

SMOOTH LOCAL ISOMETRIC EMBEDDINGS OF RIEMANNIAN MANIFOLDS IN OPTIMAL DIMENSION

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ABSTRACT. We prove that every smooth Riemannian manifold of dimension $n \geq 5$ admits a smooth local isometric embedding into $\mathbb{R}^{n(n+1)/2}$. This extends the classical Janet–Cartan theorem from real-analytic to arbitrary smooth metrics in these dimensions, achieving the optimal universal ambient dimension without curvature or genericity assumptions. The proof constructs characteristic waves around a flat immersion whose second fundamental form has full normal rank. Their Hermitian quadratic mean generates the required metric correction, while linear responses cancel oscillatory quadratic interactions and the remaining linear strain. Exact Fourier localization yields an iteration converging in C^∞ on a fixed neighborhood. The detailed geometric and microlocal constructions are proved in the appendices.

CONTENTS

1. Introduction	3
2. The reference geometry	5
3. Linear response on fixed domains	8
4. Characteristic waves and the mean equation	11
5. The normal estimate and the nonlinear mean	15
6. The metric correction	20
7. Histories and coefficient control	22
8. Choice of scales	23
9. Smooth convergence on one neighborhood	25
Appendix A. Domains, Fourier conventions, and notation	27
Appendix B. Interpolation and finite-jet estimates	30
Appendix C. Oscillatory estimates and canonical retarded integration	34
Appendix D. A fixed flat reference and its carrier geometry	38
Appendix E. The regular-sector construction	47
Appendix F. The fixed-domain inverse	57
Appendix G. Primary waves and simultaneous sampling	64
Appendix H. Ambient response to modulated waves	73
Appendix I. Transported positive backgrounds and current rows	79
Appendix J. Current correction for Fourier measures	88
Appendix K. The sampled current mean	89

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Appendix L. Signed quadratic products	92
Appendix M. Verification of the fixed scales	94
Acknowledgements	95
References	95

1. INTRODUCTION

The local isometric immersion problem asks whether a Riemannian metric g near a point of an n -manifold can be written as

$$(1.1) \quad g_{ij} = \partial_i F \cdot \partial_j F$$

for a map into a Euclidean space of prescribed dimension. The analytic local theory of Janet (1926) and Cartan (1927) [J26, C27] reaches

$$N = s_n = \frac{n(n+1)}{2},$$

the number of independent metric coefficients. For arbitrary smooth metrics, the classical local theorem gives $\mathbb{R}^{s_n+n}$ [G70]; see also [G17, §3.7] for the perturbation and scaling argument. A free reference in that construction has s_n normal directions and an algebraic normal inverse for the linearized metric equation [N56]. We reach $\mathbb{R}^{s_n}$ for every smooth metric in dimensions $n \geq 5$. Our reference has $s_n - n$ normal directions; the remaining n stress equations form a determined first-order system.

Theorem 1.1. Let $n \geq 5$, let p belong to a smooth n -dimensional manifold, and let g be a smooth Riemannian metric near p . There exist an open neighborhood U of p and a smooth embedding

$$F : U \longrightarrow \mathbb{R}^{n(n+1)/2}$$

satisfying (1.1) on U .

The same map and neighborhood work at every derivative order. The endpoint $n = 5$ gives $\mathbb{R}^{15}$; at $n = 6$ the target is $\mathbb{R}^{21}$. The theorem establishes the smooth local Schläfli assertion in dimensions $n \geq 5$, in the terminology of [G17, §1.2], without an analyticity or curvature hypothesis. The endpoint is included in the same construction: a finite-dimensional node calculation completes the characteristic menu at $n = 5$, after which all dimensions use the same response, sampled mean and convergence proof. No conclusion for $n = 4$ is asserted.

Remark 1.2 (Sharpness of the universal target dimension). The dimension s_n is necessary for a target that works for every smooth metric germ. If $M < s_n$, the polynomial induced-metric map sends $(k+1)$ -jets of maps into $\mathbb{R}^M$, a space of dimension $M \binom{n+k+1}{n}$, to k -jets of symmetric metrics, of dimension $s_n \binom{n+k}{n}$. Choose k with $(s_n - M)(k+1) > Mn$. The source dimension is then strictly smaller, so the smooth image has measure zero and misses a jet with positive definite constant term. Its Taylor polynomial is a positive definite analytic metric on a small neighborhood, and an isometric immersion into $\mathbb{R}^M$ would realize the omitted jet, a contradiction. This is a universal lower bound; individual metrics can have smaller embedding dimension.

The reduction of the linearized equations and their characteristic geometry were developed by Bryant–Griffiths–Yang [BGY83]. Smoothness of the characteristic variety cannot be assumed. Han–Khuri [HK12, Theorem 1.2] prove nonsmoothness for $n \geq 5$ in a sufficiently small parameter regime under explicit generic conditions. Our flat model

also has singular characteristics, classified in Section 2. We choose simple carriers whose sums and nonzero differences stay in protected regular sectors. The finite carrier menu detects every missing image direction, including the five-dimensional node directions and those on the multiple set: the carrier and output covectors need not coincide. The resulting quadratic means complete the missing image directions and give a positive symbol on the full stress space. Regular responses cancel the oscillatory products, while this augmented mean equation handles the full Hermitian tensor. No inverse of the determined symbol on its singular set is used.

The analytic theory is developed in [BCGGG91]. The smooth low-dimensional results and finite conjugation methods of Nakamura–Maeda [NM89] and Goodman–Yang [GY88], and the real-principal-type theory of Duistermaat–Hörmander [DH72], form the background for the quantitative matrix and ambient estimates proved here on fixed domains, with accuracy-independent losses. The three-dimensional embedding results of Nakamura–Maeda and Goodman–Yang have target $\mathbb{R}^6$ and require the Riemann curvature tensor to be nonzero at the base point.

The argument begins with a flat immersion whose second form spans the normal space. The kernel of its normal map is a smooth perturbation of an explicit n -dimensional space of quadratic forms. Projection onto this kernel gives a square first-order system with simple and multiple characteristics. A finite family of three-parameter patches on the simple locus provides the oscillations. Their signed frequency sums and differences lie in regions admitting a linear response.

The principal correction is quadratic. A real sum of complex characteristic waves has Hermitian self terms and oscillatory products. A transported positive background and a mean correction adjust the self terms; a response for the current immersion cancels the other products. The same waves and sample weights occur throughout, so the cancellations are identities for the metric of one map.

All waves and responses use the current immersion. A fixed finite correction of the continuous density prepares the positive background before sampling. High-order quadrature preserves this accuracy, and one sampled mean iteration reaches the requested precision. Its row, wave maps and continuum inverse remain fixed during the iteration.

The resulting increment satisfies a metric estimate of the form

$$(1.2) \quad \|\gamma(F + U) - g\|_{C^0} \leq C_Q (\delta R^{-Q} + \delta^3 R^{B_3}),$$

where $\|\gamma(F) - g\|_{C^0} \leq \delta^2$ and $\gamma(F) = (dF)^T dF$. The constant B_3 is independent of the requested accuracy Q . Moreover, U has exact global Fourier support below a fixed power of R , and its global L^2 norm has a similar fixed loss. These two properties allow Q to increase slowly enough to absorb its constants while ensuring convergence in every C^r norm. Thus the same sequence converges smoothly on one neighborhood.

Section 2 describes the geometric starting point. Section 3 states the required linear estimates and their uniformity. Section 4 constructs the waves and states the mean and product estimates. Section 5 develops the central normal and nonlinear mean arguments. Section 6 combines these constructions and proves (1.2). Sections 7–9

choose the scales and prove Theorem 1.1. The quantitative statements in Sections 3–5 contain the inputs used by the metric and convergence proofs, so those proofs can be read while accepting the technical appendix results. Table 1 gives their sources, domains, low topologies and dependence on accuracy. The appendices prove these inputs in separate blocks; Appendix M checks their simultaneous parameter windows.

2. THE REFERENCE GEOMETRY

2.1. The reduced metric equation. For an immersion $F : W \rightarrow \mathbb{R}^N$, write

$$D\gamma_F Y = (dF)^T dY + (dY)^T dF.$$

Its exact expansion is

$$(2.1) \quad \gamma(F + Y) = \gamma(F) + D\gamma_F Y + (dY)^T dY.$$

Let Π_F denote the second fundamental form, and set $q = N - n = n(n - 1)/2$. We regard it as a map

$$B_F : \text{Sym}^2 \mathbb{R}^n \longrightarrow \mathcal{N}_F, \quad B_F(H) = \sum_{i,j} H^{ij} \Pi_{F,ij}.$$

When B_F is onto, its kernel has dimension n . We call this kernel the stress space.

Write an ambient variation as $Y = dF \gamma(F)^{-1} u + w$, where u is a covector and w is normal. Then

$$(2.2) \quad D\gamma_F Y = 2 \text{sym} \nabla u - 2 \langle w, \Pi_F \rangle.$$

Here ∇ is the Levi-Civita connection of the induced metric $\gamma(F)$, and $\text{sym} A = (A + A^T)/2$. In particular, this connection need not be the connection of the target metric before the iteration has converged. The sign of the normal term follows by differentiating $\partial_i F \cdot w = 0$.

Use fixed smooth reference tensors to form the projected stress frame $Q_i(F) = \Pi_{\ker B_F} Q_i^{\text{ref}}$, with a positive Gram gap, and write $Q : H = \sum_{i,j} Q^{ij} H_{ij}$. Define

$$(2.3) \quad q_F(h) = (Q_i(F) : h)_{i=1}^n, \quad \mathcal{P}_F u = (Q_i(F) : \text{sym} \nabla u)_{i=1}^n.$$

Then $D\gamma_F Y = h$ reduces to

$$(2.4) \quad \mathcal{P}_F u = \frac{1}{2} q_F(h).$$

The factor $1/2$ will also be used in the mean equation. Since B_F is onto, its adjoint maps the normal space isomorphically onto $(\ker B_F)^\perp$. For any u and h , set

$$(2.5) \quad w = (B_F^*)^{-1} \Pi_{(\ker B_F)^\perp} \left(\text{sym} \nabla u - \frac{1}{2} h \right).$$

For $Y = dF \gamma(F)^{-1} u + w$ and $\mathbf{G}_F = (Q_i(F) : Q_j(F))_{ij}$, the full error is

$$(2.6) \quad D\gamma_F Y - h = 2 \sum_{i,j} Q_i(F) (\mathbf{G}_F^{-1})_{ij} (\mathcal{P}_F u - \frac{1}{2} q_F(h))_j.$$

This follows from (2.2) and the Gram formula for orthogonal projection. Thus a reduced solution solves the full tensor equation; an approximate solution has exactly the displayed stress error. We call this algebraic step *normal completion*.

2.2. An explicit flat reference. Let J be the all-ones $n \times n$ matrix, put $c_n = (n-4)^{-2}$, and define

$$(2.7) \quad Q_i = e_i e_i^T - c_n J, \quad K_0 = \text{span}\{Q_1, \dots, Q_n\}.$$

With the normalization in (2.3), the principal symbol of the frozen reduced operator is $iM_0(\xi)$ under $\partial_j \mapsto i\xi_j$. Its real contraction pencil is

$$(2.8) \quad M_0(\xi) = \text{diag}(\xi_1, \dots, \xi_n) - c_n \left(\sum_i \xi_i \right) J.$$

Put $\ell_{ab} = \mathbf{1} - 2e_a - 2e_b$, $a < b$. The q rank-one forms $\ell_{ab}\ell_{ab}^T$ are independent, annihilate every Q_i , and each satisfies the flat Gauss equation. Successive circular bendings add these forms one at a time while preserving the Euclidean metric. They give an explicit analytic flat immersion with full normal rank and center stress space K_0 . Propositions D.2 and D.4 prove these assertions.

After one fixed dilation the actual stress frames are uniformly close to (2.7); all later operators use these varying coefficients.

2.3. Characteristics and product directions. The determinant of (2.8) is

$$(2.9) \quad p_0(\xi) = \prod_i \xi_i - c_n \left(\sum_i \xi_i \right) \sum_i \prod_{j \neq i} \xi_j.$$

The nonzero set $\{p_0 = 0, dp_0 = 0\}$ is the union of

$$(2.10) \quad \{\xi_i = \xi_j = \xi_k = 0\}, \quad \{\xi_i = \xi_j = \sum_l \xi_l = 0\}, \quad \mathbb{R}\ell_{ab},$$

with the origin removed. The first two families have codimension three. The additional lines have corank one and zero determinant gradient; every other corank-one zero is regular. Proposition D.3 proves this complete classification.

At a simple characteristic covector ξ , the matrix $M_F(x, \xi)$ has a one-dimensional kernel, which determines a leading tangent polarization. After normal reconstruction, the leading derivative of its wave is a rank-one tensor of the form $\nu \otimes \xi$. Its Hermitian square is therefore $|\nu|^2 \xi \otimes \xi$. Under stress contraction this gives the vector $|\nu|^2 q_F(\xi \otimes \xi)$. The detector inequality (5.3) states how these vectors complete the image of M_F at every output direction. Section 5 constructs the positive transported background from this geometric input.

Quadratic products may leave a chosen characteristic patch. Their responses must stay in the region of the chosen microlocal construction.

Proposition 2.1 (Characteristic patches). For $n \geq 5$, one can choose finitely many three-parameter patches of simple characteristic covectors for the reference geometry, with positive angular margins, such that the following properties persist for nearby immersions on a fixed spatial chart.

- (i) For every unit output direction, the selected patches have transverse interior incidence roots and satisfy the uniform Gram inequality (5.3) with the image of M_F .

- (ii) After fixed radial labels and colors are assigned, every doubled phase and every signed product of distinct sampled phases belongs to the prescribed region of validity for a response operator. The same holds along the characteristic curves used by that operator.
- (iii) For samples separated by h , the product frequency ϱ satisfies $c\lambda h \leq \varrho \leq C\lambda$.

The Hermitian self terms are treated by the mean equation and are excluded from the product assertion.

The proof is in Propositions D.6–D.10 and Theorem F.6. Two features explain the choices. A convex three-parameter patch controls the limiting chords as well as the differences of separated samples. In addition, the source region must be protected along the entire relevant characteristic segment, in both orientations; a gap at its initial point alone is insufficient.

The carrier construction uses the same cube in every dimension $n \geq 5$. Write $d = n - 4$, $S_z = \sum_i z_i$ and $q_0(z) = (z_i^2 - d^{-2}S_z^2)_i$. For $k \neq 0$ and $0 \neq s \in \ker M_0(k)$, two disjoint coordinate pairs give transverse detecting carriers, except for a finite set of five-dimensional node directions. At each such node an explicit additional carrier supplies the same detection and normal-projection gaps. Thus all dimensions use three-parameter characteristic patches, the same signed-product separation, and the same subsequent analytic construction. Proposition D.6 proves the pair selection and the five-dimensional node calculation.

2.4. A fixed reference and a small target. The iteration accepts signed metric defects. We therefore initialize it by scaling the target toward the metric of the flat reference.

Proposition 2.2 (Fixed reference initialization). Choose coordinates with $g(0) = I$, and let F_0 be the analytic flat reference above. One sufficiently small fixed $\rho > 0$ places

$$(2.11) \quad G_{\text{ref}}(y) = \rho^{-1}[F_0(\rho y) - F_0(0)], \quad g^{(s)}(y) = g(s\rho y)$$

on a fixed chart W , with all the required reference geometric gaps positive. For $0 \leq s \leq s_*$ and every fixed r ,

$$(2.12) \quad \|g^{(s)} - \gamma(G_{\text{ref}})\|_{C^r(W)} \leq C_{r,\rho}s.$$

The reference and its geometric bounds are independent of s .

Proof. Corollary D.5 gives the fixed normalized reference, with $\gamma(G_{\text{ref}}) = I$ and a positive normal gap proportional to the fixed ρ . Since $g^{(s)}$ is jointly smooth on the compact parameter interval and $g^{(0)} = I$, the fundamental theorem of calculus gives (2.12), uniformly at every fixed order. $\square$

After all geometric, analytic and frequency constants have been fixed, $s > 0$ can thus make the initial defect smaller than any prescribed threshold. The positive transported background used by the mean equation is constructed for G_{ref} in Section 5.

3. LINEAR RESPONSE ON FIXED DOMAINS

3.1. Domains and estimates. Fix a closed ball D with positive radius, a larger physical chart W , and a flat torus $\mathbb{T}$ containing $\bar{W}$ in one coordinate cell. The limit metric equation is imposed on D . All measurements, product sources, and physical output cutoffs are contained in a fixed larger region $D_{\text{lag}} \Subset W$. For the comparison source choose cutoffs with

$$(3.1) \quad D_{\text{lag}} \Subset \text{int}\{\chi_{\text{src}} = 1\}, \quad \text{supp } \chi_{\text{src}} \Subset \text{int}\{\chi_{\text{core}} = 1\} \Subset W.$$

The appropriate linear identities hold on a neighborhood of the entire source support. Mean and product operators may have different cutoffs; their supports have positive buffers inside the relevant validity regions. This finite collection of domains is fixed before frequency, accuracy, or sampling is chosen. The full arrangement is specified in Setup A.1.

We call D the *core*. The larger cutoffs contain responses that may extend beyond their source supports. Fixing them keeps the limit neighborhood independent of accuracy.

Unqualified H^r and L^2 norms refer to $\mathbb{T}$; physical C^r norms carry their domains. Tensor estimates are componentwise. Constants may depend on the fixed dimension, chart, and geometric neighborhood. When a bound holds at every order, C_r may depend on r . An *outside loss* is a frequency exponent B in an estimate such as $C_r L^{r+B}$. The assertions below specify when B is independent of r and of the requested accuracy.

For complex matrices, T denotes transpose and $*$ conjugate transpose. We take real parts entrywise. Thus $\Re(X^*X)$ is a real symmetric tensor, whereas a cross term must be symmetrized when it stands alone. A normal operator AA^* acts on the output of A ; it is distinct from the geometric normal map B_F .

We use Fourier series on the fixed torus and the normalization

$$\hat{u}(\xi) = \frac{1}{\text{vol}(\mathbb{T})} \int_{\mathbb{T}} e^{-ix \cdot \xi} u(x) dx, \quad u(x) = \sum_{\xi \in \Lambda^*} \hat{u}(\xi) e^{ix \cdot \xi}.$$

Here Λ^* is the dual lattice. The initial variables of a wave instead use Euclidean Fourier integrals, with $\hat{f}(\xi) = (2\pi)^{-(n-1)} \int e^{-iy \cdot \xi} f(y) dy$. This convention makes $f(y) = \int e^{iy \cdot \xi} dm(\xi)$ for its Fourier measure; the unit atom at 0 represents the constant 1.

For a complex wave $Z = e^{i\lambda\phi} z$, the slowly varying quantity z is its demodulated amplitude. Estimates of z and of $e^{-i\lambda\phi} dZ$ will be carried separately. A bound for the first does not by itself provide the second at the same scale, since

$$e^{-i\lambda\phi} dZ = i\lambda z d\phi + dz.$$

This is why estimates below distinguish the amplitude scale K from the physical frequency λ .

The main parameters and fields have the following roles:

F	Current immersion for all waves and response operators.
R, λ, h	Stage scale, carrier frequency, and separation of samples.
H, K	Mean solve scale and Fourier scale of an amplitude.
T_0	Derivative scale of the current immersion.
δ, b	Square root of the defect bound and primary size $b = \delta R^A$.
P, V, S	Primary wave sum, mean field, and quadratic product correction.
Q, r	Requested finite accuracy and observation derivative order.

Each scale is specified by a power of R in Section 8. The symbols B with subscripts denote fixed loss exponents; they are not additional scales.

3.2. The response operators. On each protected regular sector, put $c = |\xi|^{1-n} \text{adj } M_F$ and $h = |\xi|^{1-n} \det M_F$, so $M_F c = hI$. For the Hamilton phase of h , one matrix transport U gives a two-term graph amplitude B with the exact identity

$$(\mathcal{P}_F/i)T_B = T_U D_t + E,$$

where E has order -1 . The weighted inverse of T_U and one characteristic integral give the sector solver. The construction includes the source cutoff and endpoint terms; the physical cutoff commutator vanishes on the inner plateau. Euler's identity gives a nonzero fiber gradient also on the elliptic region, so the same construction applies there. Equation (2.5) completes the tensor response.

Proposition 3.1 (Uniform response). Fix an order r_0 above the construction and Sobolev shifts, and the geometric jet order c_{jet} . In a sufficiently small fixed $C^{r_0+c_{\text{jet}}}$ neighborhood of the reference immersion, suppose, for some $T \geq 1$,

$$\|F\|_{C^r(W)} \leq C_r T^{(r-r_0-c_{\text{jet}})_+} \quad (r \geq 0).$$

The following choices and losses are independent of accuracy.

- (i) The normal frame, metric inverse, stress frame and reduced coefficient array a_F have positive denominator gaps and satisfy

$$\|a_F\|_{C^r(W)} + \|c(F)\|_{C^r(W)} \leq C_r T^{(r-r_0)_+},$$

where $c(F)$ denotes any coefficient in normal completion.

- (ii) On fixed solver domains $U \Subset W$ there are a real solver S_F , angular cutoff A_Γ and complete measured residual $\mathcal{R}_F$, with fixed source and measurement plateaus. For inputs supported on the source plateau, on U one has

$$\mathcal{P}_F S_F = A_\Gamma + \mathcal{R}_F, \quad \|\mathcal{R}_F\|_{L^2(W) \rightarrow H^1(U)} + \|\mathcal{R}_F\|_{H^{-1}(W) \rightarrow L^2(U)} \leq C.$$

The literal adjoint has the same gains on the reversed domains. The contraction cutoffs may be fixed so that

$$(3.2) \quad K_F = \chi_2 \mathcal{R}_F \chi_3, \quad \|K_F\|_{2 \rightarrow 2} \leq \vartheta < 1, \quad T_F = (I + K_F)^{-1}.$$

For every integer $s \geq 0$, with Sobolev norms on the construction box W ,

$$(3.3) \quad \|T_F f\|_{H^s} + \|S_F T_F f\|_{H^s} \leq C_s [\|f\|_{H^s} + T^s \|f\|_2].$$

If f is supported where $\chi_2 = 1$, then $z = T_F f$ is supported in $\text{supp } \chi_2$, and on the fixed identity core

$$\mathcal{P}_F(S_F z) = f + (A_\Gamma - I)z.$$

The same inverse acts at every Sobolev order.

- (iii) The same construction has scale $L_F = T$: its normalized phase maps $\mathcal{G}$ have bounded first jets and $|D^r \mathcal{G}| \leq C_r T^{r-1}$ for $r \geq 2$, while each actual symbol of fixed order p in this construction obeys

$$|D_{x,z}^\alpha D_\xi^\beta a_p| \leq C_{\alpha,\beta} \langle \xi \rangle^{p-|\beta|} T^{|\alpha|+|\beta|}.$$

Derivatives include both positions and normalized covariables. All cutoff and endpoint factors occur in the complete residual.

The finite construction and its complete residual are proved in Appendix E, in particular Proposition E.8 and Theorem E.1. The exact correction and common graded bounds are proved in Theorems F.3 and F.4. Setup F.7 and Lemma F.9 give the total mixed derivative requirement and protected finite-chain estimates. Their fixed shifts are included when choosing r_0 .

Fix the finite symbolic construction first. Higher precision estimates additional powers of its complete residual and one remaining exact inverse, on the same domains. The primary family below is also fixed at every finite accuracy.

3.3. Response to an oscillatory source. For a tensor source h , apply $S_M T_M$ to $\chi_2 q_M(h)/2$ and use (2.5) to define the ambient response $\mathcal{B}_M h$. Equation (2.6) expresses its complete error as the angular discrepancy $(A_\Gamma - I)T_M(\chi_2 q_M(h)/2)$ through the bounded stress-frame inverse. For each primary patch we choose its complex version $\mathcal{B}_M^\mathbb{C}$ on the whole modulation patch before sampling.

Proposition 3.2 (Preservation of amplitudes). Fix one primary patch, its compatible response $\mathcal{B}_M^\mathbb{C}$, source cutoff χ_{src} and observation cutoff χ_{core} . Their supports lie in the common buffered chart, with the source and observation supports in the identity region. Let ϕ_M be the characteristic phase with the fixed transverse Cauchy data. Assume the profiles of Proposition 3.1 and, for fixed $\nu_0 > 0$,

$$(3.4) \quad L_M \leq K, \quad 2 \leq K \leq \lambda^{1-\nu_0}.$$

Let a be a complex symmetric-tensor envelope on $\mathbb{T}$ with

$$(3.5) \quad \|a\|_{H^p} \leq C_p A K^p \quad (p \geq 0), \quad A \geq 0.$$

For the same source and response at all orders, put

$$h = \chi_{\text{src}} e^{i\lambda\phi_M} a, \quad Y = \mathcal{B}_M^\mathbb{C} h, \quad Y_0 = e^{-i\lambda\phi_M} \chi_{\text{core}} Y, \quad Y_1 = e^{-i\lambda\phi_M} \chi_{\text{core}} dY.$$

The full ambient bounds are

$$(3.6) \quad \|Y_0\|_{H^s} \leq C_s A \lambda K^s,$$

$$(3.7) \quad \|Y_1\|_{H^s} \leq C_s A \lambda^2 K^s.$$

The carrier factors λ, λ^2 are independent of s, Q . On D_{lag} ,

$$(3.8) \quad \|D\gamma_M Y - h\|_{H^s(D_{\text{lag}})} \leq C_{s,Q} A \lambda^{-Q}.$$

The constants may use finitely many source profile constants; the losses, operator, fields and domains do not change with s, Q . Exact-band data use $A = \|a\|_2$.

For each fixed S and $\nu_0 > 0$, there is a finite jet order $m = m(S, \nu_0)$ for which the two amplitude bounds also hold for $0 \leq s \leq S$ and $1 \leq K \leq \lambda^{1-\nu_0}$: it is enough that the map and coefficient jets through m are uniformly bounded and (3.5) holds through m . This finite-order assertion does not require the first window in (3.4); it includes $K = 1$.

Theorem H.1 fixes this whole-patch response. The literal graph estimates and finite expansion of its exact inverse prove Proposition H.5, including the full tensor error (3.8). These statements include the actual normal completion, endpoints, cutoffs and remaining exact inverse. Corollary H.6 proves the finite-order assertion by the same graph factors and the complete residual's negative-to-positive Sobolev gain.

Choose opposite sectors by conjugation. Their principal symbols satisfy $h_-(x, \xi) = -h_+(x, -\xi)$, so both actual components use the same canonical time and lower integral. The factor $1/i$ then gives conjugate physical responses. Literal adjoints use upper integrals with the input and output weights exchanged. Proposition E.8 proves the full pairing, including all matrix components and endpoints.

4. CHARACTERISTIC WAVES AND THE MEAN EQUATION

4.1. A separated finite family. Let R be large and write

$$(4.1) \quad \lambda = R^a, \quad h = R^{-\beta}, \quad H = R^{1+\theta}, \quad \beta = (1 + \epsilon)/3,$$

where $0 < \theta < \epsilon < 1/8$ and $1 + \theta + \beta < a < 3/2$. The three-parameter patches in Proposition 2.1 admit a separated finite sample with

$$(4.2) \quad M \leq C R^{1+\epsilon}, \quad \mu_\nu > 0, \quad \sum_\nu \mu_\nu \leq C.$$

Convolve $q_{\text{quad}} = 2n - 1$ unit-interval indicators with a fixed nonnegative smooth function of integral one. The resulting smooth window forms a positive partition under integer translations. In each weighted block use an independently translated lattice of spacing h . Coloring block indices modulo $q_{\text{quad}} + 1$ gives a fixed finite number of colors and same-color separation ch . The convolution makes each translated rule exact on polynomials of degree below q_{quad} . Taylor's formula therefore gives Hermitian quadrature error $Ch^{q_{\text{quad}}}$ in L^2 on the fixed observation support. Its required carrier jets are already within the uniform low amplitude budget. Fix the colors and radial labels, prepare the initial scalars as below, and then choose the translations. One operator event controls every dyadic input prefix; all later bands use this realization.

For the current immersion F and one characteristic phase ϕ_F , the primary rule uses two amplitudes. The leading amplitude b_0 solves $M_F(x, d\phi_F)b_0 = 0$ and $l_F^T \mathcal{P}_F b_0 = 0$, where l_F spans the left kernel of M_F . Thus $\mathcal{P}_F b_0$ lies in its image. Choose b_1 algebraically

by $M_F b_1 = i\mathcal{P}_F b_0$, with b_1 perpendicular to the right kernel. The first-order product rule gives the exact identity

$$\mathcal{P}_F(\tau^2 e^{i\phi_F/\tau}(b_0 + \tau b_1)) = \tau^3 e^{i\phi_F/\tau} \mathcal{P}_F b_1.$$

Reconstructing the ambient field from this covector gives the primary family, fixed before accuracy, with $c\lambda \leq \lambda_\nu \leq C\lambda$ and phases $\phi_\nu = \phi_{F,\nu}$. Write L_{WKB} for its common amplitude scale. The actual atom has derivative amplitude $A_\tau = e^{-i\phi_F/\tau} d\mathcal{W}_\tau^{\text{pr}}$ satisfying

$$\|D_x^r D_{z,\zeta}^k A_\tau\|_\infty \leq C_{r,k} L_{\text{WKB}}^r \quad (r \geq 0, \quad k \leq r_{\text{par}}), \quad r_{\text{par}} = 20(n+1) + 3.$$

All norms use the fixed observation supports of Section 3. The complete atomic source satisfies

$$\|\chi_{\text{src}} e^{-i\lambda_\nu \phi_\nu} D\gamma_F W_\nu(\delta_0)\|_{H^r} \leq C_r \lambda^{-3} L_{\text{WKB}}^r.$$

For a complex Fourier measure m of variation M_m , supported in $B_{C\Lambda}$, let $W_\nu(m)$ be the integral of these same atoms at $\zeta_\nu + \lambda_\nu^{-1}\xi$ in the fixed phase buffer. If $K \geq C \max(\Lambda, L_{\text{WKB}})$, the fixed observation norms obey

$$\lambda_\nu \|e^{-i\lambda_\nu \phi_\nu} W_\nu(m)\|_{H^r} + \|e^{-i\lambda_\nu \phi_\nu} dW_\nu(m)\|_{H^r} \leq C_r M_m K^r.$$

At any prescribed finite order, sufficiently many uniformly bounded map jets give these same estimates for $K \geq 1$ and $\Lambda \leq K/C$, including the zero atom at $K = 1$. Write $W_0(m_\nu) = W_\nu(m_\nu)$. The background is the zero atom. If ν_{b_0} denotes its transported leading normal amplitude, its Hermitian tensor satisfies

$$\|\text{Re}[(dW_\nu^{\text{bg}})^* dW_\nu^{\text{bg}}] - |\nu_{b_0}|^2 d\phi_\nu^2\|_{C^r} \leq C_r \lambda^{-1} L_{\text{WKB}}^r.$$

Theorem G.5 proves the atom estimates; integrating the atoms gives the measure bounds on the common supports. The leading transported tensor and its positive initial-density dependence are unchanged.

Appendix G proves the construction and separated sampling, using operator concentration as in [M17]. The physical mean and normal estimates hold for one sample realization, so they concern the same tensor.

4.2. One response for the finite primary strain. Fix an order r_* covering the geometric, response and finite density-preparation constructions. The current immersion satisfies

$$(4.3) \quad \|F\|_{C^r(W)} \leq C_r T_0^{(r-r_*)+} \quad (r \geq 0).$$

The history argument below gives this profile. All phases and responses belong to this same immersion.

Each primary wave is linear in a finite complex Fourier measure

$$(4.4) \quad m_\nu = \delta_{\{0\}} + \hat{f}_{f,\nu}(\xi) d\xi.$$

The atom gives the prepared positive background; the Fourier datum adjusts its sampled mean. For a component supported in $|\xi| \leq \Lambda$, including the zero atom, use an envelope scale K in (3.4), with

$$(4.5) \quad T_0 \leq K, \quad \Lambda \leq K/C_1, \quad \Lambda/\lambda_\nu \leq \epsilon_0.$$

Here C_1, ϵ_0 fix the modulation chart. Use $K = C_{\text{env}}T_0$ for the zero atom and a fixed larger multiple of its input band for each mean datum. For $W_0(m_\nu)$ set

$$(4.6) \quad H_\nu = \chi_{\text{src}} D\gamma_F W_0(m_\nu), \quad Z_\nu = W_0(m_\nu) - \mathcal{B}_F^{\mathbb{C}} H_\nu.$$

The response is fixed on the entire primary patch before sampling and is linear in this complete measure.

Lemma 4.1 (Current strain identity). Put $E_\nu = D\gamma_F \mathcal{B}_F^{\mathbb{C}} H_\nu - H_\nu$. On D_{lag} ,

$$(4.7) \quad D\gamma_F Z_\nu = -E_\nu.$$

Consequently, for every finite Q ,

$$(4.8) \quad \|D\gamma_F Z_\nu\|_{C^0(D_{\text{lag}})} \leq C_Q \|m_\nu\|_{\text{TV}} \lambda^{-Q}.$$

For the observation cutoff of Proposition 3.2,

$$(4.9) \quad \begin{aligned} \lambda_\nu \|\chi_{\text{core}} e^{-i\lambda_\nu \phi_{F,\nu}} Z_\nu\|_{C^r} + \|\chi_{\text{core}} e^{-i\lambda_\nu \phi_{F,\nu}} dZ_\nu\|_{C^r} \\ \leq C_r \|m_\nu\|_{\text{TV}} K^r. \end{aligned}$$

For $Z_\nu - W_0(m_\nu)$ the right side has the additional factor λ_ν^{-1} .

Proof. The source equals $D\gamma_F W_0$ on D_{lag} , giving the identity. The fixed source cutoff and response commute with integration against m_ν . The corrected atoms of Corollary H.7 have arbitrary full-strain precision on this common domain. Integrating their bounds proves (4.8). Proposition J.4 controls the central phase quotient and gives both C^r bounds in the same way. $\square$

The fixed region D_{lag} contains every mean and product measurement support used below. At fixed joint physical and carrier orders, the normalized atomic response is $O(\lambda^{-1})$. Corollary H.7 proves these bounds on the full observation support. Thus the corrected atoms have the original phase and leading tensor, and satisfy the same sampling and raw-input estimates as the primary family.

4.3. The transported positive background. The continuum family of characteristic waves has a positive normal operator after the image channel is added. Its inverse constructs one positive transported background and a response for changes in the mean, on a fixed neighborhood of Proposition 2.2. Section 5 states its full-output bounds and explains how the same row is compared with one finite sample.

Proposition 4.2 (Current mean). Let $e = g - \gamma(F)$, with $\|e\|_{C^0(D)} \leq \delta^2$, and put

$$e_R = S_R E(e|_D), \quad b = \delta R^A,$$

where E, S_R are the fixed extension and smoothing operators of Lemma 6.1. For every finite Q and the scales in Section 8, the mean construction chooses the measures (4.4) and a real ambient field $V = V_c + V_f$ such that, for the same Z_ν as in (4.6),

$$(4.10) \quad E_{\text{mean}} = D\gamma_F V + \sum_\nu \mu_\nu \Re((dZ_\nu)^* dZ_\nu) - e_R/b^2$$

satisfies $\|E_{\text{mean}}\|_{C^0(D)} \leq C_Q R^{-Q}$. Moreover,

$$\sup_{\nu} \|m_{\nu}\|_{\text{TV}} \leq C.$$

The initial Fourier inputs satisfy

$$(4.11) \quad \text{supp } \hat{f}_{f,\nu} \subset \{|\xi| \leq C K_{\max}\}, \quad K_{\max} = R^{s_f},$$

with a support constant independent of Q . On the full product source support, the same waves satisfy

$$(4.12) \quad \sup_{\nu} \|\chi_{\text{pair}} e^{-i\lambda_{\nu}\phi_{F,\nu}} dZ_{\nu}\|_{C^r} \leq C_r K_{\max}^r.$$

After the fixed physical cutoff, equal to one near D , the fields satisfy, for every $r \geq 0$,

$$(4.13) \quad \|V_c\|_{C^r(\mathbb{T})} + \|V_c\|_{H^r} \leq C_r T_0^r,$$

$$(4.14) \quad \|V_f\|_{C^r(\mathbb{T})} + \|V_f\|_{H^r} \leq C_r R^{-2A} K_{\max}^{r+n/2},$$

Here $p = (n+1)/2$ and $1 < s_f < a$. The background preparation is fixed after A . The displayed field constants are uniform once the construction threshold for Q is met; Q changes the finite mean updates and that threshold.

Section 5 proves this proposition from the prepared background and fixed canonical estimates in Appendix K. Its residual identities and normal completion are given in the main text.

4.4. Oscillatory quadratic products. The normalization can be seen with a single wave. If $dZ = e^{i\vartheta} A$, then

$$(4.15) \quad \left(d(\sqrt{2}\Re Z)\right)^T d(\sqrt{2}\Re Z) = \Re(A^* A) + \Re(e^{2i\vartheta} A^T A).$$

The first term is the Hermitian self tensor; the second has doubled phase. We call the first the *mean* to identify this algebraic part of the oscillatory expansion. It is not defined by integrating over the spatial domain. With several waves there are also sums and differences of distinct phases. These are the products canceled by the current response.

Define the real primary field

$$(4.16) \quad P = b\sqrt{2} \sum_{\nu} \sqrt{\mu_{\nu}} \Re Z_{\nu}.$$

Its Hermitian tensor and remaining quadratic tensor are

$$(4.17) \quad H_{\text{self}} = b^2 \sum_{\nu} \mu_{\nu} \Re((dZ_{\nu})^* dZ_{\nu}), \quad Q_{\text{off}} = (dP)^T dP - H_{\text{self}}.$$

The first is the tensor in Proposition 4.2. The second contains all doubled phases and all signed products of distinct samples, including products between different patches.

Proposition 4.3 (Product correction). Write $dZ_{\nu} = e^{i\lambda_{\nu}\phi_{\nu}} A_{\nu}$ on the full product source support. Suppose the amplitudes satisfy

$$\|\chi_{\text{pair}} A_{\nu}\|_{C^r} \leq C_{Q,r} K_{\text{env}}^r$$

and let

$$L_{\text{pair}} = \max\{K_{\text{env}}, T_0\}.$$

If, for a fixed $\sigma > 0$,

$$(4.18) \quad L_{\text{pair}} \leq (\lambda h)^{1-\sigma},$$

there is a real current correction S , formed from the exact signed products in Q_{off} . After its fixed physical cutoff, equal to one near D , the same field satisfies

$$(4.19) \quad \|D\gamma_F S + Q_{\text{off}}\|_{C^0(D)} \leq C_{Q,q} b^2 M (\lambda h)^{-q},$$

$$(4.20) \quad \|S\|_{H^r} \leq C_{Q,r} b^2 M \lambda^{r+1} \quad (r \geq 0).$$

The outside loss is 1, independently of r, q, Q .

Apply the current response along the full characteristic segments in Proposition 2.1, using the complete amplitudes of the corrected Z_ν . Each integration gains L_{pair}/ϱ , with $\varrho \geq c\lambda h$; the strict power in (4.18) gives arbitrary finite decay. The full proof is in Proposition L.4.

5. THE NORMAL ESTIMATE AND THE NONLINEAR MEAN

We prove Proposition 4.2 through three arguments: the continuum row has a right inverse, a finite density correction prepares the background, and one recurrence adjusts its sampled mean. The complete operator estimates and source calculations are proved in Appendices I and K.

5.1. The continuum inverse and positive background. The transport channels T_a are formed from the leading amplitudes on the fixed characteristic patches. Their full normal operators have order zero and homogeneous diagonal symbols

$$(5.1) \quad d_{0,a}(x, k) = \int_v w_a(x, v, k) q_F(\xi_a \otimes \xi_a) q_F(\xi_a \otimes \xi_a)^T dv,$$

with Fourier normalization included in w_a and positive weights near the selected incidence roots. The real channel $2\Re T_a$ uses both quadratures, so its real normal is

$$(5.2) \quad 4\Re(T_a T_a^*).$$

Its complexified principal symbol is $2[d_{0,a}(x, k) + \overline{d_{0,a}(x, -k)}]$; the present $d_{0,a}$ is real symmetric, and no evenness is assumed. The incidence change of variables and every lower-order term are proved in Appendix I.

For every unit output direction $\hat{k}$, the selected carriers have interior incidence roots with

$$(5.3) \quad \begin{aligned} \hat{k} \cdot \partial_\xi p_F(x, \xi_a) &= 0, & p_F &= \det M_F, \\ |M_F(x, \hat{k})^T w|^2 + \sum_a |w \cdot q_F(\xi_a \otimes \xi_a)|^2 &\geq c|w|^2 \quad (w \in \mathbb{R}^n). \end{aligned}$$

Here each ξ_a is evaluated at its local root. On the inner measurement region, the positive weights near these roots make (5.1) complete the image channel. For the

positive scalars constructed below, adding a fixed exterior channel gives a row A_F on the full real output $Y = L^2(\mathbb{T}; \mathbb{R}^n)$ with

$$E_0 = A_F A_F^* \geq cI, \quad Q_0 = E_0^{-1}.$$

For X^r , the sum of the raw auxiliary H^r spaces and carrier $H_{z,y}^r$ spaces, its one inverse satisfies

$$(5.4) \quad \|Q_0 f\|_{H^r} + \|A_F^* Q_0 f\|_{X^r} \leq C_r [\|f\|_{H^r} + (1 + \|F\|_{C^{r+B}(W)}) \|f\|_2].$$

The physical auxiliary realization has the same source order and a fixed coefficient shift. These are the full-output statements of Propositions 1.4, 1.5 and 1.9. For the local gap, freeze the positive principal matrix at the center of B_r . Its Fourier multiplier is positive; the coefficient difference is $\sum_i x_i T_i$ with bounded order-zero T_i . The complete lower-order error costs $C\|f\|_{H^{-1}}\|f\|_2 \leq Cr\|f\|_2^2$ for f supported in B_r . A small fixed r absorbs both errors. The exterior channel gives the full-output gap and vanishes on the mean core.

The same construction gives positive initial scalars $r_{F,a}$ and a tangent field $V_{\text{base}} = dFX_F$ on a fixed U_{fin} containing the mean output support, such that

$$(5.5) \quad q_F(D\gamma_F V_{\text{base}} + \mathcal{H}_F[r_F^2]) = 0.$$

Here $\mathcal{H}_F[r_F^2]$ is the continuum Hermitian tensor of the leading amplitudes. The scalars are bounded below on the working carrier patches, and, for one fixed B ,

$$(5.6) \quad \sum_a \|r_{F,a}\|_{C^r} + \|V_{\text{base}}\|_{C^r(U_{\text{fin}})} + \|D\gamma_F V_{\text{base}}\|_{C^r(U_{\text{fin}})} \leq C_r(1 + \|F\|_{C^{r+B}(W)}).$$

Propositions 1.6 and 1.7 prove this by one affine density correction at the fixed positive reference dilation. The row A_F and inverse Q_0 below use these corrected scalars; their gaps and shifts are independent of accuracy.

5.2. Preparation before sampling. Write $A_F = (A_0, \mathcal{T}_0)$ for the auxiliary and carrier rows. Proposition K.2 chooses positive initial scalars $r_{F,R}$ and a real field U_c before sampling. The current zero atoms $Z_{c,\nu}$ have self tensor $\mathcal{H}_c = \sum_\nu \mu_\nu \Re[(dZ_{c,\nu})^* dZ_{c,\nu}]$. For $C_c = D\gamma_F U_c + \mathcal{H}_c$, it gives a full auxiliary residual

$$(5.7) \quad r_c = q_F C_c \text{ on } D, \quad \|r_c\|_2 \leq R^{-2A}, \quad \|r_c\|_{H^r} \leq C_r T_0^r.$$

On U_{fin} , the C^r norms of U_c , its full strain, $\mathcal{H}_c$ and C_c are at most $C_r T_0^r$. The scalar change is $O(\lambda^{-1})$ in any prescribed fixed joint topology and has the same graded profile as the original scalar. All required finite orders are included in r_* before mean accuracy.

The proof uses the affine dependence of the leading mean on the initial density. Write $r(v) = r_F \sqrt{1+v}$ and let $\mathcal{E}_\lambda(v)$ be the measured difference between the colored current Hermitian mean and its leading transported mean. Linearity of each atomic wave in r gives, at any fixed order,

$$\|\mathcal{E}_\lambda(v) - \mathcal{E}_\lambda(w)\|_{H^s} \leq C_s \lambda^{-1} \|v - w\|_{C^{m_s}}, \quad \mathcal{E}_\lambda(0) = O(\lambda^{-1}),$$

with a finite input order m_s . Apply the original $A_F^* Q_0$ to the full residual e_i . The uncut density synthesis and auxiliary field change its leading part by $-A_F A_F^* Q_0 e_i = -e_i$. Consequently the exact new residual is

$$e_{i+1} = \mathcal{E}_\lambda(v_{i+1}) - \mathcal{E}_\lambda(v_i).$$

For fixed J , choosing the finite input orders backwards gives $e_J = O(\lambda^{-J-1})$ and $v_J = O(\lambda^{-1})$. Take the fixed count

$$(5.8) \quad J = \lfloor 2A/a \rfloor, \quad a(J+1) > 2A, \quad \beta(2n-1) > 2A.$$

The lattice estimate then gives (5.7). Fixed derivative shifts and interpolation give its high profiles.

For raw auxiliary cutoff π_K and carrier prefix P_K , put

$$P_K = \text{diag}(\pi_K, P_K), \quad E_{0,K} = A_F P_K^2 A_F^*, \quad \mathcal{F}_{0,K} = A_0 \pi_K^2 A_0^*.$$

The same event compares every current prefix normal E_K with the original positive row:

$$(5.9) \quad \sup_{K \leq H} \|E_K - E_{0,K}\|_{2 \rightarrow 2} \leq C \left[R^{-(\epsilon-\theta)/4} + R^{-a+1+\theta+\beta} \right].$$

The phase is unchanged; the prepared scalar changes each leading normal by only $C\lambda^{-1}$, uniformly in K . Thus Q_0 still belongs to r_F . This is a statement on the entire real output, including the identical exterior and auxiliary components.

5.3. One fixed row and one mean recurrence. Define the source directly on the torus:

$$(5.10) \quad e_f = \mathcal{C}(e_R/b^2) - r_c, \quad \mathcal{C} = \chi_{\text{meas}} q_F.$$

On D it equals $q_F(e_R/b^2 - C_c)$. For $m > \max\{r, 2A/(1-t)\}$, Fourier Hölder gives

$$\|r_c\|_{H^r} \leq C_r \|r_c\|_2^{1-r/m} \|r_c\|_{H^m}^{r/m} \leq C_r R^{-2A+r}.$$

The smoothed target therefore gives

$$(5.11) \quad \|e_f\|_{H^r} \leq C_r \epsilon_f R^r, \quad \epsilon_f = C R^{-2A}.$$

All coefficient and zero-atom profiles have scale $T_0 < R$.

Proposition 5.1 (Canonical changes). For $T_0 \leq K \leq H$ and $r \in L^2$, fix $B_\nu = e^{-i\lambda_\nu \phi_\nu} dZ_{c,\nu}$. For raw input v , let $G_{\nu,K} v$ be the demodulated derivative of its current wave and define

$$A_{\nu,K} v = \mathcal{C}[2 \text{sym } \Re(B_\nu^* G_{\nu,K} v)], \quad A_K = (\sqrt{\mu_\nu} A_{\nu,K})_\nu, \quad E_K = \mathcal{F}_{0,K} + A_K A_K^*.$$

The input map $B_{\nu,K}^{\text{in}}$ contains the detector $\alpha_\nu \chi_{\text{hi}}$ and the dyadic half-derivative sum before spatial cutoffs; the zero atom is uncut. For $q = Q_0 r$ take

$$(5.12) \quad f_{\nu,K}(r) = B_{\nu,K}^{\text{in}} A_{\nu,K}^* q.$$

Both real quadratures use the original raw Hilbert spaces. If $D_{\nu,K}(r)$ is the corresponding current derivative change, then

$$(5.13) \quad \mathcal{C} \sum_\nu \mu_\nu 2 \text{sym } \Re(B_\nu^* D_{\nu,K}(r)) = (E_K - \mathcal{F}_{0,K}) Q_0 r.$$

Every update uses this fixed zero-atom row.

Write $\|D\|_{\mu,X}^2 = \sum_{\nu} \mu_{\nu} \|D_{\nu}\|_X^2$. Use the fixed observation cutoff in all derivative norms below; data norms use the initial variables. Then

$$\|D_K(r)\|_{\mu,H^s} \leq C_s K^{s+1/2} \|r\|_2, \quad \|D_K(r)\|_{\mu,C^s} \leq C_s K^{s+n/2} \|r\|_2.$$

Individually,

$$\|f_{\nu,K}(r)\|_{C^s} + \|D_{\nu,K}(r)\|_{C^s} \leq C_s K^{s+p} \|r\|_2,$$

where $p = (n+1)/2$. The datum has Fourier support below CK and Fourier L^1 norm at most $CK^p \|r\|_2$. The raw auxiliary field has, after the fixed physical cutoff,

$$(5.14) \quad \|V_{\text{aux},K}^F(r)\|_{H^s} + \|D\gamma_F V_{\text{aux},K}^F(r)\|_{H^s} + \|\mathcal{F}_{0,K} Q_0 r\|_{H^s} \leq C_s K^s \|r\|_2.$$

The last norm includes the exterior output.

Proof. Definition K.4 constructs these exact adjoints and raw maps. The comparison (5.9) bounds A_K . Square the raw H^s and C^s bounds before summing; only the individual estimates use $\|A_{\nu,K}\| \leq C\sqrt{K}$. Proposition I.9 gives the full auxiliary bounds. $\square$

Put $K_{\max} = H^{d_f}$, with $1/(1+\theta) < d_f < 1$. The fixed choices (5.9) and (8.6) give $\nu, \nu_{\text{nl}} > 0$ such that

$$\|E_K - E_{0,K}\| \leq CH^{-\nu}, \quad \epsilon_f K_{\max}^p \leq H^{-\nu_{\text{nl}}}.$$

Lemma 5.2 (A finite mean with a fixed row). For source (5.11) and every finite Q , finitely many updates of Proposition 5.1 below $K_{\max}$ have full residual $\|r_{\text{fin}}\|_2 \leq \epsilon_f H^{-Q}$. Their source residuals have sum at most $2\epsilon_f$ in L^2 ; the accumulated data and derivative changes have C^s norm at most $C_s \epsilon_f K_{\max}^{s+p}$. In the weighted C^s norm, the derivative changes are bounded by $C_s \epsilon_f K_{\max}^{s+n/2}$. The data have Fourier variation at most $C \epsilon_f K_{\max}^p$ and support below $CK_{\max}$. The auxiliary field and full strain have H^s norm at most $C_s \epsilon_f K_{\max}^s$. These constants are independent of Q once its construction threshold is met. Any prescribed finite C^r residual accuracy follows by increasing the L^2 precision.

Proof. Fix $0 < \gamma < \min(\nu, \nu_{\text{nl}})$. For precision Q , set $I = \lfloor Q/\gamma \rfloor + 1$ and put $d_{\text{in}} = 1/(1+\theta)$,

$$\Delta = \frac{d_f - d_{\text{in}}}{I+1}, \quad d_l = d_{\text{in}} + (l+1)\Delta, \quad K_l = 2^{\lceil \log_2 H^{d_l} \rceil}, \quad 0 \leq l < I.$$

At a large threshold, $H^{d_l} \leq K_l \leq 2H^{d_l} < K_{\max}$. At step l , use (5.12) for r_l and add the auxiliary input $\pi_{K_l}^2 A_0^* Q_0 r_l$. Write D_l for its derivative change, $S_l = \sum_{i < l} D_i$ and $\mathcal{N}(S) = \mathcal{C} \sum_{\nu} \mu_{\nu} \Re(S_{\nu}^* S_{\nu})$. The exact full-output recurrence is

$$(5.15) \quad r_{l+1} = (I - E_{K_l} Q_0) r_l - [\mathcal{N}(S_l + D_l) - \mathcal{N}(S_l)], \quad r_0 = e_f.$$

Its linear error is

$$(5.16) \quad (E_{0,K_l} - E_{K_l}) Q_0 r_l + A_F (I - P_{K_l}^2) A_F^* Q_0 r_l.$$

Only raw inputs are cut off.

Inductively suppose $a_i := \|r_i\|_2 \leq \epsilon_f H^{-\gamma i}$ for $i \leq l$. Then $\sum_{i \leq l} a_i \leq 2\epsilon_f$. The weighted C^0 and L^2 bounds give

$$\|\mathcal{N}(S_l + D_l) - \mathcal{N}(S_l)\|_2 \leq C\epsilon_f K_{\max}^p a_l.$$

Leibniz and weighted Cauchy–Schwarz give, for $i \leq k$,

$$\left\| \mathcal{C} \sum_{\nu} \mu_{\nu} \Re(D_{i,\nu}^* D_{k,\nu}) \right\|_{H^s} \leq C_s a_i a_k K_i^{n/2} K_k^{s+1/2}.$$

Use the lower-band factor in weighted C^r and the higher one in weighted H^{s-r} ; coefficient derivatives cost $T_0 \leq K_i$. The cumulative identity is

$$r_l = e_f - \sum_{i < l} \mathcal{F}_{0,K_i} Q_0 r_i - \mathcal{C} \sum_{\nu} \mu_{\nu} [2 \text{sym} \Re(B_{\nu}^* S_{l,\nu}) + \Re(S_{l,\nu}^* S_{l,\nu})].$$

Since $\sum_i a_i K_i^{n/2} \leq 2\epsilon_f K_{\max}^{n/2} \leq C$, its linear and quadratic terms give

$$\|r_l\|_{H^s} \leq C_s \epsilon_f \left[R^s + \sum_{i < l} H^{-\gamma i} K_i^{s+1/2} \right].$$

Choose one finite order S with $\Delta S > d_f/2 + 2\gamma + 1$. Equation (5.4) gives $\|A_F^* Q_0 r_l\|_{X^S} \leq C_S (\|r_l\|_{H^S} + T_0^S a_l)$. Thus the raw tail in (5.16) is at most

$$C_S \epsilon_f \left[H^{-(l+1)\Delta S} + \sum_{i < l} H^{-\gamma i - (l-i)\Delta S + d_i/2} \right] \leq C_S \epsilon_f H^{-\gamma(l+1)-1}.$$

For $l \geq 1$ the geometric sum is bounded by a constant times its last term, whose exponent is $-\gamma(l-1) - \Delta S + d_{l-1}/2$; for $l = 0$ the first term suffices. The normal and nonlinear errors are at most $C(H^{-\nu} + H^{-\nu_{\text{nl}}})a_l$. At a large threshold these and the tail give $a_{l+1} \leq \epsilon_f H^{-\gamma(l+1)}$, closing the induction.

Consequently $\|r_l\|_2 \leq \epsilon_f H^{-Q}$ and $\sum_i a_i \leq 2\epsilon_f$. The raw bounds prove all accumulated data, variation and auxiliary estimates, with the stated uniform constants. The high profile is at most $C_s \epsilon_f K_{\max}^{s+1/2}$; interpolation gives each fixed C^r residual accuracy after increasing Q linearly. Choose the least such S ; both I and S are $O(Q+1)$. $\square$

5.4. One normal completion. Apply Lemma 5.2 to e_f . Sum the auxiliary fields as V_{aux} and let $\mathcal{H}$ be the final Hermitian tensor. On U_{fin} put

$$T_f = e_R/b^2 - D\gamma_F(U_c + V_{\text{aux}}) - \mathcal{H}, \quad \mathbf{N}_F = (B_F^*)^{-1}, \quad \Pi_F^{\perp} = \Pi_{(\ker B_F)^{\perp}}.$$

Define

$$(5.17) \quad V = U_c + V_{\text{aux}} - \frac{1}{2} \mathbf{N}_F \Pi_F^{\perp} T_f.$$

On D , $q_F T_f = r_I$. The metric error is $-\Pi_{\ker B_F} T_f$, so the Gram formula proves (4.10).

For the norms, split this same field as

$$\begin{aligned} V_c &= U_c + \frac{1}{2} \mathbf{N}_F \Pi_F^{\perp} C_c, \\ V_f &= V_{\text{aux}} - \frac{1}{2} \mathbf{N}_F \Pi_F^{\perp} (e_R/b^2 - D\gamma_F V_{\text{aux}} - (\mathcal{H} - \mathcal{H}_c)). \end{aligned}$$

The prepared tensor profile proves (4.13). The raw auxiliary estimate and the low residual sum give $\|V_{\text{aux}}\|_{H^r} + \|D\gamma_F V_{\text{aux}}\|_{H^r} \leq C_r \epsilon_f K_{\max}^r$. Splitting the Fourier sum at

$K_{\max}$ costs $n/2$ powers. Weighted Cauchy–Schwarz gives $\|\mathcal{H} - \mathcal{H}_c\|_{C^r} \leq C_r \epsilon_f K_{\max}^{r+n/2}$, using $\epsilon_f K_{\max}^{n/2} \leq C$ for its quadratic part. Since $R < K_{\max}$, these prove (4.14) on the full physical domain. Finally,

$$(5.18) \quad \sum_l \|\hat{f}_{\nu, K_l}\|_1 \leq C \epsilon_f K_{\max}^p \leq C.$$

Adding the unit atom proves the measure bound. Equation (4.9) gives the final amplitude bound on the full product support.

6. THE METRIC CORRECTION

6.1. Smoothing the defect. We use ordinary extension, Fourier smoothing, and tame product estimates in a form that keeps the small defect and the derivative scale separate.

Lemma 6.1. There is a real linear extension $E : C^\infty(D) \rightarrow C^\infty(\mathbb{T})$, supported in a fixed compact subset of W , with $Ef = f$ on D and $\|Ef\|_{C^r} \leq C_r \|f\|_{C^r(D)}$ for every r . Fix a smooth target tensor g on W . Let S_R , $R \geq 1$, be a family of Fourier multipliers with a fixed real-even smooth cutoff symbol, equal to one below R and zero above $2R$. Suppose $T \geq 1$, $0 < \delta \leq 1$, and that

$$\|dF\|_{C^r(W)} \leq C_r T^r, \quad \|g - \gamma(F)\|_{C^0(D)} \leq \delta^2.$$

Put $e = g - \gamma(F)$. Then $e_R = S_R E(e|_D)$ satisfies

$$(6.1) \quad \|e_R\|_{C^r} + \|e_R\|_{H^r} \leq C_r \delta^2 R^r,$$

$$(6.2) \quad \|e - e_R\|_{C^0(D)} \leq C \delta^2,$$

$$(6.3) \quad \|e - e_R\|_{C^0(D)} \leq C_N (T/R)^N.$$

The constants depend on the displayed profile constants and finitely many seminorms of g . The same operators work at all orders. If $T \leq R^t$ with $0 < t < 1$, then, for every finite Q ,

$$(6.4) \quad \|e - e_R\|_{C^0(D)} \leq C_Q \delta R^{-Q}.$$

Proof. Take the fixed extension of Lemma B.1 and the Fourier cutoff from Lemma B.3. Their kernel estimates give

$$\|S_R Ef\|_{C^r} + \|S_R Ef\|_{H^r} \leq C_r R^r \|f\|_{C^0(D)}, \quad \|(I - S_R)Ef\|_{C^0} \leq C_N R^{-N} \|f\|_{C^N(D)}.$$

The first estimate and the second at order zero give (6.1)–(6.2). Leibniz and the assumed profile imply $\|e\|_{C^N(D)} \leq C_N(g) + C_N T^N \leq C_N T^N$, proving (6.3). The geometric mean of the small and high bounds is $C_N \delta R^{-(1-t)N/2}$. Choose $(1-t)N > 2Q$. $\square$

In particular, e_R/b^2 has C^r and H^r norms at most $C_r R^{r-2A}$. The factor δ in (6.4) comes from combining the two bounds, not from the high tail. The defect e may have either sign.

6.2. A single increment. The preceding constructions give the following quantitative step.

Theorem 6.2 (Finite-stage correction). Fix the reference neighborhood, domains, and response constructions above. Let F be a current immersion in this neighborhood, and assume the wave, mean, and product hypotheses of Section 4. Suppose

$$\|g - \gamma(F)\|_{C^0(D)} \leq \delta^2, \quad T_0 = R^t,$$

with $0 < t < 1$ and $0 < \delta \leq 1$, and assume (4.3) and (4.18). Put

$$B_0 = 2A + 1 + \epsilon + a, \quad B_3 = 2B_0 + 2a(p + 1/2).$$

For every finite Q and sufficiently large R , a real global increment U satisfies

$$(6.5) \quad \|\gamma(F + U) - g\|_{C^0(D)} \leq C_Q(\delta R^{-Q} + \delta^3 R^{B_3}),$$

$$(6.6) \quad \|U\|_{L^2} \leq C_Q \delta R^{B_0},$$

$$(6.7) \quad \text{supp } \hat{U} \subset \{|\xi| \leq CR^{b_*}\}, \quad b_* = a(1 + \zeta) < 3/2.$$

Consequently the same U obeys

$$(6.8) \quad \|U\|_{C^r(\mathbb{T})} \leq C_{Q,r} \delta R^{B_0 + b_*(r + n/2)}.$$

The outside exponents and support constant are independent of Q .

Proof. Choose the sample for the fixed primary family and carry out the finite mean updates at the requested accuracy. Construct P, V, S as in Section 4, and put

$$W_2 = b^2 V + S, \quad U = \Pi(P + W_2),$$

where the fields have their accepted physical cutoffs and Π is one real-even Fourier multiplier, equal to one below $C_1 \lambda^{1+\zeta}$ and zero above $C_2 \lambda^{1+\zeta}$. All physical cutoffs equal one near D .

First, (4.2) gives

$$\sum_{\nu} \sqrt{\mu_{\nu}} \leq \left(M \sum_{\nu} \mu_{\nu} \right)^{1/2} \leq CR^{(1+\epsilon)/2}.$$

The measure bound in Proposition 4.2 and Lemma 4.1 therefore imply

$$(6.9) \quad \|D\gamma_F P\|_{C^0(D)} \leq C_Q \delta R^{A+(1+\epsilon)/2} \lambda^{-Q_p}.$$

Choose $aQ_p > Q + A + (1 + \epsilon)/2$. This gives the first term of (6.5).

By (4.17), the remaining quadratic tensor is exactly Q_{off} . Proposition 4.3, with sufficiently large internal order, gives

$$\|D\gamma_F S + Q_{\text{off}}\|_{C^0(D)} \leq C_Q \delta R^{-Q}.$$

Likewise Proposition 4.2, with accuracy greater than $Q + 2A$, gives $b^2 \|E_{\text{mean}}\|_{C^0(D)} \leq C_Q \delta R^{-Q}$.

Substitution into the metric expansion yields the exact identity

$$(6.10) \quad \begin{aligned} \gamma(F + P + W_2) - g &= (e_R - e) + D\gamma_F P + b^2 E_{\text{mean}} \\ &+ (D\gamma_F S + Q_{\text{off}}) + 2 \text{sym}((dP)^T dW_2) + (dW_2)^T dW_2. \end{aligned}$$

The physical profiles of these same cut fields give

$$(6.11) \quad \|P\|_{H^r} \leq C_{Q,r} \delta R^{B_0} \lambda^r, \quad \|W_2\|_{H^r} \leq C_{Q,r} \delta^2 R^{B_0} \lambda^r.$$

For P , use (4.9) and the weight sum. For W_2 , the pair bound costs $R^{2A+1+\epsilon}\lambda$; the prepared field costs R^{2A} , and the mean update costs $R^{s_f n/2} \leq R^{2A}$ by (8.6). All derivative slopes are at most a . The same Fourier estimate gives C^1 bounds with the factor $\lambda^{1+n/2}$. Since $\delta \leq 1$, these bound the last two terms in (6.10) by $C_Q \delta^3 R^{B_3}$. Lemma 6.1 treats its first term.

For any integer $k > 0$, the tail has H^s profile $C_{Q,k,s} \delta R^{B_0} \lambda^{s-\zeta k}$. Applying the same Fourier bound at $s = 0$ and one $s > 1 + n/2$ gives

$$\|(I - \Pi)(P + W_2)\|_{C^1} \leq C_{Q,k} \delta R^{B_0} \lambda^{p+1/2-\zeta k}.$$

Thus $E_{\text{cut}} = U - (P + W_2)$ has arbitrary finite C^1 decay with a factor δ . Its exact metric contribution is

$$D\gamma_{F+P+W_2} E_{\text{cut}} + (dE_{\text{cut}})^T dE_{\text{cut}}.$$

Choosing k large enough absorbs the fixed polynomial size of $F + P + W_2$ and bounds this by $C_Q \delta R^{-Q}$, proving (6.5).

Equation (6.11) at $r = 0$ and boundedness of Π on L^2 give (6.6); the multiplier gives (6.7). Fourier Cauchy–Schwarz and lattice counting on the fixed torus imply (6.8). Reality follows from the real fields and the even real multiplier. $\square$

The first term in (6.5) collects errors reduced by finite accuracy. The second is the nonlinear interaction between the primary and secondary fields.

7. HISTORIES AND COEFFICIENT CONTROL

Exact Fourier support gives the fixed high norm used above.

Lemma 7.1 (A band-limited history). Fix an integer $r_* \geq 1$ and $1 < b_* < v < 3/2$, and put $t = b_*/v$. For $R_0 \geq 2$ set $R_j = R_0^{v^j}$. Suppose real global increments satisfy, with $k_j \geq k_0$,

$$(7.1) \quad \text{supp } \widehat{U}_j \subset \{|\xi| \leq C R_j^{b_*}\}, \quad \|U_j\|_{L^2} \leq R_j^{-k_j+B_0+1}.$$

Put $F_j = F_{\text{init}} + \sum_{i < j} U_i$, $T_j = R_j^t$ and $b_{\text{hist}} = B_0 + b_*(r_* + n/2)$. If $\mu = k_0 - b_{\text{hist}} - 1 > 0$, then

$$(7.2) \quad \|F_j\|_{C^r(W)} \leq C_r T_j^{(r-r_*)+} \quad (r \geq 0),$$

and

$$(7.3) \quad \sum_j \|U_j\|_{C^{r_*}(\mathbb{T})} \longrightarrow 0 \quad \text{as } R_0 \longrightarrow \infty.$$

Both assertions are uniform for a compact smooth family of initial maps.

Proof. Fourier Cauchy–Schwarz gives

$$(7.4) \quad \|U_i\|_{C^{r_*}(\mathbb{T})} \leq C R_i^{-k_i+b_{\text{hist}}+1}.$$

Since $v^i \geq 1 + i(v - 1)$, their sum is at most

$$\frac{CR_0^{-\mu}}{1 - R_0^{-\mu(v-1)}}.$$

The periodic increment sum has uniformly bounded C^{r*} norm and Fourier support below $CR_j^{b_*/v} = CT_j$. Bernstein gives its higher-order profile; the fixed initial map only changes the constants. The same sum proves the smallness assertion. $\square$

Proposition 7.2 (Current coefficient scale). Enlarge the fixed order r_0 in Proposition 3.1 to include the fixed reserve $c_{\text{lin}} + 3$ of Setup H.2 and the sampling and physical-slice orders through r_{par} , with $r_0 \geq D_{\text{pr}} + r_{\text{par}} + 3$ and their fixed construction shifts. Choose r_* above the fixed response orders and the density preparation and quadrature budgets of Proposition K.2, including their positive-scalar and reconstruction shifts, so subtracting any of these shifts leaves at least r_0 . These orders and the resulting geometric neighborhood precede variable mean accuracy. If the total displacement in (7.3) stays in that C^{r*} neighborhood, the actual extended coefficient arrays satisfy

$$(7.5) \quad \|\tilde{a}(F_j)\|_{C^r} \leq C_r T_j^{(r-r_0)_+}.$$

The same bound holds for F_j and the prepared positive initial scalars, including joint data derivatives on their fixed domains. The current response construction has scale

$$L_{F,j} = T_j,$$

with the fixed outside losses of Proposition 3.1. Since $t < 1 < a - \beta$, this scale lies below $(\lambda_j h_j)^{1-\sigma}$ for some fixed $\sigma > 0$ and large R_0 .

Proof. The finite-jet bounds and fixed extension transfer the history profile to the coefficients. Proposition 3.1 gives $L_{F,j} = T_j$. The density preparation uses only orders within the uniform C^{r*} budget. Choose $t < (a - \beta)(1 - \sigma)$; its strict gap absorbs the fixed chart constants. $\square$

This coefficient scale lies below the mean input bands and the smallest product frequency.

8. CHOICE OF SCALES

We now explain why the estimates can be imposed simultaneously. All exponents in this section are fixed before choosing the initial parameter s or the increasing accuracies. The detailed preparation and mean inequalities are proved in Proposition M.1.

8.1. Frequency and amplitude gaps. Use the following fixed outer exponents:

$$(8.1) \quad \begin{aligned} \epsilon &= \frac{1}{32}, & \theta &= \frac{1}{64}, & \beta &= \frac{11}{32}, & a &= \frac{7}{5}, \\ \zeta &= \frac{1}{84}, & b_* &= \frac{17}{12}, & v &= \frac{35}{24}, & t &= \frac{b_*}{v} = \frac{34}{35}. \end{aligned}$$

Then $\beta = (1 + \epsilon)/3$, $b_* = a(1 + \zeta)$, and

$$(8.2) \quad 1 + \theta + \beta < a < b_* < v < \frac{3}{2}, \quad t = \frac{b_*}{v} < 1.$$

Choose the fine amplitude exponent

$$(8.3) \quad s_f = 1 + \theta/2.$$

Then

$$1 < s_f < 1 + \theta < a - \beta < a.$$

Here $a - \beta - (1 + \theta) = 13/320$. Thus a fine amplitude ceiling $K_{\max} = R^{s_f} = H^{d_f}$, $d_f = s_f/(1 + \theta) < 1$, lies below all product frequencies by a strict power.

The equal-gap bands of Lemma 5.2 start above R and end below $K_{\max}$. Their common positive exponent gap lets one finite derivative order control all omitted inputs. Accuracy changes only this finite list within the same sampled family.

For the primary construction fix

$$(8.4) \quad L_{\text{WKB}} = T_0 = R^t.$$

The finite transport and positive background use the current scale. Their fixed low norms cover the density preparation. The two amplitudes are independent of accuracy; their complete residual is included in the current source.

The common amplitude scale is

$$K_{\text{env}} = C_* K_{\max}$$

with one fixed C_* . Take $\sigma = 1/1000$. Since $t < 1$ and $s_f < a - \beta$,

$$(8.5) \quad C_* K_{\max}, R^t \leq (\lambda h)^{1-\sigma}$$

for large R . This verifies (4.18). The strict inequalities absorb the fixed patch and radial normalization constants.

8.2. The forcing and response size. Set the forcing exponent

$$(8.6) \quad A = \frac{p(1 + \theta)}{2}, \quad 2A - s_f p = \frac{p\theta}{2} > 0.$$

Thus the same inequality bounds the fine physical field and makes $\epsilon_f K_{\max}^p$ small. The nonlinear margin may be $\nu_{\text{nl}} = p\theta/[4(1 + \theta)]$ in H units. This also proves (5.18). Use the preparation count in (5.8). The quadrature margin is $\beta(2n - 1) - 2A = (23n - 109)/128 \geq 3/64$. The normal comparison (5.9) has positive exponents because $\theta < \epsilon$ and $1 + \theta + \beta < a$.

The current coefficient exponent is $t < 1$ and the mean input floor is R . A fixed multiple of T_0 is an envelope for the zero atom. All response envelopes satisfy $K \leq \lambda^{1-\nu_0}$ with $\nu_0 = 1/4$. Actual modulation supports are below their envelopes. These are the response and source gaps used in Proposition 3.2. Proposition M.1 verifies the individual cutoff, source, and mean inequalities without any change of the fixed operators.

8.3. Fixed orders and the initial defect. Fix the density preparation count in (5.8) first. Choose r_* in Proposition 7.2 to include these finite orders and their construction shifts, and put $b_{\text{hist}} = B_0 + b_*(r_* + n/2)$. Choose an integer $k_{\min}$ so that

$$(8.7) \quad (3 - 2v)k_{\min} > B_3 + 2v + 4,$$

$$(8.8) \quad k_{\min} > b_{\text{hist}} + 2.$$

Both are lower bounds since $3 - 2v > 0$. All orders and exponents are fixed before mean accuracy or target dilation. Use

$$(8.9) \quad Q(k) = 2k.$$

Both errors in (6.5) are at most $C_{2k}R^{-3k+B_3}$; (8.7) controls their sum.

9. SMOOTH CONVERGENCE ON ONE NEIGHBORHOOD

The target family $g^{(s)}$ of Proposition 2.2 has uniform fixed seminorms before s is chosen. Choose $\varepsilon_* > 0$ so that its C^{r_*} reference ball preserves every geometric and analytic gap.

For every finite history satisfying (7.1) with $R_i = R_0^{v_i}$ and $k_i \geq k_{\min}$, Lemma 7.1 gives uniform map profiles. Choose a finite nondecreasing threshold $\mathfrak{H}(k)$, independent of s , as follows. First enlarge $\mathfrak{H}(k_{\min})$ so that $R_0 \geq \mathfrak{H}(k_{\min})$ implies

$$(9.1) \quad \sum_{i \geq 0} C R_i^{-k_{\min} + b_{\text{hist}} + 1} < \varepsilon_*/2.$$

Every such finite history then lies in the reference ball. Proposition 7.2, the scale verification and the smoothing lemma give all step hypotheses uniformly for these histories and the compact target family.

At each fixed k , Theorem 6.2 uses finitely many uniform constants at accuracy $2k$. Enlarge $\mathfrak{H}(k)$ monotonically so that $R_j \geq \mathfrak{H}(k_j)$ meets their thresholds and bounds the metric and L^2 constants by R_j . If the current map $M_j = G_{\text{ref}} + \sum_{i < j} U_i$ has defect at most $R_j^{-2k_j}$, its increment satisfies (7.1) and

$$(9.2) \quad \|\gamma(M_j + U_j) - g^{(s)}\|_{C^0(D)} \leq 2C_{2k_j}R_j^{-3k_j+B_3} \leq R_j^{-2v(k_j+1)}.$$

The last inequality follows from (8.7).

9.1. One adaptive schedule and one target dilation. Choose $R_0 \geq \mathfrak{H}(k_{\min})$. Set $k_0 = k_{\min}$ and

$$(9.3) \quad k_{j+1} = k_j + \mathbf{1}_{\{R_{j+1} \geq \mathfrak{H}(k_{j+1})\}}.$$

Then $R_j \geq \mathfrak{H}(k_j)$ and $k_j \rightarrow \infty$: an eventual constant K would be contradicted when R_j exceeds the finite $\mathfrak{H}(K+1)$. Put $\delta_j = R_j^{-k_j}$ and $Q_j = 2k_j$. Now choose a single $s > 0$ with

$$(9.4) \quad \|g^{(s)} - \gamma(G_{\text{ref}})\|_{C^0(D)} \leq \delta_0^2,$$

using (2.12). This choice follows all thresholds.

Apply this step successively. Each increment preserves the prospective Fourier and L^2 budgets; (9.1) keeps the current maps admissible. Since $k_{j+1} \leq k_j + 1$, its metric conclusion gives inductively

$$(9.5) \quad \|g^{(s)} - \gamma(M_j)\|_{C^0(D)} \leq \delta_j^2.$$

Thus the same construction continues on the fixed core for all j .

For every fixed r , Fourier Cauchy–Schwarz gives

$$(9.6) \quad \|U_j\|_{C^r(\mathbb{T})} \leq C_r R_j^{-k_j + B_0 + 1 + b_*(r + n/2)}.$$

Here C_r is the Fourier constant, independent of Q_j . As $k_j \rightarrow \infty$, this bound is eventually $C_r R_j^{-1}$, which is summable. The same series therefore converges in every C^r to one smooth M_∞ on W . Taking C^1 limits gives $\gamma(M_\infty) = g^{(s)}$ on $\text{int } D$. On the fixed neighborhood $U = s\rho \text{ int } D$, set $F(x) = s\rho M_\infty(x/(s\rho))$. Then $\gamma(F)(x) = g^{(s)}(x/(s\rho)) = g(x)$. Positive definiteness gives $\text{rank } dF(p) = n$. Choose n ambient coordinate components whose differential minor is invertible at p . Their projection is a diffeomorphism on one smaller neighborhood by the inverse function theorem. On that neighborhood F is the graph of its other $N - n$ components, and hence an embedding. This single restriction is independent of derivative order.

APPENDIX A. DOMAINS, FOURIER CONVENTIONS, AND NOTATION

The metric and normal-map conventions are those of Section 2. Here $T_0 = T$, $K_{\text{proof}} = K_{\text{env}}$ and $B_{\text{cubic}} = B_3$ in the main text; B_0 is unchanged. Every reduced operator uses its displayed normalization, including the factor $1/2$ in (2.4). The setup below collects the common domains: the local constructions first give buffered validity regions, and Theorem H.1 allows the nested choices used by the background, mean and pair steps.

Interfaces and verification. Table 1 lists the results used in the main text. Every result in the second column is proved in this paper. A local *Setup* states the hypotheses for a block of arguments; its use in the construction is justified by the cited results. Sections 7–9 and Appendix M verify the hypotheses for the actual iterates. The continuum row first chooses the positive scalar r_F in Proposition I.7. The primary family and its fixed response are linear in each prescribed scalar. Proposition K.2 then prepares $r_{F,R}$ before sampling; the row inverse still belongs to r_F . Corollary H.7 controls this same corrected atomic family throughout.

In the table, one fixed order $d = r_*$ dominates the geometric jets, the normal order $M = 20(n+1)$, the two primary amplitudes and all calculus, density and reconstruction shifts. It also includes the atomic response budget in Corollary H.7 and the finite density and quadrature orders chosen in (5.8), after A and before Q . Membership in this fixed C^d neighborhood gives the stated gaps and low seminorms; the all-order graded profiles are separate hypotheses, verified by Lemma 7.1 and Proposition 7.2. All source sizes and scale windows referred to below are displayed in the main text.

Table 1: Interfaces used by the actual construction.

Object	Proved source	Domain and source class	Low topology and extra hypotheses	Dependence on Q
Reference and carrier geometry	Propositions D.9, D.10; Definition D.12	Common buffered charts; detection on the whole sphere; primary and central signed directions	Fixed C^d neighborhood; positive normal, detector and chord gaps	Geometry fixed before frequency and Q
Extension and finite calculus	Lemma B.1; Theorem E.6	One extension on H^r and C^r , $r \geq 0$; localized operators on W	Fixed collar and finite construction depth; fixed C^d bounds	Same extension; fixed derivative shifts
Regular inverse $S_F T_F$	Theorems F.3, F.4, F.6	Full $L^2(W)$ Neumann inverse; core equation with angular error	Gain one; fixed C^d neighborhood; graded profiles	One inverse; no extra scale power in the displayed graded bounds
Primary waves and joint sample	Theorems G.5, G.6	Whole patches; $(q_{\text{quad}} + 1)^3$ fixed colors; full L^2 outputs and all prefixes; deterministic background quadrature	$r_{\text{par}} = M + 3$; two amplitudes; fixed C^d reserve; full strain of size λ^{-3}	One family and sample event; fixed quadrature order before Q

TABLE 1. INTERFACES USED BY THE ACTUAL CONSTRUCTION (CONTINUED).

Object	Proved source	Domain and source class	Low topology and extra hypotheses	Dependence on Q
Response $\mathcal{B}_M^C$	Proposition H.5; Corollaries H.6, H.7	Source (3.5); global cutoff envelopes; tensor error on D_{lag}	Graded profiles and (3.4); fixed atomic joint jet budget	λ, λ^2 carrier factors; derivative-amplitude change λ^{-1}
Continuum row and positive mean	Propositions I.4, I.5, I.9, I.7	Full $Y = L^2(\mathbb{T}; \mathbb{R}^n)$; raw X^r ; physical fields on U_{fin}	Fixed positive reference dilation, then a fixed C^d neighborhood	Raw source order r ; density source order $r + 1$; fixed coefficient shifts
Prepared zero atoms and U_c	Proposition K.2; Corollary H.7	Fields on U_{fin} ; full auxiliary residual in (5.7)	Fixed positive class; finite density steps and lattice quadrature before sampling	Preparation fixed after A ; no extra power of T_0
Current waves Z_ν	Proposition J.4; Proposition 4.2	Complete finite Fourier measures; global cutoff envelopes; strain on D_{lag}	Same corrected atomic family; shifted support in the original chart	No envelope offset; displacement change λ^{-2} and derivative change λ^{-1}
Nonlinear mean	Lemma 5.2; Proposition 4.2	Global residual recurrence; metric error on D ; physical bounds (4.13)–(4.14)	Same current phases and sample event; canonical inverse in Proposition 5.1	$p = (n + 1)/2$, $B_{\text{mean}} = n/2$; nonlinear budget p ; finite count varies
Signed product correction	Proposition L.4	Entire protected source product; metric error on D ; cut global H^r field	Fixed C^d neighborhood; (8.5)	Source offset 0; physical loss 1 in (4.20)
Stage closure	Lemma 7.1; Proposition M.1	One fixed D and one global band for each increment	Prospective C^{r*} displacement bound and ordered scale choices	B_0, B_3 fixed; $Q_j = 2k_j$ increases

Setup A.1 (Fixed domains). Let D be a closed ball of positive radius centered at zero, $D \Subset W$ a bounded open physical neighborhood, and $\mathbb{T}$ a fixed auxiliary flat torus whose coordinate cell contains $\overline{W}$. All domains below have positive fixed buffers.

First choose the whole-patch primary responses of Theorem H.1, with actual contraction cutoffs $\chi_{2,\text{LAG}}, \chi_{3,\text{LAG}}$. Their finitely many identity cores contain a common wider core U_{LAG} , where $\chi_{2,\text{LAG}} = 1$. Choose the background core U_{BG} , frame ball B_{frame} , measurement region V_{meas} and source/observation cutoffs inside these fixed validity domains, before frequency, sampling, accuracy

and the last seed parameter, so that

$$(A.1) \quad D \Subset U_{\text{BG}} \Subset B_{\text{frame}} \Subset V_{\text{meas}} \Subset \text{Int } D_{\text{lag}}, \quad \text{supp } \chi_{\text{meas}} \Subset V_{\text{meas}},$$

with $\chi_{\text{meas}} = 1$ near $\overline{B_{\text{frame}}}$, and

$$(A.2) \quad \begin{aligned} D_{\text{lag}} &\Subset \text{Int}\{\chi_{\text{lag,src}} = 1\}, & \text{supp } \chi_{\text{lag,src}} &\Subset \text{Int}\{\chi_{\text{core,LAG}} = 1\}, \\ \text{supp } \chi_{\text{core,LAG}} &\Subset U_{\text{LAG}} \Subset W, & \text{supp } \chi_{\text{lag,src}} &\Subset \{\chi_{2,\text{LAG}} = 1\}. \end{aligned}$$

One larger finite-primary chart in W contains $\overline{U_{\text{LAG}}}$ and all these cutoffs.

Every entire measurement, protected pair-source and physical-output support lies in $\text{Int } D_{\text{lag}}$. The cutoffs $\chi_{\text{phys}}, \chi_{\text{pair}}$ equal one near D . Choose the pair cutoff inside its protected region X_{pair} , where $\chi_{2,\text{pair}} = 1$, and the mean and common final physical cutoffs inside U_{BG} for the complete primary, mean and pair sum. For the synthesis slice estimate use a separate χ_{slice} , equal to one near the entire $\text{supp } \chi_{\text{meas}}$ and supported in the larger common validity region, for example $\text{Int } D_{\text{lag}}$. This cutoff belongs only to that estimate; the actual fields use their own physical cutoffs.

The background notation $U_{\text{core}}, B_r, U_{\text{meas}}$ means $U_{\text{BG}}, B_{\text{frame}}, V_{\text{meas}}$; primary-response notation U_{core} and its control cutoff means $U_{\text{LAG}}, \chi_{\text{core,LAG}}$. Other solver cores are their own fixed validity domains.

The primary current response and protected pair correction use separate regular-sector operators. Appendix I constructs the continuum image and transport row. A literal sample in primary patch g uses one current $\mathcal{B}_F^{(g)}$ for every initial datum. The finite primary wave and its response use the same phase. Each construction has its own fixed domains, angular buffers, cutoffs, real pairings and construction depths. At each coefficient choice the actual operators and exact inverses are the same at every precision and observation order. Take the finite minimum of margins and maximum of constants and derivative shifts over the patch list.

The base strain, complete response leakage and current strain identity are estimated on D_{lag} , covering every whole measurement and pair source. Metric-defect smallness and target smoothing are imposed only on D .

Notation A.2 (Norms and Fourier transforms). For a function smooth up to the boundary of D ,

$$\|f\|_{C^r(D)} = \max_{|\alpha| \leq r} \sup_{x \in D} |\partial^\alpha f(x)|.$$

The same convention is used on W . Vector and tensor norms are taken componentwise in fixed coordinates. Unless a domain is specified otherwise, global Sobolev and L^2 norms are on $\mathbb{T}$.

Let Λ^* be the dual lattice of $\mathbb{T}$. We use

$$\widehat{u}(\xi) = \frac{1}{\text{vol}(\mathbb{T})} \int_{\mathbb{T}} e^{-ix \cdot \xi} u(x) \, dx, \quad u(x) = \sum_{\xi \in \Lambda^*} \widehat{u}(\xi) e^{ix \cdot \xi}.$$

In particular,

$$\|u\|_{L^2(\mathbb{T})}^2 = \text{vol}(\mathbb{T}) \sum_{\xi \in \Lambda^*} |\widehat{u}(\xi)|^2.$$

The integer Sobolev norm $H^r(\mathbb{T})$ is the L^2 norm of all Cartesian derivatives through order r .

For Euclidean envelope variables, the convention is

$$\widehat{f}(\xi) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-ix \cdot \xi} f(x) \, dx, \quad f(x) = \int_{\mathbb{R}^n} e^{ix \cdot \xi} \widehat{f}(\xi) \, d\xi.$$

For a finite complex Fourier measure m , its size is $\|m\|_{\text{TV}}$. The atom at the origin is denoted by δ_0 .

On local balls, positive Sobolev spaces are restriction spaces; negative estimates use the corresponding supported-distribution and restriction dual pairs. Whole-space comparisons use fixed cutoffs and the extension of Lemma B.1. Write $D = -i\partial$, $\|u\|_s = \|u\|_{H^s}$ and $H^\infty = \bigcap_{s \geq 0} H^s$. A

coefficient norm includes every matrix entry. Adjoints are complex Hilbert adjoints, restricted to real fields when the operator is real.

Notation A.3 (Stage scales). Use the scales and strict windows of Section 8:

$$\lambda = R^a, \quad h = R^{-\beta}, \quad H = R^{1+\theta}, \quad L_F = T_0 = R^t, \quad K_{\max} = H^{d_f} = R^{s_f}.$$

The zero-atom and common response envelopes are respectively $C_{\text{env}}T_0 < K_{\max}$ and

$$K_{\text{proof}} = C_* K_{\max}.$$

These constants are fixed independently of R , accuracy and derivative order. Equation (8.5) puts both below the smallest protected product frequency by a strict power.

APPENDIX B. INTERPOLATION AND FINITE-JET ESTIMATES

The estimates in this section are independent of every phase, solver and frequency choice. They will be used in the geometric and analytic constructions.

Lemma B.1 (One fixed extension at every nonnegative integer order). *Let $\Omega = B_R(c) \subset \mathbb{R}^n$ and $\bar{\Omega} \subset V$, with V open. Use the derivative Sobolev norm*

$$\|f\|_{H^m(\Omega)}^2 = \sum_{|\alpha| \leq m} \|D^\alpha f\|_{L^2(\Omega)}^2, \quad m \geq 0,$$

and the corresponding whole-space norm. There is one real linear operator, independent of m ,

$$E_\Omega: C^\infty(\bar{\Omega}) \longrightarrow C_c^\infty(\mathbb{R}^n),$$

equal to f on Ω , supported in a fixed compact subset of V , with

$$(B.1) \quad \|E_\Omega f\|_{H^m(\mathbb{R}^n)} \leq C_m \|f\|_{H^m(\Omega)},$$

$$(B.2) \quad \|E_\Omega f\|_{C^m(\mathbb{R}^n)} \leq C_m \|f\|_{C^m(\bar{\Omega})}.$$

The constants depend only on Ω, V , the fixed cutoff and m . The bounded extensions to H^m and C^m agree on their common domains. The derivative norm is equivalent to the whole-space restriction norm. The operator acts componentwise and complex linearly. No negative-order extension is asserted.

Proof. Choose $0 < d < R/2$ with $\overline{B_{R+d}(c)} \Subset V$ and a real $\phi \in C^\infty([0, \infty))$ equal to one on $[0, d/4]$ and zero on $[d/2, \infty)$. For $b_k = -2^k$ and $0 \leq k \leq N$ put

$$a_k^{(N)} = \prod_{\substack{0 \leq j \leq N \\ j \neq k}} \frac{1 + 2^j}{2^j - 2^k}.$$

Lagrange interpolation at $b_0, \dots, b_N$, evaluated at one, gives

$$(B.3) \quad \sum_{k=0}^N a_k^{(N)} b_k^\ell = 1 \quad (0 \leq \ell \leq N).$$

Separating $j < k$ and $j > k$ gives

$$(B.4) \quad |a_k^{(N)}| \leq A^2 B^{-2} 2^{-k(k+1)/2}, \quad A = \prod_{j \geq 0} (1 + 2^{-j}) < \infty, \quad B = \prod_{j \geq 1} (1 - 2^{-j}) > 0.$$

For fixed k the product converges. Its bound permits dominated convergence in the moment identities, giving $a_k = \lim_N a_k^{(N)}$ with

$$(B.5) \quad \sum_{k \geq 0} a_k b_k^\ell = 1, \quad \sum_{k \geq 0} |a_k| 2^{k\ell} < \infty \quad (\ell \geq 0).$$

Set $E_\Omega f = f$ on Ω . With $h(\omega, s) = f(c + (R - s)\omega)$, define for $t > 0$

$$(B.6) \quad E_\Omega f(c + (R + t)\omega) = \sum_{k \geq 0} a_k \phi(2^k t) h(\omega, 2^k t).$$

Terms with $2^k t \geq d/2$ are zero. The sum is finite at each $t > 0$ and supported in $\overline{B_{R+d/2}(c)} \Subset V$.

For $p = 2, \infty$ and $|\beta| + \ell \leq m$, the substitution $s = 2^k t$ in a fixed finite angular atlas gives

$$(B.7) \quad \left\| a_k 2^{k\ell} [D_\omega^\beta \partial_s^\ell (\phi h)](\omega, 2^k t) \right\|_{L^p} \leq C_m |a_k| 2^{k(\ell-1/p)} \|f\|_{W^{m,p}(\Omega)}, \quad 1/\infty = 0.$$

The same summable bound proves both norm estimates and uniform convergence of every derivative at $t = 0$. Its boundary value is $(-1)^\ell D_\omega^\beta \partial_s^\ell h(\omega, 0)$ by (B.5). These are exactly the interior jets of $h(\omega, -t)$, so the joined function is smooth, with no boundary terms in weak derivatives. For $n = 1$, use the two radial directions without angular variables.

Contraction by $x \mapsto c + (1 - \epsilon)(x - c)$, followed by mollification on the resulting larger ball, proves smooth density in H^m , C^m and their intersections. The bounds extend this same E_Ω consistently. Restriction and E_Ω give the two quotient-norm inequalities; Plancherel compares the whole-space derivative and Bessel-potential norms. The coefficients are real. $\square$

Lemma B.2 (Fourier interpolation at a prescribed scale). *On $\mathbb{R}^n$, the fixed auxiliary torus, or a fixed ball with the preceding extension, let $r \geq 0$ and $m > r + n/2$ be integers. For $K \geq 1$,*

$$(B.8) \quad \|u\|_{C^r} \leq C_{r,m} \left[K^{r+n/2} \|u\|_2 + K^{r+n/2-m} \|u\|_{H^m} \right].$$

Thus a Sobolev profile $\|u\|_{H^s} \leq C_s A K^s$ gives $\|u\|_{C^r} \leq C_r A K^{r+n/2}$, using only $s = 0, m$.

Proof. Fourier inversion bounds each order- r derivative by the frequency integral, or sum, of $|\xi|^r |\hat{u}(\xi)|$. Cauchy-Schwarz on $|\xi| \leq K$ gives the first term. On $|\xi| > K$, the squared weight $|\xi|^{2r-2m}$ has integral or lattice sum at most $C_{r,m} K^{2r+n-2m}$, giving the second. Lower derivatives obey the same bound since $K \geq 1$. The fixed extension is bounded at orders zero and m . $\square$

Lemma B.3 (Elementary interpolation, derivative products and finite jets). *On a fixed auxiliary torus, or a fixed chart with a bounded extension, let $0 \leq R < j < k$ be integers. Then*

$$\|u\|_{C^j} \leq C_{R,j,k} \|u\|_{C^R}^{1-(j-R)/(k-R)} \|u\|_{C^k}^{(j-R)/(k-R)}.$$

The endpoint and bounded low orders have their direct norm bounds.

Let f_i be finitely many primitive arrays, with possibly different nonnegative integer baselines b_i , and assume that their C^{b_i} norms are bounded. For an integer $W \geq 0$, suppose that a product of primitive derivatives, allowing repeated primitives, has orders j_h satisfying

$$\sum_h (j_h - b_{i(h)})_+ \leq W.$$

Then

$$(B.9) \quad \prod_h \|D^{j_h} f_{i(h)}\|_{L^\infty} \leq C_W \left[1 + \sum_i \|f_i\|_{C^{b_i+W}} \right].$$

Matrix factors remain in their original order. The constant may depend on the number of factors and their fixed low bounds.

Let $\mathcal{F}(J^c F)$ be a fixed smooth finite-jet expression, including finitely many matrix inversions whose singular-value gaps stay positive. If its low jet ranges in a fixed bounded set with those gaps, then

$$\|\mathcal{F}(J^c F)\|_{C^r} \leq C_r (1 + \|F\|_{C^{r+c}}).$$

Here the finite shift may be enlarged once for finitely many such operations. These are algebraic jet estimates; dependence of Hamilton flows, retarded operators and their inverses is proved separately below.

Proof. Use the fixed extension on a chart. Let $a = \|u\|_{C^R}$ and $b = \|u\|_{C^k}$, so $a \leq b$. Choose one real even $\psi \in C_c^\infty$ equal to one near zero. Its inverse Fourier kernel has mass one and all positive moments zero, by differentiation of ψ at zero. Periodizing its scale- t dilate gives P_t , for $0 < t \leq t_0$, with

$$\|P_t u\|_{C^j} \leq C t^{R-j} a, \quad \|(I - P_t)u\|_{C^j} \leq C t^{k-j} b.$$

The first bound follows by moving derivatives to the kernel; the second follows from Taylor's integral remainder and the moment conditions. If $a = 0$, then $u = 0$. Otherwise choose $t = (a/b)^{1/(k-R)}$ when it is at most t_0 . This gives

$$(B.10) \quad \|u\|_{C^j} \leq C_{R,j,k} a^{1-\theta} b^\theta, \quad \theta = \frac{j-R}{k-R}.$$

When that choice exceeds t_0 , the ratio b/a is bounded and the direct bound $\|u\|_{C^j} \leq b$ proves the same estimate. The endpoint and bounded low orders follow directly.

For $W > 0$ set $H_i = 1 + \|f_i\|_{C^{b_i+W}}$ and $\theta_h = (j_h - b_{i(h)})_+/W$. Interpolation at the indicated baselines gives

$$\prod_h \|D^{j_h} f_{i(h)}\|_\infty \leq C_W \prod_h H_{i(h)}^{\theta_h} \leq C_W \left(1 + \sum_i H_i\right), \quad \sum_h \theta_h \leq 1,$$

by weighted arithmetic-geometric mean. For $W = 0$ use the baseline bounds. This proves (B.9) for ordered matrix products as well.

For the finite-jet assertion put $U = J^c F$. Its low range and inverse gaps bound every derivative of the finite-dimensional coefficient function. Each order- r chain-rule term is a bounded coefficient times $\prod_i D^{\alpha_i} U$, where $|\alpha_i| \geq 1$ and $\sum_i |\alpha_i| = r$. The product estimate with baseline zero therefore gives $C_r(1 + \|U\|_{C^r})$. Fixed-gap inversions are smooth coefficient functions; finitely many jet operations enlarge the shift only by a fixed amount. $\square$

Corollary B.4 (A fixed profile under finite derivative loss). *Suppose $T \geq 1$ and $\|F\|_{C^j} \leq A_j T^{(j-R)_+}$ for every $j \geq 0$. Let a fixed finite family of arrays on the same buffered charts satisfy $\|\mathcal{F}_i(F)\|_{C^r} \leq C_r(1 + \|F\|_{C^{r+c_i}})$, with fixed shifts c_i . If $R \geq \max_i c_i + \nu$, then*

$$(B.11) \quad \|\mathcal{F}_i(F)\|_{C^r} \leq C_r T^{(r-\nu)_+}, \quad r \geq 0.$$

Any fixed finite list of parameter derivatives can be included among the arrays. For normalized symbols the radial order stays separate. This applies to fixed constructions, not to an accuracy-dependent number of compositions.

Proof. Substitution gives $C_r T^{(r+c_i-R)_+}$, which has the stated bound. $\square$

Lemma B.5 (Common buffered inverse domains). *Let $D_0 \Subset D_1 \Subset D$ be bounded convex boxes in $\mathbb{R}^d$ and F_0 smooth near $\overline{D}$. Suppose a constant invertible matrix G satisfies $\sup_D \|G^{-1}(DF_0 - G)\| \leq 1/4$. For all F in one sufficiently small C^1 neighborhood of F_0 , F is injective on D , with a uniform lower Lipschitz bound, and one open set V satisfies $F(D_0) \Subset V \Subset F(D_1)$ with common positive buffers. The assertion is uniform for parameter families in this same neighborhood.*

Proof. Choose the C^1 neighborhood so $\sup_D \|I - G^{-1}DF\| \leq 1/2$. The segment identity gives $|G^{-1}(F(u) - F(v))| \geq |u - v|/2$. Choose one radius $r > 0$ so every $\overline{B}_r(u_0)$, $u_0 \in \overline{D}_0$, lies in D_1 . A fixed small neighborhood V of $F_0(\overline{D}_0)$ and a small C^0 neighborhood of F_0 ensure that every $v \in \overline{V}$ has such a u_0 with $|G^{-1}(v - F(u_0))| < r/2$. Then

$$u \mapsto u - G^{-1}(F(u) - v)$$

is a $1/2$ contraction of $\overline{B}_r(u_0)$ into itself. Its fixed point gives $\overline{V} \subset F(D_1)$. Reducing the C^0 neighborhood also gives $F(\overline{D}_0) \Subset V$. All buffers and constants use only these fixed low choices. $\square$

Lemma B.6 (Uniform interpolation and joint derivative budgets). *Use finitely many fixed working balls with larger buffered coordinate neighborhoods, including every physical variable, normalized covariable and parameter. The interpolation estimate of Lemma B.3 applies jointly in these coordinates. Right-hand norms may be on the larger domains. For an entire working ball, use its single bounded extension at every order; maxima over the fixed finite family obey the same bound.*

Use (B.9) for complete ordered products on these domains.

These estimates are uniform on normalized dyadic shells. For a degree-one phase and an order- p symbol, the normalized arrays are

$$S_\mu(x, \omega) = \mu^{-1} S(x, \mu\omega), \quad a_\mu(x, \omega) = \mu^{-p} a(x, \mu\omega), \quad \mu \geq 1,$$

on fixed buffered normalized annuli and angular charts. Both physical variables are included for a two-position amplitude. The estimates control coefficient products; explicit frequency powers, shell ratios, and integration measures remain separate.

Fixed-gap matrix inverses, existing implicit and inverse charts, and fixed-time flows preserve these joint derivative budgets. All estimates use the same fixed working domains and inverse gaps.

Proof. Apply (B.9) to the complete primitive product before substituting coefficient estimates: if $\|f_i\|_{C^m} \leq C_m(1 + \|a\|_{C^{m+c_i}})$, its result uses the maximum index $b_i + W + c_i$, without multiplying high coefficient norms. The corresponding datum interpolation is

$$\|v\|_{p+k} \leq C_{p,s} \|v\|_p^{1-k/s} \|v\|_{p+s}^{k/s} \quad (0 \leq k \leq s), \quad \|f\|_{C^j} \leq C_{j,L} \|f\|_{H^{j+L}} \quad (L > d/2).$$

These follow respectively from Fourier Hölder and Fourier Cauchy–Schwarz; fixed extensions give the local versions.

Fixed-gap matrix inverses are finite-jet expressions from Lemma B.3. For compositions with bounded first jets, the derivative partitions satisfy

$$(q-1) + \sum_{i=1}^q (\alpha_i - 1) = r-1, \quad \alpha_i \geq 1, \quad \sum_i \alpha_i = r.$$

This is the baseline-one budget for a composition.

For an existing inverse $X = F^{-1}$, differentiate $F \circ X = \text{Id}$ on the same fixed working balls. The first-order inverse gap bounds DX . Inductively, all lower derivatives of X are bounded by the corresponding C^j norms of F on the fixed larger range charts. Every term other than $DF(X)D^r X$ has $q \geq 2$ and $\alpha_i < r$. Substitution of the inductive bounds leaves products of F derivatives with the displayed excess at most $r-1$. Equation (B.9) therefore bounds them by $C_r(1 + \|F\|_{C^r})$; the inverse gap proves the same bound for X . Implicit maps use the inverse of $(z, w) \mapsto (z, F(z, w))$, including all external parameters.

For the fixed-time flow $\partial_t \Psi = V(\Psi)$, the top variational term is $DV(\Psi)D^r \Psi$. The other partitions have the same budget and involve only lower flow derivatives. Induction bounds their forcing by $C_r(1 + \|V\|_{C^r})$, and Gronwall gives the same estimate for the flow on the fixed interval. Time derivatives follow from the equation. Variable initial sections are compositions with this flow and obey the same joint budget.

For $U' = -CU$, $U(0) = I$, apply the flow estimate to $-CU$ and $(U^{-1})' = U^{-1}C$, including time and the constant tube parameters as coordinates. Gronwall keeps both matrix ranges bounded, giving

$$\|U\|_{C^r} + \|U^{-1}\|_{C^r} \leq C_r(1 + \|C\|_{C^r}).$$

Variable initial values multiply this fundamental matrix. Their composition with the fixed flow obeys the same joint product budget.

For normalized symbols and phases the exact identities are

$$\partial_x^\alpha \partial_\omega^\gamma a_\mu = \mu^{|\gamma|-p} (\partial_x^\alpha \partial_\eta^\gamma a)(x, \mu\omega), \quad \partial_x^\alpha \partial_\omega^\gamma S_\mu = \mu^{|\gamma|-1} (\partial_x^\alpha \partial_\eta^\gamma S)(x, \mu\omega).$$

All interpolation domains are consequently fixed. Normalize two noncomparable frequencies separately and leave their ratios, powers and shell volumes outside the primitive norms. The fixed cutoff pullbacks of Setup E.3 obey the same composition budget. $\square$

APPENDIX C. OSCILLATORY ESTIMATES AND CANONICAL RETARDED INTEGRATION

This section proves three elementary operator estimates used later: exact normal comparison, localization of oscillatory sources, and canonical retarded integration. The phase hypotheses below are verified for the carrier phases in Proposition D.10; each result otherwise uses only its displayed phase and amplitude bounds.

C.1. Exact normal comparison.

Notation C.1. Use the Fourier convention of Notation A.2 and $\|f\|_{H^s} = \|\langle D \rangle^s f\|_2$ on $\mathbb{R}^n$, where $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$. Compactly supported chart outputs are extended by zero. Complex Hilbert inner products are linear in the first argument; operator norms are on the entire stated spaces. Derivative order means total multi-index order, and matrix dimensions are fixed.

Lemma C.2 (Two-position amplitudes on fixed supports). *Let $d \geq 1$ and fix compact physical supports inside bounded boxes of $\mathbb{R}^d$. Suppose $b(x, y, \xi)$ is smooth, has these supports in x, y , and*

$$\sup_{\xi} |D_{x,y}^{\alpha} b(x, y, \xi)| \leq A, \quad |\alpha| \leq 2(d+1).$$

Then

$$Bf(x) = (2\pi)^{-d} \iint e^{i(x-y) \cdot \xi} b(x, y, \xi) f(y) dy d\xi \quad \text{satisfies} \quad \|B\| \leq CA.$$

The frequency support may be unbounded; C depends on the fixed physical boxes and d . Fixed matrix amplitudes obey the same bound.

Let $T : L^2(D) \rightarrow L^2(\mathbb{R}^d)$ have a smooth integral kernel $t(x, u)$ on fixed compact supports. For $c(x, x', u)$ with the same physical supports and the preceding derivative bounds in x, x' , uniformly in u , the full kernel

$$\mathcal{N}(c, g)(x, x') = \int t(x, u) c(x, x', u) \overline{t(x', u)} g(u) du$$

has the bound

$$(C.1) \quad \|\mathcal{N}(c, g)\| \leq CA \|T\|^2 \|g\|_{\infty}.$$

Proof. Expand the two physical variables on one fixed larger torus, with outer cutoffs removing the periodic copies. Integration by parts with $(1 - \Delta_x - \Delta_y)^{d+1}$ gives

$$\sup_{\xi} |b_{pq}(\xi)| + \sup_u |c_{pq}(u)| \leq CA \langle (p, q) \rangle^{-2(d+1)}.$$

With the outer cutoffs included in the bounded modulations M_p, M_q , the exact operators are

$$B = \sum_{p,q} M_p b_{pq}(D) M_q, \quad \mathcal{N}(c, g) = \sum_{p,q} M_p T[c_{pq}g] T^* M_q.$$

The coefficients are summable in $2d$ dimensions. Plancherel and the bound on T prove both assertions by absolute convergence in operator norm. Compact frequency regularization, followed by its distributional limit, justifies the unbounded-frequency formula. Apply the argument entrywise for fixed matrices. $\square$

Setup C.3 (Uniform normal hypotheses). Fix $n \geq 2$, $r \geq 1$, $M = 20(n+1)$, bounded convex open sets $X, D \subset \mathbb{R}^n$, compact sets $X_0 \Subset X$, $D_0 \Subset D$, and $\epsilon_0 > 0$. Let the real phase $\phi_\epsilon(x, u)$ and the complex r -vector amplitude $a_\epsilon(x, u)$ be smooth for $0 \leq \epsilon \leq \epsilon_0$ on neighborhoods of $\overline{X} \times \overline{D}$, with smooth extensions across the parameter endpoints. Assume

$$\text{supp } a_\epsilon \subset X_0 \times D_0, \quad |D_{x,u}^\alpha \phi_\epsilon| + |D_{x,u}^\alpha a_\epsilon| + |\partial_\epsilon D_{x,u}^\alpha \phi_\epsilon| + |\partial_\epsilon D_{x,u}^\alpha a_\epsilon| \leq C_\alpha \quad (|\alpha| \leq M).$$

Define the exact averaged gradient

$$\Psi_\epsilon(x, x', u) = \int_0^1 d_x \phi_\epsilon(x' + t(x - x'), u) dt.$$

Assume each $\Psi_\epsilon(x, x', \cdot)$ is injective on D , with a common buffered inverse chart containing the images of D_0 , and $|\det d_u \Psi_\epsilon| \geq c_* > 0$ throughout these boxes.

Proposition C.4 (Exact masked normal comparison). *In Setup C.3, define for $K \geq 1$*

$$T_\epsilon^K f(x) = \left(\frac{K}{2\pi}\right)^{n/2} \int_D e^{iK\phi_\epsilon(x,u)} a_\epsilon(x, u) f(u) du.$$

This maps the entire $L^2(D; \mathbb{C})$ into $L^2(\mathbb{R}^n; \mathbb{C}^r)$, and

$$(C.2) \quad \|T_\epsilon^K\| \leq C.$$

Let $0 < h \leq 1$ and let m_h be real and smooth on D , with $0 \leq m_h \leq 1$ and $\|Dm_h\|_\infty \leq C/h$. For $E_\epsilon^K = T_\epsilon^K m_h (T_\epsilon^K)^*$,

$$(C.3) \quad \|E_\epsilon^K - E_0^K\| \leq C\epsilon/h, \quad K \geq 1.$$

The constant uses only the fixed sets, buffers, dimensions, Jacobian gap and finite phase and amplitude bounds. The comparison uses the complete kernels and only the displayed first mask derivative.

Proof. Step 1. The mask-free norm. Write $\Phi_\epsilon = \phi_\epsilon(x, u) - \phi_\epsilon(x', u)$ and $A_\epsilon = a_\epsilon(x, u) a_\epsilon(x', u)^*$. Convexity gives $\Phi_\epsilon = (x - x') \cdot \Psi_\epsilon$. Let $u_\epsilon(x, x', v)$ be the common inverse. Its fixed buffer and Jacobian gap give uniform derivative bounds by implicit differentiation. The single change $v = \Psi_\epsilon(x, x', u)$ gives the exact full kernel

$$(C.4) \quad E_\epsilon^K(x, x') = \left(\frac{K}{2\pi}\right)^n \int e^{iK(x-x') \cdot v} b_\epsilon(x, x', v) dv,$$

where

$$b_\epsilon = \frac{A_\epsilon(x, x', u_\epsilon) m_h(u_\epsilon)}{|\det D_u \Psi_\epsilon(x, x', u_\epsilon)|}.$$

The inverse buffer makes its zero extension smooth, jointly with all physical and parameter variables. For mask 1 its needed derivatives are uniformly bounded. Set $\xi = Kv$ and apply Lemma C.2 to $b_\epsilon(x, x', \xi/K)$. Its bounds are uniform for $K \geq 1$, proving $\|T_\epsilon^K\|^2 = \|T_\epsilon^K (T_\epsilon^K)^*\| \leq C$.

Step 2. Differentiation in the original input variable. Set

$$w_\epsilon = (D_u \Psi_\epsilon)^{-1} \partial_\epsilon \Psi_\epsilon.$$

Then $\partial_\epsilon \Phi_\epsilon = w_\epsilon \cdot D_u \Phi_\epsilon$ exactly. Choose fixed scalar χ_x, χ_u equal to one on the original supports, inside X, D and the common inverse buffer. Step 1 bounds the scalar channel $T_{\epsilon, \chi}^K$ with amplitude $\chi_x \chi_u$. Let $\mathcal{N}_\epsilon(c, g)$ be the second operator in Lemma C.2 for this channel. Differentiation and integration by parts in the original u give

$$\partial_\epsilon E_\epsilon^K = \mathcal{N}_\epsilon(c_0, m_h) + \sum_j \mathcal{N}_\epsilon(c_j, \partial_{u_j} m_h),$$

$$c_0 = \partial_\epsilon A_\epsilon - \text{div}_u(w_\epsilon A_\epsilon), \quad c_j = -w_{\epsilon, j} A_\epsilon.$$

The compact input support removes every boundary term. These full amplitudes have uniform spatial derivatives through $2(n+1)$: computing them uses at most $2(n+1) + 3$ phase derivatives

and one ϵ derivative, within the fixed M budget. Equation (C.1) gives $\|\partial_\epsilon E_\epsilon^K\| \leq C(1 + \|Dm_h\|_\infty) \leq C/h$. At each fixed K the compact kernel is differentiable in Hilbert–Schmidt norm. Integrating from 0 to ϵ proves (C.3). $\square$

Corollary C.5 (Both real quadratures). *Regard $L^2(D; \mathbb{C})$ as a real Hilbert space with inner product $\operatorname{Re} \int f \bar{g}$, and set*

$$C_\epsilon^K f = 2 \operatorname{Re}(T_\epsilon^K f) \in L^2(\mathbb{R}^n; \mathbb{R}^r).$$

For every real output r_0 ,

$$(C.5) \quad (C_\epsilon^K)^*_{\mathbb{R}} r_0 = 2(T_\epsilon^K)^* r_0, \quad C_\epsilon^K m_h (C_\epsilon^K)^*_{\mathbb{R}} r_0 = 4 \operatorname{Re}(E_\epsilon^K r_0).$$

The bounds of Proposition C.4 hold for these real operators with fixed numerical factors. The input still contains both complex quadratures.

Proof. For complex f and real r_0 ,

$$\langle 2 \operatorname{Re}(Tf), r_0 \rangle_{\mathbb{R}} = 2 \operatorname{Re} \langle Tf, r_0 \rangle = 2 \operatorname{Re} \langle f, T^* r_0 \rangle.$$

This proves the adjoint formula. Since the mask is real, applying C to $2m_h T^* r_0$ proves the normal identity and its factor 4. Taking norms proves the assertions. There is no restriction to real input and no extra conjugate cross operator. $\square$

Proposition C.6 (A full-space single-channel estimate). *Assume additionally that $X = J_{\text{time}} \times Y$ and $D = I_{\text{carrier}} \times D_\theta$, where $J_{\text{time}}, I_{\text{carrier}}$ are bounded open intervals and $Y, D_\theta \subset \mathbb{R}^{n-1}$ are bounded convex open sets. Write $x = (t, y)$ and $u = (z, \theta)$. Suppose, uniformly in t, z, ϵ , the maps*

$$\theta \mapsto \int_0^1 d_y \phi_\epsilon(t, y' + s(y - y'), z, \theta) ds$$

satisfy the buffered inverse, Jacobian and derivative assumptions of Setup C.3 in dimension $n - 1$. Define

$$L_{\epsilon, z}^K f(t, y) = \left(\frac{K}{2\pi} \right)^{n/2} \int e^{iK\phi_\epsilon(t, y, z, \theta)} a_\epsilon(t, y, z, \theta) f(\theta) d\theta.$$

Then, on the entire indicated input and output spaces,

$$\|L_{\epsilon, z}^K\|_{L^2(D_\theta; \mathbb{C}) \rightarrow L^2(\mathbb{R}^n; \mathbb{C}^r)}^2 \leq CK.$$

The realification $2 \operatorname{Re} L_{\epsilon, z}^K$ has the same squared bound with a factor 4.

Proof. Put $d = n - 1$. For each fixed t, z, ϵ , the slice

$$A_{t, z, \epsilon}^K f(y) = \left(\frac{K}{2\pi} \right)^{d/2} \int e^{iK\phi_\epsilon(t, y, z, \theta)} a_\epsilon(t, y, z, \theta) f(\theta) d\theta$$

has a uniformly bounded L^2 norm by the mask-free proof of Proposition C.4, now in dimension d . At time t , the operator $L_{\epsilon, z}^K$ is $\sqrt{K/(2\pi)}$ times this slice. Integrating the squared slice estimate over the fixed bounded interval J_{time} proves the claim. The inequality $\|2 \operatorname{Re} v\| \leq 2\|v\|$ gives the real estimate. $\square$

For the actual carrier families, Proposition D.10 verifies these hypotheses for the exact finite-difference phase (D.14), including its smooth value at zero from (D.16).

C.2. Exact phase localization and Sobolev tails.

Proposition C.7 (A radial shell and negative Sobolev gain). *Fix $n \geq 1$, a bounded box $X \subset \mathbb{R}^n$, a compact $K \Subset X$, a finite complex measure μ on a parameter set D , $0 < c < C$, $b \geq 0$, and $0 < \sigma < 1$. For $\rho \geq 2$ and $L \geq 1$, let $\phi(x, u)$ be a smooth real phase and $a(x, u)$ a smooth complex amplitude of fixed finite dimension, supported in K in x . Assume uniformly in $u \in D$*

$$c \leq |d_x \phi| \leq C, \quad |D_x^r \phi| \leq C_r L^{r-1} \quad (r \geq 2), \quad |D_x^r a| \leq C_r A L^{r+b} \quad (r \geq 0),$$

where $A \geq 0$. Set

$$h_\rho(x) = \int_D e^{i\rho\phi(x,u)} a(x, u) d\mu(u).$$

Choose one fixed rescaled smooth radial multiplier Π_ρ equal to 1 on $c\rho/2 \leq |\xi| \leq 2C\rho$, with support in $c\rho/4 \leq |\xi| \leq 4C\rho$. For integers $m \geq 0$ and $q > m + n/2$,

$$(C.6) \quad \|(I - \Pi_\rho)h_\rho\|_{H^m} \leq C_{m,q} A \|\mu\|_{\text{TV}} L^b \rho^{m+n/2} (L/\rho)^q.$$

If $L \leq \rho^{1-\sigma}$, then for every integer $M \geq 0$,

$$(C.7) \quad \|h_\rho\|_{H^{-M}} \leq C_{M,\sigma} A \|\mu\|_{\text{TV}} L^b \rho^{-M}.$$

The source, cutoff family and derivative exponent b are fixed for all requested observation and tail orders. More generally, the tail estimate uses only the upper phase bounds and the gap $|\rho d_x \phi - \xi| \geq c_1(\rho + |\xi|)$ on the entire complement of a fixed bounded rescaled Fourier mask. If that mask is supported in a fixed annulus, the negative-norm conclusion follows under the same scale gap. For $d\mu = F du$ on a set of measure V , $\|\mu\|_{\text{TV}} \leq V^{1/2} \|F\|_2$.

Proof. For fixed u , the Fourier kernel is

$$\hat{h}_u(\xi) = (2\pi)^{-n} \int_K e^{i[\rho\phi(x,u) - x \cdot \xi]} a(x, u) dx.$$

On the entire support of $1 - \Pi_\rho$, either $|\xi| < c\rho/2$ or $|\xi| > 2C\rho$, including its transition regions. Thus

$$|\rho d_x \phi - \xi| \geq c_1(\rho + |\xi|).$$

Integrate by parts with

$$\mathcal{V}_\xi = \frac{(\rho d_x \phi - \xi) \cdot \partial_x}{i|\rho d_x \phi - \xi|^2}.$$

Reciprocal differentiation and the phase bounds show that its coefficients have derivatives of order j bounded by $C_j L^j / (\rho + |\xi|)$. An amplitude derivative costs one factor L . Induction on the number q of transposed integrations gives

$$|(1 - \pi_\rho(\xi)) \hat{h}_u(\xi)| \leq C_q A L^{q+b} (\rho + |\xi|)^{-q}.$$

All boundary terms vanish because K is a common compact support. Integrate against $|\mu|$. For $q > m + n/2$, rescaling $\xi = \rho\omega$ gives

$$\int_{\mathbb{R}^n} \langle \xi \rangle^{2m} (\rho + |\xi|)^{-2q} d\xi \leq C_{m,q} \rho^{2m+n-2q}.$$

Taking square roots proves (C.6).

The amplitude bound and fixed output support give $\|h_\rho\|_2 \leq C A \|\mu\|_{\text{TV}} L^b$. The annular split and the tail estimate at $m = 0$ therefore give

$$\|h_\rho\|_{H^{-M}} \leq C_q A \|\mu\|_{\text{TV}} L^b [\rho^{-M} + \rho^{n/2} (L/\rho)^q].$$

Take $q > n/2$ with $\sigma q \geq M + n/2$ to prove (C.7). Only the complementary gap was used for the tail, and annular support for the negative norm. $\square$

Remark C.8. The true frequency in Proposition C.7 may be $\rho = \lambda|z - z'|$ for a normalized same-carrier difference phase. The estimate applies when the normalized phase satisfies the displayed gradient and graded derivative hypotheses. The phase is used exactly, rather than expanded in the source parameter.

C.3. Canonical retarded integration. The literal integral gives all integer Sobolev bounds.

Definition C.9 (The fixed retarded operator). Fix real smooth cutoffs $\chi_{\text{in}}, \chi_{\text{out}}$ compactly supported in one rectangle $(t_-, t_+) \times Y \subset \mathbb{R} \times \mathbb{R}^d$, $n = d + 1$, with a fixed positive buffer after t_- for $\text{supp } \chi_{\text{in}}$. Put

$$Jf(t, y) = \chi_{\text{out}}(t, y) \int_{t_-}^t \chi_{\text{in}}(s, y) f(s, y) ds.$$

Proposition C.10 (The fixed retarded bounds). *The literal adjoint is*

$$(C.8) \quad J^*v(s, y) = \chi_{\text{in}}(s, y) \int_s^{t_+} \chi_{\text{out}}(t, y) v(t, y) dt.$$

Both operators are bounded on $H^m(\mathbb{R}^n)$ for every integer m ; on buffered rectangles the negative bounds use the reversed supported/restriction pairs. For real λ and transverse ζ ,

$$(C.9) \quad e^{-i\lambda\zeta \cdot y} J[e^{i\lambda\zeta \cdot y} f] = Jf.$$

Proof. Fubini gives the adjoint. Cauchy–Schwarz on the fixed time interval gives both L^2 bounds. For $j \geq 1$,

$$\partial_t^j \int_{t_-}^t g(s, y) ds = \partial_t^{j-1} g(t, y),$$

and transverse derivatives pass under the integral. Leibniz' rule gives the H^m bounds for $m \geq 0$, with the same argument for the upper integral J^* . Their literal adjoint relation gives all negative indices on the reversed supported/restriction pairs. The transverse multiplier commutes with both weights and the integral. $\square$

APPENDIX D. A FIXED FLAT REFERENCE AND ITS CARRIER GEOMETRY

The flat reference, finite carrier menu and common Hamilton charts are constructed here. Setup D.8 specifies the current-map neighborhood; its metric, normal and stress gaps hold near the reference by Corollary D.5. Every later use of this geometry refers to these proved constructions.

D.1. An explicit flat cube and its characteristics.

Definition D.1 (Cube notation). Let $n \geq 5$, $N = n(n+1)/2$, $q = n(n-1)/2$, $d = n-4$ and $c_n = d^{-2}$. Let $E = \text{Sym}^2 \mathbb{R}^n$ have its Frobenius pairing, and write $S_z = \sum_i z_i$ and $J = \mathbf{11}^T$. Set

$$Q_i^0 = e_i e_i^T - c_n J, \quad K_0 = \text{span}\{Q_i^0\}_{i=1}^n, \quad \ell_{ab} = \mathbf{1} - 2e_a - 2e_b \quad (a < b).$$

The model pencil and its quadratic map are

$$M_0(\xi) = \text{diag}(\xi_i) - c_n S_\xi J, \quad p_0(\xi) = \det M_0(\xi), \quad q_0(\xi) = (\xi^T Q_i^0 \xi)_i = M_0(\xi) \xi.$$

For an immersion F , put $\gamma(F) = (dF)^T dF$ and regard its second form as $B_F : E \rightarrow N_F$. Its stress space is $\ker B_F$. Full normal rank means that B_F is onto; then B_F^* is injective.

Proposition D.2 (A flat cube form). *The form*

$$B_0(u, v) = ((\ell_{ab} \cdot u)(\ell_{ab} \cdot v))_{a < b}$$

is onto, has kernel K_0 and satisfies the flat Gauss equations.

Proof. The matrices $(e_a + e_b)(e_a + e_b)^T$, $a < b$, are independent by their distinct off-diagonal entries. Since

$$\ell_{ab} = -2(I - J/4)(e_a + e_b), \quad \det(I - J/4) = 1 - n/4 \neq 0,$$

the q matrices $\ell_{ab}\ell_{ab}^T$ are independent. Every Q_i^0 annihilates them: each coordinate of ℓ_{ab} has square one and $S_{\ell_{ab}} = d$. The n matrices Q_i^0 are independent, so dimension gives $\ker B_0 = K_0$. Each rank-one component satisfies the Gauss equations separately.

The rank-one determinant formula gives

$$(D.1) \quad p_0(\xi) = \prod_i \xi_i - c_n S_\xi \sum_i \prod_{j \neq i} \xi_j.$$

□

Proposition D.3 (The complete bad set). *The nonzero bad set $\mathfrak{M} = \{p_0 = 0, dp_0 = 0\} \setminus \{0\}$ is the union of*

$$L_{ijl} = \{\xi_i = \xi_j = \xi_l = 0\}, \quad L_{ijS} = \{\xi_i = \xi_j = S_\xi = 0\},$$

and the lines $\mathbb{R}\ell_{ab}$, with the origin removed. These lines are also precisely the nonzero common zeros of q_0 . Let $\mathcal{L}$ consist of these codimension-three spaces and node lines, so $\mathfrak{M} = (\bigcup_{L \in \mathcal{L}} L) \setminus \{0\}$.

Proof. If $S_\xi = 0$, the matrix is diagonal. Otherwise its kernel equation is

$$\xi_i v_i = c_n S_\xi \sum_j v_j.$$

With no zero coordinates this has at most one free scalar. A zero coordinate forces $\sum_j v_j = 0$ and confines the kernel to the zero positions. Thus corank at least two is equivalent to membership in the displayed spaces, and makes the adjugate zero.

At corank one, $\text{adj } M_0 = a v v^T$ with $a \neq 0$, so $dp_0(\xi) = a q_0(v)$. A nonzero common zero of the quadrics satisfies $v_i = \pm S_v/d$ and $S_v \neq 0$. Summing gives sign sum $d = n - 4$, hence exactly two negative signs: $v \in \mathbb{R}\ell_{ab}$. The kernel equation then gives $\xi = (S_\xi/S_v)v$. Conversely each node line has corank one and zero determinant gradient. This proves both assertions. □

D.2. The actual strain and an analytic flat reference. Successive circular bendings realize the flat form and fix the later geometry.

For an immersion F and variation V , put $u_i = \langle \partial_i F, V \rangle$. The proved identity (2.2) gives $\frac{1}{2}Q_a : D\gamma_F V = Q_a : \text{sym } \nabla^{\gamma(F)} u$ for every stress Q_a . Thus the reduced principal row is $(Q_a \xi)^T$ and its zero-order term is $-\sum_{ij} (Q_a)_{ij} \Gamma_{ij}^k u_k$.

Proposition D.4 (An explicit analytic flat reference). *There is a real analytic local embedding $F_0 : B_r \rightarrow \mathbb{R}^N$ such that*

$$F_0(0) = 0, \quad dF_0(0) = (I, 0), \quad \gamma(F_0) = I, \quad B_{F_0}(0) = B_0.$$

Its center stress space is K_0 .

Proof. For a unit vector $v \in \mathbb{R}^m$ and $t = v \cdot Y$, define

$$\mathcal{C}_v(Y) = \left(Y + \left[\frac{\sin(nt)}{n} - t \right] v, \frac{1 - \cos(nt)}{n} \right) \in \mathbb{R}^{m+1}.$$

Its derivative is an isometry: it fixes $v^\perp$ and sends v to the unit vector $(\cos(nt)v, \sin(nt))$. At zero its first derivative is the standard inclusion and its second derivative is $n v \otimes v$ in the new normal direction.

Start with the identity on $\mathbb{R}^n$. In a fixed order of the q pairs $a < b$, compose with $\mathcal{C}_v$ using $v = (\ell_{ab}/\sqrt{n}, 0) \in \mathbb{R}^m$. All maps have center zero and standard first derivative there. Each composition leaves the preceding center second forms unchanged and adds exactly $\ell_{ab} \otimes \ell_{ab}$. The resulting analytic map has the stated metric and second form. Its projection to the first n coordinates is a

local diffeomorphism, so it is an embedding after shrinking. Proposition D.2 gives full normal rank at the center, which persists nearby. $\square$

Corollary D.5 (One fixed normalized flat reference). *For any prescribed finite kernel topology and sufficiently small neighborhood of the constant cube, one can choose a fixed $\rho > 0$ so that*

$$G_{\text{ref}}(y) = \rho^{-1}(F_0(\rho y) - F_0(0))$$

is defined on a fixed ball, has $\gamma(G_{\text{ref}}) = I$, and has an actual stress frame in that neighborhood. Its normal gap is at least $c\rho > 0$. After ρ is fixed, all its geometric seminorms are finite, and the required metric, normal and kernel gaps persist on one fixed finite-topology neighborhood of G_{ref} .

Proof. Full normal rank gives a smooth stress frame with $Q_i(0) = Q_i^0$. The dilated frame is $Q_i(\rho y)$. Its zeroth difference from Q_i^0 tends to zero; every positive-order derivative of order j has the factor ρ^j . Thus its difference tends to zero in any one prescribed finite topology.

Moreover $dG_{\text{ref}} = dF_0(\rho y)$ and $d^2G_{\text{ref}} = \rho d^2F_0(\rho y)$. The normal spaces agree, so $B_{G_{\text{ref}}} = \rho B_{F_0}(\rho y)$ and its least singular value is at least $c\rho$ on a smaller fixed ball. The metric remains exactly I . Smooth dependence of the normal projector and kernel frame gives the stated neighborhood, using two additional map derivatives for the chosen kernel topology. All later constants may depend on this fixed ρ ; the actual stress frame need not be constant away from zero. $\square$

D.3. Frozen three-parameter carrier geometry.

Proposition D.6 (A finite three-parameter menu). *Let $n \geq 5$ and use the cube pencil of Definition D.1. There are finitely many simple-characteristic patches*

$$\xi_a: Z_a \longrightarrow \mathbb{R}^n,$$

where Z_a is a fixed closed convex three-dimensional parameter cube, and a finite open cover V_b of $\mathbb{S}^{n-1}$, with the following uniform properties.

- (i) *Every carrier has all coordinates nonzero, $p_0(\xi_a) = 0$, $\text{rank } M_0(\xi_a) = n-1$, and norm bounded above and below. With $r_{a,i} = 1/\xi_{a,i}$, the vector $v_a = q_0(r_a)$ is a nonzero characteristic normal. For every $L \in \mathcal{L}$ and $z, z' \in Z_a$,*

$$(D.2) \quad |\pi_{L^\perp}(\xi_a(z) - \xi_a(z'))| \geq c|z - z'|, \quad |\xi_a(z) - \xi_a(z')| \leq C|z - z'|.$$

Every nonzero same-patch difference and every same-patch sum has a normalized direction uniformly separated from $\mathfrak{M}$.

- (ii) *For each V_b there is a finite selected subcollection and smooth incidence points $z_a(k)$ in the patch interiors such that*

$$k \cdot v_a(z_a(k)) = 0, \quad |\partial_{z_1}(k \cdot v_a(z))| \geq c$$

near the selected roots, for one chosen first parameter on each patch. Moreover,

$$(D.3) \quad |M_0(k)^T s|^2 + \sum_a |s \cdot q_0(\xi_a(z_a(k)))|^2 \geq c|s|^2, \quad s \in \mathbb{R}^n, \quad k \in V_b.$$

- (iii) *For any fixed finite grouping into color groups, positive radial labels R_g can be fixed so that every cross-group sum or difference $R_g \xi_a(z) \pm R_{g'} \xi_{a'}(z')$ has normalized direction uniformly separated from $\mathfrak{M}$. Labels are independent of later sampling spacing and accuracy. Within one patch and group, (D.2) controls every nonzero difference and the sum bound controls every sum.*

This proposition concerns frozen geometry, not a current-field parametrix or an analytic frame lower bound.

Proof. Step 1. Paired detecting carriers. For all-nonzero r and $\xi_i = r_i^{-1}$, the determinant formula gives $p_0(r) = p_0(\xi) = 0$ exactly when $S_r S_\xi = d^2$. In that case $M_0(\xi)$ and $M_0(r)$ have respective kernels $\mathbb{R}r$ and $\mathbb{R}\xi$. Fix $k \neq 0$ and $0 \neq s \in \ker M_0(k)$. We seek disjoint coordinate pairs I, J such that, with

$$K_I = \sum_{i \in I} k_i, \quad S_I = \sum_{i \in I} s_i, \quad K_J = \sum_{i \in J} k_i, \quad S_J = \sum_{i \in J} s_i,$$

one has

$$(D.4) \quad (S_I, S_J) \neq (0, 0), \quad [K_I = K_J = 0 \quad \text{or} \quad K_I K_J (K_I + K_J) \neq 0].$$

The kernel equation is

$$(D.5) \quad k_i s_i = d^{-2} S_k S_s.$$

If k has a zero coordinate, this confines s to its zero coordinates; if also $S_k \neq 0$, it gives $S_s = 0$.

First let $n \geq 6$. With at least four zero coordinates, choose a pair of zeros whose s -sum is nonzero, and another disjoint pair of zeros. Such a first pair exists because vanishing of all pair sums on at least three positions forces every component to vanish. With two or three zero coordinates, at least three nonzero entries of k remain. Two have nonzero sum; pair them with different zero coordinates, including one where $s_i \neq 0$. With one zero coordinate, at least five nonzero entries contain three of one sign. Pair two of these together and the third with the zero coordinate. These choices satisfy (D.4). If k has no zero coordinate, (D.5) implies $S_k \neq 0$ and $s_i = C/k_i$ with $C \neq 0$. Four entries of one sign give the required disjoint pairs. The remaining case has $n = 6$ with three entries of each sign. Over all positive pairs I and negative pairs J ,

$$\sum_{I, J} (K_I + K_J) = 6S_k \neq 0.$$

Some summand is nonzero, and the corresponding same-sign pair sums of both k and s are nonzero. This proves the criterion for $n \geq 6$.

Now let $n = 5$, so $d = 1$, and let z be the number of zero coordinates of k . For $z = 4$, use two pairs of zero coordinates, choosing the first with nonzero s -sum. For $z = 3$ and $S_k = 0$, the two nonzero entries of k form one pair with $K_I = 0$; choose as the other pair two zeros with nonzero s -sum. For $z = 3$ and $S_k \neq 0$, pair the two nonzero entries separately with two zeros, choosing at least one zero where $s_i \neq 0$. The two K -sums are nonzero and their sum is $S_k \neq 0$. For $z = 2$, choose two of the three nonzero entries of k whose sum is nonzero, and pair them separately with the two zeros. For $z = 1$, (D.5) and $s \neq 0$ force $S_k = 0$. Choose two nonzero entries with nonzero sum as one pair. Pair the unique zero with one of the two remaining nonzero entries. The two K -sums are nonzero; their sum is the negative of the last unused entry and is therefore nonzero. The pair containing the zero has nonzero s -sum.

It remains to consider $z = 0$ in dimension five. Again $S_k \neq 0$, $s_i = C/k_i$, and

$$(D.6) \quad S_k \sum_i \frac{1}{k_i} = 1.$$

If four entries have one sign, choose two disjoint pairs of that sign. Otherwise, after replacing k by $-k$ if necessary and permuting coordinates, write $k = (a, b, c, u, v)$ with $a, b, c > 0$ and $u, v < 0$. Take $J = \{4, 5\}$ and any pair $I \subset \{1, 2, 3\}$. Unless one of these choices satisfies $K_I + K_J \neq 0$, all three positive pair sums equal $-(u + v)$; hence $a = b = c = A > 0$ and $u + v = -2A$. Then $S_k = A$, and (D.6) gives

$$1 = A \left(\frac{3}{A} + \frac{u + v}{uv} \right) = 3 - \frac{2A^2}{uv}.$$

Thus $uv = A^2$, and $u = v = -A$. Consequently the pair criterion holds in dimension five except possibly when k belongs to one of the node lines $\mathbb{R}\ell_{ab}$.

Whenever the pair criterion holds, choose α, β with $K_I\alpha + K_J\beta = 0$ and $\alpha\beta(\alpha - \beta) \neq 0$. Use $(\alpha, \beta) = (1, 2)$ when $K_I = K_J = 0$, and $(K_J, -K_I)$ otherwise. Put $A = t\alpha$, $B = t\beta$. On I let r have entries $-\sqrt{1+A}, \sqrt{1+A}$; on J use $-\sqrt{1+B}, \sqrt{1+B}$; elsewhere put $r_i = 1$. For all sufficiently small t , these entries are nonzero and, with $\xi_i = r_i^{-1}$,

$$S_r = S_\xi = d, \quad k \cdot q_0(r) = 0, \quad s \cdot q_0(\xi) = -\frac{t[\alpha S_I + \beta S_J + t\alpha\beta(S_I + S_J)]}{(1+t\alpha)(1+t\beta)}.$$

The numerator is not identically zero: if both coefficients vanished, then $S_I + S_J = 0$ and $(\alpha - \beta)S_I = 0$, contradicting the criterion. Choose small nonzero t with nonzero pairing. Then ξ is regular and simple, and $q_0(r)$ has entries A, A, B, B on the two pairs and zero elsewhere. It has four nonzero coordinates and no $n - 2$ equal coordinates. It therefore lies in none of the spaces $L_{ijl}^\perp$ or $L_{ijs}^\perp$.

Step 2. The five-dimensional node carriers. In dimension five consider the node direction $\ell = (1, 1, 1, -1, -1)$; all other node lines follow by permutation. Its kernel is $\ker M_0(\ell) = \mathbb{R}\ell$. Set

$$(D.7) \quad \begin{aligned} r &= \left(1, 1, -\frac{1}{2}, \frac{-1+\sqrt{3}}{2}, \frac{-1-\sqrt{3}}{2}\right), \\ \xi &= \left(1, 1, -2, 1+\sqrt{3}, 1-\sqrt{3}\right) = r^{-1}. \end{aligned}$$

Here $S_r = 1/2$, $S_\xi = 2$, so both vectors are all-nonzero simple characteristics. Direct calculation gives

$$\begin{aligned} q_0(r) &= \left(\frac{3}{4}, \frac{3}{4}, 0, \frac{3}{4} - \frac{\sqrt{3}}{2}, \frac{3}{4} + \frac{\sqrt{3}}{2}\right), \\ q_0(\xi) &= (-3, -3, 0, 2\sqrt{3}, -2\sqrt{3}), \\ \ell \cdot q_0(r) &= 0, \quad \ell \cdot q_0(\xi) = -6. \end{aligned}$$

Thus this carrier detects every nonzero stress in the kernel at every $k \in \mathbb{R}\ell \setminus \{0\}$ and has the required incidence. Its normal $q_0(r)$ has four nonzero coordinates and no three equal coordinates, and therefore again avoids every $L_{ijl}^\perp$ and $L_{ijs}^\perp$. Both characteristic normals are nonzero, so both characteristic points are regular.

Step 3. Transversality and three-dimensional patches. The preceding constructions give, for every relevant k, s , all-nonzero reciprocal characteristics r, ξ with $k \cdot q_0(r) = 0$, $s \cdot q_0(\xi) \neq 0$ and a normal $q_0(r)$ outside every codimension-three bad-normal space. Incidence is transverse on the characteristic hypersurface. Indeed, in reciprocal coordinates its defining function is $E_k(r) = k \cdot q_0(r) = r^T M_0(k)r$, with gradient $2M_0(k)r$. This gradient annihilates s , whereas the characteristic normal $q_0(\xi)$ does not. The incidence gradient cannot vanish: if $M_0(k)r = 0$, the coordinate equations and $r_i \neq 0$ give $k = d^{-2}S_k S_r \xi$, with a nonzero scalar factor. Then $s \in \ker M_0(\xi) = \mathbb{R}r$, contradicting $s \cdot q_0(\xi) \neq 0$, since $r \cdot q_0(\xi) = \xi^T M_0(r)\xi = 0$. Thus the two hypersurfaces meet transversely. Reciprocal inversion is a local diffeomorphism, so this proves the claimed transversality in the ξ coordinates as well.

At a detecting carrier c for k_0 , put $v = q_0(c^{-1})$ and $T = v^\perp$. For a codimension-three L , the projection $T \rightarrow L^\perp$ is onto since $v \notin L^\perp$. For a node line L , that projection has rank at least $n - 2 \geq 3$. Consequently each Gram determinant

$$\det[\pi_{L^\perp} e_i \cdot \pi_{L^\perp} e_j]_{i,j=1}^3$$

is a nonzero polynomial on T^3 . Their finite product with $d(k_0 \cdot v)(e_1)$ is also a nonzero polynomial, by incidence transversality. Choose a triple at which this product does not vanish. Its span E_3 projects injectively to every $L^\perp$ and has the required incidence derivative.

For a transverse direction e with $dp_0(c)e \neq 0$, the implicit function theorem parameterizes a characteristic patch as

$$\xi(z) = c + E_3 z + u(z)e, \quad D\xi(0) = E_3.$$

Here E_3 also denotes the injective linear map with the chosen three columns; $u(0) = 0$ and $Du(0) = 0$. Shrink a convex parameter cube so every $\pi_{L^\perp} D\xi(z)$ stays within half the least singular value of $\pi_{L^\perp} E_3$ in operator norm. For z, z' in the cube, subtract the linear map $\pi_{L^\perp} E_3$ inside the segment identity; the resulting error is at most half its lower singular bound times $|z - z'|$. This proves (D.2). Combining its lower and upper bounds separates every normalized nonzero difference from every L . On a smaller cube all sums have directions close to the fixed nonzero regular carrier c , hence have a fixed gap from $\mathfrak{M}$. Rename the first parameter so that its incidence derivative is nonzero, and shrink to keep a positive lower bound.

Step 4. Finite covers and radial labels. At each k_0 , the preceding detecting linear forms have no common nonzero nullvector on $\ker M_0(k_0)$. Compactness of its unit sphere selects finitely many carriers with a positive detector gap there; since $M_0(k_0)$ is symmetric, the Gram form in (D.3) is positive on $\mathbb{R}^n$. For invertible $M_0(k_0)$ no detector is needed. The implicit function theorem continues the transverse incidence roots as $z_1 = z_1(k)$, with $z_2 = z_3 = 0$, in a neighborhood of k_0 . Continuity preserves the full Gram gap on a smaller neighborhood. Compactness of $\mathbb{S}^{n-1}$ supplies a finite subcover; take the minimum of its finitely many positive margins.

All carriers lie in a compact nonzero regular set. Choose radial labels successively with ratios sufficiently large that a cross-group sum or difference has normalized direction as close as desired to the carrier of the larger group, up to sign. A finite ratio suffices because carrier norms have common positive upper and lower bounds. Each carrier and its negative have a positive gap from $\mathfrak{M}$, so the cross-group directions have one as well. The labels are fixed independently of later sampling or accuracy. The same-patch assertions were already proved by the chord and sum estimates. $\square$

D.4. Actual Hamilton geometry and buffered phase charts. We deform the frozen menu to actual current immersions and verify the common inverse-chart hypotheses for exact full-dispersion phases.

Lemma D.7 (A common homogeneous Hamilton phase). *Let h_a be a real degree-one principal symbol on fixed physical and nonzero conic boxes, in a sufficiently small fixed C^3 neighborhood of h_0 , with $|\partial_{\xi_1} h_a| \geq c > 0$. Assume its normalized order- m seminorm is bounded by $C_m(1 + \|a\|_{C^{m+c_0}})$, with fixed c_0 . There are common buffered physical and canonical boxes and a homogeneous phase, for $\eta = (e, \eta')$, satisfying*

$$(D.8) \quad h_a(x, S_x(x, \eta)) = e, \quad S(0, x', \eta) = x' \cdot \eta'.$$

Its coordinates $S_\eta = (t, u')$ are Hamilton time and the initial section point. The entire canonical preimage of the inner physical box lies in a fixed larger source rectangle. The matrix $S_{x\eta}$ has a uniform inverse gap, and

$$(D.9) \quad \Xi(x, \tilde{x}, \eta) = \int_0^1 S_x(\tilde{x} + \tau(x - \tilde{x}), \eta) d\tau$$

is injective in η on the buffered convex chart and its conic extension, with a common buffered inverse domain. All joint phase, flow and inverse derivatives through order m have a one-high bound at coefficient order $m+c$, for fixed c . The construction is continuous from the normalized C^{m+1} symbol topology to C^m phases, at each fixed m . All phase assertions concern the homogeneous principal symbol; low-frequency and lower-order operator terms are separate.

Proof. The map $\xi \mapsto (h_a(x, \xi), \xi')$ has Jacobian h_{a, ξ_1} . Euler's identity excludes zero from its image on a nonzero transverse patch. Its homogeneous inverse therefore has $c|\eta| \leq |\xi| \leq C|\eta|$ on common buffered convex conic boxes, also when $e \neq 0$ and $\eta' = 0$. The transverse gap gives the unique homogeneous root $h_a(x, g(x, e, \eta'), \eta') = e$, with $g_e = (h_{a, \xi_1})^{-1}$. Use the Hamilton field $X_h = (h_{a, \xi}, -h_{a, x})$ starting at $(0, u', g(0, u', e, \eta'), \eta')$. Its low bounds and homogeneity give, for $\omega = \xi/|\xi|$,

$$|x(t) - x(0)| + |\omega(t) - \omega(0)| \leq C|t|, \quad e^{-C|t|} \leq |\xi(t)|/|\xi(0)| \leq e^{C|t|}.$$

Choose one short interval for both signs inside the spatial and angular buffers; a first-exit argument gives the entire flow there. At $t = 0$, the differential of $(t, u') \mapsto x$ has columns $h_{a,\xi}, e_2, \dots, e_n$ and determinant of magnitude $|h_{a,\xi_1}| \geq c$. After one refinement it stays close to a central invertible matrix. Lemma B.5 gives the common inverse projection and preimage buffer, uniformly in (e, η') .

For a variation of initial data at fixed time, Hamilton's equations and degree-one homogeneity give

$$\frac{d}{dt}(\xi \cdot \delta x) = (-h_x + \xi \cdot h_{\xi x})\delta x + (\xi \cdot h_{\xi \xi})\delta \xi = 0.$$

Initially $\xi \cdot \delta x = \eta' \cdot \delta u'$. Since $\xi \cdot \dot{x} = h_a = e$, allowing time to vary gives $\xi \cdot dx = e dt + \eta' \cdot du'$. Therefore

$$S(x, \eta) = e t(x, \eta) + u'(x, \eta) \cdot \eta', \quad dS = \xi \cdot dx + t de + u' \cdot d\eta'.$$

This proves the eikonal and canonical-coordinate identities. Positive covector scaling leaves the base flow unchanged, so S is homogeneous. The joint implicit, flow and inverse estimates of Lemma B.6 use symbol orders at most $m + 1$, and give the claimed fixed-shift bound. Subtracting the differentiated equations also proves the asserted finite-order continuity on the same domains.

On the section, $S_{x'\eta'} = I$, $S_{x'e} = 0$ and $S_{x_1e} = (h_{a,\xi_1})^{-1}$. Shrink the boxes and strip once so $S_{x\eta}$ and hence Ξ_η stay close to a central invertible matrix. Lemma B.5 gives the common inverse domain. Since Ξ_η is degree zero, the same matrix gap holds on the convex conic extension; its segment identity proves injectivity there. Inverse differentiation and the joint estimates give the one-high bounds. $\square$

Setup D.8 (A finite current neighborhood). Fix the patches, cover, roots and labels of Proposition D.6, and put $M_{\text{an}} = 20(n+1)$. Let $F: B_1 \rightarrow \mathbb{R}^N$ be a real smooth immersion with $\|F\|_{C^4} \leq C_0$, induced metric $g = (dF)^T dF \geq \kappa_{\text{met}} I$, and actual second form $B_F: E \rightarrow N_F$ onto with least normal singular value at least $\kappa > 0$. Use throughout the projected cube frame

$$P^\perp = I - dF g^{-1} (dF)^T, \quad \mathcal{A}_F = B_F B_F^* + (I - P^\perp), \\ \Pi_{K_F} = I - B_F^* \mathcal{A}_F^{-1} B_F, \quad Q_i = \Pi_{K_F} Q_i^0,$$

and assume $\max_i \|Q_i - Q_i^0\|_{C^6(B_1)} < \epsilon_0$. The two gaps are separate hypotheses; they bound $\|g^{-1}\| \leq \kappa_{\text{met}}^{-1}$ and $\|\mathcal{A}_F^{-1}\| \leq \max(1, \kappa^{-2})$. The small low neighborhood makes Q_i a frame of $K_F = \ker B_F$. Define

$$M_F(x, \xi)_i = (Q_i(x) \xi)^T, \quad p_F(x, \xi) = \det M_F(x, \xi), \quad q_F(x, \xi) = (Q_i(x) : \xi \xi^T)_i.$$

Proposition D.9 (Actual carrier phases, signed pairs and detection). *There are fixed balls $B_{\text{in}} \Subset B_{\text{out}} \Subset B_1$ and a fixed choice of $\epsilon_0 > 0$ such that every F in Setup D.8 has the following properties on the same B_{out} .*

- (i) *For every frozen patch there is a jointly smooth real family $\Theta_a(x, z)$ on its fixed convex parameter cube, with*

$$p_F(x, d_x \Theta_a) = 0, \quad \text{rank } M_F(x, d_x \Theta_a) = n - 1.$$

Its gradient is uniformly close to $\xi_a(z)$ in $C_z^1 C_x^0$, and has a uniform upper parameter Lipschitz bound. For all $L \in \mathcal{L}$, all x and all z, z' in one patch,

$$(D.10) \quad |\pi_{L^\perp}(d\Theta_a(x, z) - d\Theta_a(x, z'))| \geq c|z - z'|.$$

All nonzero same-patch differences, same-patch sums, and the cross-group signed pairs with the fixed radial labels have normalized directions in one compact set separated from $\mathfrak{M}$. On those directions the $(n-1)$ st singular value of M_F has a positive lower bound. At any actual determinant zero there, $|d_\xi p_F| \geq c$. Thus the directions are elliptic or simple real principal type.

- (ii) On each member of a fixed smaller cover V_b of $\mathbb{S}^{n-1}$, the selected patches have jointly smooth interior roots $z_a(F; x, k)$ satisfying

$$k \cdot d_\xi p_F(x, d\Theta_a(x, z_a)) = 0, \quad |\partial_{z_1}[k \cdot d_\xi p_F(x, d\Theta_a(x, z))]| \geq c.$$

The same roots satisfy

$$(D.11) \quad |M_F(x, k)^T s|^2 + \sum_a |s \cdot q_F(x, d\Theta_a(x, z_a(F; x, k)))|^2 \geq c|s|^2, \quad s \in \mathbb{R}^n.$$

All low domains and gaps depend only on the frozen finite data, $\kappa, \kappa_{\text{met}}, C_0$ and the fixed C^6 neighborhood, not on higher norms or accuracy. The same zero-energy phase S_F used below, carrier gradients and incidence roots obey one estimate on their stated fixed buffered domains:

$$(D.12) \quad \|\mathcal{G}_F\|_{C^r} \leq C_r(1 + \|F\|_{C^{r+6}}), \quad \mathcal{G}_F = (S_F, d\Theta_a, z_a), \quad r \geq 0.$$

The norm is the maximum over the finite family, jointly in physical variables, normalized covectors and its parameters. The right-hand norm is on the fixed construction neighborhood.

Proof. Step 1. We use the zero-energy part of the common phase construction. On each enlarged frozen patch choose adapted physical coordinates $x = (t, y)$, $\xi = (\tau, \zeta)$ with a fixed $\partial_\tau p_0 > c > 0$ gap. Put $h_F = p_F|\xi|^{1-n}$. Lemma D.7 applies directly to h_F . On its zero set the Hamilton field is $|\xi|^{1-n}X_{p_F}$; finite minima give common buffers for the menu. Restrict the full-energy phase to $e = 0$ and call it S_F . It satisfies

$$\partial_t S_F = H_F(t, y, d_y S_F), \quad p_F(x, (H_F(x, \zeta), \zeta)) = 0, \quad S_F(0, y, \zeta) = y \cdot \zeta.$$

The root is the unique homogeneous continuation of H_0 , and $S_0 = tH_0(\zeta) + y \cdot \zeta$. Thus the primary characteristic phase is exactly the zero-energy restriction of the current full-energy phase, with the same section data and sign convention.

The continuity in Lemma D.7 gives C^5 closeness of these phases from the C^6 coefficient closeness. The displayed projector is a fixed-gap rational function of the second jet of F , so Lemma B.3 gives

$$(D.13) \quad \max_i \|Q_i\|_{C^r} \leq C_r(1 + \|F\|_{C^{r+2}}).$$

That lemma uses symbol derivatives through $r + 1$, so $\|S_F\|_{C^r} \leq C_r(1 + \|F\|_{C^{r+3}})$. All choices use the fixed low bounds, without high-norm smallness.

Step 2. We keep the complete signed-pair geometry. Let $\zeta_a(z)$ be the spatial projection of the frozen patch and set

$$\Theta_a(x, z) = S_F(x, \zeta_a(z)).$$

The frozen gradient is $\xi_a(z)$. The C^5 comparison gives $C_z^1 C_x^0$ closeness of the actual gradients to it. The characteristic equation is exact, and a frozen nonzero rank- $(n - 1)$ minor persists.

For every x, L , the actual projected parameter map is C^1 close to the frozen one on its convex cube. The same segment estimate therefore preserves the lower chord bound (D.10). Together with the upper derivative bound it gives a fixed angular gap, including arbitrarily short nonzero chords. The compact gaps for sums and fixed-label cross-group signed pairs persist by low continuity.

Take the closure of all normalized pair directions, including limits of normalized derivative chords. It is compact and avoids $\mathfrak{M}$. The $(n - 1)$ st singular value of M_0 has a positive minimum on it. Proposition D.3 makes the determinant gradient nonzero on its reference zero subset, and compactness gives a positive minimum there. On a slightly larger compact angular set, low C^1 closeness of p_F and M_F preserves the singular-value bound and ensures that every actual determinant zero keeps $|d_\xi p_F| \geq c$. This proves the claimed elliptic-or-simple classification.

Step 3. We continue incidence roots and stress detection. The actual $d_\xi p_F(x, d\Theta_a)$ is C_z^1 close to a nonzero scalar multiple of v_a . At incidence, differentiating that scalar contributes zero because $k \cdot v_a = 0$. The implicit function theorem therefore continues the transverse frozen roots to actual interior roots, uniformly for all x on the common ball and k in a fixed smaller V_b . The phase bound

and joint implicit estimate use coefficient order at most $r + 4$, proving (D.12). The full-matrix gap (D.3) persists by continuity of M_F , q_F and the roots, giving (D.11). Take a finite minimum over the cover. $\square$

Proposition D.10 (Exact full-dispersion buffered charts). *Under Setup D.8, fix an incidence patch and a base direction $k_b = (\omega_b, \eta_b)$ in its adapted coordinates. There are fixed compact convex spatial-covector and data charts, nested common convex physical boxes inside B_{out} , and a fixed $\epsilon_* > 0$, with the following property. With spectator parameters z_2, z_3 in a fixed sufficiently small box, write*

$$D_{\text{chart}} = I_{z_1} \times B_\eta, \quad D_0 \subseteq D_{\text{chart}},$$

where $B_\eta \subset \mathbb{R}^{n-1}$ is convex and contains η_b . For $0 \leq \epsilon \leq \epsilon_*$ define

$$(D.14) \quad \phi_{F,\epsilon}(x, z_1, \eta) = \frac{S_F(x, \zeta_a(z) + \epsilon\eta) - S_F(x, \zeta_a(z))}{\epsilon},$$

with its integral-derivative value at zero. Put

$$\Psi_{F,\epsilon}(x, x', u) = \int_0^1 d_x \phi_{F,\epsilon}(x' + t(x - x'), u) dt, \quad u = (z_1, \eta).$$

The normalized data charts stay in a fixed annulus and satisfy

$$(D.15) \quad |\Psi_{F,\epsilon,y} - \eta| \leq \frac{1}{2}|\eta|, \quad c \leq |\Psi_{F,\epsilon}| \leq C$$

on the entire indicated physical and data boxes. For every x, x' in those boxes, this map is globally one-to-one on D_{chart} and has a uniform positive absolute Jacobian determinant. It has an inverse on one fixed open V_{chart} , with all images $\Psi_{F,\epsilon}(D_0)$ in a fixed compact subset of V_{chart} . The inverse and its first ϵ derivative satisfy the same bound (D.12) at every order r , jointly on these fixed charts.

Uniformly in t, z_1, ϵ , the transverse maps

$$\eta \mapsto \int_0^1 d_y \phi_{F,\epsilon}(t, y' + s(y - y'), z_1, \eta) ds$$

have analogous common buffered inverse charts and the same bounds on fixed convex η boxes. Include these maps, their inverses and first ϵ derivatives in the same finite family $\mathcal{G}_F$; all estimates are joint in physical, data, spectator and ϵ variables. These are the phase and chart hypotheses of Setup C.3 and Proposition C.6.

Proof. At the frozen reference, $d_\xi p_0 = p_{0,\tau}(1, -\nabla H_0)$. Incidence gives $\omega_b = \eta_b \cdot \nabla H_0(\zeta_a(z))$, so $\eta_b \neq 0$; its transversality gives $\partial_{z_1}[\eta_b \cdot \nabla H_0(\zeta_a(z))] \neq 0$. On sufficiently small convex data and spectator boxes, put

$$G(z_1, \eta) = \nabla H_0(\zeta_a(z)) \cdot \eta, \quad \Psi_0 = (G, \eta).$$

Then $\phi_0 = tG + y \cdot \eta$ and $\det D\Psi_0 = \partial_{z_1} G$ has a fixed nonzero sign. Shrink once so $D\Psi_0$ stays close to its central matrix.

The exact identity

$$(D.16) \quad \phi_{F,\epsilon}(x, z_1, \eta) = \int_0^1 \partial_\zeta S_F(x, \zeta_a(z) + s\epsilon\eta) \cdot \eta ds$$

includes $\epsilon = 0$ smoothly. The C^5 phase comparison in Proposition D.9 makes $\Psi_{F,\epsilon}$ uniformly C^1 close to Ψ_0 ; these derivatives and their first ϵ derivatives use phase jets through order four. First choose the low kernel neighborhood, then ϵ_* small. The transverse averaged-gradient map is likewise C^1 close to $\eta \mapsto \eta$. Since η stays in a fixed annulus, the same comparison proves (D.15) on the full boxes. Lemma B.5, applied to both families with all physical and spectator parameters, gives their global injectivity, Jacobian gaps and common buffered inverse domains.

The integral quotient and its gradient use two phase derivatives. One ϵ derivative uses a third. The joint inverse estimate therefore uses S_F through order $r + 3$, hence F through $r + 6$. This proves the stated bound, on domains fixed by low data alone. $\square$

Remark D.11 (Finite map topology). Equation (D.13) makes a fixed C^8 map neighborhood sufficient for the low C^6 kernel condition. The fixed order $C^{M_{\text{an}}+11}$ controls phase jets through $M_{\text{an}}+5$, carrier gradients and roots through $M_{\text{an}}+3$, and chart inverse jets through $M_{\text{an}}-2$ with one ϵ derivative. Corollary B.4 applies to this same finite geometric family, without changing its domains. For the reference of Corollary D.5, $\gamma(G_{\text{ref}}) = I$, so $\|dF - dG_{\text{ref}}\|_{C^0} < 1/2$ gives $|dF(x)v| \geq |v|/2$ and $\gamma(F) \geq I/4$. Choose this neighborhood together with the normal and kernel neighborhoods. The positive ρ stays fixed thereafter.

D.5. Signed carrier groups. The actual menu fixes the central signed directions before frequencies and sampling.

Definition D.12 (Exhaustive literal carrier groups). Freeze the finite list of full carrier-patch specifications before frequencies and sampling. An unmerged duplicate detector entry remains distinct. A group consists of that identifier and a block color modulo $q_{\text{col}} = q_{\text{quad}} + 1$ in Theorem G.6; each chart has q_{col}^3 colors. The quadrature order is fixed by (5.8) before the radial labels. Apply the finite-group choice in Proposition D.6 to this entire list. Different specifications or colors have radially separated labels, fixed before frequencies and sampling.

Within one block the translated lattice has spacing h . Different blocks of one color have centers at least $(q_{\text{quad}} + 1)h$ apart in some coordinate and support width $(q_{\text{quad}} + 1/2)h$. Their sample distance is therefore at least $h/2$ in every realization. All band updates at a literal carrier enter its one final Fourier measure and complex wave; they are not new sample indices.

Remark D.13 (Exhaustion of all signed terms). For two distinct sample indices in the same full group, the strong chord gives the protected minus phase with true beat $\lambda|z - z'| \geq c\lambda h$, and the same-patch sum bound gives the plus phase with beat comparable to λ . For indices in different full groups, prior radial domination gives both signs with beat comparable to λ , even for equal parameter values or equal unscaled carrier directions. For a repeated single index $j = j$, the doubled plus phase is protected; its entire Hermitian minus tensor is the self contribution assigned to the mean. These cases exhaust the ordered real quadratic expansion. There is no distinct-index zero-beat copy of a literal carrier. These are central phase directions; all modulation enters the complete amplitudes in Appendix L. The finite group choice is available before the sample event, so this convention introduces neither a new random event nor an additional separation claim.

APPENDIX E. THE REGULAR-SECTOR CONSTRUCTION

E.1. The selected sectors and their estimates. Consider the real matrix system

$$(E.1) \quad P_a = \sum_{j=1}^n A_j(x) \partial_{x_j} + B(x), \quad H_a(x, \xi) = \sum_{j=1}^n A_j(x) \xi_j,$$

where the $r \times r$ coefficient array $a = (A_1, \dots, A_n, B)$ is smooth near $x_0 \in \mathbb{R}^n$. Set $L_a = P_a/i$ and

$$(E.2) \quad p_a = \det H_a, \quad q_a = p_a |\xi|^{1-r}, \quad \xi \neq 0.$$

A characteristic is simple when $\partial_\xi p_a \neq 0$. We construct a finite inverse on selected regular sectors, keeping both the angular complement and characteristic integration. The latter can turn a symbolic error into a nonpseudodifferential residual. Finite conjugation and composition have the classical background [NM89, GY88, DH72]; the actual-domain estimates used here are proved below.

Theorem E.1 (Regular-sector approximate inverse). *For the system (E.1), fix a smooth reference a_0 and write $H_0 = H_{a_0}$, $p_0 = \det H_0$. Let $\Gamma_{\text{in}}, \Gamma_{\text{out}} \subset S^{n-1}$ be fixed antipodally symmetric compact sets, with $\Gamma_{\text{in}} \subset \text{int } \Gamma_{\text{out}}$. Assume, on a fixed spatial neighborhood of x_0 , that*

$$p_0(x, \omega) = 0, \quad \omega \in \Gamma_{\text{out}} \implies \partial_\xi p_0(x, \omega) \neq 0.$$

There is no hypothesis on characteristic directions outside Γ_{out} .

After one spatial shrinking, there exist fixed balls $U \Subset W$, an integer k_0 , a finite C^{k_0} neighborhood of a_0 , a real even order-zero angular/source cutoff A_Γ , and one fixed nonnegative integer β , with the following properties. The operator A_Γ is the identity microlocally on Γ_{in} at high frequency and has angular microsupport in Γ_{out} .

For each smooth real a in this neighborhood, one real linear operator

$$S_{a,\Gamma}: C_c^\infty(W, \mathbb{R}^r) \longrightarrow C^\infty(W, \mathbb{R}^r)$$

and one complete residual $R_{a,\Gamma}$ satisfy

$$(E.3) \quad P_a S_{a,\Gamma} h = A_\Gamma h + R_{a,\Gamma} h \quad \text{on } U.$$

For every integer $s \geq 0$,

$$(E.4) \quad \|S_{a,\Gamma} h\|_{H^s(W)} \leq C_s \left[\|h\|_{H^s(W)} + (1 + \|a\|_{H^{s+\beta}(W)}) \|h\|_{L^2(W)} \right].$$

For fixed real $\chi_{\text{in}} \in C_c^\infty(W)$ and $\chi_{\text{meas}} \in C_c^\infty(U)$, define the measured pair by zero extension:

$$(E.5) \quad \mathcal{R}_a = \chi_{\text{meas}} R_{a,\Gamma} \chi_{\text{in}}, \quad \mathcal{R}_a^* = \chi_{\text{in}} R_{a,\Gamma}^* \chi_{\text{meas}}.$$

For $T = R_{a,\Gamma}, \mathcal{R}_a$ use the domain pair $W \rightarrow U$; for $T = \mathcal{R}_a^*$ use $U \rightarrow W$. On these input and output domains, every integer $s \geq 0$ satisfies

$$(E.6) \quad \|Th\|_{H^s} \leq C_s \left[\|h\|_{H^{\max(s-1,0)}} + (1 + \|a\|_{H^{s+\beta}(W)}) \|h\|_{L^2} \right].$$

Both measured operators also map H^t to H^{t+1} for every fixed negative integer t , on the corresponding supported/restriction pairs. In particular, one fixed finite low coefficient neighborhood controls

$$(E.7) \quad \mathcal{R}_a^{(*)}: L^2 \longrightarrow H^1, \quad \mathcal{R}_a^{(*)}: H^{-1} \longrightarrow L^2.$$

The construction has one fixed finite accuracy. Its domains, cutoffs, low coefficient neighborhood, and derivative losses are independent of s . The residual includes characteristic integration, finite conjugation, inverse-coordinate errors, all source and spatial-cutoff terms, and the low-frequency term.

E.2. Adjugate amplitudes and domains.

Proposition E.2 (Adjugates and regular charts). *The protected unit set in Theorem E.1 has a finite cover by buffered conic patches. On each patch a fixed unit vector v satisfies, in one common coefficient neighborhood,*

$$v \cdot \partial_\xi q_a \geq c > 0$$

and, for $\xi \neq 0$, the adjugate amplitudes are

$$(E.8) \quad c_a = |\xi|^{1-r} \text{adj } H_a, \quad h_a = |\xi|^{1-r} \det H_a = q_a.$$

The matrix c_a has order zero, and

$$(E.9) \quad H_a c_a = h_a I.$$

These symbols have fixed-shift one-high coefficient bounds. All low and angular cutoffs will be in the graph amplitudes. Opposite sectors use the exact conjugate factors chosen below.

Proof. Euler's identity $\xi \cdot \partial_\xi q_a = q_a$ excludes a zero fiber gradient wherever $q_a \neq 0$; the theorem's hypothesis excludes it on $q_a = 0$. After one spatial shrinking, compactness of the protected unit set gives a uniform gradient gap for q_0 . A finite cover and fixed rotations give common transverse gaps in one low coefficient neighborhood, including nonzero-energy patches, each inside one larger phase chart. The adjugate identity gives $H_a c_a = h_a I$; the polynomial adjugate and normalized power of $|\xi|$ have the stated bounds by Lemma B.3. No inverse of c_a is used. $\square$

E.3. Finite calculus on common charts.

Setup E.3 (Actual short section-energy charts). Use the symbols of Proposition E.2 and the common buffered physical and canonical charts of Lemma D.7, in their fixed finite coefficient neighborhood. The ordinary pseudodifferential formulas also apply independently. Write $x = (x_1, x')$, $\eta = (e, \eta')$ and $y = (t, z)$. On those boxes fix real physical cutoffs χ_s, χ_t, χ_e and source and target conic Fourier multipliers Π_s, Π_t . The source cutoff χ_s equals one on the inner source patch and the canonical preimage of the inner target patch. The target cutoff χ_t equals one near $\text{supp } \chi_e$. The microsupports of Π_s, Π_t lie strictly inside their corresponding conic boxes. All transitions have positive fixed buffers.

The graph operators use this actual type-I phase $S(x, \eta) - y \cdot \eta$. Their inverse-coordinate factors are the weighted adjoints V constructed below from the full matrix transport and its inverse. A fixed low-frequency cutoff is included in every homogeneous construction; its complementary terms are kept.

All physical, proper-support, angular and frequency cutoffs are fixed smooth profiles on the common boxes, rescaled with their indicated frequencies. Their phase and inverse-coordinate pullbacks, including $\chi_s(S_\eta)$, obey the joint composition estimate of Lemma B.6. On dyadic shells λ and μ , each external η derivative gains λ^{-1} , and each derivative in the other covariable gains μ^{-1} . These actual profiles and pullbacks have fixed support buffers at every order.

For a full symbol a of order m use

$$|a|_{m,j} = \max_{|\alpha|+|\beta|+|\gamma|\leq j} \sup \langle \eta \rangle^{-m+|\gamma|} |\partial_x^\alpha \partial_z^\beta \partial_\eta^\gamma a|,$$

with absent variables omitted, and the degree-one seminorm for the phase.

Two-position pseudodifferential amplitudes keep their physical and proper-support masks. Their diagonal terms and exact order-lower remainders are used only on the stated plateaus. Physical multipliers and finite products stay in their literal order.

Lemma E.4 (Fixed ordered products). *Use the fixed whole-space representatives of the buffered Sobolev domain pairs. Let $H_s \geq 1$ be nondecreasing. Suppose finitely many operators T_j and their literal adjoints have uniformly bounded L^2 norms and, for fixed nonnegative integers d_j , satisfy*

$$\|T_j f\|_{H^s} \leq C_s [\|f\|_{H^{\max(s-d_j, 0)}} + H_s \|f\|_2], \quad s \geq 0,$$

for integers $s \geq 0$, with the same estimate for T_j^* on its reversed domain pair. Every fixed compatible product has this bound with gain $d = \sum_j d_j$, as does its literal adjoint. Fixed sums have the minimum gain. The high quantity remains H_s .

Proof. For two factors put $t = \max(s - d_1, 0)$. Applying the first estimate at order s and the second at orders t and zero gives

$$\|T_1 T_2 f\|_{H^s} \leq C_s [\|f\|_{H^{\max(s-d_1-d_2, 0)}} + (H_t + H_s) \|f\|_2].$$

Here the high term uses only the uniform L^2 bound of T_2 . Induction and reverse order prove the product and adjoint assertions. $\square$

Lemma E.5 (Exact cutoff errors on an inner plateau). *Let T_A have phase $S(x, \eta) - y \cdot \eta$ and a full order- p amplitude $A(x, y, \eta)$ with fixed compact physical supports. If a scalar cutoff b vanishes near every $S_\eta(x, \eta)$ on the inner output and covector supports, then*

$$(E.10) \quad \chi_e T_A b = T_{r_b}, \quad r_b = -i \sum_j \partial_{\eta_j} \left[\chi_e A \int_0^1 (\partial_j b)(S_\eta + t(y - S_\eta)) dt \right].$$

This full amplitude has order $p - 1$. For a compactly supported b , a fixed source cutoff equal to one on $\text{supp } b$ may first be included in A .

The identity keeps the actual cutoffs inside the displayed amplitude. Its normalized derivatives have one-high bounds with fixed shifts whenever the original primitives do. Literal adjunction reverses this exact kernel.

Proof. Expand $b(y)$ to first order at S_η and use $(y_j - S_{\eta_j})e^{i(S-y\cdot\eta)} = i\partial_{\eta_j}e^{i(S-y\cdot\eta)}$. One integration by parts proves the identity; derivatives of S_η in η have order -1 . The joint derivative budgets of Lemma B.6 prove the fixed-shift bounds, including every position and mask. $\square$

Theorem E.6 (Finite calculus on an inner physical plateau). *Fix the boxes and low coefficient neighborhood of Setup E.3. After one reduction of these boxes and that finite neighborhood, the following assertions hold.*

- (i) *The phase and boxes are those of Lemma D.7, applied to h_a . Its eikonal, canonical preimage, mixed inverse and averaged-gradient gaps hold on the full energy chart, with fixed-shift one-high bounds.*
- (ii) *Let T_p have phase $S(x, \eta) - y \cdot \eta$ and a full matrix amplitude $A_p(x, y, \eta)$ of order $p = -d$, $d \in \{0, 1\}$, with fixed compact physical and conic supports. Source-independent amplitudes are also allowed. Declare a finite low phase and amplitude bound M_{low} large enough for the low L^2 estimates. For $A = T_p$, and separately for its literal Hilbert adjoint $A = T_p^*$,*

$$(E.11) \quad \|Av\|_s \leq C_s [\|v\|_{\max(s-d, 0)} + H_s \|v\|_0], \quad s \geq 0.$$

Here one may take

$$H_s = 1 + |A_p|_{s+\mu} + |S|_{s+\mu},$$

where the bars are the symbol and homogeneous phase seminorms on the fixed buffered boxes and μ is independent of s . The constants may depend on M_{low} , but not on higher seminorms. For the amplitudes constructed below,

$$(E.12) \quad H_s \leq C_s (1 + \|a\|_{H^{s+\beta}(W)})$$

with fixed β . An order -1 error and its literal adjoint gain one derivative in this form. At every fixed integer index the corresponding bounds hold on the buffered supported/restriction pairs, with a finite coefficient bound.

- (iii) *Every complete two-position pseudodifferential amplitude is its diagonal principal term, with a fixed source cutoff, plus an exact one-order-lower full amplitude. The exact graph normal becomes such an amplitude by an averaged-gradient change. Right multiplication of a source-independent graph by a fixed scalar source cutoff has the exact formula below; right Fourier multiplication is exact amplitude multiplication. These formulas handle the actual localizations. The pure type-I normal is computed before source multiplication; the source-cut normal includes its exact source-mismatch term. Principal terms, complete remainders and literal adjoints have the indicated orders and one-high bounds, using only index $s+c$ at output index s , with fixed c . All angular, low-frequency and separated-support terms are included.*

The transported matrix and its inverse define the order-zero weighted adjoint V in (E.21). It has the strong tame estimate, and

$$(E.13) \quad \Pi_t \chi_e (TV - \text{Id}) \chi_e \Pi_t$$

has order at most -1 , including the full source-cutoff error.

- (iv) For $L_a = P_a/i$, one full matrix transport constructs the preceding T , an order-zero graph $\tilde{T}$ and an order- -1 graph error E such that

$$(E.14) \quad \chi_e(L_a \tilde{T} - T D_{y_1}) \Pi_s = \chi_e E \Pi_s + R_{\text{cut}},$$

where R_{cut} is a finite sum of full cutoff factors with at least one order- -1 graph or diagonal factor. The physical transition term is $[L_a, \chi_t] \tilde{T}_0 \chi_s$; its inner evaluation is exactly zero since L_a is differential and $\chi_t = 1$ near $\text{supp } \chi_e$.

The complete products after characteristic integration are specified in Proposition E.8.

The operator calculus in items (2)–(3) also holds for any real degree-one type-I phase on these buffered boxes with the stated mixed gap and injective averaged-gradient chart. Its bounds use the phase and amplitude seminorms in H_s ; the weighted inverse uses the stated matrix inverse gap. The coefficient version (E.12) applies when the primitives have those coefficient bounds. Steps 1–4 below use only these assumptions.

The regular construction uses fixed finite depths. Its buffers, low topology and derivative shifts are fixed before s and correction precision. Larger integration-by-parts orders only estimate the same operators.

Proof. Step 1. The exact normal and the low operator bound. Before source multiplication write

$$T_0 v(x) = (2\pi)^{-n} \iint e^{i(S(x,\eta) - y \cdot \eta)} A_0(x, \eta) v(y) dy d\eta.$$

Its literal normal is

$$(E.15) \quad (T_0 T_0^*)(x, \tilde{x}) = (2\pi)^{-n} \int e^{i(S(x,\eta) - S(\tilde{x},\eta))} A_0(x, \eta) A_0(\tilde{x}, \eta)^* d\eta.$$

The exact identity $S(x, \eta) - S(\tilde{x}, \eta) = (x - \tilde{x}) \cdot \Xi(x, \tilde{x}, \eta)$ and the change $\xi = \Xi$ give the full pseudodifferential amplitude

$$(E.16) \quad b(x, \tilde{x}, \xi) = \frac{A_0(x, \eta(x, \tilde{x}, \xi)) A_0(\tilde{x}, \eta(x, \tilde{x}, \xi))^*}{|\det \Xi_\eta(x, \tilde{x}, \eta(x, \tilde{x}, \xi))|}.$$

The same identity holds for a cross-product, with the second amplitude replaced by another amplitude of the same phase. The compact amplitude support lies inside its inverse chart, so zero extension is smooth; the low-frequency pieces are smooth kernels.

Lemma C.2 bounds this order-zero full amplitude on L^2 from finitely many derivatives. The declared low bound and inverse gap control them, hence $\|T_0\|_{0 \rightarrow 0} \leq C$. The literal adjoint has the same L^2 norm.

For a full amplitude compactly supported in y , expand that variable in a Fourier series on one fixed larger box, with a fixed cutoff inside the box. Thus $T_A = \sum_k T_{a_k} M_{\beta e^{ik \cdot y}}$ exactly. Integration in y gives $|a_k|_{0, \mu_0} \leq C_N \langle k \rangle^{-N} |A|_{0, \mu_0 + N}$. The source multipliers are uniformly bounded on L^2 ; choose one $N > n + 1$. The series and its literal adjoint give the same low bound from a fixed total derivative reserve. This Fourier series is used only for the norm estimate; the full amplitudes and their original supports remain the operators used below.

For an order- -1 amplitude, one output derivative of the graph or its literal adjoint has order zero. These same bounds therefore give $L^2 \rightarrow H^1$ in both directions, with a fixed finite derivative reserve.

Step 2. Exact diagonal and source formulas. For a full order- q amplitude $c(x, y, \xi)$ on the fixed physical boxes, choose $\beta \in C_c^\infty$ equal to one near both its physical supports. Taylor's formula

in y , followed by one integration by parts in ξ , gives the exact identity

$$(E.17) \quad \begin{aligned} P_c &= \text{Op}(c(x, x, \xi))\beta + P_r, \\ r(x, y, \xi) &= -i \sum_j \partial_{\xi_j} \left[\beta(y) \int_0^1 \partial_{y_j} c(x, x + t(y - x), \xi) dt \right]. \end{aligned}$$

Indeed, $\beta(y)c(x, y, \xi) = c(x, y, \xi)$ and $(y_j - x_j)e^{i(x-y)\cdot\xi} = i\partial_{\xi_j}e^{i(x-y)\cdot\xi}$. All segments lie in one fixed larger convex box. The remainder has compact physical supports, order $q - 1$ and seminorm bounded by $C_s|c|_{q,s+2}$. Compact frequency regularization proves the identity distributionally. Its literal adjoint reverses the full kernel and has the same bounds. Choose $\beta = 1$ near every actual physical input cutoff. This applies directly to the normal amplitude in (E.16).

For fixed $m \in C_c^\infty$, choose $\Pi_{\text{in}} = 1$ on $\text{supp } \chi_g$ inside the larger phase chart. Fixed angular and low-frequency cutoffs extend S to a real smooth symbol, agreeing on both buffered supports, with the same one-high bounds; extend A_p by zero. Fourier inversion gives the absolutely convergent formula

$$(E.18) \quad \begin{aligned} (T_{A_p}m)\Pi_{\text{in}} &= T_b\Pi_{\text{in}}, \\ b(x, \eta) &= \int \widehat{m}(v)e^{i[S(x, \eta+v) - S(x, \eta)]} A_p(x, \eta + v) dv \\ &= A_p(x, \eta)m(S_\eta(x, \eta)) + r_m(x, \eta). \end{aligned}$$

For $|v| \leq \delta\langle\eta\rangle$, choose δ so the whole covector segment stays in the buffered chart. Apply the fundamental theorem of calculus to

$$e^{i[S(x, \eta+tv) - S(x, \eta) + (1-t)S_\eta(x, \eta)\cdot v]} A_p(x, \eta + tv), \quad 0 \leq t \leq 1.$$

Its differentiated amplitude is $v \cdot A_{p, \eta}$ plus $i[S_\eta(x, \eta + tv) - S_\eta(x, \eta)] \cdot v A_p$. Both lower the order by one, with respective factors $|v|$ and $|v|^2$. External covector derivatives gain their radial powers; joint product differentiation uses only $s + c$ primitive derivatives. The Schwartz moments of $\widehat{m}$ give the full order- $p - 1$ bound with one fixed shift. On $|v| \geq c\langle\eta\rangle$, the same moments absorb every frequency power, including all cutoff transitions and bounded frequencies. This exact complement has arbitrary negative order. For phase $x \cdot \xi$, the same formula makes each fixed physical/Fourier commutator order -1 . Thus the added β can be moved past a fixed Fourier multiplier and removed against its actual physical input cutoff. Every commutator is kept.

Finally, substitute the coefficient estimates into these exact amplitudes. Lemma D.7 bounds the phases in the normal and source formulas; the symbols have the stipulated coefficient bounds; and the full matrix transport and its inverse in Steps 4–5 give $|A_p|_{p,m} \leq C_m(1 + \|a\|_{C^{m+c_A}})$ for these fixed constructions. The joint budget above therefore gives $C_s(1 + \|a\|_{C^{s+\beta}})$ for every kept coefficient and complete remainder. The Schwartz moments depend only on the fixed source cutoff. Fixed extensions and Sobolev embedding replace this by $C_s(1 + \|a\|_{H^{s+\beta}(W)})$, with fixed β .

Step 3. Strong forward and literal adjoint estimates. Let μ_0 include every fixed low derivative cost in Step 1. First consider order zero. Take as primitives the normalized amplitude, $S_{x\eta}$, its inverse B , and all cutoff profiles; choose μ to cover μ_0 and their fixed jet shifts. Their C^{μ_0} norms are uniformly bounded on the normalized boxes.

For the forward graph, Euler's identity $S_x = S_{x\eta}\eta$ trades each phase derivative for a datum derivative. For its literal adjoint the exact identity is

$$(E.19) \quad i\eta_i e^{i(y\eta - S(x, \eta))} = - \sum_j B_{ij} \partial_{x_j} e^{i(y\eta - S(x, \eta))}.$$

Integrate by parts in x ; compact support removes the boundary terms. Thus, for either $A = T_p$ or $A = T_p^*$,

$$(E.20) \quad D^\alpha A = \sum_{|\beta| \leq s} A_{\alpha\beta} D^\beta, \quad |\alpha| = s,$$

where the full amplitudes of $A_{\alpha\beta}$ have order zero and primitive derivative excess at most $s - |\beta|$. Derivatives in the second physical position enter this same count. This induction uses only the displayed phase-gradient identity and the low L^2 bound for the complete modified amplitudes. Including a low operator seminorm costs only μ_0 additional derivatives. If r_h are the resulting primitive orders and $k = |\beta|$, then $\sum_h r_h \leq s - k + \mu_0$, hence $\sum_h (r_h - \mu_0)_+ \leq s - k$. Interpolation from the bounded C^{μ_0} baseline to $C^{s+\mu_0}$ therefore gives total high exponent at most $(s - k)/s$. Step 1 yields

$$\|A_{\alpha\beta}\|_{L^2 \rightarrow L^2} \leq C_s H_s^{1-|\beta|/s}, \quad H_s = 1 + |A_p|_{s+\mu} + |S|_{s+\mu}.$$

For $k = |\beta|$, Fourier interpolation and Young's inequality give

$$H_s^{1-k/s} \|v\|_k \leq (H_s \|v\|_0)^{1-k/s} \|v\|_s^{k/s} \leq H_s \|v\|_0 + \|v\|_s.$$

Summing proves (E.11) for $d = 0$; $s = 0$ is Step 1. Fixed low-frequency pieces have the same estimate directly.

The coefficient bounds at the end of Step 2 give (E.12), after enlarging β by the fixed μ .

For order -1 and $s \geq 1$, apply the order-zero estimate at $s - 1$ to each first output derivative of the graph or its literal adjoint. Together with the low L^2 bound, this proves (E.11) with gain one. Increasing μ once includes that first derivative. The opposite positive estimate and supported/restriction duality give gain one at every fixed negative integer index. The full cutoff amplitudes in Lemma E.5 have these same order- -1 bounds and literal reversed domain pairs.

Step 4. A weighted adjoint with identity principal amplitude. Let U be the invertible full transport matrix constructed in Step 5. Choose one real conic cutoff $\chi_g(\eta)$, including the fixed low-frequency cutoff, equal to one on the smaller source and target frequency regions and supported inside the common larger phase chart. Choose $\chi_s = 1$ near all canonical preimages of the inner output for every frequency in $\text{supp } \chi_g$. The amplitude of T_0 is $\chi_g U$. Put $j_S = |\det S_{x\eta}|$ and define

$$(E.21) \quad \begin{aligned} V &= \chi_s V_0 \chi_t, \\ V_0 f(y) &= (2\pi)^{-n} \iint e^{i(y\eta - S(x,\eta))} \chi_g(\eta) j_S(x, \eta) U(x, \eta)^{-1} f(x) dx d\eta. \end{aligned}$$

It is the literal adjoint graph whose forward amplitude is $\chi_g j_S U^{-*}$, where $U^{-*} = (U^{-1})^*$. The inverse and determinant gaps and Step 3 give its order-zero strong estimates, with a fixed coefficient shift.

The exact cross-normal formula (E.16) gives $T_0 V_0$ the diagonal amplitude

$$b^\vee(x, x, \xi) = \chi_g(\eta(x, x, \xi))^2 I.$$

It equals I on the inner frequency plateau. Step 2 gives the complete order- -1 reduction error. Its source multiplier is one on the inner input. Move a fixed physical multiplier past the diagonal principal operator by the source formula of Step 2 with phase $x \cdot \xi$; its commutator has order -1 . Right multiplication by Π_t then multiplies the symbol exactly, so the principal defect vanishes. The low-frequency term is smooth. This proves the paired gain-one bounds for the pure localized defect, with the same fixed shift.

For source-cut factors $T = T_t \chi_s$, $W = W_t \chi_s$ of the same phase, the exact identity is

$$(E.22) \quad TW^* = T_t W_t^* - T_t (1 - \chi_s^2) W_t^*.$$

It includes the literal normal when $W = T$ and the weighted product TV for the preceding choice. Formula (E.18) gives $T_t \chi_s^2$ the principal amplitude $A_t(x, \eta) \chi_s^2(S_\eta(x, \eta))$ and an exact order- -1

remainder. On the inner output the first multiplier is one. Here $\Pi_{\text{in}} V_0 = V_0$ exactly, since its canonical Fourier support lies in $\text{supp } \chi_g$ inside the common larger phase chart. The remaining cross-normal is therefore a full order-1 amplitude by Step 1. Consequently $TV = I - E_0$ on the stated inner localizations, with all source, physical, angular and low-frequency terms included.

Step 5. Two graph amplitudes and the physical plateau. Put $\Xi = S_x$. With the symbol derivatives evaluated at (x, Ξ) , define the real matrix and vector field

$$(E.23) \quad \begin{aligned} C_{\text{tr}} &= \sum_j A_j \partial_{x_j} c_a + B c_a + \frac{1}{2} \sum_{j,l} h_{a,\xi_j \xi_l} S_{x_j x_l} I, \\ X &= \sum_j h_{a,\xi_j} \partial_{x_j}. \end{aligned}$$

Here $\partial_{x_j} c_a$ is taken at fixed ξ . The eikonal gives $X S_{\eta_1} = 1$ and $X S_{\eta_j} = 0$ for $j \geq 2$, so $X = \partial_t$, with zero canonical time on the section.

Solve the full matrix equations

$$(E.24) \quad U' = -C_{\text{tr}} U, \quad U(0) = I, \quad (U^{-1})' = U^{-1} C_{\text{tr}}.$$

Fixed-time Gronwall and the matrix transport estimate of Lemma B.6 give

$$|U|_{0,s} + |U^{-1}|_{0,s} \leq C_s (1 + \|a\|_{C^{s+c}})$$

with a fixed shift and uniform low inverse bounds. The matrix U is real. Define, with every derivative of c_a evaluated at (x, Ξ) ,

$$(E.25) \quad \begin{aligned} b_0 &= c_a(x, \Xi) U, \\ b_{-1} &= -i \sum_j c_{a,\xi_j} \partial_{x_j} U - \frac{i}{2} \sum_{j,l} c_{a,\xi_j \xi_l} S_{x_j x_l} U. \end{aligned}$$

These have orders 0 and -1. Put $\tilde{T}_0 = T_{\chi_g(b_0+b_{-1})}$ and $T_0 = T_{\chi_g U}$. The following identity is exact:

$$(E.26) \quad L_a \tilde{T}_0 - T_0 D_t = T_{\chi_g r_{-1}}, \quad r_{-1} = \sum_j A_j D_{x_j} b_{-1} - i B b_{-1}.$$

Indeed, $T_U D_t = T_{eU}$ by integration in canonical time. Differentiating $H_a c_a = h_a I$ in ξ gives

$$H_a c_{a,\xi_j} = h_{a,\xi_j} I - A_j c_a, \quad H_a c_{a,\xi_j \xi_l} = h_{a,\xi_j \xi_l} I - A_j c_{a,\xi_l} - A_l c_{a,\xi_j}.$$

The product rule in $L_a T_{b_0+b_{-1}}$ now gives the order-one term eU and the order-zero term $-i(X + C_{\text{tr}})U = 0$; the remaining term is precisely r_{-1} . Matrix factors stay in the displayed order. Since χ_g depends only on η , it preserves the identity. The explicit formulas and joint product bounds give r_{-1} and $\tilde{T}_0$ fixed-shift one-high bounds. No inverse of c_a is needed.

For $\tilde{T} = \chi_t \tilde{T}_0 \chi_s$ and $T = \chi_t T_0 \chi_s$, put $E = T_{\chi_g r_{-1}}$. Then

$$(E.27) \quad L_a \tilde{T} - T D_{y_1} = \chi_t E \chi_s + [L_a, \chi_t] \tilde{T}_0 \chi_s + \chi_t T_0 [D_{y_1}, \chi_s].$$

The target cutoff is a function of physical x , and $[L_a, \chi_t] = -i \sum_j A_j \partial_{x_j} \chi_t$. Thus its inner evaluation is exactly zero. The last term has the cutoff form (E.10) after a fixed source localization equal to one on its support. For angular localizations, right Fourier multiplication is exact; moving a fixed physical multiplier uses Step 2 with phase $x \cdot \xi$. Its order-1 error and the inner-cutoff errors remain in their original order. Thus (E.14) follows on the stated plateau, as a fixed finite chain of full graph and diagonal factors. $\square$

A negative-order error before characteristic integration does not identify the complete residual. Proposition E.8 includes the ordered integrated products and their physical transition terms.

E.4. Characteristic integration and the complete residual.

Notation E.7 (Construction domains and characteristic integral). Use the finite regular cover of Proposition E.2 and the common buffered charts of Theorem E.6. Take W containing all construction supports. After one fixed inner shrinking, choose all canonical rectangles with their complete physical flow images in the prescribed $W_0 \subseteq W$, before the final core and accuracy. All graph, source and low-frequency cutoff pieces stay in their literal order. Extend $a - a_0$ by the single componentwise operator of Lemma B.1, and add a fixed extension of a_0 . This agrees with a on the construction region and preserves every nonnegative Sobolev and C^m order. Negative estimates use the supported/restriction dual pairs.

On each canonical rectangle use J from Definition C.9, with lower endpoint $-T$. Choose both χ_{in} and χ_{out} equal to one near $\text{supp } \chi_s$, inside one larger fixed rectangle. The input still vanishes near the lower endpoint. Put

$$(E.28) \quad \mathcal{J} = iJ, \quad D_t = -i\partial_t.$$

These choices are fixed at every observation order.

Proposition E.8 (The complete finite construction). *Use the preceding buffered cover and the actual factors $\tilde{T}_j, T_j, V_j$ of Theorem E.6, with the cutoffs in (E.21) and (E.27). Write*

$$(E.29) \quad L_a \tilde{T}_j - T_j D_{t_j} = E_j + \mathcal{C}_j + \mathcal{S}_j, \quad T_j V_j = I - E_{0,j}.$$

Here $E_j, \mathcal{C}_j, \mathcal{S}_j$ are respectively the graph term, target commutator and source commutator of (E.27). The target commutator vanishes on the inner measurement plateau. The graph and localized identity errors have measured order -1 , including every localization difference. All factors and their literal adjoints have the preceding strong estimates at their respective orders, with one fixed common coefficient shift.

Choose a fixed Fourier angular partition with fixed physical multipliers, $A_\Gamma = A_{\text{low}} + \sum_j A_j$, inside the matched target frequency plateaus. Each localized sector copy is accompanied by its exact conjugate. With $\mathfrak{c}$ denoting conjugation on each physical or canonical space, set

$$\tilde{T}_- = \mathfrak{c} \tilde{T}_+ \mathfrak{c}, \quad A_- = \mathfrak{c} A_+ \mathfrak{c},$$

and choose the entire source-cut factors by

$$(E.30) \quad T_- = \mathfrak{c}_x T_+ \mathfrak{c}_{\text{can}},$$

$$(E.31) \quad V_- = \mathfrak{c}_{\text{can}} V_+ \mathfrak{c}_x,$$

$$(E.32) \quad \mathcal{J}_- = -\mathfrak{c}_{\text{can}} \mathcal{J}_+ \mathfrak{c}_{\text{can}} = \mathcal{J}_+.$$

All partner cutoffs have the same physical and canonical roles. Put

$$(E.33) \quad Q_N = \sum_j \tilde{T}_j \mathcal{J}_j V_j A_j.$$

All factors include their fixed buffered representatives. The complete physical residual is

$$(E.34) \quad \mathcal{E}_a = A_\Gamma - L_a Q_N.$$

With $\mathcal{F}_j = E_j + \mathcal{C}_j + \mathcal{S}_j$, including all exact source and target localization differences, one has

$$(E.35) \quad \mathcal{E}_a = E_{\text{part}} + \sum_j [E_{0,j} - \mathcal{F}_j \mathcal{J}_j V_j] A_j.$$

Here $E_{\text{part}} = A_{\text{low}}$, including any further low-frequency partition adjustments. Each $\mathcal{F}_j$ is a finite sum of full matrix graph and diagonal products, with measured order at most -1 before its integral and the preceding strong bounds. Its source commutators and angular differences are estimated on the inner measurement and actual source supports. The physical transition vanishes there; no global gain at that transition is asserted.

Proof. Choose $c_-(x, \xi) = c_+(x, -\xi)$. The principal adjugate has parity $(-1)^{r-1}$, so $h_- = (-1)^{r-1}q_a = -h_+(x, -\xi)$ and its type-I phase is

$$(E.36) \quad S_-(x, e, \eta') = -S_+(x, -e, -\eta').$$

At paired covectors $S_{-, \eta_-} = S_{+, \eta_+}$; hence the canonical base point and time are unchanged. The weighted amplitude $j_S U^{-1}$ and the source-cut cross-normal conjugate with all their cutoffs. Thus the physical source gaps and separations persist with the same nested canonical localizers on both full branches of the antipodal regular buffer.

Since $\mathfrak{c}L_a\mathfrak{c} = -L_a$ and $\mathcal{J} = iJ$ with J real, the paired factors give $Q_- = -\mathfrak{c}Q_+\mathfrak{c}$. Thus

$$(E.37) \quad \mathfrak{c}Q_N\mathfrak{c} = -Q_N, \quad Q_N/i \text{ is real}, \quad \mathfrak{c}\mathcal{E}_a\mathfrak{c} = \mathcal{E}_a.$$

The nesting gives $\chi_{\text{in}}V = V$, $T\chi_{\text{out}} = T$ and $T[D_t, \chi_{\text{out}}] = 0$ as exact operator identities. Direct differentiation therefore yields

$$(E.38) \quad TD_t\mathcal{J}V = TV.$$

The endpoint term vanishes by support, as does its literal adjoint. The exact graph identity and source-cut inverse now give

$$(E.39) \quad I - L_a\tilde{T}_j\mathcal{J}_jV_j = E_{0,j} - \mathcal{F}_j\mathcal{J}_jV_j.$$

Formula (E.10) handles the source commutator with its cutoffs in their displayed order. Substitution in (E.33) gives (E.35). These formulas use nested inner localizers. Fixed physical/Fourier commutators use Step 2 of Theorem E.6; graph cutoff errors use (E.10). Each exact order-1 term stays before the next nonlocal factor. Additional copies of a sector index distinguish different localized inputs. Hence every complete component is an exact finite product of the selected graph, weighted adjoint, diagonal and buffered integral factors, with at least one order-1 factor. Derivatives of an amplitude do not change its phase. Lemma E.4 therefore gives the paired gain-one bounds for these actual products and their literal reverse products. Literal adjunction reverses every full product and gives the upper integral (E.42), with its input and output weights exchanged and no time reflection. The bounded-frequency terms and their conjugates are smooth directly. $\square$

Proof of Theorem E.1. Use the fixed finite construction of Proposition E.8, with its complete conjugate pairs. For real h , define

$$(E.40) \quad S_{a,\Gamma}h = Q_N h/i,$$

with a fixed output cutoff equal to one near $\bar{U}$ and supported in W . Since $P_a = iL_a$ is real,

$$P_a S_{a,\Gamma}h = A_\Gamma h - \mathcal{E}_a h \quad \text{on } U.$$

Thus the theorem's plus-residual convention is

$$(E.41) \quad R_{a,\Gamma} = -\mathcal{E}_a \quad \text{on real inputs and their complexifications.}$$

Proposition C.10 bounds $\mathcal{J}_j$ and its literal adjoint

$$(E.42) \quad (\mathcal{J}_j^* v)(\sigma, z) = -i\chi_{\text{in},j}(\sigma, z) \int_\sigma^T \chi_{\text{out},j}(t, z) v(t, z) dt$$

on every integer supported/restriction pair. Theorem E.6 and Lemma E.4 apply in both factor orders to the finite factorization (E.33), proving (E.4). Every complete residual term and its literal adjoint has a gain-one factor on its measured supports. The same lemma therefore proves (E.6), including the low-frequency, integrated and measured boundary terms, with one maximum fixed shift β . In particular,

$$(E.43) \quad (\mathcal{F}_j\mathcal{J}_jV_jA_j)^* = A_j^*V_j^*\mathcal{J}_j^*\mathcal{F}_j^*.$$

The actual source and measurement cutoffs and the literal upper integral stay in this reverse product. For $T = \mathcal{R}_a$ or $\mathcal{R}_a^*$ and $t = -k - 1$, supported/restriction duality gives

$$(E.44) \quad \|T\|_{H^{-k-1} \rightarrow H^{-k}} = \|T^*\|_{H^k \rightarrow H^{k+1}}, \quad k \geq 0,$$

with equivalent norms under the fixed extensions. Each fixed k uses finitely many coefficient seminorms. All domains and low choices are fixed before s . $\square$

Remark E.9 (The kept angular complement). For an arbitrary source h , the identity on U is

$$P_a S_{a,\Gamma} h = h - (\text{Id} - A_\Gamma)h + R_{a,\Gamma} h.$$

The term $(\text{Id} - A_\Gamma)h$ is not a smoothing error under the hypotheses of Theorem E.1. In particular, inversion of a localized residual alone does not remove this term. Repeated residual corrections also require separate control of their angular support. No accuracy-independent angular tail estimate is asserted.

APPENDIX F. THE FIXED-DOMAIN INVERSE

F.1. The fixed inverse and its inputs.

The complete residual. Apply Appendix E to the real first-order system

$$(F.1) \quad P_a = \sum_{\nu=1}^n A_\nu(x) \partial_{x_\nu} + B(x).$$

The coefficient array is denoted by a . The matrices have size $r \times r$. Its principal matrix, determinant, and homogeneous normalization are

$$(F.2) \quad H_a(x, \xi) = \sum_{\nu=1}^n A_\nu(x) \xi_\nu, \quad p_a(x, \xi) = \det H_a(x, \xi), \quad q_a(x, \xi) = p_a(x, \xi) |\xi|^{1-r}.$$

We work on angular sectors where every characteristic is simple real principal type and H_a is invertible away from the characteristic set.

Setup F.2 specifies the proved operator bounds used here. The localized identity is

$$(F.3) \quad P_a S_{a,\Gamma} = A_\Gamma + R_{a,\Gamma}.$$

Here $R_{a,\Gamma}$ is the complete residual. It includes the direct graph and coordinate-inverse errors, the canonical source commutators and the low-frequency partition. The physical cutoff commutators vanish on the measurement core. The correction below uses this entire operator. Quantitative localization also requires the actual coefficient profile and protected source product stated below; Proposition 7.2 and Theorem F.6 verify them in the iteration.

F.2. The fixed regular family.

Notation F.1. Sobolev norms are taken on the fixed construction box, on W , or on the indicated core. Compactly supported functions are extended by zero when this is required by the displayed operator. All cutoff multiplications used with a zero extension have support strictly inside the relevant domain. Matrix-valued norms are taken componentwise.

An angular sector is a compact subset of the unit sphere together with its positive conic lift. All sectors are antipodally paired. The notation $E \subseteq F$ for angular sets means containment with a positive angular gap. Fourier transforms use Notation A.2. For a family with size H_ρ , an arbitrary-accuracy tail means a bound $C_{m,Q} H_\rho \rho^{-Q}$ in each fixed H^m . Its input precision may be increased before propagation through a fixed finite operator chain.

Setup F.2 (The proved regular-sector construction). Fix a smooth reference coefficient array a_0 for (F.1). Fix compact antipodally symmetric sectors

$$\Gamma_{\text{in}} \subseteq \Gamma_{\text{out}}$$

such that, on the construction neighborhood, every zero of $p_0 = \det H_0$ in Γ_{out} has nonzero fiber gradient. Assume that H_0 is invertible at every other point of this sector. Shrink one finite coefficient neighborhood once so that the corresponding simple-characteristic, elliptic, chart-projection, and normalization gaps remain uniformly positive.

Theorems E.6 and E.1, with Proposition E.8, give the following construction on this coefficient neighborhood.

- (i) There are fixed domains $U \subseteq W$, a real angular source cutoff A_Γ , a real localized parametrix $S_{a,\Gamma}$, and a complete residual $R_{a,\Gamma}$ satisfying (F.3) on U . The cutoff equals the identity microlocally on a fixed neighborhood of the protected inner sector at sufficiently high frequencies. Any fixed low-frequency modification belongs to the complete residual. Before choosing χ_2, χ_3 , fix a real source cutoff equal to one near $\bar{U}$ and a larger measurement plateau containing $\bar{U}$. Include that source cutoff on the right of $S_{a,\Gamma}, A_\Gamma, R_{a,\Gamma}$, and use the buffered restriction for the low maps below. The identity on U is unchanged, and every later χ_3 lies in this source plateau.

- (ii) The complete residual has uniform one-gain bounds

$$(F.4) \quad R_{a,\Gamma} : L^2(W) \longrightarrow H^1(U), \quad R_{a,\Gamma} : H^{-1}(W) \longrightarrow L^2(U).$$

Its literal adjoint has the corresponding bounds on the reversed supported/restriction domains. At every fixed order both obey the additive one-gain estimates of the cited theorem. After fixed physical cutoffs, (E.6) gives, for one fixed nonnegative integer b and every integer $s \geq 1$,

$$(F.5) \quad \|Kv\|_{H^s} \leq C_s [\|v\|_{H^{s-1}} + (1 + \|a\|_{H^{s+b}})\|v\|_{L^2}].$$

The constants are uniform in the declared low neighborhood. The actual solver obeys (E.4):

$$(F.6) \quad \|S_{a,\Gamma}v\|_{H^s} \leq C_s [\|v\|_{H^s} + (1 + \|a\|_{H^{s+b_S}})\|v\|_{L^2}], \quad s \geq 0,$$

with one fixed shift b_S . The fixed low class includes the $s = 0$ coefficient norm, so $\|S_{a,\Gamma}\|_{2 \rightarrow 2} \leq C$. All selected cutoffs are included in these estimates.

- (iii) The complete residual is $R_{a,\Gamma} = -\mathcal{E}_a$, with $\mathcal{E}_a$ exactly as in (E.35); in particular $R_{\text{low}} = -A_{\text{low}}$. Every measured component is a fixed product of full graph, weighted-coordinate-inverse, diagonal and buffered integral factors, with one order-1 factor. All physical cutoffs, localized sector copies, coordinate errors, source commutators and angular discrepancies are included. The outer physical commutator vanishes on the core; the differentiated integral cutoff vanishes by (E.38).
- (iv) For a positive representative each canonical integration chart has coordinates $(t, y; \tau, \eta)$. Its integral is retarded on one fixed interval $[t_-, t_+]$. Writing the actual two cutoffs in that chart as α, β , its localized form is

$$(F.7) \quad Jf(t, y) = i\alpha(t, y) \int_{t_-}^t \beta(s, y) f(s, y) ds.$$

The input factor βf vanishes on one fixed interval immediately after t_- . Both weights are one near the canonical source support, as in (E.38). The source-cut commutator has the fixed preimage gap on the final physical core. The full signed identities (E.30)–(E.37) give

$$(F.8) \quad J_- = -\mathbf{c}J_+\mathbf{c} = J_+,$$

where $\mathfrak{c}$ is conjugation. Both partners have the same lower endpoint and input buffer. Literal adjunction instead exchanges the weights and reverses the limits, as in (E.42), with the literal Hilbert densities on each supported/restriction pair.

- (v) The two graph factors $T_j, \tilde{T}_j$, inverse graphs, matrix transport, weighted adjoints and exact Taylor-integral remainders use one fixed finite set of operations. Their low inverses, first flow derivatives, and graph Jacobians are uniformly controlled by one finite norm $C^{k_{\text{lin}}}$. There is one fixed extra coefficient-derivative count c_{lin} for their tame high derivative estimates, covering the preceding Sobolev shifts and the fixed low symbol seminorms for full two-position L^2 bounds. No operation is repeated to an order depending on Fourier-tail accuracy.

The diagonal and characteristic relations, with quantified off-relation errors, follow from the full kernels in Lemma F.9.

The additional coefficient profile for quantitative localization is Setup F.7.

F.3. The exact small-support inverse. The first result identifies what remains after the complete residual has been inverted. All domains and all cutoffs in this statement are fixed before an accuracy is requested.

Theorem F.3 (Exact small-support correction). *Use the fixed real operators of Setup F.2 on $U \Subset W$. There are real cutoffs $\chi_2, \chi_3 \in C_c^\infty(U)$, with*

$$0 \leq \chi_2, \chi_3 \leq 1, \quad \chi_3 = 1 \text{ on } \text{supp } \chi_2, \quad \chi_2 = 1 \text{ on } U_{\text{core}},$$

such that the zero extension to W of

$$K_a = \chi_2 R_{a,\Gamma} \chi_3$$

satisfies

$$\|K_a\|_{L^2(W) \rightarrow L^2(W)} \leq \theta < 1$$

uniformly on the fixed sufficiently small finite coefficient neighborhood. Consequently

$$T_a = (I + K_a)^{-1}$$

exists on the entire real space $L^2(W)$. It preserves smooth input functions for each fixed smooth a . There is a fixed integer β_* , independent of requested accuracy, such that

$$(F.9) \quad \|T_a f\|_{H^s(W)} \leq D_s \left[\|f\|_{H^s(W)} + (1 + \|a\|_{H^{s+\beta_*}(W)}) \|f\|_{L^2(W)} \right], \quad s \geq 0.$$

For every smooth real h supported in U , set

$$z = T_a(\chi_2 h), \quad u = S_{a,\Gamma} z.$$

Then $\text{supp } z \subset \text{supp } \chi_2$, and the exact physical identity on U_{core} is

$$(F.10) \quad P_a u = h + (A_\Gamma - I)z.$$

The same $S_{a,\Gamma}, R_{a,\Gamma}, K_a, T_a$, cutoffs, and core apply to all such sources.

Proof of Theorem F.3. Choose $p > 2$ with $H^1(U) \hookrightarrow L^p(U)$. Equation (F.4) bounds $R_{a,\Gamma} : L^2(W) \rightarrow L^p(U)$ uniformly before the final supports are chosen. For the stated cutoffs, Hölder gives

$$\|\chi_2 R_{a,\Gamma} \chi_3 f\|_2 \leq |\text{supp } \chi_2|^{1/2-1/p} \|R_{a,\Gamma} \chi_3 f\|_{L^p(U)} \leq C |\text{supp } \chi_2|^{1/2-1/p} \|f\|_2.$$

Fix the output support so this operator bound is at most $\theta < 1$. Their derivatives affect subsequent constants. The real operator

$$(F.11) \quad T_a = \sum_{j=0}^{\infty} (-K_a)^j$$

converges on the entire real $L^2(W)$, with norm at most $(1 - \theta)^{-1}$.

Fix smooth a and $f \in H^s$, $s \geq 1$. Starting with $z = T_a f \in L^2$, the identity $z = f - K_a z$ and (F.5) at orders $1, \dots, s$ give $z \in H^s$ and

$$\|z\|_{H^s} \leq \|f\|_{H^s} + C_s \|z\|_{H^{s-1}} + C_s (1 + \|a\|_{H^{s+b}}) \|z\|_2.$$

Induction from $\|z\|_2 \leq (1 - \theta)^{-1} \|f\|_2$ proves (F.9) with $\beta_* = b$, since the preceding coefficient norms are bounded by the one at order $s + b$. The same operator preserves smooth input. No parameter-tame dependence is asserted.

Finally $z = T_a(\chi_2 h)$ satisfies $z = \chi_2 h - K_a z$, so $\text{supp } z \subset \text{supp } \chi_2$ and $\chi_3 z = z$. On U_{core} this becomes $z + R_{a,\Gamma} z = h$. Applying the localized identity to this compactly supported z yields

$$P_a S_{a,\Gamma} z = A_\Gamma z + R_{a,\Gamma} z = h + (A_\Gamma - I)z,$$

which proves (F.10). $\square$

F.4. Common graded profiles.

Theorem F.4 (Common graded bounds). *Use the fixed finite construction of Setup F.2. Suppose its actual coefficient array satisfies*

$$(F.12) \quad \|a\|_{C^r(W)} \leq D_r T^{(r-R_*)+}, \quad r \geq 0, \quad T \geq 1,$$

with one fixed $R_* > c_{\text{lin}} + 2$ above the construction orders. Set $L = T$. Every normalized phase map $\mathcal{G} = \Phi_x$ or Φ_ξ has bounded zeroth and first jets and

$$(F.13) \quad |D^r \mathcal{G}| \leq C_r L^{r-1}, \quad r \geq 2.$$

Derivatives are joint in physical and normalized covariable coordinates. Every actual full symbol of fixed order p , including the finite remainders, satisfies, with $r = |\alpha| + |\beta|$,

$$(F.14) \quad |D_{x,z}^\alpha D_\xi^\beta a_p(x, z, \xi)| \leq C_{\alpha,\beta} \langle \xi \rangle^{p-|\beta|} L^r.$$

All domains, operators, cutoffs and derivative reserves are fixed before accuracy.

Proof. Theorem E.6 and Proposition E.8 bound the normalized phase maps and full symbols by $C_r(1 + \|a\|_{C^{r+c_{\text{lin}}}})$. This includes both amplitude positions, inverse charts, matrix transport, cutoff jets and the exact inner-cutoff factors. Corollary B.4 gives directly

$$(F.15) \quad 1 + \|a\|_{C^{r+c_{\text{lin}}}} \leq C_r T^{(r-1)+}.$$

The phase and normalized symbol bounds follow; radial rescaling gives (F.14). $\square$

F.5. Protection by characteristic phase graphs.

Definition F.5 (Protected source product). Fix $X \Subset W_0 \Subset W$, with X compact and every complete physical flow tube of the selected factors contained in W_0 . A compact annular covector set E_0 over X is protected if its characteristic part lies in a conic set $\mathcal{P} \subset \{q_a = 0\}$ invariant under Hamilton segments wholly in W_0 , in both orientations. The normalized restriction of $\mathcal{P}$ over X , together with the elliptic points of E_0 , must be compactly inside Γ_{in} , with a positive gap from the nonprincipal set. Uniform protection means uniform compact buffers on the fixed low coefficient neighborhood.

Proposition D.9 puts all normalized signed pair directions, including limiting difference chords and antipodes, in one compact set $\mathcal{C}_{\text{quad}}$ separated from the frozen nonprincipal set $\mathfrak{M}$.

Theorem F.6 (Protection of the complete source product). *Use that current immersion and pair family in a sufficiently small fixed C^{k_*} neighborhood of the cube, with k_* above the regular construction orders. There are fixed balls $X \Subset W_0 \Subset W$, with X closed, a real smooth coefficient extension $\tilde{a}$ equal to a_F on W_0 , and symmetric angular sets*

$$\Omega_0 \Subset \Omega'_0 \Subset \Gamma_{\text{in}} \Subset \Gamma_{\text{out}}$$

with these properties. Every actual normalized pair covector on X lies inside Ω_0 ; its magnitude is comparable to its true beat $\lambda|z - z'|$, or to λ for sums and cross-group pairs. The entire product $X \times \Omega'_0$ is uniformly protected for $\tilde{a}$. All actual flow tubes lie in W_0 and the operator agrees with P_F on the core. The extension stays in the fixed low class and preserves every stated coefficient profile.

Proof. Let a_0 have the frozen cube principal matrices and a fixed smooth lower-order completion. Choose small fixed balls and the affine extension

$$(F.16) \quad \tilde{a} = a_0 + \mathcal{E}(a_F - a_0), \quad \|\tilde{a}\|_{C^r} \leq C_r(1 + \|a_F\|_{C^r(W_0)}),$$

using Lemma B.1. Shrink the current neighborhood by its fixed low operator norm. Choose $\Omega_0 \Subset \Omega'_0$ around $\mathcal{C}_{\text{quad}}$; the signed chord and sum bounds give the true beats.

Apply Lemma D.7 to $q_{\tilde{a}}$ on a finite cover of its regular directions. Write these comparison phases as $\Phi_\alpha(x, e, \eta')$. Choose compact annular parameter sets V_α so that every characteristic point of $X \times \Omega'_0$ has the form

$$(x, \Phi_{\alpha,x}(x, 0, \eta')), \quad \eta' \in V_\alpha,$$

and these graphs for all $x \in \overline{W_0}$ lie in fixed compact regular angular sets D_α , with norms bounded above and below. This is possible with uniform buffers: the frozen phase gradient is independent of x , and the actual maps and their inverses are uniformly close in the fixed low topology. Include the antipodes. Choose $\Gamma_{\text{in}}, \Gamma_{\text{out}}$ around these images and Ω'_0 . Choose the actual finite regular factors with their complete canonical rectangles having physical flow images inside W_0 ; their source, target and both integral weights use the inner buffers. These are fixed shrinkages in the same phase construction.

For fixed η' , put $\psi = \Phi_\alpha(\cdot, 0, \eta')$. Differentiating $q_{\tilde{a}}(x, d\psi) = 0$ gives

$$\psi_{xx} q_{\tilde{a},\xi} = -q_{\tilde{a},x}.$$

Hamilton's equations and uniqueness therefore keep each segment wholly in W_0 on this same graph, in either orientation. Let $\mathcal{P}$ be the positive conic lift of the union of these graphs over the compact V_α . Its restriction over X has compact normalized image inside Γ_{in} . Every complete residual relation is diagonal or one of the actual characteristic segments, so each finite chain preserves $\mathcal{P}$. This proves protection of the whole source product, including elliptic identity images, with no condition on exterior trajectories. $\square$

F.6. Finite powers and arbitrary accuracy.

Setup F.7 (Actual coefficients and total mixed grades). Use Setup F.2 and the actual profile (F.12) with $L = T = T_0$. The operators and cutoffs are those of Theorem F.3. For a compact annular frequency set V'_0 , require uniform protection of the whole $X \times V'_0$, where $X = \text{supp } \chi_2$, in Definition F.5. The entire invariant set over X lies in the selected buffered charts. For $\rho \geq 2$ and fixed $0 < \sigma < 1$, assume

$$(F.17) \quad 1 \leq L \leq \rho^{1-\sigma}.$$

The product protection is Definition F.5; Theorem F.4 gives the total mixed phase and full amplitude grades from the stated coefficient bounds.

Theorem F.8 (Arbitrary-accuracy correction). *Use Setup F.7 and its fixed operators and cutoffs. Choose a compact annular set $V_0 \Subset V'_0$ and one fixed smooth rescaled Fourier cutoff $\Pi_{0,\rho}$, equal to one on the actual input frequency set in ρV_0 and supported in $\rho V'_0$. Let h_ρ be real and smooth, supported where $\chi_2 = 1$, and satisfy*

$$(F.18) \quad \|h_\rho\|_{L^2} \leq CH_\rho,$$

$$(F.19) \quad \|(I - \Pi_{0,\rho})h_\rho\|_{H^m} \leq C_{m,Q} H_\rho \rho^{-Q} \quad \text{for every fixed } m, Q.$$

The same exact inverse gives

$$z_\rho = T_a h_\rho, \quad u_\rho = S_{a,\Gamma} z_\rho,$$

with, for every fixed m, Q ,

$$(F.20) \quad \|P_a u_\rho - h_\rho\|_{H^m(U_{\text{core}})} \leq C_{m,Q} H_\rho \rho^{-Q}.$$

For every fixed s ,

$$(F.21) \quad \|u_\rho\|_{H^s} \leq C_s H_\rho \rho^s.$$

The operator, its finite symbolic order, its physical domains, and its angular source cutoff are unchanged when Q changes. The conclusions keep genuine outgoing branches and make no assertion near excluded nonprincipal characteristic directions.

F.6.1. Finite-chain localization.

Lemma F.9 (Base-aware localization through the actual finite chain). *Use Setup F.7. For each fixed finite chain of graph, inverse-graph, diagonal, retarded and complete residual factors, one can choose buffered proof localizers in the factors' own canonical coordinates. Across an off-relation gap their full composed kernels satisfy*

$$(F.22) \quad \|(I - M_{\text{out}})AM_{\text{in}}\|_{L^2 \rightarrow H^m} \leq C_{m,q} \rho^{m+C_A} L^b (L/\rho)^q.$$

Here C_A, b are fixed independently of q and requested accuracy. For each fixed number of complete residual powers the final physical mask lies in an annulus $[c_j \rho, C_j \rho]$, $0 < c_j < C_j < \infty$, with angular support strictly inside Γ_{in} . Intermediate canonical masks need not lie in that physical cone. The discarded output has arbitrary prescribed H^m decay. All actual operators, cutoffs and their finite symbolic order remain the same.

Proof. The joint grades (F.13)–(F.14) hold at $L = T_0$ for both amplitude positions, all cutoff jets and fixed Taylor parameters. The buffered proof masks have the same grades. A fixed product adds only one fixed amplitude offset b .

A common kernel estimate. On comparable normalized annuli, a localized composed kernel has phase ρF and compact base support. Off its full stationary relation, compactness gives $|\nabla F| \geq \delta > 0$. The field $\nabla F \cdot \nabla / (i\rho |\nabla F|^2)$ then has transpose cost CL/ρ at each application, since $|D^r \nabla F| \leq C_r L^r$. After q integrations, the fixed symbol orders, Fourier measures and m output derivatives give $C_{m,q} \rho^{m+C_A} L^b (L/\rho)^q$. Schur's inequality proves (F.22). For a half-space kernel, use a tangential field in a fixed collar where the boundary tangential gradient has a gap, and a full-gradient field in the interior. This gives the same count without differentiating the boundary jump. We verify the relations and gaps for every factor.

Graph factors. Write the actual kernel as

$$Af(u) = \int e^{i[\Phi(u,\xi) - z \cdot \xi]} a_A(u, z, \xi) f(z) dz d\xi.$$

Use the right input symbol $m_{\text{in}}(y, \eta/\rho)$ and left output symbol $m_{\text{out}}(x, \zeta/\rho)$. The former has compact base and annular support; the latter is one near the complete canonical image, with base support compact on the actual output. After frequency normalization their exact joint phase is

$$(F.23) \quad F = \phi(u, \xi) - z \cdot \xi + (z - y) \cdot \eta + (x - u) \cdot \zeta, \quad \Phi(u, \rho\xi) = \rho\phi(u, \xi).$$

Stationarity in (u, z, ξ, η, ζ) means

$$(F.24) \quad \phi_u = \zeta, \quad \eta = \xi, \quad \phi_\xi = z, \quad z = y, \quad x = u.$$

These equations describe precisely that full image. Thus $1 - m_{\text{out}}$, including its identity contribution, has the required joint gap; the input-base equation is essential. For the literal reverse phase $z \cdot \xi - \Phi(u, \xi)$ the equations are $\eta = \phi_u$, $\zeta = \xi$, $z = \phi_\xi$, $u = y$, $x = z$, giving the inverse full graph relation. The weighted coordinate inverse has this phase and its actual two-position amplitude.

Diagonal factors use $\phi(u, \xi) = u \cdot \xi$; the two-term $\tilde{T}$ and every negative-order graph use the same argument.

Outside comparable annuli, forward graphs have the larger-frequency gap $\eta - \xi$ in z , or $\Phi_u - \zeta$ in u ; reverse graphs use $\eta - \Phi_u$ or $\xi - \zeta$. Homogeneity and the inverse mixed gap give $|\Phi_u| \asymp |\xi|$. Each integration costs L and one inverse larger frequency. Taking q above a fixed integrability threshold sums all dyadic shells with only a fixed enlargement of C_A . This includes bounded frequencies and all source shoulders, with both amplitude positions unchanged.

Retarded and signed factors. The actual lower input buffer removes the term $H(t_- - s)$ from the primitive. Its remaining kernel is

$$i\alpha(t, y)H(t - s)\beta(s, y)\delta(y - y').$$

With input (t_i, y_i) and output (t_o, y_o) its localized phase is

$$(t_o - t)\kappa + (y_o - y) \cdot \zeta + (s - t_i)\tau + (y - y_i) \cdot \eta.$$

Put $d = t - s \geq 0$. At $d = 0$, tangential stationarity gives the full diagonal: $t_o = t_i$, $y_o = y_i$ and $(\kappa, \zeta) = (\tau, \eta)$. In $d > 0$ the additional equation gives $\kappa = \tau = 0$, $t_o \geq t_i$, $y_o = y_i$ and $\zeta = \eta$. This is the oriented characteristic translation; its transverse covector is nonzero by the input annulus. A chart with $|\tau| \geq c|(\tau, \eta)|$ has only the diagonal branch. Away from buffered neighborhoods of these images, compactness gives the boundary tangential gap and the interior full gap used above. For noncomparable frequencies the tangential derivatives $\tau - \kappa$ and $\eta - \zeta$ give the larger-frequency gap. The same dyadic count proves the full tail and nonzero output annulus, without division by τ or an omitted endpoint value. Literal J^* exchanges the weights and reverses the translation; its upper input buffer removes the fixed upper-endpoint term.

The exact signed identities (E.30)–(E.37) have $A_- = c_A \mathbf{c}_{\text{out}} A_+ \mathbf{c}_{\text{in}}$, $|c_A| = 1$. Use conjugate proof masks. Conjugation preserves supported integer Sobolev norms and radial annuli, reflects covectors and cancels between adjacent factors. Hence it proves the same bounds for every negative chain. The complete residual $R = -\mathcal{E}$, including its source-cut branches, consists of these actual ordered factors by Lemma E.5 and Theorem E.1; every cutoff and negative-order factor is included.

The protected finite chain. Partition the entire source product $X \times \rho V'_0$. At each factor choose a buffered full image in its own canonical coordinates. Since $L/\rho \leq \rho^{-\sigma}$, a sufficiently large q gives any prescribed tail in (F.22); increase its accuracy by the polynomial Sobolev costs of the remaining finitely many factors. No datum is differentiated.

Every complete residual relation is diagonal or a zero-characteristic segment wholly in W_0 . In each chart $h_j = \pm q_a$ by (E.8), so either orientation preserves $\mathcal{P}$. The actual χ_2 puts every next source base in X , where $\chi_3 = 1$. Thus final physical covectors lie strictly inside Γ_{in} ; intermediate canonical masks need not. The compact buffer permits finitely many nested proof masks at each fixed depth. Homogeneity and the nonzero transverse retarded covector give positive annular endpoints. Induction proves the assertion, including original Fourier tails and localizer leakage at a higher internal precision. No actual operator, cutoff or order depends on that precision. $\square$

F.6.2. *One estimate for the remaining inverse.*

Lemma F.10 (The remaining exact inverse). *Use the fixed complete K_a and $T_a = (I + K_a)^{-1}$ above. Suppose the actual graded profile gives*

$$1 + \|a\|_{H^{r+b}} \leq C_r L^r, \quad 1 \leq L \leq \rho,$$

where b covers their fixed tame shifts. If $\|h\|_{H^{-N}} \leq C_N A \rho^{-N}$, then for fixed integers $N, s \geq 0$,

$$(F.25) \quad \|T_a K_a^{N+s} h\|_{H^s} \leq C_{N,s} A L^s (L/\rho)^N.$$

For bounded coefficient jets through the finitely many proof orders, take $L = 1$. All products use the original supported/restriction domains.

Proof. Both K_a and $K_a^* = \chi_3 R_{a,\Gamma}^* \chi_2$ have uniform L^2 bounds and the same additive one-gain estimate on their fixed reversed support pairs. Induction gives, for $B = K_a, K_a^*$,

$$\|B^j v\|_{H^j} \leq C_j(1 + \|a\|_{H^{j+b}})\|v\|_2.$$

At each step the new coefficient term multiplies only $\|B^{j-1}v\|_2$; the other term uses the preceding Sobolev order. Dualize the entire $(K_a^*)^N$ to obtain

$$\|K_a^N h\|_2 \leq C_N(1 + \|a\|_{H^{N+b}})\|h\|_{H^{-N}} \leq C_N A(L/\rho)^N.$$

The same induction at order s gives

$$\|K_a^s v\|_{H^s} \leq C_s L^s \|v\|_2.$$

Apply it to $v = K_a^N h$. The tame estimate for T_a , using its uniform L^2 bound in the high term, gives (F.25). $\square$

For every integer $d \geq 1$, the same L^2 inverse satisfies

$$(F.26) \quad T_a h = \sum_{j=0}^{d-1} (-K_a)^j h + (-1)^d T_a K_a^d h.$$

If the original source obeys $\|h\|_{H^{-N}} \leq C_N A \rho^{-N}$ for all N and $L \leq \rho^{1-\sigma}$, any operator B_ρ with fixed polynomial bounds $\|B_\rho v\|_{H^s} \leq C_s \rho^{c_s} \|v\|_{H^{t_s}}$, with fixed nonnegative integers t_s , satisfies

$$(F.27) \quad B_\rho T_a h = \sum_{j < d} (-1)^j B_\rho K_a^j h + O_{s,Q}(A \rho^{-Q}) \quad \text{in } H^s$$

for one sufficiently large proof depth $d = d(s, Q)$. Indeed, the lemma with $d = N + t_s$ bounds the last term by $C A \rho^{c_s + t_s - \sigma N}$. The operators and domains are unchanged.

Proof of Theorem F.8. The original annular source and its complementary tail give

$$\|h_\rho\|_{H^s} \leq C_s H_\rho \rho^s, \quad \|h_\rho\|_{H^{-N}} \leq C_N H_\rho \rho^{-N}.$$

The actual profile, with its reserve covering the fixed inverse and solver shifts, bounds their coefficient terms by $C_s L^s$. Their tame estimates prove (F.21) directly, since $L \leq \rho$.

Apply Lemma F.9 to each fixed complete $K_a^j h_\rho$. Every following residual starts over $X = \text{supp } \chi_2$, where $\chi_3 = 1$. Its diagonal or characteristic relation stays in the protected set $\mathcal{P}$. The resulting physical mask is supported strictly inside Γ_{in} , and its complementary tail has arbitrary precision. The actual source plateau therefore gives

$$(A_\Gamma - I) K_a^j h_\rho = O_{j,m,Q}(H_\rho \rho^{-Q}) \quad \text{in } H^m,$$

including the fixed low-frequency modification. Intermediate proof masks stay in their own canonical coordinates; every original cutoff and complete error is included.

Apply (F.27) with $B_\rho = A_\Gamma - I$. The preceding finite-chain tails give

$$(F.28) \quad \|(A_\Gamma - I) T_a h_\rho\|_{H^m} \leq C_{m,Q} H_\rho \rho^{-Q}.$$

The exact identity (F.10) proves (F.20) on the same physical core. $\square$

APPENDIX G. PRIMARY WAVES AND SIMULTANEOUS SAMPLING

G.1. The primary family and its inputs. We construct one finite primary family and separated finite-band samples on the entire output space. The same samples approximate the Hermitian background, while individual corrections have no sample-count loss. Throughout, the stress space is $\ker B_G$ and tangent covectors are raised by the actual metric in the ambient reconstruction. The finite transport and sampling are completed in this section. Appendix H then constructs the response used to correct their full strain for the current immersion.

G.2. Geometric and oscillatory estimates.

Notation G.1. Use the dimensions, frozen stresses Q_i^0 and Frobenius pairing of Definition D.1. All tensor complements and projections use that pairing. The common Fourier and adjoint conventions are in Notation A.2. Spatial and carrier supports have fixed larger buffers. Constants may depend on those supports, the fixed low class, inverse gaps and finitely many declared profile constants, but not on frequencies or sample counts unless displayed. An exact Fourier-band hypothesis refers to the canonical datum before any physical initial-variable cutoff. Such a cutoff stays in the wave amplitude; if it is applied to the datum, its Fourier leakage is estimated explicitly. In particular, $\kappa(y)P_K f(y)$ is not declared band limited.

Definition G.2 (Actual second-form reconstruction). Let G be an actual smooth immersion into $\mathbb{R}^N$, and put $g = \gamma(G)$. Write

$$\Pi_{ij} = \Pi_{\text{Nor}_G} \partial_{ij} G.$$

Define

$$B_G: \text{Sym}^2 \mathbb{R}^n \longrightarrow \text{Nor}_G, \quad B_G(T) = \sum_{i,j} T_{ij} \Pi_{ij}.$$

Assume that B_G is surjective. Its stress space and its orthogonal projection are

$$K_G = \ker B_G, \quad \Pi_{K_G} = I - B_G^* (B_G B_G^*)^{-1} B_G.$$

In particular, K_G is not $\ker B_G^*$. The inverse of $B_G^*: \text{Nor}_G \rightarrow K_G^\perp$ is

$$N_G = (B_G^*)^{-1} = (B_G B_G^*)^{-1} B_G|_{K_G^\perp}.$$

For formulas on the fixed ambient space, extend B_G^* by zero on tangent vectors and put

$$(G.1) \quad \begin{aligned} P_{\text{tan},G} &= dG g^{-1} (dG)^T, & A_G &= B_G B_G^* + P_{\text{tan},G}, & N_G &= A_G^{-1} B_G, \\ \Pi_{K_G} &= I - B_G^* N_G, & N_G|_{K_G^\perp} &= N_G. \end{aligned}$$

The ambient matrix A_G is invertible on the orthogonal normal and tangent summands. These are rational fixed-jet formulas. For a tangent covector $u = (u_i)$, set

$$(G.2) \quad \mathcal{V}_G(u) = \sum_{i,j} g^{ij} u_i \partial_j G + N_G \Pi_{K_G^\perp} (\text{sym } \nabla^g u).$$

Thus the tangential term is literally $dG g^{-1} u$.

For a real frame $Q_1, \dots, Q_n$ of K_G , put

$$(G.3) \quad P_G u = (Q_i : \text{sym } \nabla^g u)_{i=1}^n, \quad M_G(x, \xi)_{i\bullet} = (Q_i(x) \xi)^T, \quad p_G(x, \xi) = \det M_G(x, \xi).$$

The operator P_G includes the actual Christoffel terms. We write $\xi \text{sym } b = (\xi b^T + b \xi^T)/2$.

Full carrier groups and all literal sample pairs use Definition D.12 and Remark D.13.

Setup G.3 (The common oscillatory interface). Use the full and slice inverse charts of Proposition D.10, with $x = (t, y)$, $z = (s, v)$, $v \in \mathbb{R}^2$, and normalized annular $\theta \in \mathbb{R}^{n-1}$. They satisfy Setup C.3 and Proposition C.6 with $u = (s, \theta)$ and fixed spectator v . Use their common larger supports and buffers at $M = 20(n+1)$; the bounds include first parameter derivatives and the literal full-dispersion phases.

For the normalized full operator T_ϵ^K and fixed-carrier slice $L_{\epsilon,z}^K$ of those results, the entire-space bounds are

$$\|T_\epsilon^K\| \leq C \|a_\epsilon\|_{C^M}, \quad \|L_{\epsilon,z}^K\| \leq C \sqrt{K} \|a_\epsilon\|_{C^M}.$$

The input is the entire complex chart L^2 space and the output the entire physical L^2 space. On a fixed phase, an amplitude difference of C^M norm δ therefore costs $C\delta$ and $C\sqrt{K}\delta$, respectively: for $\delta > 0$, apply the cited mask-free proof to that amplitude divided by δ . Fixed matrix components and spectator integration obey the same bounds.

For $0 < h \leq 1$ and real $0 \leq m_h(s, v) \leq 1$ with $\|Dm_h\|_\infty \leq C/h$, Proposition C.4 gives

$$(G.4) \quad \|T_\epsilon^K m_h(T_\epsilon^K)^* - T_0^K m_h(T_0^K)^*\| \leq C\epsilon/h, \quad K \geq 1.$$

Both complex quadratures are used. By Corollary C.5, the real channel $2 \operatorname{Re} T$ has, on real outputs, the exact normal

$$(G.5) \quad (2 \operatorname{Re} T)(2 \operatorname{Re} T)^* = 4 \operatorname{Re}(TT^*).$$

G.3. Hamilton phases and transport amplitudes.

Theorem G.4 (Phases, transport and ambient reconstruction). *Use an actual three-parameter primary patch, its adapted coordinates $x = (t, y)$, carrier map $\zeta(z)$ and larger full modulation charts from Propositions D.9 and D.10. Use the projected frame $Q_i(G) = \Pi_{K_G} Q_i^0$. Fix the immersion, normal-rank, null-minor and root gaps and the convex buffered full and slice inverse charts on the entire larger source boxes. One sufficiently small finite C^m neighborhood fixes all required low bounds; the boxes and gaps are independent of frequencies and accuracy.*

For a real positive initial scalar $r_z(y)$ with fixed positive lower and C^m upper bounds, there are a real phase and amplitude $b_G = \alpha_G k_G$ satisfying

$$p_G(x, d_x S_G) = 0, \quad S_G(0, y, \zeta) = y \cdot \zeta, \quad \alpha_G|_{t=0} = r_z, \quad l_G^T P_G b_G = 0.$$

Here k_G, l_G are real nonzero right and left nullvectors of $M_G(x, dS_G)$, and α_G has a positive lower bound. The reconstruction (G.2) obeys

$$(G.6) \quad \frac{1}{2} D\gamma_G \mathcal{V}_G(u) = \Pi_{K_G}(\operatorname{sym} \nabla^g u).$$

For any amplitude b , put $\xi = d_x S_G$ and

$$(G.7) \quad \nu_b = N_G \Pi_{K_G^\perp}(\xi \operatorname{sym} b), \quad A_0(b) = -\nu_b \otimes \xi.$$

The exact product identity is

$$(G.8) \quad \mathcal{V}_G(\tau^2 e^{iS_G/\tau} b) = e^{iS_G/\tau} (i\tau \nu_b + \tau^2 \mathcal{V}_G(b)).$$

For $b = b_G$, $|\nu_b|, |\xi| \geq c > 0$ and $A_0(b)^* A_0(b) = |\nu_b|^2 \xi \xi^T$. This transported tensor is linear in the initial density r_z^2 .

One fixed shift D gives, for every integer $r \geq 0$,

$$(G.9) \quad \|S_G\|_{C^r(x, \zeta)} + \|b_G\|_{C^r(x, z, \zeta)} + \|\nu_b\|_{C^r(x, z, \zeta)} + \|\mathcal{V}_G(b)\|_{C^r(x, z, \zeta)} + \|A_0(b)\|_{C^r(x, z, \zeta)} \leq C_r \left(1 + \|G\|_{C^{r+D}(B_1)} + \|r_z\|_{C^{r+D}(y, z)}\right).$$

These are joint physical and normalized carrier norms on the same larger domains at every order.

Proof. The stated propositions give the phases, full modulation charts and fixed inverse gaps. The metric, normal inverse and nullvectors are smooth fixed-jet formulas on these gaps. Choose real nullvectors with the reference orientation. Since $\operatorname{adj} M_G$ is a nonzero multiple of $k_G l_G^T$,

$$\partial_{\xi_t} p_G = (\text{nonzero scalar}) l_G^T C_t k_G \neq 0,$$

where C_t is the time coefficient of P_G . Thus the equation for α_G is the noncharacteristic scalar transport

$$\sum_a (l_G^T C_a k_G) \partial_a \alpha_G + (l_G^T P_G k_G) \alpha_G = 0.$$

Its characteristic formula is $\alpha_G = a_{\text{flow}} r_z(y_{\text{foot}})$, where the positive flow exponential a_{flow} is independent of r_z . This proves positivity and the density assertion.

The Gauss formula (2.2) with $\nu = N_G \Pi_{K_G^\perp}(\operatorname{sym} \nabla^g u)$ proves (G.6); the product rule gives (G.8). For b_G , the null relation puts $\xi \operatorname{sym} b_G$ in $K_G^\perp$, and

$$|\xi \operatorname{sym} b_G|^2 \geq \frac{1}{2} |\xi|^2 |b_G|^2, \quad B_G^* \nu_b = \xi \operatorname{sym} b_G.$$

The fixed gaps give the asserted lower bounds.

Equation (D.12) gives the joint one-high phase bound. The jet and characteristic formulas, with Lemma B.6, give one high norm of G or r_z for b_G ; other factors use their fixed low bounds. Reconstruction uses only a fixed number of derivatives, so one fixed reserve D proves (G.9). $\square$

One algebraic correction reduces the strain by one further power. The same current response will correct the complete residual.

G.4. A two-term primary wave.

Theorem G.5 (One finite primary family). *Use Theorem G.4. Set $M = 20(n+1)$ and $r_{\text{par}} = M+3$. Assume*

$$\|a_G\|_{C^r} + \|G\|_{C^r} + \|r_z\|_{C^r} \leq D_r L^{(r-R_*)+}, \quad r \geq 0, \quad L \geq 1.$$

Choose R_* above the construction orders and $D_{\text{pr}} + r_{\text{par}} + 2$, where D_{pr} is one fixed one-high derivative reserve for the algebraic operations, complete strain and differentiated reconstruction below. It is independent of accuracy. All estimates hold on the larger primary boxes containing every later source and measurement support. Put $b_0 = b_G$ and

$$(G.10) \quad \mathcal{R} = (M_G(x, dS_G)|_{k_G^\perp \rightarrow l_G^\perp})^{-1} \Pi_{l_G^\perp}, \quad b_1 = i\mathcal{R}P_G b_0.$$

For sufficiently large $\lambda = \tau^{-1}$, define

$$(G.11) \quad v_\tau(x, z, \zeta') = \tau^2 e^{iS_G(x, \zeta')/\tau} (b_0 + \tau b_1),$$

$$(G.12) \quad \mathcal{W}_\tau^{\text{pr}} = \mathcal{V}_G(v_\tau), \quad W_{\tau, z}^{\text{pr}}(\mu) = \int \mathcal{W}_\tau^{\text{pr}}(x, z, \zeta(z) + \tau\eta) d\mu(\eta).$$

Here μ is any finite complex Fourier measure whose shifted carriers lie in the original chart; the background uses $\mu = \delta_0$. The fields are separately linear in the initial scalar and measure. Ordinary data mean $d\mu = \hat{f} d\eta$; on B_{CK} their variation is at most $CK^{(n-1)/2} \|f\|_2$.

Put $A_\tau^{\text{pr}} = e^{-iS_G/\tau} d\mathcal{W}_\tau^{\text{pr}}$. For every physical order r and joint carrier/patch parameter order $k \leq r_{\text{par}}$,

$$(G.13) \quad \begin{aligned} \|D_x^r D_{z, \zeta'}^k A_\tau^{\text{pr}}\|_\infty &\leq C_{r, k} L^r, \\ \|D_x^r D_{z, \zeta'}^k (A_\tau^{\text{pr}} - A_0(b_0))\|_\infty &\leq C_{r, k} \tau L^r. \end{aligned}$$

The displacement amplitude has the first bound times τ . The bounds are uniform over the fixed positive radial labels. The strain amplitude $\mathcal{E}_\tau = e^{-iS_G/\tau} D\gamma_G \mathcal{W}_\tau^{\text{pr}}$ satisfies

$$(G.14) \quad \|D_x^r D_{z, \zeta'}^k \mathcal{E}_\tau\|_\infty \leq C_{r, k} \lambda^{-3} L^r.$$

The current response corrects this full atomic strain.

The phase maps and frame satisfy

$$(G.15) \quad \|D^r(S_{G, x}, S_{G, \zeta})\| \leq C_r L^{r-1}, \quad r \geq 1, \quad \|Q_i(G)\|_{C^r} \leq C_r L^r.$$

The real Hermitian background satisfies

$$(G.16) \quad \left\| \text{Re}[(dW_{\tau, z}^{\text{pr}, \text{bg}})^* dW_{\tau, z}^{\text{pr}, \text{bg}}] - |\nu_{b_0}|^2 dS_G^2 \right\|_{C^r} \leq C_r \tau L^r.$$

Its derivative stays nonzero, and its leading transported mean is linear in the initial density r_z^2 .

Fix smooth compact physical and input cutoffs χ, χ_{in} . For $K \geq 1$ and $\epsilon = \tau K \leq \epsilon_0$, the actual Hermitian channel, with input half-derivative and unitary dilation $\eta = K\theta$, is, up to one fixed

Fourier scalar,

$$(G.17) \quad T_{\tau,v}^{K,\text{pr}} F(x) = \left(\frac{K}{2\pi}\right)^{n/2} \int e^{iK\phi_\epsilon(x,s,v,\theta)} a_{\tau,\epsilon}^{\text{pr}}(x,s,v,\theta) F(s,\theta) ds d\theta,$$

$$(G.18) \quad \phi_\epsilon = \frac{S_G(x, \zeta(z) + \epsilon\theta) - S_G(x, \zeta(z))}{\epsilon},$$

$$(G.19) \quad a_{\tau,\epsilon}^{\text{pr}} = \chi \chi_{\text{in}} |\theta|^{1/2} (Q_i : A_\tau^{\text{pr}}(x, z, \zeta(z))^* A_\tau^{\text{pr}}(x, z, \zeta(z) + \epsilon\theta))_{i=1}^n.$$

The quotient has its smooth value at zero. The input is the entire complex $L^2(s, \theta)$ and the output the entire $L^2(x; \mathbb{C}^n)$. Measurement and initial cutoffs occur in this channel, not in the ambient field. For $0 < h \leq 1$ and real masks $0 \leq m_h \leq 1$, $\|Dm_h\|_\infty \leq C/h$, put $E_{\tau,v}^{K,\text{pr}} = T_{\tau,v}^{K,\text{pr}} m_h (T_{\tau,v}^{K,\text{pr}})^*$. Let $E_{0,v}^K$ set $\tau = \epsilon = 0$, so its amplitude is $A_0(b_0)$. The whole-space channels are bounded, and their normals satisfy

$$(G.20) \quad \|E_{\tau,v}^{K,\text{pr}} - E_{0,v}^K\| \leq C\tau K/h, \quad K \geq 1.$$

The squared whole-space slice norm is at most CK . Spectator integration and realification have the same conclusions. All fields and constants use only finitely many profile constants at each order.

Proof. Write $P = P_G$. The leading transport gives $M_G(x, dS_G)b_0 = 0$ and $l_G^T P b_0 = 0$, so the algebraic correction satisfies

$$iM_G(x, dS_G)b_1 + P b_0 = 0.$$

The finite list consists of $b_0, b_1, P b_1$, together with $\nu_{b_j}, \mathcal{V}_G(b_j)$ and their first physical derivatives, $j = 0, 1$. It uses the fixed-gap jet formulas and the already constructed leading transport. Lemma B.6 gives one common one-high reserve D_{pr} . Corollary B.4, after at most r_{par} parameter derivatives, gives

$$(G.21) \quad \|D_x^r D_{z,\zeta'}^k c\|_\infty \leq C_{r,k} L^r, \quad k \leq r_{\text{par}},$$

for every member c . The same argument gives the phase and frame bounds and uniform joint low norms through $M+2$.

Put $\text{Gram}_G = (Q_i : Q_j)$. The two algebraic cancellations and exact half-strain formula give

$$(G.22) \quad P v_\tau = \tau^3 e^{iS_G/\tau} P b_1, \quad \mathcal{E}_\tau = 2\tau^3 \sum_{i,j} Q_i (\text{Gram}_G^{-1})_{ij} (P b_1)_j.$$

The fixed frame and inverse Gram have the same grades, proving (G.14).

Substitute $b = b_0 + \tau b_1$ in (G.8) and differentiate once. The leading derivative amplitude is $A_0(b_0)$; every other term has a factor τ . This proves the amplitude bounds. Their product estimate, with $A_0(b_0)^* A_0(b_0) = |\nu_{b_0}|^2 dS_G^2$, proves (G.16) and nonvanishing.

The exact differentiated fields give the displayed channel. Its phase and the full and slice inverse charts obey Setup G.3 on the larger supports. The phase and amplitude bounds through M , and inverse-chart bounds through $M-2$, are uniform in L , including first ϵ derivatives; the primary reserve covers the two extra phase derivatives in the quotient. On this same phase, replacing A_τ^{pr} by $A_0(b_0)$ changes the C^M amplitude by $O(\tau)$, hence its channel and normal by $C\tau$. The remaining homotopy from $\epsilon = \tau K$ to zero costs $C\epsilon/h$ at every $K \geq 1$. Since $\tau \leq \tau K/h$, this proves the normal comparison. The same setup gives the slice and spectator bounds and the real normal $4 \text{Re } E_{\tau,v}^{K,\text{pr}}$ on real outputs. $\square$

G.5. Separated finite-band sampling.

Theorem G.6 (Separated lattice sampling). *Use finitely many buffered three-parameter charts $z = (s, v)$ and the finite primary family of Theorem G.5, with fixed smooth nonnegative compact weights $\psi(z)$ and one amplitude scale L for all radial labels. Its fixed low topology and parameter orders give the uniform C^{M+2} amplitude and inverse-chart bounds used below, including the ϵ derivatives of the*

leading homotopy. All spatial, angular and annular measurement and input cutoffs are fixed inside those larger charts; the finite collection covers the selected angular support. Frequencies below one are omitted. Put $q_{\text{quad}} = 2n - 1$ and $q_{\text{col}} = q_{\text{quad}} + 1$; use q_{col}^3 radial colors per chart, fixed before frequency and accuracy.

For $\Lambda \geq 2$, $\lambda \geq 1$, with Λ/λ below the phase buffer, fix finitely many positive radial labels R_c bounded above and below and set $\tau_c = (R_c\lambda)^{-1}$. Use one smooth dyadic square partition of $1 \leq |\eta| \leq C\Lambda$, with scales $K \geq 1$, and the same angular cutoffs for every color. Define actual fields by (G.11)–(G.12), with $\tau = \tau_c$, keeping the entire phase, and define backgrounds directly at $\eta = 0$.

Let $T_{c,z,K}$ be their actual background-modulation Hermitian contraction with Q_i , or any selected smooth tensor contraction satisfying the same fixed low amplitude bounds, with input half-derivative $|\eta|^{1/2}$ and the fixed localizations. Choose the fixed Fourier scalar so that dilation $\eta = K\theta$ gives prefactor $(K/(2\pi))^{n/2}$ and phase $K\phi_{\tau_c K}$. Let

$$T_{c,z} = (T_{c,z,K})K$$

be the row on the Hilbert direct sum of entire annular input spaces. Its output is the entire

$$\mathcal{H} = L^2(\mathbb{R}^n; \mathbb{C}^n),$$

with compact measurement. Arbitrary row data correspond to finite sums of actual modulated ambient fields.

Choose $\kappa \geq 0$ smooth of integral one, supported in $(-1/4, 1/4)$, and put $w = \kappa * \mathbf{1}_{[-1/2, 1/2]}^{*q_{\text{quad}}}$. For $0 < h \leq h_0 \leq 1$, define

$$c_q = hq, \quad \rho_q(z) = \prod_{\ell=1}^3 w(z_\ell/h - q_\ell), \quad N_s = h^{-3}.$$

The fixed h_0 respects the enlarged block supports and chart buffers. For each block meeting $\text{supp } \psi$, independently draw U_q uniformly in $[0, h)^3$ and, for $m \in \mathbb{Z}^3$, set

$$Z_{q,m} = c_q + hm + U_q, \quad \mu_{q,m} = h^3(\psi\rho_q)(Z_{q,m}).$$

Omit zero weights; each block has at most $C_{q_{\text{quad}}}$ nodes. Color them by $q \bmod q_{\text{col}}$ and assign each color its fixed radial label. Distinct nodes of one color are separated by ch . There are at most CN_s samples, their weights are at most C/N_s , and their total mass is bounded. Number the literal samples by j and set

$$m_c = \sum_{q: q \bmod q_{\text{col}} = c} \rho_q,$$

and let $\mathcal{D}_\Lambda$ be the fixed dyadic scale set. For each prefix define, with finite chart sums understood,

$$\begin{aligned} T_{c,z}^{\leq K} &= (T_{c,z,\kappa})_{\kappa \in \mathcal{D}_\Lambda, \kappa \leq K}, \\ E_{\text{sample}}^{\leq K} &= \sum_j \mu_j T_{c(j), Z_j}^{\leq K} (T_{c(j), Z_j}^{\leq K})^*, \\ E_h^{\leq K} &= \sum_c \int \psi m_c T_{c,z}^{\leq K} (T_{c,z}^{\leq K})^* dz. \end{aligned} \tag{G.23}$$

Write E_{sample}, E_h for the largest prefix. Each normal is exactly the sum of its annular normals, since its input is the same Hilbert direct sum restricted to that prefix. The actual wave maps, radial labels, colors, weights and samples are unchanged.

The following whole-space bounds hold:

$$\begin{aligned} \|T_{c,z}\|^2 &\leq C\Lambda, & \|T_{c,z}\|_{\text{HS}}^2 &\leq C\Lambda^n, \\ \|E_h\| &\leq C, & \text{tr } E_h &\leq C\Lambda^n. \end{aligned} \tag{G.24}$$

For fixed $\epsilon > 0$, set $h = \Lambda^{-(1+\epsilon)/3}$. With probability tending to one,

$$(G.25) \quad \max_{K \in \mathcal{D}_\Lambda} \|E_{\text{sample}}^{\leq K} - E_h^{\leq K}\| \leq \Lambda^{-\epsilon/4}.$$

The ceiling and complete dyadic partition are fixed before sampling; later choices of prefixes do not change the event or its constants. Both input quadratures remain physical real data; $2\text{Re}T$ has normal $4\text{Re}E$, with the same conclusions up to fixed constants.

Define

$$H_{c,z} = \text{Re}[(dW_{\tau_c,z}^{\text{pr,bg}})^* dW_{\tau_c,z}^{\text{pr,bg}}], \quad H_{\text{transport},z} = |\nu_{b_0}|^2 dS(x, \zeta(z))^2.$$

Localize tensor norms by a fixed compact spatial cutoff and put

$$H_{\text{col}} = \sum_c \int \psi m_c H_{c,z} dz.$$

Since $q_{\text{quad}} < r_{\text{par}}$, the stated low amplitude budget already covers its carrier derivatives. Every realization satisfies

$$(G.26) \quad \left\| \sum_j \mu_j H_{c(j),Z_j} - H_{\text{col}} \right\|_{L^2} \leq Ch^{q_{\text{quad}}}.$$

The primary amplitude comparison also gives $\|H_{\text{col}} - \int \psi H_{\text{transport},z} dz\|_{L^2} \leq C\lambda^{-1}$. For each integer $r \geq 0$ it also satisfies

$$(G.27) \quad \left\| \sum_j \mu_j H_{c(j),Z_j} - H_{\text{col}} \right\|_{C^r} \leq C_r L^r.$$

The primary family and common band ceiling are fixed before sampling. Only the all-prefix normal estimate requires a probability event; the background estimate and same-color spacing hold for every realization.

Let $E_{\text{transport,band}}$ be the continuous uncolored normal on the same bands and cutoffs, with $\phi_{\tau_c K}$ replaced by ϕ_0 and $A_{\tau_c}^{\text{pr}}$ by $A_0(b_0)$. Then

$$(G.28) \quad \|E_h - E_{\text{transport,band}}\| \leq C\lambda^{-1}\Lambda/h.$$

Thus one separated sample set approximates that normal on the entire output space within the right side plus $\Lambda^{-\epsilon/4}$. A separately established lower bound on a specified output class transfers by this perturbation estimate.

The conclusions apply to any same-phase ambient family satisfying (G.13), the fixed full and slice seminorms, and the stated background jets, with the same leading tensor. In particular the proof uses no formula for its linear strain.

Finally,

$$(G.29) \quad \mathcal{P} = \sqrt{2} \sum_j \sqrt{\mu_j} \text{Re} W_j^{\text{bg}}$$

is an actual field in $\mathbb{R}^N$, whose same-carrier Hermitian metric contribution is $\sum_j \mu_j H_j$. Doubled same-carrier oscillations and all cross-carrier products remain as unremoved metric terms.

Proof. Operator bounds and separated blocks. Theorem G.5 and Setup G.3 give, on entire chart spaces,

$$\|T_{c,z,K}\|^2 \leq CK, \quad \|T_{c,z,K}\|_{\text{HS}}^2 \leq CK^n.$$

The second inequality is the squared-kernel integral after dilation: the prefactor is $(K/(2\pi))^{n/2}$ and the amplitude and its physical and normalized-frequency supports are uniformly bounded. Since $T_{c,z}T_{c,z}^* = \sum_K T_{c,z,K}T_{c,z,K}^*$, geometric dyadic sums give the first two bounds in (G.24). First discard the masks by positivity. By (D.15) and the exact change (C.4), each unmasked band normal has a full two-position amplitude supported in $cK \leq |\xi| \leq CK$, with uniform physical derivatives

through $2(n+1)$; the fixed smooth weight ψ is part of this amplitude. At any ξ only a uniformly bounded number of bands contribute. Lemma C.2 bounds their sum, giving $\|E_h\| \leq C$ after the fixed spectator integral and color sum. Tonelli gives

$$\mathrm{tr} E_h = \sum_c \int \psi m_c \|T_{c,z}\|_{\mathrm{HS}}^2 dz \leq C\Lambda^n.$$

Convolution with κ makes w smooth and nonnegative. Fubini, the unit-box periodization and the unit masses give $\sum_{k \in \mathbb{Z}} w(u-k) = 1$. The q_{quad} box factors give Fourier zeros of order at least q_{quad} at every nonzero $2\pi k$. The support length is at most $q_{\mathrm{quad}} + 1/2$. There are $O(h^{-3})$ blocks, each with a fixed number of nodes at spacing h and total weight Ch^3 . Distinct blocks of one color have gap at least $h/2$. This proves separation, count and mass bounds. Uniform lattice translation has mean Lebesgue measure, so (G.23) is its exact expectation, with $0 \leq m_c \leq 1$ and $\sum_c m_c = 1$.

Concentration on the entire output. Use the independent positive blocks $X_q = \sum_m \mu_{q,m} T_{q,m} T_{q,m}^*$. Their total weight is at most Ch^3 . Put $Y_q = X_q - \mathbb{E}X_q$ and $R = C\Lambda h^3$. Then

$$0 \leq X_q \leq RI, \quad \|Y_q\| \leq R, \quad V = \sum_q \mathbb{E}Y_q^2 \leq RE_h,$$

$$\|V\| \leq v := CR, \quad \mathrm{tr} V \leq CRA^n.$$

These bounds hold for every prefix with the same upper values.

Equations (3.6)–(3.9) of [M17], with the positive variance upper bound v , give

$$\mathbb{P}\left(\left\|\sum_q Y_q\right\| > t\right) \leq \frac{14 \mathrm{tr} V}{v} \exp\left(-\frac{t^2}{2(v + Rt/3)}\right), \quad t^2 \geq v + Rt/3.$$

To justify this bound on the whole output, apply its finite-dimensional form to $P_\ell Y_q P_\ell$ for finite-rank orthogonal $P_\ell \uparrow I$. Their variance is at most $P_\ell V P_\ell$, so the same v and trace bound apply. Each X_q and its expectation are trace class, so Y_q is compact; the finite centered sum therefore converges in norm under these compressions, and Fatou's lemma passes the bound to the full sum. This also covers zero variance.

For $t = \Lambda^{-\epsilon/4}$ and $h^{-3} = \Lambda^{1+\epsilon}$, the threshold holds at large Λ . The bound and a union over the fixed prefixes give failure probability at most

$$C\Lambda^{n+1} \exp[-c\Lambda^{\epsilon/2}],$$

proving (G.25) for the unchanged channels and samples. Equation (G.5) gives the real-channel assertion.

Background quadrature. For $W = w \otimes w \otimes w$, every nonzero integer lattice Fourier coefficient of pW vanishes when $\deg p < q_{\mathrm{quad}}$. Indeed it differentiates the product $\prod_\ell \hat{w}$ fewer than q_{quad} times; a nonzero coordinate keeps its zero. Poisson summation therefore gives, for every shift u ,

$$\sum_{m \in \mathbb{Z}^3} W(m+u)p(m+u) = \int_{\mathbb{R}^3} W(z)p(z) dz.$$

The Hermitian phase cancels. Taylor-expand the L^2 -valued function $\psi H_{c,z}$ at c_q through degree $q_{\mathrm{quad}} - 1$. The displayed identity cancels its polynomial part exactly in block q . The remainder is $Ch^{q_{\mathrm{quad}}}$ on that block; its sampled and integral masses are Ch^3 . Summing $O(h^{-3})$ blocks proves (G.26), uniformly in all shifts. Bounded total mass and the amplitude profiles give the rough bound. Replacing each actual Hermitian tensor by its leading one costs $C\lambda^{-1}$ in the prescribed fixed norm.

The carrier masks satisfy $\sum_c m_c = 1$ and $\|Dm_c\|_\infty \leq C/h$. Sum (G.20) on all dyadic bands. Since

$$\sum_{K \leq C\Lambda} K/h \leq C\Lambda/h,$$

this proves (G.28). At transport order phase, amplitude and input cutoffs are independent of the radial label, so the masks sum to the uncolored normal. The triangle inequality transfers any already known lower bound on its specified output class.

Finally, for a complex derivative matrix D ,

$$2(\operatorname{Re} D)^T(\operatorname{Re} D) = \operatorname{Re}(D^* D) + \operatorname{Re}(D^T D).$$

Applied to (G.29), this identifies its same-carrier Hermitian term as $\sum_j \mu_j H_j$. The second term is the doubled oscillation, and every distinct pair gives a cross term; these metric contributions remain separate. $\square$

Remark G.7. At fixed bandwidth the sampled normals are compact on the entire output space. Their norm approximation therefore transfers coercivity only on an output class for which a lower bound has already been established.

G.6. Raw inputs and primary fields.

Theorem G.8 (Raw primary fields). *Choose a finite family from Theorem G.5 on its larger buffered boxes, and a compact inner box X . Write $W_j, S_j, \zeta_j, \tau_j$ for its maps, phases, carriers and inverse frequencies. Fix a band ceiling Λ whose shifted covectors stay in the original charts. The exact representation is*

$$(G.30) \quad \begin{aligned} e^{-iS_j(x, \zeta_j)/\tau_j} dW_j(f) &= \int e^{i\phi_j(x, \eta)} a_j(x, \eta) \hat{f}(\eta) d\eta, \\ \phi_j(x, \eta) &= \frac{S_j(x, \zeta_j + \tau_j \eta) - S_j(x, \zeta_j)}{\tau_j}. \end{aligned}$$

Equations (G.13) and (G.15) give

$$(G.31) \quad |D_x^r a_j| \leq C_r L^r, \quad |D_x^r \partial_\zeta S_j| \leq C_r L^{r-1} \quad (r \geq 1).$$

The amplitudes and the required low slice seminorms are uniformly bounded. Constants are independent of sample count and weights.

For $1 \leq K \leq \Lambda$, let X_K be the real Hilbert space underlying the finite direct sum of complex initial Fourier L^2 spaces on dyadic annuli up to CK , with fixed bounded-overlap cutoffs β_ℓ . Put

$$\widehat{\mathcal{B}}_K v(\eta) = \sum_\ell |\eta|^{1/2} \beta_\ell(\eta) v_\ell(\eta).$$

This is the Fourier input before any spatial initial cutoff, and for every integer $m \geq 0$ it satisfies

$$(G.32) \quad \|\mathcal{B}_K v\|_{C^m} \leq C_m K^{m+n/2} \|v\|_{X_K}, \quad \|\widehat{\mathcal{B}}_K v\|_1 \leq CK^{n/2} \|v\|_{X_K}.$$

The same bounds hold after fixed detector and high-pass masks, with uniform rescaled seminorms, are included in each β_ℓ . These masks act on the annular datum; the separate zero atom is unchanged. The corresponding raw map may depend on j . Choose a fixed smooth χ_{slice} inside the larger wave validity region, equal to one near X and the measurement support. Then

$$(G.33) \quad \mathcal{G}_j^K v = \chi_{\text{slice}} e^{-iS_j/\tau_j} dW_j(\mathcal{B}_K v), \quad \|\mathcal{G}_j^K v\|_{H^s(\mathbb{R}^n; \mathbb{C}^{N \times n})} \leq C_s K^{s+1/2} \|v\|_{X_K}, \quad K \geq L.$$

At order zero the bound holds for all $K \geq 1$. This cutoff is used only in the norm; the actual fields are unchanged. At any prescribed finite order the raw slice bound holds for $K \geq 1$ under bounded coefficient and initial-data jets through the fixed shifted orders. For every f with $\operatorname{supp} \hat{f} \subset B_{CK}$, the physical bound is

$$(G.34) \quad \|e^{-iS_j/\tau_j} dW_j(f)\|_{C^m(\operatorname{supp} \chi_{\text{slice}})} \leq C_m (K + L)^m \|\hat{f}\|_1.$$

These are the same smooth integral fields for arbitrary $v \in X_K$.

Proof. On an initial annulus of radius κ , dimension $n - 1$, Fourier Cauchy–Schwarz with the input half-derivative gives $\|D^m \mathcal{B}_\kappa v\|_\infty \leq C_m \kappa^{m+n/2} \|v\|_2$. Cauchy–Schwarz over the annuli and their geometric squared-power sum prove (G.32).

Use the componentwise slice estimate of Setup G.3 on the entire larger support. On an initial annulus $\kappa \leq CK$, a term with q phase factors after s physical derivatives has prefactor κ^q . Its primitive excess through the additional M slice derivatives is at most $s - q + M + D_{\text{pr}}$. The fixed reserve $D_{\text{pr}} + r_{\text{par}} + 2$ therefore gives the C^M bound $C_s \kappa^q L^{s-q} \leq C_s K^s$. The squared slice bound is $C_s K^{2s} \kappa$; Cauchy–Schwarz and $\sum_{\kappa \leq CK} \kappa \leq CK$ prove (G.33). At order zero only uniform low slice seminorms enter; bounded jets at prescribed finite orders allow $L = 1$.

The quotient formula (G.18) gives $|D_x^r \phi_j| \leq C_r K L^{r-1}$ for $r \geq 1$. A term with q phase derivatives in an order- m product costs $K^q L^{m-q} \leq (K+L)^m$. Integrating against $|\hat{f}|$ proves (G.34). Compactly Fourier-supported L^2 data have all polynomial moments in L^1 , proving smoothness at every order. $\square$

APPENDIX H. AMBIENT RESPONSE TO MODULATED WAVES

H.1. The fixed response and its inputs. We select the regular construction of Setup F.2 on an entire primary patch, using the full graphs and literal weighted coordinate inverse of Appendix E.

Theorem H.1 (A fixed real response on an entire primary patch). *Use one compact pointed primary patch of Theorem G.4, with section $t = 0$, phase ϕ_G , a fixed compact neighborhood Z of every allowed shifted carrier, and the larger convex modulation chart. Write $\mathbf{h}_G = p_G |\xi|^{1-n}$. The simple-root branch gives*

$$\partial_t \phi_G = H_G(t, y, \partial_y \phi_G), \quad \phi_G(0, y, \zeta) = y \cdot \zeta.$$

Choose symmetric regular sectors $\Gamma_{\text{in}} \Subset \Gamma_{\text{out}}$ inside the common energy chart of this branch and its antipode, containing the entire positive family and its antipode with compact buffers on the larger physical box. All root and projection gaps are uniform on the fixed low neighborhood.

The real regular family of Setup F.2 can use just these two conjugate charts. The positive chart has section-energy phase $\Phi_G(x, e, \eta')$ satisfying

$$\begin{aligned} \mathbf{h}_G(x, d_x \Phi_G(x, e, \eta')) &= e, & \Phi_G(0, y, e, \eta') &= y \cdot \eta', \\ \Phi_G(x, 0, \zeta) &= \phi_G(x, \zeta) & (\zeta \in Z). \end{aligned}$$

Both canonical projections have uniform inverse gaps. The literal coordinate inverse is the weighted adjoint of (E.21), with its original source and target cutoffs; the central inverse carrier is $(0, \zeta)$. For either primary sign, the opposite source piece and $A_\pm - I$ have a fixed normalized source gap, including all shoulders. The full low-frequency part and source-cut errors remain in the selected residual.

This family obeys Theorem F.3: its fixed complete residual R_G and cutoffs give $\mathcal{T} = (I + \chi_2 R_G \chi_3)^{-1}$ on the entire L^2 space. The signed partners and literal adjoints use the full identities in Setup F.2, including both integral weights and endpoint buffers.

For finitely many primary patches, choose common finite-low bounds, buffer minima and one response ball U_{resp} inside all patch cores. All source and observation plateaus are fixed there before frequencies, samples and requested accuracies. Every Fourier atom or measure datum in a patch uses that patch's same selected map.

Proof. On the compact primary family, $\partial_{\xi_t} \mathbf{h}_G = |\xi|^{1-n} \partial_{\xi_t} p_G$ has a uniform nonzero gap. Its transverse covector is also bounded away from zero: if $\xi' = 0$, Euler's identity would give $0 = \mathbf{h}_G = \xi_t \partial_{\xi_t} \mathbf{h}_G$. Lemma D.7 therefore gives the unique homogeneous root and phase

$$(H.1) \quad \mathbf{h}_G(x, h_G(x, e, \eta'), \eta') = e, \quad |e| \leq \epsilon_e |\eta'|,$$

$$(H.2) \quad \partial_t \Phi_G = h_G(t, y, e, \partial_y \Phi_G), \quad \Phi_G(0, y, e, \eta') = y \cdot \eta',$$

with common buffered projection gaps and joint one-high bounds. At $e = 0$ this is the primary Cauchy problem; uniqueness proves phase equality on the whole shifted-carrier compact.

Shrink the physical strip and coefficient neighborhood once. Choose a smooth angular a_+ , independent of the source base, equal to one near that positive family and supported in its common energy chart, with support disjoint from its antipode. Put

$$(H.3) \quad a_-(\omega) = a_+(-\omega), \quad a_\Gamma = a_+ + a_-.$$

The same $T, \tilde{T}, V, J$ handles the whole positive support, including $e \neq 0$; its exact partner handles the negative support. Use the common high-frequency cutoff and a right source plateau near $\text{supp } \chi_3$, leaving the low piece in the complete residual. Both inactive differences have the stated full source gap.

Choose the literal coordinate inverse's source plateau to contain the full canonical preimage of the inner target. At $(e, \eta') = (0, \zeta)$, phase equality and Euler's identity give

$$d_x \Phi_G = d_x \phi_G, \quad \Phi_G = \zeta \cdot \Phi_{G, \eta'} = z' \cdot \zeta, \quad z = \Phi_{G, \eta}.$$

The mixed inverse gap makes this the unique central inverse carrier. Proposition E.8 applied to this phase and partition gives the actual weighted inverse, its source-cut error, signed factors and complete residual, with all original cutoffs and integral buffers. Theorem F.3 gives $\mathcal{T}$. Finite buffer minima and a ball inside every patch core give the common response choices. $\square$

The operator and all carrier-compatible geometric choices are now fixed. Theorem H.3 below proves the quantitative packet estimates for this selection from its literal kernels and the separate profile hypotheses.

Setup H.2 (The proved response and its profile). Use the immersion and reduced system of Definition G.2, and the per-patch real response just fixed on a buffered chart X . The metric, normal and Gram gaps are positive. Write

$$\mathcal{K} = \chi_2 R_G \chi_3, \quad \mathcal{T} = (I + \mathcal{K})^{-1}, \quad V_G = \mathcal{V}_G.$$

Fix a source cutoff $\chi \in C_c^\infty(X)$ inside $\text{Int}\{\chi_2 = 1\}$ and the primary phase chart. Source and observation supports are those of Setup A.1; the primary identity region is U_{LAG} .

The additional quantitative hypothesis is the actual profile

$$(H.4) \quad \|a_G\|_{C^v} + \|G\|_{C^v} \leq D_v T^{(v-R_*)+}, \quad v \geq 0, \quad T \geq 1.$$

Here a_G contains the principal and connection coefficients of P_G ; it does not determine G . Choose $R_* > c_{\text{lin}} + 2$, also above the fixed geometric jet reserves, before observation and tail orders. The reconstruction coefficients, including one derivative of V_G , are smooth functions of a fixed jet $j^{q_{\text{geo}}} G$ with the one-high bounds of Lemma B.3.

Theorem E.6 and Proposition E.8 give the selected full graphs, weighted coordinate inverses, diagonal and source factors, and exact inner-cutoff products uniform L^2 bounds and fixed-shift Sobolev estimates. Supported-adjoint duality gives negative integer orders; the signed integrals and their literal adjoints preserve these orders with the original weights and endpoint buffers. The inverse and solver obey (F.9) and (F.6). Together with the profile, these give polynomial costs at every fixed order used below. Ambient differentiation is counted separately.

With the original cutoffs, the complex-linear extension of the real solver in (E.37) is

$$(H.5) \quad S_{G, \Gamma}^{\mathbb{C}} = \frac{\chi_{\text{out}}}{i} \sum_{\pm} \tilde{T}_{\pm} \mathcal{J}_{\pm} V_{\pm} A_{\pm}.$$

Only the transport graph T , with invertible matrix amplitude U , is inverted by V . The complete antipodal sum is real; all low and cutoff errors belong to R_G .

Lemma 7.1 and Proposition 7.2 verify this profile for the iteration.

H.2. Carrier transfer through the fixed factor family. The actual profiles in Setup H.2 give the common scale $L_* = T$ by Theorem F.4.

Theorem H.3 (Carrier transfer through the fixed factor family). *Use that setup and its whole-patch choice. On every active chart, with $\zeta_c = (0, \zeta)$, the section-energy phase satisfies*

$$(H.6) \quad S(x, \zeta_c) = \phi_G(x, \zeta).$$

For fixed $0 < \nu < 1$, suppose

$$L_* \leq K, \quad 2 \leq K \leq \lambda^{1-\nu}.$$

Let A be $T, \tilde{T}, V$, a full graph error from this active chart, or any literal adjoint, with order at most zero and its actual cutoffs. The forward carriers are $E_{\text{in}} = e^{i\lambda z \cdot \zeta_c}$ and $E_{\text{out}} = e^{i\lambda \phi_G(x)}$; adjoint-phase graphs use the reverse pair. If $\|f\|_{H^p} \leq C_p A_0 K^p$ for $p \geq 0$, then

$$(H.7) \quad \|E_{\text{out}}^{-1} A E_{\text{in}} f\|_{H^s} \leq C_s A_0 K^s.$$

The same profile holds through every fixed finite sum or chain with matching carriers, including physical multipliers, the buffered signed integrals and their literal adjoints. It holds for the complete residual and solver chains after the source cutoff. Every inactive source component and $A_+ - I$ applied to the original supported carrier input has norm $C_{s,Q} A_0 \lambda^{-Q}$ in physical and demodulated H^s . All statements use full supports and literal Hilbert measures; constants may use finitely many profile constants, while factors, cutoffs and their orders are fixed.

These conclusions hold jointly on any fixed compact smooth parameter box c , with $\zeta_c = (0, \zeta(c))$ in the full shifted-covector chart. The frequency and all actual operators are fixed when differentiating c . If $\|\partial_c^\alpha f_c\|_{H^p} \leq C_{p,\alpha} A_0 K^{p+|\alpha|}$, the left side of (H.7), after ∂_c^α , is at most $C_{s,\alpha} A_0 K^{s+|\alpha|}$. Parameter derivatives of the complete complementary errors have the same arbitrary precision.

Proof. Put $L = L_*$. Phase equality follows from the common Cauchy problem. The full symbols have the bounds (F.14). Each order-at-most-zero factor has uniform L^2 norm by the cited calculus; its required low seminorms lie in the fixed C^{R_*} bound, independently of K, λ . We first estimate each near-carrier graph on the whole envelope Sobolev space and then bound its complete complement.

Derivative transfer in both graph directions. Set $\eta = \lambda \zeta_c + \delta$ and $\Delta S = S(x, \lambda \zeta_c + \delta) - S(x, \lambda \zeta_c)$. In the proof select $|\delta| \leq c\lambda$ inside the larger convex carrier chart. For one small fixed $c > 0$,

$$B(x, \delta) = \int_0^1 S_{x\xi}(x, \lambda \zeta_c + t\delta) dt, \quad D_x \Delta S = B\delta$$

has a uniform inverse: B is uniformly close to $S_{x\xi}(x, \lambda \zeta_c)$ on every physical support. All other supports belong to the complete complement below. Homogeneity gives the parameter identity

$$\partial_c \Delta S = C_c \delta, \quad C_c = (\partial_c \zeta_c)^T \int_0^1 S_{\xi\xi}(x, \zeta_c + t\delta/\lambda) dt.$$

These coefficients have normalized order zero and the joint grades of B .

The exact demodulated forward phase is $\Delta S - z \cdot \delta$. An output derivative produces $B\delta$; integration in z transfers δ to an input derivative, or differentiates the full amplitude. For the reverse phase $z \cdot \delta - \Delta S$, use $\delta = B^{-1} D_x \Delta S$ and integrate in x . The parameter term $C_c \delta$ transfers in the same way; $\partial_c a(x, z, \lambda \zeta_c + \delta)$ contributes only normalized covector derivatives of order zero. The actual compact weights remove boundary terms. The induction (E.20) now applies with B in place of $S_{x\xi}$. Each modified amplitude has primitive derivative excess $s - |\beta|$, including the second physical position. Its low L^2 seminorm costs the fixed c_{lin} extra orders. The actual profile and $R_* > c_{\text{lin}} + 2$ bound it by $C_s L^{s-|\beta|}$. Covector derivatives are normalized on $\eta \asymp \lambda$; the selector costs λ^{-1} per derivative, so every modified amplitude still has order zero. The actual L^2 calculus, in both directions, gives

$$(H.8) \quad \|A_\lambda f\|_{H^s} \leq C_s \sum_{r=0}^s L^{s-r} \|f\|_{H^r}.$$

This includes the weighted amplitude $j_S U^{-1}$ of the literal coordinate inverse. Since $L \leq K$, it proves the stated profile for the near-carrier graph, without projecting the datum. Counting physical and parameter derivatives together gives the joint assertion: each derivative lands on the source or adds one graded amplitude derivative. With finitely many bounded jets, take $L = 1$, $K \geq 1$ at the prescribed joint orders.

For the complete complement put $K_{\text{far}} = c\lambda^{1-\nu/2}$ and split the source envelope there. On the lower part, the full source-gradient gap outside the preceding carrier neighborhood is $c(\lambda + |\eta|)$: use z for a forward graph and x for its adjoint phase. Source integration and dyadic summation, as in Lemma F.9, give the bound

$$C_{s,q} A_0 \lambda^{c_s} K_{\text{far}}^{n/2} (L/\lambda)^q, \quad L/\lambda \leq \lambda^{-\nu}.$$

Here c_s is fixed before q . Every radial or angular shoulder and bounded-frequency piece has this same gap. The upper source has

$$\|f_{>K_{\text{far}}}\|_{H^t} \leq C_{t,p} A_0 K^p K_{\text{far}}^{t-p} \leq C_{t,p} A_0 \lambda^{t-\nu p/2}.$$

Increasing q, p pays every fixed tame, carrier and demodulation cost, giving any prescribed precision in λ for the full complement. Proposition C.7 gives the same bounds for the pure Fourier inactive source factors and $A_+ - I$. A fixed cutoff transfers torus profiles to the physical chart by Leibniz's rule, so this argument also covers torus data.

Physical multipliers preserve the profile since their jets have grade $L^r \leq K^r$. The zero-time canonical carrier commutes exactly with the signed integrals. Their direct Sobolev estimates in Proposition C.10 preserve the same profile, including the literal adjoints with exchanged weights and upper input buffers. Thus every near-carrier factor preserves the profile, and every complement has arbitrary λ precision. Propagate each complement through the remaining fixed factors using their polynomial Sobolev costs, choosing its precision higher beforehand. This proves the assertion for fixed chains without successive envelope projections. The complete signed identities preserve all original cutoff orders. A fixed number of parameter derivatives in the original complementary expressions adds only a fixed power of λ ; increasing the internal precision gives their stated joint bounds.

For the actual solver and residual, first apply their literal source operators. Every input is first multiplied by χ_3 or is supported in $\text{supp } \chi_2$, where the right physical source cutoff is one. Thus A_+ is the identity up to the preceding λ -small error and every other source term has the same precision. In the positive residual keep the exact ordered terms $(TV - I)A_+$ and $(L_G \tilde{T} - T D_t) J V A_+$ from the differential identity and the buffered $D_t J$ identity. Estimate TV and I separately for carrier transfer; their difference is never set to zero. Its gain-one bound is the independent complete-normal estimate of Appendix E. The remaining measured error is a full order-1 graph or the exact source-cut graph product. Thus the graph and integral estimates apply to these complete chains, with every original cutoff in its original order. $\square$

H.3. Full ambient strain reconstruction.

Proposition H.4 (Exact full-tensor identity). *Let $\text{Gram}_G = (Q_i : Q_j)$ and define, for a symmetric tensor h supported where $\chi_2 = 1$,*

$$(H.9) \quad e_i = \frac{1}{2} Q_i : h, \quad z = \mathcal{T}(\chi_2 e), \quad u = S_{G,\Gamma} z, \quad \mathcal{B}_G h = V_G(u) - \frac{1}{2} \mathbf{N}_G h.$$

This is one real linear map, extended complex linearly when needed. On U_{core} ,

$$(H.10) \quad D\gamma_G(\mathcal{B}_G h) - h = 2 \sum_{i,j} Q_i (\text{Gram}_G^{-1})_{ij} (A_\Gamma - I) z_j.$$

All complete residual components remain inside this same z .

Proof. The map is exactly the normal completion in (2.5). Substitute $P_G u = e + (A_\Gamma - I)z$ from (F.10) into (2.6) to obtain the stated identity. The real solver and normal inverse give real linearity; complex linear extension preserves the formula. $\square$

For a reduced source e satisfying Theorem F.8, (H.10) converts its angular error into the full tensor error. This uses that theorem's actual coefficient profile and protection of the entire product $\text{supp } \chi_2 \times V'_0$. The fixed frame multipliers cost only a polynomial at each observation order, absorbed by a higher internal accuracy.

H.4. Graded sources and the full ambient response.

Proposition H.5 (Displacement and derivative of the same graded response). *Keep the same fixed real ambient map, source cutoff and buffered observation cutoff χ_{core} . Let $\lambda, K \geq 2$ obey*

$$(H.11) \quad L_* \leq K \leq \lambda^{1-\nu}, \quad 0 < \nu < 1.$$

Let a be a smooth complex symmetric-tensor envelope on the fixed auxiliary torus or on $\mathbb{R}^n$, with

$$(H.12) \quad \|a\|_{H^p} \leq C_p A K^p, \quad p \geq 0,$$

where $A \geq 0$. Put $h = \chi e^{i\lambda\phi_G} a$, $Y = \mathcal{B}_G^{\mathbb{C}} h$, and

$$(H.13) \quad U = e^{-i\lambda\phi_G} \chi_{\text{core}} Y, \quad D = e^{-i\lambda\phi_G} \chi_{\text{core}} dY.$$

The explicit bounds are

$$(H.14) \quad \|U\|_{H^s} \leq C_s A \lambda K^s,$$

$$(H.15) \quad \|D\|_{H^s} \leq C_s A \lambda^2 K^s,$$

for every fixed s . On each fixed compact inner identity core,

$$(H.16) \quad \|D\gamma_G Y - h\|_{H^s} \leq C_{s,Q} A \lambda^{-Q}.$$

All orders use the same field and map. In particular exact-band data $\text{supp } \hat{a} \subset B_K$ satisfy these conclusions with $A = \|a\|_2$. Physical derivatives keep their carrier factors; D is the demodulated physical derivative. For the parameter families of Theorem H.3, assume $\|\partial_c^\alpha a_c\|_{H^p} \leq C_{p,\alpha} A K^{p+|\alpha|}$. The same envelope bounds hold after ∂_c^α , with an additional $K^{|\alpha|}$ on the right.

Proof. The profile in Setup H.2, with $R_* > q_{\text{geo}}$, gives by Corollary B.4

$$(H.17) \quad \|c(G)\|_{C^r} + \|d\phi_G\|_{C^r} \leq C_r L_*^r.$$

Leibniz's rule and $L_* \leq K$ give $f = (Q_i : a/2)_i$ the profile $\|f\|_{H^p} \leq C_p A K^p$. Thus $z = \mathcal{T}(\chi e^{i\lambda\phi_G} f)$ and $u = S_{G,\Gamma} z$.

Theorem H.3 applies directly to this graded f . For $h_0 = \chi e^{i\lambda\phi_G} f$, every fixed $\mathcal{K}^j h_0$ and $S_{G,\Gamma} \mathcal{K}^j h_0$ has demodulated profile $C_{j,s} A K^s$. Append the angular source factor to each finite chain. Its input is supported in $\text{supp } \chi_2$, inside the original right source plateau, so

$$(A_\Gamma - I) \mathcal{K}^j h_0 = O_{j,s,Q}(A \lambda^{-Q}) \quad \text{in } H^s.$$

Only for the original source, the same full source gap and high-envelope split give $\|h_0\|_{H^{-N}} \leq C_N A \lambda^{-N}$. Apply (F.27) with $\rho = \lambda$ and the operators $e^{-i\lambda\phi_G} S_{G,\Gamma}$ and $A_\Gamma - I$, whose Sobolev costs are fixed polynomials. Since $L_*/\lambda \leq \lambda^{-\nu}$, the finite-chain estimates prove

$$(H.18) \quad \|e^{-i\lambda\phi_G} u\|_{H^s} \leq C_s A K^s, \quad \|(A_\Gamma - I)z\|_{H^s} \leq C_{s,Q} A \lambda^{-Q}.$$

The fixed Gram and frame multipliers cost only a polynomial at each observation order. Increase the internal precision by that fixed cost and apply (H.10) to prove the strain estimate.

Write $u = e^{i\lambda\phi_G} g$. The literal reconstruction gives $du = e^{i\lambda\phi_G}(i\lambda d\phi_G g + dg)$. By (H.17) and $K \leq \lambda$, $V_G(u)$ costs at most λ and $dV_G(u)$ at most λ^2 beyond the reduced profile. Every additional derivative acts on an envelope or slow coefficient and costs K . The term $-N_G h/2$ obeys the same

bounds, also after the fixed observation cutoff. For parameter derivatives apply the joint factor bounds to the same finite expansion. In its last term the fixed $\mathcal{K}, \mathcal{T}$ are unchanged; derivatives hit the original source or final carrier. At total order k their additional cost is at most λ^k , absorbed by a larger finite depth. Reconstruction then gives the stated joint bounds. Real data give real responses by the unchanged conjugate pairing. $\square$

Corollary H.6 (A fixed finite jet budget). *Fix a nonnegative integer S and $0 < \nu < 1$. There is a finite integer $m = m(S, \nu)$, depending also on the fixed construction, with the following property. Suppose $\|a_G\|_{C^m} + \|G\|_{C^m}$ is uniformly bounded and the joint source bounds of the preceding proposition hold through total order m (with no parameter derivatives for a single source). If*

$$\lambda \geq 2, \quad 1 \leq K \leq \lambda^{1-\nu},$$

then the same response and envelopes in (H.13) satisfy

$$\|\partial_c^\alpha U\|_{H^s} \leq C_{s,\alpha} \lambda K^{s+|\alpha|}, \quad \|\partial_c^\alpha D\|_{H^s} \leq C_{s,\alpha} \lambda^2 K^{s+|\alpha|}, \quad s + |\alpha| \leq S.$$

The budget is fixed before frequency and changes none of the operators, fields or domains. It is independent of any later strain accuracy.

Proof. The preceding proof through joint order $s_1 = S + 2$ uses only finitely many source and coefficient derivatives. At those orders use (H.8) with $L = 1$, $K \geq 1$, and the same complete source split. In (F.25), take $N > s_1 + 2$; apply the bounded fixed-order solver and demodulate. The last term is at most $C_{N,s_1} \lambda \lambda^{s_1-N}$ in H^{s_1} . The finite sum has the proved envelope profiles, and the same reconstruction adds at most two derivatives and carrier factors. Choose the internal complement precision once to absorb these fixed costs, and let m be the largest derivative order used. Only S, ν and the fixed construction enter this choice. $\square$

Corollary H.7 (The corrected atomic family). *Use the two-term primary on its whole compact shifted-carrier patch. Choose the fixed history reserve to include $m(M + 2 + \ell + r_{\text{par}}, \nu)$, the primary shifts and the initial-data orders, where $\ell = \lfloor n/2 \rfloor + 1$ and $r_{\text{par}} = M + 3$. For the same source and observation cutoffs, put*

$$\mathcal{Y}_\tau = \mathcal{B}_G^{\mathbb{C}}(\chi_{\text{src}} D \gamma_G \mathcal{W}_\tau^{\text{pr}}), \quad \mathcal{W}_\tau^{\text{cur}} = \mathcal{W}_\tau^{\text{pr}} - \mathcal{Y}_\tau.$$

Write $A_\tau^{\text{cur}} = e^{-iS_G/\tau} d\mathcal{W}_\tau^{\text{cur}}$. On the observation support, through parameter orders $k \leq r_{\text{par}}$ and uniformly over the fixed radial labels,

$$(H.19) \quad \lambda \|D_x^r D_c^k (e^{-iS_G/\tau} \mathcal{Y}_\tau)\|_\infty + \|D_x^r D_c^k (e^{-iS_G/\tau} d\mathcal{Y}_\tau)\|_\infty \leq C_{r,k} \lambda^{-1} T^{(r-M)_+}.$$

Thus the corrected family has the original phase and the mixed amplitude bounds (G.13), with the same leading tensor. Its full strain has arbitrary precision on the inner identity core. The raw slice and sampling conclusions of Theorems G.8 and G.6 apply to this same corrected family. All choices precede sampling and accuracy.

There is also one fixed shift D_{at} . If an initial scalar has $\|r\|_{C^s} \leq C_s T^{s+B}$ with $B \geq 0$ and the prescribed bounded low jets, then

$$(H.20) \quad \|D_x^s D_c^k (A_\tau^{\text{cur}}(r) - A_0(r))\|_\infty \leq C_{s,k} \lambda^{-1} T^{s+k+B+D_{\text{at}}}.$$

The shift is independent of s, k, B and accuracy.

The atomic maps and A_0 are linear in their initial scalar r . At every prescribed joint order s , one finite jet budget d_s gives, for $\mathcal{E}_\tau(r) = \Re[(A_\tau^{\text{cur}}(r))^* A_\tau^{\text{cur}}(r)] - A_0(r)^* A_0(r)$,

$$(H.21) \quad \|\mathcal{E}_\tau(r) - \mathcal{E}_\tau(\tilde{r})\|_{C^s} \leq C_s \lambda^{-1} (\|r\|_{C^{d_s}} + \|\tilde{r}\|_{C^{d_s}}) \|r - \tilde{r}\|_{C^{d_s}}.$$

The same coefficient budget is imposed on G ; d_s is independent of frequencies and any later accuracy.

Proof. The complete atomic strain is λ^{-3} times the graded amplitude in (G.22). For each $k \leq r_{\text{par}}$, the finite-jet corollary at $K = 1$ gives both normalized response envelopes in C^{M+2} with size $C\lambda^{-1}$. The joint graded response at $K = T$ gives their C^j norm at most $C_{j,k}\lambda^{-1}T^{j+k+\ell}$. For $r > M + 2$, choose $j \geq r + (M + k + \ell)(r - M - 1)$. Interpolation from C^{M+1} in x has exponent $(j + k + \ell)(r - M - 1)/(j - M - 1) \leq r - M$. The low estimate covers the remaining orders. The full response identity proves the strain assertion.

The corrected field is still an integral on the exact original phase; linearity of the fixed response permits integration of its atoms. Its full and slice amplitudes on the shifted-carrier charts satisfy the hypotheses used in the two cited proofs; the reserve M pays the additional slice seminorms after physical differentiation. Their operator bounds, normal homotopy and independent lattice sampling apply without changing cutoffs or data; $q_{\text{quad}} < r_{\text{par}}$ also covers the background quadrature. No derivative of the actual inverse with respect to a parameter is used.

For the offset profile, the fixed one-high primary bounds give the complete source size $\lambda^{-3}T^{B+D_{\text{pr}}}$ at scale $K = T$, including joint derivatives. Apply the graded response and the fixed physical Sobolev embedding, with $D_{\text{at}} = D_{\text{pr}} + \ell$. This proves (H.20). Linearity follows from the scalar transport, algebraic correction and fixed response. The finite-jet estimates applied to an initial scalar difference give $\|A_r^{\text{cur}}(r) - A_0(r)\|_{C^s} \leq C_s\lambda^{-1}\|r\|_{C^{d_s}}$. Expand the two Hermitian products and use bilinearity to obtain (H.21). $\square$

APPENDIX I. TRANSPORTED POSITIVE BACKGROUNDS AND CURRENT ROWS

I.1. The common geometry and row. We use the actual reference and carrier charts of Appendix D and the primary maps of Appendix G on the domains of Setup A.1. The choices below fix their common finite-low neighborhood and define the continuum row. We prove its full-output lower bound, one inverse and positive density correction, then compare that same row with the sampled waves. Appendix M verifies the later frequency and history windows.

Definition I.1 (Second form and reduced strain). Use the actual geometry and ambient formulas of Definition G.2, with $g_G = \gamma(G)$ and $\mathcal{N}_G = \text{Nor}_G$. For the fixed cube tensors of Appendix D, put

$$Q_{G,i} = \Pi_{K_G} Q_i^0, \quad \mathcal{G}_G = (Q_{G,i} : Q_{G,j})_{i,j=1}^n.$$

Proposition D.9 gives this frame and its positive Gram gap on one fixed C^2 neighborhood. Use P_G of (G.3) with this frame, and write

$$q_G(H) = (Q_{G,i} : H)_{i=1}^n, \quad q_G(x, \xi) = q_G(\xi \otimes \xi).$$

The rational second-jet formulas and Lemma B.3 give

$$(I.1) \quad \|\Pi_{K_G}\|_{C^r} + \sum_i \|Q_{G,i}\|_{C^r} + \|\mathcal{G}_G^{-1}\|_{C^r} + \|\text{coeff}(P_G)\|_{C^r} \leq C_r(1 + \|G\|_{C^{r+2}}).$$

Replacing B_G by B_G/σ leaves its kernel projection unchanged. Uniform normalized gaps and fixed-order bounds for dG , B_G/σ and the metric connection therefore give uniform frame and reduced-operator bounds. The tangent, normal and reconstruction identities are (2.2) and (G.6), pointwise on W .

The actual flat reference and its fixed neighborhood are given by Proposition D.4, Corollary D.5 and Proposition D.9.

Definition I.2 (Cube geometry and Hamilton carriers). Use the analytic flat F_0 of Proposition D.4, with surjective actual second form and projected frame $Q_{F_0,i} = \Pi_{K_{F_0}} Q_i^0$, $Q_{F_0,i}(0) = Q_i^0$. Keep these fixed reference tensors and put

$$G_\sigma(x) = \frac{F_0(\sigma x) - F_0(0)}{\sigma}.$$

For small positive σ , its metric is I , its normal gap is positive and its actual frame is C^6 close to

$$Q_i^0 = e_i e_i^T - c_n \mathbf{J}, \quad M_0(\xi) = \text{diag}(\xi_i) - c_n \left(\sum_i \xi_i \right) \mathbf{J},$$

where $c_n = (n-4)^{-2}$ and $\mathbf{J}$ is the all-ones matrix. Use the finite full-sphere cover, simple-characteristic patches $z = (s, v) \in \mathbb{R} \times \mathbb{R}^2$ and buffered phase charts of Propositions D.6–D.10. These constructions hold on one fixed finite C^d neighborhood. For the normalized family use common boxes and normalized gaps down to $\sigma = 0$.

On each patch use adapted coordinates $x = (t, y)$, transverse covector $\zeta \in \mathbb{R}^{n-1}$, and an actual Hamilton phase satisfying

$$p_G(x, d_x S_G(x, \zeta)) = 0, \quad S_G(0, y, \zeta) = y \cdot \zeta.$$

The carrier map is $\zeta_a(z)$. Write

$$\xi_{G,a} = d_x S_G(x, \zeta_a(z)), \quad Y_{G,a} = d_\zeta S_G(x, \zeta_a(z)), \quad \Phi_{G,a}(x, z, \eta) = Y_{G,a}(x, z) \cdot \eta.$$

The matrix $M_G(x, \xi_{G,a})$ has rank $n-1$. Smooth real right and left nullvectors k_G, l_G are selected by fixed nonzero minors. A positive real scalar transport with prescribed positive initial scalar gives $b_G = \alpha_G k_G$ and satisfies $l_G^T P_G b_G = 0$. Its normal field and transport derivative are

$$\nu_G = N_G(\xi_G \text{Sym } b_G), \quad A_0 = -\nu_G \otimes \xi_G.$$

Here $\xi \text{Sym } b = (\xi b^T + b \xi^T)/2$. Both ν_G and ξ_G have positive lower bounds on the selected compact supports. Their mixed spatial, carrier and transverse-covector derivatives satisfy

$$(I.2) \quad \|D^m(S_G, A_0, \nu_G)\| \leq C_m(1 + \|G\|_{C^{m+D}})$$

with one fixed D , under the fixed finite low coefficient bounds. For variable initial scalar r , the right side also includes one $\|r\|_{C^{m+D}}$; this follows from its initial value evaluated at Y_G times the scalar flow exponential.

Use the interior incidence roots $z_a(G; x, \hat{k})$ of Proposition D.9. They satisfy

$$\hat{k} \cdot d_\xi p_G(x, \xi_{G,a}) = 0, \quad |\partial_s(\hat{k} \cdot d_\xi p_G(x, \xi_{G,a}))| \geq c,$$

and the augmented Gram bound

$$|M_G(x, \hat{k})^T w|^2 + \sum_a |w \cdot q_G(x, \xi_{G,a})|^2 \geq c|w|^2 \quad (w \in \mathbb{R}^n)$$

on each neighborhood of the full-sphere cover.

For fixed spectator v , the averaged-gradient map

$$(I.3) \quad \Psi_G(x, x', s, v, \eta) = \int_0^1 d_x \Phi_G(x' + t(x - x'), s, v, \eta) dt$$

is injective on one fixed buffered annular chart in (s, η) and has a common inverse output chart. Its determinant is bounded away from zero on that chart. The corresponding averaged y -gradient charts are also buffered and injective. The full-dispersion phases

$$\phi_\epsilon(x, z, \theta) = \frac{S_G(x, \zeta_a(z) + \epsilon\theta) - S_G(x, \zeta_a(z))}{\epsilon} = \int_0^1 d_\zeta S_G(x, \zeta_a(z) + t\epsilon\theta) \cdot \theta dt$$

have the same buffered charts for $0 \leq \epsilon \leq \epsilon_0$; at zero they give the transport phase. Proposition D.10 gives the finite-low inverse-chart bounds, and Theorem G.4 gives (I.2). Use their common parameter neighborhoods, containing every shifted covector in Definition I.10.

Use the full-amplitude bounds and diagonal reduction of Theorem E.6, with its finite-low derivative requirements included in the fixed neighborhood. The transport normal becomes a PDO

amplitude under the exact conic change (I.3); fixed-jet substitution and (B.9) give its one-high estimates, also for literal adjoints. For $C = 2\Re T$, the raw inner product is $\Re \int f \bar{h}$ on both quadratures, and

$$(I.4) \quad C^*w = 2T^*w, \quad CC^*w = 4\Re(TT^*w) \quad (w \text{ real}).$$

All derivative shifts precede observation and accuracy.

I.2. The full continuum normal and its low-frequency completion. We prove a full-output lower bound from the global incidence Gram estimate.

Definition I.3 (Continuum row). Fix one sufficiently rescaled reference $G_{\text{ref}} = G_\sigma$ and a small fixed C^d neighborhood of it. Fix domains

$$U_{\text{core}} \subseteq B_r \subseteq U_{\text{meas}} \subseteq W$$

and a flat auxiliary torus containing W . The radius r will be chosen sufficiently small. All extensions and cutoffs are fixed before G varies. Take $\chi_{\text{meas}} = 1$ throughout B_r , with support strictly inside U_{meas} , and $\chi_{\text{src}} = 1$ near $\overline{B_r}$, with support in W . For every carrier patch choose a smooth nonnegative $\rho_a(z, \hat{\eta})$, positive near the selected incidence roots and their modulation directions, compactly supported in the buffered input charts. Let $\chi_{\text{hi}} = 0$ for $|\eta| \leq 1$ and $\chi_{\text{hi}} = 1$ for $|\eta| \geq 2$.

With one fixed Fourier normalization, define the complex channel by

$$(I.5) \quad T_a f(x) = \int e^{i\Phi_{G,a}(x,z,\eta)} \chi_{\text{meas}} \rho_a \chi_{\text{hi}}(|\eta|) |\eta|^{1/2} |\nu_a(x, z)|^2 q_G(x, \xi_a(x, z)) f(z, \eta) dz d\eta.$$

The output is zero outside the chart, and the input is the full complex $L^2(z, \eta)$. Set $C_a = 2\Re T_a$. The physical row is the direct-sum row

$$A_{\text{phys}} = (\chi_{\text{meas}} P_G \langle D \rangle^{-1} \chi_{\text{src}}, (C_a)_a).$$

Take a real ψ with $\psi = 1$ on U_{core} and $\text{supp } \psi \subseteq B_r$. Append $A_{\text{out}} = 1 - \psi$ and write

$$A_{\text{aug}} = (A_{\text{phys}}, A_{\text{out}}), \quad E_{\text{aug}} = A_{\text{aug}} A_{\text{aug}}^*.$$

The exterior input is auxiliary data on the torus. It is not a Euclidean ambient coordinate.

Proposition I.4 (Continuum lower bound and auxiliary inverse). *For one sufficiently small fixed r and one sufficiently small finite C^d neighborhood of G_{ref} , every row in Definition I.3 is bounded on its full L^2 spaces. Uniformly for real f supported in B_r ,*

$$(I.6) \quad \|A_{\text{phys}}^* f\|_2^2 \geq c \|f\|_2^2.$$

On the entire real auxiliary torus,

$$(I.7) \quad E_{\text{aug}} \geq c_0 I.$$

It has one bounded self-adjoint L^2 inverse. The exterior channel vanishes on U_{core} . The same choices work after multiplying each prescribed initial scalar by $\sqrt{1 + v_a(z, y)}$, with $1/2 \leq 1 + v_a \leq 3/2$ and $\|v_a\|_{C^{d_0}} \leq 1$, where d_0 includes the complete normal proof's fixed low derivative reserves.

Proof. Fix $\sigma > 0$ and one common finite neighborhood preserving the stated gaps, chart buffers and amplitude bounds. In this proposition the geometric constants may depend on σ . All choices include the stated fixed-low class of initial multipliers.

Step 1. The exact transport normal. The transport derivative has $Q_{G,i} : A_0^* A_0 = |\nu_a|^2 Q_{G,i} : (\xi_a \otimes \xi_a)$, which gives the amplitude in (I.5). For fixed spectator v , $\Phi_a(x, s, v, \eta) - \Phi_a(x', s, v, \eta) = (x - x') \cdot \Psi_a(x, x', s, v, \eta)$. The annular inverse extends homogeneously to the cone over a smaller compact support. The transverse output block is close to η , so equal outputs have input-magnitude ratio close to one. Reduce the fixed angular support and perturbation size to fit the annular buffer.

A common normalization then puts both inputs in that buffered chart, where injectivity holds. Homogeneity and the inverse identity give

$$(I.8) \quad J_a = |\det D_{(s,\eta)} \Psi_a|, \quad c|\eta| \leq J_a \leq C|\eta|.$$

Changing exactly to $k = \Psi_a$ in $T_a T_a^*$ gives its kernel

$$K_{T_a T_a^*}(x, x') = \int e^{i(x-x') \cdot k} b_{T,a}(x, x', k) dk,$$

where, at $(s, \eta) = u_a(x, x', v, k)$,

$$(I.9) \quad b_{T,a}(x, x', k) = \chi_{\text{meas}}(x) \chi_{\text{meas}}(x') \int_v \frac{\rho_a(s, v, \hat{\eta})^2 \chi_{\text{hi}}(|\eta|)^2 |\eta|}{J_a} \cdot |\nu_a(x, z)|^2 |\nu_a(x', z)|^2 q_G(x, \xi_a(x, z)) q_G(x', \xi_a(x', z))^T dv.$$

Thus the normalized PDO amplitude in Theorem E.6 is $(2\pi)^n b_{T,a}$. The input support lies strictly inside the inverse chart, so zero extension is smooth. The two half weights cancel the degree-one Jacobian. Differentiating the inverse identity gives order-zero full-amplitude estimates, with one high geometric norm; each k derivative lowers order. The finite-low full-space bound therefore controls T_a and C_a .

For $k \neq 0$, define its homogeneous diagonal by

$$(I.10) \quad d_{0,a}(x, k) = b_{T,a}(x, x, k)|_{\chi_{\text{hi}}=1}.$$

Since $|\eta| \asymp |k|$, the part containing $\chi_{\text{hi}}^2 - 1$ is a bounded-frequency smoothing term. The identity $C_a C_a^* = 4\Re(T_a T_a^*)$ and full-amplitude reduction in Theorem E.6 give the real principal matrix

$$(I.11) \quad b_a(x, k) = 2(2\pi)^n [d_{0,a}(x, k) + d_{0,a}(x, -k)] \geq 2(2\pi)^n d_{0,a}(x, k).$$

No evenness of the input mask is required. There are an actual order-minus-one reduction error and the bounded-frequency error. Their literal adjoints have the same gains and fixed shifts.

For the image normal, the fixed-source formula at phase $x \cdot \xi$ moves χ_{src}^2 past $\langle D \rangle^{-1}$ with an order-minus-two error. The next inverse multiplier and the two first-order factors P_G, P_G^* leave an order-minus-one error. The leading full amplitude is $M_G(x, \xi) M_G(x', \xi)^* \langle \xi \rangle^{-2}$ times the fixed cutoffs. The same direct diagonal reduction therefore applies.

Step 2. Principal positivity. Differentiating $p_G(x, d_x S_G) = 0$ in ζ shows that $D_\zeta d_x S_G$ maps onto the characteristic tangent hyperplane; its transverse block has rank $n - 1$. At each incidence root there is therefore a unique nonzero η_a with $d_x \Phi_a = \hat{k}$. The incidence derivative bounds $J_a/|\eta_a|$ above and below. Choose ρ_a positive near these root/modulation pairs.

The image normal has principal matrix $M_G M_G^T$ up to a fixed positive normalization. If it and all $d_{0,a}$ annihilate w , then $M_G^T w = 0$. The homogeneous positive weights in (I.10) and continuity also give $w \cdot q_G(x, \xi_{G,a}) = 0$ at every selected incidence root. The augmented Gram bound forces $w = 0$. Compactness and continuity therefore give a uniform positive gap on the entire sphere. For the allowed initial multiplier, scalar transport multiplies ν_a by $\sqrt{1 + v_a(z, Y_{G,a})}$. Every detector symbol weight is therefore multiplied by $(1 + v_a)^2 \geq 1/4$, and its finite-low full-amplitude bounds remain uniform. The gap persists under the chosen finite-low perturbations.

Step 3. Low frequencies and small support. Fix a plateau $\chi = 1$ near the eventual B_r , inside the positive principal-symbol region. Full-amplitude reduction gives

$$\chi A_{\text{phys}} A_{\text{phys}}^* = \chi \text{Op}(e_0) + R_0, \quad R_0 : H^{-1} \longrightarrow L^2.$$

Choose a fixed smooth low-frequency extension with $e_0(0, k) \geq cI$. Near B_r , Taylor's formula gives

$$\text{Op}(e_0) - e_0(0, D) = \sum_i x_i T_i.$$

The order-zero T_i have symbols $\int_0^1 \partial_{x_i} e_0(tx, k) dt$, extended with fixed cutoffs, and are uniformly bounded. For f supported in B_r , positivity of the frozen Fourier multiplier gives

$$\begin{aligned} \|A_{\text{phys}}^* f\|_2^2 &\geq c\|f\|_2^2 - \sum_i \|x_i f\|_2 \|T_i f\|_2 - C\|f\|_{H^{-1}} \|f\|_2 \\ &\geq (c - Cr)\|f\|_2^2. \end{aligned}$$

Here $\|f\|_{H^{-1}} \leq Cr\|f\|_2$ follows from Sobolev duality and Hölder on the small support. Choose r inside the fixed plateaus to absorb both errors. This proves (I.6).

Step 4. Exterior completion. Apply the local estimate to ψf . Global row boundedness gives

$$\|f\|_2 \leq \|\psi f\|_2 + \|(1 - \psi)f\|_2 \leq C\|A_{\text{phys}}^* f\|_2 + C'\|(1 - \psi)f\|_2.$$

Squaring proves (I.7). This bounded self-adjoint operator has closed range and zero kernel, so its range is also dense. Its inverse is bounded and self-adjoint. The exterior row vanishes on U_{core} . $\square$

I.3. One inverse and its initial-variable adjoint. The L^2 inverse, derivative commutators and one derivative trade give the raw canonical data and physical density correction.

Proposition I.5 (The inverse and initial-variable adjoint). *Use the common finite-low class of Proposition I.4, including its positive initial scalars r_a , and write*

$$A = A_{G,r}, \quad E = AA^*, \quad \mathcal{C}_m = 1 + \|G\|_{C^m} + \sum_a \|r_a\|_{C^m}.$$

Let X^s be the direct sum of auxiliary H^s inputs and carrier initial-variable $H_{z,y}^s$ inputs before their physical cutoffs. For one fixed B , the same inverse satisfies

$$(I.12) \quad \|E^{-1}e\|_{H^s} + \|A^*E^{-1}e\|_{X^s} \leq C_s[\|e\|_{H^s} + \mathcal{C}_{s+B}\|e\|_2], \quad s \geq 0.$$

It preserves H^∞ , and A^*E^{-1} is a right inverse, whose exterior component is zero on U_{core} . For the real carrier adjoint C_a^* set

$$D_a f = 2\Re\{\chi_y(z, y)\mathcal{F}_\eta^{-1}[\rho_a(z, \hat{\eta})\chi_{\text{hi}}(|\eta|)|\eta|^{1/2}f(z, \eta)](y)\},$$

where the fixed χ_y equals one at the core's footpoints. After enlarging B once,

$$(I.13) \quad \sum_a \|D_a C_a^* E^{-1}e\|_{H_{z,y}^s} \leq C_s[\|e\|_{H^{s+1}} + \mathcal{C}_{s+B}\|e\|_2].$$

This includes all carrier and initial coordinates and also holds before χ_y .

Proof. The image, transport and exterior normals make E an order-zero PDO, with the full-amplitude one-high bounds. For $|\alpha| = s \geq 1$, these bounds give

$$(I.14) \quad \|[D^\alpha, E]v\|_2 \leq C_s[\|v\|_{H^{s-1}} + \mathcal{C}_{s+B}\|v\|_2].$$

In the commutator, at least one derivative hits the full amplitude; interpolation from its fixed first-derivative baseline, jointly with v at H^{s-1} , gives the displayed additive bound.

Let $z = E^{-1}e$. At order one, difference quotients justify (I.14): the PDO translation commutator is uniformly bounded on L^2 by the fundamental theorem of calculus. Thus E^{-1} preserves H^1 ; self-adjunction extends the same inverse to H^{-1} . If $z \in H^{s-1}$, the distributional identity

$$ED^\alpha z = D^\alpha e - [D^\alpha, E]z$$

has right side in L^2 . Uniqueness in H^{-1} gives $D^\alpha z = E^{-1}(D^\alpha e - [D^\alpha, E]z) \in L^2$. Integer induction proves the first bound in (I.12), with the same B .

Image and exterior adjoints are order zero. For a carrier put $A = D_\zeta d_x S_G$ and $L = (A^T A)^{-1} A^T$. The transverse gap gives

$$(I.15) \quad d_x \Phi = A\eta, \quad \eta = L d_x \Phi, \quad \partial_{z_\ell} \Phi = V_\ell \cdot d_x \Phi, \quad V_\ell = (\partial_{z_\ell} d_\zeta S_G)L.$$

Thus every initial or carrier phase derivative of T_a^*v can be transposed to an x derivative. Compact measurement support removes boundary terms. After s derivatives, a term with k derivatives on v has coefficient excess $s - k$ plus one fixed reserve. Its amplitude still has order $|\eta|^{1/2}$; its paired normal has order zero by the degree-one conic Jacobian. The full-amplitude bound, Plancherel and interpolation give

$$\mathcal{C}_{s-k+c}\|v\|_{H^k} \leq C_s[\|v\|_{H^s} + \mathcal{C}_{s+c}\|v\|_2].$$

Sum these terms and insert $v = E^{-1}e$, using the inverse at the high order and only its L^2 bound in the coefficient term. This proves the raw estimate. Scalar transport is $r(z, Y_G)$ times its flow factor, so the same calculation has one high G or r norm when the initial scalar varies.

Finally D_a is a fixed half-order multiplier in y , followed by real part and χ_y . Its carrier derivatives act on the fixed mask, so its H^s norm is bounded by the raw H^{s+1} norm, also before χ_y . This proves (I.13). The identity $AA^*E^{-1} = I$ proves the right-inverse assertion; its exterior component is $(1 - \psi)E^{-1}e$. $\square$

Proposition I.6 (Uniform normalization as the second form shrinks). *For $0 < \sigma \leq \sigma_0$, take G_σ from Definition I.2 and use initial scalar σ for the null transports. All domains, cutoffs and carrier charts in Definition I.3 can be chosen independently of σ . The fields*

$$\sigma^{-1}B_{G_\sigma}, \quad \sigma N_{G_\sigma}, \quad \sigma^{-1}b_{\sigma,a}, \quad S_{G_\sigma}, \quad \nu_{\sigma,a}, \quad A_{0,\sigma}$$

extend smoothly through $\sigma = 0$ in every fixed mixed space/carrier derivative. The normal fields $\nu_{\sigma,a}$ keep a uniform positive lower bound. For fixed smooth compact nonnegative carrier weights w_a , put

$$H_\sigma(x) = \sum_a \int w_a(z) |\nu_{\sigma,a}(x, z)|^2 \xi_{\sigma,a}(x, z) \otimes \xi_{\sigma,a}(x, z) dz.$$

Its limit H_0 is constant and positive semidefinite, and

$$(I.16) \quad \|H_\sigma - H_0\|_{C^s} \leq C_s \sigma.$$

The normalized augmented normal satisfies, on the entire auxiliary L^2 ,

$$(I.17) \quad E_\sigma \geq cI, \quad \|E_\sigma^{-1}f\|_{H^s} + \|A_\sigma^* E_\sigma^{-1}f\|_{X^s} \leq C_s \|f\|_{H^s},$$

with c, C_s independent of σ . The density estimate (I.13) is uniform as well, with right side $C_s \|f\|_{H^{s+1}}$. The algebraic limit at zero is used for these estimates only; G_0 is not asserted to have full normal rank.

Proof. On one small box for F_0 , construct the phase $\hat{S}$ and null transport $\hat{b}$ with initial scalar 1. Choose a fixed normalized box with σx in that domain. Corollary D.5 gives its flat metric and the scaled second form and stress frame. The ambient formula and common Hamilton Cauchy data then give

$$(I.18) \quad \begin{aligned} N_{G_\sigma}(x) &= \sigma^{-1} N_{F_0}(\sigma x), & S_\sigma(x, \zeta) &= \sigma^{-1} \hat{S}(\sigma x, \zeta), \\ b_\sigma(x, \zeta) &= \sigma \hat{b}(\sigma x, \zeta), & \xi_\sigma(x, \zeta) &= \hat{\xi}(\sigma x, \zeta), & \nu_\sigma(x, \zeta) &= \hat{\nu}(\sigma x, \zeta). \end{aligned}$$

Indeed the displayed phase has initial value $y \cdot \zeta$, and $P_{G_\sigma}[\sigma \hat{b}(\sigma x)] = \sigma^2 (P_{F_0} \hat{b})(\sigma x)$. Thus it has the same leading scalar transport and initial scalar σ ; uniqueness proves the identities.

Since $\hat{S}(0, \zeta) = 0$,

$$S_\sigma(x, \zeta) = \int_0^1 x \cdot d_x \hat{S}(t\sigma x, \zeta) dt.$$

Every normalized field therefore extends smoothly in all mixed variables through zero. There the nonzero tensor $\xi_0 \text{Sym } k_0$ lies in $K_0^\perp$; its inverse image under $B_{F_0}(0)^*$ is nonzero. Compact detector supports give a common positive lower bound for ν . For the fixed carrier weights, the same identities give $H_\sigma(x) = \hat{H}(\sigma x)$ and $H_0 = \hat{H}(0)$. The fundamental theorem of calculus proves (I.16) at every fixed order.

The normalized image and detector symbols have the frozen positive joint gap at zero. Their full amplitudes, chart inverses and literal adjoints have uniform fixed-order bounds. Apply the proofs of Propositions I.4 and I.5 to this compact normalized family, including its fixed-low initial multiplier class: one common small ball absorbs the local errors, and the same exterior row gives the full L^2 gap. The inverse, raw and density estimates have uniform coefficients at every fixed order. The physical domains stay fixed, with zero exterior row on U_{core} . No norm continuity of individual graph operators is used. $\square$

I.4. The positive transported mean. The mean is affine in its initial density. One density correction and tangent field cancel its reduced strain, at the normalized reference and throughout its fixed neighborhood.

Proposition I.7 (A positive mean at the reference and nearby immersions). *Use the normalized family of Proposition I.6. Choose smooth nonnegative cutoffs χ_a compactly supported in the carrier charts and equal to one near every detector projection. Fix the external carrier measure $\psi_a dz = \chi_a^2 dz$ and put $\alpha_a = \rho_a$, so $\rho_a = \chi_a \alpha_a$. Let H_0 be the corresponding constant limiting mean and fix $X_0(x) = -H_0 x/2$, a source cutoff $\theta \in C_c^\infty(B_r)$ equal to one on U_{core} , and $U_{\text{fin}} \Subset U_{\text{core}}$.*

For a positive σ and an immersion G near G_σ , use initial scalar σ and its uncorrected continuum row $A = A_{G,\sigma}$, $E = AA^*$. For the leading normal $\nu_{G,\sigma,a}$ with initial scalar σ , define the transported mean with this fixed measure:

$$(I.19) \quad H_G[\sigma^2(1+v)] = \sum_a \int \psi_a |\nu_{G,\sigma,a}|^2 [1 + v_a(z, Y_{G,a})] \xi_{G,a} \otimes \xi_{G,a} dz.$$

Define, with the same source and raw-input cutoffs,

$$(I.20) \quad \begin{aligned} R_0 &= D\gamma_G(dGX_0) + H_G[\sigma^2], \\ e &= -\frac{1}{2}\theta q_G(R_0), \quad h = A^*E^{-1}e, \\ u_{\text{corr}} &= \langle D \rangle^{-1} \chi_{\text{src}} h_{\text{im}}, \\ X_G &= X_0 + g_G^{-1} u_{\text{corr}}, \end{aligned}$$

$$(I.21) \quad v_a = 4\Re \mathcal{F}_\eta^{-1}[\alpha_a \chi_{\text{hi}}(|\eta|)|\eta|^{1/2} h_a].$$

The source is smoothly zero-extended to the torus; the density synthesis is uncut in the initial spatial variables. Put

$$r_{G,a} = \sigma \sqrt{1 + v_a}, \quad V_{\text{base}} = dGX_G.$$

For all sufficiently small $\sigma > 0$, and then on one sufficiently small fixed finite C^d neighborhood of G_σ , these fields are real smooth, $1 + v_a \geq 1/2$, the scalar $r_{G,a}$ is positive on detector supports, and

$$(I.22) \quad q_G(D\gamma_G V_{\text{base}} + H_G[r_G^2]) = 0 \quad \text{on } U_{\text{fin}}.$$

The corrected positive row has a full L^2 gap and one all-Sobolev inverse. For one fixed B ,

$$(I.23) \quad \sum_a \|r_{G,a}\|_{C^s} + \|V_{\text{base}}\|_{C^s(U_{\text{fin}})} + \|D\gamma_G V_{\text{base}}\|_{C^s(U_{\text{fin}})} \leq C_s(1 + \|G\|_{C^{s+B}(W)}).$$

The scalar norm includes all carrier and initial variables (z, y) on the fixed buffered patches; B includes the one derivative in the tangent strain. These choices precede accuracy and any later target dilation; fixed-order bounds are uniform on smooth compact families.

Along $G = G_\sigma$, the construction satisfies

$$(I.24) \quad \sum_a \|v_a\|_{C^s} + \|X_G - X_0\|_{C^s(W)} \leq C_s \sigma,$$

the identity holds on U_{core} , and the corrected normal gap and fixed-order inverse bounds are uniform as $\sigma \rightarrow 0$.

Proof. The density v_a has carrier support inside $\{\chi_a = 1\}$, so $\psi_a v_a = v_a$. Scalar transport evaluates it at $Y_{G,a}$. The factor 4 in the uncut synthesis and $C_a = 2\Re T_a$ give, on the entire measurement support,

$$(I.25) \quad \chi_{\text{meas}} q_G (H_G[\sigma^2(1+v)] - H_G[\sigma^2]) = 2 \sum_a C_a h_a.$$

The same identity holds for any fixed positive initial scalar and its row. On U_{core} , $\theta = \chi_{\text{meas}} = \psi = 1$, so covector raising gives

$$\begin{aligned} q_G(D\gamma_G V_{\text{base}} + H_G[r_G^2]) &= q_G R_0 + 2P_G u_{\text{corr}} + 2 \sum_a C_a h_a \\ &= q_G R_0 + 2e = 0. \end{aligned}$$

At G_σ the metric is I , so $D\gamma_G(dGX_0) = -H_0$ and $R_0 = H_\sigma - H_0 = O_s(\sigma)$. The uncorrected row is the normalized row. Its uniform inverse, raw-adjoint and density bounds give $h = O_s(\sigma)$, $v = O_s(\sigma)$ and $X_G - X_0 = O_s(\sigma)$. Choose σ so that v lies in the fixed-low multiplier class of Proposition I.4. Its normalized proof gives a uniform corrected gap. Since $v = O_s(\sigma)$, the same inverse proof has uniform coefficients at every fixed order.

Now fix this positive σ . Choose the neighborhood order above d_0 plus the density and inverse shifts, keeping the preselected support. A small neighborhood in that finite topology preserves the uncorrected gap and the needed low smallness of e, v . The fixed-jet and transport formulas give $\|R_0\|_{C^s} + \|e\|_{H^s} \leq C_s(1 + \|G\|_{C^{s+B_1}})$. Apply the inverse, adjoint, density and physical realization estimates; every high coefficient multiplies a bounded low source norm. Covector raising, the positive square root and the one tangent-strain derivative enlarge this fixed shift once, proving the profile. The same fixed-low multiplier class gives the corrected gap and inverse throughout this neighborhood. $\square$

I.5. The row and its auxiliary fields. The continuum inverse uses one positive row. We truncate every raw input before its physical realization.

Definition I.8 (One row and its input cuts). For the positive base of Proposition I.7, write

$$A_G = (A_{0,G}, T_G), \quad E_G = A_G A_G^*, \quad Q_G = E_G^{-1},$$

where $A_{0,G}$ contains the image and exterior channels, and T_G the real carrier channels. Every geometric coefficient in this row belongs to G, r_G .

Let π_K be one real even Fourier contraction on each auxiliary torus input, equal to one below K and zero above $2K$. On the carrier input let $P_K^2 = \sum_{\kappa \leq K} \beta_\kappa^2$ be the fixed dyadic square prefix, from an annular partition that sums to one on $\text{supp } \chi_{\text{hi}}$. Set

$$P_K = \text{diag}(\pi_K, P_K), \quad \mathcal{F}_{G,K} = A_{0,G} \pi_K^2 A_{0,G}^*, \quad E_{G,K} = A_G P_K^2 A_G^*.$$

All squares act on raw inputs. The spatial source and footpoint cutoffs remain in the original operators after these multipliers. The raw norm X^s is the direct sum of auxiliary H^s norms and carrier initial-variable $H_{z,y}^s$ norms, before the footpoint cutoff. Each carrier component of A_G^* contains χ_{hi} , and $P_K^2 = 1$ on that support below cK . Hence

$$\|(I - P_K^2)v\|_{X^0} \leq C_s K^{-s} \|v\|_{X^s} \quad (v \in \text{Ran } A_G^* \cap X^s).$$

Proposition I.5, composed with π_K^2 and the same strong order-zero auxiliary row, gives

$$(I.26) \quad \begin{aligned} &\|Q_G e\|_{H^s} + \|A_G^* Q_G e\|_{X^s} + \|\mathcal{F}_{G,K} Q_G e\|_{H^s} \\ &\leq C_s [\|e\|_{H^s} + \mathcal{C}_{s+B}(G, r_G) \|e\|_2]. \end{aligned}$$

Here $\mathcal{C}_m(G, r_G) = 1 + \|G\|_{C^m} + \sum_a \|r_{G,a}\|_{C^m}$. The fixed loss and constants are uniform in K ; at each prescribed low order the coefficient norm is bounded. The inverse and raw cutoff order are unchanged.

Proposition I.9 (Physical realization of truncated data). *For $h = A_{0,G}^* Q_G e$, put*

$$(I.27) \quad \begin{aligned} u_K &= \langle D \rangle^{-1} \chi_{\text{src}} \pi_K^2 h_{\text{im}}, \\ V_{\text{aux},K}^G(e) &= \frac{1}{2} dG g_G^{-1} u_K. \end{aligned}$$

On the physical core, $q_G D \gamma_G V_{\text{aux},K}^G = P_G u_K$; the exterior output is zero. For one fixed shift B_{phys} , uniformly in K ,

$$(I.28) \quad \|V_{\text{aux},K}^G(e)\|_{H^s(W)} \leq C_s [\|e\|_{H^s} + \mathcal{C}_{s+B_{\text{phys}}}(G, r_G) \|e\|_2].$$

The same ordered physical factors, started after the raw Fourier cut, also give

$$(I.29) \quad \|V_{\text{aux},K}^G(e)\|_{H^s(W)} + \|\mathcal{F}_{G,K} Q_G e\|_{H^s} \leq C_s [K^s + \mathcal{C}_{s+B_{\text{phys}}}(G, r_G)] \|e\|_2.$$

In the field estimates here and below one may also include $\|D \gamma_G V_{\text{aux},K}^G(e)\|_{H^s(W)}$ on the left. A fixed physical cutoff gives the same field and strain bounds on the torus. In particular its C^2 norm is at most $C\|e\|_{H^{s_0}}$ at one fixed s_0 under the prescribed fixed-low bounds.

Suppose

$$\mathcal{C}_m(G, r_G) \leq C_m T^{(m-R_*)+}, \quad T \geq 2,$$

with one fixed R_* above every displayed shift. Then

$$(I.30) \quad \|A_G^* Q_G e\|_{X^s} + \|\mathcal{F}_{G,K} Q_G e\|_{H^s} \leq C_s [\|e\|_{H^s} + T^s \|e\|_2],$$

and, if $T \leq K$,

$$(I.31) \quad \|V_{\text{aux},K}^G(e)\|_{H^s(W)} + \|\mathcal{F}_{G,K} Q_G e\|_{H^s} \leq C_s K^s \|e\|_2.$$

At any fixed finite set of orders where the coefficient norms are bounded, this last conclusion holds for every $K \geq 2$.

Proof. Equation (I.26) controls h . The tangent identity gives $D \gamma_G V_{\text{aux},K}^G = \text{Sym} \nabla^{g_G} u_K$. Its first derivative is paired with $\langle D \rangle^{-1}$, so the field and full strain in (I.27) have order at most zero. Their one-high bounds prove (I.28); a final fixed cutoff adds only a lower-order term to the strain. Starting instead with $\|\pi_K^2 h\|_{H^s} \leq C_s K^s \|e\|_2$ proves (I.29), including the complete auxiliary row and its exterior output.

Corollary B.4 gives $\mathcal{C}_{s+c} \leq C_s T^s$ for every displayed fixed shift c . Apply this to the one-high shifts in (I.26) and (I.29). This proves both graded bounds; bounded coefficient norms give the finite-order conclusion directly. $\square$

I.6. The same primary row and separated samples. The fixed external weights identify the positive continuum row with the leading normal of its actual primary waves.

Definition I.10 (Primary realization of the positive row). Use the positive scalar $r_{G,a}$ of Proposition I.7, its fixed measure $\psi_a dz = \chi_a^2 dz$ and detector mask $\alpha_a = \rho_a$, so $\chi_a \alpha_a = \rho_a$. The primary atom $\mathcal{W}_{\tau,a,z}^{\text{pr}}(x, \zeta')$ is (G.12), with the common Hamilton Cauchy data and one algebraic correction. Use its proved mixed grades at the same scale L as G, r_G , including the fixed parameter order $r_{\text{par}} = M + 3$.

For $\zeta = \zeta_a(z)$ define the background-modulation tensor and real channel on both quadratures by

$$\begin{aligned} \mathcal{K}_{\tau,a,z}(x, \eta) &= \text{Sym}[(d\mathcal{W}_\tau(x, \zeta))^* d\mathcal{W}_\tau(x, \zeta + \tau\eta)], \\ \mathcal{U}_{\tau,a,z,K} f &= 2\Re \int \chi_{\text{meas}} \alpha_a \chi_{\text{hi}} |\eta|^{1/2} q_G(\mathcal{K}_{\tau,a,z}) \beta_K f(\eta) d\eta. \end{aligned}$$

Its exact phase is $[S_G(x, \zeta + \tau\eta) - S_G(x, \zeta)]/\tau$; at zero, $\mathcal{K}_{0,a,z} = e^{i\Phi_G} |\nu_G|^2 \xi_G \otimes \xi_G$. Use the fixed dyadic square partition and radial labels of Theorem G.6, with $\tau_c = (R_c \lambda)^{-1}$ and

$$(I.32) \quad \Lambda/\lambda < \text{the fixed phase margin.}$$

Every shifted covector stays in its common buffered chart. Write $H_G = H_G[r_G^2]$ for this same positive transported mean.

The square weights satisfy $\rho_a^2 = \psi_a \alpha_a^2$. Thus the transport normal of these channels is exactly $T_G P_K^2 T_G^*$; real adjunction gives $CmC^* = 4\Re(TmT^*)$. Summing the fixed radial colors removes their masks at transport order. The auxiliary normal is appended with the same raw cut. The fixed-low full-amplitude bounds also show that an initial-scalar change of size η in those seminorms changes every continuum prefix normal by $C\eta$, uniformly in its ceiling.

APPENDIX J. CURRENT CORRECTION FOR FOURIER MEASURES

J.1. One current patch and its two envelopes. The finite primary and its complete response use the same immersion. We apply the response once to the finite strain of each Fourier measure, on the common source and observation supports.

Setup J.1 (The current phase and full response). Use the actual geometry of Definition I.1 at F , in its fixed finite-low neighborhood. On one buffered simple patch let $\phi_F(x) = S_F(x, \zeta)$ be the Hamilton phase with the fixed transverse Cauchy data. Use the finite primary of Theorem G.5, including its positive initial scalar, and the compatible complex-linear response $\mathcal{B}_F^C$ of Theorem H.1. This is the fixed full ambient map (H.9), including its literal inverse and normal reconstruction.

Use the wider response core U_{LAG} of Setup A.1, with $D \in \text{Int } D_{\text{lag}} \in U_{\text{LAG}}$. The entire $\text{supp } \chi_{\text{meas}}$ and every eventual mean, protected pair-source and physical-output support lie compactly inside $\text{Int } D_{\text{lag}}$. Fix real source and observation cutoffs with

$$(J.1) \quad \text{supp } \chi_{\text{src}} \in \{\chi_{\text{core}} = 1\}, \quad \text{supp } \chi_{\text{core}} \in U_{\text{LAG}}, \quad \text{supp } \chi_{\text{src}} \subset \{\chi_2 = 1\}.$$

Require $D_{\text{lag}} \in \text{Int}\{\chi_{\text{src}} = 1\}$ and the full observation support inside the primary's validity chart. These choices precede sampling, frequency and accuracy. All envelopes have fixed extensions to the auxiliary torus.

Setup J.2 (Source scales and finite measures). For a fixed r_* above the finite construction shifts and the joint budget of Corollary H.7, suppose

$$(J.2) \quad \|F\|_{C^r} \leq C_r T^{(r-r_*)+}, \quad T \geq 2.$$

The phase, finite primary and response have their proved grades at this same scale. Let $\lambda = \tau^{-1} \geq 2$ and choose $K \geq 2$ with fixed $\nu > 0$ and

$$(J.3) \quad T \leq K, \quad T \leq \lambda^{1-\nu}.$$

Let m be a finite complex measure in the $(n-1)$ initial Fourier variables, with $M = \|m\|_{\text{TV}}$, and

$$\text{supp } m \subset B_\Lambda, \quad 2 \leq \Lambda \leq K/C_{\text{mod}}, \quad \Lambda/\lambda < \text{the fixed phase margin}.$$

The allowed modulation chart includes this entire support and any atom at zero. No relation between T and Λ is required. For fixed radial labels $\lambda_j = c_j R^a$, strict power gaps absorb the positive constants c_j uniformly.

J.2. Finite measures and one current response.

Definition J.3 (The finite primary and its strain correction). Put

$$(J.4) \quad W_0(m) = \int \mathcal{W}_\tau^{\text{pr}}(F; x, \zeta + \tau\xi) dm(\xi),$$

and define

$$(J.5) \quad H = \chi_{\text{src}} D\gamma_F W_0, \quad Y = \mathcal{B}_F^C H, \quad Z = W_0 - Y.$$

Every map is complex linear in the complete measure. The finite primary, response, inverse and cutoffs precede the datum and accuracy.

Proposition J.4 (Finite-measure envelopes and current strain). *Under the preceding setups, the same fields satisfy*

$$(J.6) \quad \lambda \|\chi_{\text{core}} e^{-i\lambda\phi_F} Z\|_{C^s} + \|\chi_{\text{core}} e^{-i\lambda\phi_F} dZ\|_{C^s} \leq C_s M K^s,$$

$$(J.7) \quad \lambda \|\chi_{\text{core}} e^{-i\lambda\phi_F} Y\|_{C^s} + \|\chi_{\text{core}} e^{-i\lambda\phi_F} dY\|_{C^s} \leq C_s M \lambda^{-1} K^s.$$

On D_{lag} , for every finite r, Q ,

$$(J.8) \quad \|D\gamma_F Z\|_{C^r(D_{\text{lag}})} \leq C_{r,Q} M \lambda^{-Q}.$$

The fields and domains are independent of these proof orders.

Proof. The full source cutoff and the response are fixed and linear. Compact shifted support permits integration in every fixed Sobolev space; continuity of the same response therefore gives

$$(Y, Z) = \int (\mathcal{Y}_\tau, \mathcal{W}_\tau^{\text{cur}})(F; x, \zeta + \tau\xi) dm(\xi).$$

These are the original complete fields, including all source leakage. The central demodulation phase is

$$(J.9) \quad \Psi(x, \xi) = \frac{S_F(x, \zeta + \tau\xi) - S_F(x, \zeta)}{\tau} = \int_0^1 d_\zeta S_F(x, \zeta + u\tau\xi) \cdot \xi du.$$

Its joint grades give $|D_x^r \Psi| \leq C_r \Lambda T^{r-1}$. A derivative monomial of order s in $e^{i\Psi}$, with k phase factors, costs $C_s \Lambda^k T^{s-k} \leq C_s K^s$. Multiply the atomic envelopes of Corollary H.7 by this phase and integrate against $|m|$ to prove both envelope bounds. The same atomic theorem gives full strain $C_{r,Q} \lambda^{-Q}$ on the common inner core, uniformly throughout the shifted compact support. Its finite higher precision includes the physical C^r derivatives. Integration proves (J.8) on D_{lag} . $\square$

APPENDIX K. THE SAMPLED CURRENT MEAN

K.1. Preparation and the original positive row.

Setup K.1 (One row and the stage window). Use Definitions I.1 and I.8 at the current F on $D \subseteq U_{\text{fin}} \subseteq U_{\text{core}} \subseteq B_r \subseteq U_{\text{meas}} \subseteq W$ and the common auxiliary torus. Write $\mathcal{C} = \chi_{\text{meas}} q_F$ and $\ell = \lfloor n/2 \rfloor + 1$. For the original positive scalar r_F , set

$$A_F = (A_0, \mathcal{T}_0), \quad E_0 = A_F A_F^*, \quad Q_0 = Q_F, \quad \mathcal{F}_{0,K} = A_0 \pi_K^2 A_0^*, \quad E_{0,K} = A_F \mathcal{P}_K^2 A_F^*.$$

Use $A = (n+1)(1+\theta)/4$ from (8.6) and

$$(K.1) \quad \begin{aligned} 0 < \theta < \epsilon < \frac{1}{8}, \quad \beta = (1+\epsilon)/3, \quad 1+\theta+\beta < a < \frac{3}{2}, \\ \lambda = R^a, \quad H = R^{1+\theta}, \quad h = R^{-\beta}, \quad T = R^t, \quad 0 < t < 1. \end{aligned}$$

The current history profile is $\|F\|_{C^r} \leq C_r T^{(r-r_*)+}$; its positive base, phases and finite primary have the same derivative scale. Choose r_* above the fixed shifts in Proposition I.9, which gives

$$(K.2) \quad \|A_F^* Q_0 e\|_{X^s} + \|\mathcal{F}_{0,K} Q_0 e\|_{H^s} \leq C_s [\|e\|_{H^s} + T^s \|e\|_2],$$

and, for $T \leq K$,

$$(K.3) \quad \|V_{\text{aux},K}^F(e)\|_{H^s(W)} + \|\mathcal{F}_{0,K} Q_0 e\|_{H^s} \leq C_s K^s \|e\|_2.$$

The latter includes the complete strain and a fixed physical cutoff. All raw inputs precede physical source and footpoint cutoffs. The exterior output is included on the torus and vanishes on D .

The fixed order $q_{\text{quad}} = 2n - 1$ precedes the finite colors, radial labels and positive-row neighborhood. The preparation orders below depend only on n and the fixed power margins; put their budget into r_* before sampling, fine accuracy and target dilation.

Proposition K.2 (A prepared positive background). *Use the positive scalar r_F , base field and row $A_F = (A_0, \mathcal{T}_0)$ of Appendix I, its inverse Q_0 , the current atomic family, and Setup K.1. Put $J = \lfloor 2A/a \rfloor$. Then $a(J+1) > 2A$ and $\beta q_{\text{quad}} > 2A$. Use the convolution weights, their fixed radial colors and spacing h of Theorem G.6. The weights and colors precede the following density preparation and the random shifts.*

There are positive initial scalars $r_{F,R}$ and a real field U_c , chosen before sampling, such that one sample event gives the current zero-atom fields $Z_{c,j} = \mathcal{W}_{\tau_j, z_j}^{\text{cur}}(r_{F,R})$ and

$$H_c = \sum_j \mu_j \Re[(dZ_{c,j})^* dZ_{c,j}], \quad C_c = D\gamma_F U_c + H_c,$$

with a full auxiliary residual satisfying

$$(K.4) \quad r_c = q_F C_c \text{ on } D, \quad \|r_c\|_2 \leq R^{-2A}, \quad \|r_c\|_{H^s} \leq C_s T^s.$$

The initial scalar stays in the fixed positive class of the original row. For every fixed reserve R_0 , the history order can be chosen so that

$$(K.5) \quad \|r_{F,R} - r_F\|_{C^{R_0}} \leq C\lambda^{-1}, \quad \|r_{F,R}\|_{C^s} \leq C_s T^{(s-R_0)_+}.$$

All carrier and initial variables are included. The field and full self tensor obey

$$(K.6) \quad \|U_c\|_{C^s} + \|H_c\|_{C^s} + \|C_c\|_{C^s} \leq C_s T^s \quad \text{on } U_{\text{fin}},$$

also after a fixed physical cutoff. The same event has at most $CR^{1+\epsilon}$ nodes, weights at most $CR^{-1-\epsilon}$, bounded mass and same-color separation ch . For every prefix $K \leq H$, its current normal satisfies

$$(K.7) \quad \|E_K - E_{0,K}\| \leq C[R^{-(\epsilon-\theta)/4} + R^{-a+1+\theta+\beta}].$$

The inverse Q_0 still belongs to r_F . All preparation orders, profiles and domains precede the accuracy of the later mean solve.

Proof. Write $r(v) = r_F \sqrt{1+v}$ and let $\mathcal{H}_\lambda(v)$ be the colored continuum Hermitian mean of its current zero atoms. Put

$$\mathcal{E}_\lambda(v) = \mathcal{C}[\mathcal{H}_\lambda(v) - H_{\text{transport}}[r(v)^2]].$$

For each fixed s , the current-atom estimates and linearity in the initial scalar give, on a bounded finite-jet positive class,

$$(K.8) \quad \|\mathcal{E}_\lambda(v) - \mathcal{E}_\lambda(w)\|_{H^s} \leq C_s \lambda^{-1} \|v - w\|_{C_{z,y}^{m_s}}, \quad \|\mathcal{E}_\lambda(0)\|_{H^s} \leq C_s \lambda^{-1}.$$

Here m_s is finite. Indeed both the finite primary and its fixed response are linear in r . Their derivative amplitude is its transport amplitude plus a C^s error of size λ^{-1} with a finite initial-data shift. Subtract the two Hermitian squares and apply the positive square root. The masks are only integrated; $0 \leq m_c \leq 1$ and $\sum_c m_c = 1$ make these constants uniform in h .

Start $v_0 = 0$, $U_0 = V_{\text{base}}$ and $e_0 = \mathcal{E}_\lambda(0)$. For $i < J$, put $h_i = A_F^* Q_0 e_i$ and use the uncut density synthesis of (I.21):

$$\begin{aligned} \delta v_{i,a} &= -2\Re \mathcal{F}^{-1}[\alpha_a \chi_{\text{hi}} |\eta|^{1/2} h_{i,a}], \\ v_{i+1} &= v_i + \delta v_i, \quad U_{i+1} = U_i - \frac{1}{2} dF g_F^{-1} \langle D \rangle^{-1} \chi_{\text{src}} h_{i,\text{im}}. \end{aligned}$$

Define the full residual, including the exterior component, by

$$e_i = \mathcal{C}[\mathcal{H}_\lambda(v_i) - H_{\text{transport}}[r_F^2]] - \sum_{k < i} A_0 A_0^* Q_0 e_k.$$

Equation (I.25) gives the full measured density change $-\mathcal{T}_0 \mathcal{T}_0^* Q_0 e_i$; the auxiliary change is $-A_0 A_0^* Q_0 e_i$. Since $A_F A_F^* Q_0 = I$ on the whole torus,

$$(K.9) \quad e_{i+1} = \mathcal{E}_\lambda(v_{i+1}) - \mathcal{E}_\lambda(v_i).$$

No source is cut again between these steps.

The inverse and density estimates have fixed shifts. Start above the initial-scalar orders of Corollary H.7, then choose their finite input orders and m_s backwards through the fixed J steps, then put this finite budget into r_* . Induction gives $e_i = O(\lambda^{-i-1})$ and $v_i = O(\lambda^{-1})$ at all prescribed low orders. Thus $1 + v_i \geq 1/2$ at a large threshold, and $r_{F,R} = r(v_J)$ is defined. For the high profiles, a scalar bound $C_s T^{s+b}$ gives a primary strain of size $\lambda^{-3} T^{s+b+D_{\text{pr}}}$. The graded response at $K = T$ and fixed Sobolev embedding give current envelopes $C_s T^{s+b+D_{\text{pr}}+\ell}$. One-high products, the density inverse and the auxiliary field add only fixed shifts. After J steps all fields therefore have bounds $C_s T^{s+B_J}$ with B_J independent of s . Include a fixed low order above $R_0 + B_J + 1$, and interpolate with a sufficiently high order; this proves the scalar profile in (K.5). The same argument gives grade T^s for U_J and its full strain. These are finite choices independent of later accuracy.

Now fix this family and sample. The deterministic lattice estimate at order $q_{\text{quad}} < r_{\text{par}}$, gives $\|H_c - \mathcal{H}_\lambda(v_J)\|_2 \leq Ch^{q_{\text{quad}}}$. Set $U_c = U_J$ and

$$r_c = e_J + \mathcal{C}[H_c - \mathcal{H}_\lambda(v_J)].$$

The strict choices of J, q_{quad} give its L^2 bound. On D , the original positive-base identity and vanishing exterior row give $r_c = q_F C_c$. Bounded mass and the current envelopes prove all field and global residual profiles.

The sampling theorem and current-atom theorem compare every current prefix with the leading row at $r_{F,R}$ by the right side of (K.7). The phase is unchanged, and the scalar difference is $O(\lambda^{-1})$ in the fixed full-amplitude seminorms. Thus replacing that leading row by the original $E_{0,K}$ adds only $C\lambda^{-1}$, uniformly in K , by the exact normal formula of Appendix I. This is absorbed by the displayed bound. The random event is chosen only after the complete prepared family has been fixed. $\square$

K.2. One fixed current row. Fix the prepared scalar and the sample event above. All subsequent wave maps use this scalar, while A_F, Q_0 keep the original r_F .

Definition K.3 (The fixed current measure maps). At each literal sample let $\mathcal{L}_j(m)$ be the current map $Z(m)$ of Definition J.3, with initial scalar $r_{F,R}$ and

$$\tau_j = \lambda_j^{-1}, \quad \vartheta_j = S_F(x, \zeta_j)/\tau_j.$$

Function data mean $dm = \hat{f} d\xi$; the zero atom evaluates the original integrand at $\xi = 0$. Use the full domains and cutoffs of Setup J.1. The region D_{lag} contains every measurement, extension, pair-source and physical-output support. Corollary H.7 proves the joint low bounds for the raw slice spaces, with both quadratures. These wave maps and responses stay fixed in every update.

Definition K.4 (One current canonical row). Use the graded response window and the current zero-atom background $B_j = e^{-i\vartheta_j} d\mathcal{L}_j(\delta_0)$. Use the raw space X_K of Theorem G.8 with both real quadratures, and set $B_{j,K}^{\text{in}} = \alpha_j \chi_{\text{hi}} \mathcal{B}_K$ in the initial Fourier variables. These are the detector and high-pass masks of the positive row; the zero atom above is uncut. Define

$$G_{j,K}v = e^{-i\vartheta_j} d\mathcal{L}_j(B_{j,K}^{\text{in}}v), \quad A_{j,K}v = \mathcal{C}[2 \text{sym } \Re(B_j^* G_{j,K}v)],$$

and put

$$A_K = (\sqrt{\mu_j} A_{j,K})_j, \quad E_K = \mathcal{F}_{0,K} + A_K A_K^*.$$

The slice cutoff, equal to one on every measurement support, is understood only in norm estimates. Spatial initial cutoffs remain inside the physical wave map, after the Fourier input. Every mean step uses this same background and canonical row.

The current-atom theorem and the raw slice bound give

$$(K.10) \quad \|G_{j,K}v\|_{H^s} \leq C_s K^{s+1/2} \|v\|_2.$$

Proposition K.2 compares all sampled prefixes with $E_{0,K}$ on the entire output. Hence A_K is bounded. Only the continuum normal is inverted.

For $q = Q_0 r$, define the canonical data by

$$(K.11) \quad f_{j,K}[q] = B_{j,K}^{\text{in}} A_{j,K}^* q, \quad D_{j,K}[q] = e^{-i\vartheta_j} d\mathcal{L}_j(f_{j,K}[q]).$$

Adjoints use the real Hilbert products; the row weight and physical normalization cancel, with no inverse sample weight. The complete linear response at B is exactly $E_K q$. The individual raw bound $\|A_{j,K}\| \leq C\sqrt{K}$, (G.32) and the current version of (G.34) give

$$(K.12) \quad \|f_{j,K}[Q_0 r]\|_{C_{\text{initial}}^m} + \|D_{j,K}[Q_0 r]\|_{C^m} \leq C_m K^{m+p} \|r\|_2, \quad p = (n+1)/2.$$

The initial Fourier support is below CK , and its Fourier L^1 norm is at most $CK^p \|r\|_2$. Apply the raw Sobolev and physical bounds before summing, then use boundedness of A_K :

$$(K.13) \quad \sum_j \mu_j \|D_{j,K}[Q_0 r]\|_{H^s}^2 \leq C_s K^{2s+1} \|A_K^* Q_0 r\|_2^2 \leq C_s K^{2s+1} \|r\|_2^2,$$

$$(K.14) \quad \sum_j \mu_j \|D_{j,K}[Q_0 r]\|_{C^s}^2 \leq C_s K^{2s+n} \|A_K^* Q_0 r\|_2^2 \leq C_s K^{2s+n} \|r\|_2^2.$$

These bounds hold when $T \leq K$. For a complete derivative change D , the exact self-tensor increment from B is

$$(K.15) \quad 2 \text{sym } \Re(B_j^* D_j) + \Re(D_j^* D_j).$$

Equal phases cancel before differentiation.

For the mean solve the fixed normal comparison is

$$(K.16) \quad \|E_K - E_{0,K}\|_{2 \rightarrow 2} \leq CH^{-\nu}, \quad \nu > 0.$$

Proposition K.2 proves it for all bands at once, before any canonical update.

APPENDIX L. SIGNED QUADRATIC PRODUCTS

L.1. The complete current sources and one solver. The geometric and linear inputs below have been proved in Appendices D and F. Proposition 4.2 gives the complete current derivative envelopes used here.

Setup L.1 (Current geometry and derivative envelopes). Use the actual geometry of Definition I.1, the domains of Setup A.1, and the fixed regular solver of Theorems F.3 and F.8 at F . Suppose

$$(L.1) \quad \|F\|_{C^r} \leq D_r T^{(r-r_*)+}, \quad L_F = T,$$

where the fixed r_* exceeds the coefficient shifts in that solver and in ambient reconstruction. Corollary B.4 gives, for every reconstruction coefficient and fixed cutoff c_F ,

$$(L.2) \quad \|D^r c_F\|_{\infty} \leq C_r T^r.$$

Use the exact current Hamilton phases and literal groups of Definition D.12, with $\tau_j^{-1} \asymp \lambda$. The same-group spacing is ch . The protected signed classes and their true frequencies are

$$\left. \begin{array}{l} \text{distinct same-group differences} \\ \text{sums, doubled carriers, and either sign between groups} \end{array} \right| \begin{array}{l} \rho = \lambda |z - z'| \\ \rho = \lambda. \end{array}$$

Distinct radial labels protect different groups even when their unscaled directions agree. Only the equal-index Hermitian difference is a self term. All band updates at one literal carrier are combined in one field. Thus every off-self term has $2 \leq c\lambda h \leq \rho_{jk} \leq C\lambda$.

The mixed phase grades of Theorem G.5 give $|D_x^r S| + |D_x^r D_{\zeta} S| + |D_x^r D_z S| \leq C_r T^{r-1}$ for $r \geq 1$. The segment formula on each convex carrier patch gives the same bound for a divided difference. Proposition D.9 and Theorem F.6 place every normalized pair covector in the fixed buffered regular region. The whole source product is protected there, and every complete characteristic segment stays in W_0 in either orientation. Use their original source cutoff χ_{src} , supported where $\chi_2 = 1$ and equal to one near D ; all measurement and output supports lie in the common validity charts.

Let W_j^{lag} be the complete complex fields, with

$$(L.3) \quad dW_j^{\text{lag}} = e^{i\theta_j} A_j, \quad \theta_j = S_F(x, \zeta_j)/\tau_j.$$

After one fixed envelope cutoff equal to one on every pair source, assume

$$(L.4) \quad \|A_j\|_{C^s} \leq C_s K_{\text{env}}^s, \quad L_{\text{pair}} = \max(K_{\text{env}}, T) \leq \rho_{jk}^{1-\sigma}, \quad K_{\text{env}} \geq 2, \quad \sigma > 0.$$

For a fixed mean construction, these fields and coefficients serve every product accuracy and observation order. Equation (4.12) proves this envelope for the final measures on the whole protected source support; Proposition 7.2 verifies the separate history bound in the iteration.

Definition L.2 (The one complete regular solve). The preceding real operators and literal source cutoffs give

$$\mathcal{C}_F = \chi_2 \mathcal{R}_F \chi_3, \quad \|\mathcal{C}_F\|_{2 \rightarrow 2} \leq \kappa_0 < 1, \quad \mathcal{T}_F = (I + \mathcal{C}_F)^{-1}, \quad u = \mathcal{S}_F \mathcal{T}_F h.$$

All outgoing branches, endpoints, cutoff and low-frequency terms are included. The quantitative theorem applies to a source with its full complementary tail; the actual inverse and physical cutoffs do not depend on the beat or proof accuracy.

Write $N_F = \mathbf{N}_F$, $V_F = \mathcal{V}_F$ for the reconstruction of (G.2). For a real metric source q , put $h_i = -\frac{1}{2}Q_i(F) : q$ and define

$$(L.5) \quad U = V_F(u) + \frac{1}{2}N_F \Pi_{K_F^\perp} q.$$

Equation (2.6), with target $-q$, converts $P_F u - h$ to the full error $D\gamma_F U + q$ on D through the actual Gram inverse.

For ambient derivative matrices use $\mathbf{B}(A, B) = (A^T B + B^T A)/2$.

L.2. Exact signed expansion and weighted correction. We sum the actual pair corrections with the literal physical coefficients and leave the complete Hermitian self contribution for the current mean.

Lemma L.3 (The exact signed expansion). *Let $b > 0$, $\mu_j \geq 0$, $dW_j = e^{i\theta_j} A_j$, and set*

$$P = b\sqrt{2} \sum_{j=1}^M \sqrt{\mu_j} \text{Re } W_j, \quad H_{\text{self}} = b^2 \sum_j \mu_j \text{Re}(A_j^* A_j).$$

For the ordered signed pairs put

$$q_{jk}^+ = \text{Re}(e^{i(\theta_j + \theta_k)} \mathbf{B}(A_j, A_k)), \quad q_{jk}^- = \text{Re}(e^{i(\theta_j - \theta_k)} \mathbf{B}(A_j, \overline{A_k})).$$

Then

$$(L.6) \quad Q_{\text{off}} := (dP)^T dP - H_{\text{self}} = b^2 \left[\sum_{j,k} \sqrt{\mu_j \mu_k} q_{jk}^+ + \sum_{j \neq k} \sqrt{\mu_j \mu_k} q_{jk}^- \right].$$

The first sum includes doubled carriers. The omitted difference diagonal is the complete Hermitian self tensor, with every modulation and current correction.

Proof. For complex matrices D, E ,

$$2\mathbf{B}(\Re D, \Re E) = \Re \mathbf{B}(D, E) + \Re \mathbf{B}(D, \overline{E}), \quad \mathbf{B}(D, \overline{D}) = \Re(D^* D).$$

Expand dP using the first identity and subtract its difference diagonal using the second. $\square$

Proposition L.4 (One current solve for the complete off-self tensor). *Use Setup L.1, Definition L.2, and $\sum_j \mu_j \leq C_\mu$. Use the exact tensor Q_{off} of (L.6). Put*

$$q = \chi_{\text{src}} Q_{\text{off}}, \quad h_i = -\frac{1}{2} Q_i(F) : q, \quad z = \mathcal{T}_F h, \quad u = \mathcal{S}_F z,$$

and define one real current field

$$(L.7) \quad S_{\text{lag}} = V_F(u) + \frac{1}{2} N_F \Pi_{K_F^\perp} q.$$

With $\mathcal{W}_\mu = (\sum_j \sqrt{\mu_j})^2 \leq M C_\mu$,

$$(L.8) \quad \|D\gamma_F S_{\text{lag}} + Q_{\text{off}}\|_{H^s(D)} \leq C_{s,Q} b^2 \mathcal{W}_\mu (\lambda h)^{-Q},$$

$$(L.9) \quad \|S_{\text{lag}}\|_{H^s(W)} \leq C_s b^2 \mathcal{W}_\mu \lambda^{s+1}.$$

For a fixed physical cutoff $\chi_{\text{out}} = 1$ near D and supported in the observation chart, the same size bound holds for $\chi_{\text{out}} S_{\text{lag}}$ on the torus. The literal field and operators serve all s, Q . All doubled and distinct signed terms are included; the complete Hermitian self tensor is excluded. The sample count enters only through $\mathcal{W}_\mu \leq M C_\mu$.

Proof. Step 1. One signed source at its true scale. Let $U_{jk}^\pm$ and $u_{jk}^\pm$ be its ambient and reduced fields at $q = \chi_{\text{src}} q_{jk}^\pm$. Leibniz, including the source cutoff and current reduced multiplier, bounds each product amplitude directly by

$$\|a_{jk}\|_{C^s} + \|a_{jk}\|_{H^s} \leq C_s I_{\text{pair}}^s.$$

The geometric coefficients have the derivative scale (L.2); only the total derivative order enters.

Write the product as $e^{i\rho\varphi} a_{jk}$ at its true $\rho = \rho_{jk}$. The normalized phase has $|D^r \varphi| \leq C_r L_{\text{pair}}^{r-1}$ and a fixed nonzero gradient. Its compact protected covector set gives $|\rho d\varphi - \xi| \geq c(\rho + |\xi|)$ on the entire complementary mask. Proposition C.7 therefore bounds the off-mask source in H^s by

$$C_{s,k} \rho^{s+n/2} (L_{\text{pair}}/\rho)^k.$$

Since $L_{\text{pair}} \leq \rho^{1-\sigma}$, a finite k gives any requested decay. The source plateau and protected complete segments are those of Definition L.2. The fixed solve and (2.6) give

$$(L.10) \quad \|D\gamma_F U_{jk}^\pm + q_{jk}^\pm\|_{H^s(D)} \leq C_{s,Q} \rho_{jk}^{-Q}.$$

A higher finite proof accuracy pays the fixed frame and Gram derivative costs, without changing the field.

Step 2. The exact weighted sum. The solver, inverse, frame and cutoffs are fixed across all pair scales. By Cauchy–Schwarz, the total absolute coefficient in (L.6) is at most $2b^2 \mathcal{W}_\mu \leq 2b^2 M C_\mu$. By linearity, (L.10) and $\rho_{jk} \geq c\lambda h$ give the full tensor error on D . For the size, put $\mathfrak{A} = b^2 \mathcal{W}_\mu$. Leibniz and the same coefficient sum give

$$\|q\|_{H^s} + \|h\|_{H^s} \leq C_s \mathfrak{A} \lambda^s.$$

Corollary B.4 absorbs the fixed shifts in the strong tame estimates for $\mathcal{S}_F \mathcal{T}_F$; since $L_F \leq \lambda$, $\|u\|_{H^s} \leq C_s \mathfrak{A} \lambda^s$. The geometric profile and one derivative in V_F prove the ambient size bound, also after physical cutoff. The operators are real and uniform on the fixed chart menu. $\square$

APPENDIX M. VERIFICATION OF THE FIXED SCALES

We check the choices of Section 8 on the fixed domains, with the same operators and sample family.

Proposition M.1 (The fixed scale windows). *The stated choices satisfy all primary, current-response, mean and product hypotheses. Their exponents and finite low topologies are independent of stage accuracy and the final target dilation.*

Proof. 1. Preparation and one sample. The fixed quadrature order is $q_{\text{quad}} = 2n - 1 < r_{\text{par}}$. Its color list depends only on n and changes only constants. Choose A by (8.6) and $J = \lfloor 2A/a \rfloor$. Then

$$a(J+1) > 2A, \quad \beta(2n-1) - 2A \geq 3/64.$$

The radial labels, common geometric neighborhood and finite backward input orders then precede sampling and mean accuracy. The existing joint amplitude budget covers the quadrature. The prepared scalars stay in the original positive class. Every translated lattice has the required background error; the full-prefix normal event has positive probability at large R . In R units, the two normal-error exponents are

$$(\epsilon - \theta)/4 = 1/256, \quad a - 1 - \theta - \beta = 13/320.$$

The $O(\lambda^{-1})$ change from the original leading row is smaller. The inverse and all auxiliary input cuts are unchanged.

2. Modulation and product bands. Use a fixed multiple of T_0 for the zero atom and a fixed larger multiple of the actual input band for each mean datum. With $\sigma_{\text{pair}} = 1/1000$ and $\nu = 1/4$, direct calculation gives

$$1 - t = 1/35, \\ (1 - \nu)a - s_f = 27/640, \quad (1 - \sigma_{\text{pair}})(a - \beta) - s_f = 7581/160000.$$

These verify $T_0 \leq K \leq \lambda_j^{1-\nu}$ and the product window at every true beat $c\lambda h \leq \rho \leq C\lambda$. The input support lies below K/C_1 , and its ratio to λ_j tends to zero. The primary and coefficient scale is T_0 ; all physical slopes are at most $a < b_*$. Strict gaps absorb the fixed support and radial constants.

3. The sampled mean. For $p = (n+1)/2$ and $A = p(1+\theta)/2$,

$$\epsilon_f K_{\text{max}}^p = CR^{-p\theta/2} \leq H^{-\nu_{\text{nl}}}, \quad \nu_{\text{nl}} = \frac{p\theta}{4(1+\theta)} > 0.$$

Choose a contraction exponent below this and the comparison exponents in Step 1. The prepared residual has L^2 size R^{-2A} and global H^s profile $C_s R^{ts}$. Interpolation with $t < 1$ gives the source bound $C_s \epsilon_f R^s$ in (5.11). Its slope $1/(1+\theta)$ and the coefficient slope $t/(1+\theta)$ lie below the first input band. At every finite update count the equal-gap bands have a positive gap, so one finite derivative order controls every omitted input. The low residual sum proves the variation and physical field bounds.

4. Increasing precision. Choose the fixed high history order after the preparation orders, then k_{min} by (8.7)–(8.8); $3 - 2\nu = 1/12 > 0$. Set $Q(k) = 2k$. Take $aQ_{\text{primary}} > Q(k) + A + (1+\epsilon)/2$, mean accuracy above $Q(k) + 2A + 2$, and smoothing order $(1-t)N > 2Q(k)$. The product and physical-cutoff gaps $\sigma_{\text{pair}}(a - \beta) > 0$ and $b_* - a > 0$ give every finite precision for the same primary family and responses. Only the finite mean updates vary with accuracy. $\square$

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