the cost of its norm. □

**Setup 8.3** (Phases with conic structure)**.** Write $x=(t,y)\in\mathbb{R}\times\mathbb{R}^{n-1}$ and $z=(s,v)$, where $v$ ranges over a fixed compact set of remaining parameters. On fixed physical and covector boxes let $S(x,\zeta)$ and $\zeta(z)$ be smooth and real, and put

$$\Phi_\tau(x,z,\eta)=\frac{S(x,\zeta(z)+\tau\eta)-S(x,\zeta(z))}{\tau}=\int_0^1 d_\zeta S(x,\zeta(z)+r\tau\eta)\cdot\eta\,\mathrm{d}r,\qquad \phi_\epsilon=\Phi_\epsilon,$$

with the integral giving the value at $\tau=0$. The physical boxes are convex. Fix convex product boxes $D_0\Subset D_1\Subset D$ in the variables $(s,\theta)$, whose $\theta$-factors lie in one annulus around $(s_b,\eta_b)$, $\eta_b\neq0$, and a smooth map $\Psi_0(s,\theta)=(G(s,\theta),\theta)$ with $|\partial_sG|\ge c>0$. The boxes are chosen so that $D\Psi_0$ stays close to one constant invertible matrix, as in Lemma 7.4. We assume that the averaged $x$-gradients

$$\Psi_\epsilon^{\mathrm{ann}}(x,x',s,v,\theta)=\int_0^1 d_x\phi_\epsilon\big(x'+r(x-x'),s,v,\theta\big)\,\mathrm{d}r$$

are uniformly close to $\Psi_0$ in $C^1(s,\theta)$ on $D$, and have fixed joint $C^M$ bounds in the physical, data, $v$ and $\epsilon$ variables, where $M=M_{\mathrm{an}}=20(n+1)$ and $0\le\epsilon\le\epsilon_0$. The closeness is measured relative to the fixed domains and their margins. For the phases of our construction these hypotheses are parts (i)–(iii) of Proposition 6.5; the closeness uses only the low-order conditions (L) of Setup 6.3, and the $C^M$ bounds use the high-order bound (H).

The amplitudes $a_\tau(x,z,\eta)$ have fixed compact physical and parameter supports, with $(s,|\eta_b|\widehat{\eta})$ in a fixed compact subset of $D_0$. For $|\eta|\ge 1/2$ and $\tau|\eta|$ sufficiently small we assume

$$|D^\alpha_{x,z}D^\beta_\eta a_\tau|\le A|\eta|^{-|\beta|},\qquad |\alpha|+|\beta|\le M.$$

Finally, $p_\Lambda$ is a real multiplier whose square is smooth, supported in $\{1/2\le|\eta|\le C\Lambda\}$, with $|D^\beta_\eta(p_\Lambda^2)|\le C_\beta|\eta|^{-|\beta|}$; its lower cutoff does not depend on $\Lambda$.

**Proposition 8.4** (Normal operators and fixed-parameter estimates)**.** *Assume Setup 8.3. The averaged full and transverse gradients have inverse charts on common smaller domains, with fixed joint $C^M$ bounds. For $\Lambda\ge 1$ and $\tau\Lambda$ sufficiently small define*

$$T_\tau f(x)=(2\pi)^{-n/2}\int e^{\mathrm{i}\Phi_\tau(x,z,\eta)}|\eta|^{1/2}p_\Lambda(\eta)a_\tau(x,z,\eta)f(z,\eta)\,\mathrm{d}z\,\mathrm{d}\eta,$$

*and let $T_{\tau,z}$ denote the operator obtained by fixing $z$, acting on functions of $\eta$. On the entire complex input and output spaces,*

$$\|T_\tau\|\le CA,\qquad \|T_{\tau,z}\|^2\le CA^2\Lambda,\qquad \|T_{\tau,z}\|_{\mathrm{HS}}^2\le CA^2\Lambda^n. \tag{8.2}$$

*Without the factor $|\eta|^{1/2}$ the norm of $T_{\tau,z}$ is at most $CA$. The first bound is also uniform when $v$ is fixed, and at $\tau=0$ it holds without an upper frequency cutoff. On an unchanged phase, an amplitude difference of size $d$ (in the norms above) changes $T_\tau$ by at most $Cd$ in norm and $T_{\tau,z}$ by at most $C\sqrt{\Lambda}d$.*

*The normal operator $T_\tau T_\tau^*$ has an exact kernel of the form (8.7), with an amplitude $b_\tau$ of order zero, and it depends continuously on the data in operator norm: if two sets of data (phases and amplitudes) satisfy the hypotheses with the same constants and differ by at most $d$ in the joint $C^{2\ell+2}$ norms of the normalized phases $\Psi^{\mathrm{ann}}_\epsilon$ and of the normalized amplitudes $a_\tau(x,z,|\eta|\omega)$, $\omega$ in the unit sphere, with $\ell=\lfloor n/2\rfloor+1$, then the two normal operators differ by at most $Cd$ in norm, where $C$ depends only on the constants in the hypotheses. No such continuity is asserted for $T_\tau$ itself when the phase changes: a phase difference of size $d$ is multiplied by the unbounded covector $\eta$ in $e^{\mathrm{i}\Phi_\tau}$, and $T_\tau$ is in general not continuous in operator norm with respect to the phase. Only the normal operators and the uniform bounds are used below.*

*Assume in addition that $|D^\alpha_{x,z}D^\beta_\eta\partial_\tau a_\tau|\le A|\eta|^{1-|\beta|}$ for $|\alpha|+|\beta|\le M-1$. Then for every fixed smooth real weight $m(s,v)$,*

$$\|T_\tau m T_\tau^*-T_0 m T_0^*\|\le CA^2\tau\Lambda, \tag{8.3}$$

*where $C$ depends on finitely many derivatives of $m$.*

*Proof. Step 1 (inverse charts).* Lemma 7.4 gives inverse charts of $(s,\theta)\mapsto\Psi^{\mathrm{ann}}_\epsilon$ on common domains. When $t'=t$, the $y$-component of $\Psi^{\mathrm{ann}}_\epsilon$ is the averaged transverse gradient, which is uniformly $C^1$ close to $\theta$; the same lemma gives its inverse charts at fixed time. Implicit differentiation (Lemma 7.5) gives the joint $C^M$ bounds of the inverses. We choose the closeness so that

$$|\Psi^{\mathrm{ann}}_{\epsilon,y}-\theta|\le\delta|\theta|, \tag{8.4}$$

with $\delta$ small compared with the margins of the data boxes.

*Step 2 (scaling).* Let $\Psi_\tau$ be the averaged $x$-gradient of $\Phi_\tau$. Substituting $\eta = \rho\theta$ in the integral formula for $\Phi_\tau$ gives $\Phi_\tau(x, z, \rho\theta) = \rho\, \Phi_{\tau\rho}(x, z, \theta)$, hence

$$\Psi_\tau(x, x', s, v, \rho\theta) = \rho\, \Psi^{\mathrm{ann}}_{\tau\rho}(x, x', s, v, \theta). \tag{8.5}$$

Suppose $\Psi_\tau(s, \eta) = \Psi_\tau(s', \eta')$ at fixed $(x, x', v)$. By (8.4) and (8.5) the $y$-components satisfy $|\Psi_{\tau,y} - \eta| \le \delta|\eta|$, so $|\eta - \eta'| \le \delta(|\eta| + |\eta'|)$, and $|\eta|$, $|\eta'|$ are comparable. Hence one common $\rho > 0$ puts both $\eta/\rho$ and $\eta'/\rho$ in the annular chart, with the same parameter $\tau\rho$, and injectivity of the annular map gives $(s, \eta) = (s', \eta')$. Thus $(s, \eta) \mapsto \Psi_\tau$ is injective on the whole cone; the same argument applies at fixed time. The support margins and a small threshold for $\tau\Lambda$ keep $(s, \eta/\rho)$ in $D_0$ for the choice $\rho = |k_y|/|\eta_b|$ made below, which depends only on the output point $k$.

For fixed $\rho$, differentiating (8.5) in $(s, \theta)$ shows that $\det D_{(s,\eta)}\Psi_\tau = \rho \det D_{(s,\theta)}\Psi^{\mathrm{ann}}_{\tau\rho}$, and the second determinant is bounded above and below. Since $\rho \asymp |\eta|$,

$$J_\tau = |\det D_{(s,\eta)}\Psi_\tau| \asymp |\eta|, \qquad |\det D_\eta \Psi_{\tau,y}| \asymp 1. \tag{8.6}$$

By the same scaling, each derivative of the inverse map in the covector variable lowers its order of homogeneity by one, while physical derivatives preserve it, uniformly on the cone.

*Step 3 (the kernel of the normal operator).* The kernel of $T_\tau T_\tau^*$ is

$$(2\pi)^{-n} \int e^{\mathrm{i}(\Phi_\tau(x,z,\eta) - \Phi_\tau(x',z,\eta))} |\eta| p_\Lambda^2(\eta) a_\tau(x, z, \eta) a_\tau(x', z, \eta)^*\, \mathrm{d}z\, \mathrm{d}\eta.$$

By the fundamental theorem of calculus along the segment from $x'$ to $x$, the phase equals $(x - x') \cdot \Psi_\tau(x, x', z, \eta)$ exactly. Changing variables $(s, \eta) \mapsto k = \Psi_\tau(x, x', s, v, \eta)$, which is legitimate by Step 2, we obtain the kernel

$$(2\pi)^{-n} \int e^{\mathrm{i}(x-x')\cdot k} b_\tau(x, x', k)\, \mathrm{d}k, \qquad b_\tau = \int \frac{|\eta|\, p_\Lambda^2\, a_\tau(x, z, \eta) a_\tau(x', z, \eta)^*}{J_\tau}\bigg|_{(s,\eta) = \Psi_\tau^{-1}(k)} \mathrm{d}v. \tag{8.7}$$

The factor $|\eta|$ cancels the Jacobian, so $b_\tau$ is bounded by $CA^2$. Its $(x, x')$-derivatives are bounded by $CA^2$ as well: a physical derivative falling on $a_\tau$ costs $A$, one falling on the inverse point $(s, \eta)$ produces a derivative of $\eta$ of order $|\eta|$, which is compensated by the decay $|\eta|^{-1}$ of the $\eta$-derivative of $a_\tau$, $p_\Lambda^2$ or $|\eta|/J_\tau$. The amplitude vanishes near the boundary of the inverse charts, so its extension by zero is smooth.

*Step 4 (the full bound).* Part (ii) of Lemma 8.2, with $d = n$ and $2\ell \le M$, gives $\|T_\tau T_\tau^*\| \le CA^2$, that is, $\|T_\tau\| \le CA$. At $\tau = 0$ the amplitude bounds do not use the upper cutoff, and the estimate without it follows by inserting a frequency cutoff and letting it tend to one.

*Step 4′ (continuity of the normal operator).* Let primed quantities refer to a second set of data. Both normal operators have kernels of the form (8.7) with the same exponential $e^{\mathrm{i}(x-x')\cdot k}$, so by Lemma 8.2(ii) it suffices to show $\sup_k \|b_\tau - b'_\tau\|_{C^{2\ell}_{x,x'}} \le Cd$. By (8.5), the inverse point $(s, \eta)$ of $k$ is $(s, \rho\theta)$, where $(s, \theta)$ is the inverse of the normalized map $\Psi^{\mathrm{ann}}_{\tau\rho}$ at $k/\rho$ and $\rho = |k_y|/|\eta_b|$ depends only on $k$. Implicit differentiation, with the common lower bound for the Jacobian, shows that the normalized inverses of the two data sets differ by $O(d)$ in $C^{2\ell+1}$. Hence $\eta - \eta' = O(d|\eta|)$ at the two inverse points, together

with $2\ell$ physical derivatives. Each factor of the integrand of $b_\tau$, namely $a_\tau$, $p_\Lambda^2$ and $|\eta|/J_\tau$, has $\eta$-derivative $O(|\eta|^{-1})$, so it changes by $O(d)$ when $\eta$ is replaced by $\eta'$, and by $O(d)$ when the data are replaced at a fixed point. This proves the continuity assertion. The argument concerns the kernel of $T_\tau T_\tau^*$ after the change of variables $k = \Psi_\tau$; no comparison of the operators $T_\tau$ with different phases is involved.

*Step 5 (fixed parameters).* Fix $z$ and $t$. The operator $f \mapsto T_{\tau,z} f(t,\cdot)$ acts from $L^2(d\eta)$ to $L^2(dy)$. Its normal operator has the kernel $(2\pi)^{-n}\int e^{\mathrm{i}(y-y')\cdot\Psi_{\tau,y}}|\eta|p_\Lambda^2 a_\tau a_\tau^*\,\mathrm{d}\eta$, and the change of variables $\eta \mapsto \kappa = \Psi_{\tau,y}$, whose Jacobian is comparable to one by (8.6), brings it to the form of Lemma 8.2(ii) in dimension $n-1$, with amplitude bounded by $CA^2|\eta| \le CA^2\Lambda$. Hence $\|T_{\tau,z} f(t,\cdot)\|_{L^2_y}^2 \le CA^2\Lambda\|f\|_2^2$, and integration over the fixed $t$-interval gives $\|T_{\tau,z}\|^2 \le CA^2\Lambda$. Without the factor $|\eta|^{1/2}$ the same argument gives $CA^2$. The Hilbert–Schmidt norm is the $L^2$ norm of the kernel:

$$\|T_{\tau,z}\|_{\mathrm{HS}}^2 \le CA^2 \int_{\{|\eta|\le C\Lambda\}} |\eta|\,\mathrm{d}\eta \le CA^2\Lambda^n,$$

since $\eta \in \mathbb{R}^{n-1}$. Amplitude differences are estimated by the same argument applied to the difference divided by $d > 0$.

*Step 6 (comparison).* Insert the weight $m(s,v)$ in the numerator of (8.7), and call the resulting amplitude $b_\tau^m$. At fixed $k$ the inverse point is obtained from the annular inverse at parameter $\tau\rho$, with $\rho \asymp |\eta|$ independent of $\tau$. Hence $\partial_\tau s = O(|\eta|)$ and $\partial_\tau \eta = O(|\eta|^2)$, together with their physical derivatives. Consequently

$$\Big|\partial_\tau\Big[a_\tau(x,z,\eta(\tau))\Big]\Big| \le |\partial_\tau a_\tau| + |\partial_s a_\tau||\partial_\tau s| + |\partial_\eta a_\tau||\partial_\tau \eta| \le CA|\eta|,$$

and the same bound holds for $p_\Lambda^2$, the weight $m$ and $|\eta|/J_\tau$, each of which loses one power of $|\eta|$ under an $\eta$-derivative. On the support, $|k| \asymp |\eta| \le C\Lambda$. The mean value theorem in $\tau$ therefore gives

$$\sup_k |D_{x,x'}^\alpha (b_\tau^m - b_0^m)| \le CA^2\tau\Lambda, \qquad |\alpha| \le 2(n+1),$$

and Lemma 8.2(ii), applied to the difference of amplitudes, proves (8.3). □

**Corollary 8.5** (Real and imaginary parts of the input)**.** *Regard the complex input space as a real Hilbert space with inner product* $\mathrm{Re}\langle\cdot,\cdot\rangle$*, and put* $C = 2\,\mathrm{Re}\,T$*. For real outputs* $r$*,*

$$C^* r = 2T^* r, \qquad CmC^* r = 4\,\mathrm{Re}(TmT^* r). \tag{8.8}$$

*Thus the preceding bounds hold for the real operators, with fixed numerical factors, when both the real and the imaginary parts of the input are used.*

*Proof.* For real $r$ and complex $f$, $\langle 2\,\mathrm{Re}\,Tf, r\rangle = 2\,\mathrm{Re}\langle Tf, r\rangle = \mathrm{Re}\langle f, 2T^* r\rangle$, which is the first identity. Applying $C = 2\,\mathrm{Re}\,T$ to $2mT^* r$ gives the second. □

## 8.2. **Fourier localization of oscillatory functions.**

**Proposition 8.6** (Fourier tails of oscillatory functions)**.** *Fix* $n \ge 1$*, a bounded box* $X \subset \mathbb{R}^n$*, a compact set* $K \Subset X$ *and* $0 < c < C$*. For* $\rho \ge 2$ *and* $L \ge 1$ *let* $\phi$ *be a smooth*

*real phase and $a$ a smooth complex amplitude of fixed finite dimension, supported in $K$, with*

$$c \le |d_x\phi| \le C, \qquad |D_x^r\phi| \le C_r L^{r-1}\ (r \ge 2), \qquad |D_x^r a| \le C_r A L^r\ (r \ge 0),$$

*where $A \ge 0$. Put $h_\rho = e^{\mathrm{i}\rho\phi}a$. Let $\Pi_\rho = \pi(D/\rho)$, where $\pi$ is a fixed smooth radial function equal to 1 on $\{c/2 \le |\xi| \le 2C\}$ and supported in $\{c/4 \le |\xi| \le 4C\}$. Then for all integers $m \ge 0$ and $q > m + n/2$,*

$$\|(I - \Pi_\rho)h_\rho\|_{H^m} \le C_{m,q} A \rho^{m+n/2}(L/\rho)^q. \tag{8.9}$$

*More generally, the same estimate holds for any fixed bounded rescaled Fourier cutoff, provided that on the support of $1 - \Pi_\rho$ one has $|\rho d_x\phi(x) - \xi| \ge c_1(\rho + |\xi|)$ for all $x \in K$.*

*Proof.* The Fourier transform is

$$\widehat{h}_\rho(\xi) = (2\pi)^{-n} \int_K e^{\mathrm{i}[\rho\phi(x) - x\cdot\xi]} a(x)\,\mathrm{d}x.$$

On the support of $1 - \pi(\xi/\rho)$ either $|\xi| < c\rho/2$ or $|\xi| > 2C\rho$. In the first case $|\rho d\phi - \xi| \ge c\rho/2$ and $\rho + |\xi| \le (1 + c/2)\rho$. In the second case $|\rho d\phi - \xi| \ge |\xi| - C\rho \ge |\xi|/2$, and $|\xi|/2 > C\rho \ge c\rho$. In both cases, with $c_1 = \min\{c/(2+c), 1/4, c/2\}$,

$$|\rho d_x\phi(x) - \xi| \ge c_1(\rho + |\xi|) \qquad (x \in K).$$

Let $w = \rho d_x\phi - \xi$ and

$$\mathcal{V}_\xi = \frac{w \cdot \partial_x}{\mathrm{i}|w|^2}, \qquad \mathcal{V}_\xi e^{\mathrm{i}[\rho\phi - x\cdot\xi]} = e^{\mathrm{i}[\rho\phi - x\cdot\xi]}.$$

Since $a$ vanishes near $\partial K$, integration by parts $q$ times gives

$$\widehat{h}_\rho(\xi) = (2\pi)^{-n} \int e^{\mathrm{i}[\rho\phi - x\cdot\xi]} (\mathcal{V}_\xi^T)^q a\,\mathrm{d}x, \qquad \mathcal{V}_\xi^T f = -\partial_x \cdot \Big(\frac{w}{\mathrm{i}|w|^2} f\Big).$$

The coefficient $w/|w|^2$ is bounded by $C/(\rho + |\xi|)$. Its $j$th derivative is a sum of terms involving derivatives of $w$, that is $\rho D^{k+1}\phi$, divided by powers of $|w|$; since $\rho|D^{k+1}\phi| \le C\rho L^k \le CL^k(\rho + |\xi|)$, an induction on $j$ shows that the $j$th derivative is bounded by $C_j L^j(\rho + |\xi|)^{-1}$. Each derivative of $a$ costs a factor $L$. By induction on $q$, $(\mathcal{V}_\xi^T)^q a$ is a sum of terms bounded by $C_q A L^q(\rho + |\xi|)^{-q}$, so

$$\big|(1 - \pi(\xi/\rho))\widehat{h}_\rho(\xi)\big| \le C_q A L^q(\rho + |\xi|)^{-q}.$$

For $q > m + n/2$ the substitution $\xi = \rho\omega$ gives

$$\int_{\mathbb{R}^n} \langle\xi\rangle^{2m}(\rho + |\xi|)^{-2q}\,\mathrm{d}\xi \le C\rho^{2m+n-2q} \int_{\mathbb{R}^n} (1 + |\omega|)^{2m-2q}\,\mathrm{d}\omega \le C_{m,q}\rho^{2m+n-2q}.$$

By Plancherel, $\|(I - \Pi_\rho)h_\rho\|_{H^m}^2 \le C_{m,q} A^2 L^{2q} \rho^{2m+n-2q}$, which is (8.9). For a general mask the proof used only the lower bound for $|\rho d\phi - \xi|$ on its complement. □

*Remark* 8.7. Proposition 8.6 is applied with $\rho = \lambda|z - z'|$ to normalized phase differences $\lambda[\Theta(x,z) - \Theta(x,z')]/\rho$ of two waves in the same characteristic patch. The phase difference is used exactly, not expanded in the patch parameter; the hypotheses of the proposition are verified for it in Proposition 6.4.

## 9. A parametrix on regular patches

In this section we construct a parametrix for a real first-order $r \times r$ system on a conic region of covectors where the system is elliptic, or has one simple real characteristic root. The construction is used in Section 10 to build finite corrections, which in turn correct the characteristic waves (Section 11) and the oscillatory quadratic terms (Section 15).

### 9.1. The system and the main result.

Consider the real system

$$P_a = \sum_{j=1}^{n} A_j(x)\partial_{x_j} + B(x), \qquad H_a(x,\xi) = \sum_{j=1}^{n} A_j(x)\xi_j, \tag{9.1}$$

whose coefficient array $a = (A_1, \dots, A_n, B)$ of real $r \times r$ matrices is smooth near a point $x_0 \in \mathbb{R}^n$. Put

$$p_a = \det H_a. \tag{9.2}$$

A real zero $\xi \neq 0$ of $p_a(x,\cdot)$ is a simple characteristic if $\partial_\xi p_a(x,\xi) \neq 0$. On a characteristic patch we use one retarded oscillatory integral with a scalar Hamilton phase and two matrix amplitudes, together with an algebraic inverse on the complementary directions; on an elliptic patch we invert the principal symbol.

**Theorem 9.1** (A regular patch)**.** *Fix a smooth reference coefficient array $a_0$ for (9.1) and nested closed conic sets $\Gamma_{\mathrm{in}} \Subset \operatorname{int}\Gamma_{\mathrm{out}}$. Assume that on a fixed spatial neighborhood of $x_0$ one of the following holds:*

(i) *$H_{a_0}(x,\xi)$ is invertible for $\xi \in \Gamma_{\mathrm{out}} \setminus 0$;*

(ii) *in a coordinate system $x = (t,y)$, $\xi = (\tau,\eta)$, the equation $p_{a_0} = 0$ has on $\Gamma_{\mathrm{out}}$ exactly one root $\tau = h_0(x,\eta)$, which is simple, with $|\eta| \asymp |\xi|$ and $|\partial_\tau p_{a_0}|$ bounded below on unit covectors.*

*No condition is imposed outside $\Gamma_{\mathrm{out}}$. Then, after one spatial shrinking, there are fixed balls $U \Subset W$, a neighborhood of $a_0$ in a fixed $C^k$ norm, and a scalar pseudodifferential cutoff $A_\Gamma$, equal to the identity microlocally on $\Gamma_{\mathrm{in}}$ at high frequency and with conic support in $\Gamma_{\mathrm{out}}$, such that the following holds for every smooth real $a$ in that neighborhood. For $h \in C_c^\infty(W,\mathbb{C}^r)$ there are complex linear operators $S_{a,\Gamma}$, $R_{a,\Gamma}$ with smooth outputs, satisfying*

$$P_a S_{a,\Gamma} h = A_\Gamma h + R_{a,\Gamma} h \quad \text{on } U, \tag{9.3}$$

*and*

$$\|S_{a,\Gamma} h\|_{L^2(W)} + \|R_{a,\Gamma} h\|_{H^1(U)} \le C \|h\|_{L^2(W)}. \tag{9.4}$$

*For every integer $s \ge 0$ the operators $S_{a,\Gamma} : H^s(W) \to H^s(W)$ and $R_{a,\Gamma} : H^{\max(s-1,0)}(W) \to H^s(U)$ have norms bounded by polynomials in finitely many coefficient seminorms. The residual $R_{a,\Gamma}$ is a finite sum of pseudodifferential operators of order $-1$ and of operators containing one retarded oscillatory integral with an amplitude of order $-1$; it includes all source and low-frequency terms. The domains, the coefficient neighborhood, the derivative losses and the orders of the operators are fixed before $s$ and before any later accuracy. All statements include fixed input and output cutoffs.*

The proof is given at the end of the section, after the necessary estimates.

### 9.2. **Pseudodifferential estimates.**

**Setup 9.2** (Symbols on fixed boxes)**.** All evaluations of coefficients and all Hamilton trajectories lie in fixed relatively compact subdomains of $W$. Local symbols and amplitudes are multiplied by their cutoffs before being extended by zero, so their seminorms are controlled by the coefficient norms on $W$. Every homogeneous symbol is multiplied by one fixed cutoff vanishing near $\xi = 0$, and the complementary low-frequency part is kept in the formulas. We write

$$T_b f(x) = (2\pi)^{-n} \iint e^{\mathrm{i}(x-z)\cdot\xi} b(x,z,\xi) f(z)\,\mathrm{d}z\,\mathrm{d}\xi,$$

and $\mathrm{Op}(b) = T_b$ when $b$ does not depend on $z$. For an amplitude of order $p$ we use the seminorms

$$|b|_{p,j} = \max_{|\alpha|+|\beta|+|\gamma|\le j} \sup \langle\xi\rangle^{-p+|\gamma|} |\partial_x^\alpha \partial_z^\beta \partial_\xi^\gamma b|.$$

**Lemma 9.3** (Pseudodifferential bounds)**.** *An amplitude of order* $0$ *defines a bounded operator on* $L^2$*, and an amplitude of order* $-1$ *defines an operator which, together with its adjoint, maps* $L^2$ *to* $H^1$*. For each integer* $s \ge 0$ *these operators map* $H^{\max(s+p,0)}$ *to* $H^s$*, where* $p \in \{0,-1\}$ *is the order. The bounds involve finitely many symbol seminorms, a fixed number of them beyond* $s$*, and depend polynomially on finitely many coefficient seminorms.*

*Proof.* Lemma 8.2(ii) gives $\|T_b\|_{2\to 2} \le C|b|_{0,2(n+1)}$ (only $x$- and $z$-derivatives are used). For order $-1$, an output derivative gives $D_{x_j} T_b = T_{\xi_j b + D_{x_j} b}$, which has order zero, so $T_b : L^2 \to H^1$; the adjoint of $T_b$ is $T_{b^\dagger}$ with $b^\dagger(x,z,\xi) = b(z,x,\xi)^*$, an amplitude of the same order. Integration by parts in $z$ gives the exact identity

$$D_x^\alpha T_b f = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} T_{D_{x+z}^\beta b} D^{\alpha-\beta} f, \qquad D_{x+z} = D_x + D_z,$$

from which the $H^s$ bounds follow by the $L^2$ estimate; for order $-1$ we first take one output derivative and then argue at order $s-1$. The symbols are fixed algebraic expressions in the coefficients, which gives the polynomial dependence. □

### 9.3. **Retarded oscillatory integrals.**

**Lemma 9.4** (Bounds for retarded kernels)**.** *Let* $d = n-1$ *and let* $\Psi(t,s,y,\eta)$ *be homogeneous of degree one in* $\eta$*, with* $\Psi(s,s,y,\eta) = y\cdot\eta$ *and* $\Psi_{y\eta}$ *uniformly close to* $I$ *on fixed boxes with interior margins. For a spatial amplitude* $b(t,s,y,\eta)$ *of order* $p \in \{0,-1\}$ *put*

$$U_b(t,s) f(y) = (2\pi)^{-d} \iint e^{\mathrm{i}(\Psi(t,s,y,\eta) - z\cdot\eta)} b(t,s,y,\eta) f(z)\,\mathrm{d}z\,\mathrm{d}\eta,$$

*and*

$$\mathcal{E}_b f(t) = \int_{t_-}^{t} U_b(t,s) f(s)\,\mathrm{d}s.$$

*Assume that $b$ has fixed physical and conic margins, contains the fixed low-frequency cutoff, and vanishes for $s$ near the initial time $t_-$. Let $B = \beta(D_t, D_y)$ be a fixed Fourier multiplier of order zero supported in $\{|\tau| \le C|\eta|\}$. Then $\mathcal{E}_b B : H^{\max(s+p,0)} \to H^s$ for every integer $s \ge 0$, with bounds given by finite products of seminorms of the phase and the amplitude.*

*Proof. Fixed times.* For fixed $t, s$, the operator $U_b(t,s)U_b(t,s)^*$ has the kernel

$$(2\pi)^{-d} \int e^{\mathrm{i}(\Psi(t,s,y,\eta)-\Psi(t,s,y',\eta))} b(t,s,y,\eta) b(t,s,y',\eta)^* \,\mathrm{d}\eta.$$

The phase difference equals $(y-y')\cdot\widetilde{\Psi}$, where $\widetilde{\Psi} = \int_0^1 \Psi_y(t,s,y'+\theta(y-y'),\eta)\,\mathrm{d}\theta$ is homogeneous of degree one in $\eta$ and, since $\Psi_{y\eta}$ is close to $I$, defines a homogeneous change of variables $\eta \mapsto \widetilde{\Psi}$ with Jacobian close to one. After this change of variables the kernel has the form of Lemma 8.2(ii) with an amplitude of order zero, so $\|U_b(t,s)\|_{L^2\to L^2} \le C$ uniformly.

*A fixed-time representation.* Put $x = (t,y)$, $\xi = (\tau,\eta)$ and $r = t-s$. Since $b$ vanishes near $s = t_-$, we may extend it by zero to $s < t_-$, and on the fixed output support we obtain

$$\begin{aligned} \mathcal{E}_b B &= \int_0^{\Delta} \mathcal{U}_{r,b}\,\mathrm{d}r, \qquad \Theta_r(t,y,\tau,\eta) = (t-r)\tau + \Psi(t,t-r,y,\eta), \\ \mathcal{U}_{r,b} f(x) &= (2\pi)^{-n} \iint e^{\mathrm{i}[\Theta_r(x,\xi)-z\cdot\xi]} b(t,t-r,y,\eta)\beta(\xi) f(z)\,\mathrm{d}z\,\mathrm{d}\xi, \end{aligned} \tag{9.5}$$

with a fixed $\Delta > 0$. Indeed, $\mathcal{U}_{r,b} f(t,\cdot)$ is $U_b(t,t-r)$ applied to $(Bf)(t-r,\cdot)$, whose time Fourier variable produces the factor $e^{\mathrm{i}(t-r)\tau}$. By time translation and the fixed-time bound, each $\mathcal{U}_{r,b}$ of order zero is bounded on $L^2$ uniformly in $r$. For amplitudes depending on both positions we use the last assertion of Lemma 8.2, with outer operator $\mathcal{U}_{r,*}\mathcal{F}^{-1}$ and fixed larger spatial and Fourier cutoffs equal to one on the support of the amplitude.

*Derivatives.* On the support of $\beta$ we have $|\tau| \le C|\eta|$, so the amplitude has full spacetime order $p$. An output derivative of the phase factor produces an order-zero coefficient times $\xi$, and integration by parts in $z$ transfers this factor to a derivative of the input. For $p = -1$ one output derivative first makes the amplitude of order zero. The full-amplitude bound then gives the $H^s$ estimates uniformly in $r$, and integration over $[0,\Delta]$ completes the proof. □

## 9.4. **A characteristic patch.**

**Lemma 9.5** (A characteristic parametrix)**.** *On a regular characteristic patch write $x = (t,y)$, $\xi = (\tau,\eta)$, with $|\eta| \asymp |\xi|$ and $|\partial_\tau p_a|$ bounded below. Choose a fixed Fourier cutoff $\beta(\tau,\eta)$ supported in this cone, including the low-frequency cutoff, and a spatial source cutoff $\sigma(x,\eta)$ whose forward Hamilton flowout over the time interval stays in the patch. Put*

$$B = \beta(D_t, D_y), \qquad \alpha = \sigma\beta, \qquad A = \mathrm{Op}(\alpha).$$

*There is a fixed operator*

$$S = \chi_{\mathrm{out}}\Big(\mathcal{E}_W B + \frac{1}{\mathrm{i}}\operatorname{Op}(b_{-1}\alpha)\Big),$$

*where* $W = W_0 + W_1$ *is an amplitude with terms of orders* $0$ *and* $-1$ *and* $b_{-1}$ *is a symbol of order* $-1$*, such that* $P_a S = A + R$ *on the inner physical box, with*

$$R = \operatorname{Op}(A_t W_1(t,t)\beta) + \mathcal{E}_{P_a W_1} B + \frac{1}{\mathrm{i}}\operatorname{Op}\Big(P_x(b_{-1}\alpha)\Big). \tag{9.6}$$

*Here* $A_t$ *is the coefficient of* $\partial_t$*,* $W_1(t,t)$ *denotes the amplitude at* $s = t$*, and* $P_x$ *applies* $P_a$ *to the physical variable of a symbol. The construction uses a Hamilton phase and scalar transport. If the coefficients satisfy* $\|a\|_{C^q} \le C_q T^{(q-R_*)_+}$*, then the phase gradients, flow maps and amplitudes satisfy the same normalized mixed derivative bounds with* $R_*$ *replaced by* $R_* - c$*, for a fixed loss* $c$*.*

*Proof.* Write $P = P_a$, $H = H_a$, and let $\tau = h(x,\eta)$ be the simple root.

*Null vectors.* Since the root is simple, $\partial_\tau p = \operatorname{tr}(\operatorname{adj} H \cdot A_t) \neq 0$ at $\tau = h$, so $\operatorname{adj} H \neq 0$ and $H(x,h,\eta)$ has corank one. Using a fixed nonvanishing minor, choose smooth real vectors $k(x,\eta)$, $l(x,\eta)$, homogeneous of degree zero, with

$$H(x,h,\eta)k = 0, \qquad l^T H(x,h,\eta) = 0, \qquad l^T A_t k = 1.$$

The last normalization is possible because $\operatorname{adj} H(x,h,\eta) = c\, k l^T$ with $c \neq 0$, so $\partial_\tau p = c\, l^T A_t k \neq 0$. Since $H$ is linear in $\tau$,

$$l^T H(x,\tau,\eta) = (\tau - h)\, l^T A_t. \tag{9.7}$$

*The algebraic inverse.* For $\tau$ near $h$, $H(x,\tau,\eta)$ maps the hyperplane $\ker(l^T A_t)$ into $\ker l^T$, by (9.7). At $\tau = h$ this map is an isomorphism: its kernel is $\mathbb{R}k \cap \ker(l^T A_t) = 0$, and both spaces have dimension $r - 1$. Hence it remains an isomorphism on a fixed conic neighborhood of the root, and we choose $\beta$ supported in that neighborhood. Put

$$\Pi = A_t k l^T, \qquad b_{-1}(x,\tau,\eta) = \Big(H(x,\tau,\eta)|_{\ker(l^T A_t)\to\ker l^T}\Big)^{-1}(I - \Pi). \tag{9.8}$$

Since $\Pi^2 = \Pi$ and $l^T(I - \Pi) = 0$, the matrix $I - \Pi$ takes values in $\ker l^T$, and by construction

$$H(x,\tau,\eta)\, b_{-1}(x,\tau,\eta) = I - \Pi. \tag{9.9}$$

The restricted matrix is homogeneous of degree one, so $b_{-1}$ is homogeneous of degree $-1$. Its product with $\alpha$ is extended by zero outside the fixed conic neighborhood and is a spacetime symbol of order $-1$.

*The phase.* On a common short time interval, Lemma 6.2, with initial time $s$, gives

$$\Psi_t = h(t,y,\Psi_y), \qquad \Psi(s,s,y,\eta) = y\cdot\eta, \qquad \Psi_{y\eta} \text{ uniformly close to } I.$$

Below, $k$ and $l$ are evaluated at $(t,y,\Psi_y(t,s,y,\eta))$. We choose $\sigma$ equal to one on the inner source region, with its complete flowout inside the larger chart, and vanishing near the initial time. Since $|\Psi_y| \asymp |\eta|$, nested low-frequency cutoffs leave every homogeneous formula unchanged near the transported support of $\sigma$.

*Transport.* Let the row vector $a(t,s,y,\eta)$ solve

$$(\partial_t - h_\eta \cdot \partial_y)a + l^T(Pk)\,a = 0, \qquad a(s,s,y,\eta) = l(s,y,\eta)^T\sigma(s,y,\eta),$$

where $h_\eta$ is evaluated at $(t,y,\Psi_y)$ and $Pk$ denotes $P$ applied to the function $(t,y)\mapsto k(t,y,\Psi_y)$. Put $W_0 = ka$. Differentiating $H(x,h(x,\eta),\eta)k(x,\eta) = 0$ in $\eta_j$ and multiplying by $l^T$ gives $h_{\eta_j} + l^T A_{y_j} k = 0$, where $A_{y_j}$ is the coefficient of $\partial_{y_j}$. Therefore

$$l^T P W_0 = l^T A_t k\,\partial_t a + \sum_j l^T A_{y_j} k\,\partial_{y_j} a + l^T(Pk)a = (\partial_t - h_\eta\cdot\partial_y)a + l^T(Pk)a = 0.$$

Moreover, since $\Psi_t = h(t,y,\Psi_y)$,

$$P(e^{\mathrm{i}\Psi}W_0) = e^{\mathrm{i}\Psi}\big(\mathrm{i}H(x,\Psi_t,\Psi_y)W_0 + PW_0\big) = e^{\mathrm{i}\Psi}PW_0.$$

Put $W_1 = \mathrm{i}\,b_{-1}(t,y,\Psi_t,\Psi_y)PW_0$. By (9.9) and $l^TPW_0 = 0$ (which gives $\Pi PW_0 = 0$), we get $HW_1 = \mathrm{i}(I-\Pi)PW_0 = \mathrm{i}PW_0$, and hence

$$P(e^{\mathrm{i}\Psi}W_1) = e^{\mathrm{i}\Psi}\big(\mathrm{i}HW_1 + PW_1\big) = e^{\mathrm{i}\Psi}(-PW_0 + PW_1).$$

Adding, we obtain the exact cancellation

$$P\{e^{\mathrm{i}\Psi}(W_0+W_1)\} = e^{\mathrm{i}\Psi}PW_1. \tag{9.10}$$

No pseudodifferential operator acts on the phase. The transported support stays in the patch, and all symbols are extended by zero only outside a fixed larger margin.

*The residual.* For $W = W_0 + W_1$, differentiation of the retarded integral gives

$$P\mathcal{E}_W g(t) = A_t U_W(t,t)g(t) + \int_{t_-}^{t} PU_W(t,s)g(s)\,\mathrm{d}s.$$

By (9.10), $PU_W(t,s) = U_{PW_1}(t,s)$, and since $\Psi(t,t,y,\eta) = y\cdot\eta$, the operator $U_W(t,t)$ is the spatial pseudodifferential operator $\mathrm{Op}_y(W(t,t))$. Because $B$ is a Fourier multiplier and $W(t,t)$ does not depend on $\tau$, $\mathrm{Op}_y(A_tW(t,t))B = \mathrm{Op}(A_tW(t,t)\beta)$ exactly. Moreover $A_tW_0(t,t) = A_tk\,l^T\sigma = \Pi\sigma$, so

$$P\mathcal{E}_W B = \mathrm{Op}(\Pi\alpha) + \mathrm{Op}(A_tW_1(t,t)\beta) + \mathcal{E}_{PW_1}B.$$

For the algebraic term, $P\,\mathrm{Op}(c) = \mathrm{Op}(\mathrm{i}Hc + P_xc)$, so by (9.9)

$$\frac{1}{\mathrm{i}}P\,\mathrm{Op}(b_{-1}\alpha) = \mathrm{Op}((I-\Pi)\alpha) + \frac{1}{\mathrm{i}}\,\mathrm{Op}\big(P_x(b_{-1}\alpha)\big).$$

Adding these identities gives $PS = A + R$ with $R$ as in (9.6), on the set where $\chi_{\mathrm{out}} = 1$; there the commutator of $P$ with $\chi_{\mathrm{out}}$ vanishes. Both pseudodifferential terms in $R$ have spacetime order $-1$, because $\beta$ restricts to $|\tau| \le C|\eta|$, and the retarded term gains one derivative by Lemma 9.4. These are complete formulas, including low frequencies, and a common input cutoff can be placed on the right of all terms.

*Derivative bounds.* Every division has a fixed lower bound. Lemma 7.5, applied to the Hamilton flow, its inverse charts and the scalar transport equation, gives the stated derivative bounds, and they pass to $W_0$, $W_1$, $PW_1$ and $b_{-1}$ through the explicit formulas and the fixed cutoffs. One fixed loss in $R_*$ covers all these factors, independently of later compositions. □

9.5. **Elliptic patches and the proof of the main result.** On an elliptic patch choose a scalar source symbol $\alpha$, equal to one on the inner region and containing the fixed low-frequency cutoff, and put $S = \chi_{\mathrm{out}}\,\mathrm{Op}(H_a^{-1}\alpha)/\mathrm{i}$. As above, $PS = \mathrm{Op}(\alpha) + R$ where $\chi_{\mathrm{out}} = 1$, with the residual $R = \mathrm{Op}(P_x(H_a^{-1}\alpha))/\mathrm{i}$ of order $-1$.

*Proof of Theorem 9.1.* In case (i) use the elliptic construction just described, and in case (ii) the parametrix of Lemma 9.5; in both cases $A_\Gamma = \mathrm{Op}(\alpha)$ with $\alpha$ equal to one on $\Gamma_{\mathrm{in}}$ at high frequency and supported in $\Gamma_{\mathrm{out}}$. All hypotheses are open conditions on a fixed finite jet of the coefficients, which gives the coefficient neighborhood. This proves (9.3), with the residual of the stated form. The bounds (9.4) and the $H^s$ bounds follow from Lemmas 9.3 and 9.4, whose constants are polynomial in finitely many coefficient seminorms. All boxes, cutoffs, phases and amplitudes are fixed before any later accuracy. □

## 10. Finite correction on regular patches

We now cover a compact regular set of covectors by finitely many characteristic and elliptic charts and show that a finite Neumann series of the parametrices of Section 9 solves the system to any prescribed finite order for oscillatory sources.

### 10.1. The choice of charts.

**Notation 10.1.** Compactly supported functions are extended by zero, and all cutoffs have supports strictly inside the stated domains. Conic sectors are compact subsets of the unit sphere together with their conic extensions; the source cover is taken symmetric under $\xi \mapsto -\xi$. We write $\mathfrak{c}f = \overline{f}$ for complex conjugation. For a size $A_\rho$, a *tail of arbitrary order* means a term bounded by $C_{m,Q}A_\rho\rho^{-Q}$ in each fixed $H^m$, for every $Q$.

**Setup 10.2** (The local solution operators)**.** We consider the system of Section 9 on a compact set of nonzero covectors which are elliptic or simple characteristics. The finite cover by charts and the source regions are chosen in Theorem 10.3. On each selected chart we use Theorem 9.1, with an input cutoff equal to one near $\overline{U}$. The residual has the terms (9.6), or their elliptic counterparts, and satisfies $R_j : L^2(W) \to H^1(U)$; order-zero factors are bounded on $L^2$, and every fixed Sobolev norm has a polynomial cost in finitely many coefficient seminorms. Output cutoff derivatives vanish on $U$, and all other cutoff terms are included. One finite order $k_{\mathrm{lin}}$ of coefficient derivatives controls the $L^2$ and $L^2 \to H^1$ constants, and one fixed loss $c_{\mathrm{lin}}$ covers the higher symbol, flow and Fourier estimates. These choices are fixed before every later accuracy.

### 10.2. Correction on each chart.

**Theorem 10.3** (A common choice of charts)**.** *Let $V_0'$ be a compact conic set of regular covectors as in Setup 10.2. After one common shrinking of the physical domains there are scalar source operators $A_j^0$ with*

$$\sum_j (A_j^0 + \mathfrak{c}A_j^0\mathfrak{c}) = A_\Gamma,$$

*where $A_\Gamma$ is the identity microlocally near $U \times V_0'$. Each index $j$ selects one member of a pair of characteristic or elliptic charts symmetric under $\xi \mapsto -\xi$. There are larger source cutoffs $\widetilde{A}_j$ and fixed complex linear operators with*

$$P_a S_j = \widetilde{A}_j + R_j \quad \text{on } U,$$

*satisfying the bounds of Setup 10.2. On each characteristic chart put $\ell_j = \tau - h_j(x,\eta)$. The forward Hamilton flowout of $\ell_j$, over the entire fixed time interval, of every point in the conic support of $A_j^0$ stays in a comparable annulus and in the region where $\widetilde{A}_j$ is the identity. The physical source region contains $\overline{U}$.*

*Having fixed these data, choose $0 \le \chi_2 \le 1$ in $C_c^\infty(U)$, equal to one on a fixed set $U_{\mathrm{core}}$, and put*

$$K_j = \chi_2 R_j, \qquad \mathcal{T}_{j,N} = \sum_{0 \le \nu < N} (-K_j)^\nu.$$

*For $f = \chi_2 h$ and $z_{j,N} = \mathcal{T}_{j,N} f$ we have the exact identity*

$$P_a S_j z_{j,N} = h + (\widetilde{A}_j - I) z_{j,N} + (-1)^{N-1} K_j^N f \quad \text{on } U_{\mathrm{core}}. \tag{10.1}$$

*The set of normalized sums and differences of phase gradients of Proposition 6.4 admits one such choice on a fixed neighborhood of the reference immersion. All flowouts lie in one fixed $W_0 \Subset W$, where the operators use the coefficients of the immersion $F$. These choices precede the accuracy; only the finite depth $N$ changes with it.*

*Proof.* At a simple characteristic Euler's identity gives $\xi \cdot \partial_\xi p = (\deg p)\, p = 0$, so in coordinates with $\partial_\tau p \neq 0$ we have $\eta \neq 0$, and the root is of the form required in Theorem 9.1(ii). Cover $V_0'$ by finitely many root charts and elliptic charts, symmetric under $\xi \mapsto -\xi$, and select one member of each symmetric pair. By compactness there are nested conic sets $\Gamma_j^0 \Subset \Gamma_j^1 \Subset \Gamma_j^2$ inside larger charts, with uniform margins. Choose $A_j^0$ with conic support in $\Gamma_j^0$ forming the stated partition, and $\widetilde{A}_j$ equal to one on $\Gamma_j^1$ and supported in $\Gamma_j^2$, with fixed low-frequency and physical cutoffs.

On a characteristic chart, the Hamilton equations of $\ell_j$, restricted to the unit sphere in $\xi$, have bounded angular velocity, and $|\xi|$ changes by a bounded logarithmic rate. We choose the time interval so short that a trajectory starting in $\Gamma_j^0$ stays in $\Gamma_j^1$ and in a comparable annulus. The same shrinking puts the transported support from $\Gamma_j^2$ and all its physical supports in a fixed $W_0 \Subset W$ inside the larger chart. Elliptic factors involve only the identity relation.

Apply Theorem 9.1 on each selected chart, with source symbol $\widetilde{A}_j$ (equal to $\sigma_j \beta_j$ on characteristic charts); this defines $S_j$ and $R_j$ with all cutoff errors. Then choose $\chi_2$ inside the common source region. On $U$,

$$P_a S_j z_{j,N} = \widetilde{A}_j z_{j,N} + R_j z_{j,N} = z_{j,N} + (\widetilde{A}_j - I) z_{j,N} + R_j z_{j,N},$$

and on $U_{\mathrm{core}}$, where $\chi_2 = 1$, we have $R_j z_{j,N} = K_j z_{j,N}$. The telescoping identity

$$(I + K_j) z_{j,N} = \sum_{0 \le \nu < N} (-K_j)^\nu f - \sum_{1 \le \nu \le N} (-K_j)^\nu f = f + (-1)^{N-1} K_j^N f$$

and $f = h$ on $U_{\mathrm{core}}$ give (10.1). The source $f$ is supported in $\operatorname{supp} \chi_2$, and the fixed Sobolev bounds compose with polynomial costs; we take the maximum of the finitely

many derivative losses once. The set of normalized sums and differences is compact and lies at a positive distance from the singular set by Proposition 6.4, so the same construction applies to all of them. The construction preserves the derivative bounds of the coefficients, with a fixed loss. □

10.3. **Finite powers and arbitrary accuracy.**

**Setup 10.4** (Coefficient estimates)**.** We use the fixed chart operators of Theorem 10.3 and assume

$$\|a\|_{C^r(W)} \le D_r T^{(r-R_*)_+}, \qquad r \ge 0, \quad R_* > c_{\mathrm{lin}} + 2, \tag{10.2}$$

and put $L = T \ge 1$. By the derivative bounds of Lemma 9.5, the normalized flow maps and their inverses have bounded derivatives of orders zero and one, and the phase function $\mathcal{G}$ of each retarded factor satisfies

$$|D^r \mathcal{G}| \le C_r L^{r-1} \quad (r \ge 2). \tag{10.3}$$

Every symbol or amplitude of order $p$ satisfies

$$|D^\alpha_{x,z} D^\beta_\xi b_p| \le C_{\alpha,\beta} \langle \xi \rangle^{p-|\beta|} L^{|\alpha|+|\beta|}, \tag{10.4}$$

including both positions and all cutoffs; the covariable is $\eta$ for spatial symbols, and spacetime bounds are used only on their cones. At each fixed Sobolev order the operator norms of the factors are at most $C_s L^{B_s}$ for some finite $B_s$. Since $R_* > c_{\mathrm{lin}} + 2$, the $L^2$ and $L^2 \to H^1$ bounds do not depend on $L$. For $\rho \ge 2$ and a fixed $0 < \sigma < 1$ we assume

$$1 \le L \le \rho^{1-\sigma}. \tag{10.5}$$

The index $R_*$, the orders of the construction and all factors are fixed before the accuracy.

**Theorem 10.5** (Approximate solutions to any finite order)**.** *Assume Setup 10.4. Choose a compact conic set $V_0 \Subset V_0'$ and a fixed smooth rescaled Fourier cutoff $\Pi_{0,\rho}$, equal to one on $\rho V_0 \cap \{|\xi| \asymp \rho\}$ and supported in $\rho V_0'$. Let $h_\rho$ be real and smooth, supported where $\chi_2 = 1$, with*

$$\|h_\rho\|_2 \le C H_\rho, \tag{10.6}$$

$$\|(I - \Pi_{0,\rho}) h_\rho\|_{H^m} \le C_{m,Q} H_\rho \rho^{-Q} \quad \textit{for all fixed } m, Q. \tag{10.7}$$

*For each finite $N \ge 1$ define*

$$\mathcal{D}_{j,N} = S_j \mathcal{T}_{j,N} \chi_2 A_j^0, \qquad \mathcal{D}_{a,N} = \sum_j (\mathcal{D}_{j,N} + \mathfrak{c} \mathcal{D}_{j,N} \mathfrak{c}),$$

$$u_{\rho,N} = \mathcal{D}_{a,N} h_\rho.$$

*Then $\mathcal{D}_{a,N}$ is complex linear and preserves real fields. For all fixed $m, Q$ and every integer $N \ge m + Q$,*

$$\|P_a u_{\rho,N} - h_\rho\|_{H^m(U_{\mathrm{core}})} \le C_{N,m,Q} H_\rho \rho^{-Q}, \tag{10.8}$$

*and for every fixed $s$*

$$\|u_{\rho,N}\|_{H^s} \le C_{N,s} H_\rho \rho^s. \tag{10.9}$$

*A higher accuracy changes only the finite depth $N$; the derivative losses, physical domains, conic source cutoffs and orders of the kernels are fixed. The constants and the frequency threshold may depend on $N$, $m$, $Q$ and on the constants $D_r$ in (10.2), but not on $\rho$ or $L$ within (10.5).*

*Proof. Step 1 (localization of one factor).* Let $f$ satisfy $\|f\|_{H^s} \le C_s A \rho^s$ for all $s$, and suppose that, up to a tail of arbitrary order, its Fourier transform is concentrated in a compact conic subset, with a margin, of one source chart, within a comparable annulus $\{|\xi| \asymp \rho\}$. We show that the image of $f$ under each parametrix factor or residual factor is, up to a tail of arbitrary order, concentrated in any prescribed open conic neighborhood of the forward Hamilton flowout of this set (including the set itself), within a comparable annulus. The constants depend on the chosen neighborhoods but not on $\rho$.

For a retarded factor we use the representation (9.5). At a fixed elapsed time $r$, the critical points of the phase $\Theta_r(x,\xi) - z\cdot\xi$ relate the input point and the output covector by

$$(s,z) = (t-r, \Psi_\eta), \qquad (\tau_{\mathrm{out}}, \eta_{\mathrm{out}}) = (\tau + \Psi_t + \Psi_s, \Psi_y).$$

The Hamilton–Jacobi identities $\Psi_t = h(t,y,\Psi_y)$ and $-\Psi_s = h(s,z,\eta)$ show that this relation is the forward Hamilton flow of $\ell = \tau - h$ for the time $r$, at every value of $\tau$. Pseudodifferential terms and fixed Fourier cutoffs have the identity relation, corresponding to $r = 0$.

Compose a fixed-$r$ factor with an input cutoff concentrated near the given conic set and an output cutoff vanishing near its flowout (these cutoffs are used only in this proof), and rescale comparable covectors by $\rho$. Away from the flowout, the combined phase, as a function of the integration variables, has a gradient bounded below by $c\rho$, uniformly in $0 \le r \le \Delta$. Each integration by parts, at fixed $r$, gains $\rho^{-1}$ and costs at most a factor $L$ by (10.3) and (10.4). Hence for all fixed $m, q$ the resulting operator is bounded in the relevant Sobolev norms by

$$C_{m,q} \rho^{m+C} (L/\rho)^q. \tag{10.10}$$

The integrations include all amplitude positions and the source masks. Frequencies in non-comparable annuli are separated from the output by a frequency gap in a physical derivative of the phase; their dyadic sum converges after increasing $q$. Fixed physical supports, Schur's test and the bounded $r$-interval give the Sobolev bounds. Since $L/\rho \le \rho^{-\sigma}$ by (10.5), increasing $q$ gives a tail of arbitrary order. By the choice of charts, the short flow keeps the covectors in a comparable annulus.

*Step 2 (finite powers).* Put $X = \operatorname{supp}\chi_2$, $f_j = \chi_2 A_j^0 h_\rho$ and $v_{j,d} = K_j^d f_j$. We show that, for $d \le N+1$, $v_{j,d}$ is concentrated, up to a tail of arbitrary order, in a fixed conic region where $\widetilde{A}_j$ is the identity, and that

$$\|v_{j,d}\|_{H^s} \le C_{d,s} H_\rho \rho^{s-d} \qquad (d, s \ge 0). \tag{10.11}$$

On a characteristic chart we use its time coordinate $t$, which ranges over a fixed interval $[t_-, t_+]$, and the Hamilton flow $\Phi^r$ of $\ell_j = \tau - h_j$; along this flow $\dot{t} = 1$. Let $\Gamma_j^0$ be the

conic support of $A_j^0$ and put

$$\Lambda_j = \{\Phi^r(p) : p \in \Gamma_j^0,\ 0 \le r \le t_+ - t(p)\}.$$

By the choice of charts in Theorem 10.3, $\Lambda_j$ lies in the open conic region where $\widetilde{A}_j$ is the identity, and $|\xi|$ varies on $\Lambda_j$ only within a fixed factor of its initial value.

*(a) The total elapsed time.* By Step 1, the relation of each factor of $K_j$, and of $S_j$, is contained in $\{(\Phi^r(p), p) : 0 \le r \le t_+ - t(p)\}$, the identity relation corresponding to $r = 0$. Since $\Phi^{r_2} \circ \Phi^{r_1} = \Phi^{r_1+r_2}$ and the time coordinate increases by $r$ along $\Phi^r$, the composition of any number of such relations is contained in the same set: the total elapsed time is $t_{\text{out}} - t_{\text{in}} \le t_+ - t_-$, independently of the number of factors. Hence $\Lambda_j$ is invariant under the relation of every factor. In particular the frequency band reached by the iterates is the fixed band of $\Lambda_j$, determined by the bounded logarithmic rate of change of $|\xi|$ over the time $t_+ - t_-$; it does not widen with $d$. On an elliptic chart all relations are the identity and $\Lambda_j = \Gamma_j^0$.

*(b) Auxiliary sets depending on the depth.* The relations are closed, and their restrictions to the unit cosphere bundle are compact. Hence for every open conic set $O \supset \Lambda_j$ there is an open conic set $O'$ with $\Lambda_j \subset O'$ whose closure is mapped by every relation into $O$. Starting from an open conic set $O_{N+1} \supset \Lambda_j$, over a neighborhood of $X$, on which $\widetilde{A}_j$ is the identity, we choose successively $O_N, \dots, O_0$ with $\Lambda_j \subset O_k$, $\overline{O_k} \subset O_{k+1}$, and with the closure of $O_k$ mapped by every relation into $O_{k+1}$. These sets, and the cutoffs of Step 1 equal to one near $\overline{O_k}$ and vanishing outside $O_{k+1}$, depend on $N$ but not on $\rho$. They are used only in this proof and affect only the constants and the frequency threshold.

*(c) Induction.* The source $f_j$ is concentrated in $O_0$ up to a tail of arbitrary order, by the conic support of $A_j^0$ and (10.7). If $v_{j,d}$ is concentrated in $O_d$, Step 1 shows that $v_{j,d+1} = K_j v_{j,d}$ and $S_j v_{j,d}$ are concentrated in $O_{d+1}$, up to tails; at each step we increase the order of the tail to absorb the polynomial Sobolev costs of the subsequent factors. Elliptic factors have only the identity relation, and low-frequency pieces are separated by the annular gap. Thus, for $d \le N+1$, all these functions are concentrated in $O_{N+1}$, where $\widetilde{A}_j$ is the identity. This proves the localization, including all source and cutoff terms.

Fix $j$ and write $v_d = v_{j,d}$, so that $\|v_0\|_2 \le C H_\rho$. For $d \ge 1$, since $v_d$ is concentrated in an annulus of radius comparable to $\rho$ up to a tail, and $K_j : L^2 \to H^1$ with a bound independent of $L$,

$$\|v_d\|_2 \le C\rho^{-1}\|v_d\|_{H^1} + O(H_\rho \rho^{-\infty}) \le C\rho^{-1}\|v_{d-1}\|_2 + O(H_\rho \rho^{-\infty}).$$

Induction gives $\|v_d\|_2 \le C_d H_\rho \rho^{-d}$. In the other direction, frequency localization gives $\|v_d\|_{H^s} \le C_s \rho^s \|v_d\|_2 + O(H_\rho \rho^{-\infty})$, which proves (10.11).

*Step 3 (the finite correction).* Put $z_{j,N} = \mathcal{T}_{j,N} f_j$. By the localization just proved,

$$(\widetilde{A}_j - I) K_j^\nu f_j = O_{\nu,m,Q}(H_\rho \rho^{-Q}) \quad \text{in } H^m.$$

Each $S_j$ is bounded on $L^2$, and by Step 1 every $S_j K_j^\nu f_j$ is concentrated in a comparable annulus up to a tail; the same frequency localization and the finite sum give (10.9).

For real $h_\rho$ we have $\mathfrak{c}h_\rho = h_\rho$, so $\mathfrak{c}\mathcal{D}_{j,N}\mathfrak{c}h_\rho = \overline{\mathcal{D}_{j,N}h_\rho}$ and $u_{\rho,N} = 2\,\mathrm{Re}\sum_j S_j z_{j,N}$. Since $P_a$ is real, applying (10.1) with $f = f_j$ (on $U_{\mathrm{core}}$, $f_j = A_j^0 h_\rho$) and taking real parts gives, on $U_{\mathrm{core}}$,

$$P_a u_{\rho,N} - h_\rho = (A_\Gamma - I)h_\rho + 2\,\mathrm{Re}\sum_j \left[(\widetilde{A}_j - I)z_{j,N} + (-1)^{N-1}K_j^N f_j\right],$$

where we used $\sum_j 2\,\mathrm{Re}(A_j^0 h_\rho) = \sum_j (A_j^0 + \mathfrak{c}A_j^0\mathfrak{c})h_\rho = A_\Gamma h_\rho$. The source tail (10.7) and the choice of $A_\Gamma$ make the first term a tail of arbitrary order, and the localization makes the second one such a tail. The last term is bounded in $H^m$ by $C_{N,m}H_\rho\rho^{m-N}$ by (10.11), which is at most $C_{N,m}H_\rho\rho^{-Q}$ when $N \geq m + Q$. □

## 11. Corrected characteristic waves

In this section we construct, for an immersion $F$ near the reference, families of complex waves whose linearized metric change vanishes to any prescribed order on a fixed domain, whose leading amplitude is an explicit rank-one tensor, and which depend linearly on Fourier data. The construction has three stages: a WKB approximation with a Hamilton phase and a transported amplitude; a correction of the remaining residual by a finite Neumann series of the parametrix of Section 9; and a superposition over shifted covectors against a finite Fourier measure. The main result is Proposition 11.10 at the end of the section.

### 11.1. **Normal reconstruction and conventions.**

**Notation 11.1.** We use the dimensions, the tensors $Q_i^0$ and the Frobenius pairing of Definition 4.3; all orthogonal complements and projections of tensors refer to this pairing. Spatial and parameter supports have fixed larger margins. Constants may depend on these supports, on the fixed neighborhood of the reference, on lower bounds for inverses and on finitely many constants in the derivative bounds, but not on frequencies or the number of samples unless this is indicated. A statement that a datum has Fourier support in a ball refers to the datum before any cutoff in the initial variables; such a cutoff is kept in the amplitude of the wave, and if it is applied to the datum, its Fourier tail is estimated separately. In particular, $\kappa(y)P_K f(y)$ is not regarded as band limited.

**Definition 11.2** (Normal reconstruction)**.** Let $G$ be an immersion as in Setup 6.3, with induced metric $g$, normal map $N_G$ and projection $\Pi_{K_G}$ as in (6.5), and projected frame $Q_i = \Pi_{K_G}Q_i^0$. We use the objects of Definition 4.1 and (4.7) for this frame, and write them out for later reference:

$$\mathcal{V}_G(u) = \sum_{i,j} g^{ij}u_i\partial_j G + N_G(\mathrm{sym}\,\nabla^g u), \tag{11.1}$$

$$\begin{aligned} P_G u = \mathcal{P}_G u &= (Q_i : \mathrm{sym}\,\nabla^g u)_{i=1}^n, \\ M_G(x,\xi)_{i\bullet} = (Q_i(x)\xi)^T, &\qquad p_G(x,\xi) = \det M_G(x,\xi). \end{aligned} \tag{11.2}$$

The operator $P_G$ includes the Christoffel terms of $g$. For a covector $\xi$ and a vector $b$ we write $\xi\,\mathrm{sym}\,b = (\xi b^T + b\xi^T)/2$; note that $Q_i : (\xi\,\mathrm{sym}\,b) = \xi^T Q_i b = (M_G(x,\xi)b)_i$.

The patches and samples are indexed as in Definition 6.7, and their quadratic interactions are listed in Remark 6.8.

**Setup 11.3** (Oscillatory estimates for phase differences)**.** We use Setup 8.3 with the full and fixed-time charts of Proposition 6.5, $x = (t, y)$, $z = (s, v)$ and $v \in \mathbb{R}^2$. Their lower bounds, larger supports and margins are fixed with the order $M = M_{\mathrm{an}} = 20(n + 1)$, and the immersions satisfy (L) and (H) of Setup 6.3, so that Proposition 8.4 applies uniformly to the scalar and matrix amplitudes below, on the entire complex input and output spaces. Both the real and the imaginary parts of the input are used; by Corollary 8.5, on real outputs,

$$(2\operatorname{Re} T)(2\operatorname{Re} T)^* = 4\operatorname{Re}(TT^*). \tag{11.3}$$

## 11.2. **Hamilton phases and transport amplitudes.**

**Theorem 11.4** (Transport amplitudes and normal reconstruction)**.** *Consider one of the three-parameter characteristic patches, with its adapted coordinates $x = (t, y)$, its map $z \mapsto \zeta(z)$ of base covectors and the larger charts for shifted covectors of Propositions 6.4 and 6.5. Use the projected frame $Q_i(G) = \Pi_{K_G} Q_i^0$. Fix lower bounds for the metric, for the singular values of the second fundamental form, for the minors defining the null vectors and for $\partial_\tau p_G$, and use full and fixed-time inverse charts on the larger nested convex source domains. One sufficiently small finite $C^m$ neighborhood of the reference gives all these bounds; the boxes and lower bounds do not depend on frequencies or on the accuracy.*

*For a real initial amplitude $r_z(y)$ with fixed positive lower bound and fixed $C^m$ upper bound, there are a real phase $S_G$ and an amplitude $b_G = \alpha_G k_G$ satisfying*

$$p_G(x, d_x S_G) = 0, \qquad S_G(0, y, \zeta) = y \cdot \zeta, \qquad \alpha_G|_{t=0} = r_z, \qquad l_G^T P_G b_G = 0,$$

*where $k_G$, $l_G$ are real nonzero right and left null vectors of $M_G(x, d_x S_G)$ and $\alpha_G$ has a positive lower bound. The reconstruction (11.1) satisfies*

$$\tfrac{1}{2} D\gamma_G \mathcal{V}_G(u) = \Pi_{K_G}(\operatorname{sym} \nabla^g u). \tag{11.4}$$

*For any amplitude $b$ put $\xi = d_x S_G$ and*

$$\nu_b = N_G(\xi \operatorname{sym} b), \qquad A_0(b) = -\nu_b \xi^T. \tag{11.5}$$

*Then, for $\tau > 0$,*

$$\mathcal{V}_G(\tau^2 e^{\mathrm{i}S_G/\tau} b) = e^{\mathrm{i}S_G/\tau}(\mathrm{i}\tau\nu_b + \tau^2 \mathcal{V}_G(b)), \tag{11.6}$$

*and $e^{-\mathrm{i}S_G/\tau} d\,\mathcal{V}_G(\tau^2 e^{\mathrm{i}S_G/\tau} b) = A_0(b) + O(\tau)$. For $b = b_G$ we have $|\nu_b|, |\xi| \geq c > 0$ and $A_0(b)^* A_0(b) = |\nu_b|^2 \xi\xi^T$; this transported tensor is linear in the initial density $r_z^2$.*

*Suppose moreover that $T \geq 1$ and $\|G\|_{C^r} + \|r_z\|_{C^r} \leq C_r T^{(r - R_*)_+}$ for all $r$. There is one fixed loss $D$ such that, if $R_* > D$, then for all $r \geq 0$*

$$\begin{aligned} &\|S_G\|_{C^r(x,\zeta)} + \|b_G\|_{C^r(x,z,\zeta)} + \|\nu_b\|_{C^r(x,z,\zeta)} \\ &\qquad + \|\mathcal{V}_G(b)\|_{C^r(x,z,\zeta)} + \|A_0(b)\|_{C^r(x,z,\zeta)} \leq C_r T^{(r - R_* + D)_+}, \end{aligned} \tag{11.7}$$

*as joint norms in the physical variables and the normalized covectors, on the same larger domains.*

*Proof. Phase and transport.* The phase is given by Proposition 6.4 and Lemma 6.2; the inverse metric, the normal map and the null vectors are smooth functions of a fixed jet of $G$ under the stated lower bounds. We use the real null vectors and the scalar transport equation of Lemma 9.5, with initial time 0 and initial datum $r_z$. As computed there, the transport equation is equivalent to $l_G^T P_G(\alpha_G k_G) = 0$, and since it is a linear first-order equation along the projected Hamilton flow, its solution is

$$\alpha_G = a_{\text{flow}}\, r_z(y_{\text{foot}}),$$

where $y_{\text{foot}}$ is the initial point of the trajectory through $x$ and $a_{\text{flow}} > 0$ is the exponential of the integral of $-l^T(Pk)$ along it. Since $a_{\text{flow}}$ does not depend on $r_z$, $\alpha_G$ is positive and linear in $r_z$, so $\alpha_G^2$ is linear in $r_z^2$.

*Reconstruction.* By (4.4) with the normal field $w = N_G(\operatorname{sym}\nabla^g u)$,

$$D\gamma_G \mathcal{V}_G(u) = 2\operatorname{sym}\nabla^g u - 2B_G^* N_G(\operatorname{sym}\nabla^g u) = 2\Pi_{K_G}(\operatorname{sym}\nabla^g u),$$

which is (11.4). For $u = \tau^2 e^{\mathrm{i}S_G/\tau} b$ the tangential part of $\mathcal{V}_G(u)$ is $\tau^2 e^{\mathrm{i}S_G/\tau} dG\, g^{-1} b$, and

$$\operatorname{sym}\nabla^g\Big(e^{\mathrm{i}S_G/\tau} b\Big) = e^{\mathrm{i}S_G/\tau}\Big(\operatorname{sym}\nabla^g b + \frac{\mathrm{i}}{\tau}\,\xi \operatorname{sym} b\Big).$$

Multiplying by $\tau^2$ and applying $N_G$ gives (11.6). Differentiating it,

$$e^{-\mathrm{i}S_G/\tau} d\Big(e^{\mathrm{i}S_G/\tau}(\mathrm{i}\tau\nu_b + \tau^2 \mathcal{V}_G(b))\Big) = -\nu_b \xi^T + \tau\Big(\mathrm{i}\, d\nu_b + \mathrm{i}\mathcal{V}_G(b)\xi^T\Big) + \tau^2 d\mathcal{V}_G(b),$$

whose leading term is $A_0(b)$. For real $\nu_b$, $A_0(b)^* A_0(b) = \xi \nu_b^T \nu_b \xi^T = |\nu_b|^2 \xi\xi^T$.

*Lower bounds.* For $b = b_G$ the null relation gives $q_G(\xi \operatorname{sym} b_G) = M_G(x,\xi) b_G = 0$, so $\xi \operatorname{sym} b_G \in K_G^{\perp}$ and $B_G^* \nu_b = \Pi_{K_G^{\perp}}(\xi \operatorname{sym} b_G) = \xi \operatorname{sym} b_G$. Moreover

$$|\xi \operatorname{sym} b|^2 = \tfrac{1}{4}|\xi b^T + b\xi^T|^2 = \tfrac{1}{2}\Big(|\xi|^2|b|^2 + (\xi\cdot b)^2\Big) \geq \tfrac{1}{2}|\xi|^2|b|^2.$$

Hence $|\nu_b| \geq \|B_G^*\|^{-1}|\xi \operatorname{sym} b_G| \geq c$, using the lower bounds for $|\xi|$, $\alpha_G$ and $|k_G|$.

*Derivative bounds.* The phase, the algebraic formulas and the transport amplitudes satisfy the derivative bounds with a fixed loss, by Lemmas 7.2, 7.5 and Proposition 6.4; in $\alpha_G = a_{\text{flow}} r_z(y_{\text{foot}})$, joint composition and multiplication are covered by Lemma 7.5. The reconstruction uses finitely many further derivatives, so one common loss $D$ proves (11.7). □

11.3. **A two-term WKB approximation.** One algebraic correction reduces the reduced residual by one more power of $\tau$. With the phase and leading transport of Theorem 11.4, put $b_0 = b_G$ and, using the symbol (9.8) of the system $P_G$,

$$b_1 = \mathrm{i}\, b_{-1,G}(x, dS_G) P_G b_0. \tag{11.8}$$

For $\lambda = \tau^{-1}$ define, on the fixed compact set of shifted covectors $\zeta'$,

$$v_\tau(x,z,\zeta') = \tau^2 e^{\mathrm{i}S_G(x,\zeta')/\tau}(b_0 + \tau b_1). \tag{11.9}$$

It is linear in the initial amplitude.

**Lemma 11.5.**

$$P_G v_\tau = \tau^3 e^{\mathrm{i}S_G/\tau} P_G b_1. \tag{11.10}$$

*Proof.* Apply (9.10) with $\Psi = S_G/\tau$, $W_0 = \tau^2 b_0$ and $W_1 = \tau^3 b_1$. Its hypotheses hold: $\Psi$ solves $\Psi_t = h(t, y, \Psi_y)$ because $S_G$ does and $h$ is homogeneous of degree one; $l^T P_G W_0 = \tau^2 l^T P_G b_0 = 0$; and, since $b_{-1}$ is homogeneous of degree $-1$,

$$\mathrm{i}\, b_{-1}(x, d\Psi) P_G W_0 = \mathrm{i}\,\tau\, b_{-1}(x, dS_G)\,\tau^2 P_G b_0 = \tau^3 b_1 = W_1.$$

Hence $P_G(e^{\mathrm{i}\Psi}(W_0 + W_1)) = e^{\mathrm{i}\Psi} P_G W_1 = \tau^3 e^{\mathrm{i}S_G/\tau} P_G b_1$. □

The correction constructed in the next subsections removes this residual; the joint amplitude bounds, including those of $v_\tau$, are proved together in Corollary 11.8.

## 11.4. **The solution operator.**

**Setup 11.6** (A common solution operator)**.** We use the immersion $G$ and the reduced system of Definition 11.2. On a compact characteristic patch, the phase of the waves and the characteristic parametrix use the same simple root $\tau = h_G(t, y, \eta)$, so that

$$\partial_t \phi_G = h_G(t, y, \partial_y \phi_G). \tag{11.11}$$

On one chart containing the whole compact set of shifted covectors, Lemma 9.5 gives $P_G S_+ = A_+ + R_+$ on the inner region, with

$$S_+ = \chi_{\mathrm{out}}\Big(\mathcal{E}_W B_+ + \frac{1}{\mathrm{i}}\,\mathrm{Op}(b_{-1}\alpha_+)\Big), \qquad B_+ = \beta_+(D). \tag{11.12}$$

The conic cutoff $\alpha_+$ and $\beta_+$ are chosen equal to one near all these phase gradients throughout $\operatorname{supp}\chi_2$. By compactness, the finitely many patches have common source and time margins, common lower bounds and one common ball on which the identities hold. The larger physical and conic domains are convex and contain the source and observation boxes, all short flowouts and all initial points. For finite powers of the residual we use an observation domain containing $\operatorname{supp}\chi_2$ and the core cutoff. All factors are fixed independently of the patch parameters and of the accuracy. Put

$$\mathcal{K} = \chi_2 R_+, \qquad \mathcal{T}_N = \sum_{\nu=0}^{N-1} (-\mathcal{K})^\nu.$$

The source cutoff of the primary waves is $\chi_{\mathrm{src}} = \chi_{\mathrm{lag,src}}$, supported inside the primary chart and in $\operatorname{int}\{\chi_2 = 1\}$; the source and observation supports are those of Setup 3.3, and the region where the identities hold is $U_{\mathrm{LAG}}$.

The additional quantitative hypothesis is

$$\|a_G\|_{C^v} + \|G\|_{C^v} \le D_v T^{(v - R_*)_+}, \qquad v \ge 0, \quad T \ge 1, \tag{11.13}$$

where $a_G$ consists of the principal and Christoffel coefficients of the reduced system. The metric, normal and Gram lower bounds are positive, and the reconstruction, including one derivative of $\mathcal{V}_G$, is a smooth function of a fixed jet of $G$. The parametrix has the fixed factors of Setup 10.2. The waves are complex; real fields are obtained by taking real parts. We choose $R_*$ above the fixed geometric and analytic losses, so that every finite composition has polynomial costs at each physical or parameter order.

Lemma 18.1 and Proposition 18.2 verify (11.13) along the iteration.

11.5. **Oscillatory inputs depending on parameters.**

**Lemma 11.7** (Oscillatory inputs depending on parameters)**.** *Let $\mathcal{A}$ be one of the retarded or pseudodifferential factors of the parametrix of Lemma 9.5, of order $p \in \{0, -1\}$. Let $\phi(x; c)$ solve (11.11) on the source flowout, with phase gradients in the interior of the region where the source cutoff equals one, depending smoothly on parameters $c$. Let $T \le \lambda^{1-\nu}$ for a fixed $\nu > 0$. There is a fixed loss $b$ such that the coefficient and phase bounds with index $R + b$ imply the following. If $a_c$ is supported in the fixed source chart and*

$$\|\partial_c^\alpha a_c\|_{H^s} \le C_{s,\alpha} A T^{(s+|\alpha|-R)_+}, \qquad |\alpha| \le k_0,$$

*then, on the fixed observation support,*

$$\left\|\partial_c^\alpha \left(e^{-\mathrm{i}\lambda\phi}\mathcal{A}(e^{\mathrm{i}\lambda\phi}a_c)\right)\right\|_{H^s} \le C_{s,\alpha} A\lambda^p T^{(s+|\alpha|-R)_+}, \qquad |\alpha| \le k_0.$$

*If the amplitude of $\mathcal{A}$ vanishes near the phase gradients (a fixed nonstationary gap), the left side is $O(A\lambda^{-Q})$ at each fixed joint order, for every $Q$. The loss $b$ does not depend on $s$, $k_0$ or the length of a finite composition, and finite compositions satisfy the same bounds with the product of the factors $\lambda^p$.*

*Proof.* For bounded $\lambda$ the claim follows from the ordinary estimates, so let $\lambda$ be large. Include the fixed input cutoff in $a_c$ and put $u_c = e^{\mathrm{i}\lambda\phi}a_c$.

*Frequency localization.* Choose a fixed spacetime annular multiplier $P_\lambda = \pi(D/\lambda)$ with $\pi = 1$ near all phase gradients. By Sobolev embedding the hypothesis gives $|D_x^r \partial_c^\alpha a_c| \le C_r A T^\ell T^r$, so Proposition 8.6, with $\rho = \lambda$, $L = T \le \lambda^{1-\nu}$ and amplitude constant $AT^\ell\lambda^{|\alpha|} \le A\lambda^{\ell+k_0}$ (the factor $\lambda^{|\alpha|}$ comes from parameter derivatives of the phase factor), gives $(I - P_\lambda)u_c = O(A\lambda^{-Q})$ for every $Q$, at every fixed joint order. The powers of $\lambda$ produced by derivatives and by the ordinary bounds for $\mathcal{A}$ are absorbed by increasing $Q$; this uses higher physical derivatives of $a_c$, but does not change $R$ or the order $k_0$. In (9.5) the multiplier $P_\lambda$ enters the amplitude as the factor $\pi(\xi/\lambda)$, so all frequency integrals may be restricted to $|\xi| \asymp \lambda$; the multiplier $\beta_+$ remains in the amplitude.

*Stationary structure.* Fix $r$, write $\xi = \lambda\theta$ and remove the factor $e^{\mathrm{i}\lambda\phi(x)}$. The amplitude of order $p$ contributes a factor $\lambda^p$, and the phase becomes $\lambda f$,

$$f(x, c; z, \theta) = \Theta_r(x, \theta) - z \cdot \theta + \phi(z; c) - \phi(x; c),$$

with $\Theta_0 = x \cdot \theta$ for a pseudodifferential factor. The critical points in $w = (z, \theta)$ are given by $\theta = d\phi(z; c)$ and $z = \partial_\theta \Theta_r(x, \theta)$, that is, $z$ is the initial point of the Hamilton trajectory of time $r$ ending at $x$, with initial covector $d\phi(z)$. Since $\phi$ solves the eikonal equation, the Hamilton flow preserves the graph of $d\phi$, and the projection of the flow is close to the identity on the short interval; hence there is exactly one critical point $w_0(x, c)$, and by the Hamilton–Jacobi identities of Lemma 6.2 the critical value is $f(w_0) = 0$. The Hessian in $w$ is

$$\begin{pmatrix} \phi_{zz}(z; c) & -I \\ -I & (\Theta_r)_{\theta\theta}(x, \theta) \end{pmatrix},$$

with $\|\phi_{zz}\| \le C$ and $(\Theta_r)_{\theta\theta} = \operatorname{diag}(0, \Psi_{\eta\eta}) = O(r)$. On a fixed short interval every segment average of this Hessian has the same form and is uniformly invertible, and the amplitude support and the critical point lie in one fixed convex product domain.

*Derivatives in $x$ and $c$.* Let $\partial_e$ be a derivative in an output variable or a parameter. Since the critical value vanishes identically, $\partial_e f(w_0) = 0$, and the identities

$$\nabla_w f = \Big(\int_0^1 f_{ww}(w_0 + \theta(w - w_0))\,\mathrm{d}\theta\Big)(w - w_0),$$
$$\partial_e f = \Big(\int_0^1 \nabla_w \partial_e f(w_0 + \theta(w - w_0))\,\mathrm{d}\theta\Big) \cdot (w - w_0)$$

give $\partial_e f = A_e \cdot \nabla_w f$, with

$$A_e = \Big(\int_0^1 f_{ww}(w_0 + \theta(w - w_0))\,\mathrm{d}\theta\Big)^{-T} \int_0^1 \nabla_w \partial_e f(w_0 + \theta(w - w_0))\,\mathrm{d}\theta.$$

Hence $\partial_e(e^{\mathrm{i}\lambda f}) = A_e \cdot \nabla_w e^{\mathrm{i}\lambda f}$, and one integration by parts in $w = (z, \theta)$ gives

$$\partial_e \int e^{\mathrm{i}\lambda f} \alpha\, a_c\,\mathrm{d}w = \int e^{\mathrm{i}\lambda f}\Big[\partial_e\alpha\, a_c + \alpha\,\partial_e a_c - \nabla_w \cdot (A_e\alpha)\, a_c - \alpha A_e \cdot \nabla_w a_c\Big]\,\mathrm{d}w, \tag{11.14}$$

where $\alpha$ is the amplitude of the factor, multiplied by the input cutoff, and $\partial_e a_c = 0$ unless $e$ is a parameter. No power of $\lambda$ is produced: a derivative in $\theta$ is $\lambda\partial_\xi$, which preserves the order of the amplitude in the annulus $|\xi| \asymp \lambda$, and $a_c$ depends on $w$ only through $z$. The new integrand has the same phase.

We now count derivatives. Let $\kappa$ be the fixed number of derivatives of the coefficients of $P$ needed to write down $f_{ww}$, $\nabla_w \partial_e f$ and the amplitude $\alpha$; it consists of the losses of the Hamilton–Jacobi phases and their inverse charts (Lemma 6.2 and Proposition 6.4), of the null vectors and of the transport equation, plus the two derivatives in $f_{ww}$ and $\nabla_w \partial_e f$. With $\ell = \lfloor n/2 \rfloor + 1$ we take

$$b = \kappa + 2\ell.$$

If the coefficients satisfy bounds with index $R + b$, then, by Lemmas 7.2 and 7.5 (the averaged Hessian has a uniformly bounded inverse), every factor $\beta$ among $A_e$ and $\alpha$ satisfies

$$\|\partial^\delta \partial^\gamma \beta\|_\infty \le C_\gamma T^{(|\gamma| - R)_+}, \qquad |\delta| \le 2\ell, \tag{11.15}$$

where $\gamma$ counts the derivatives produced by the integrations by parts and $\delta$ the derivatives used below for the $L^2$ bound. After $q$ derivatives $\partial_{e_1} \cdots \partial_{e_q}$, repeated use of (11.14) writes the derivative of the output as a finite sum of terms

$$\int e^{\mathrm{i}\lambda f} \beta\, \partial_z^\gamma \partial_c^{\alpha_1} a_c\,\mathrm{d}w, \qquad j = |\gamma| + |\alpha_1| \le q,$$

in which $\beta$ is a product of derivatives of the factors $A_{e_i}$ and $\alpha$ of total order $q - j$: each of the $q$ derivatives falls either on the input or on one of these factors. By (11.15) and (7.5), the product $\beta$ satisfies (11.15) with $|\gamma|$ replaced by $q - j$. Restoring the phase factor, the term is $e^{-\mathrm{i}\lambda\phi}\mathcal{A}[\beta](e^{\mathrm{i}\lambda\phi}\partial_z^\gamma \partial_c^{\alpha_1} a_c)$, where $\mathcal{A}[\beta]$ is the factor with its amplitude multiplied by $\beta$. The fixed-time bound in the proof of Lemma 9.4, in the form for amplitudes depending on both positions (Lemma 8.2), uses at most $2\ell$ derivatives of

$\beta$ in the physical variables, so $\|\mathcal{A}[\beta]\|_{L^2\to L^2} \le C\lambda^p T^{(q-j-R)_+}$. The hypothesis gives $\|\partial_z^\gamma \partial_c^{\alpha_1} a_c\|_2 \le CAT^{(j-R)_+}$ when $|\alpha_1| \le k_0$. By (7.5),

$$C_q A\lambda^p T^{(q-j-R)_+} T^{(j-R)_+} \le C_q A\lambda^p T^{(q-R)_+}.$$

Integration over the fixed $r$-interval proves the assertion for a retarded factor, including time derivatives.

The loss $b$ is thus consumed entirely by the coefficient factors $A_e$ and $\alpha$, each of which is differentiated at most $2\ell$ times beyond its share $\gamma$. The input enters only through its derivatives of total order $j$, at the same index $R$ as in the hypothesis. This is why $b$ does not depend on $s$, on $k_0$, or on the number of factors in a composition.

*Nonstationary case and compositions.* Fixed low-frequency pieces contribute only tails. If the amplitude vanishes near the critical set, $|\nabla_w f|$ is bounded below on its support, and $J$ integrations by parts give $C_{q,J} A\lambda^{C_q}(T/\lambda)^J$ at joint order $q$; choosing $J > (Q + C_q)/\nu$ gives $O(A\lambda^{-Q})$. For a finite composition $\mathcal{A}_m \cdots \mathcal{A}_1$, the output $a_c' = e^{-\mathrm{i}\lambda\phi}\mathcal{A}_1(e^{\mathrm{i}\lambda\phi} a_c)$, multiplied by the fixed cutoff in front of the next factor, satisfies by the estimate just proved the hypothesis of the lemma with $A$ replaced by $CA\lambda^{p_1}$, with the same index $R$ and the same $k_0$. The coefficient index $R + b$ required for the next factor is the same as for the first one. Hence the estimate iterates, with the product of the factors $\lambda^{p_i}$ and constants depending on the number of factors, but without any increase of the required derivative order. □

## 11.6. **The corrected fields.**

**Corollary 11.8** (The corrected fields)**.** *Use the two-term approximation (11.9), with a real initial amplitude, on the whole compact set of shifted covectors, and assume*

$$\|a_G\|_{C^r} + \|G\|_{C^r} + \|r_z\|_{C^r} \le D_r T^{(r-R_*)_+}, \qquad r \ge 0, \qquad 1 \le T \le \lambda^{1-\nu}, \quad \nu > 0.$$

*Recall $M_{\mathrm{an}} = 20(n+1)$, and put $r_{\mathrm{par}} = M_{\mathrm{an}} + 3$ and $\ell = \lfloor n/2 \rfloor + 1$. Before the sample and the later accuracy of the mean are chosen, fix*

$$R_{\mathrm{env}} \ge M_{\mathrm{an}} + \ell + 2 + r_{\mathrm{par}}, \qquad k_0 = r_{\mathrm{par}}.$$

*There is a fixed $B_{\mathrm{at}}$ such that $R_* \ge R_{\mathrm{env}} + B_{\mathrm{at}}$ covers the losses in the coefficients, in $P_G b_1$, in the reconstruction and in the transfer to shifted covectors, including the finite orders required by the iteration for the mean; the two derivatives of the reconstruction and the Sobolev embedding loss are included in $R_{\mathrm{env}}$. After $R_{\mathrm{env}}$ is fixed, prescribe finitely many derivative orders for the linearized metric and an accuracy, and choose a sufficiently large finite $N$. Suppressing $N$ from the notation, define*

$$\begin{aligned} u_\tau &= S_+ \mathcal{T}_N(\chi_{\mathrm{src}} P_G v_\tau), \qquad \mathcal{Y}_\tau = \mathcal{V}_G(u_\tau), \\ \mathcal{W}_\tau^{\mathrm{cur}} &= \mathcal{V}_G(v_\tau - u_\tau), \qquad A_\tau^{\mathrm{cur}} = e^{-\mathrm{i}S_G/\tau} d\mathcal{W}_\tau^{\mathrm{cur}}. \end{aligned} \tag{11.16}$$

*Write $c = (z, \zeta')$. On the full observation support, for $k \le r_{\mathrm{par}}$ and uniformly over the fixed radial factors,*

$$\lambda\|D_x^r D_c^k(e^{-\mathrm{i}S_G/\tau}\mathcal{Y}_\tau)\|_\infty + \|D_x^r D_c^k(e^{-\mathrm{i}S_G/\tau} d\mathcal{Y}_\tau)\|_\infty \le C_{r,k}\lambda^{-1}T^{(r-M_{\mathrm{an}})_+}, \tag{11.17}$$

$$\lambda\|D_x^r D_c^k(e^{-\mathrm{i}S_G/\tau}\mathcal{W}_\tau^{\mathrm{cur}})\|_\infty + \|D_x^r D_c^k A_\tau^{\mathrm{cur}}\|_\infty \le C_{r,k}T^{(r-M_{\mathrm{an}})_+},$$

$$\|D_x^r D_c^k(A_\tau^{\mathrm{cur}} - A_0)\|_\infty \le C_{r,k}\lambda^{-1}T^{(r-M_{\mathrm{an}})_+}, \tag{11.18}$$

*where $A_0 = A_0(b_0)$ is the leading transported tensor. More precisely, for every $r \ge 0$ and $k \le k_0$ the bounds for $A_\tau^{\mathrm{cur}}$, $A_0$ and their difference hold with $T^{(r+k+\ell+2-R_{\mathrm{env}})_+}$ in place of $T^{(r-M_{\mathrm{an}})_+}$, the difference carrying the factor $\lambda^{-1}$. The linearized metric $D\gamma_G\mathcal{W}_\tau^{\mathrm{cur}}$ is bounded by the prescribed accuracy on $D_{\mathrm{lag}}$, where the source cutoff equals one and the reduced identity holds. These are the joint bounds assumed in Theorem 12.4, which therefore applies to this family. The construction is linear in the initial amplitude and is fixed before the sample. Constants may depend on $N$; the lower bounds at the finitely many orders used in the sampling theorem are uniform above a frequency threshold depending on $N$, because the corrections carry a factor $\lambda^{-1}$.*

*Proof. Step 1 (the reduced residual).* By (11.4), $D\gamma_G\mathcal{V}_G(v) = 2\Pi_{K_G}(\mathrm{sym}\,\nabla^g v)$, whose components in the frame are determined by $P_G v$ through (4.7). Thus the normal completion has no source, and it suffices to correct the reduced residual $P_G v_\tau$. The finite array $b_0, b_1, P_G b_1$, together with $\nu_{b_j}$, $\mathcal{V}_G(b_j)$ and their first physical derivatives, $j = 0, 1$, is given by the transport equation and fixed algebraic formulas, and satisfies the joint derivative bounds with index $R_{\mathrm{env}}$ after one fixed increase of $B_{\mathrm{at}}$. Put $h_\tau = \chi_{\mathrm{src}} P_G v_\tau$. By (11.10), $h_\tau = e^{\mathrm{i}S_G/\tau}\tau^3\chi_{\mathrm{src}}P_G b_1$, an oscillatory function with amplitude of size $\lambda^{-3}$ and derivative bounds with index $R_{\mathrm{env}}$.

*Step 2 (the correction $u_\tau$).* Lemma 11.7 gives the same bounds for every $\mathcal{K}^j h_\tau$, $j < N$, and then for $u_\tau = S_+\mathcal{T}_N h_\tau$; each new residual starts inside the original source region. Hence

$$\|\partial_c^\alpha(e^{-\mathrm{i}S_G/\tau}\chi_{\mathrm{core}}u_\tau)\|_{H^s} \le C_{N,s,\alpha}\lambda^{-3}T^{(s+|\alpha|-R_{\mathrm{env}})_+}.$$

*Step 3 (amplitude bounds).* The lift $\mathcal{V}_G$ and the differential use at most two derivatives, and Sobolev embedding uses $\ell$ further orders. After $r$ physical and $k$ parameter derivatives the exponent is at most $(r+k+2+\ell-R_{\mathrm{env}})_+ \le (r-M_{\mathrm{an}})_+$, because $k \le r_{\mathrm{par}}$ and $R_{\mathrm{env}} \ge M + \ell + 2 + r_{\mathrm{par}}$. Each derivative of the phase factor costs at most $\lambda$, and at most two such factors occur, which proves (11.17). For $v_\tau$ substitute $b = b_0 + \tau b_1$ in (11.6) and differentiate once: the leading term of the demodulated derivative is $A_0(b_0)$, and every other term carries a factor $\tau$; the demodulated field itself carries a factor $\tau$. The same bounds for the finite array and the subtraction of $\mathcal{Y}_\tau$ prove (11.18).

*Step 4 (the linearized metric).* On the region where the reduced identity holds,

$$P_G(v_\tau - u_\tau) = (1 - \chi_{\mathrm{src}})P_G v_\tau - (A_+ - I)\mathcal{T}_N h_\tau - (-1)^{N-1}\mathcal{K}^N h_\tau,$$

by the telescoping identity of Theorem 10.3. The first term vanishes on $D_{\mathrm{lag}}$. Each term of the residual (9.6) has one factor of order $-1$, so the preceding lemma, after restoring the phase factor, gives

$$\|\partial_c^\alpha\mathcal{K}^N h_\tau\|_{H^s} \le C_{N,s,\alpha}\lambda^{s+|\alpha|-N-3}.$$

The conic complement $A_+ - I$ vanishes near the phase gradients, so the nonstationary case of the lemma makes the second term $O(\lambda^{-Q})$. By (4.7) and the Gram formula, with $T \le \lambda$, increasing $N$ gives the prescribed accuracy for the linearized metric. The index $R_{\mathrm{env}}$ and all factors are independent of $N$. □

11.7. **Finite Fourier measures.** The same correction applies to every Fourier modulation of the initial scalar. We integrate the corrected fields over the fixed set of shifted covectors; the source and observation cutoffs are those of Setup 3.3.

**Proposition 11.9** (Bounds for Fourier measures)**.** *Apply Corollary 11.8 at the immersion $F$, with coefficient scale $T \ge 2$ and phase $S_F(x,\zeta)$, and with its finite depth chosen for a prescribed derivative order $r_{\mathrm{obs}}$ and accuracy $Q$. Let $\lambda = \tau^{-1} \ge 2$, $T \le \lambda^{1-\nu}$ for a fixed $\nu > 0$, and let $m$ be a finite complex measure supported in $B_\Lambda \subset \mathbb{R}^{n-1}$. Suppose $K \ge \max(T, C_{\mathrm{mod}}\Lambda, 2)$ and that $\zeta + \tau B_\Lambda$ lies in the fixed set of shifted covectors. Put $\|m\| = \|m\|_{\mathrm{TV}}$, $\phi_F = S_F(\cdot,\zeta)$ and*

$$Z(m) = \int \mathcal{W}_\tau^{\mathrm{cur}}(F; x, \zeta + \tau\xi)\, \mathrm{d}m(\xi). \tag{11.19}$$

*Then for every $s \ge 0$*

$$\lambda \|\chi_{\mathrm{core}} e^{-\mathrm{i}\lambda\phi_F} Z\|_{C^s} + \|\chi_{\mathrm{core}} e^{-\mathrm{i}\lambda\phi_F} dZ\|_{C^s} \le C_s \|m\| K^s, \tag{11.20}$$

*and on $D_{\mathrm{lag}}$, for $0 \le r \le r_{\mathrm{obs}}$,*

$$\|D\gamma_F Z\|_{C^r(D_{\mathrm{lag}})} \le C_{r,Q} \|m\| \lambda^{-Q}. \tag{11.21}$$

*The map $m \mapsto Z(m)$ is complex linear. The depth is chosen before $m$ and the sample; the geometry, the derivative losses and the domains do not depend on it. The fixed radial factors affect only the constants.*

*Proof.* Since the shifted covectors range over a compact set, we may differentiate under the integral. Demodulating by the central phase,

$$e^{-\mathrm{i}\lambda\phi_F} \mathcal{W}_\tau^{\mathrm{cur}}(F; x, \zeta + \tau\xi) = e^{\mathrm{i}\Psi(x,\xi)}\, e^{-\mathrm{i}S_F(x,\zeta+\tau\xi)/\tau} \mathcal{W}_\tau^{\mathrm{cur}}(F; x, \zeta + \tau\xi),$$

where

$$\Psi(x,\xi) = \frac{S_F(x,\zeta+\tau\xi) - S_F(x,\zeta)}{\tau} = \int_0^1 d_\zeta S_F(x, \zeta + u\tau\xi) \cdot \xi \, \mathrm{d}u. \tag{11.22}$$

By the joint phase bounds of Proposition 6.4, $|D_x^j \Psi| \le C_j \Lambda T^{j-1}$ for $j \ge 1$ and $|\xi| \le \Lambda$. By the Leibniz and chain rules, $D_x^s(e^{\mathrm{i}\Psi} w)$ is a sum of terms $e^{\mathrm{i}\Psi}\, D^{j_1}\Psi \cdots D^{j_k}\Psi\, D^i w$ with $j_1 + \cdots + j_k + i = s$ and $j_\iota \ge 1$. The amplitude bounds of Corollary 11.8 give $\|D^i w\|_\infty \le C_i T^i$ for both demodulated amplitudes (the field with the factor $\lambda$, and its differential). Since $\sum_\iota (j_\iota - 1) = s - i - k$, such a term is bounded by

$$C\Lambda^k T^{s-i-k} \|D^i w\|_\infty \le C\Lambda^k T^{s-k} \le C_s K^s,$$

using $\Lambda \le K$ and $T \le K$. Integrating against $|m|$ proves (11.20). The linearized metric of each field is bounded by the prescribed accuracy on $D_{\mathrm{lag}}$, uniformly over the shifted covectors and including all source terms, in every $C^r$ norm with $r \le r_{\mathrm{obs}}$; integration proves (11.21). □

11.8. **The waves at one step.** We now state the result in the form used at one step of the iteration. Fix a finite order $r_*$ containing all geometric, analytic and density derivative orders used below. The immersion $F$ lies in a fixed $C^{r_*}$ neighborhood of $G_{\mathrm{ref}}$ and satisfies

$$\|F\|_{C^r(W)} \le C_r T_0^{(r-r_*)_+}, \qquad T_0 \ge 2. \tag{11.23}$$

The estimates of Section 18 verify both conditions for every iterate. All constructions use the same immersion $F$, at which the metric is linearized.

On a whole characteristic patch we use $x = (t, y)$ and $\zeta \in \mathbb{R}^{n-1}$, and the Hamilton phase with

$$p_F(x, d_x\phi_F) = 0, \qquad \phi_F(0, y, \zeta) = y \cdot \zeta.$$

For each prescribed integer $q \ge 1$, Corollary 11.8 gives a family $\mathcal{W}^{\mathrm{cur}}_\tau(F; x, \zeta)$: a transported characteristic wave corrected by a finite Neumann series applied to its reduced residual. The correction preserves the phase and the leading transported tensor. The family depends linearly on a prescribed real initial amplitude, with complex extension for Fourier data; its depth may depend on $q$ but not on the frequency or the base covector. The family is chosen on the entire compact set of shifted covectors before the sample is drawn. For a sample $\nu$ with $\lambda_\nu = \tau_\nu^{-1}$ and central phase $\phi_\nu = \phi_F(\cdot, \zeta_\nu)$ we define the complex linear map

$$Z_\nu(m) = \int \mathcal{W}^{\mathrm{cur}}_{\tau_\nu}(F; x, \zeta_\nu + \tau_\nu \xi)\, \mathrm{d}m(\xi). \tag{11.24}$$

The Dirac mass $m = \delta_0$ gives the unmodulated wave; a finite Fourier measure changes its initial data without changing the correction operator.

**Proposition 11.10** (Corrected characteristic waves)**.** *Fix $q \ge 1$. Assume (11.23), the fixed domains of Setup 3.3, and use the positive initial amplitude $r_F$ of Proposition 13.7, whose derivative losses are included in $r_*$. The family has the original phase and leading transported tensor, and $\|d\phi_\nu\|_{C^r} \le C_r T_0^r$ on the full observation support. Let $m$ have finite total variation and support in $B_\Lambda$, and choose a fixed $\nu_0 > 0$ with*

$$C_{\mathrm{env}} \max(T_0, \Lambda) \le K \le \lambda^{1-\nu_0}, \qquad c\lambda \le \lambda_\nu \le C\lambda. \tag{11.25}$$

*Then, for large $\lambda$, the shifted covectors stay in the domain of the phase, and on the full observation support*

$$\lambda_\nu \|\chi_{\mathrm{core}} e^{-\mathrm{i}\lambda_\nu \phi_\nu} Z_\nu(m)\|_{C^r} + \|\chi_{\mathrm{core}} e^{-\mathrm{i}\lambda_\nu \phi_\nu} dZ_\nu(m)\|_{C^r} \le C_r \|m\|_{\mathrm{TV}} K^r. \tag{11.26}$$

*The depth of the correction can be chosen so that*

$$\|D\gamma_F Z_\nu(m)\|_{C^0(D_{\mathrm{lag}})} \le C_q \|m\|_{\mathrm{TV}} \lambda^{-q}. \tag{11.27}$$

*The domains, the order $r_*$ and the frequency exponents do not depend on $q$.*

*Proof.* Apply Proposition 11.9 with $T = T_0$, $\tau = \tau_\nu$, $\zeta = \zeta_\nu$ and accuracy $Q$ with $Q \ge q + 1$. Its hypotheses hold: by (11.25), $T_0 \le K \le \lambda^{1-\nu_0} \le C\lambda_\nu^{1-\nu_0}$, so $T \le \lambda_\nu^{1-\nu}$ for any fixed $\nu < \nu_0$ and large $\lambda$; $K \ge C_{\mathrm{env}}\Lambda \ge C_{\mathrm{mod}}\Lambda$ after enlarging $C_{\mathrm{env}}$; and $\tau_\nu \Lambda \le C\lambda^{-1}K \le C\lambda^{-\nu_0}$, so the shifted covectors $\zeta_\nu + \tau_\nu B_\Lambda$ stay in the fixed domain of the phase for large $\lambda$. The coefficient bounds required in Corollary 11.8 follow from

(11.23) by Proposition 18.2 and the bounds for $r_F$ in Proposition 13.7. Then (11.20) is (11.26) (with $\lambda$ replaced by the comparable $\lambda_\nu$), and (11.21) with $r = 0$ gives (11.27). The bound for $d\phi_\nu$ is part of (6.7): no derivative loss is hidden in passing from a single shifted covector to finite measures, because the phase quotient (11.22) varies on the scale $\max(T_0, |\xi|) \le K$. □

## 12. Separated sampling

A continuous superposition of waves over the patch parameters is replaced by a finite sum over sample points which are mutually separated, so that the products of distinct waves oscillate at frequencies at least comparable to $\lambda h$. In this section we show that such a sample reproduces the continuous normal operator in operator norm, uniformly on the whole output space, and the continuous Hermitian mean in every Sobolev norm.

### 12.1. **Separated finite-band sampling.**

**Construction 12.1** (The sampling procedure)**.** Let finitely many patches with parameters $z = (s, v)$ in buffered three-dimensional charts be given, together with fixed smooth nonnegative compactly supported weights $\psi_a(z)$. In each chart and for $0 < h \le h_0$ consider the cubes $C_q = h(q + [0,1)^3)$, $q \in \mathbb{Z}^3$. Take all cubes meeting $\operatorname{supp}\psi_a$ together with their neighbors, and draw independent uniformly distributed points $U_q \in C_q$; different charts use independent draws. Discard a point if another original point of the same chart has $\ell^\infty$ distance less than $h/4$ from it. For $z \in C_q$ put

$$p_q(z) = \prod_{q' \ne q} \left(1 - \frac{|C_{q'} \cap (z + [-h/4, h/4]^3)|}{h^3}\right);$$

only the 26 neighboring cubes contribute. To each surviving point $U_q$ of a cube meeting $\operatorname{supp}\psi_a$ assign the weight

$$\mu_q = h^3 \psi_a(U_q)/p_q(U_q),$$

and omit zero weights. Number the resulting samples by $j$, with patch index $a(j)$ and parameter $Z_j$. The fixed $h_0$ keeps all points inside the chart margins.

**Lemma 12.2** (Separation and unbiasedness)**.** *For every realization there are at most $Ch^{-3}$ samples, the weights are at most $Ch^3$, the total mass is bounded, and distinct samples in the same chart are separated by at least $h/4$. Moreover $p_q \ge 7/8$, and for every integrable family $\Phi$ with values in a Banach space,*

$$\mathbb{E}\left[\mathbf{1}_{\{U_q \text{ survives}\}} \frac{h^3 \psi_a(U_q)}{p_q(U_q)} \Phi(U_q)\right] = \int_{C_q} \psi_a(z)\Phi(z)\,\mathrm{d}z.$$

*Proof.* Conditioned on $U_q = z$, the point survives if and only if no other point $U_{q'}$ lies in $z + (-h/4, h/4)^3$. These events are independent, and the probability that $U_{q'}$ avoids the box is $1 - |C_{q'} \cap (z + [-h/4, h/4]^3)|/h^3$ (the boundary has measure zero). Hence the conditional survival probability is exactly $p_q(z)$. The box has volume $(h/2)^3 = h^3/8$, so

the excluded volumes in the other cubes sum to at most $h^3/8$, and $p_q(z) \geq 1 - \sum_{q'} x_{q'} \geq 7/8$, where $x_{q'}$ are the factors subtracted from one. Since $U_q$ has density $h^{-3}$ on $C_q$,

$$\mathbb{E}\Big[\mathbf{1}_{\mathrm{surv}} \frac{h^3 \psi_a(U_q)}{p_q(U_q)} \Phi(U_q)\Big] = \int_{C_q} h^{-3} p_q(z) \frac{h^3 \psi_a(z)}{p_q(z)} \Phi(z)\, \mathrm{d}z = \int_{C_q} \psi_a \Phi \, \mathrm{d}z.$$

If two surviving points were at distance less than $h/4$, both would have been discarded, which gives the separation. There is at most one sample per cube, and there are $O(h^{-3})$ cubes; the weights are at most $\frac{8}{7} h^3 \|\psi_a\|_\infty$, and the total mass is at most a constant. No derivative of $p_q$ is used. □

We use the following form of the operator Bernstein inequality with intrinsic dimension; see [M17] and [T15, Theorem 7.3.1].

**Lemma 12.3** (Operator Bernstein inequality)**.** *Let $Y_1, \dots, Y_m$ be independent random self-adjoint operators on a finite-dimensional Hilbert space with $\mathbb{E} Y_q = 0$ and $\|Y_q\| \leq \mathsf{R}$ almost surely. Let $V$ be a positive semidefinite operator with $\sum_q \mathbb{E} Y_q^2 \leq V$, and put $d = \operatorname{tr} V / \|V\|$. Then for $u \geq \|V\|^{1/2} + \mathsf{R}/3$,*

$$\mathbb{P}\Big(\Big\|\sum_q Y_q\Big\| \geq u\Big) \leq 8d \exp\Big(-\frac{u^2/2}{\|V\| + \mathsf{R}u/3}\Big).$$

**Theorem 12.4** (Separated sampling)**.** *Use finitely many buffered three-parameter charts with weights $\psi_a$ as in Construction 12.1, and a smooth family of ambient waves $\mathcal{W}_\tau(x, z, \zeta')$ with the phase $S_G$ and the leading tensor $A_0 = A_0(b_G)$ of Theorem 11.4. Put $A_\tau = e^{-\mathrm{i}S_G/\tau} d\mathcal{W}_\tau$, and assume, for one scale $L \geq 1$, every physical order $r$ and every joint order $k \leq M_{\mathrm{an}} + 3$ in the patch parameters and shifted covectors,*

$$\|D_x^r D_{z,\zeta'}^k A_\tau\|_\infty + \tau^{-1} \|D_x^r D_{z,\zeta'}^k (A_\tau - A_0)\|_\infty \leq C_{r,k} L^r. \tag{12.1}$$

*Assume also uniform joint $C^{M_{\mathrm{an}}+2}$ bounds for the amplitudes and the inverse charts, with an $O(\tau)$ difference of amplitudes in that norm, including the $\epsilon$-derivatives of the leading homotopy. No hypothesis on the linearized metric of the waves is needed. All physical, parameter and conic cutoffs lie inside the larger charts, and frequencies below one are omitted.*

*Fix the band $\Lambda \geq 2$ before sampling, with $\Lambda/\lambda$ inside the phase margin. Each indexed patch $a$ has a fixed positive radial factor $R_a$; put $\tau_a = (R_a \lambda)^{-1}$. Fix a smooth radial cutoff $0 \leq \vartheta \leq 1$, equal to one on $B_1$ and zero outside $B_2$, and put*

$$P_K(\eta) = \vartheta(\eta/K), \qquad K \geq 1. \tag{12.2}$$

*The background uses the unshifted wave $\mathcal{W}_{\tau_a}(x, z, \zeta(z))$. With fixed measurement and frame cutoffs $\chi, \chi_{\mathrm{in}}$ define, at each fixed $z$,*

$$T_{a,z} F(x) = \int e^{\mathrm{i}\Phi_{\tau_a}(x,z,\eta)} |\eta|^{1/2} \chi_{\mathrm{hi}} P_\Lambda(\eta) a_{\tau_a}(x, z, \eta) F(\eta)\, \mathrm{d}\eta, \tag{12.3}$$

$$\Phi_\tau = \frac{S_G(x, \zeta(z) + \tau\eta) - S_G(x, \zeta(z))}{\tau}, \tag{12.4}$$

$$a_\tau = \chi \chi_{\mathrm{in}} \Big(Q_i : A_\tau(x, z, \zeta(z))^* A_\tau(x, z, \zeta(z) + \tau\eta)\Big)_{i=1}^n, \tag{12.5}$$

*the quotient being extended by its integral formula at $\tau = 0$. The input space is the complex $L^2(d\eta)$ and the output space is $\mathcal{H} = L^2(\mathbb{R}^n; \mathbb{C}^n)$. The cutoffs belong to these operators; the ambient fields are not cut. The same random sample may serve finitely many further operators of this type with the same lower bounds, or scalar operators with amplitude 1 and any fixed physical cutoff in the common larger phase chart. Define*

$$E_{\mathrm{sample}} = \sum_j \mu_j T_{a(j),Z_j} T_{a(j),Z_j}^*, \qquad E_{\mathrm{cont}} = \sum_a \int \psi_a T_{a,z} T_{a,z}^* \,\mathrm{d}z. \tag{12.6}$$

*Then the following hold.*

(i) *The operators satisfy, on the whole spaces,*

$$\|T_{a,z}\|^2 \le C\Lambda, \quad \|T_{a,z}\|_{\mathrm{HS}}^2 \le C\Lambda^n, \quad \|E_{\mathrm{cont}}\| \le C, \quad \operatorname{tr} E_{\mathrm{cont}} \le C\Lambda^n. \tag{12.7}$$

(ii) *For fixed $\epsilon > 0$ and $h^{-3} \asymp \Lambda^{1+\epsilon}$, with probability tending to one as $\Lambda \to \infty$,*

$$\|E_{\mathrm{sample}} - E_{\mathrm{cont}}\| \le \Lambda^{-\epsilon/4}. \tag{12.8}$$

*The same holds for the real operators $2\operatorname{Re} T$, whose normal operators are $4\operatorname{Re} E$, with fixed constants.*

(iii) *Put, with a fixed compact spatial cutoff,*

$$H_{a,z} = \operatorname{Re}\Big[A_{\tau_a}(x,z,\zeta(z))^* A_{\tau_a}(x,z,\zeta(z))\Big], \qquad H_{\mathrm{cont}} = \sum_a \int \psi_a H_{a,z} \,\mathrm{d}z.$$

*If $\sup_{a,z} \|H_{a,z}\|_{H^s} \le B_s$ for all integers $s \ge 0$, the event in (ii) can be chosen so that, simultaneously for all $s$,*

$$\Big\| \sum_j \mu_j H_{a(j),Z_j} - H_{\mathrm{cont}} \Big\|_{H^s} \le C 2^{s/2} h B_s, \tag{12.9}$$

*at the cost of an additional failure probability at most $Ch$. Moreover, with $H_{\mathrm{transport},a,z} = |\nu_{b_0}|^2 \, dS(x,\zeta(z))^{\otimes 2}$,*

$$\Big\| H_{\mathrm{cont}} - \sum_a \int \psi_a H_{\mathrm{transport},a,z} \,\mathrm{d}z \Big\|_{L^2} \le C\lambda^{-1}.$$

(iv) *Let $E_{\mathrm{transport,band}}$ be the continuous normal operator with the same weights and cutoffs, but with $\Phi_{\tau_a}$ replaced by $\Phi_0$ and $A_{\tau_a}$ by $A_0(b_0)$. Then*

$$\|E_{\mathrm{cont}} - E_{\mathrm{transport,band}}\| \le C\lambda^{-1}\Lambda. \tag{12.10}$$

*In particular one separated sample approximates the transported normal operator on the whole output space within $C\lambda^{-1}\Lambda + \Lambda^{-\epsilon/4}$, so that a lower bound for the latter on a given class of outputs passes to the former. The wave family and the band are fixed before the sample; separation, count, weight and mass bounds hold for every realization.*

*Proof. Operator bounds.* The joint amplitude bounds give the hypotheses of Setup 11.3, and Proposition 8.4, with $p_\Lambda^2 = \chi_{\mathrm{hi}}^2 P_\Lambda^2$, gives the bounds for $T_{a,z}$. Since $E_{\mathrm{cont}} = \sum_a T_a \psi_a T_a^*$ with the full operators $T_a$, the full bound gives $\|E_{\mathrm{cont}}\| \le C$, and $\operatorname{tr} E_{\mathrm{cont}} = \sum_a \int \psi_a \|T_{a,z}\|_{\mathrm{HS}}^2 \,\mathrm{d}z \le C\Lambda^n$. For several operator families we use the block-diagonal normal operator on the finite direct sum of the output spaces.

*Decomposition into independent classes.* Let $X_q$ be the positive operator contributed by the cube $q$, that is $\mu_q T_{a,U_q} T_{a,U_q}^*$ if $U_q$ survives and 0 otherwise, and put $Y_q = X_q - \mathbb{E}X_q$. By Lemma 12.2, $\sum_q \mathbb{E}X_q = E_{\mathrm{cont}}$. Each $X_q$ depends only on the draws in the cubes $q + \{-1,0,1\}^3$. Partition the indices according to $q \bmod 3 \in (\mathbb{Z}/3)^3$; for two distinct indices in the same class the sets $q + \{-1,0,1\}^3$ are disjoint, so within each of the 27 classes the operators $X_q$ are independent. Fix a class $\mathcal{I}$ and put $\mathsf{R} = C\Lambda h^3$. Then $0 \le X_q \le \mathsf{R}I$, so $\|Y_q\| \le \mathsf{R}$, and $X_q^2 \le \mathsf{R}X_q$ gives

$$\sum_{q\in\mathcal{I}} \mathbb{E}Y_q^2 \le \sum_{q\in\mathcal{I}} \mathbb{E}X_q^2 \le \mathsf{R}\sum_{q\in\mathcal{I}} \mathbb{E}X_q \le \mathsf{R}E_{\mathrm{cont}}.$$

*Concentration.* Let $P_\ell \uparrow I$ be finite-rank orthogonal projections, and let $E_1$ be a rank-one projection with range in that of $P_\ell$. Apply Lemma 12.3 to the compressions $P_\ell Y_q P_\ell$, with $V_\ell = \mathsf{R}P_\ell E_{\mathrm{cont}} P_\ell + \mathsf{R}E_1$. The variance condition holds, since $(P_\ell Y_q P_\ell)^2 \le P_\ell Y_q^2 P_\ell$ for self-adjoint $Y_q$. The term $\mathsf{R}E_1$ makes $V_\ell \ne 0$, so that the intrinsic dimension is defined even when the variance of the compressed sum vanishes. Then $\mathsf{R} \le \|V_\ell\| \le C\mathsf{R}$ and $\operatorname{tr} V_\ell \le \mathsf{R}(C\Lambda^n + 1)$, so $d \le C\Lambda^n$. Take $u = \Lambda^{-\epsilon/4}/27$. Since $h^{-3} \asymp \Lambda^{1+\epsilon}$, $\mathsf{R} \asymp \Lambda^{-\epsilon}$, and for large $\Lambda$ we have $u \ge (C\mathsf{R})^{1/2} + \mathsf{R}/3$ and

$$\frac{u^2/2}{C\mathsf{R} + \mathsf{R}u/3} \ge c\Lambda^{-\epsilon/2}\Lambda^{\epsilon} = c\Lambda^{\epsilon/2}.$$

Hence $\mathbb{P}(\|P_\ell \sum_{q\in\mathcal{I}} Y_q P_\ell\| \ge u) \le C\Lambda^n \exp(-c\Lambda^{\epsilon/2})$, uniformly in $\ell$. The sum $\sum_{q\in\mathcal{I}} Y_q$ is a finite sum of trace class operators, hence compact, so its compressions converge to it in norm; by Fatou's lemma the same bound holds for $\|\sum_{q\in\mathcal{I}} Y_q\|$. A union bound over the 27 classes and the triangle inequality give (12.8) with failure probability at most $C\Lambda^n \exp(-c\Lambda^{\epsilon/2})$. The assertion for the real operators follows from (11.3).

*Background quadrature.* The tensor contributed by one cube has $H^s$ norm at most $Ch^3 B_s$. Centering these vectors and summing within one class, independence and the Hilbert space identity $\mathbb{E}\|\sum_q Z_q\|^2 = \sum_q \mathbb{E}\|Z_q\|^2$ for independent centered $Z_q$ give a bound $Ch^{-3}(Ch^3 B_s)^2 = Ch^3 B_s^2$, and the triangle inequality in $L^2(\mathbb{P}; H^s)$ over the 27 classes gives

$$\mathbb{E}\Big\|\sum_j \mu_j H_{a(j),Z_j} - H_{\mathrm{cont}}\Big\|_{H^s}^2 \le Ch^3 B_s^2.$$

By Chebyshev's inequality, the probability that the norm exceeds $C2^{s/2}hB_s$ is at most $C2^{-s}h$. Summing over $s \ge 0$ and intersecting with the event of (ii) proves (12.9). By (12.1), $|A_\tau - A_0| \le C\tau \le C\lambda^{-1}$, and $A_0^* A_0 = |\nu_{b_0}|^2 \xi\xi^T$ by Theorem 11.4; this gives the comparison with the transported tensor.

*Comparison of normal operators.* First replace the amplitudes by the leading ones on the unchanged phase. Their difference is $O(\tau_a)$ in the relevant norm, so by Proposition 8.4 the normal operator changes by at most $C\tau_a$. The leading amplitudes along the homotopy in $\tau$ have $\tau$-derivatives of order $|\eta|$, so (8.3), applied with each weight $\psi_a$ and summed over the finitely many charts, bounds the change of phase by $C\tau_a\Lambda$. Since $\tau_a \asymp \lambda^{-1}$, this proves (12.10). □

*Remark* 12.5. For a fixed band the sampled normal operators are compact on the whole output space. Their approximation in norm therefore transfers a lower bound only on a class of outputs for which such a bound has already been established, as in Section 13.

## 13. The continuous normal operator and a positive mean

In this section we combine the image of the reduced operator with the quadratic tensors of characteristic waves into one operator $A$, and show that $E = AA^* + 1 - \chi^2_{\mathrm{meas}}$ is coercive; this is the analytic form of the Gram inequality (6.1). We then estimate the inverse of $E$ in Sobolev spaces, and use it to construct a positive initial amplitude $r_F$ and a tangent field $V_{\mathrm{base}}$ such that the transported mean of the unmodulated waves with density $r_F^2$, together with $D\gamma_F V_{\mathrm{base}}$, has zero component in $K_F$. This is what permits metric defects of either sign. Finally we identify the continuous operator with the leading part of the normal operator of the waves, which is approximated by the sample in Section 12.

In summary, the input of this section is an immersion $G$ in a fixed $C^{r_*}$ neighborhood of the reference, with derivative bounds of the form (3.1); the output consists of the coercive operator $E_G$ (Proposition 13.5), Sobolev bounds for its inverse with a fixed loss (Proposition 13.6), and the positive initial amplitude $r_G$ with the base field $V_{\mathrm{base}}$ (Proposition 13.7). In the proof of Theorem 1.1, $E_G^{-1}$ produces the modulations of the waves in Section 14, and $r_G$, $V_{\mathrm{base}}$ cancel the positive mean of the unmodulated waves.

We use the reference immersion and phase charts of Section 6, the transported fields of Section 11 and the domains of Setup 3.3. All derivative orders used here are included in $r_*$.

### 13.1. **Geometry and the associated operators.**

**Definition 13.1** (Second fundamental form and reduced operator)**.** For an immersion $G$ near the reference we use the normal map $N_G$ and the projected frame $Q_{G,i} = \Pi_{K_G} Q_i^0$ of (6.5), and the reduced operator $P_G$ of (11.2). Put

$$g_G = \gamma(G), \qquad \mathscr{G}_G = (Q_{G,i} : Q_{G,j})_{i,j=1}^n, \qquad q_G(H) = (Q_{G,i} : H)_{i=1}^n,$$

and $q_G(x,\xi) = q_G(\xi \otimes \xi)$. The neighborhood of Setup 6.3 gives a lower bound for the Gram matrix $\mathscr{G}_G$. These are smooth functions of the second jet of $G$, so by Corollary 7.3 they satisfy the derivative bounds of $G$ with a loss of two. Replacing $B_G$ by $B_G/\sigma$ does not change the kernel projection; hence uniform normalized lower bounds and fixed-order bounds for $dG$, $B_G/\sigma$ and the metric connection give uniform bounds for the frame and the reduced operator. The identities (4.4) and (11.4) hold pointwise on $W$.

**Definition 13.2** (Normalized reference and characteristic phases)**.** We use the flat family $G_\sigma$ of Corollary 4.8, whose projected frames tend to the constant frame of Definition 4.3, and the finite cover of the sphere, the characteristic patches with parameters $z = (s, v) \in \mathbb{R} \times \mathbb{R}^2$, and the phase charts of Propositions 6.1–6.5. These constructions hold on a fixed $C^{r_*}$ neighborhood, and for the normalized family they hold on common boxes with normalized lower bounds down to $\sigma = 0$.

For each such immersion $G$ we use the phase $S_G$, the transported amplitude $b_G$, the normal field $\nu_G$ and the tensor $A_0 = A_0(b_G)$ of Theorem 11.4, with the projected frame and a prescribed positive initial amplitude; their lower bounds and derivative bounds are those of that theorem. In the adapted coordinates $x = (t, y)$ put

$$\xi_{G,a} = d_x S_G(x, \zeta_a(z)), \quad Y_{G,a} = d_\zeta S_G(x, \zeta_a(z)), \quad \Phi_{G,a}(x, z, \eta) = Y_{G,a}(x, z) \cdot \eta.$$

Thus $\Phi_{G,a}$ is the phase $\Phi_\tau$ of Setup 8.3 at $\tau = 0$, and Proposition 6.5 gives its normalized full and fixed-time inverse charts, together with those of the exact difference quotients. The parameter neighborhoods are taken large enough to contain every shifted covector in Definition 13.9.

13.2. **The limit $\sigma \to 0$.** The normalized family has a regular limit. Let $\widehat{S}$ and $\widehat{b}$ be the phase and the transported amplitude for $F_0$ with initial amplitude 1, and use the initial amplitude $\sigma$ for $G_\sigma$.

**Lemma 13.3.** *For $x$ in the fixed ball,*

$$\begin{aligned} B_{G_\sigma}(x) &= \sigma B_{F_0}(\sigma x), & N_{G_\sigma}(x) &= \sigma^{-1} N_{F_0}(\sigma x), \\ S_\sigma(x, \zeta) &= \sigma^{-1} \widehat{S}(\sigma x, \zeta), & b_\sigma(x, \zeta) &= \sigma \widehat{b}(\sigma x, \zeta), \\ \xi_\sigma(x, \zeta) &= \widehat{\xi}(\sigma x, \zeta), & \nu_\sigma(x, \zeta) &= \widehat{\nu}(\sigma x, \zeta). \end{aligned} \tag{13.1}$$

*Moreover*

$$S_\sigma(x, \zeta) = \int_0^1 x \cdot d_x \widehat{S}(\theta \sigma x, \zeta) \, \mathrm{d}\theta,$$

*so the phases, and with them the normalized second fundamental form, its inverse, the transported amplitudes and the normal fields, extend smoothly to $\sigma = 0$ in all fixed mixed derivatives. At $\sigma = 0$ the tensors $Q_i^0$, $\xi_0$ and $\nu_0$ do not depend on $x$, $Y_0$ is linear in $x$, and $\nu_0 \neq 0$.*

*Proof.* The first identity was proved in Corollary 4.8. The normal spaces of $G_\sigma$ at $x$ and of $F_0$ at $\sigma x$ coincide, so $K_{G_\sigma}(x) = K_{F_0}(\sigma x)$, and since $N_G$ vanishes on $K_G$ and inverts $B_G^* = \sigma B_{F_0}^*$ on $K_G^\perp$, we get $N_{G_\sigma}(x) = \sigma^{-1} N_{F_0}(\sigma x)$. Since $Q_i(G_\sigma)(x) = Q_i(\sigma x)$, the principal symbols satisfy $M_{G_\sigma}(x, \xi) = M_{F_0}(\sigma x, \xi)$, and both metrics are the Euclidean one, so there are no Christoffel terms. The function $\sigma^{-1} \widehat{S}(\sigma x, \zeta)$ has gradient $(d_x \widehat{S})(\sigma x, \zeta)$, hence solves the eikonal equation for $G_\sigma$, and at $t = 0$ it equals $\sigma^{-1}(\sigma y \cdot \zeta) = y \cdot \zeta$; by uniqueness for the Hamilton–Jacobi problem it is $S_\sigma$, and $\xi_\sigma(x) = \widehat{\xi}(\sigma x)$. The null vectors of $G_\sigma$ at $x$ are those of $F_0$ at $\sigma x$, and

$$P_{G_\sigma}\big[\sigma \widehat{b}(\sigma x)\big] = \Big(Q_i(\sigma x) : \operatorname{sym} \partial_x[\sigma \widehat{b}(\sigma x)]\Big)_i = \sigma^2 (P_{F_0} \widehat{b})(\sigma x),$$

so $\sigma \widehat{b}(\sigma x)$ solves the transport equation for $G_\sigma$ with initial amplitude $\sigma$; by uniqueness it equals $b_\sigma$. Then $\nu_\sigma = N_{G_\sigma}(\xi_\sigma \operatorname{sym} b_\sigma) = \sigma^{-1} N_{F_0}(\sigma x)\big(\widehat{\xi} \operatorname{sym} \sigma \widehat{b}\big)(\sigma x) = \widehat{\nu}(\sigma x)$.

Since $\widehat{S}(0, \zeta) = 0$, the fundamental theorem of calculus along the segment from 0 to $\sigma x$ gives the integral formula, which is smooth in $\sigma$ including $\sigma = 0$, where $S_0(x, \zeta) = x \cdot d_x \widehat{S}(0, \zeta)$ is linear in $x$. The remaining quantities are of the form $f(\sigma x)$ and are smooth in $(\sigma, x)$. Finally, $\xi_0 \neq 0$ and the null vector $k_0 \neq 0$ give $\xi_0 \operatorname{sym} k_0 \neq 0$

(see the proof of Theorem 11.4); this tensor lies in $K_0^{\perp}$, and its preimage under the injective map $B_{F_0}(0)^*$ is a nonzero multiple of $\nu_0$. Compact parameter supports give a uniform positive lower bound for $|\nu|$. □

Only these normalized algebraic coefficients are used at $\sigma = 0$; $G_0$ itself is not an immersion of full normal rank. All derivative losses are fixed before the observation order and the accuracy.

### 13.3. **Coercivity of the continuous normal operator.**

**Definition 13.4** (The continuous operator)**.** We first use the normalized family of coefficients, including its limit at $\sigma = 0$, and then a small fixed $C^{r_*}$ neighborhood of a chosen positive $G_{\mathrm{ref}} = G_\sigma$. Fix domains $U_{\mathrm{core}} \Subset U_{\mathrm{meas}} \Subset W$ and a cutoff $0 \le \chi_{\mathrm{meas}} \le 1$ equal to one near $\overline{U_{\mathrm{core}}}$ and supported in $U_{\mathrm{meas}}$; the measurement region is chosen small below, and these cutoffs are fixed before $\sigma$ and the neighborhood of $G_\sigma$. For every patch choose a smooth function $\rho_a(z, \widehat{\eta}) \ge 0$, positive on fixed neighborhoods of the model incidence points and of the corresponding directions $\widehat{\eta}$, and compactly supported in the input charts. Let $\chi_{\mathrm{hi}}(|\eta|)$ vanish for $|\eta| \le 1$ and equal one for $|\eta| \ge 2$.

With a fixed Fourier normalization, define the complex operators

$$T_a f(x) = \int e^{\mathrm{i}\Phi_{G,a}(x,z,\eta)} \chi_{\mathrm{meas}} \rho_a \chi_{\mathrm{hi}}(|\eta|) |\eta|^{1/2} |\nu_a(x,z)|^2 q_G(x, \xi_a(x,z)) f(z,\eta)\, \mathrm{d}z\, \mathrm{d}\eta, \tag{13.2}$$

acting on the complex space $L^2(z,\eta)$, with values in $L^2(\mathbb{R}^n;\mathbb{C}^n)$, and put $C_a = 2\,\mathrm{Re}\, T_a$. The operator

$$A = \Big( \chi_{\mathrm{meas}} P_G \langle D \rangle^{-1},\ (C_a)_a \Big)$$

acts on the direct sum of $L^2(\mathbb{R}^n;\mathbb{R}^n)$ (the image component) and the real Hilbert spaces underlying the inputs of the $T_a$, with values in $L^2(\mathbb{R}^n;\mathbb{R}^n)$. Put

$$E = AA^* + 1 - \chi_{\mathrm{meas}}^2.$$

The image component corresponds to tangential corrections, whose linearized metric is $2\,\mathrm{sym}\,\nabla u$, and the operators $C_a$ correspond to modulations of the waves: by Theorem 11.4, changing the initial density of a wave by a real modulation changes its transported mean by $|\nu_a|^2 \xi_a \otimes \xi_a$ times the modulation evaluated at the initial point $Y_{G,a}$, and $q_G$ of this tensor is the amplitude in (13.2).

**Proposition 13.5** (Coercivity)**.** *Fix a finite order $k_{\mathrm{gap}}$ containing the derivatives used in the operator norm estimates of Proposition 8.4. One sufficiently small measurement region and fixed cutoffs work for every $0 \le \sigma \le \sigma_0$ in the normalized family. All the operators above are bounded, and*

$$E \ge c_0 I; \tag{13.3}$$

*in particular $E$ has a bounded self-adjoint inverse on $L^2$. The same conclusions hold on a small finite $C^{r_*}$ neighborhood of each fixed positive $G_\sigma$, and after multiplying the initial amplitude by $\sqrt{1 + v_a(z,y)}$ with $\|v_a\|_{C^{k_{\mathrm{gap}}}}$ sufficiently small.*

*Proof. Boundedness.* Proposition 8.4, at $\tau = 0$ with $p_\infty = \chi_{\mathrm{hi}}$, applies to $T_a$; with $J_a$ as in (8.6), the change of variables $k = \Psi_a$ shows that the kernel of $T_a T_a^*$ is $\int e^{\mathrm{i}(x-x')\cdot k} b_{T,a}(x, x', k)\,\mathrm{d}k$, where, at the inverse point $(s, \eta) = u_a(x, x', v, k)$,

$$b_{T,a}(x, x', k) = \chi_{\mathrm{meas}}(x)\chi_{\mathrm{meas}}(x') \int \frac{\rho_a^2 \chi_{\mathrm{hi}}^2 |\eta|}{J_a} |\nu_a(x, z)|^2 |\nu_a(x', z)|^2 \cdot q_G(x, \xi_a(x, z)) q_G(x', \xi_a(x', z))^T \,\mathrm{d}v. \tag{13.4}$$

This is an amplitude of order zero. Hence $\|T_a\| \le C$, uniformly on the coefficient neighborhood, and by the continuity statement of Proposition 8.4 the normal operator $T_a T_a^*$ depends continuously in operator norm on finitely many normalized derivatives of the data. By Corollary 8.5, $C_a C_a^* = 4\,\mathrm{Re}(T_a T_a^*)$ on real outputs, so the same holds for $C_a C_a^*$. The image component is a product of coefficient multiplications, the Fourier multipliers $D_j\langle D\rangle^{-1}$ and $\langle D\rangle^{-1}$, and the measurement cutoff; it is bounded and depends continuously on the coefficients, and so does its normal operator. Since

$$E = \chi_{\mathrm{meas}} P_G \langle D\rangle^{-2} P_G^* \chi_{\mathrm{meas}} + \sum_a C_a C_a^* + 1 - \chi_{\mathrm{meas}}^2,$$

the operator $A$ is bounded uniformly on the neighborhood, and $E$ depends continuously on the coefficients in operator norm. We do not claim, and do not use, continuity of $A$ itself in operator norm: the phases $\Phi_{G,a}$ enter $T_a$ through $e^{\mathrm{i}\Phi_{G,a}}$ with unbounded $\eta$. All later perturbation arguments are applied to $E$.

*The limit $\sigma = 0$.* By Lemma 13.3 the coefficients are constant in $x$ at $\sigma = 0$, and the phases are linear in $x$. The reduced operator is then $P_0 = M_0(D)$ (up to the factor i), and the kernels of $T_a T_a^*$, apart from the measurement cutoffs, depend only on $x - x'$. Hence

$$AA^* = \chi_{\mathrm{meas}}\, e_0(D)\, \chi_{\mathrm{meas}},$$

with a matrix-valued multiplier $e_0(k) \ge 0$: the image component contributes $M_0(k)M_0(k)^T\langle k\rangle^{-2}$, and each wave component contributes the (realified) amplitude (13.4), which is a positive semidefinite matrix; realification adds the nonnegative contributions at $k$ and $-k$. For $|k| \ge 2$ the multiplier is homogeneous of degree zero up to the factor $|k|^2/\langle k\rangle^2$.

Suppose $s \in \mathbb{R}^n$ and $e_0(k)s \cdot s = 0$ for some large $|k|$. Then $M_0(k)^T s = 0$, and the wave contributions give

$$\int \frac{\rho_a^2 |\eta|}{J_a} |\nu_a|^4 \left| s \cdot q_0(\xi_a(z)) \right|^2 \mathrm{d}v = 0$$

at the inverse points of $k$. At an incidence point of $\widehat{k}$ the map $\eta \mapsto D_\zeta d_x S_0 \cdot \eta$ is onto the tangent hyperplane of the characteristic variety, which contains $\widehat{k}$; hence there is a unique nonzero $\eta$ with $d_x \Phi_a = \widehat{k}$, and the weight $\rho_a$ is positive there. Therefore $s \cdot q_0(\xi_a(z_a(\widehat{k}))) = 0$ for every selected patch, and the Gram inequality (6.3) gives $s = 0$. Since $e_0$ is continuous and asymptotically homogeneous, compactness of the sphere gives $e_0(k) \ge c$ for $|k| \ge R_0$, with fixed $R_0$; we used $\chi_{\mathrm{hi}} = 1$ there and $|k|^2/\langle k\rangle^2 \ge 1/2$.

For $u = \chi_{\mathrm{meas}} f$ we obtain, by Plancherel,

$$\|A^* f\|_2^2 = \langle e_0(D)u, u\rangle \geq c\|u\|_2^2 - C\int_{|k|\leq R_0} |\widehat{u}(k)|^2\,\mathrm{d}k \geq c\|u\|_2^2 - C\|u\|_1^2,$$

using $\|\widehat{u}\|_\infty \leq C\|u\|_1$. By the Cauchy–Schwarz inequality $\|u\|_1^2 \leq |\operatorname{supp}\chi_{\mathrm{meas}}|\,\|u\|_2^2$, hence

$$\|A^* f\|_2^2 \geq \big(c - C|\operatorname{supp}\chi_{\mathrm{meas}}|\big)\|\chi_{\mathrm{meas}} f\|_2^2 \geq \tfrac{c}{2}\|\chi_{\mathrm{meas}} f\|_2^2$$

if the measurement support is small enough. Since $\|\chi_{\mathrm{meas}} f\|_2^2 + \langle(1-\chi_{\mathrm{meas}}^2)f, f\rangle = \|f\|_2^2$,

$$\langle Ef, f\rangle = \|A^* f\|_2^2 + \langle(1-\chi_{\mathrm{meas}}^2)f, f\rangle \geq \min\Big(\tfrac{c}{2}, 1\Big)\|f\|_2^2.$$

*Perturbation.* With these cutoffs fixed, the continuity of $E$ in operator norm, proved above, gives $\sigma_0 > 0$ such that (13.3) holds for $0 \leq \sigma \leq \sigma_0$ with $c_0$ halved: by Lemma 13.3 the normalized data of $G_\sigma$ converge in every fixed $C^m$ norm as $\sigma \to 0$. Then a finite-order neighborhood of each fixed positive $G_\sigma$ gives the same bound with a common constant. A bounded self-adjoint operator with $E \geq c_0$ is invertible, with $\|E^{-1}\| \leq c_0^{-1}$.

*Change of the initial amplitude.* The transported amplitude is linear in the initial amplitude, and $|\nu_a|^2$ is quadratic in it. Multiplying the initial amplitude by $\sqrt{1+v_a}$ therefore multiplies the amplitude of the kernel of $T_a T_a^*$ by $[1 + v_a(z, Y_a(x,z))][1 + v_a(z, Y_a(x',z))]$, which changes the normal operator by at most $C\max_a \|v_a\|_{C^{k_{\mathrm{gap}}}}$ in operator norm. Small $v$ therefore preserves coercivity with a smaller constant. □

13.4. **The inverse and its realization.** We realize every image input by a tangential field:

$$\mathcal{J}_G h = \tfrac{1}{2} dG\, g_G^{-1}\langle D\rangle^{-1} h. \tag{13.5}$$

By (4.4) with $u = \frac{1}{2}\langle D\rangle^{-1}h$ and $w = 0$,

$$D\gamma_G \mathcal{J}_G h = \operatorname{sym}\nabla^{g_G}(\langle D\rangle^{-1}h), \qquad q_G D\gamma_G \mathcal{J}_G h = P_G\langle D\rangle^{-1}h.$$

The single derivative is compensated by $\langle D\rangle^{-1}$, so the field and its linearized metric have order at most zero.

The following elementary inequality is used repeatedly: for $L \geq 1$, $r_0 \geq 0$ and integers $0 \leq i \leq s$,

$$L^{(s-i-r_0)_+}\|u\|_{H^i} \leq C_s\Big[\|u\|_{H^s} + L^{(s-r_0)_+}\|u\|_2\Big]. \tag{13.6}$$

If $s - i - r_0 \leq 0$ this is $\|u\|_{H^i} \leq \|u\|_{H^s}$. Otherwise split $u$ in frequency at $|\xi| = L$: the high part satisfies $\|u_{\mathrm{hi}}\|_{H^i} \leq CL^{i-s}\|u\|_{H^s}$ and $L^{s-i-r_0}L^{i-s} \leq 1$, while the low part satisfies $\|u_{\mathrm{lo}}\|_{H^i} \leq CL^i\|u\|_2$ and $L^{s-i-r_0}L^i = L^{s-r_0}$.

**Proposition 13.6** (Inverse estimates and realization)**.** *Work in the coefficient neighborhood of Proposition 13.5, including its positive initial amplitudes $r_a$, and write*

$$A = A_{G,r} = (A_{\mathrm{im}}, T), \qquad \mathcal{F} = A_{\mathrm{im}}A_{\mathrm{im}}^* + 1 - \chi_{\mathrm{meas}}^2, \qquad E = \mathcal{F} + TT^*,$$

*where $A_{\mathrm{im}}$ is the image component and $T$ collects the real wave components. Let $X^s$ be the direct sum of $H^s$ for the image input and of $H^s_{z,y}$ for the wave inputs, the latter*

*measured in the initial variables after inverse Fourier transform in $\eta$. With the cutoff $P_K$ of (12.2) define*

$$E_K = \mathcal{F} + TP_K^2T^*, \qquad V_{\mathrm{aux}}(e) = \mathcal{J}_G A_{\mathrm{im}}^* E^{-1} e.$$

*Suppose that, for some $L \geq 1$,*

$$\|G\|_{C^m} + \sum_a \|r_a\|_{C^m} \leq C_m L^{(m-R_*)_+}, \qquad m \geq 0.$$

*There is a fixed $B$ such that, whenever $R_* \geq r_0 + B$ with $r_0 \geq 0$, for every integer $s \geq 0$,*

$$\begin{aligned} &\|E^{-1}e\|_{H^s} + \|A^*E^{-1}e\|_{X^s} + \|\mathcal{F}E^{-1}e\|_{H^s} + \|V_{\mathrm{aux}}(e)\|_{H^s(W)} \\ &\quad \leq C_s\Big[\|e\|_{H^s} + L^{(s-r_0)_+}\|e\|_2\Big]. \end{aligned} \tag{13.7}$$

*The constants depend on the constants in the hypothesis but not on $L$ or $K$. The bound for $V_{\mathrm{aux}}$ includes its linearized metric and holds on $\mathbb{R}^n$ after a fixed physical cutoff. Before that cutoff, $q_G D\gamma_G V_{\mathrm{aux}}(e) = P_G\langle D\rangle^{-1}A_{\mathrm{im}}^*E^{-1}e$, and on $U_{\mathrm{core}}$, where $\chi_{\mathrm{meas}} = 1$, this equals $A_{\mathrm{im}}A_{\mathrm{im}}^*E^{-1}e = \mathcal{F}E^{-1}e$. The inverse preserves $H^\infty$, and $AA^*E^{-1}e = e$ on $U_{\mathrm{core}}$. For every integer $N \geq 0$,*

$$\|(E - E_K)E^{-1}e\|_2 \leq C_N K^{-N}\Big[\|e\|_{H^N} + L^{(N-r_0)_+}\|e\|_2\Big]. \tag{13.8}$$

*For the normalized family $G_\sigma$ with initial amplitude $\sigma$, the lower bound and all constants are uniform as $\sigma \to 0$, and the right side of (13.7) becomes $C_s\|e\|_{H^s}$.*

*Proof. Derivative losses.* The algebraic formulas, fixed-time flows and inverse charts preserve the derivative bounds by Lemma 7.5 and (7.6). We choose $B$ to include their fixed losses and all fixed amplitude tests below. Then an $s$th derivative of an operator applied to a source, with $i$ of the derivatives falling on the source, carries the coefficient factor $L^{(s-i-r_0)_+}$, and (13.6) converts all such terms into the right side of (13.7).

*The inverse.* Let $U_z f(x) = f(x+z)$ and $E_z = U_z E U_{-z}$. Conjugation by translations translates the coefficients; Fourier multipliers commute with translations. In the amplitude of the wave component, a derivative in $z$ acts as $\partial_x + \partial_{x'}$, and the image component and $1 - \chi_{\mathrm{meas}}^2$ are finite products of smooth coefficients and fixed multipliers. Taylor's formula and the amplitude bounds show that $z \mapsto E_z$ is smooth in operator norm, with

$$\|\partial_z^\alpha E_z\|_{2\to 2} \leq C_\alpha L^{(|\alpha|-r_0)_+}.$$

Since $E_z \geq c_0$ for all $z$, the resolvent identity shows that $z \mapsto E_z^{-1}$ is smooth in operator norm, and differentiating $E_z E_z^{-1} = I$ gives

$$\partial_z^\alpha E_z^{-1} = \sum (-1)^m \frac{\alpha!}{\alpha_1! \cdots \alpha_m!} E_z^{-1}(\partial^{\alpha_1}E_z)E_z^{-1}\cdots(\partial^{\alpha_m}E_z)E_z^{-1},$$

summed over ordered decompositions $\alpha = \alpha_1 + \cdots + \alpha_m$ with $\alpha_i \neq 0$. By (7.5),

$$\|\partial_z^\alpha E_z^{-1}\|_{2\to 2} \leq C_\alpha L^{(|\alpha|-r_0)_+}.$$

For smooth $e$, the identity $U_z(E^{-1}e) = E_z^{-1}U_z e$ shows that $z \mapsto U_z(E^{-1}e)$ is smooth with values in $L^2$, and its derivatives at $z = 0$ are the weak derivatives of $E^{-1}e$:

$$\partial^\alpha(E^{-1}e) = \sum_{\beta\le\alpha} \binom{\alpha}{\beta} (\partial_z^\beta E_z^{-1})|_{z=0}\, \partial^{\alpha-\beta} e.$$

Hence $\|\partial^\alpha E^{-1}e\|_2 \le C\sum_\beta L^{(|\beta|-r_0)_+}\|e\|_{H^{|\alpha|-|\beta|}}$, and (13.6) gives (13.7) for $E^{-1}e$. By density the bound extends to all $e \in H^s$; in particular $E^{-1}$ preserves $H^\infty$.

*The adjoint.* The image adjoint has order zero. For a phase $\Phi = Y(x,z)\cdot\eta$ with $B = D_y Y$ invertible (rows indexed by the physical $y$ variables), the exact identities

$$d_y\Phi = B\eta, \qquad \eta = B^{-1}d_y\Phi, \qquad \partial_{z_\ell}\Phi = V_\ell \cdot d_y\Phi, \quad V_\ell = B^{-T}\partial_{z_\ell} Y, \tag{13.9}$$

hold. Here $Y = d_\zeta S_G$, and the common charts make $B$ uniformly close to $I$. Consequently $\eta_j e^{-\mathrm{i}\Phi} = \mathrm{i}(B^{-1}\partial_y)_j e^{-\mathrm{i}\Phi}$ and $\partial_{z_\ell} e^{-\mathrm{i}\Phi} = V_\ell\cdot\partial_y e^{-\mathrm{i}\Phi}$, so every factor $\eta$ and every parameter derivative of the phase factor in $T_a^* v$ can be transposed to a $y$-derivative falling on $v$ and on the amplitude. The compact measurement support removes boundary terms. After $s$ such operations, a term with $i$ derivatives on $v$ has coefficient factor $L^{(s-i-r_0)_+}$. Its amplitude still has the factor $|\eta|^{1/2}$, and its normal operator has order zero because of the degree-one Jacobian (8.6). The amplitude bounds, Plancherel and (13.6) give

$$\|A^* v\|_{X^s} \le C_s\Big[\|v\|_{H^s} + L^{(s-r_0)_+}\|v\|_2\Big],$$

and we insert $v = E^{-1}e$. The transported amplitude equals the initial scalar $r(z, Y_G)$ times a flow factor, so these estimates include variable initial amplitudes.

*The remaining bounds.* The image component $A_{\mathrm{im}}$ has order zero, and (13.5) gives the bounds for $V_{\mathrm{aux}}$ and its linearized metric; a fixed output cutoff adds only a lower-order term. The identity $AA^*E^{-1} = I - (1-\chi_{\mathrm{meas}}^2)E^{-1}$ gives $AA^*E^{-1}e = e$ on $U_{\mathrm{core}}$. Every wave adjoint contains the factor $\chi_{\mathrm{hi}}$, and $1 - P_K^2$ vanishes for $|\eta| \le K$, so

$$\|(I-P_K^2)T^*E^{-1}e\|_2 \le K^{-N}\||\eta|^N T^*E^{-1}e\|_2 \le C_N K^{-N}\Big[\|e\|_{H^N} + L^{(N-r_0)_+}\|e\|_2\Big],$$

by the adjoint estimate. Composing with the bounded operator $T$ gives (13.8).

*The normalized family.* For $G_\sigma$, Lemma 13.3 bounds every fixed derivative of the normalized amplitudes and of the coefficients in (13.9) uniformly in $\sigma$, and Proposition 13.5 gives a common lower bound. The proof above therefore has uniform constants, and since all derivatives are bounded uniformly, the factor $L^{(s-r_0)_+}$ can be replaced by a constant. □

13.5. **The positive transported mean.** The transported mean is affine in the initial density. One correction of the density and one tangential field cancel its component in $K_G$.

**Proposition 13.7** (A positive mean at the reference and nearby immersions)**.** *Use the normalized family of Definition 13.2. Choose smooth weights $\psi_a \ge 0$, compactly supported in the parameter charts and equal to one on a neighborhood of the projection $\pi_z(\operatorname{supp}\rho_a)$. For $\sigma > 0$ and an immersion $G$ near $G_\sigma$, let $A = A_{G,\sigma}$ and $E = AA^* + 1 -$*

$\chi^2_{\mathrm{meas}}$ *be the operators with constant initial amplitude* $\sigma$. *For the leading normal field* $\nu_{G,\sigma,a}$ *with initial amplitude* $\sigma$, *define the transported mean*

$$H_G[\sigma^2(1+v)] = \sum_a \int \psi_a |\nu_{G,\sigma,a}|^2 \Big[1 + v_a(z, Y_{G,a})\Big] \xi_{G,a} \otimes \xi_{G,a} \, \mathrm{d}z. \tag{13.10}$$

*Write* $H_\sigma = H_{G_\sigma}[\sigma^2]$, *and let* $H_0$ *be its constant limit. Fix* $X_0(x) = -H_0 x/2$ *and* $U_{\mathrm{fin}} \Subset U_{\mathrm{core}}$, *and define*

$$\begin{aligned} R_0 &= D\gamma_G(dG\, X_0) + H_G[\sigma^2], \\ e &= -\chi_{\mathrm{meas}} q_G(R_0), \qquad h = A^* E^{-1} e, \\ v_a &= 2\,\mathrm{Re}\, \mathcal{F}_\eta^{-1}\Big[\rho_a \chi_{\mathrm{hi}}(|\eta|) |\eta|^{1/2} h_a\Big], \end{aligned} \tag{13.11}$$

*where* $h = (h_{\mathrm{im}}, (h_a)_a)$, *the source is extended smoothly by zero to* $\mathbb{R}^n$, *and the inverse Fourier transform in* $\eta$ *is taken without a cutoff in the initial variables. Put*

$$r_{G,a} = \sigma\sqrt{1+v_a}, \qquad V_{\mathrm{base}} = dG\, X_0 + \mathcal{J}_G h_{\mathrm{im}}.$$

*Then, for all sufficiently small* $\sigma > 0$, *and on one sufficiently small fixed* $C^{r_*}$ *neighborhood of* $G_\sigma$, *these fields are real and smooth,* $1 + v_a \geq 1/2$, *the scalar* $r_{G,a}$ *is positive on the supports of the cutoffs* $\rho_a$, *and*

$$q_G\Big(D\gamma_G V_{\mathrm{base}} + H_G[r_G^2]\Big) = 0 \quad \text{on } U_{\mathrm{fin}}. \tag{13.12}$$

*The normal operator for this positive scalar is coercive, and its inverse satisfies the estimates of Proposition 13.6. For one fixed* $B$, *if* $\|G\|_{C^m(W)} \leq C_m L^{(m-R_*)_+}$ *with* $L \geq 1$, $r_0 \geq 0$ *and* $R_* \geq r_0 + B$, *then*

$$\sum_a \|r_{G,a}\|_{C^s} + \|V_{\mathrm{base}}\|_{C^s(U_{\mathrm{fin}})} + \|D\gamma_G V_{\mathrm{base}}\|_{C^s(U_{\mathrm{fin}})} \leq C_s L^{(s-r_0)_+}, \tag{13.13}$$

*where the norm of the scalar is taken in all variables* $(z, y)$ *on the fixed patches. These choices precede the accuracy and the final dilation, and fixed-order bounds are uniform on smooth compact families. Along* $G = G_\sigma$,

$$\sum_a \|v_a\|_{C^s} + \|V_{\mathrm{base}} - dG\, X_0\|_{C^s(W)} \leq C_s \sigma, \tag{13.14}$$

*the identity holds on* $U_{\mathrm{core}}$, *and the coercivity and the fixed-order inverse bounds for the corrected scalar are uniform as* $\sigma \to 0$.

*Proof. The identity.* The density $v_a$ has parameter support in $\{\psi_a = 1\}$, so $\psi_a v_a = v_a$, and the transport evaluates it at the initial point $Y_{G,a}$. Writing out the inverse Fourier transform,

$$v_a(z, Y_{G,a}(x,z)) = 2\,\mathrm{Re} \int e^{\mathrm{i}\Phi_{G,a}(x,z,\eta)} \rho_a \chi_{\mathrm{hi}} |\eta|^{1/2} h_a(z,\eta) \, \mathrm{d}\eta.$$

Multiplying by $\chi_{\mathrm{meas}} \psi_a |\nu_{G,\sigma,a}|^2 q_G(\xi_{G,a} \otimes \xi_{G,a})$, integrating in $z$ and using $\psi_a \rho_a = \rho_a$, we get, on the whole measurement support,

$$\chi_{\mathrm{meas}} q_G\Big(H_G[\sigma^2(1+v)] - H_G[\sigma^2]\Big) = \sum_a C_a h_a. \tag{13.15}$$

The same identity holds for any fixed positive initial amplitude and the corresponding operators. On $U_{\mathrm{core}}$ we have $\chi_{\mathrm{meas}} = 1$, so, by (13.5),

$$q_G\Big(D\gamma_G V_{\mathrm{base}} + H_G[r_G^2]\Big) = q_G R_0 + P_G\langle D\rangle^{-1} h_{\mathrm{im}} + \sum_a C_a h_a = q_G R_0 + Ah.$$

Since $Ah = AA^* E^{-1} e = e = -q_G R_0$ on $U_{\mathrm{core}}$, the right side vanishes.

*Size at the reference.* By Lemma 13.3, $H_\sigma(x) = \widehat{H}(\sigma x)$ with $\widehat{H}$ smooth, so $H_0 = \widehat{H}(0)$ is constant and $H_\sigma - H_0 = O_s(\sigma)$ in every $C^s$. At $G_\sigma$ the metric is $I$ and the connection is flat, so by (4.4)

$$D\gamma_G(dG\, X_0) = 2\,\mathrm{sym}\, dX_0 = -H_0,$$

and $R_0 = H_\sigma - H_0 = O_s(\sigma)$. The density formula is a multiplier of order one half in $y$ with fixed smooth masks in $z$, so, before any spatial cutoff,

$$\|v\|_{H^s_{z,y}} \le C_s \|h\|_{X^{s+1}}.$$

The uniform inverse and adjoint bounds for the normalized family, together with Sobolev embedding, give $h = O_s(\sigma)$, $v = O_s(\sigma)$ and $V_{\mathrm{base}} - dG\, X_0 = O_s(\sigma)$. At the single output order $k_{\mathrm{gap}}$ these estimates give $\|v\|_{C^{k_{\mathrm{gap}}}} \le C\|e\|_{H^m}$ for one fixed $m$, with a constant depending only on the lower bound and finitely many normalized seminorms; these input orders are included in $r_*$. We choose $\sigma$ so small that this bound is below the threshold of Proposition 13.5 with a fixed margin; then the corrected operator is coercive, $1+v_a \ge 1/2$, and $v = O_s(\sigma)$ gives uniform bounds for the corrected coefficients at every fixed order.

*Nearby immersions.* Now fix this positive $\sigma$. On a small $C^{r_*}$ neighborhood of $G_\sigma$, the finite-jet and flow formulas preserve these bounds and keep $\|e\|_{H^m}$ small, so $\|v\|_{C^{k_{\mathrm{gap}}}}$ stays small and the supports are unchanged.

*Derivative bounds.* Let $c$ include the losses of the density multiplier and of Sobolev embedding. After increasing $B$ once, the finite-jet and flow formulas give $\|e\|_{H^s} \le C_s L^{(s-r_0-c)_+}$. Proposition 13.6, applied with $r_0 + c$ in place of $r_0$ to the operator with constant initial amplitude $\sigma$, the density estimate and Sobolev embedding give $\|v\|_{C^s} + \|\mathcal{J}_G h_{\mathrm{im}}\|_{C^s} \le C_s L^{(s-r_0)_+}$, also for the linearized metric of the field. The positive square root and $dG\, X_0$ satisfy the same bounds by Lemma 7.2. This proves (13.13). Continuity of the normal operator gives the lower bound for the corrected scalar, and Proposition 13.6 applies to it. □

**Definition 13.8** (The operators with the corrected scalar)**.** We use the preceding operators with the positive scalar $r_G$ of Proposition 13.7, and write

$$A_G = (A_{\mathrm{im},G}, T_G), \quad \mathcal{F}_G = A_{\mathrm{im},G} A^*_{\mathrm{im},G} + 1 - \chi^2_{\mathrm{meas}}, \quad E_G = \mathcal{F}_G + T_G T_G^*,$$
$$E_{G,K} = \mathcal{F}_G + T_G P_K^2 T_G^*,$$

and $V^G_{\mathrm{aux}}$ for the field $V_{\mathrm{aux}}$ of these operators. The inverse estimates and the tail estimate of Proposition 13.6 apply to them.

13.6. **Comparison with the sampled waves.** The fixed weights identify the continuous operator with the leading part of the normal operator of the waves.

**Definition 13.9** (The associated family of waves)**.** Use the positive scalar $r_{G,a}$ of Proposition 13.7, the measure $\psi_a\,\mathrm{d}z$ and the cutoff $\rho_a$. In Theorem 12.4 take the phase $S_G$, the frame $Q_{G,i}$, the measurement cutoff $\chi_{\mathrm{meas}}$ and the cutoff $\rho_a$. The wave family has leading tensor $A_0(b_G)$ and satisfies the hypotheses of the sampling theorem by Corollary 11.8. We use the fixed radial factors, $\tau_a = (R_a\lambda)^{-1}$, and

$$\Lambda/\lambda < \text{the fixed margin of the phase charts.} \tag{13.16}$$

Every shifted covector stays in its common chart. We write $H_G = H_G[r_G^2]$ for the positive transported mean.

At $\tau = 0$, the amplitude in (12.3) is $\chi_{\mathrm{meas}}\rho_a|\nu_{G,a}|^2 q_G(x,\xi_{G,a})$, because $A_0^*A_0 = |\nu|^2\xi\xi^T$. Since $\psi_a\rho_a^2 = \rho_a^2$, summing the weighted real normal operators gives exactly $T_G P_\Lambda^2 T_G^*$, the factor $4\,\mathrm{Re}$ coming from real adjunction (Corollary 8.5). The image part $\mathcal{F}_G$ is added unchanged.

## 14. Correction of the mean

In this section we construct the part of the metric correction which is not oscillatory. The primary field of the metric correction is a weighted sum of real parts of complex characteristic waves $Z_\nu$. If $dZ = e^{\mathrm{i}\vartheta}A$ with $\vartheta$ real, then

$$\frac{1}{2\pi}\int_0^{2\pi}\left(d\sqrt{2}\,\mathrm{Re}(e^{\mathrm{i}\theta}Z)\right)^T d\sqrt{2}\,\mathrm{Re}(e^{\mathrm{i}\theta}Z)\,d\theta = \mathrm{Re}(A^*A),$$

since the remaining term $\mathrm{Re}(e^{2\mathrm{i}(\vartheta+\theta)}A^TA)$ has mean zero in $\theta$. We call the Hermitian tensor $\mathrm{Re}(A^*A)$ the *mean* of the wave; it is the average over the phase, not an average over the spatial domain. The quadratic tensor of the primary field is the weighted sum of these means plus oscillatory products; the latter are treated in Section 15. Here we choose the initial data of the waves and a real ambient field $V$ so that the linearized metric $D\gamma_F V$ plus the weighted sum of the means equals the rescaled smoothed defect up to an error $O(R^{-Q})$. The result is Proposition 14.9, which is stated and proved in Section 14.8.

14.1. **Scales and the sample.** Let $R$ be large and put

$$\lambda = R^a, \qquad h = R^{-\beta}, \qquad K_{\max} = R^{s_f}, \qquad \beta = (1+\epsilon)/3, \tag{14.1}$$

where $0 < \epsilon < 1/8$, $1 < s_f < 1+\epsilon$ and $1+\beta < a < 3/2$. Here $\lambda$ is the frequency of the waves, $h$ is the separation of the sample, and $K_{\max}$ is a fixed upper bound for the Fourier support of the initial data. The condition $s_f > 1$ places $K_{\max}$ above the derivative scale $R$ of the sources constructed below, while $s_f < 1+\epsilon$ places it below the number $h^{-3} = R^{1+\epsilon}$ of samples, which is needed for the concentration of the sampled normal operator. Section 19 verifies all required inequalities between these exponents at once.

*The positive initial amplitude and the sample.* We use the positive initial amplitude $r_F$ and the tangent field $V_{\mathrm{base}}$ of Proposition 13.7, for which the leading transported mean

of the waves, together with $D\gamma_F V_{\mathrm{base}}$, has zero projection onto $K_F$; this is what permits target defects of either sign. We sample the unmodulated waves, that is, the waves with initial data the Dirac mass $\delta_0$ and this initial amplitude, by Construction 12.1 on the three-parameter characteristic patches. This produces $M$ samples with weights $\mu_\nu$ such that

$$M \le CR^{1+\epsilon}, \qquad \mu_\nu > 0, \qquad \sum_\nu \mu_\nu \le C, \tag{14.2}$$

separated by at least $h/4$ within each patch. One realization in the event of Theorem 12.4 approximates the continuous normal operator with the input cutoff at $K_{\max}$ on the whole output space, and the continuous mean with error $O(h)$ in every Sobolev norm.

14.2. **The fixed operators and scales.** In the rest of this section the symbol $Q$ denotes the requested accuracy, $Q_i(F)$ the projected frame, and $\mathcal{Q}$ the quadratic operator defined below. Three normal operators occur: the continuous operator $E_F$ of Definition 13.8 at $G = F$, without input cutoff, which is inverted; the continuous operator $E_{F,K}$ with the input cutoff $P_K$; and the sampled operator $\widetilde{E}_K$ defined in (14.4) below, which carries the same input cutoff.

**Setup 14.1** (Operators and scales)**.** We use Definitions 13.1 and 13.8 at the immersion $F$, on the domains

$$D \Subset U_{\mathrm{fin}} \Subset U_{\mathrm{core}} \Subset U_{\mathrm{meas}} \Subset W$$

of Setup 3.3, with the same auxiliary spaces. Write $\mathcal{C} = \chi_{\mathrm{meas}} q_F$. For the positive initial amplitude $r_F$ of Proposition 13.7 (with $G = F$), set

$$A_F = (A_{\mathrm{im},F}, T_F), \qquad \mathcal{F}_F = A_{\mathrm{im},F} A_{\mathrm{im},F}^* + 1 - \chi_{\mathrm{meas}}^2, \qquad E_F = \mathcal{F}_F + T_F T_F^*,$$
$$E_{F,K} = \mathcal{F}_F + T_F P_K^2 T_F^*,$$

where $P_K$ is the initial-frequency cutoff (12.2). Let $H_F = H_F[r_F^2]$ be the positive transported mean and $V_{\mathrm{base}}$ the base field of Proposition 13.7, and let $V_{\mathrm{aux}}^F$ be the auxiliary field of Proposition 13.6, together with the bounds for its linearized metric $D\gamma_F V_{\mathrm{aux}}^F$ and its fixed physical cutoff.

Let $m_0 = \lfloor n/2 \rfloor + 2$, $s_0 = 2m_0$ and $A = m_0 + \frac{1}{2}$, as in (19.3), and assume

$$0 < \epsilon < \tfrac{1}{8}, \quad \beta = (1+\epsilon)/3, \quad 1 < s_f < 1 + \epsilon, \quad 1 + \beta < a < \tfrac{3}{2},$$
$$\lambda = R^a, \quad h = R^{-\beta}, \qquad T = R^t, \quad 2/3 < t < 1. \tag{14.3}$$

Before the sample is drawn we fix the input frequency bound

$$K = K_{\max} = R^{s_f};$$

all wave data below have Fourier support in this range. The immersion satisfies $\|F\|_{C^r} \le C_r T^{(r-r_*)_+}$, and its positive initial scalar, its phases and its corrected waves satisfy derivative bounds of the same form, with fixed shifts of the index $r_*$. We choose $r_*$ so large that (13.13) gives $r_F$ the derivative bounds required below. The data in the patch variables are fixed before the spatial source cutoffs and the cutoffs in the initial variables are applied, and $1 - \chi_{\mathrm{meas}}^2 = 0$ on a neighborhood of $\overline{U_{\mathrm{fin}}}$, in particular on $D$.

The order of choices is the following. The patch labels and the coefficient neighborhood are fixed first. The low derivative orders and the indices in the derivative bounds depend only on $n$ and on the fixed margins between exponents. The integer $r_*$ is chosen above all required orders before the sample, the accuracy $Q$ and the target dilation are chosen.

14.3. **The sampled operators and their corrections.** The $\nu$-th sample has patch label $a(\nu)$, patch parameter $z_\nu$, base covector $\zeta_\nu = \zeta_{a(\nu)}(z_\nu)$, weight $\mu_\nu$ and frequency $\lambda_\nu = \tau_\nu^{-1}$, $\tau_\nu = (R_{a(\nu)}\lambda)^{-1}$. We write $\mathcal{L}_\nu$ for the complex-linear map $m \mapsto Z_\nu(m)$ of (11.24).

**Definition 14.2** (The sampled corrections)**.** The number of terms of the finite correction in Corollary 11.8 is fixed before sampling, and the positive initial amplitude $r_F$ is used in the resulting maps $\mathcal{L}_\nu$ of Section 11. In this section constants may depend on this number and on the finite number of iteration steps; the input frequency bound, $r_*$, $R_{\mathrm{env}}$ and the exponents do not. Set

$$\vartheta_\nu = S_F(x,\zeta_\nu)/\tau_\nu = \lambda_\nu\phi_\nu, \qquad B_\nu = e^{-\mathrm{i}\vartheta_\nu} d\mathcal{L}_\nu(\delta_0),$$

so that $B_\nu$ is the demodulated derivative of the unmodulated wave. A function $f$ of the initial variables is identified with the measure $dm = \widehat{f}\,d\eta$. On the real Hilbert space underlying $L^2(d\eta)$ define $\mathcal{B}_K$ by $\widehat{\mathcal{B}_K v} = |\eta|^{1/2} P_K v$. These compactly supported Fourier densities are integrable by the Cauchy–Schwarz inequality, so Proposition 11.9 defines the corresponding waves for every $L^2$ input. Write $\rho_\nu = \rho_{a(\nu)}(z_\nu, \widehat{\eta})$ and, with the high-pass multiplier $\chi_{\mathrm{hi}}$ of Definition 13.4, put

$$B^{\mathrm{in}}_{\nu,K} = \rho_\nu \chi_{\mathrm{hi}} \mathcal{B}_K, \qquad G_{\nu,K} v = e^{-\mathrm{i}\vartheta_\nu} d\mathcal{L}_\nu(B^{\mathrm{in}}_{\nu,K} v), \qquad A_{\nu,K} v = \mathcal{C}[2\,\mathrm{sym}\,\mathrm{Re}(B_\nu^* G_{\nu,K} v)].$$

Thus $A_{\nu,K}$ is the linearization at $\delta_0$ of the reduced mean of the $\nu$-th wave. For any family $V = (V_\nu)_\nu$ of derivative amplitudes, let $\widetilde{A}_K[V]$ be the operator from the direct sum of the input spaces to the reduced output space given by

$$\widetilde{A}_K[V](v_\nu)_\nu = \sum_\nu \sqrt{\mu_\nu}\,\mathcal{C}[2\,\mathrm{sym}\,\mathrm{Re}(V_\nu^* G_{\nu,K} v_\nu)],$$

and put

$$\widetilde{A}_K = \widetilde{A}_K[(B_\nu)_\nu], \qquad \widetilde{E}_K = \mathcal{F}_F + \widetilde{A}_K \widetilde{A}_K^* = \mathcal{F}_F + \sum_\nu \mu_\nu A_{\nu,K} A_{\nu,K}^*. \tag{14.4}$$

The operator $\widetilde{A}_K[V]$ is an aggregated operator: it acts on the whole family of inputs at once, and its norm is controlled by the sampling theorem, whereas each individual operator $G_{\nu,K}$ has norm of order $K^{1/2}$ (Proposition 8.4). This distinction is used in the proof of Proposition 14.6. For a real source $r$, define the initial data and the demodulated derivatives

$$f_\nu[r] = B^{\mathrm{in}}_{\nu,K} A^*_{\nu,K} E_F^{-1} r, \qquad D_\nu r = e^{-\mathrm{i}\vartheta_\nu} d\mathcal{L}_\nu(f_\nu[r]) = G_{\nu,K} A^*_{\nu,K} E_F^{-1} r. \tag{14.5}$$

If the initial data of the $\nu$-th wave are changed from $\delta_0$ to $\delta_0 + \widehat{f_\nu[r]}\,d\eta$, the weighted mean changes by $\sum_\nu \mu_\nu$ times the exact tensor increment

$$2\,\mathrm{sym}\,\mathrm{Re}(B_\nu^* D_\nu r) + \mathrm{Re}((D_\nu r)^* D_\nu r). \tag{14.6}$$

Together with the auxiliary field $V^F_{\mathrm{aux}}(r)$, the linear part of this change equals $\widetilde{E}_K E_F^{-1} r$ on $U_{\mathrm{core}}$ after application of $q_F$; see Lemma 14.3. Both real quadratic forms are taken with respect to the stated Hilbert space inner products. No inverse of a sample weight occurs. The spatial cutoffs in the initial variables are applied after these data are formed, and observation cutoffs are used in norms only. We use the domains of Setup 3.3; the set $D_{\mathrm{lag}}$ contains the measurement supports, the product source supports and the physical output supports. The maps $\mathcal{L}_\nu$, the unmodulated waves, the adjoints and the operator $\widetilde{A}_K$ are the same throughout the iteration below.

14.4. **The equation for the source.** We now derive the equation that the source $r$ must satisfy. Put

$$\mathcal{E} = \widetilde{E}_K E_F^{-1}, \qquad \mathcal{Q}(u,v) = \mathcal{C} \sum_\nu \mu_\nu \operatorname{sym} \operatorname{Re}((D_\nu u)^* D_\nu v), \tag{14.7}$$

and

$$r_0 = \mathcal{C}\Big[H_F - \sum_\nu \mu_\nu \operatorname{Re}(B_\nu^* B_\nu)\Big], \qquad g = r_0 + \mathcal{C}(e_R/b^2), \tag{14.8}$$

where $e_R = S_{R,m}(e|_D)$ and $b = \delta R^A$ are the smoothed defect and the amplitude of Proposition 14.9. The operator $\mathcal{Q}$ is real bilinear and symmetric, because $D_\nu$ is real linear and $\operatorname{Re}(X^*Y)^T = \operatorname{Re}(Y^*X)$ for complex matrices, so that $\operatorname{sym}\operatorname{Re}(X^*Y) = \frac{1}{2}[\operatorname{Re}(X^*Y) + \operatorname{Re}(Y^*X)]$ is symmetric in $X, Y$. In particular

$$\mathcal{Q}(u,u) - \mathcal{Q}(v,v) = \mathcal{Q}(u-v, u+v). \tag{14.9}$$

**Lemma 14.3** (The reduced mean)**.** *Let $r \in H^\infty(\mathbb{R}^n;\mathbb{R}^n)$ be real, let the $\nu$-th wave have the initial data $m_\nu = \delta_0 + \widehat{f}_\nu[r]\,d\eta$, and put*

$$\mathcal{H} = \sum_\nu \mu_\nu \operatorname{Re}\Big((dZ_\nu)^* dZ_\nu\Big), \qquad Z_\nu = \mathcal{L}_\nu(m_\nu), \qquad V_0 = V_{\mathrm{base}} + V^F_{\mathrm{aux}}(r).$$

*Then, on $U_{\mathrm{fin}}$,*

$$q_F\Big(D\gamma_F V_0 + \mathcal{H}\Big) = \mathcal{E} r + \mathcal{Q}(r,r) - r_0. \tag{14.10}$$

*Proof.* By linearity of $\mathcal{L}_\nu$ and (14.5), $e^{-\mathrm{i}\vartheta_\nu} dZ_\nu = B_\nu + D_\nu r$. Since $\vartheta_\nu$ is real, the phase factor cancels in $(dZ_\nu)^* dZ_\nu$, and

$$\operatorname{Re}\Big((dZ_\nu)^* dZ_\nu\Big) = \operatorname{Re}(B_\nu^* B_\nu) + 2\operatorname{sym}\operatorname{Re}(B_\nu^* D_\nu r) + \operatorname{Re}((D_\nu r)^* D_\nu r),$$

which is (14.6). On $U_{\mathrm{fin}}$ we have $\chi_{\mathrm{meas}} = 1$, hence $q_F = \mathcal{C}$ there. By the definitions of $A_{\nu,K}$ and $D_\nu$,

$$\mathcal{C}\sum_\nu \mu_\nu 2\operatorname{sym}\operatorname{Re}(B_\nu^* D_\nu r) = \sum_\nu \mu_\nu A_{\nu,K} A_{\nu,K}^* E_F^{-1} r = \widetilde{A}_K \widetilde{A}_K^* E_F^{-1} r,$$

and $\mathcal{C}\sum_\nu \mu_\nu \operatorname{Re}((D_\nu r)^* D_\nu r) = \mathcal{Q}(r,r)$. By Proposition 13.6, applied to the operators of Definition 13.8, $q_F D\gamma_F V^F_{\mathrm{aux}}(r) = \mathcal{F}_F E_F^{-1} r$ on $U_{\mathrm{core}}$ (here, and in the lemma, $V^F_{\mathrm{aux}}(r)$ is the field before its fixed physical cutoff; this cutoff equals one near $D$, so nothing

changes on $D$). By (13.12), $q_F D\gamma_F V_{\text{base}} = -q_F H_F$ on $U_{\text{fin}}$. Adding these identities gives, on $U_{\text{fin}}$,

$$q_F(D\gamma_F V_0 + \mathcal{H}) = -q_F H_F + q_F \sum_\nu \mu_\nu \operatorname{Re}(B_\nu^* B_\nu) + (\mathcal{F}_F + \widetilde{A}_K \widetilde{A}_K^*) E_F^{-1} r + \mathcal{Q}(r,r),$$

which is (14.10). □

By (14.10), the projection of the mean error onto $K_F$ vanishes exactly when the source solves the quadratic equation

$$g = \mathcal{E} r + \mathcal{Q}(r,r). \tag{14.11}$$

The component of the error orthogonal to $K_F$ is then removed algebraically by a normal field; see Section 14.7. We do not solve (14.11) exactly; a finite number of steps of a Picard iteration suffices. The reasons are the following.

(i) The operator $\mathcal{E}$ is close to the identity on the sources that occur, but not on all of $L^2$. Indeed $E_F E_F^{-1} = I$, so

$$I - \mathcal{E} = (E_F - E_{F,K}) E_F^{-1} + (E_{F,K} - \widetilde{E}_K) E_F^{-1}.$$

The second term has norm $O(R^{-\omega})$ on $L^2$ for a fixed $\omega > 0$, by the concentration of the sampled normal operator and the comparison of phases in Theorem 12.4; both operators carry the same input cutoff $P_K$. The first term is the part of the continuous operator above the input frequency $K$. It is not small in operator norm, but by (13.8) it is $O(h(R/K)^q)$ in $L^2$, for every $q$, on sources $u$ with $\|u\|_{H^q} \le C_q h R^{(q-s_0)_+}$.

(ii) The operator $\mathcal{Q}$ is quadratic and small: it satisfies $\|\mathcal{Q}(u,v)\|_2 \le C\|u\|_2 \|v\|_{H^{s_0/2}}$, and the source $g$ has size $O(h)$.

(iii) Consequently the map $r \mapsto g + (I - \mathcal{E})r - \mathcal{Q}(r,r)$, whose fixed points are the solutions of (14.11), contracts differences of iterates in $L^2$ by a factor $CR^{-\omega}$, up to the frequency tail, as long as the iterates stay in the class of (i). Each step gains $R^{-\omega}$, and $J$ steps give an $L^2$ error $O(h[R^{-\omega J} + (R/K)^q])$.

(iv) High Sobolev norms are not contracted, but they remain bounded: every iterate satisfies $\|r^{(j)}\|_{H^s} \le C_{j,s} h R^{(s-s_0)_+}$ for all $s$, with constants depending on $j$. Since only finitely many steps are taken, this suffices. An interpolation inequality between $L^2$ and $H^{s_0}$, $s_0 > n$, then turns the $L^2$ smallness into uniform smallness.

The bounds in (i) and (ii), and their higher-order versions, are proved in Proposition 14.6; the iteration is carried out in Proposition 14.7.

14.5. **Estimates for the sampled corrections.** We first prove two elementary inequalities.

**Lemma 14.4.** *Let $s_0 \ge 0$, and let $s, i, j \ge 0$ be real numbers with $i + j \le s$ and $j \le s/2$. Then*

$$(s - i - j - s_0)_+ + (i - s_0)_+ + (j - s_0/2)_+ \le (s - s_0)_+.$$

*Proof.* Put $x = s-j-s_0$. We first show $(x-i)_+ + (i-s_0)_+ \le x_+$. If $i \le s_0$, the left side is $(x-i)_+ \le x_+$, since $i \ge 0$. If $i > s_0$ and $x \ge i$, the left side is $x-i+i-s_0 = x-s_0 \le x_+$. If $i > s_0$ and $x < i$, the left side is $i-s_0$, and $i-s_0 \le s-j-s_0 = x$ because $i+j \le s$.

It remains to show $(s-j-s_0)_+ + (j-s_0/2)_+ \le (s-s_0)_+$. If $j \le s_0/2$, this follows from $(s-j-s_0)_+ \le (s-s_0)_+$. If $j > s_0/2$, then $s \ge 2j > s_0$. When $s-j-s_0 \ge 0$ the left side is $s - \frac{3}{2}s_0 \le s-s_0$; when $s-j-s_0 < 0$ it is $j - \frac{s_0}{2} \le \frac{s-s_0}{2} \le s-s_0$. □

**Lemma 14.5.** *Let $s_0 > n$. There is a constant $C$ such that, for all $f \in H^{s_0}(\mathbb{R}^n)$,*

$$\|f\|_{C^0} \le C\|f\|_2^{1/2}\|f\|_{H^{s_0}}^{1/2}.$$

*Proof.* We use $\|f\|_{H^{s_0}} = \|\langle D\rangle^{s_0} f\|_2$; integer Sobolev norms are equivalent. We may assume $f \neq 0$. Put $X = \|f\|_2$ and $Y = \|f\|_{H^{s_0}}$, so that $X \le Y$. With the Fourier normalization of Section 3, $f(x) = \int e^{\mathrm{i}x\cdot\xi}\widehat{f}(\xi)\,d\xi$ and $\|\widehat{f}\|_2 = (2\pi)^{-n/2}\|f\|_2$; we show below that $\widehat{f} \in L^1$, so $f$ has a continuous representative with $\|f\|_{C^0} \le \|\widehat{f}\|_{L^1}$. For $\Theta > 0$ we split the frequencies. By the Cauchy–Schwarz inequality,

$$\int_{|\xi|\le\Theta} |\widehat{f}|\,d\xi \le |B_\Theta|^{1/2}\|\widehat{f}\|_2 \le C\Theta^{n/2}X,$$

and, since $2s_0 > n$,

$$\int_{|\xi|>\Theta} |\widehat{f}|\,d\xi \le \Big(\int_{|\xi|>\Theta} |\xi|^{-2s_0}\,d\xi\Big)^{1/2}\Big(\int |\xi|^{2s_0}|\widehat{f}|^2\,d\xi\Big)^{1/2} \le C\Theta^{n/2-s_0}Y.$$

Choose $\Theta = (Y/X)^{1/n}$. The first bound becomes $CX^{1/2}Y^{1/2}$, and the second becomes

$$C(Y/X)^{1/2-s_0/n}Y = CX^{1/2}Y^{1/2}(X/Y)^{s_0/n-1} \le CX^{1/2}Y^{1/2},$$

since $s_0/n > 1$ and $X \le Y$. □

**Proposition 14.6** (Estimates for the sampled corrections)**.** *Put $\ell = m_0 - 1 = \lfloor n/2\rfloor + 1$. There is a fixed $B_{\mathrm{can}}$ such that, if*

$$r_* \ge s_0 + B_{\mathrm{can}}, \qquad R_{\mathrm{env}} \ge s_0 + B_{\mathrm{can}} + \ell + 2 + r_{\mathrm{par}},$$

*then the following bounds hold for every realization of the sample in the event of Theorem 12.4.*

(i) *For all smooth sources $u, v$,*

$$\|\mathcal{Q}(u,v)\|_2 \le C\|u\|_2\|v\|_{H^{s_0/2}}. \tag{14.12}$$

(ii) *For all integers $s \ge 0$,*

$$\|\mathcal{E}u\|_{H^s} \le C_s[\|u\|_{H^s} + T^{(s-s_0)_+}\|u\|_2]. \tag{14.13}$$

(iii) *If $u, v$ satisfy $\|u\|_{H^s} + \|v\|_{H^s} \le C_s h R^{(s-s_0)_+}$ for all $s$, then*

$$\|\mathcal{Q}(u,v)\|_{H^s} \le C_s h^2 R^{(s-s_0)_+}, \qquad s \ge 0, \tag{14.14}$$

*where $C_s$ depends on the constants in the hypothesis.*

(iv) *For all integers* $s \ge 0$,

$$\begin{aligned}\sup_{\nu} \|D_\nu u\|_{H^s} + \|V^F_{\mathrm{aux}}(u)\|_{H^s} + \|D\gamma_F V^F_{\mathrm{aux}}(u)\|_{H^s} \\ \le C_s[\|u\|_{H^{s+1}} + T^{(s+1-s_0)_+}\|u\|_2].\end{aligned} \tag{14.15}$$

(v) *The measures* $\widehat{f}_\nu[u]\,d\eta$ *have support in* $\{|\eta| \le CK\}$ *and total variation at most* $C\|u\|_{H^{m_0}}$.

*The bounds hold on the fixed supports and are uniform in* $K$. *No regularity with respect to the sample index* $\nu$ *is used.*

*Proof. Step 1 (Operator bounds).* We use the scalar auxiliary operator of Theorem 12.4, with a cutoff $\chi_*$ equal to one near every measurement support and near the supports of the fixed-parameter operators. Its sampled operator $\mathbb{T}_K$ is bounded on the event of that theorem. By positivity this bounds every operator with a smaller cutoff. For an amplitude $b(x,\nu,\eta)$ supported where $\chi_* = 1$, let $\mathbb{T}_K[b]$ be the operator whose kernel is multiplied by $b$. Lemma 8.2 gives

$$\|\mathbb{T}_K[b]\| \le C \sup_{\nu,\eta} \|b(\cdot,\nu,\eta)\|_{H^\ell}. \tag{14.16}$$

Only physical derivatives of $b$ occur; no derivatives in the initial frequency or in the sample index are needed. In particular $\|\widetilde{A}_K\| = \|\widetilde{A}_K^*\| \le C$, and $\|E_F^{-1}\| \le C$ by Proposition 13.5.

*Step 2 (Moment estimates for the adjoint).* The exact phase of the $\nu$-th operator is $\Phi_\nu = Y_\nu \cdot \eta$, where

$$Y_\nu = \int_0^1 d_\zeta S_F(x, \zeta_\nu + t\tau_\nu\eta)\,dt,$$

and its $y$-derivative is uniformly close to the identity. Hence the identities (13.9) hold, with a matrix $\mathsf{B}_\nu$, uniformly invertible, such that $d_y\Phi_\nu = \mathsf{B}_\nu\eta$. Consequently

$$\eta_k e^{-\mathrm{i}\Phi_\nu} = \mathrm{i}\sum_l (\mathsf{B}_\nu^{-1})_{kl}\,\partial_{y_l} e^{-\mathrm{i}\Phi_\nu}.$$

The adjoint $A^*_{\nu,K}w$ is an integral of $e^{\mp\mathrm{i}\Phi_\nu}$ against an amplitude times $\mathcal{C}^*w = \chi_{\mathrm{meas}} q_F^* w$, which has compact support in the measurement region. Multiplying by $\eta_k$ and integrating by parts in $y$ therefore produces no boundary terms and moves one derivative onto $\mathsf{B}_\nu^{-1}$, onto the amplitude, or onto $w$. Thus one factor of the input frequency is exchanged for one physical derivative of the source or of the coefficients.

For an integer $q \ge 0$ we have $|\eta|^{2q} = \sum_{|\alpha|=q} \frac{q!}{\alpha!}\eta^{2\alpha}$, so

$$\||\eta|^q f\|_2 \le C_q \sum_{|\alpha|=q} \|\eta^\alpha f\|_2,$$

and it suffices to treat the monomials $\eta^\alpha$. After $q$ integrations by parts, a term with $i$ derivatives on $w$ carries $q-i$ derivatives on the coefficients. It is an operator of the same form as $\widetilde{A}_K^*$ with a modified amplitude, applied to a derivative of order $i$ of $w$. We bound it by (14.16), which costs $\ell$ further derivatives of the amplitude. The coefficients satisfy derivative bounds of the form $C_m T^{(m-r_*+B)_+}$ with fixed shifts $B$; including these shifts,

the amplitude test and the shifts of the inverse and of the corrected waves in $B_{\mathrm{can}}$, the condition $r_* \ge s_0 + B_{\mathrm{can}}$ gives the coefficient factor $T^{(q-i-s_0)_+}$. With $w = E_F^{-1}u$ and the inverse estimate (13.7) (with $L = T$ and $r_0 = s_0$), $\|E_F^{-1}u\|_{H^i} \le C_i[\|u\|_{H^i} + T^{(i-s_0)_+}\|u\|_2]$. Since $(q-i-s_0)_+ + (i-s_0)_+ \le (q-s_0)_+$, we obtain

$$\||\eta|^q \widetilde{A}_K^* E_F^{-1}u\|_2 \le C_q \sum_{i=0}^{q} T^{(q-i-s_0)_+}\|u\|_{H^i}. \tag{14.17}$$

For an individual adjoint $A_{\nu,K}^*$ we remove the weight $|\eta|^{1/2}$ and use the bound for the fixed-parameter operator without this weight in Proposition 8.4. The same argument shows that the right side of (14.17) also bounds $\sup_\nu \||\eta|^{q-1/2}A_{\nu,K}^* E_F^{-1}u\|_2$; we call this the individual moment of order $q$.

*Step 3 (Proof of (14.13)).* A derivative of order $s$ of the forward operator $\widetilde{A}_K$ produces at most $s$ factors of the input frequency, from derivatives of $e^{\mathrm{i}\Phi_\nu}$; a term with $k$ such factors carries a coefficient factor $T^{(s-k-s_0)_+}$. By (14.16),

$$\|\widetilde{A}_K \widetilde{A}_K^* E_F^{-1}u\|_{H^s} \le C_s \sum_{k=0}^{s} T^{(s-k-s_0)_+}\||\eta|^k \widetilde{A}_K^* E_F^{-1}u\|_2.$$

The products of the differentiated coefficients obey (7.6), so these operations use the same $B_{\mathrm{can}}$. Inserting (14.17) and using $(s-k-s_0)_+ + (k-i-s_0)_+ \le (s-i-s_0)_+$ for $0 \le i \le k \le s$, we obtain terms $C_s T^{(s-i-s_0)_+}\|u\|_{H^i}$, $0 \le i \le s$. By (13.6) with $L = T$ and $r_0 = s_0$, each of them is at most $C_s[\|u\|_{H^s} + T^{(s-s_0)_+}\|u\|_2]$. The term $\mathcal{F}_F E_F^{-1}u$ satisfies the same bound by (13.7). This proves (14.13).

*Step 4 (Proof of (14.15) and of (v)).* By (14.5), $D_\nu u = G_{\nu,K}A_{\nu,K}^* E_F^{-1}u$, and $G_{\nu,K}$ is a fixed-parameter operator with input weight $|\eta|^{1/2}$. By the bound for the unweighted fixed-parameter operator in Proposition 8.4, a term of an order-$s$ derivative of $D_\nu u$ with $k$ frequency factors is bounded by $C_s T^{(s-k-s_0)_+}\||\eta|^{k+1/2}A_{\nu,K}^* E_F^{-1}u\|_2$, that is, by the individual moment of order $k+1$. Step 2 and the endpoint inequality (13.6) at order $s+1$ give the bound for $D_\nu u$ in (14.15). The auxiliary field and its linearized metric obey the same bound by Proposition 13.6.

For the measures, $\widehat{f}_\nu[u] = \rho_\nu \chi_{\mathrm{hi}}|\eta|^{1/2}P_K A_{\nu,K}^* E_F^{-1}u$ vanishes for $|\eta| \le 1$ and for $|\eta| \ge 2K$, which gives the support assertion. Since the initial variables lie in $\mathbb{R}^{n-1}$ and $2(m_0-1) = 2\lfloor n/2\rfloor + 2 > n-1$, the Cauchy–Schwarz inequality gives

$$\int |\widehat{f}_\nu[u]|\,d\eta \le C\Big(\int_{|\eta|\ge 1} |\eta|^{2(1-m_0)}d\eta\Big)^{1/2}\||\eta|^{m_0-1/2}A_{\nu,K}^* E_F^{-1}u\|_2 \le C\|u\|_{H^{m_0}},$$

by the individual moment of order $m_0$ and $m_0 < s_0$.

*Step 5 (Proof of (14.12)).* By the definitions and the symmetry of $\mathcal{Q}$,

$$\begin{aligned}\widetilde{A}_K[(D_\nu v)_\nu]\widetilde{A}_K^* E_F^{-1}u &= \sum_\nu \mu_\nu \mathcal{C}[2\,\mathrm{sym}\,\mathrm{Re}((D_\nu v)^* G_{\nu,K}A_{\nu,K}^* E_F^{-1}u)] \\ &= 2\mathcal{Q}(v,u) = 2\mathcal{Q}(u,v).\end{aligned}$$

The operator $\widetilde{A}_K[(D_\nu v)_\nu]$ is obtained from the aggregated operator of Step 1 by multiplying its amplitude by $(D_\nu v)^*$ and by coefficient factors with uniform $C^\ell$ bounds; the

input $\widetilde{A}_K^* E_F^{-1} u$ is the aggregated adjoint, which is bounded on $L^2$, and no individual operator $G_{\nu,K}$ is estimated separately. By (14.16) and (14.15) with $s = \ell$,

$$\|\widetilde{A}_K[(D_\nu v)_\nu]\| \le C \sup_\nu \|D_\nu v\|_{H^\ell} \le C\|v\|_{H^{m_0}},$$

since $\ell + 1 = m_0 < s_0$. Together with $\|\widetilde{A}_K^*\| \le C$ and $\|E_F^{-1}\| \le C$ this proves (14.12), because $m_0 = s_0/2$.

*Step 6 (Proof of (14.14)).* Fix $s$ and a multi-index $\sigma$ with $|\sigma| = s$. By the Leibniz rule, $\partial^\sigma \mathcal{Q}(u, v)$ is a finite sum of terms

(14.18)
$$(\partial^{\sigma_1}\mathcal{C}) \sum_\nu \mu_\nu \operatorname{sym} \operatorname{Re}\Big((\partial^{\sigma_2} D_\nu u)^* \, \partial^{\sigma_3} D_\nu v\Big), \qquad c = |\sigma_1|,\ r = |\sigma_2|,\ t = |\sigma_3|, \quad c+r+t = s.$$

Since $\mathcal{Q}$ is symmetric and the hypothesis is symmetric in $u$ and $v$, we may assume $t \le r$, hence $t \le s/2$.

The two factors are treated differently. The factor with $r$ derivatives cannot be estimated sample by sample: by Step 4, $\sup_\nu \|D_\nu u\|_{H^r}$ is controlled only through the individual moment of order $r+1$, that is, with one more derivative of $u$, and the resulting bound $h^2 R^{(s+1-s_0)_+}$ is not sufficient, since $hR \ge 1$. We therefore keep this factor inside the aggregated operator of Step 1 and place the other factor in its amplitude.

*The factor with $r$ derivatives.* Write $w_\nu = A_{\nu,K}^* E_F^{-1} u$, so that $D_\nu u = G_{\nu,K} w_\nu$. Each physical derivative of the phase factor $e^{\mathrm{i}\Phi_\nu}$ in $G_{\nu,K}$ produces one factor of the input frequency $\eta$, multiplied by a coefficient with bounded physical derivatives, because $\Phi_\nu = Y_\nu \cdot \eta$ with $Y_\nu$ as in Step 2, so that $d_x \Phi_\nu = (D_x Y_\nu)^T \eta$. Hence

$$\partial^{\sigma_2} D_\nu u = \sum_{|\kappa| \le r} G_{\nu,K}[\gamma_{\nu,\kappa}]\Big(\eta^\kappa w_\nu\Big),$$

where $G_{\nu,K}[\gamma]$ is the operator $G_{\nu,K}$ with its amplitude multiplied by $\gamma$, and the coefficient $\gamma_{\nu,\kappa}$ is a product of derivatives of the phase, of the amplitude of the wave and of the cutoffs, of total order $r - |\kappa|$. Including the shifts in $B_{\mathrm{can}}$, its first $\ell$ physical derivatives are bounded by $CT^{(r-|\kappa|-s_0)_+}$, uniformly in $\nu$ and $\eta$.

*The aggregated form.* Substituting this into (14.18) and recalling the definition of $\widetilde{A}_K[V]$ and $\widetilde{A}_K^* E_F^{-1} u = (\sqrt{\mu_\nu}\, w_\nu)_\nu$, the term with a given $\kappa$, $k = |\kappa|$, equals

$$\tfrac{1}{2}\, \widetilde{A}_K[b_\kappa]\Big(\eta^\kappa \widetilde{A}_K^* E_F^{-1} u\Big),$$

where $\widetilde{A}_K[b_\kappa]$ denotes the operator $\widetilde{A}_K[(\partial^{\sigma_3} D_\nu v)_\nu]$ in which $\mathcal{C}$ is replaced by $\partial^{\sigma_1}\mathcal{C}$ and $G_{\nu,K}$ by $G_{\nu,K}[\gamma_{\nu,\kappa}]$. It differs from the aggregated operator of Step 1 only by an amplitude factor $b_\kappa(x, \nu, \eta)$, built from the coefficients of $\partial^{\sigma_1}\mathcal{C}$, from $\gamma_{\nu,\kappa}$ and from $(\partial^{\sigma_3} D_\nu v)^*$. Thus the factor with $t$ derivatives, together with the differentiated coefficients, has become part of the amplitude of the aggregated operator. By (14.16), which uses $\ell$ physical derivatives of the amplitude, and by Step 4 at order $t + \ell$,

$$\|\widetilde{A}_K[b_\kappa]\| \le CT^{(c+r-k-s_0)_+} \sup_\nu \|D_\nu v\|_{H^{t+\ell}} \le CT^{(c+r-k-s_0)_+}\Big[\|v\|_{H^{t+m_0}} + T^{(t+m_0-s_0)_+}\|v\|_2\Big],$$

since $\ell+1=m_0$. By the hypothesis on $v$, $T<R$ and $s_0=2m_0$, the bracket is at most $C_t h R^{(t+m_0-s_0)_+} = C_t h R^{(t-s_0/2)_+}$. The input is controlled by the aggregated moment estimate (14.17) and the hypothesis on $u$:

$$\|\eta^\kappa \widetilde{A}_K^* E_F^{-1} u\|_2 \le C_k \sum_{i\le k} T^{(k-i-s_0)_+}\|u\|_{H^i} \le C_k h \sum_{i\le k} T^{(k-i-s_0)_+} R^{(i-s_0)_+}.$$

*The exponents.* Using $T<R$ and (7.5) to combine $(c+r-k-s_0)_+ + (k-i-s_0)_+ \le (c+r-i-s_0)_+$, each term is bounded by $C_s h^2 R^{\varepsilon_i}$ with

$$\varepsilon_i = (s-i-t-s_0)_+ + (i-s_0)_+ + (t-s_0/2)_+,$$

because $c+r=s-t$. Here $i\le k\le r$, so $i+t\le s$, and $t\le s/2$. Lemma 14.4, with $j=t$, gives $\varepsilon_i \le (s-s_0)_+$. Summing the finitely many terms proves (14.14).

The index $s_0/2$ in the last summand of $\varepsilon_i$ is what makes this distribution of derivatives possible: the factor carried in the amplitude loses $m_0=s_0/2$ derivatives (one for the individual operator and $\ell$ for the amplitude test), and this loss is compensated by $t\le s/2$. Only source norms of order at most $\max(s,s_0)$ occur, so the source regularity required in the iteration is fixed. □

## 14.6. **A finite iteration.**

**Proposition 14.7** (Finite correction of the mean)**.** *We work under Setup 14.1 and Definition 14.2, with the positive initial amplitude $r_F$, the base field $V_{\mathrm{base}}$, the continuous operator $A_F$ and the inverse $E_F^{-1}$, and we sample the unmodulated waves with the input frequency bound $K$. Let $\mathcal{E}$, $\mathcal{Q}$, $r_0$ and $g$ be as in (14.7)–(14.8). For every finite $Q$ and all sufficiently large $R$, the finite iteration*

$$r^{(0)}=0, \qquad r^{(j+1)} = g + (I-\mathcal{E})r^{(j)} - \mathcal{Q}(r^{(j)}, r^{(j)}), \tag{14.19}$$

*stopped after a suitable number $J$ of steps, gives a real source $r=r^{(J)}$ whose residual satisfies*

$$r_{\mathrm{rem}} = g - \mathcal{E}r - \mathcal{Q}(r,r), \qquad \|r_{\mathrm{rem}}\|_{C^0} \le R^{-Q}. \tag{14.20}$$

*The measures $\widehat{f}_\nu[r]\,d\eta$ have support in $\{|\eta|\le CK\}$ and total variation at most $C$. On the fixed supports, for every $s\ge 0$,*

$$\sup_\nu \|D_\nu r\|_{C^s} + \|V^F_{\mathrm{aux}}(r)\|_{H^s} + \|D\gamma_F V^F_{\mathrm{aux}}(r)\|_{H^s} \le C_{s,Q} R^s. \tag{14.21}$$

*The input frequency bound, $r_*$ and $R_{\mathrm{env}}$ do not depend on $Q$. The family of waves is fixed before the sample is drawn, and the sample does not depend on the number $J$ of Picard steps; $J$, the constants and the threshold for $R$ depend on $Q$.*

*Proof. Step 1 (The sample).* We apply Theorem 12.4 with $\Lambda = K = R^{s_f}$. Then $h^{-3} = R^{1+\epsilon} = \Lambda^{1+\epsilon_{\mathrm{samp}}}$ with $\epsilon_{\mathrm{samp}} = (1+\epsilon)/s_f - 1 > 0$, and $\Lambda^{-\epsilon_{\mathrm{samp}}/4} = R^{-(1+\epsilon-s_f)/4}$. We fix a realization of the sample in the event of that theorem; this event is nonempty for large $R$, since its probability tends to one. The concentration bound (12.8), the

comparison of phases (12.10), and the identification of the continuous operator at $\tau = 0$ with $T_F P_K^2 T_F^*$ (end of Section 13) give

$$\|\widetilde{E}_K - E_{F,K}\| \le C[R^{-(1+\epsilon-s_f)/4} + R^{-a+s_f}] \le CR^{-\omega}, \qquad \omega = \frac{1+\epsilon-s_f}{4}, \tag{14.22}$$

where $\omega$ is the exponent of (19.6); the second inequality uses $a - s_f > \omega$, verified in Section 19. The sample has at most $CR^{1+\epsilon}$ points, bounded total mass, weights at most $CR^{-1-\epsilon}$ and separation $ch$ within each patch. The orders $s_0 + B_{\mathrm{can}}$ and the shifts of the primary waves and of the background are included in the fixed order $r_*$.

*Step 2 (Bounds for the source).* We take $R_{\mathrm{env}} \ge s_0 + B_{\mathrm{can}} + \ell + 2 + r_{\mathrm{par}}$ in Corollary 11.8. The quadrature estimate (12.9), valid on the same event at all orders, and the comparison of the corrected and the leading amplitudes (11.18), together with $\lambda^{-1} \le h$, give

$$\|r_0\|_{H^s} \le C_s h T^{(s-s_0)_+}, \qquad \sup_\nu \|B_\nu\|_{C^s} + \|\mathcal{C}\|_{C^s} \le C_s T^{(s-s_0)_+}. \tag{14.23}$$

By (17.1) and $b^2 = \delta^2 R^{2A}$, the rescaled defect satisfies $\|e_R/b^2\|_{H^s} \le C_s R^{s-2A}$, and $2A = s_0 + 1$. By the Leibniz rule and (14.23), the zero extension of $\mathcal{C}(e_R/b^2)$ satisfies

$$\|\mathcal{C}(e_R/b^2)\|_{H^s(\mathbb{R}^n)} \le C_s \sum_{i \le s} T^{(s-i-s_0)_+} R^{i-s_0-1} \le C_s R^{s-s_0-1}.$$

Since $T < R$ and $R^{s-s_0-1} \le R^{-1} R^{(s-s_0)_+} \le h R^{(s-s_0)_+}$ (because $\beta < 1$), the complete source obeys

$$\|g\|_{H^s} \le C_s h R^{(s-s_0)_+} \qquad (s = 0, 1, \ldots). \tag{14.24}$$

*Step 3 (Bounds for the iterates).* Fix a finite $J$. We claim that

$$\|r^{(j)}\|_{H^s} \le C_{J,s} h R^{(s-s_0)_+}, \qquad 0 \le j \le J+1,\ s \ge 0. \tag{14.25}$$

This holds for $j = 0$. If it holds for $j$, then (14.13) and $T < R$ give $\|\mathcal{E} r^{(j)}\|_{H^s} \le C_s h R^{(s-s_0)_+}$, and (14.14) gives $\|\mathcal{Q}(r^{(j)}, r^{(j)})\|_{H^s} \le C_s h^2 R^{(s-s_0)_+}$. With (14.24) and (14.19) this gives the claim for $j+1$. Each of the finitely many steps changes only the constants. All iterates are real and belong to $H^\infty$.

*Step 4 (The difference equation).* Put $d_j = r^{(j+1)} - r^{(j)}$. Since $r^{(0)} = 0$ and $\mathcal{Q}(0,0) = 0$, $d_0 = r^{(1)} = g$. Subtracting two consecutive steps of (14.19) and using (14.9),

$$\begin{aligned} d_{j+1} &= (I - \mathcal{E}) d_j - \mathcal{Q}(r^{(j+1)}, r^{(j+1)}) + \mathcal{Q}(r^{(j)}, r^{(j)}) \\ &= (I - \mathcal{E}) d_j - \mathcal{Q}(d_j, r^{(j+1)} + r^{(j)}). \end{aligned} \tag{14.26}$$

For the linear term we use $E_F E_F^{-1} = I$:

$$I - \mathcal{E} = (E_F - E_{F,K}) E_F^{-1} + (E_{F,K} - \widetilde{E}_K) E_F^{-1}.$$

By (14.22) and the frequency tail estimate (13.8) with $L = T$ and $r_0 = s_0$, for every integer $q \ge 0$,

$$\begin{aligned} \|(I - \mathcal{E}) d_j\|_2 &\le C R^{-\omega} \|d_j\|_2 + C_q K^{-q} [\|d_j\|_{H^q} + T^{(q-s_0)_+} \|d_j\|_2] \\ &\le C R^{-\omega} \|d_j\|_2 + C_{J,q} h (R/K)^q, \end{aligned}$$

where in the second line we used (14.25) for $r^{(j+1)}$ and $r^{(j)}$ and $T < R$. For the quadratic term, (14.12) and (14.25) at order $s_0/2 < s_0$ give

$$\|\mathcal{Q}(d_j, r^{(j+1)} + r^{(j)})\|_2 \le C\|d_j\|_2\|r^{(j+1)} + r^{(j)}\|_{H^{s_0/2}} \le C_J h\|d_j\|_2.$$

Hence, for $j < J$,

$$\|d_{j+1}\|_2 \le [CR^{-\omega} + C_J h]\|d_j\|_2 + C_{J,q}h(R/K)^q.$$

Since $\beta > \omega$, we have $h \le R^{-\omega}$, and the bracket is at most $C_J R^{-\omega} \le 1$ for large $R$. Iterating from $\|d_0\|_2 = \|g\|_2 \le Ch$ and absorbing the finite powers of $C_J$ into $C_{J,q}$ gives

$$\|d_J\|_2 \le C_{J,q}h[R^{-\omega J} + (R/K)^q]. \tag{14.27}$$

*Step 5 (The residual and the choice of $J$ and $q$).* By (14.19), $d_J = r^{(J+1)} - r^{(J)} = g - \mathcal{E}r^{(J)} - \mathcal{Q}(r^{(J)}, r^{(J)})$, so the residual is exactly $r_{\mathrm{rem}} = d_J$. By (14.25), $\|d_J\|_{H^{s_0}} \le C_J h$. Since $s_0 > n$, Lemma 14.5 and (14.27) give

$$\|r_{\mathrm{rem}}\|_{C^0} \le C\|d_J\|_2^{1/2}\|d_J\|_{H^{s_0}}^{1/2} \le C_{J,q}h[R^{-\omega J} + (R/K)^q]^{1/2}.$$

Now $(R/K)^q = R^{-q(s_f-1)}$. We choose the number of steps $J$ with $\omega J > 2Q$. The integer $q$ is not part of the construction; it is only the order of the tail estimate, and we choose it with $q(s_f - 1) > 2Q$. Then the right side is at most $C_{J,q}R^{-Q-\kappa}$ for some $\kappa > 0$, and after increasing the threshold for $R$ it is at most $R^{-Q}$. This proves (14.20).

*Step 6 (The fields and the measures).* By (14.15) at order $s + \ell$, the Sobolev embedding $H^{s+\ell} \hookrightarrow C^s$ and (14.25),

$$\sup_\nu \|D_\nu r\|_{C^s} \le C_s[\|r\|_{H^{s+m_0}} + T^{(s+m_0-s_0)_+}\|r\|_2] \le C_{J,s}hR^{(s+m_0-s_0)_+} \le C_{s,Q}R^s,$$

since $\ell + 1 = m_0$ and $s_0 = 2m_0$, so that $(s + m_0 - s_0)_+ = (s - m_0)_+ \le s$. The auxiliary field and its linearized metric obey the same bounds in $H^s$. By Proposition 14.6(v) the total variation of the data is at most $C\|r\|_{H^{m_0}} \le C_J h$, hence bounded, and the support is still determined by the cutoff $P_K$. This proves (14.21). □

14.7. **The normal component.** Let $r$ be the source of Proposition 14.7. Put $m_\nu = \delta_0 + \widehat{f}_\nu[r]\,d\eta$, let $\mathcal{H} = \sum_\nu \mu_\nu \operatorname{Re}((dZ_\nu)^* dZ_\nu)$ be the Hermitian tensor of the waves $Z_\nu = \mathcal{L}_\nu(m_\nu)$, and write $V_0 = V_{\mathrm{base}} + V_{\mathrm{aux}}^F(r)$. On $U_{\mathrm{fin}}$ set

$$T_f = e_R/b^2 - D\gamma_F V_0 - \mathcal{H},$$

and define

$$V = V_0 - \tfrac{1}{2}N_F T_f, \tag{14.28}$$

where $N_F$ is the normal map of (4.2).

**Lemma 14.8** (The error of the mean)**.** *On $U_{\mathrm{fin}}$, in particular on $D$,*

$$q_F T_f = r_{\mathrm{rem}}, \tag{14.29}$$

*and the tensor $E_{\mathrm{mean}}$ of (14.31) satisfies*

$$E_{\mathrm{mean}} = -\Pi_{K_F} T_f = -\sum_{i,j} Q_i(F)(\mathsf{G}_F^{-1})_{ij}(r_{\mathrm{rem}})_j. \tag{14.30}$$

*Proof.* On $U_{\mathrm{fin}}$ we have $q_F = \mathcal{C}$. By Lemma 14.3 and the definition of $g$,

$$q_F T_f = q_F(e_R/b^2) - \Big[\mathcal{E}r + \mathcal{Q}(r,r) - r_0\Big] = g - \mathcal{E}r - \mathcal{Q}(r,r) = r_{\mathrm{rem}}.$$

Next, by the definitions of $E_{\mathrm{mean}}$ and $T_f$,

$$E_{\mathrm{mean}} = D\gamma_F V_0 - \tfrac{1}{2} D\gamma_F(N_F T_f) + \mathcal{H} - e_R/b^2 = -T_f + D\gamma_F\Big(-\tfrac{1}{2} N_F T_f\Big).$$

The field $w = -\frac{1}{2} N_F T_f$ is normal, and by (4.4) with $u = 0$ its linearized metric is the algebraic expression $D\gamma_F w = -2\langle w, \mathrm{II}_F\rangle = -2B_F^* w$. Since $N_F = (B_F^*)^{-1}\Pi_{K_F^\perp}$, where $B_F^*$ maps the normal space isomorphically onto $K_F^\perp = \operatorname{im} B_F^*$,

$$D\gamma_F\Big(-\tfrac{1}{2} N_F T_f\Big) = B_F^*(B_F^*)^{-1}\Pi_{K_F^\perp} T_f = \Pi_{K_F^\perp} T_f.$$

Hence $E_{\mathrm{mean}} = -T_f + \Pi_{K_F^\perp} T_f = -\Pi_{K_F} T_f$. (This is also (4.5) with $u = 0$ and $h = T_f$.) Finally, the Gram formula for the orthogonal projection onto the span of the frame $Q_1(F), \dots, Q_n(F)$ reads

$$\Pi_{K_F} H = \sum_{i,j} Q_i(F)(\mathsf{G}_F^{-1})_{ij}(Q_j(F) : H),$$

since the right side lies in $K_F$ and its pairing with each $Q_k(F)$ equals

$$\sum_{i,j} (\mathsf{G}_F)_{ki}(\mathsf{G}_F^{-1})_{ij}(Q_j(F) : H) = Q_k(F) : H.$$

With $H = T_f$ and $(Q_j(F) : T_f)_j = q_F T_f = r_{\mathrm{rem}}$, this proves (14.30). □

14.8. **Correction of the mean.**

**Proposition 14.9** (Correction of the mean)**.** *There is a positive initial amplitude $r_F$, depending only on $F$, for which Proposition 11.10 applies and the following conclusions hold. Let $e = g - \gamma(F)$, with $\|e\|_{C^0(D)} \le \delta^2$ and $0 < \delta \le 1$. Fix a finite smoothing order $m \ge 1$ and put*

$$e_R = S_{R,m}(e|_D), \qquad b = \delta R^A,$$

*where $S_{R,m}$ is the operator of Lemma 17.1. Given a finite $Q$, at the scales of Section 19 and for sufficiently large $R$, there are a family of measures $m_\nu$ and a real ambient field $V$ with the following properties, where $Z_\nu$ is given by (11.24) with $m = m_\nu$.*

(i) *The tensor*

$$E_{\mathrm{mean}} = D\gamma_F V + \sum_\nu \mu_\nu \operatorname{Re}\Big((dZ_\nu)^* dZ_\nu\Big) - e_R/b^2 \tag{14.31}$$

*satisfies $\|E_{\mathrm{mean}}\|_{C^0(D)} \le C_Q R^{-Q}$.*

(ii) *The measures satisfy $\sup_\nu \|m_\nu\|_{\mathrm{TV}} \le C$ and*

$$\operatorname{supp} m_\nu \subset \{|\xi| \le C K_{\max}\}, \qquad K_{\max} = R^{s_f}, \tag{14.32}$$

*with a support constant independent of every accuracy choice.*

(iii) *On the whole product source support, the waves satisfy*

$$\sup_\nu \|\chi_{\mathrm{pair}} e^{-\mathrm{i}\lambda_\nu \phi_\nu} dZ_\nu\|_{C^r} \le C_r R^r. \tag{14.33}$$

(iv) *After the fixed physical cutoff, equal to one near $D$, the field $V$ satisfies, for every $r \geq 0$,*

$$\|V\|_{H^r} \leq C_r R^r. \tag{14.34}$$

The source order $s_0$ and the coefficient index $r_*$ are fixed before the accuracy. The number of iteration steps may depend on $Q$; the input frequency bound and the exponents in the derivative bounds do not. The constants $C_r$ may depend on $m$ and $Q$.

*Proof of Proposition 14.9. Step 1 (The initial amplitude and the waves).* Let $r_F$ be the positive initial amplitude of Proposition 13.7 with $G = F$; it is determined by $F$ and the fixed choices of Section 13. Its derivative bounds (13.13) are of the form required in Proposition 11.10, whose joint coefficient shifts are included in $r_*$. The measures below have Fourier support in $\{|\xi| \leq CK_{\max}\}$, and $C_* K_{\max} \leq \lambda^{1-\nu_0}$ by (19.5), so Proposition 11.10 applies to the waves $Z_\nu$.

*Step 2 (Choice of the data and of $V$).* We work under Setup 14.1; its conditions (14.3) are those of (14.1) together with $T = T_0 = R^t$. We apply Proposition 14.7 with accuracy $Q$, and obtain the sample with (14.2) and the source $r$. We put

$$m_\nu = \delta_0 + \widehat{f}_\nu[r]\, d\eta,$$

and define $V$ by (14.28). It is real, since $r$, $V_{\mathrm{base}}$, $V^F_{\mathrm{aux}}$, $e_R$ and $\mathcal{H}$ are real.

*Step 3 (The error).* By Lemma 14.8, on $D$,

$$E_{\mathrm{mean}} = -\sum_{i,j} Q_i(F)(\mathsf{G}_F^{-1})_{ij}(r_{\mathrm{rem}})_j.$$

The frame $Q_i(F)$ is bounded on $D$, and $\mathsf{G}_F^{-1}$ is bounded by the uniform Gram gap of Definition 13.1. Hence, by (14.20),

$$\|E_{\mathrm{mean}}\|_{C^0(D)} \leq C\|r_{\mathrm{rem}}\|_{C^0} \leq CR^{-Q}.$$

Thus the mean equation holds up to $O(R^{-Q})$: the iteration solves its projection onto $K_F$ up to the residual $r_{\mathrm{rem}}$, and the normal field $-\frac{1}{2}N_F T_f$ removes the component orthogonal to $K_F$ exactly.

*Step 4 (The measures).* Since $\|\delta_0\|_{\mathrm{TV}} = 1$, Proposition 14.7 gives $\|m_\nu\|_{\mathrm{TV}} \leq 1 + C \leq C$. The Dirac mass is supported at the origin and $\widehat{f}_\nu[r]$ vanishes for $|\eta| \geq 2K$; this proves (14.32), with a support constant fixed by the cutoff $P_K$.

*Step 5 (The amplitudes).* We have $e^{-\mathrm{i}\lambda_\nu\phi_\nu} dZ_\nu = B_\nu + D_\nu r$. The amplitude $B_\nu$ of the unmodulated wave has derivative scale $T < R$ by (14.23) (equivalently, by (11.26) with $m = \delta_0$ and $K = C_{\mathrm{env}}T_0$), and (14.21) bounds $D_\nu r$ in $C^s$ by $C_{s,Q}R^s$ on the fixed supports, which contain $\operatorname{supp}\chi_{\mathrm{pair}}$. This proves (14.33). We remark that this bound does not follow from (11.26) with the Fourier support of $m_\nu$, which would give the larger scale $K_{\max} = R^{s_f}$; it uses the fact that the data $f_\nu[r]$ are produced by the adjoint from a source with bounds $\|r\|_{H^s} \leq C_s h R^{(s-s_0)_+}$.

*Step 6 (The size of $V$).* By Step 5, the Leibniz rule and $\sum_\nu \mu_\nu \le C$,

$$\|\mathcal{H}\|_{C^s} \le \sum_\nu \mu_\nu \sum_{i\le s} C_s \|B_\nu + D_\nu r\|_{C^i} \|B_\nu + D_\nu r\|_{C^{s-i}} \le C_{s,Q} R^s$$

on the fixed supports. By (13.13), $V_{\mathrm{base}}$ and $D\gamma_F V_{\mathrm{base}}$ have $C^s(U_{\mathrm{fin}})$ norms at most $C_s T^s \le C_s R^s$; by (14.21), $V^F_{\mathrm{aux}}(r)$ and $D\gamma_F V^F_{\mathrm{aux}}(r)$ have $H^s$ norms at most $C_{s,Q}R^s$; and $\|e_R/b^2\|_{H^s} \le C_s R^{s-2A}$. Hence $T_f$ is bounded by $C_{s,Q}R^s$ in $H^s(U_{\mathrm{fin}})$, after the fixed physical cutoff. The coefficients of the normal map $N_F$ are smooth second-jet expressions in $F$ with derivative bounds $C_s T^s \le C_s R^s$ on $U_{\mathrm{fin}}$. The Leibniz rule in $H^s$ for a product with a smooth coefficient $c$, $\|cu\|_{H^s} \le C_s \sum_{i\le s} \|c\|_{C^{s-i}} \|u\|_{H^i}$, therefore gives $\|N_F T_f\|_{H^s} \le C_{s,Q} R^s$ after the fixed physical cutoff, which is supported in $U_{\mathrm{fin}}$. Together with the bounds for $V_0$ this proves (14.34).

*Step 7 (Order of choices).* The input frequency bound $K_{\max}$, $r_*$, $R_{\mathrm{env}}$ and the exponents do not depend on $Q$. The accuracy $Q$ enters only through the number $J$ of iteration steps in Proposition 14.7, the constants and the threshold for $R$; the depth of the wave correction is fixed by the accuracy of Theorem 17.3 before the sample is drawn (Section 21). □

## 15. Correction of the oscillatory quadratic terms

In this section we construct a correction, linearized at the immersion $F$, which cancels all oscillatory terms in the quadratic part $(dP)^T dP$ of the metric of $F+P$. The Hermitian part of $(dP)^T dP$ is not treated here; it is the tensor corrected by Proposition 14.9. The input is the family of sampled waves of Proposition 14.9, whose amplitudes vary on the scale $R$, and the output is a real field $S$, constructed at $F$, whose linearized metric cancels all products of distinct phases and all doubled phases up to $O(b^2 M(\lambda h)^{-q})$ (Proposition 15.4). The construction applies the finite corrections of Theorem 10.5 to each product separately; this is possible because every product has its frequency in a regular region (Proposition 6.9) and at least $c\lambda h$, which exceeds the amplitude scale $R$ by a fixed power.

15.1. **The average over the phase.** The normalization of the primary field is already visible for a single wave. Let $Z$ be a complex ambient field with $dZ = e^{\mathrm{i}\vartheta} A$, where $\vartheta$ is a real phase and $A$ is a complex $N \times n$ matrix. Then

$$\Big(d(\sqrt{2}\,\mathrm{Re}\,Z)\Big)^T d(\sqrt{2}\,\mathrm{Re}\,Z) = \mathrm{Re}(A^*A) + \mathrm{Re}(e^{2\mathrm{i}\vartheta} A^T A). \tag{15.1}$$

Indeed, $d(\mathrm{Re}\,Z) = \mathrm{Re}(dZ) = \frac{1}{2}(e^{\mathrm{i}\vartheta}A + e^{-\mathrm{i}\vartheta}\overline{A})$, so

$$\Big(d(\sqrt{2}\,\mathrm{Re}\,Z)\Big)^T d(\sqrt{2}\,\mathrm{Re}\,Z) = \tfrac{1}{2}\Big(e^{2\mathrm{i}\vartheta} A^T A + e^{-2\mathrm{i}\vartheta}\,\overline{A}^T\overline{A} + A^T\overline{A} + \overline{A}^T A\Big).$$

Since $\overline{A}^T A = A^*A$ and $A^T\overline{A} = \overline{A^*A}$, the last two terms add up to $2\,\mathrm{Re}(A^*A)$, while the first two are complex conjugate to each other and add up to $2\,\mathrm{Re}(e^{2\mathrm{i}\vartheta}A^TA)$. This proves (15.1).

The first term on the right of (15.1) is the mean of the wave in the sense of Section 14; the second oscillates with the doubled phase $2\vartheta$. For a sum of several waves

the quadratic tensor contains, in addition to the means and the doubled phases, the sums and differences of distinct phases. All these oscillatory terms are cancelled by the correction constructed in this section.

Let $Z_\nu$ be the waves and $\mu_\nu$ the weights of Proposition 14.9, and let $b = \delta R^A$. Define the real primary field

$$P = b\sqrt{2}\sum_\nu \sqrt{\mu_\nu}\,\mathrm{Re}\,Z_\nu. \tag{15.2}$$

Its Hermitian tensor and the remaining quadratic tensor are

$$H_{\mathrm{self}} = b^2\sum_\nu \mu_\nu\,\mathrm{Re}\Big((dZ_\nu)^* dZ_\nu\Big), \qquad Q_{\mathrm{off}} = (dP)^T dP - H_{\mathrm{self}}. \tag{15.3}$$

The tensor $H_{\mathrm{self}}$ is the one appearing in the mean equation (14.31). The tensor $Q_{\mathrm{off}}$ contains all doubled phases and all signed products of distinct samples, including the products between different patches; Lemma 15.2 below gives its exact expansion.

15.2. **The quadratic sources.** The geometric and linear results used below have been proved in Sections 6 and 10.

**Setup 15.1** (Geometry and amplitude bounds)**.** We work with the geometry of Definition 13.1, the domains of Setup 3.3, and the finite regular corrections of Theorem 10.5 at the immersion $F$.

(i) *Coefficients.* Assume

$$\|F\|_{C^r} \le D_r T^{(r-r_*)_+}, \qquad L_F = T, \tag{15.4}$$

where the fixed index $r_*$ exceeds the derivative shifts of the coefficients in Theorem 10.5 and in the normal reconstruction. By Corollary 7.3, every reconstruction coefficient $c_F$ (the inverse metric $\gamma(F)^{-1}$, the normal map $N_F$, the frame $Q_i(F)$ and $\mathsf{G}_F^{-1}$), multiplied by any fixed cutoff, satisfies

$$\|D^r c_F\|_\infty \le C_r T^r. \tag{15.5}$$

(ii) *Phases and frequencies.* We use the Hamilton phases $S_F$ and the groups of patches of Definition 6.7. The samples are indexed by $j = 1, \dots, M$, with base covectors $\zeta_j$ and frequencies $\tau_j^{-1}$ satisfying $c\lambda \le \tau_j^{-1} \le C\lambda$, and two distinct samples in the same group have patch parameters at distance at least $ch$. For an ordered pair $(j,k)$ and a sign $\pm$ we write $\theta_j \pm \theta_k = \rho_{jk}\varphi$, where the frequency $\rho = \rho_{jk}$ is chosen as follows:

| | |
|---|---|
| distinct same-group differences | $\rho = \lambda\lvert z - z'\rvert$ |
| sums, doubled phases, and either sign between groups | $\rho = \lambda$. |

Here $z, z'$ are the patch parameters of the two samples. The distinct radial labels separate different groups even when their unscaled directions agree. The equal-index differences $\theta_j - \theta_j = 0$ form the Hermitian diagonal, which is excluded. All Fourier modulations at the same patch parameter are combined in one field. Thus every oscillatory quadratic term has

$$2 \le c\lambda h \le \rho_{jk} \le C\lambda.$$

(iii) *Phase bounds.* The joint phase bounds of Proposition 6.4, with the fixed geometric shifts included in $r_* \ge 8$, give

$$|D_x^r S_F| + |D_x^r D_\zeta S_F| + |D_x^r D_z S_F| \le C_r T^{r-1} \qquad (r \ge 1).$$

The segment formula on each convex patch gives the same bound for the divided differences of the phase in the patch parameter. Proposition 6.4 and Theorem 10.3 place every normalized covector $d\varphi$ of a sum or a difference in the fixed buffered regular region. All source terms use the chart partition and the larger sets where the cutoffs of that theorem equal one.

(iv) *Cutoffs.* The source cutoff is $\chi_{\mathrm{src}} = \chi_{\mathrm{pair,src}}$ of Setup 3.3; it is supported where $\chi_2 = 1$ and equals one near $D$. It lies compactly inside $D_{\mathrm{lag}}$ and is distinct from the source cutoff of the primary waves in Section 11. All measurement and output supports lie in the common charts of validity.

(v) *Amplitudes.* Let $W_j$ be the complete complex fields, with

$$dW_j = e^{\mathrm{i}\theta_j} A_j, \qquad \theta_j = S_F(x, \zeta_j)/\tau_j. \tag{15.6}$$

After a fixed cutoff equal to one on every quadratic source, assume

$$\|A_j\|_{C^s} \le C_s K_{\mathrm{env}}^s, \qquad L_{\mathrm{pair}} = \max(K_{\mathrm{env}}, T) \le \rho_{jk}^{1-\sigma}, \quad K_{\mathrm{env}} \ge 2, \quad \sigma > 0. \tag{15.7}$$

For the final measures of Proposition 14.9, (14.33) proves this amplitude bound with $K_{\mathrm{env}} = C_* R$ on the full source support; along the iteration the coefficient bounds are verified in Proposition 18.2.

For ambient derivative matrices we write

$$\mathsf{B}(A, B) = \tfrac{1}{2}(A^T B + B^T A).$$

This is a symmetric bilinear form with values in symmetric matrices, and $\overline{\mathsf{B}(A,B)} = \mathsf{B}(\overline{A}, \overline{B})$.

15.3. **Quadratic expansion and its correction.** The correction below is linear in the coefficients of the quadratic expansion. The Hermitian diagonal term is handled by the correction of the mean.

**Lemma 15.2** (The quadratic expansion)**.** *Let $b > 0$, $\mu_j \ge 0$, $dW_j = e^{\mathrm{i}\theta_j} A_j$ with real phases $\theta_j$, and set*

$$P = b\sqrt{2} \sum_{j=1}^{M} \sqrt{\mu_j}\, \mathrm{Re}\, W_j, \qquad H_{\mathrm{self}} = b^2 \sum_j \mu_j\, \mathrm{Re}(A_j^* A_j).$$

*For ordered pairs $(j,k)$ put*

$$q_{jk}^+ = \mathrm{Re}\Big(e^{\mathrm{i}(\theta_j+\theta_k)} \mathsf{B}(A_j, A_k)\Big), \qquad q_{jk}^- = \mathrm{Re}\Big(e^{\mathrm{i}(\theta_j-\theta_k)} \mathsf{B}(A_j, \overline{A_k})\Big).$$

*Then*

$$Q_{\mathrm{off}} := (dP)^T dP - H_{\mathrm{self}} = b^2 \Big[ \sum_{j,k} \sqrt{\mu_j \mu_k}\, q_{jk}^+ + \sum_{j \ne k} \sqrt{\mu_j \mu_k}\, q_{jk}^- \Big]. \tag{15.8}$$

*The first sum includes the doubled phases $j = k$. The omitted diagonal of the differences is the complete Hermitian tensor, including every Fourier modulation and every correction contained in the fields $W_j$.*

*Proof.* Let $D, E$ be complex $N \times n$ matrices. Since $\operatorname{Re} D = \frac{1}{2}(D + \overline{D})$ and $\mathsf{B}$ is bilinear,

$$4\mathsf{B}(\operatorname{Re} D, \operatorname{Re} E) = \mathsf{B}(D, E) + \mathsf{B}(\overline{D}, \overline{E}) + \mathsf{B}(D, \overline{E}) + \mathsf{B}(\overline{D}, E).$$

The first two terms are complex conjugate, and so are the last two. Hence

$$2\mathsf{B}(\operatorname{Re} D, \operatorname{Re} E) = \operatorname{Re} \mathsf{B}(D, E) + \operatorname{Re} \mathsf{B}(D, \overline{E}). \tag{15.9}$$

Moreover,

$$\mathsf{B}(D, \overline{D}) = \tfrac{1}{2}\Big(D^T\overline{D} + \overline{D}^T D\Big) = \tfrac{1}{2}\Big(\overline{D^*D} + D^*D\Big) = \operatorname{Re}(D^*D).$$

Now put $D_j = dW_j = e^{\mathrm{i}\theta_j} A_j$. Then $dP = b\sqrt{2}\sum_j \sqrt{\mu_j}\operatorname{Re} D_j$, and since the double sum is symmetric in $(j, k)$,

$$(dP)^T dP = 2b^2 \sum_{j,k} \sqrt{\mu_j \mu_k}\,(\operatorname{Re} D_j)^T \operatorname{Re} D_k = 2b^2 \sum_{j,k} \sqrt{\mu_j \mu_k}\,\mathsf{B}(\operatorname{Re} D_j, \operatorname{Re} D_k).$$

By (15.9) and the reality of the phases,

$$2\mathsf{B}(\operatorname{Re} D_j, \operatorname{Re} D_k) = \operatorname{Re}\Big(e^{\mathrm{i}(\theta_j+\theta_k)}\mathsf{B}(A_j, A_k)\Big) + \operatorname{Re}\Big(e^{\mathrm{i}(\theta_j-\theta_k)}\mathsf{B}(A_j, \overline{A_k})\Big) = q^+_{jk} + q^-_{jk}.$$

For $j = k$ the second term is $q^-_{jj} = \mathsf{B}(A_j, \overline{A_j}) = \operatorname{Re}(A_j^* A_j)$. Summing over $j$ with the weights $b^2\mu_j$ gives exactly $H_{\mathrm{self}}$, and subtracting it proves (15.8). □

With $W_j = Z_\nu$ and $\vartheta = \theta_j$, the case $M = 1$, $b\sqrt{\mu_1} = 1$ of the lemma is (15.1), since $\mathsf{B}(A, A) = A^T A$.

**Proposition 15.3** (Correction of all oscillatory quadratic terms)**.** *We work under the assumptions of Setup 15.1 and assume $\sum_j \mu_j \le C_\mu$. Let $N_F$ and $V_F = \mathcal{V}_F$ be the reconstruction maps of Definition 11.2. Given finite $s$ and $Q$, choose one depth $N = N(s, Q)$ in the real operator $\mathcal{D}_{F,N}$, that is, the operator $\mathcal{D}_{a,N}$ of Theorem 10.5 for the coefficients of $\mathcal{P}_F$, with all its fixed cutoffs. Let $Q_{\mathrm{off}}$ be the exact tensor (15.8), put*

$$q = \chi_{\mathrm{src}} Q_{\mathrm{off}}, \qquad h_i = -\tfrac{1}{2} Q_i(F) : q, \qquad u = \mathcal{D}_{F,N} h,$$

*and define the real field*

$$S_{\mathrm{lag}} = V_F(u) + \tfrac{1}{2} N_F q. \tag{15.10}$$

*With $\mathcal{W}_\mu = (\sum_j \sqrt{\mu_j})^2 \le M C_\mu$,*

$$\|D\gamma_F S_{\mathrm{lag}} + Q_{\mathrm{off}}\|_{H^s(D)} \le C_{s,Q} b^2 \mathcal{W}_\mu (\lambda h)^{-Q}, \tag{15.11}$$

$$\|S_{\mathrm{lag}}\|_{H^r(W)} \le C_{r,N} b^2 \mathcal{W}_\mu \lambda^{r+1} \qquad (r \ge 0). \tag{15.12}$$

*For a fixed physical cutoff $\chi_{\mathrm{out}}$, equal to one near $D$ and supported in the observation chart, the same size bound holds for $\chi_{\mathrm{out}} S_{\mathrm{lag}}$ on $\mathbb{R}^n$.*

The depth $N$ may depend on $s$ and $Q$; the additional power of $\lambda$ in (15.12) and all derivative shifts are fixed. All doubled phases and all signed terms of distinct samples are included, and the complete Hermitian diagonal tensor is excluded. The number of samples enters only through $\mathcal{W}_\mu \le MC_\mu$, which follows from the Cauchy–Schwarz inequality $(\sum_j \sqrt{\mu_j})^2 \le M \sum_j \mu_j$.

*Proof. Step 1 (One signed source).* Fix an ordered pair $(j,k)$ and a sign, and apply the construction in the statement to the single source $q = \chi_{\mathrm{src}} q^{\pm}_{jk}$ in place of $\chi_{\mathrm{src}} Q_{\mathrm{off}}$. Denote the resulting reduced and ambient fields by $u^{\pm}_{jk}$ and $U^{\pm}_{jk}$. Write the complex product defining $q^{\pm}_{jk}$ as $e^{\mathrm{i}\rho\varphi} a$, with $\rho = \rho_{jk}$ as in Setup 15.1, $\rho\varphi = \theta_j \pm \theta_k$, and $a = \mathsf{B}(A_j, A_k)$ or $a = \mathsf{B}(A_j, \overline{A_k})$. The reduced source is

$$h^{\pm}_{jk,i} = -\tfrac{1}{2} Q_i(F) : \chi_{\mathrm{src}} q^{\pm}_{jk} = \mathrm{Re}\big(e^{\mathrm{i}\rho\varphi} a_{jk,i}\big), \qquad a_{jk,i} = -\tfrac{1}{2}\chi_{\mathrm{src}}\, Q_i(F) : a,$$

because $Q_i(F)$ and $\chi_{\mathrm{src}}$ are real. By the Leibniz rule, (15.5) and (15.7),

$$\|a_{jk}\|_{C^s} + \|a_{jk}\|_{H^s} \le C_s \sum_{s_1+s_2+s_3=s} T^{s_1} K_{\mathrm{env}}^{s_2+s_3} \le C_s L_{\mathrm{pair}}^{s},$$

where the $H^s$ bound follows from the $C^s$ bound and the fixed compact support. Only the total derivative order enters.

*Step 2 (Fourier localization).* By Setup 15.1(iii), the normalized phase satisfies $|D^r\varphi| \le C_r L_{\mathrm{pair}}^{r-1}$ for $r \ge 2$, and $d\varphi$ takes values in a fixed compact set of regular covectors, in particular $c \le |d\varphi| \le C$. Choose a fixed compact annular Fourier mask containing a neighborhood of this compact set and of its antipodal set, and let $\Pi_{0,\rho}$ be the corresponding rescaled multiplier of Theorem 10.5. On the complement of the rescaled mask, $|\rho\, d\varphi - \xi| \ge c(\rho + |\xi|)$. The general form of Proposition 8.6, applied with amplitude constant $A = 1$ and scale $L = L_{\mathrm{pair}}$, gives for every $m \ge 0$ and every integer $k > m + n/2$

$$\|(I - \Pi_{0,\rho})(e^{\mathrm{i}\rho\varphi} a_{jk})\|_{H^m} \le C_{m,k} \rho^{m+n/2} (L_{\mathrm{pair}}/\rho)^k \le C_{m,k} \rho^{m+n/2-\sigma k},$$

where the last step uses $L_{\mathrm{pair}} \le \rho^{1-\sigma}$. Choosing $k \ge (m + n/2 + Q')/\sigma$ gives the decay $\rho^{-Q'}$ for any requested $Q'$. Conjugation gives the same estimate for the term with the opposite phase. Hence $h^{\pm}_{jk}$ satisfies the hypotheses (10.6)–(10.7) of Theorem 10.5 with $H_\rho$ equal to a fixed constant. The scale condition (10.5) holds because $1 \le T \le L_{\mathrm{pair}} \le \rho^{1-\sigma}$, and the sources are supported where $\chi_2 = 1$. The chart source sets and complete flowouts are those of Setup 15.1.

*Step 3 (Approximate solution and normal reconstruction).* By (4.5), applied to the tensor $-q$, the field $U = V_F(u) + \frac{1}{2} N_F q = dF\, \gamma(F)^{-1} u + N_F(\mathrm{sym}\, \nabla u + \frac{1}{2} q)$ satisfies the exact identity

$$D\gamma_F U + q = 2 \sum_{i,l} Q_i(F) (\mathsf{G}_F^{-1})_{il}\, (\mathcal{P}_F u - h)_l, \qquad h_l = -\tfrac{1}{2} Q_l(F) : q. \tag{15.13}$$

Theorem 10.5, with a depth $N \ge s + Q'$, gives $\|\mathcal{P}_F u^{\pm}_{jk} - h^{\pm}_{jk}\|_{H^s(U_{\mathrm{core}})} \le C_{N,s,Q'} \rho^{-Q'}$. Multiplication by the coefficients in (15.13), whose $C^s$ norms are at most $C_s T^s \le C_s \rho^s$,

costs at most a factor $C_s\rho^s$ in $H^s(D)$. Taking $Q' = Q + s$ and using $D \subset U_{\mathrm{core}}$ and $\chi_{\mathrm{src}} = 1$ near $D$, we obtain

$$\|D\gamma_F U_{jk}^{\pm} + q_{jk}^{\pm}\|_{H^s(D)} \le C_{s,Q}\rho_{jk}^{-Q}. \tag{15.14}$$

One such depth works for the compact sets of regular covectors and the finite list of charts; it is independent of the frequency $\rho_{jk}$.

*Step 4 (The exact weighted sum).* The operator $\mathcal{D}_{F,N}$, the frame and the cutoffs are the same for all pairs. By linearity, $S_{\mathrm{lag}}$ is the sum of the fields $U_{jk}^{\pm}$ with the coefficients of (15.8), and $D\gamma_F S_{\mathrm{lag}} + Q_{\mathrm{off}}$ is, on $D$, the same combination of the left-hand sides of (15.14). The total absolute coefficient is

$$b^2\Big[\sum_{j,k}\sqrt{\mu_j\mu_k} + \sum_{j\neq k}\sqrt{\mu_j\mu_k}\Big] \le 2b^2\Big(\sum_j \sqrt{\mu_j}\Big)^2 = 2b^2\mathcal{W}_\mu \le 2b^2 M C_\mu.$$

Since $\rho_{jk} \ge c\lambda h$ for every oscillatory term, (15.14) gives (15.11), with $c^{-Q}$ absorbed in the constant.

*Step 5 (Size bounds).* Put $\mathfrak{A} = b^2\mathcal{W}_\mu$. Each derivative of $e^{\mathrm{i}\rho\varphi}$ costs at most $C(\rho + L_{\mathrm{pair}}) \le C\lambda$, and each derivative of $a_{jk}$ costs at most $CL_{\mathrm{pair}} \le C\lambda$. The Leibniz rule and the coefficient sum of Step 4 therefore give

$$\|q\|_{H^s} + \|h\|_{H^s} \le C_s\mathfrak{A}\lambda^s.$$

For each signed source, (10.9) gives $\|u_{jk}^{\pm}\|_{H^r} \le C_{r,N}\rho_{jk}^r$. Since $\rho_{jk} \le C\lambda$, the same coefficient sum gives $\|u\|_{H^r} \le C_{r,N}\mathfrak{A}\lambda^r$. The field $V_F(u)$ contains $u$ and one derivative of $u$, multiplied by coefficients whose derivatives of order $i$ are bounded by $C_i T^i \le C_i\lambda^i$; the Leibniz rule gives $\|V_F(u)\|_{H^r(W)} \le C_{r,N}\mathfrak{A}\lambda^{r+1}$. Similarly $\|N_F q\|_{H^r(W)} \le C_r\mathfrak{A}\lambda^r$. This proves (15.12); multiplication by the fixed cutoff $\chi_{\mathrm{out}}$ changes only the constants. The operators are real and uniform on the fixed family of charts. □

15.4. **The product correction for the sampled waves.** We now apply Proposition 15.3 to the waves constructed in Proposition 14.9.

**Proposition 15.4** (Product correction)**.** *Let $Z_\nu$ and $\mu_\nu$ be the waves and weights of Proposition 14.9, and let $P$, $Q_{\mathrm{off}}$ be given by (15.2) and (15.3). Assume that, for a fixed $\sigma > 0$ and a fixed amplitude constant $C_*$,*

$$C_* R \le (\lambda h)^{1-\sigma}. \tag{15.15}$$

*Then, for every finite $q \ge 1$, there is a real correction $S$, constructed at $F$ from the exact signed products in $Q_{\mathrm{off}}$, such that, after its fixed physical cutoff equal to one near $D$,*

$$\|D\gamma_F S + Q_{\mathrm{off}}\|_{C^0(D)} \le C_q b^2 M(\lambda h)^{-q}, \tag{15.16}$$

$$\|S\|_{H^r} \le C_r b^2 M\lambda^{r+1} \qquad (r \ge 0). \tag{15.17}$$

*The finite depth of the construction depends on $q$; the same field satisfies the size bound at every $r$, with the same additional power of $\lambda$.*

The mechanism is the following. The correction solves the linearized equation along the full characteristic segments of Proposition 6.9, using the complete amplitudes of the corrected waves $Z_\nu$, whose derivative scale is $R > T_0$. Each step of the finite Neumann series in Theorem 10.5 gains a factor $C_*R/\varrho$, with $\varrho \geq c\lambda h$, and the strict power in (15.15) turns this into arbitrary finite decay.

*Proof of Proposition 15.4.* We apply Lemma 15.2 and Proposition 15.3 with the samples $j = \nu$, the fields $W_\nu = Z_\nu$, the phases $\theta_\nu = \lambda_\nu \phi_\nu = S_F(x, \zeta_\nu)/\tau_\nu$, and the amplitudes $A_\nu = e^{-\mathrm{i}\theta_\nu} dZ_\nu$. Since $\theta_\nu$ is real, $(dZ_\nu)^* dZ_\nu = A_\nu^* A_\nu$, so the fields $P$, $H_{\mathrm{self}}$ and $Q_{\mathrm{off}}$ of (15.2)–(15.3) are those of Lemma 15.2. By (14.2), $\mu_\nu > 0$ and $\sum_\nu \mu_\nu \leq C =: C_\mu$.

We verify the hypotheses of Setup 15.1. The coefficient bound (15.4) is (11.23) with $T = T_0 = R^t$. The phases, groups and frequencies are those of the sampling in Proposition 14.9; by Proposition 6.9(iii), every oscillatory term has frequency $c\lambda h \leq \varrho \leq C\lambda$, and $c\lambda h \geq 2$ for large $R$ because $a > \beta$. For the amplitudes, (14.33) gives $\|\chi_{\mathrm{pair}} A_\nu\|_{C^r} \leq C_r R^r$, where $\chi_{\mathrm{pair}}$ equals one on every quadratic source (Setup 3.3). Thus (15.7) holds with $K_{\mathrm{env}} = C_0 R$ for any fixed $C_0 \geq 2$. Since $t < 1$, we have $T_0 = R^t \leq C_0 R$, so $L_{\mathrm{pair}} = \max(K_{\mathrm{env}}, T_0) = C_0 R$. For every oscillatory term, (15.15) gives

$$\rho_{jk}^{1-\sigma} \geq (c\lambda h)^{1-\sigma} = c^{1-\sigma}(\lambda h)^{1-\sigma} \geq c^{1-\sigma} C_* R \geq C_0 R = L_{\mathrm{pair}},$$

provided the amplitude constant satisfies $C_* \geq c^{\sigma-1} C_0$; this is how the constant $C_*$ in (15.15) is understood. Hence all hypotheses of Setup 15.1 hold.

Apply Proposition 15.3 with $s = \lfloor n/2 \rfloor + 1$ and $Q = q$, and put $S = \chi_{\mathrm{out}} S_{\mathrm{lag}}$. Since $\chi_{\mathrm{out}} = 1$ near $D$ and $D\gamma_F$ is a first-order differential operator, $D\gamma_F S = D\gamma_F S_{\mathrm{lag}}$ on $D$. Since $s > n/2$ and $D$ is a ball, the Sobolev embedding $H^s(D) \subset C^0(D)$ and (15.11) give

$$\|D\gamma_F S + Q_{\mathrm{off}}\|_{C^0(D)} \leq C \|D\gamma_F S_{\mathrm{lag}} + Q_{\mathrm{off}}\|_{H^s(D)} \leq C_q b^2 \mathcal{W}_\mu (\lambda h)^{-q} \leq C_q b^2 M (\lambda h)^{-q},$$

which is (15.16). The bound (15.17) follows from the cut form of (15.12) and $\mathcal{W}_\mu \leq MC_\mu$. The depth $N = N(s, q)$ depends only on $q$, so the constants $C_r$ may depend on $q$. □

## 16. The conformal equation in the common analysis

The preceding technical statements have been written for the isometric equation. We now prove their projected counterparts, using the conformal reference and the same reduced system. The algebraic completion and physical tensor are (4.5) and (4.8). In this section $\mathsf{P} = \mathsf{P}_g$, $h_F = \gamma(F)$, and $\Xi_F(w, t) = \mathrm{II}_F^* w + tg$. In particular

$$q_F \mathsf{P} = q_F. \tag{16.1}$$

The characteristic patches and detection estimate are those of Proposition 6.1. Both reference immersions give the same frozen stress frame and the same pencil with $c_n = (n-4)^{-2}$. We use these common model data and the augmented normal inverse in the conformal correction.

16.1. **The projection and normal reconstruction.** The target metric is fixed during a correction step. Its projection $\mathsf{P}$ has kernel $\mathbb{R}g$ and satisfies $\mathsf{P}^2 = \mathsf{P}$. It is generally not orthogonal for the coordinate Frobenius pairing. The stress projection $\Pi_{K_F}$ is orthogonal for that pairing. These are different maps, and their order in (4.5) is part of the formula.

The augmented normal equation separates this distinction from the differential analysis. For any symmetric tensor $T$, write

$$(w_T, t_T) = \Xi_F^{-1} \Pi_{K_F^\perp} T.$$

Then $w_T = N_F T$ and

$$B_F^* N_F T + t_T g = T - \Pi_{K_F} T.$$

Because $g \in K_F^\perp$ and $\Xi_F(0,1) = g$, the tensor $g$ has zero physical normal component. Consequently

$$q_F \mathsf{P} = q_F, \qquad N_F \mathsf{P} = N_F, \qquad Y_F(u, \mathsf{P}H) = Y_F(u, H). \tag{16.2}$$

Thus a scalar multiple of the target can be removed from a source before or after reconstruction, with exactly the same ambient field. This assertion holds for an approximate reduced solution as well.

To see the residual explicitly, put $T = \operatorname{sym} \nabla^{h_F} u - H/2$. Normal elimination gives

$$D\gamma_F Y_F(u, H) - H = 2\Pi_{K_F} T + 2t_T g,$$

and therefore

$$\mathsf{P}(D\gamma_F Y_F(u, H) - H) = 2\mathsf{P} \sum_{i,j} Q_i (\mathsf{G}_F^{-1})_{ij} (P_F u - q_F(H)/2)_j. \tag{16.3}$$

The coefficients on the right are smooth on the fixed rank and Gram gaps. Every finite-order bound for the reduced residual hence gives the corresponding bound for the projected metric residual by the product rule. There is no additional differential equation for $t_T$. Only the normal vector $w_T$ enters the ambient variation. The connection remains that of the induced metric $h_F$ throughout.

16.2. **Normalized physical amplitudes.** Let $F_0$ be either reference, let $g_0$ be its limiting target metric, and put

$$F_\sigma(x) = \sigma^{-1}(F_0(\sigma x) - F_0(0)).$$

In the conformal case the target is $g_{\sigma,s}(x) = g(s\sigma x)$, where $g(0) = g_0 = H_0$. Choose smooth identifications of the normal spaces along this family. The second form scales as

$$\mathrm{II}_{F_\sigma} = \sigma \widehat{B}_\sigma, \qquad \widehat{\Xi}_{\sigma,s}(w, t) = \widehat{B}_\sigma^* w + t g_{\sigma,s}.$$

The normalized normal map is injective down to $\sigma = 0$, by Proposition 4.4 in the isometric case and Proposition 5.1 in the conformal case. Its inverse on its image and the orthogonal stress projection consequently extend smoothly to that endpoint. All endpoint assertions concern these normalized maps. The actual second form vanishes at $\sigma = 0$ and is not inverted there.

If $T \in \operatorname{im} \widehat{\Xi}_{\sigma,s}$ and $\widehat{\Xi}_{\sigma,s}^{-1} T = (\widehat{w}, \widehat{t})$, then exactly

$$\Xi_{\sigma,s}^{-1} T = (\sigma^{-1}\widehat{w}, \widehat{t}), \qquad \Xi_{\sigma,s}^{-1}(\sigma T) = (\widehat{w}, \sigma\widehat{t}). \tag{16.4}$$

The scalar term is omitted in the isometric case. In particular, the actual physical normal inverse has a factor $\sigma^{-1}$. This factor is cancelled by choosing the initial tangential wave amplitude to have size $\sigma$.

More explicitly, use the normalized stress frame to solve the eikonal and transport equations. Their coefficients extend smoothly in $(\sigma, s)$ to the constant model at $\sigma = 0$. The transport equation is linear in its initial amplitude, so the transported polarization has the form $b_\sigma = \sigma \widehat{b}_\sigma$, where $\widehat{b}_\sigma$ is smooth down to zero. If $\xi_\sigma = dS_\sigma$ and $\widehat{N}_{\sigma,s} = \operatorname{pr}_{\mathcal{N}} \widehat{\Xi}_{\sigma,s}^{-1} \Pi_{K_{\sigma,s}^\perp}$, the leading physical normal is therefore

$$\nu_\sigma = N_{F_\sigma}(\xi_\sigma \odot b_\sigma) = \widehat{N}_{\sigma,s}(\xi_\sigma \odot \widehat{b}_\sigma). \tag{16.5}$$

Here $\odot$ denotes the symmetric product used in (4.8). The null-vector equation puts its argument in $K_{\sigma,s}^\perp$. At the limit this is a nonzero tensor of rank at most two. It cannot equal a nonzero multiple of the positive definite $g_0$, so its physical normal component is nonzero. The same conclusion holds in the isometric case by injectivity. Compactness of the chosen characteristic patches gives a uniform positive lower bound for $|\nu_\sigma|$.

Thus the leading physical tensor is still $|\nu_\sigma|^2 \xi_\sigma \xi_\sigma^T$, with a smooth nonzero constant-model limit. The scalar component of the augmented inverse contributes no squared length to this tensor. At a fixed positive $\sigma$ the inverse and its derivatives have finite bounds, which may depend on $\sigma$. The uniform limit needed for the continuous normal operator is instead the normalized physical formula (16.5).

16.3. **Transfer of the analytic estimates.** The analytic estimates above concern square first-order systems, their characteristic phases, and the physical fields reconstructed from them. The next proposition checks how each of these objects changes for the conformal equation. Its four hypotheses separate the finite-dimensional geometry from the common analytic proofs. They are verified for the actual reference in Corollary 16.3 below.

**Proposition 16.1** (The analytic argument for the projected equation)**.** *Assume the following geometric facts on fixed nested physical and parameter domains.*

(i) *The metric, augmented rank, stress Gram, null-vector minor and simple-root gaps are positive on a finite-order neighborhood of an actual reference. The coefficients of $N_F$, $q_F$, $P_F$ and $\mathsf{P}$ are smooth functions of a fixed jet of the immersion and target, with fixed derivative losses.*

(ii) *A finite menu satisfies detection at every output direction, incidence transversality, protected signed interactions, and common buffered full and fixed-time inverse charts for the phase differences.*

(iii) *At the reference the normalized physical normal amplitudes have a smooth nonzero limit as in (16.4). The induced metric, target, normalized augmented contraction and stress frame tend in every fixed norm to their constant model data; the Hamilton phases tend to affine phases and the normalized transport amplitudes to the*

*corresponding constant-model amplitudes. These normalized data extend smoothly in the dilation parameter down to zero. The current geometric data lie in a sufficiently small finite-order neighborhood of this reference after a positive dilation is fixed.*

(iv) *For a sufficiently large fixed $R_*$ the full coefficient array and initial amplitudes satisfy $\|c\|_{C^r} \le C_r L^{(r-R_*)_+}$ for all $r$, on the same enlarged domains. The target metric is included in this array.*

*Then the regular parametrices, corrected waves, separated sampling, continuous inverse, finite correction of the mean, product correction and one-step estimates of the common technical body hold for $\mathsf{P}\gamma$, with the same frequency exponents independent of the requested finite accuracy. They use the current immersion in every field and every linear response. No coarse immersion or lagged phase is needed.*

*Proof.* We give every change in the geometric identities and indicate why the analytic estimates are unaffected.

*Regular response.* The regular-patch theorem 9.1 is stated for an arbitrary real square first-order system. Apply it to $P_F$. The fixed domains and complete residual in Theorem 10.3, and the finite powers in Theorem 10.5, depend only on this system, its regular covectors and its coefficient bounds. The proof controls total elapsed Hamilton time independently of the finite depth. None of these arguments uses an ambient normal dimension. For a symmetric source $H$ solve the reduced equation with source $q_F(H)/2$ and use $Y_F(u, H)$. Formula (4.5) turns its reduced residual into the projected physical residual. The additional factors $\mathsf{P}, Q_i, \mathsf{G}_F^{-1}$ have the bounds in (i) and (iv).

*Waves.* Use the same eikonal and scalar transport equations as in Theorem 11.4. Their coefficients are those of $P_F$. In the two-term WKB construction the reduced residual is still $e^{\mathrm{i}S/\tau}\tau^3 P_F b_1$. The finite response correction and the parameter estimate of Lemma 11.7 concern the same square system. They therefore give the same demodulated reduced amplitudes before physical reconstruction. After reconstruction, Lemma 4.2 gives

$$\mathsf{P} D\gamma_F \mathcal{V}_F(u) = 2\mathsf{P}\Pi_{K_F}(\operatorname{sym} \nabla^{h_F} u).$$

This replaces the unprojected strain identity wherever it is used in Corollary 11.8 and Proposition 11.10. The exact physical oscillatory expansion and its leading tensor are (4.8). Hypotheses (i), (iii) and (iv) give the same fixed losses in the remainder and its joint parameter derivatives. The whole shifted-covector family is chosen before sampling.

*Normal operators and sampling.* The inverse charts and amplitude bounds give the normal-operator estimate in Proposition 8.4. The cross amplitude used in the sampling theorem is the physical tensor

$$(Q_i : A_\tau(x, z, \zeta)^* A_\tau(x, z, \zeta + \tau\eta))_i.$$

The concentration and quadrature argument in Theorem 12.4 therefore gives one separated sample for the required operator families and the background mean. The real and imaginary components come from the ambient field.

*The continuous inverse.* Use exactly the row

$$A = (\chi P_F \langle D\rangle^{-1}, (2\,\mathrm{Re}\, T_a)_a), \qquad E = AA^* + 1 - \chi^2,$$

where the wave amplitude of $T_a$ is $|\nu_a|^2 q_F(\xi_a \xi_a^T)$, with the initial-variable half derivative and its Jacobian as in Definition 13.4. Detection in (ii) and the physical gap in (iii) give positivity of the constant limiting multiplier at high frequency. The low-frequency error is bounded by $C\|\chi f\|_1^2$ and is absorbed by shrinking the measurement support. The proof of Proposition 13.5 then gives $E \geq cI$. Its perturbation argument concerns the normal operator $E$; continuity of the phase-dependent row $A$ in operator norm is neither needed nor asserted.

For completeness, the Sobolev mechanism in Proposition 13.6 is unchanged: translation conjugation gives $E_z = U_z E U_{-z}$ with smooth operator-norm derivatives, and differentiating $E_z E_z^{-1} = I$ bounds the derivatives of the inverse. The adjoint estimate transfers each initial-variable derivative using $\eta = (D_y Y)^{-1} d_y \Phi$ and $\partial_z \Phi = V_z \cdot d_y \Phi$. The coefficients of these identities and their inverses satisfy (iv). Hence, for one fixed loss $B$ and $R_* \geq r_0 + B$,

$$\|E^{-1} e\|_{H^s} + \|A^* E^{-1} e\|_{X^s} \leq C_s \Big( \|e\|_{H^s} + L^{(s-r_0)_+} \|e\|_2 \Big),$$

together with the same adjoint-frequency tail bound. The image realization is the actual tangential field $\mathcal{J}_F h = \frac{1}{2} dF h_F^{-1} \langle D\rangle^{-1} h$; by (16.1) its reduced projected strain is $P_F \langle D\rangle^{-1} h$.

*Positive background.* Let $H_{\mathrm{mean},0}$ be the constant limiting physical mean. The induced metric $g_0$ of the flat reference is constant. For the affine field $X_0(x) = -g_0^{-1} H_{\mathrm{mean},0} x/2$, we have

$$D\gamma_{F_\sigma}(dF_\sigma X_0) = \mathcal{L}_{X_0} g_0 = g_0\, dX_0 + (dX_0)^T g_0 = -H_{\mathrm{mean},0}.$$

The normalized phases and amplitudes depend smoothly on $\sigma$ by (iii). Their transported mean is therefore $H_{\mathrm{mean},0} + O(\sigma)$ in every fixed norm. The residual source is therefore $O(\sigma)$ in every fixed norm. The preceding uniform inverse and adjoint estimates give an initial density change $v = O(\sigma)$, so $1 + v \geq 1/2$ after fixing a small $\sigma > 0$. The affine dependence on the initial density and $AA^* E^{-1} = I$ on the core prove the same exact reduced cancellation as in Proposition 13.7. This argument uses the actual reference and its physical mean. It does not require a positive flat metric deficit.

*Finite mean correction.* By (16.1), the reduced identity of Lemma 14.3 is unchanged. Keep the same physical sample, wave maps, adjoints and continuum inverse throughout the finite iteration. Its linear part is

$$\mathcal{E} = \widetilde{E}_K E^{-1}.$$

Although $E$ and $\widetilde{E}_K$ are self-adjoint, their product need not be self-adjoint. No positivity or self-adjointness of $\mathcal{E}$ is used. The estimate is instead obtained from the ordered identity

$$I - \mathcal{E} = (E - E_K) E^{-1} + (E_K - \widetilde{E}_K) E^{-1}.$$

The second term is small in operator norm by sampling; the first is small only on the actual smooth sources, by the adjoint-frequency tail estimate. The quadratic term uses the same physical fields as the linear part. These are precisely the hypotheses in the

finite Picard proof of Proposition 14.7. In particular no inverse of a sample weight, no inverse of a compact finite-band operator, and no commutation of $E^{-1}$ with a cutoff is introduced.

For algebraic completion put $\mathcal{H} = H_{\mathrm{self}}/b^2 = \sum_\nu \mu_\nu \operatorname{Re}((dZ_\nu)^* dZ_\nu)$, and define $T_f = e_R/b^2 - D\gamma_F V_0 - \mathcal{H}$ and $V = V_0 - N_F T_f/2$. In place of Lemma 14.8 the exact error is

$$\mathsf{P}(D\gamma_F V + \mathcal{H} - e_R/b^2) = -\mathsf{P}\Pi_{K_F} T_f = -\mathsf{P}\sum_{i,j} Q_i(\mathsf{G}_F^{-1})_{ij}(q_F T_f)_j. \tag{16.6}$$

The reduced residual therefore gives the same error estimate. The normal completion and all of its derivatives have the fixed coefficient losses of (i); the fixed positive $\sigma$ is allowed in their constants.

*Products and the one-step identity.* Use the complete physical signed products of the same complex waves. The protected phases are those in (ii). For each source apply the regular reduced response and (4.5); hence Proposition 15.4 gives the projected product cancellation. The primary self tensor includes the entire complex sample, including its modulations. With primary field $P$ and quadratic correction $W_2$, the nonlinear remainder remains exactly

$$2\operatorname{sym}((dP)^T dW_2) + (dW_2)^T dW_2.$$

Apply $\mathsf{P}$ to the exact quadratic metric identity. Its $C^0$ norm is bounded on the fixed target class, so all estimates in Theorem 17.3 hold with the same frequency exponents. For smoothing one may use $e_R = \mathsf{P}S_{R,m}e$ with $e = -\mathsf{P}\gamma(F)$ in the conformal problem. The error is $\mathsf{P}(S_{R,m}e - e)$ and the derivative estimates include the fixed target coefficients. No commutation of $\mathsf{P}$ and $S_{R,m}$ is claimed. Fourier truncation acts on the ambient field; first estimate the resulting metric difference and then apply the bounded map $\mathsf{P}$. This also requires no commutation.

*Uniformity and convergence.* The exact Fourier support and global $L^2$ bounds of the increment are statements about the ambient field. The proofs of Lemma 18.1, Proposition 18.3 and the accuracy schedule therefore apply without alteration. The target coefficients are fixed throughout that history and satisfy (iv) with their fixed smooth seminorms. Positivity of the final conformal scalar follows from immersion: if $\mathsf{P}\gamma(F) = 0$, then $\gamma(F) = \rho g$ with $\rho = \operatorname{tr}_g \gamma(F)/n > 0$. □

*Remark* 16.2 (Independent checks). The regular kernel construction and finite powers are proved in Sections 9 and 10; the oscillatory estimates and parameter bounds in Sections 8 and 11; the sampling, continuous inverse and nonlinear mean estimates in Sections 12, 13 and 14. Their geometric inputs are the two reference constructions and the common characteristic-patch proposition. Thus these parts can be checked separately through their stated inputs and outputs for every $n \geq 5$.

**Corollary 16.3.** *For the conformal reference of Proposition 5.1, the hypotheses of Proposition 16.1 hold on fixed domains and a sufficiently small finite-order neighborhood. They hold uniformly for the compact target family $g(s\sigma x)$ used in Proposition 20.1.*

*Proof.* The normalized augmented map is $(\widehat{B}_\sigma, g(s\sigma x))$. Its kernel projection, projected frame, inverse on the complementary space, and the coefficients of $P_F$ are smooth

functions of this map and of the induced metric. By Proposition 5.1, adjoining $H_0$ to the retained normal forms recovers the full span of the model normal forms. This gives the rank gap. Projection of the independent frozen frame gives the Gram gap by continuity and proves (i).

At the reference the normalized data converge to $(C_0, H_0, K_0)$, the connection tends to zero, and the constant coefficient Hamilton flow has affine phase. Smooth dependence, jointly in $(\sigma, s)$ down to $\sigma = 0$, of the characteristic ODEs and their scalar transport equations gives the convergence in (iii). Its nonzero physical gap follows from (4.8) and (16.4).

Proposition 6.1 gives all strict incidence, detection and chord bounds for the common pencil. Continue its simple roots and Hamilton phases as in Propositions 6.4 and 6.5. Their proofs use only the stress frame and the strict bounds. The full and fixed-time inverse charts have the same buffered domains by their inverse-function arguments. This proves (ii). Fix a positive $\sigma$ and then take the current neighborhood small enough to preserve these bounds.

Finally include the fixed target and the normalized inverse in the coefficient array of Proposition 18.2. Differentiating an inverse on its fixed positive gap and applying the same product and interpolation estimates adds one finite derivative reserve. Enlarge $r_*$ to cover it before choosing $k_{\min}$. The Fourier history argument then gives (iv), since the target family has uniformly bounded seminorms of each fixed order. All choices precede the final $s$ and the accuracy of any stage. □

16.4. **The projected correction at one step.** We spell out the metric identity used in the conformal iteration. This also fixes where the target projection is applied. Put

$$e = -\mathsf{P}\gamma(F), \qquad e_R = \mathsf{P}S_{R,m}(e|_D), \qquad b = \delta R^A.$$

The source $e$ satisfies $\mathsf{P}e = e$. Hence

$$e_R - e = \mathsf{P}\Big(S_{R,m}(e|_D) - e\Big).$$

All target coefficients and their derivatives are bounded on the fixed chart. Lemma 17.1, with the same product estimates for the projected error, gives

$$\|e_R\|_{C^r} + \|e_R\|_{H^r} \le C_{r,m}\delta^2 R^r, \qquad \|e_R - e\|_{C^0(D)} \le C_Q \delta R^{-Q}.$$

The second bound uses the minimum of the smallness and high-order approximation bounds, exactly as in that lemma. Neither this step nor reconstruction requires compact Fourier support for $e_R$.

Use the waves, sample and modulation measures from the preceding construction, and let

$$P = b\sqrt{2}\sum_\nu \sqrt{\mu_\nu}\,\mathrm{Re}\,Z_\nu, \qquad W_2 = b^2 V + S.$$

Here $V$ corrects the mean and $S$ corrects all oscillatory products. Define their projected mean error by

$$b^2 E_{\mathrm{mean}} = \mathsf{P}\Big(b^2 D\gamma_F V + H_{\mathrm{self}} - e_R\Big).$$

On $D$ all physical cutoffs equal one. Apply $\mathsf{P}$ to the quadratic metric identity and use $(dP)^T dP = H_{\text{self}} + Q_{\text{off}}$. Since $\mathsf{P}\gamma(F) = -e$, this gives the exact equality

$$\begin{aligned}\mathsf{P}\gamma(F+P+W_2) = (e_R - e) + \mathsf{P}D\gamma_F P + b^2 E_{\text{mean}} \\ + \mathsf{P}(D\gamma_F S + Q_{\text{off}}) \\ + \mathsf{P}\Big(2\,\mathrm{sym}((dP)^T dW_2) + (dW_2)^T dW_2\Big).\end{aligned} \tag{16.7}$$

The first four terms have the prescribed finite accuracy by the smoothing, wave, mean and product estimates. The last line is bounded by $C_Q\delta^3 R^{B_3}$, using the same physical $C^1$ bounds as in Theorem 17.3.

Finally set $U = \Pi(P + W_2)$, with the ambient Fourier multiplier $\Pi$ used in that theorem. The Fourier tail is estimated on the complete ambient field. The exact difference between $\gamma(F+U)$ and $\gamma(F+P+W_2)$ is bounded first; applying the bounded pointwise map $\mathsf{P}$ then gives its projected bound. No interchange of $\mathsf{P}$ with $\Pi$ or $S_{R,m}$ is used. We obtain, for every finite $Q$,

$$\begin{aligned}\|\mathsf{P}\gamma(F+U)\|_{C^0(D)} &\le C_Q(\delta R^{-Q} + \delta^3 R^{B_3}), \\ \|U\|_2 &\le C_Q \delta R^{B_0}, \qquad \operatorname{supp}\widehat{U} \subset \{|\xi| \le C R^{b_*}\}.\end{aligned} \tag{16.8}$$

The hypotheses and exponents are those of Theorem 17.3; only its metric defect is replaced by $\|\mathsf{P}\gamma(F)\|_{C^0(D)} \le \delta^2$. The constants may depend on the fixed positive reference dilation and on finitely many target seminorms. Their frequency exponents and the support constant are independent of $Q$. These are the three estimates used by the common convergence argument.

## 17. The metric correction at one step

In this section we combine the primary waves, the correction of the mean and the correction of the products into one increment. The input is an immersion $F$ with derivative bounds (11.23), a metric defect of size $\delta^2$ on $D$, a frequency $R$ and an accuracy $Q$. The output is a real increment $U$ with exact Fourier support in $\{|\xi| \le CR^{b_*}\}$, an $L^2$ bound with a fixed loss, and new defect $O(\delta R^{-Q} + \delta^3 R^{B_3})$ (Theorem 17.3). The exponents $b_*$, $B_0$, $B_3$ and the support constant do not depend on $Q$; this is what makes the iteration of Sections 18–20 possible.

### 17.1. Smoothing the defect.

The smoothing operator acts on the defect on $D$ and gives two separate bounds for the approximation error: one by the size of the defect, and one by its derivatives.

**Lemma 17.1.** *For each integer $m \ge 1$, let $S_{R,m}$ be the real smoothing operator of Lemma 7.1 for the ball $\Omega = D$, with output supported in a fixed compact subset of $W$. Fix a smooth target tensor $g$ on $W$. Suppose that $T \ge 1$, $0 < \delta \le 1$, and*

$$\|dF\|_{C^r(W)} \le C_r T^r, \qquad \|g - \gamma(F)\|_{C^0(D)} \le \delta^2.$$

*Put $e = g - \gamma(F)$. For all sufficiently large $R$, the tensor $e_R = S_{R,m}(e|_D)$ is smooth and satisfies, for every integer $r \geq 0$,*

$$\|e_R\|_{C^r} + \|e_R\|_{H^r} \leq C_{r,m}\delta^2 R^r, \tag{17.1}$$

$$\|e - e_R\|_{C^0(D)} \leq C_m \delta^2, \tag{17.2}$$

$$\|e - e_R\|_{C^0(D)} \leq C_m (T/R)^m. \tag{17.3}$$

*The constants depend on the constants $C_r$ above and on finitely many seminorms of $g$. If $T \leq R^t$ with $0 < t < 1$, then for every finite $Q$ the choice $(1-t)m > 2Q$ gives*

$$\|e - e_R\|_{C^0(D)} \leq C_Q \delta R^{-Q}. \tag{17.4}$$

*Proof.* By (7.1) and $\|e\|_{C^0(D)} \leq \delta^2$,

$$\|e_R\|_{C^r} + \|e_R\|_{H^r} \leq C_{r,m} R^r \|e\|_{C^0(D)} \leq C_{r,m}\delta^2 R^r,$$

which is (17.1). Taking $r = 0$ and using the triangle inequality, $\|e - e_R\|_{C^0(D)} \leq \|e\|_{C^0(D)} + \|e_R\|_{C^0} \leq (1 + C_{0,m})\delta^2$, which is (17.2). For (17.3), the Leibniz rule applied to $\gamma(F) = (dF)^T dF$ gives

$$\|D^m \gamma(F)\|_{C^0(D)} \leq \sum_{i=0}^{m} \binom{m}{i} \|D^i dF\|_{C^0} \|D^{m-i} dF\|_{C^0} \leq C_m T^m,$$

and hence $\|e\|_{C^m(D)} \leq \|g\|_{C^m(D)} + C_m T^m \leq C_m T^m$, since $T \geq 1$. Now (7.2) gives $\|e - e_R\|_{C^0(D)} \leq C_m R^{-m} T^m$.

Finally, if $T \leq R^t$, then the minimum of the two bounds (17.2) and (17.3) is at most their geometric mean:

$$\|e - e_R\|_{C^0(D)} \leq \left(C_m \delta^2 \cdot C_m (T/R)^m\right)^{1/2} \leq C_m \delta R^{-(1-t)m/2} \leq C_m \delta R^{-Q},$$

because $(1-t)m/2 > Q$ and $R \geq 1$. □

In particular, since $b^2 = \delta^2 R^{2A}$, the normalized tensor $e_R/b^2$ has $C^r$ and $H^r$ norms at most $C_{r,m} R^{r-2A}$. The factor $\delta$ in (17.4) comes from combining the two bounds, not from the high-order bound alone. The defect $e$ may have either sign.

17.2. **Two Fourier inequalities.** We use two elementary consequences of the Fourier inversion formula and the Cauchy–Schwarz inequality.

**Lemma 17.2.** *Let $m_0 = \lfloor n/2 \rfloor + 2$.*

(i) *For every $f \in H^{m_0}(\mathbb{R}^n)$, $\|f\|_{C^1(\mathbb{R}^n)} \leq C\|f\|_{H^{m_0}}$.*

(ii) (Bernstein inequality.) *If $f \in L^2(\mathbb{R}^n)$ and $\operatorname{supp} \widehat{f} \subset \{|\xi| \leq \Lambda\}$ with $\Lambda \geq 1$, then for every integer $r \geq 0$*

$$\|f\|_{C^r(\mathbb{R}^n)} \leq C_r \Lambda^{r+n/2} \|f\|_{L^2}.$$

*Proof.* For a multi-index $\alpha$, the inversion formula gives $|\partial^\alpha f(x)| \leq \int |\xi|^{|\alpha|} |\widehat{f}(\xi)|\, d\xi$. For (i) and $|\alpha| \leq 1$, the Cauchy–Schwarz inequality bounds this by

$$\left(\int \langle \xi \rangle^{2(|\alpha| - m_0)} d\xi\right)^{1/2} \left(\int \langle \xi \rangle^{2m_0} |\widehat{f}|^2 d\xi\right)^{1/2} \leq C\|f\|_{H^{m_0}},$$

since $2(m_0-1)=2\lfloor n/2\rfloor+2>n$. For (ii), with $|\alpha|\le r$,

$$|\partial^\alpha f(x)|\le \Lambda^{|\alpha|}\Big|\{|\xi|\le\Lambda\}\Big|^{1/2}\|\widehat{f}\|_{L^2}=C\Lambda^{|\alpha|+n/2}\|f\|_{L^2},$$

by the Plancherel identity $\|f\|_2^2=(2\pi)^n\|\widehat{f}\|_2^2$. □

17.3. **A single step.** The preceding constructions give the following quantitative step.

**Theorem 17.3** (Correction at one step)**.** *Fix the reference neighborhood, the domains, and the correction operators constructed above. Let $F$ be an immersion in this neighborhood, and assume the hypotheses of Propositions 11.10, 14.9 and 15.4. Suppose that*

$$\|g-\gamma(F)\|_{C^0(D)}\le\delta^2,\qquad T_0=R^t,$$

*with $0<t<1$ and $0<\delta\le1$, and that (11.23) and (15.15) hold. Put $m_0=\lfloor n/2\rfloor+2$ and*

$$B_0=2A+1+\epsilon+a,\qquad B_3=2B_0+2am_0.$$

*For every finite $Q$ and all sufficiently large $R$, there is one real global increment $U=U_Q$ satisfying*

$$\|\gamma(F+U)-g\|_{C^0(D)}\le C_Q\Big(\delta R^{-Q}+\delta^3R^{B_3}\Big), \tag{17.5}$$

$$\|U\|_{L^2}\le C_Q\delta R^{B_0}, \tag{17.6}$$

$$\operatorname{supp}\widehat{U}\subset\{|\xi|\le CR^{b_*}\},\qquad b_*=a(1+\zeta)<3/2. \tag{17.7}$$

*Consequently,*

$$\|U\|_{C^r(\mathbb{R}^n)}\le C_{r,Q}\delta R^{B_0+b_*(r+n/2)}. \tag{17.8}$$

*The frequency exponents and the support constant $C$ in (17.7) are independent of $Q$.*

The exponent $B_0$ is the size at order zero of the quadratic correction: by (15.17), (14.2) and $b=\delta R^A$,

$$b^2M\lambda\le C\delta^2R^{2A}\cdot R^{1+\epsilon}\cdot R^a=C\delta^2R^{B_0}.$$

The exponent $B_3$ arises from the product of the $C^1$ norms of the primary field and of the quadratic correction, each estimated through Lemma 17.2(i) by $\delta^\ell R^{B_0}\lambda^{m_0}$ with $\ell=1,2$; see Step 5 below. The first term in (17.5) comes from the finite depths of the constructions and from Fourier tails; the second is the nonlinear interaction between the primary field and the quadratic correction.

*Proof. Step 1 (Choice of the finite accuracies).* Fix $Q$. Choose integers $m$, $Q_p$, $Q_m$, $q$ and $k$ with

$$\begin{gathered}(1-t)m>2Q,\qquad aQ_p>Q+A+\tfrac{1+\epsilon}{2},\qquad Q_m>Q+2A,\\ (a-\beta)q>Q+2A+1+\epsilon,\qquad a\zeta k>Q+B_3.\end{gathered} \tag{17.9}$$

These depend only on $Q$ and the fixed exponents. We use $m$ for the smoothing $e_R=S_{R,m}(e|_D)$, the depth $Q_p$ in Proposition 11.10 (the wave families are chosen before sampling), the accuracy $Q_m$ in Proposition 14.9, the accuracy $q$ in Proposition 15.4, and $k$ in Step 6. Then we choose the sample on the fixed input band, the measures

$m_\nu$ and the field $V$ of Proposition 14.9, the primary field $P$ of (15.2) and the field $S$ of Proposition 15.4. All these fields carry their fixed physical cutoffs, which equal one near $D$. Put

$$W_2 = b^2 V + S, \qquad U = \Pi(P + W_2),$$

where $\Pi$ is the Fourier multiplier with symbol $\pi_0(\xi/\lambda^{1+\zeta})$, and $\pi_0$ is a fixed smooth real even function equal to one for $|\xi| \le C_1$ and to zero for $|\xi| \ge C_2$. Note that $\lambda^{1+\zeta} = R^{a(1+\zeta)} = R^{b_*}$. Since the symbol is real and even, $\Pi$ maps real fields to real fields. Finally $R$ is taken large enough for all propositions at these accuracies.

*Step 2 (The exact identity).* Put $Y = P + W_2$. Since the physical cutoffs equal one near $D$ and the metric involves only first derivatives, we may compute on $D$ with the uncut fields. By (4.3),

$$\gamma(F + Y) = \gamma(F) + D\gamma_F P + b^2 D\gamma_F V + D\gamma_F S + (dY)^T dY,$$

and

$$(dY)^T dY = (dP)^T dP + 2\,\mathrm{sym}\Big((dP)^T dW_2\Big) + (dW_2)^T dW_2.$$

By (15.3), $(dP)^T dP = H_{\mathrm{self}} + Q_{\mathrm{off}}$, and by (14.31),

$$b^2 E_{\mathrm{mean}} = b^2 D\gamma_F V + H_{\mathrm{self}} - e_R.$$

Substituting these identities and $\gamma(F) - g = -e$, we obtain on $D$ the exact identity

$$\begin{aligned}\gamma(F + P + W_2) - g &= (e_R - e) + D\gamma_F P + b^2 E_{\mathrm{mean}} \\ &\quad + (D\gamma_F S + Q_{\mathrm{off}}) + 2\,\mathrm{sym}\Big((dP)^T dW_2\Big) + (dW_2)^T dW_2. \end{aligned} \tag{17.10}$$

*Step 3 (The linear terms).* We bound the first four terms of (17.10) by $C_Q \delta R^{-Q}$. The first is bounded by Lemma 17.1, applied with $T = T_0 = R^t$; its hypothesis $\|dF\|_{C^r(W)} \le C_r T_0^r$ follows from (11.23), since $(r + 1 - r_*)_+ \le r$ for $r_* \ge 1$.

For the second term, (14.2) and the Cauchy–Schwarz inequality give

$$\sum_\nu \sqrt{\mu_\nu} \le \Big(M \sum_\nu \mu_\nu\Big)^{1/2} \le C R^{(1+\epsilon)/2}.$$

Since $D\gamma_F$ is real, $D\gamma_F P = b\sqrt{2}\sum_\nu \sqrt{\mu_\nu}\,\mathrm{Re}(D\gamma_F Z_\nu)$. The measure bound $\|m_\nu\|_{\mathrm{TV}} \le C$ of Proposition 14.9 and (11.27) with depth $Q_p$ (recall $D \subset D_{\mathrm{lag}}$ and $\lambda = R^a$) give

$$\|D\gamma_F P\|_{C^0(D)} \le C_Q \delta R^{A+(1+\epsilon)/2} \lambda^{-Q_p} = C_Q \delta R^{A+(1+\epsilon)/2 - aQ_p} \le C_Q \delta R^{-Q}, \tag{17.11}$$

by the choice of $Q_p$ in (17.9).

For the third term, Proposition 14.9 at accuracy $Q_m$ gives

$$b^2 \|E_{\mathrm{mean}}\|_{C^0(D)} \le C_Q \delta^2 R^{2A - Q_m} \le C_Q \delta R^{-Q},$$

since $\delta^2 \le \delta$ and $Q_m > Q + 2A$.

For the fourth term, Proposition 15.4 at accuracy $q$, together with $M \le C R^{1+\epsilon}$ and $\lambda h = R^{a-\beta}$, gives

$$\|D\gamma_F S + Q_{\mathrm{off}}\|_{C^0(D)} \le C_Q \delta^2 R^{2A} R^{1+\epsilon} R^{-(a-\beta)q} \le C_Q \delta R^{-Q},$$

since $(a - \beta)q > Q + 2A + 1 + \epsilon$.

*Step 4 (Sobolev bounds for $P$ and $W_2$).* We show that

$$\|P\|_{H^r} \le C_r \delta R^{B_0} \lambda^r, \qquad \|W_2\|_{H^r} \le C_r \delta^2 R^{B_0} \lambda^r. \tag{17.12}$$

For $P$, put $z_\nu = \chi_{\mathrm{core}} e^{-\mathrm{i}\lambda_\nu \phi_\nu} Z_\nu$. By (11.26), with an amplitude scale $K \le \lambda^{1-\nu_0} \le \lambda$, we have $\|z_\nu\|_{C^r} \le C_r \lambda^{-1} K^r \le C_r \lambda^{r-1}$. Since $\|d\phi_\nu\|_{C^r} \le C_r T_0^r$ and $T_0 \le \lambda$, every term in the expansion of a derivative of order $j$ of $e^{\mathrm{i}\lambda_\nu \phi_\nu}$ is a product of factors $\lambda_\nu D^i \phi_\nu$ with total order $j$, each bounded by $C\lambda T_0^{i-1} \le C\lambda^i$. Hence $\|\chi_{\mathrm{core}} Z_\nu\|_{C^r} \le C_r \lambda^{r-1}$. The physical cutoff of $P$ is supported in $D_{\mathrm{lag}}$, where $\chi_{\mathrm{core}} = 1$ by (3.2). Using the weight sum of Step 3 and the fixed support,

$$\|P\|_{H^r} \le C_r b \sum_\nu \sqrt{\mu_\nu}\, \lambda^{r-1} \le C_r \delta R^{A+(1+\epsilon)/2} \lambda^{r-1} \le C_r \delta R^{B_0} \lambda^r.$$

For $W_2$, (14.34) and $R \le \lambda$ give $b^2 \|V\|_{H^r} \le C_r \delta^2 R^{2A} R^r \le C_r \delta^2 R^{B_0} \lambda^r$, and (15.17) gives

$$\|S\|_{H^r} \le C_r \delta^2 R^{2A} R^{1+\epsilon} \lambda^{r+1} = C_r \delta^2 R^{2A+1+\epsilon+a} \lambda^r = C_r \delta^2 R^{B_0} \lambda^r.$$

In all cases each derivative costs at most a factor $\lambda = R^a$.

*Step 5 (The quadratic terms).* By Lemma 17.2(i) and (17.12) with $r = m_0$,

$$\|P\|_{C^1} \le C\delta R^{B_0 + a m_0}, \qquad \|W_2\|_{C^1} \le C\delta^2 R^{B_0 + a m_0}.$$

Therefore, using $\delta \le 1$,

$$\Big\|2\,\mathrm{sym}\Big((dP)^T dW_2\Big)\Big\|_{C^0(D)} + \|(dW_2)^T dW_2\|_{C^0(D)} \le C\Big(\delta^3 + \delta^4\Big) R^{2B_0 + 2a m_0} \le C_Q \delta^3 R^{B_3}.$$

Together with Step 3, the identity (17.10) gives

$$\|\gamma(F + P + W_2) - g\|_{C^0(D)} \le C_Q\Big(\delta R^{-Q} + \delta^3 R^{B_3}\Big). \tag{17.13}$$

*Step 6 (The Fourier truncation).* Since $1 - \pi_0(\xi/\lambda^{1+\zeta})$ vanishes for $|\xi| \le C_1 \lambda^{1+\zeta}$, we have $\langle \xi \rangle^{m_0} \le (C_1 \lambda^{1+\zeta})^{-k} \langle \xi \rangle^{m_0 + k}$ on its support. By Lemma 17.2(i), (17.12) and $\delta \le 1$,

$$\begin{aligned} \|(I - \Pi)(P + W_2)\|_{C^1} &\le C \|(I - \Pi)(P + W_2)\|_{H^{m_0}} \le C_k \lambda^{-(1+\zeta)k} \|P + W_2\|_{H^{m_0 + k}} \\ &\le C_k \delta R^{B_0} \lambda^{m_0 - \zeta k}. \end{aligned}$$

Thus $E_{\mathrm{cut}} = U - (P + W_2) = -(I - \Pi)(P + W_2)$ is small in $C^1$, with a factor $\delta$. Put $G = F + P + W_2$. By (4.3),

$$\gamma(F + U) - \gamma(G) = D\gamma_G E_{\mathrm{cut}} + (dE_{\mathrm{cut}})^T dE_{\mathrm{cut}}, \qquad D\gamma_G E = (dG)^T dE + (dE)^T dG.$$

By Step 5, $\|G\|_{C^1(D)} \le C + C\delta R^{B_0 + a m_0} \le C_Q R^{B_0 + a m_0}$. Hence

$$\begin{aligned} \|\gamma(F + U) - \gamma(G)\|_{C^0(D)} &\le C_Q R^{B_0 + a m_0} \cdot \delta R^{B_0 + a m_0 - a\zeta k} + C_Q \delta^2 R^{2B_0 + 2a m_0 - 2a\zeta k} \\ &\le C_Q \delta R^{B_3 - a\zeta k} \le C_Q \delta R^{-Q}, \end{aligned}$$

since $a\zeta k > Q + B_3$. Combined with (17.13), this proves (17.5).

*Step 7 (The $L^2$, band and $C^r$ bounds).* The multiplier $\Pi$ has norm at most $\sup|\pi_0| = 1$ on $L^2$ (we take $0 \le \pi_0 \le 1$), so (17.12) at $r = 0$ gives $\|U\|_{L^2} \le \|P\|_{L^2} + \|W_2\|_{L^2} \le C_Q \delta R^{B_0}$,

which is (17.6). The symbol of $\Pi$ vanishes for $|\xi| \geq C_2\lambda^{1+\zeta} = C_2 R^{b_*}$, which gives (17.7); and $b_* < 3/2$ by (19.1). Lemma 17.2(ii) with $\Lambda = C_2 R^{b_*}$ gives

$$\|U\|_{C^r(\mathbb{R}^n)} \leq C_r R^{b_*(r+n/2)} \|U\|_{L^2} \leq C_{r,Q}\delta R^{B_0+b_*(r+n/2)},$$

which is (17.8). The field $U$ is real because $P$, $V$ and $S$ are real and $\Pi$ preserves real fields. □

## 18. Uniform estimates along the iteration

The exact Fourier support of the increments gives the high-order bounds on the iterates which are required in the hypotheses of Theorem 17.3. We prove these bounds (Lemma 18.1), transfer them to the coefficients of all constructions (Proposition 18.2), and conclude that the single-step theorem applies at every iterate with uniform constants (Proposition 18.3). The parameters are fixed in Section 19, and the iteration is carried out in Section 20.

**Lemma 18.1** (Estimates for sums of Fourier-localized increments)**.** *Fix an integer $r_* \geq 1$ and numbers $1 < b_* < v < 3/2$, and put $t = b_*/v$. For $R_0 \geq 2$ set $R_j = R_0^{v^j}$. Suppose that real global increments $U_j$ satisfy, with integers $k_j \geq k_0$,*

$$\operatorname{supp} \widehat{U}_j \subset \{|\xi| \leq C R_j^{b_*}\}, \qquad \|U_j\|_{L^2} \leq R_j^{-k_j+B_0+1}. \tag{18.1}$$

*Put $F_j = F_{\mathrm{init}} + \sum_{i<j} U_i$ on $W$, $T_j = R_j^t$ and $b_\sharp = B_0 + b_*(r_* + n/2)$. If $\mu = k_0 - b_\sharp - 1 > 0$, then*

$$\|F_j\|_{C^r(W)} \leq C_r T_j^{(r-r_*)_+} \qquad (r \geq 0), \tag{18.2}$$

*and*

$$\sum_j \|U_j\|_{C^{r_*}(\mathbb{R}^n)} \longrightarrow 0 \quad \text{as } R_0 \longrightarrow \infty. \tag{18.3}$$

*Both assertions are uniform for a compact smooth family of initial maps $F_{\mathrm{init}}$.*

*Proof.* By the Bernstein inequality, Lemma 17.2(ii), with $\Lambda = C R_i^{b_*}$, and by (18.1),

$$\|U_i\|_{C^{r_*}(\mathbb{R}^n)} \leq C R_i^{b_*(r_*+n/2)} R_i^{-k_i+B_0+1} = C R_i^{-k_i+b_\sharp+1}. \tag{18.4}$$

The constant $C$ depends only on $n$, $r_*$ and the support constant. Since $k_i \geq k_0$, the exponent is at most $-\mu$, so $\|U_i\|_{C^{r_*}} \leq C R_0^{-\mu v^i}$. Bernoulli's inequality $v^i \geq 1 + i(v-1)$ and $R_0 > 1$ give $R_0^{-\mu v^i} \leq R_0^{-\mu}\big(R_0^{-\mu(v-1)}\big)^i$, and hence

$$\sum_{i\geq 0} \|U_i\|_{C^{r_*}(\mathbb{R}^n)} \leq \frac{C R_0^{-\mu}}{1 - R_0^{-\mu(v-1)}}. \tag{18.5}$$

The right-hand side tends to zero as $R_0 \to \infty$, which proves (18.3).

For $r \leq r_*$, (18.5) gives $\|F_j\|_{C^r(W)} \leq \|F_{\mathrm{init}}\|_{C^{r_*}(W)} + C$, uniformly in $j$. Let $r > r_*$, and put $\Psi_j = \sum_{i<j} U_i$. For $i < j$ we have $R_i^{b_*} \leq R_{j-1}^{b_*} = R_j^{b_*/v} = T_j$, so $\operatorname{supp} \widehat{\Psi}_j \subset \{|\xi| \leq C T_j\}$. Fix $\chi \in C_c^\infty(\mathbb{R}^n)$ equal to one on $\{|\xi| \leq C\}$, put

$$\kappa(x) = (2\pi)^{-n} \int e^{\mathrm{i}x\cdot\xi} \chi(\xi)\, \mathrm{d}\xi, \qquad \kappa_j(x) = T_j^n \kappa(T_j x).$$

With the normalization of Notation 3.1 the convolution theorem reads $\widehat{f * g} = (2\pi)^n \widehat{f}\,\widehat{g}$, and $\widehat{\kappa}_j(\xi) = (2\pi)^{-n}\chi(\xi/T_j)$; hence $\widehat{\kappa_j * \Psi_j} = \chi(\cdot/T_j)\widehat{\Psi}_j = \widehat{\Psi}_j$, that is, $\Psi_j = \kappa_j * \Psi_j$. For $|\alpha| = r$ write $\alpha = \beta + \gamma$ with $|\beta| = r_*$; then $\partial^\alpha \Psi_j = (\partial^\gamma \kappa_j) * \partial^\beta \Psi_j$ and $\|\partial^\gamma \kappa_j\|_{L^1} = T_j^{r-r_*}\|\partial^\gamma \kappa\|_{L^1}$. Young's inequality gives

$$\|\partial^\alpha \Psi_j\|_{L^\infty} \le C_r T_j^{r-r_*}\|\Psi_j\|_{C^{r_*}} \le C_r T_j^{r-r_*}.$$

Since $F_{\mathrm{init}}$ is smooth on $\overline{W}$ and $T_j \ge 1$, also $\|F_{\mathrm{init}}\|_{C^r(W)} \le C_r \le C_r T_j^{r-r_*}$. This proves (18.2). The initial map is used only on $W$, and all constants depend on finitely many seminorms of $F_{\mathrm{init}}$, which gives the uniformity. □

The derivative bounds (18.2) for the iterates imply the coefficient bounds used in Sections 10–15, uniformly along such sequences.

**Proposition 18.2** (Coefficient estimates along the iteration)**.** *Choose a fixed order $\bar{r}_0$ above the orders used in the constructions and the source order $s_0$, and choose $r_*$ above $\bar{r}_0$ by the fixed shifts of the coefficients and of the reconstruction. For the sequences of Lemma 18.1, the reduced coefficients $a(F_j)$ on $W$ and the positive initial amplitudes satisfy*

$$\|a(F_j)\|_{C^r(W)} + \sum_a \|r_{F_j,a}\|_{C^r_{z,y}} \le C_r T_j^{(r-\bar{r}_0)_+}. \tag{18.6}$$

*Consequently the estimates for phases, symbols and waves apply with $L = T_j$, and these derivative bounds give the bounds for the source and for the coefficients used in Proposition 14.7 for all fixed-parameter operators. The shifts are fixed before the accuracy; the finite number of correction steps for the waves and the number of iteration steps enter only the constants and the thresholds.*

*Proof.* The inverse metric $g_F^{-1}$, the normal map $N_F$ and the projected frame $Q_i = \Pi_{K_F} Q_i^0$ are smooth functions of the second jet of $F$ on a set where the rank and Gram gaps are bounded below. By Lemma 7.2, such functions transfer the derivative bounds (18.2) of $F_j$, with a fixed shift of the index. The same holds for the positive initial amplitudes by (13.13). This proves (18.6). The kernel estimates of Setup 10.4 and Corollary 11.8 then give the conclusions for the phases and for the corrected waves with initial data $\delta_0$. No regularity with respect to the sample index is required. □

**Proposition 18.3** (Uniformity of the correction estimates)**.** *Choose $r_*$ larger than the fixed derivative orders of Propositions 11.10, 14.9 and 15.4, including the source order of Proposition 14.9. Consider a sequence satisfying the hypotheses of Lemma 18.1 whose total displacement stays in the corresponding $C^{r_*}$ neighborhood. Then these propositions apply successively at $F_j$, with $T_0 = T_j$ and the scales of Section 19. At each fixed accuracy their constants are uniform over all such sequences. The same $r_*$ works for every accuracy.*

*Proof.* Equation (18.2) is (11.23) with $T_0 = T_j$, and (18.3) keeps $F_j$ in the fixed $C^{r_*}$ neighborhood. This neighborhood preserves the low-order conditions (L) of Setup 6.3 and all lower bounds, and, since $r_* \ge M_{\mathrm{an}} + 11$, also the high-order bound (H), by

Remark 6.6. Proposition 18.2 transfers (18.2) to the coefficient bounds used by the three propositions. These constants depend only on the constants in (18.2), which are uniform by Lemma 18.1. □

Since $t < 1$, the coefficient scale $T_j = R_j^t$ lies below the input frequency bound $K_{\max} = R_j^{s_f}$ of the mean and below the smallest product frequency $c\lambda h$. The next section verifies both strict gaps.

## 19. Choice of parameters

The estimates require only an open set of exponents. The following choice verifies all of them, before the accuracy or the target dilation is chosen.

**Proposition 19.1** (Compatible scales)**.** *Choose*

$$0 < \epsilon < \tfrac{1}{8}, \quad \beta = \frac{1+\epsilon}{3}, \quad 1 < s_f < 1+\epsilon, \quad 1+\beta < a < b_* < v < \tfrac{3}{2}. \tag{19.1}$$

*Set*

$$\frac{2}{3} < t = \frac{b_*}{v} < 1, \qquad \zeta = \frac{b_*}{a} - 1 > 0, \tag{19.2}$$

*and*

$$m_0 = \lfloor n/2 \rfloor + 2, \qquad s_0 = 2m_0, \qquad A = m_0 + \tfrac{1}{2}. \tag{19.3}$$

*Use $\lambda = R^a$, $h = R^{-\beta}$, $K_{\max} = R^{s_f}$ and*

$$T_0 = R^t. \tag{19.4}$$

*There are fixed $\sigma, \nu_0 > 0$ such that the conditions (11.25) of Proposition 11.10, (15.15) of Proposition 15.4, and the conditions of Proposition 14.9 hold for all large $R$. The fixed derivative orders are chosen before the accuracy and the target dilation.*

*Proof. Step 1 (The ordering).* Since $\epsilon < 1/8$, we have $\beta = (1+\epsilon)/3 < (9/8)/3 = 3/8$, so $1+\beta < 11/8 < 3/2$ and the interval $(1+\beta, 3/2)$ is nonempty; hence $a, b_*, v$ can be chosen. Next, $\epsilon < \beta$ is equivalent to $3\epsilon < 1+\epsilon$, that is $\epsilon < 1/2$; thus $s_f < 1+\epsilon < 1+\beta < a$. Since $b_* > a > 1$ and $v < 3/2$, we get $t = b_*/v > 2/3$, and $t < 1$ because $b_* < v$; also $\zeta > 0$ because $b_* > a$, and $b_* = a(1+\zeta)$. Altogether

$$2/3 < t < 1 < s_f < a < b_* < v < 3/2, \qquad a - \beta > 1.$$

*Step 2 (The strict gaps).* Since $a - \beta > 1$ and $s_f < a$, we may choose $\sigma, \nu_0 > 0$ so small that $(1-\sigma)(a-\beta) > 1$ and $s_f < (1-\nu_0)a$. Then $(\lambda h)^{1-\sigma} = R^{(1-\sigma)(a-\beta)}$ and $\lambda^{1-\nu_0} = R^{(1-\nu_0)a}$ are strictly larger powers of $R$ than $R$ and $K_{\max} = R^{s_f}$, respectively, so every fixed constant is absorbed for large $R$:

$$C_* R \le (\lambda h)^{1-\sigma}, \qquad C_* K_{\max} \le \lambda^{1-\nu_0}. \tag{19.5}$$

The first inequality is (15.15): the amplitude scale $R$ of the mean lies below the product frequencies. The second puts the input frequency bound below the wave frequency, as

required in (11.25) with $\Lambda = CK_{\max}$ and $K = C_*K_{\max}$; there also $C_{\mathrm{env}}T_0 \le K$ because $t < 1 < s_f$. Moreover $c\lambda h = cR^{a-\beta} \ge 2$ for large $R$.

*Step 3 (The exponents of the mean).* At the fixed input band the concentration exponent is

$$\omega = (1 + \epsilon - s_f)/4 > 0, \tag{19.6}$$

which is positive because $s_f < 1 + \epsilon$. The phase error is smaller: by Step 1,

$$a - s_f > (1 + \beta) - (1 + \epsilon) = \beta - \epsilon = \frac{1 - 2\epsilon}{3} > \frac{1 - 1/4}{3} = \frac{1}{4},$$

while $\omega < \epsilon/4 < 1/32$ because $s_f > 1$. Hence $\omega < a - s_f$. The source order satisfies $s_0 = 2m_0 \ge 2\left(\frac{n-1}{2} + 2\right) = n + 3 > n$, and

$$2A - s_0 = 2m_0 + 1 - 2m_0 = 1 > \beta.$$

The smoothing estimate of Lemma 17.1 gives $\|e_R/b^2\|_{H^{s_0}} \le CR^{-1}$, while the sampled background has error $O(h) = O(R^{-\beta})$. Their sum is $O(R^{-\beta})$ in $H^{s_0}$, since $\beta < 1$. The normal error and the quadratic estimate give a small $L^2$ factor for successive differences of the iteration in Proposition 14.9; the tail of the initial-frequency cutoff is controlled by the same derivative bounds at higher orders. The amplitude scale $R$ and the coefficient scale $T_0$ lie below the fixed input band $K_{\max} = R^{s_f}$, because $t < 1 < s_f$; this permits arbitrary finite precision for this sample. □

19.1. **The accuracy as a function of the defect.** We fix $r_*$ after these derivative orders and after all geometric and analytic shifts. With $b_\sharp = B_0 + b_*(r_* + n/2)$, choose an integer $k_{\min}$ such that

$$(3 - 2v)k_{\min} > B_3 + 2v + 4, \tag{19.7}$$

$$k_{\min} > b_\sharp + 2. \tag{19.8}$$

These are two lower bounds for $k_{\min}$, and they are compatible because $3 - 2v > 0$, by $v < 3/2$. We take the accuracy

$$Q(k) = 2k. \tag{19.9}$$

At defect size $\delta = R^{-k}$, both terms in (17.5) are bounded by $C_{2k}R^{-3k+B_3}$:

$$\delta R^{-Q(k)} = R^{-3k} \le R^{-3k+B_3}, \qquad \delta^3 R^{B_3} = R^{-3k+B_3}.$$

For $k \ge k_{\min}$ and $R \ge 1$, (19.7) gives

$$R^{-3k+B_3} = R^{-2v(k+1)}\, R^{-(3-2v)k+B_3+2v} \le R^{-2v(k+1)}\, R^{-4}. \tag{19.10}$$

The strict gaps proved above permit all the internal accuracy choices of Theorem 17.3 without changing the domains, the derivative orders or the frequency exponents.

## 20. A fixed reference and a small target

For the rest of the proof write

$$\mathcal{E}_g(F) = \mathsf{P}_g\Big(\gamma(F) - g\Big), \qquad \mathsf{P}_g = I \text{ or } \mathsf{P}_g H = H - \frac{\operatorname{tr}(g^{-1}H)}{n} g.$$

The target $g$ is fixed along an iteration. In the second case $\mathcal{E}_g(F) = \mathsf{P}_g\gamma(F)$.

**Proposition 20.1** (Fixed reference initialization)**.** *In the isometric case use the analytic reference of Proposition 4.7 and set $H = I$. In the conformal case use Proposition 5.1 and set $H = H_0$. Choose normal coordinates for the prescribed metric, followed by a constant linear change, so that $g(0) = H$ and $Dg(0) = 0$. For one fixed sufficiently small $\sigma > 0$ put*

$$F_\sigma(y) = \sigma^{-1}\Big(F_*(\sigma y) - F_*(0)\Big), \qquad g^{(s)}(y) = g(s\sigma y).$$

*On one fixed chart, and for $0 \le s \le s_*$, this family has common metric, normalized rank and stress bounds. Its positive transported background and continuous inverse can be chosen on common domains. For every fixed $r$,*

$$\|\mathcal{E}_{g^{(s)}}(F_\sigma)\|_{C^r} \le C_{r,\sigma} s^2. \tag{20.1}$$

*The reference $F_\sigma$ is independent of $s$.*

*Proof.* Taylor's formula gives

$$g(s\sigma y) = H + s^2\sigma^2 T(s,\sigma,y), \qquad T = \int_0^1 (1-t) D^2 g(ts\sigma y)[y,y]\, dt,$$

with $T$ jointly smooth. In both cases $\gamma(F_\sigma) = H$. Consequently

$$\mathcal{E}_{g^{(s)}}(F_\sigma) = -s^2\sigma^2 \mathsf{P}_{g^{(s)}} T.$$

This identity proves (20.1) in every fixed norm. The metrics have a common positive lower bound.

The actual second form is $\sigma B_*(\sigma y)$. In the conformal case its normalized augmented contraction is $(\widehat{B}_\sigma, g^{(s)})$, which tends to $(C_0, H_0)$ as $\sigma \to 0$. The rank and stress gaps therefore persist uniformly on a compact parameter interval. The normalized inverse and physical amplitudes are those of (16.4). The full-output detection estimate and its variable-coefficient continuation give the continuous inverse. The positive background follows from the affine cancellation in the proof of Proposition 16.1.

Fix the finite derivative orders, choose $\sigma$ once, and then restrict to a finite-order neighborhood of this compact family. All subsequent domains and constants can be chosen before $s > 0$ is made smaller. This uses smooth compact families and strict gaps; it imposes no condition on higher Taylor coefficients of $g$. □

## 21. One sequence and the smooth limit

The following argument uses the one-step theorem only through its three quantitative conclusions. It applies to the isometric estimates of Theorem 17.3 and their projected versions in Proposition 16.1.

Fix the compatible exponents of Proposition 19.1, so that $b_* < v < 3/2$ and $t = b_*/v < 1$. Fix all geometric and analytic derivative orders, including $r_*$, before choosing $k_{\min}$. Write $B_0, B_3$ for the fixed outside exponents in the one-step theorem. Choose $k_{\min}$ large enough for its coefficient hypotheses and

$$\begin{aligned} k_{\min} &> B_0 + 2 + b_*(r_* + n/2), \\ (3-2v)k_{\min} &> B_3 + 2v + 4. \end{aligned} \tag{21.1}$$

Both are finite requirements. Set $Q(k) = 2k$ for $k \geq k_{\min}$. The one-step estimates are

$$\begin{aligned} \|\mathcal{E}_g(F+U)\|_{C^0(D)} &\leq C_{2k}\Big(R^{-3k} + R^{-3k+B_3}\Big), \\ \|U\|_2 \leq C_{2k}R^{-k+B_0}, &\qquad \operatorname{supp}\widehat{U} \subset \{|\xi| \leq CR^{b_*}\}, \end{aligned} \tag{21.2}$$

when $\|\mathcal{E}_g(F)\|_{C^0(D)} \leq R^{-2k}$. The estimates are uniform over the admitted finite histories and the compact target family of Proposition 20.1.

For each $k$ choose a finite threshold $H(k)$ covering the finitely many accuracies used at that index and satisfying $C_{2k} \leq R$ and $2C_{2k} \leq R^4$ for $R \geq H(k)$. Take finite successive maxima to make $H$ nondecreasing. Enlarge $H(k_{\min})$ so that the Fourier displacement estimate below stays inside the fixed geometric neighborhood and preserves half the reference immersion gap. No supremum over derivative orders is used.

Choose $R_0 \geq H(k_{\min})$, and define

$$R_j = R_0^{v^j}, \qquad k_0 = k_{\min}, \qquad k_{j+1} = k_j + \mathbf{1}_{\{R_{j+1} \geq H(k_j+1)\}}. \tag{21.3}$$

Then $R_j \geq H(k_j)$ and $k_{j+1} \leq k_j + 1$. The integers $k_j$ tend to infinity: eventual constancy at $K$ would contradict $R_j \to \infty$ and finiteness of $H(K+1)$.

Only now choose $s \in (0, s_*]$ with $C_{0,\sigma}s^2 \leq R_0^{-2k_{\min}}$. Let $F_0 = F_\sigma$ and, inductively, choose the increment in (21.2) at $F_j$, $R_j$, $k_j$, and put $F_{j+1} = F_j + U_j$. We verify

$$\|U_j\|_2 \leq R_j^{-k_j+B_0+1}, \qquad \|\mathcal{E}_{g^{(s)}}(F_j)\|_{C^0(D)} \leq R_j^{-2k_j}.$$

The first follows from $C_{2k_j} \leq R_j$. The second holds at zero by initialization. At the next step, (21.1) gives

$$\|\mathcal{E}_{g^{(s)}}(F_{j+1})\|_{C^0(D)} \leq 2C_{2k_j}R_j^{-3k_j+B_3} \leq R_j^{-2v(k_j+1)} \leq R_{j+1}^{-2k_{j+1}}.$$

The history estimates of Section 18 check the coefficient hypotheses uniformly. Indeed Fourier Cauchy–Schwarz gives

$$\|U_j\|_{C^r} \leq C_r R_j^{-k_j+B_0+1+b_*(r+n/2)}. \tag{21.4}$$

At $r = r_*$ the exponent is less than $-1$ by (21.1). Its sum is as small as desired by increasing $R_0$, since $v^j \geq 1 + j(v-1)$. This proves the prospective displacement bound used in the choice of $H(k_{\min})$ and closes the induction. At a finite stage choose one realization of its sampling event and use it in the whole correction.

For every fixed $r$, $k_j \to \infty$ makes the tail in (21.4) bounded by $C_r R_j^{-1}$. Hence the single series

$$F_\infty = F_\sigma + \sum_{j \geq 0} U_j$$

converges with all derivatives on the same chart. Its $C^1$ bound preserves immersion, and continuity of the metric equation gives $\mathcal{E}_{g^{(s)}}(F_\infty) = 0$ on $\operatorname{int} D$. For isometry this is the required metric identity. For conformality put

$$\rho_\infty = \frac{1}{n} \operatorname{tr}\Big((g^{(s)})^{-1}\gamma(F_\infty)\Big).$$

The projection identity gives $\gamma(F_\infty) = \rho_\infty g^{(s)}$. The factor is smooth and positive because $F_\infty$ is an immersion and $g^{(s)}$ is positive definite.

On $s\sigma \operatorname{int} D$ define

$$\widetilde{F}(x) = s\sigma F_\infty(x/(s\sigma)).$$

Its differential is $dF_\infty(x/(s\sigma))$; thus the original metric is recovered exactly, with factor $\rho_\infty(x/(s\sigma))$ in the conformal case. Undo the one initial coordinate map. At the base point project to an ambient tangent $n$-plane. The composition has invertible differential and is a local diffeomorphism. On one smaller neighborhood the original immersion is therefore an embedding. This proves Theorems 1.1 and 1.2.

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