

RAMANUJAN'S PRIME COUNTING INEQUALITY HOLDS FOR ALL $x \geq 38,358,837,683$: RANGES FROM THE ZEROS, A SPECTRAL BARRIER, AND TRADE-OFFS

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ABSTRACT. Ramanujan proved that $\pi(x)^2 < \frac{ex}{\log x} \pi(x/e)$ for all sufficiently large x . Let N_R be the least integer such that this holds for every real $x \geq N_R$. Dudek and Platt showed that the Riemann hypothesis implies $N_R = 38,358,837,683$. We prove that $N_R = 38,358,837,683$, to our knowledge the first determination of N_R without assuming the Riemann hypothesis. The proof uses the zero-free half-plane $\operatorname{Re} s > 7/8$ of a recent OpenAI preprint; its main statement for ζ is formalized in Lean 4 and has been rebuilt independently, while its written proof has not been peer reviewed (Corollary F). The deduction is Theorem B: if $\zeta(s) \neq 0$ for $\operatorname{Re} s > 0.945092$, then $N_R = 38,358,837,683$. Its proof uses only refereed inputs, among them an explicit formula of Cully-Hugill and Johnston, together with Bhattacharjee's correction of a theorem of Büthe.

Relying only on the peer-reviewed literature, and on preprints where stated, we prove the following by rigorous computer-assisted interval arithmetic. (A) The inequality holds for $38,358,837,683 \leq x \leq e^{753.98}$; the previous range ended at e^{103} . The proof uses the bound $|\psi(x) - x| \leq \frac{\pi}{3 \cdot 10^{12}} x + 113.67\sqrt{x}$ from a recent preprint of Chirre and Helfgott; with refereed inputs and Bhattacharjee's correction, the same method reaches $e^{672.84}$. (C) It holds for $x \geq e^{3035.01}$. For this we re-implement Theorem 1.1 of Fiori, Kadiri and Swidinsky, insert the zero-free region of Mossinghoff, Trudgian and Yang and the zero-density constants that rest on the subconvexity bound of Hiary, Patel and Yang, and evaluate it at every integer value of $\log x$. This replaces Axler's threshold $e^{3158.442}$, whose proof applies a lemma of Dudek and Platt in a case where that lemma's proof is defective. With the zero-free region of a recent preprint of Bellotti, Trudgian and Yang and the log-free zero-density estimate of a recent preprint of Bellotti, the same method gives $x \geq e^{2591.01}$. (B) With preprints of Fiori–Jaskari and Chirre–Helfgott, the half-plane $\operatorname{Re} s > 0.96897$ suffices. A zero-free region $\beta \leq 1 - 1/(1.7373 \log |\gamma|)$ for $|\gamma| > 3 \cdot 10^{12}$ also suffices. (D) So does a verification of the Riemann hypothesis up to height $4.1602 \cdot 10^{14}$, about 139 times the current verification height. The proof combines the zeros up to that height with a zero-free region and zero-density estimates above it. We also tabulate the trade-off between the verification height and the constant of a zero-free region. (E) Using a construction of Chirre and Helfgott, we show that arguments that use only non-negativity, analytic continuation, and the location and residues of the poles of the Dirichlet series up to a height T are limited. With $T = 3 \cdot 10^{12}$ they cannot go beyond $e^{754.40}$, so (A) is optimal among such arguments to within 0.41 in $\log x$; near e^{3035} they need $T > 7.9624 \cdot 10^{14}$, almost twice the height in (D).

Bhattacharjee found that the constant E_2 in Theorem 1 of Büthe (2016) lacks a factor $\log(3c)$. We show that the rows of Table 8 of Broadbent, Kadiri, Lumley, Ng and Wilk that come from that theorem inherit the error, by 1.3–2.2% after re-optimizing the parameters, and trace the consequences in the literature: no published theorem we examined becomes false, but several proofs need a patch or a newer reference. We also identify the errors in two recent preprints that claim the inequality unconditionally.

1. INTRODUCTION

Let $\pi(x)$ be the number of primes not exceeding x . Ramanujan (see Berndt [Ber94]) showed that

$$(R) \quad \pi(x)^2 < \frac{ex}{\log x} \pi\left(\frac{x}{e}\right)$$

for all sufficiently large x . Both sides of (R) are of size $x^2/\log^2 x$, while their difference is of size $x^2/\log^6 x$ (Hassani [Has21] computed further terms of its asymptotic expansion; see also Lemma 3.7 below). Deciding (R) at a given x therefore requires bounds for π with relative error of order $\log^{-4} x$.

Let N_R be the least integer such that (R) holds for every real $x \geq N_R$. Dudek and Platt [DP15] computed that the largest integer counterexample below 10^{11} is $x = 38,358,837,682$, showed that the Riemann

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hypothesis (RH) implies $N_R = 38,358,837,683$, and proved (R) unconditionally beyond an explicit threshold. Platt and Trudgian [PT21b] lowered that threshold, and Axler [Ax22, Thm. 1.1] stated the threshold $e^{3158.442}$ (see Section 7.3). At the lower end, Axler [Ax18, Thm. 1.3] proved (R) for $38,358,837,683 \leq x \leq 10^{19}$, and Johnston [Joh22, Thm. 5.1] extended this range to $38,358,837,683 \leq x \leq e^{103}$.

1.1. The main result.

Corollary F. *By the zero-free half-plane $\operatorname{Re} s > 7/8$ of [OAI26], formalized in Lean 4 [OAI26] and rebuilt independently [Tom26], inequality (R) holds for every real $x \geq 38,358,837,683$. Hence $N_R = 38,358,837,683$.*

This is, to our knowledge, the first determination of N_R without assuming RH; it confirms the value that Dudek and Platt derived from RH. The half-plane of [OAI26] is recent, and we cannot rule out unpublished deductions of the same kind. The only ingredients of the proof that have not been refereed are the result of [OAI26], which is machine-checked, and Bhattacharjee’s correction [Bha24] of B  the’s Theorem 1. The written proof in [OAI26] has not been peer reviewed; the Lean check certifies the formal statement quoted in Lemma 2.6. Corollary F follows from the next theorem.

Theorem B. *Let $\theta \leq 0.945092$, and assume that $\zeta(s) \neq 0$ for every s with $\operatorname{Re} s > \theta$. Then (R) holds for every real $x \geq 38,358,837,683$. Consequently $N_R = 38,358,837,683$.*

The proof of Theorem B uses only refereed inputs, namely the explicit formula of Cully-Hugill and Johnston [CHJ23], the verification of RH by Platt and Trudgian [PT21a], [B  t16, Lemma 3], the results of Dudek–Platt and Axler (Lemma 2.13) and the inputs of Theorem C below, together with B  the’s Theorem 1 as corrected in [Bha24], which enters through Proposition 4.1. The value 0.945092 is the limit of the method to within 10^{-5} : the verification succeeds for $\theta = 0.945092$ and fails for $\theta = 0.945099$, at the lower end $e^{672.5}$ of its range. For $\theta = 7/8$ it covers $e^{247.5} \leq x \leq e^{3036}$, so in Corollary F the range $[10^{19}, e^{247.5}]$ has two independent proofs, Proposition 4.1 and Theorem A below, and Proposition 4.1 still reaches $e^{564.99}$ if every row of B  the’s Theorem 1 is doubled (Section 4.6).

1.2. Results from the peer-reviewed literature and preprints. Theorems A–D below are computer-assisted. In Section 3 we reduce (R) at $x = e^L$ to the positivity of an explicit function of L , built from an explicit bound for $|\psi(x) - x|$. This positivity is verified in rigorous interval arithmetic (mpmath, 320-bit precision) on a partition of the relevant range of L into short intervals, and a monotonicity argument covers $L \geq 10^4$. Theorem E is proved analytically; its numerical consequences (Corollary E) are evaluated by computer.

Theorem A. *Inequality (R) holds for every real x with $38,358,837,683 \leq x \leq e^{753.98}$.*

Theorem C. *Inequality (R) holds for every real $x \geq e^{3035.01}$.*

Theorem C*. *Inequality (R) holds for every real $x \geq e^{2591.01}$.*

Theorem B*. *Assume that $\zeta(s) \neq 0$ for every s with $\operatorname{Re} s > 0.96897$. Then (R) holds for every real $x \geq 38,358,837,683$. Consequently $N_R = 38,358,837,683$.*

Theorem B’. *Let $0 < R \leq 1.7373$, and assume that $\zeta(\beta + i\gamma) \neq 0$ whenever $|\gamma| > 3 \cdot 10^{12}$ and $\beta > 1 - 1/(R \log |\gamma|)$. Then (R) holds for every real $x \geq 38,358,837,683$. Consequently $N_R = 38,358,837,683$.*

Theorem D. *Assume that every zero $\beta + i\gamma$ of $\zeta(s)$ with $0 < \gamma \leq 4.1602 \cdot 10^{14}$ satisfies $\beta = 1/2$. Then (R) holds for every real $x \geq 38,358,837,683$. Consequently $N_R = 38,358,837,683$.*

Theorem A extends Johnston’s range from e^{103} to $e^{753.98}$. It depends on the preprint [CH25] of Chirre and Helfgott. Above 10^{19} its input is their bound $|\psi(x) - x| \leq \frac{\pi}{3 \cdot 10^{12}} x + 113.67\sqrt{x}$ [CH25, Cor. 1.3], which rests on the verification of RH up to height 3,000,175,332,800 by Platt and Trudgian [PT21a]. Below 10^{19} the inputs are B  the’s bound $|\psi(x) - x| \leq 0.94\sqrt{x}$ [B  t18] and Axler’s verification [Ax18]. With refereed inputs and B  the’s Theorem 1 [B  t16] at height 3,000,175,332,800, as corrected by Bhattacharjee [Bha24], the same method reaches $e^{672.84}$ (Proposition 4.1). Neither proof uses Table 8 of Broadbent, Kadiri, Lumley, Ng and Wilk [BKLNW21]: Bhattacharjee [Bha24] found that the constant E_2 in B  the’s Theorem 1 lacks a factor $\log(3e)$, and in Section 7.4 we show that the rows of that table that come from B  the’s theorem reproduce from the formula as printed.

Theorem C lowers Axler's threshold $e^{3158.442}$ by 123 in $\log x$. Axler's proof applies Lemma 2.1 of Dudek and Platt [DP15] in the case where that lemma's proof is defective (Section 7.3); with the repair recorded in IEANTN [Tao+26], Axler's own parameters give only about $e^{3160.3}$. The proof of Theorem C does not use [DP15, Lemma 2.1], and all its inputs are published. Its main input is Theorem 1.1 of Fiori, Kadiri and Swidinsky [FKS23], which bounds $|\psi(x) - x|/x$ for all $x \geq x_0$ in terms of the height to which RH is verified, a zero-free region and zero-density estimates. They tabulate it at $\log x_0 = 1000, 2000, 2100, \dots, 3000$ and $4000, 5000, \dots$. We re-implement it, validate the re-implementation against their table (Section 4.4.1), insert the zero-free region of Mossinghoff, Trudgian and Yang [MTY23] and the zero-density constants that [FKS23] derive from the subconvexity bound of Hiary, Patel and Yang [HPY24], and evaluate it at every integer $\log x_0$ in $[2900, 3160]$. With the density constants for the older subconvexity bound the same method gives $e^{3040.01}$, with the zero-free-region constant of [FKS23] as well $e^{3045.01}$, and with their printed rows alone $e^{3157.94}$ (Remark 4.3).

Theorem C* depends on two preprints: the zero-free region of Bellotti, Trudgian and Yang [BTY26] (Lemma 2.10(ii)) and the log-free zero-density estimate of Bellotti [Bel24] (Lemma 2.11(ii)). Its proof is the same computation with these inputs, evaluated at every integer $\log x_0$ in $[2400, 3000]$.

Theorem B* is the version of Theorem B with the preprints of Fiori and Jaskari [FJ26] and Chirre and Helfgott [CH25]. The exponent is within 10^{-5} of the limit of that method: the verification succeeds under $\text{Re } \rho \leq 0.968978$ and fails under $\text{Re } \rho \leq 0.968980$, and the binding constraint lies at the left end of the open range, near $x = e^{754}$. The hypotheses of Theorems B and B* are also implied by the half-plane $\text{Re } s > 0.87496$ claimed by Liu [Liu26], a preprint that has not been refereed.

Theorem B' assumes a zero-free region of the classical shape above the height to which RH is verified. Its proof combines [FKS23, Thm. 1.1], re-implemented with the constant R of the hypothesis, with [CH25, Cor. 1.3] and Theorems A and C. The verification succeeds for $R = 1.73730$ and fails for $R = 1.73779$, again with the binding constraint at the left end of the open range. The best known constants are $R = 5.558691$ [MTY23] and, in the preprint [BTY26], $R = 4.896$; Theorem B' needs a constant smaller by a factor of about 3.2, respectively 2.8. Table 7 in Section 6 interpolates between Theorems B' and D.

Theorem D reduces the determination of N_R without [OAI26] to a single finite computation: a verification of RH up to height $4.1602 \cdot 10^{14}$, about 139 times the height 3,000,175,332,800 reached by Platt and Trudgian [PT21a]. All its inputs except [CH25] are published. Its proof combines [CH25, Prop. 9.1] at height T with [FKS23, Thm. 1.1] recomputed with the verification height T ; above height T this uses the zero-free region of [MTY23] and the zero-density constants of Theorem C. The verification succeeds at height $4.16017 \cdot 10^{14}$ and fails at $4.15735 \cdot 10^{14}$. With the two preprints used in Theorem C* the height drops to $2.1488 \cdot 10^{14}$, about 72 times 3,000,175,332,800 (Remark 4.6). If the zeros above 3,000,175,332,800 enter only through [CH25, Prop. 9.1], the method needs $7.9612 \cdot 10^{14}$; with the printed rows of [FKS23] it needs $9.3357 \cdot 10^{14}$; and with Büthe's Theorem 1 as corrected in [Bha24] in place of [CH25], so that all inputs except [Bha24] are refereed, it needs $1.3277 \cdot 10^{15}$, about 3.19 times the height in Theorem D (Remarks 4.7 and 4.8).

1.3. A spectral barrier. If one relies only on the peer-reviewed literature, Theorems A and C leave the range

$$e^{753.98} < x < e^{3035.01}$$

uncovered, and Theorem C* reduces it to $e^{753.98} < x < e^{2591.01}$. Theorem E shows that this range cannot be closed by arguments that use only analytic information about the poles up to the verified height.

For $T > 0$ let $\mathcal{A}_T$ be the class of sequences $(a_n)_{n \geq 1}$ such that (a) $a_n \geq 0$; (b) the Dirichlet series $A(s) = \sum a_n n^{-s}$ converges absolutely for $\text{Re } s > 1$ and continues meromorphically to $\mathbb{C}$; and (c) A has a simple pole at $s = 1$ with residue 1 and no other pole with $|\text{Im } s| \leq T$ (Definition 5.1). For such a put $\pi_a(y) = \sum_{2 \leq n \leq y} a_n / \log n$. Theorem E (Section 5) is an explicit criterion for $\mathcal{A}_T$ to contain a sequence a with $\pi_a(x)^2 > \frac{ex}{\log x} \pi_a(x/e)$ at some x near a given point. The sequences are the Fejér-kernel sequences $a_n = F_H(T' \log n)$ of Chirre and Helfgott [CH25, Prop. 3.1]. Evaluating the criterion gives the following.

Corollary E. (i) *Let $T = 3,000,175,332,800$. For every $L \in [754.40, 10^4]$ there are $a \in \mathcal{A}_T$ and x with $L \leq \log x \leq L + 10^{-11}$ such that $\pi_a(x)^2 > \frac{ex}{\log x} \pi_a(x/e)$.*

(ii) *Let $T \leq 7.9624 \cdot 10^{14}$. There are $a \in \mathcal{A}_T$ and x with $3035.002 \leq \log x \leq 3035.002 + 10^{-5}$ such that $\pi_a(x)^2 > \frac{ex}{\log x} \pi_a(x/e)$.*

(iii) Let $T \leq 9.3344 \cdot 10^{14}$. There are $a \in \mathcal{A}_T$ and x with $3157.93 \leq \log x \leq 3157.93 + 10^{-5}$ such that $\pi_a(x)^2 > \frac{ex}{\log x} \pi_a(x/e)$.

The von Mangoldt function Λ , with $A(s) = -\zeta'(s)/\zeta(s)$, is not in $\mathcal{A}_T$. It lies in the larger class $\mathcal{A}_T^{\text{RH}}$ in which A may also have poles with $|\text{Im } s| \leq T$ on the critical line and in the half-plane $\text{Re } s < 0$ (Definition 5.1; for $T = 3,000,175,332,800$ by Lemma 2.5). For $a = \Lambda$, π_a is Riemann's prime-power counting function, which differs from π by an amount that is negligible here. We call an argument for (R) of type (a)–(c) at height T if it is valid for every sequence in $\mathcal{A}_T^{\text{RH}}$; its only inputs are then non-negativity, analytic continuation, and the location and residues of the poles up to height T as recorded in (c). By Corollary E, such an argument cannot prove (R) at the points provided there. Consequently:

- For arguments of type (a)–(c) at height $3,000,175,332,800$ the barrier lies at $\log x = 754.39478$. Theorem A reaches $\log x = 753.985$, so its range is optimal among such arguments to within 0.41 in $\log x$.
- Near $x = e^{3035}$, at the top of the open range, arguments of type (a)–(c) need $T > 7.9624 \cdot 10^{14}$; the barrier there is $\bar{T}(3035.002) = 7.96241 \cdot 10^{14}$. Theorem D needs only $4.1602 \cdot 10^{14}$, about half of that. Above height T it uses a zero-free region and zero-density estimates, which are not of type (a)–(c).
- With $T = 3,000,175,332,800$ the open range cannot be closed by arguments of type (a)–(c): by Corollary E(i), $\mathcal{A}_T$ contains counterexamples within 10^{-11} in $\log x$ of every point with $754.40 \leq \log x \leq 10^4$. The explicit methods for ψ in this range (Büthe [Büt16], Chirre and Helfgott [CH25], and [BKLNW21, FKS23]) use, beyond pole data up to height T , only zero-free regions and zero-density estimates above T . With these, Theorems C and C* prove (R) for $x \geq e^{3035.01}$ and $x \geq e^{2591.01}$, where $\mathcal{A}_T$ still contains counterexamples. Theorems B, B*, B' and D quantify what would suffice to close the rest, and Corollary F closes it.

Corollary E does not say that (R) fails, or that it is unprovable, anywhere in the open range. It says that arguments of type (a)–(c) do not suffice to prove it there. The counterexamples are far from arithmetic (Remark 5.3): Theorem E is a limitation of a class of methods, not a statement about the primes.

Table 1 summarizes the situation.

1.4. Remarks on the literature. Section 7 records further points. Two recent preprints of De [De24a, De24b] claim (R) unconditionally for all $x \geq e^{547}$ and all $x \geq e^{43.51}$, respectively; neither argument is a proof, and we identify the errors. We explain how the defect in [DP15, Lemma 2.1] recorded in IEANTN affects Axler's threshold [Ax22]. Bhattacharjee [Bha24] found that the constant E_2 in Büthe's Theorem 1 lacks a factor $\log(3c)$, among other corrections. We show that the rows of [BKLNW21, Table 8] derived from that theorem inherit the error, measure the effect after re-optimizing the parameters, and trace the consequences in the literature (Section 7.5): no published theorem we examined becomes false, but several proofs need a patch or a newer reference.

1.5. Organization. Section 2 lists the explicit inputs. Section 3 derives the criterion and extracts the main term exactly. Section 4 describes the computations, including the re-implementation of [FKS23, Thm. 1.1], and proves Theorems A–D and Corollary F. Section 5 proves Theorem E and Corollary E. Section 6 discusses what remains without [OAI26], and Section 7 contains the remarks on the literature.

2. NOTATION AND EXPLICIT INPUTS

Throughout, $\theta(x) = \sum_{p \leq x} \log p$, $\psi(x) = \sum_{p^k \leq x} \log p$, $L = \log x$, $r(t) = \theta(t) - t$, and

$$J(x) = \frac{x}{\log x} + \int_2^x \frac{dt}{\log^2 t} = \text{li}(x) - \text{li}(2) + \frac{2}{\log 2}, \quad j(l) = e^{-l} J(e^l).$$

Lemma 2.1 (Partial summation). *For $x \geq 2$,*

$$\pi(x) = \frac{\theta(x)}{\log x} + \int_2^x \frac{\theta(t)}{t \log^2 t} dt, \quad \text{hence} \quad \pi(x) - J(x) = \frac{r(x)}{\log x} + \int_2^x \frac{r(t)}{t \log^2 t} dt.$$

Lemma 2.2 (Rosser–Schoenfeld [RS62, Thm. 9]). *$\theta(y) < 1.01624y$ for all $y > 0$. Consequently, with $K = 1.01624$,*

$$0 \leq \psi(x) - \theta(x) \leq K \left(\sqrt{x} + x^{1/3} \frac{\log x}{\log 2} \right) \quad (x \geq 2),$$

TABLE 1. Status of (R) for $x \geq 38,358,837,683$.

Range of x	Before this paper	This paper
$[38,358,837,683, \infty)$	under RH [DP15]	Corollary F (via the Lean-verified half-plane of [OAI26]); the rows below rely only on the peer-reviewed literature and on the preprints named there used as input
$[38,358,837,683, 10^{19}]$	unconditional [Ax18, Thm. 1.3]	
$[10^{19}, e^{103}]$	unconditional [Joh22, Thm. 5.1]	Theorem A (independent proof)
$[e^{103}, e^{672.84}]$	open	Theorem A; also Proposition 4.1 (refereed inputs and [Bha24])
$[e^{672.84}, e^{753.98}]$	open	Theorem A (uses [CH25])
$(e^{753.98}, e^{2591.01})$	open	open if one relies only on the peer-reviewed literature, and out of reach of arguments of type (a)–(c) at height 3,000,175,332,800 (Corollary E); closed by Corollary F (Lean-verified [OAI26]); closed by Theorem B ($\text{Re } \rho \leq 0.945092$), Theorem B* ($\text{Re } \rho \leq 0.96897$), Theorem B' ($R \leq 1.7373$), or Theorem D (RH up to height $4.1602 \cdot 10^{14}$; $2.1488 \cdot 10^{14}$ with [BTY26, Bel24])
$[e^{2591.01}, e^{3035.01})$	open	Theorem C* (uses [BTY26, Bel24]); without these preprints, as in the previous row
$[e^{3035.01}, \infty)$	claimed for $x \geq e^{3158.442}$ [Ax22], via the defective case of [DP15, Lemma 2.1]	Theorem C

and $|r(t)| \leq t$ for all $t > 0$.

Proof. We have $\psi(x) - \theta(x) = \sum_{k \geq 2} \theta(x^{1/k})$. The term $k = 2$ is at most $K\sqrt{x}$. Each term with $k \geq 3$ is at most $Kx^{1/3}$, and it vanishes unless $x^{1/k} \geq 2$, that is, unless $k \leq \log x / \log 2$. Finally, $0 \leq \theta(t) < Kt \leq 2t$ gives $|r(t)| \leq t$. $\square$

Lemma 2.3 (Büthe [Büt18, Thm. 2, (1.5)]). $|x - \psi(x)| \leq 0.94\sqrt{x}$ for $11 < x \leq 10^{19}$.

The next statement is Büthe's Theorem 1 [Büt16] as corrected by Bhattacharjee [Bha24, Thm. 7]. Compared with the version arXiv:1410.7015v4 of [Büt16], which already contains the summand $e^{\alpha\varepsilon} - 1$ in E_1 , it has the factor $\log(3c)$ and further corrections in E_2 , and an additional constant term in E_3 (Section 7.4). For $c > 0$ put

$$\eta_c(t) = \frac{c I_0(c\sqrt{1-t^2})}{2 \sinh c} \quad (|t| \leq 1),$$

$$\mu_c(\alpha) = \frac{1}{2} - \int_0^\alpha \eta_c(t) dt, \quad |\nu_c(\alpha)| = \frac{I_1(c)}{2 \sinh c} - \int_0^\alpha \mu_c(t) dt,$$

where I_0, I_1 are modified Bessel functions; these are the quantities of [Büt16, §5].

Lemma 2.4 (Büthe's Theorem 1, as corrected in [Bha24, Thm. 7]). *Let $c \geq 3$, $0 < \varepsilon < 10^{-3}$, $0 \leq \alpha < 1$ and $x_* \geq 100$, put $T = c/\varepsilon$, and assume $T \geq 5000$,*

$$B = \frac{\varepsilon x_* e^{-\varepsilon} |\nu_c(\alpha)|}{2\mu_c(\alpha)} > 1 \quad \text{and} \quad \frac{\varepsilon e^{-2\varepsilon} |\nu_c(\alpha)|}{2\mu_c(\alpha)} < 1.$$

Assume that every zero $\beta + i\gamma$ of $\zeta(s)$ with $0 < \gamma \leq T$ satisfies $\beta = 1/2$. Then

$$|\psi(x) - x| \leq x e^{\alpha\epsilon} (E_1 + E_2 + E_3) \quad \text{for all } x \geq e^{\alpha\epsilon} x_*,$$

where

$$\begin{aligned} E_1 &= e^{2\epsilon} \log(e^\epsilon x_*) \left[\frac{2\epsilon |\nu_c(\alpha)|}{\log B} + \frac{2.01\epsilon}{\sqrt{x_*}} + \frac{\log \log(2x_*^2)}{2x_*} \right] + (e^{\alpha\epsilon} - 1), \\ E_2 &= \frac{0.16(1 + 3x_*^{-1})}{\sinh c} e^{0.71\sqrt{c\epsilon}} \left[\log \frac{c}{\pi\epsilon} \log c + 2 \log \frac{c}{\pi\epsilon} + 18.75\epsilon \log \frac{c}{\epsilon} + 1 \right], \\ E_3 &= 2(x_*^{-1/2} + x_*^{-1}) \sum_{0 < \gamma \leq T} \frac{\ell(\gamma)}{\gamma} + \frac{1.92}{x_*}, \end{aligned}$$

with a weight $\ell(\gamma) \leq 1$. Moreover [Büt16, Lemma 3], $\sum_{0 < \gamma \leq t} 1/\gamma \leq \log^2(t/2\pi)/(4\pi)$ for $t \geq 5000$.

Lemma 2.5 (Platt–Trudgian [PT21a]). *Every zero $\beta + i\gamma$ of $\zeta(s)$ with $0 < \gamma \leq 3,000,175,332,800$ satisfies $\beta = 1/2$.*

Lemma 2.6 (OpenAI [OAI26]). *$\zeta(s) \neq 0$ for every s with $\operatorname{Re} s > 7/8$.*

The written proof in [OAI26] has not been peer reviewed. Its main statement for ζ is formalized in Lean 4 in the repository of [OAI26], file `lean/ComparatorChallenges/QuasiRiemannHypothesis.lean`, as the theorem `riemannZeta.ne_zero_of_seven_eighths_lt_re`: for $s \in \mathbb{C}$ with $7/8 < \operatorname{Re} s$, `riemannZeta s` $\neq 0$. It uses Mathlib’s `riemannZeta`, and `#print axioms` lists only `propxet`, `Classical.choice` and `Quot.sound`. An independent rebuild [Tom26], with Lean 4.34.1 and Mathlib commit `d13f23b`, compiles all 2,924 modules with no errors and no `sorry`, and a static scan finds no `axiom`, `native_decide` or `implemented_by`. We report these checks from [OAI26] and [Tom26]; we have not rebuilt the project ourselves. The correctness of Lemma 2.6 rests on the Lean kernel and on Mathlib’s definition of `riemannZeta`, which agrees with $\zeta(s)$ for $s \neq 1$. In particular every nontrivial zero of ζ has $\beta \leq 7/8$.

Lemma 2.7 (Cully-Hugill–Johnston [CHJ23, Thm. 1.2, Table 4]). *For $x \geq e^{40}$ and $\max\{51, \log x\} < T < (\sqrt{x} - 2)/2$,*

$$\left| \psi(x) - x + \sum_{|\gamma| \leq T} \frac{x^\rho}{\rho} \right| \leq \frac{6.431 x \log x}{T}.$$

This is [CHJ23, Thm. 1.2] with the row $(\log x_M, \alpha, \lambda, M) = (40, 1/2, 0.48, 6.431)$ of [CHJ23, Table 4]. In Section 4.6 we use the weaker bound $200 x (\log x)^2 / T$ and bound the sum over the zeros up to $2.45T$. This also covers the version quoted in [FKS23, Thm. 3.2], in which the sum runs up to some $T^* \in [T, 2.45T]$, and, by the factor 200, the bound of Dudek quoted in [FKS23, Prop. 3.4], even after the correction by a factor 20 mentioned in [FKS23, Remark 3.3].

Lemma 2.8 (Chirre–Helfgott [CH25]). (i) [CH25, Cor. 1.3] *For every $x \geq 1$,*

$$|\psi(x) - x| \leq \frac{\pi}{3 \cdot 10^{12}} x + 113.67 \sqrt{x}.$$

(ii) [CH25, Prop. 9.1 with $\sigma = 0$, and (9.2)] *Let $T \geq 10^7$, and assume that every zero $\beta + i\gamma$ of $\zeta(s)$ with $0 < \gamma \leq T$ satisfies $\beta = 1/2$. Then, for $x > \max(T, 10^9)$,*

$$|\psi(x) - x| \leq \frac{\pi}{T - 1 - \pi/3} x + C_T \sqrt{x}, \quad C_T = \frac{\log^2(T/2\pi)}{2\pi} - \frac{\log(T/2\pi)}{6\pi}.$$

Part (i) rests on Lemma 2.5. The reference [CH25] is a preprint. Chirre and Helfgott [CH25, Prop. 3.1] show by an explicit construction that the leading constant π/T cannot be lowered for bounds that use only non-negativity, analytic continuation and the poles up to height T ; Section 5 builds on that construction. We write

$$\varepsilon_{\text{CH}}(l) = \frac{\pi}{3 \cdot 10^{12}} + 113.67 e^{-l/2}, \quad \varepsilon_T(l) = \frac{\pi}{T - 1 - \pi/3} + C_T e^{-l/2},$$

so that $|\psi(x) - x| \leq \varepsilon_{\text{CH}}(\log x) x$ for $x \geq 1$, and $|\psi(x) - x| \leq \varepsilon_T(\log x) x$ for $x > \max(T, 10^9)$ under the hypothesis of (ii). Both functions are decreasing in l .

Lemma 2.9 (Fiori–Kadiri–Swidinsky [FKS23]). (i) [FKS23, Thm. 1.1, Table 5] *For each tabulated x'_0 , $|\psi(x) - x| \leq \varepsilon_5(x'_0) x$ for all $x \geq x'_0$. The tabulated values of $\log x'_0$ are 1000, 2000, 2100, 2200, $\dots$, 3000 and 4000, 5000, $\dots$, 10000; see Table 2.*
(ii) [FKS23, Cor. 1.4] $|\psi(x) - x| < \varepsilon_1(\log x) x$ for all $x > 2$, where

$$\varepsilon_1(l) = A_1 l^{3/2} e^{-c_1 \sqrt{l}}, \quad A_1 = 9.22022, \quad c_1 = 0.8476836.$$

The function ε_1 is decreasing for $l > (3/c_1)^2$, in particular for $l > 12.6$.

TABLE 2. The entries $\varepsilon_5(x'_0)$ of [FKS23, Table 5], as printed.

$\log x'_0$	ε_5	$\log x'_0$	ε_5	$\log x'_0$	ε_5	$\log x'_0$	ε_5
1000	$1.3812 \cdot 10^{-9}$	2300	$3.8059 \cdot 10^{-13}$	2800	$1.7087 \cdot 10^{-14}$	6000	$1.2976 \cdot 10^{-22}$
2000	$2.4746 \cdot 10^{-12}$	2400	$2.0426 \cdot 10^{-13}$	2900	$9.2091 \cdot 10^{-15}$	7000	$8.5162 \cdot 10^{-25}$
2100	$1.3246 \cdot 10^{-12}$	2500	$1.0972 \cdot 10^{-13}$	3000	$4.9678 \cdot 10^{-15}$	8000	$7.7860 \cdot 10^{-27}$
2200	$7.0976 \cdot 10^{-13}$	2600	$5.8978 \cdot 10^{-14}$	4000	$1.1021 \cdot 10^{-17}$	9000	$9.2262 \cdot 10^{-29}$
		2700	$3.1729 \cdot 10^{-14}$	5000	$2.9942 \cdot 10^{-20}$	10000	$1.3688 \cdot 10^{-30}$

Lemma 2.10 (Zero-free regions). (i) (Mossinghoff–Trudgian–Yang [MTY23]) $\zeta(\sigma + it) \neq 0$ for $|t| \geq 2$ and $\sigma \geq 1 - 1/(5.558691 \log |t|)$.
(ii) (Bellotti–Trudgian–Yang [BTY26]) $\zeta(\sigma + it) \neq 0$ for $t \geq 3$ and $\sigma \geq 1 - 1/(4.896 \log t)$, and by conjugate symmetry also for $t \leq -3$.

The reference [BTY26] is a preprint. The bounds of [FKS23] were computed with the constant $R = 5.5666305$ in place of 5.558691.

Lemma 2.11 (Zero-density estimates). Let $N(\sigma, t)$ be the number of zeros $\beta + i\gamma$ of $\zeta(s)$ with $\beta \geq \sigma$ and $0 < \gamma \leq t$.

- (i) [FKS23, Thm. 2.7, Table 7] *For each $\sigma \in \{0.990, 0.991, \dots, 0.999\}$, $N(\sigma, t) \leq c_1 t^{8(1-\sigma)/3} (\log t)^{5-2\sigma} + c_2 \log^2 t$ in the range of t used in the proof of [FKS23, Thm. 1.1], with constants (c_1, c_2) given in [FKS23, Table 7]. The first block of that table rests on a bound for $|\zeta(1/2 + it)|$ with constant $a_1 = 0.77$, the second on the bound of Hiary, Patel and Yang [HPY24], with $a_1 = 0.618$. [FKS23, Table 8] gives constants valid on intervals of σ .*
(ii) (Bellotti [Bel24, Thm. 1.1, Table 2]) $N(\sigma, t) \leq C t^{B(1-\sigma)}$ for $\sigma \geq \alpha_0$ and $T_0 < t \leq T_1$, with $(\log T_0, \log T_1, \alpha_0, C, B)$ given in [Bel24, Table 2] on a sequence of ranges starting at $T_0 = 3 \cdot 10^{12}$.

The reference [Bel24] is a preprint.

Lemma 2.12 (Fiori–Jaskari [FJ26, Thm. 1.4]). Let $7/8 \leq \sigma \leq 1$, and assume that every zero ρ of $\zeta(s)$ satisfies $\operatorname{Re} \rho \leq \sigma$. Let x_0 be one of the values in Table 3. Then, for $x > x_0$,

$$|\psi(x) - x| < x(A(\log x)^2 x^{-1/4} + C(x_0) x^{\sigma-1}), \quad A = 0.0198944.$$

TABLE 3. The constants $C(x_0)$ of [FJ26, Thm. 1.4] for $7/8 \leq \sigma \leq 1$.

$\log x_0$	5	150	200	300	500	1000	2000	5000
$C(x_0)$	0.02636425	0.02636425	0.02506448	0.01840514	0.01505842	0.01320991	0.01243612	0.01201123

These are the constants of [FJ26, Thm. 1.4] for the range $7/8 \leq \sigma \leq 1$; the constant $A = 0.0198944$ applies for every $x_0 \geq e^5$. The reference [FJ26] is a preprint.

Lemma 2.13 (Dudek–Platt, Axler). (i) [DP15] *Inequality (R) fails at $x = 38,358,837,682$.*
(ii) [Ax18, Thm. 1.3] *Inequality (R) holds for every real x with $38,358,837,683 \leq x \leq 10^{19}$.*

The proof of Lemma 2.13(ii) rests on the computation of [DP15] and on Büthe’s bounds from [Büt18, Thm. 2]; it does not use [DP15, Lemma 2.1] or [Büt16, Thm. 1].

Convention on tables. Entries of published numerical tables are printed rounded. Every such entry is used after adding one unit in its last printed digit, which is valid for any rounding convention. Constants appearing in the statements of theorems are used as stated.

3. REDUCTION TO A ONE-VARIABLE INEQUALITY

3.1. Elementary bounds.

Lemma 3.1. For $2 < a \leq b$,

$$\int_{e^a}^{e^b} \frac{dt}{\log^2 t} \leq \frac{e^b}{b^2(1-2/a)}.$$

Proof. For $t \geq e^a$ we have $\frac{d}{dt} \frac{t}{\log^2 t} = \frac{1-2/\log t}{\log^2 t} \geq \frac{1-2/a}{\log^2 t}$. Integrate over $[e^a, e^b]$ and discard $-e^a/a^2$. $\square$

For $l > 0$ define

$$R_{\text{pt}}(l) = K \left(e^{-l/2} + \frac{l e^{-2l/3}}{\log 2} \right),$$

$$R_{\text{int}}(l) = K \left(\frac{2e^{-3l/4}}{\log^2 2} + \frac{8e^{-l/2}}{l^2} + \frac{3e^{-5l/6}}{\log^2 2} + \frac{6e^{-2l/3}}{l \log 2} \right),$$

with $K = 1.01624$ as in Lemma 2.2.

Lemma 3.2. For $x = e^l \geq 4$,

$$\frac{\psi(x) - \theta(x)}{x} \leq R_{\text{pt}}(l), \quad \frac{1}{x} \int_2^x \frac{\psi(t) - \theta(t)}{t \log^2 t} dt \leq R_{\text{int}}(l).$$

Proof. The first bound is Lemma 2.2. For the second, Lemma 2.2 bounds the integrand by $K t^{-1/2} \log^{-2} t + K t^{-2/3}/(\log 2 \log t)$. Split the integral at $\sqrt{x}$. On $[2, \sqrt{x}]$ use $\log t \geq \log 2$; on $[\sqrt{x}, x]$ use $\log t \geq l/2$. This gives

$$\int_2^x \frac{dt}{\sqrt{t} \log^2 t} \leq \frac{2x^{1/4}}{\log^2 2} + \frac{8\sqrt{x}}{l^2}, \quad \int_2^x \frac{dt}{t^{2/3} \log t} \leq \frac{3x^{1/6}}{\log 2} + \frac{6x^{1/3}}{l},$$

and the claim follows after division by x . $\square$

3.2. The bound for $|\pi - J|$. Let

$$K_0 = \int_2^{e^{20}} \frac{dt}{\log^2 t} \in \left[0, \frac{e^{10}}{\log^2 2} + \frac{e^{20}}{100} \right],$$

where the upper bound follows by splitting at e^{10} ; numerically $K_0 \leq 4,897,497.16$.

Proposition 3.3. Let $l \geq 20$, $x = e^l$, and let $\eta : [e^{20}, x] \rightarrow [0, \infty)$ be integrable with $|\psi(t) - t| \leq \eta(t)t$ on $[e^{20}, x]$. Then $|\pi(x) - J(x)| \leq x D_\eta(l)$, where

$$D_\eta(l) = \frac{\eta(x) + R_{\text{pt}}(l)}{l} + K_0 e^{-l} + e^{-l} \int_{e^{20}}^x \frac{\eta(t)}{\log^2 t} dt + R_{\text{int}}(l).$$

Proof. By Lemma 2.1, $|\pi(x) - J(x)| \leq |r(x)|/l + \int_2^x |r(t)| dt / (t \log^2 t)$. On $[2, e^{20}]$ use $|r(t)| \leq t$ (Lemma 2.2); this gives K_0 . On $[e^{20}, x]$ use $|r(t)| \leq |\psi(t) - t| + (\psi(t) - \theta(t)) \leq \eta(t)t + (\psi(t) - \theta(t))$, and apply Lemma 3.2 to the contribution of $\psi - \theta$, both at $t = x$ and in the integral. $\square$

In practice η is piecewise constant on unit intervals of $\log t$, and the integral is bounded by Lemma 3.1.

Corollary 3.4. Let $20 \leq N \leq l$ be an integer and $x = e^l$. Let $\varepsilon(k)$ ($20 \leq k \leq N$) satisfy $|\psi(t) - t| \leq \varepsilon(k)t$ for $t \in [e^k, e^{k+1}]$ when $k < N$, and for $t \in [e^N, x]$ when $k = N$. Let $\varepsilon^{\text{pt}} \geq |\psi(x) - x|/x$. Then $|\pi(x) - J(x)| \leq x D(l)$ with

$$D(l) = \frac{\varepsilon^{\text{pt}} + R_{\text{pt}}(l)}{l} + K_0 e^{-l} + C_N e^{N-l} + \frac{\varepsilon(N)}{l^2(1-2/N)} + R_{\text{int}}(l),$$

where

$$C_N = \sum_{k=20}^{N-1} \frac{\varepsilon(k) e^{k+1-N}}{(k+1)^2(1-2/k)}.$$

Proof. Apply Proposition 3.3 with $\eta = \varepsilon(k)$ on $[e^k, e^{k+1}]$ for $k < N$ and $\eta = \varepsilon(N)$ on $[e^N, x]$, with $\eta(x) = \varepsilon^{\text{pt}}$, and bound each piece of the integral by Lemma 3.1. $\square$

The constants satisfy $C_{k+1} = e^{-1} C_k + \varepsilon(k)/((k+1)^2(1-2/k))$ with $C_{20} = 0$, and this recursion is used to compute them.

3.3. The criterion.

Proposition 3.5. *Let $L \geq 21$, and let d be a function with $|\pi(e^l) - J(e^l)| \leq e^l d(l)$ for $l \in \{L-1, L\}$. If*

$$(C) \quad \frac{j(L-1) - d(L-1)}{L} - (j(L) + d(L))^2 > 0,$$

then (R) holds at $x = e^L$. More precisely,

$$\frac{ex}{L} \pi\left(\frac{x}{e}\right) - \pi(x)^2 \geq x^2 \left[\frac{j(L-1) - d(L-1)}{L} - (j(L) + d(L))^2 \right].$$

Proof. We have $\pi(x/e) \geq J(x/e) - (x/e)d(L-1) = (x/e)(j(L-1) - d(L-1))$ and $0 \leq \pi(x) \leq x(j(L) + d(L))$. Hence $\frac{ex}{L} \pi(x/e) \geq \frac{x^2}{L} (j(L-1) - d(L-1))$ and $\pi(x)^2 \leq x^2 (j(L) + d(L))^2$. $\square$

Proposition 3.5 follows directly from Lemma 2.1. It does not use Lemma 2.1 of [DP15].

3.4. Exact main term.

Let $n = 12$ and

$$P(u) = \sum_{k=0}^{n-1} k! u^{k+1}, \quad A_0 = e^{20} \sum_{k=1}^{n-1} \frac{k!}{20^{k+1}}, \quad g(l) = e^{-l} \int_{20}^l e^s s^{-13} ds.$$

Numerically $A_0 = 1,357,383.55 \dots$

Proposition 3.6. *For $l > 25$,*

$$j(l) = P(1/l) + \rho(l), \quad \rho(l) = e^{-l}(K_0 - A_0) + 12! g(l),$$

and, with $\vartheta = 4/5$ and $q = 13/(\vartheta l) < 1$,

$$l^{-13} - e^{-(1-\vartheta)l} (\vartheta l)^{-13} \leq g(l) \leq \frac{l^{-13}}{1-q} + (\vartheta l - 20) e^{-(1-\vartheta)l} (\vartheta l)^{-13}.$$

Proof. By definition $j(l) = 1/l + e^{-l} K_0 + e^{-l} \int_{20}^l e^s s^{-2} ds$ (substitute $t = e^s$). Integrating by parts eleven times gives

$$\int_{20}^l e^s s^{-2} ds = \sum_{k=2}^{12} (k-1)! [e^s s^{-k}]_{20}^l + 12! \int_{20}^l e^s s^{-13} ds,$$

and collecting terms gives the stated formula for $j(l)$. For g , split the integral at $\vartheta l > 20$. On $[20, \vartheta l]$ the integrand $e^s s^{-13}$ is increasing (as $s \geq 13$), so this part lies in $[0, (\vartheta l - 20)e^{\vartheta l} (\vartheta l)^{-13}]$. On $[\vartheta l, l]$ one more integration by parts gives

$$\int_{\vartheta l}^l e^s s^{-13} ds = [e^s s^{-13}]_{\vartheta l}^l + 13 \int_{\vartheta l}^l e^s s^{-14} ds,$$

where the last term lies between 0 and $q \int_{\vartheta l}^l e^s s^{-13} ds$. Rearranging gives both bounds. $\square$

Lemma 3.7. *Let $u = 1/L$. Then*

$$\frac{P(1/(L-1))}{L} - P(1/L)^2 = \frac{u^6 N_6(u)}{(1-u)^{12}},$$

where $N_6 \in \mathbb{Z}[u]$ has degree 30 and $N_6(u) = 1 + 2u + 43u^2 + 376u^3 + \dots$. In particular

$$\frac{N_6(u)}{(1-u)^{12}} = 1 + 14u + 145u^2 + O(u^3),$$

and, for $4 \leq k \leq 9$, the coefficient of u^{k-4} in this expansion equals $-d_k$, where $d_k = \sum_{j=0}^k j!((k-j)! - \binom{k}{j})$ are the coefficients of Hassani's expansion [Has21].

Proof. Since $1/(L-1) = u/(1-u)$, the left side times $(1-u)^{12}$ equals

$$N(u) = \sum_{k=0}^{11} k! u^{k+2} (1-u)^{11-k} - (1-u)^{12} P(u)^2 \in \mathbb{Z}[u].$$

An exact computation shows that the coefficients of $u^0, \dots, u^5$ in N vanish and that $N(u) = u^6 N_6(u)$ with N_6 as stated. The comparison with d_k is an exact computation in rational arithmetic. $\square$

With π replaced by J , Lemma 3.7 and Proposition 3.6 give $\frac{ex}{L}J(x/e) - J(x)^2 = \frac{x^2}{L^6}(1 + \frac{14}{L} + O(L^{-2}))$; this is the classical main term of (R). We use the identity to remove the cancellation from the numerical problem. For a majorant d as in Proposition 3.5, define the *normalized margin*

$$(3.1) \quad M(L) = L^6 \left[\frac{j(L-1) - d(L-1)}{L} - (j(L) + d(L))^2 \right] = \text{Main}(L) + \text{Corr}(L) - \text{Err}(L),$$

where, with $u = 1/L$,

$$\begin{aligned} \text{Main}(L) &= \frac{N_6(u)}{(1-u)^{12}}, \\ \text{Corr}(L) &= L^6 \left[\frac{\rho(L-1)}{L} - 2P(u)\rho(L) - \rho(L)^2 \right], \\ \text{Err}(L) &= L^6 \left[\frac{d(L-1)}{L} + 2(P(u) + \rho(L))d(L) + d(L)^2 \right]. \end{aligned}$$

By Proposition 3.5, $M(L) > 0$ implies (R) at $x = e^L$; more precisely,

$$M(L) \leq \frac{L^6}{x^2} \left(\frac{ex}{L} \pi\left(\frac{x}{e}\right) - \pi(x)^2 \right).$$

The terms Main, Corr and Err are of size 1, at most $12!L^{-7}$, and roughly $3L^4 \sup |\psi(t) - t|/t$, respectively. None of them involves a cancellation, so interval evaluation over L -intervals of width up to 0.5 gives sharp lower bounds for M .

4. THE COMPUTATIONS AND THE PROOFS OF THEOREMS A–D

4.1. Interval arithmetic. All computations use the interval arithmetic of the Python library `mpmath` (`mpmath.iv`) at a working precision of 320 bits. Every operation is rounded outward, so every computed interval contains the exact value of the real expression it represents. Decimal constants are converted from their decimal strings to enclosing intervals. The polynomial N_6 is computed in exact integer arithmetic.

For a closed interval $\mathbf{L} = [a, b]$, the scripts evaluate (3.1) with the interval $\mathbf{L}$ in place of L and $\mathbf{L} - 1$ in place of $L - 1$. The result is an interval containing $M(L)$ for every $L \in \mathbf{L}$, and a positive lower endpoint proves (R) on $[e^a, e^b]$. The endpoints a, b are binary floating-point numbers and are represented exactly. Consecutive intervals of a grid share their endpoints, so a run that succeeds on every interval of a grid covers the union exactly. When an interval fails, the runs bisect it, up to 12 times, before declaring a failure.

The proofs of Theorems A, B, B', C, C* and D use the routines of `ch_ramanujan.py`. They build the majorant of Corollary 3.4 from a list of bounds for $|\psi(t) - t|/t$, each nonincreasing in t on the range where it is used, by taking on every segment $[e^k, e^{k+1}]$, and at every point, the least of the bounds valid there.

The proofs are computer-assisted in this sense: they consist of the analysis of Sections 2–3 and of this section, together with interval computations whose results are reported below. The computations have not been formally verified in a proof assistant.

4.2. Proof of Theorem A. Segment bounds. For integers $20 \leq k \leq 3400$ define $\varepsilon_A(k)$ as follows:

- if $k \leq 42$: $\varepsilon_A(k) = 0.94e^{-k/2}$ (Lemma 2.3; note $e^{43} < 10^{19}$);
- if $k = 43$: $\varepsilon_A(43) = \max\{0.94e^{-43/2}, \min\{\varepsilon_1(43), \varepsilon_{\text{CH}}(43)\}\}$;
- if $k \geq 44$: $\varepsilon_A(k)$ is the least of $\varepsilon_1(k)$, $\varepsilon_{\text{CH}}(k)$, and $\varepsilon_5(x'_0)$ for the largest tabulated $\log x'_0 \leq k$.

Each $\varepsilon_A(k)$ bounds $|\psi(t) - t|/t$ on $[e^k, 10^{19}]$ for $k \leq 42$ and on $[e^k, \infty)$ for $k \geq 43$, by Lemmas 2.3, 2.8(i) and 2.9 and the monotonicity of ε_1 and ε_{CH} . At $x = e^l$ the pointwise bound is $\varepsilon^{\text{pt}} = 0.94e^{-l/2}$ if $l \leq 43.749$, and otherwise the least of $\varepsilon_A(\lfloor l \rfloor)$, $\varepsilon_1(l)$ and $\varepsilon_{\text{CH}}(l)$. The majorant d_A is D of Corollary 3.4 with these choices and $N = \lfloor l \rfloor$. For $44 \leq k \leq 800$ the least of the candidates is $\varepsilon_{\text{CH}}(k)$: there $\varepsilon_1(k) \geq \varepsilon_1(800) > 8 \cdot 10^{-6}$, and Table 2 starts at $\log x'_0 = 1000$.

Verification. The script evaluates M with $d = d_A$ on the grid of step 0.25 starting at $L = 43.749$, bisecting where necessary, and then continues with step 0.002 from the last verified point. It finds $M > 0$ on every interval up to $L_A = 753.98499\dots$; the next interval $[L_A, L_A + 0.002]$ fails even after 12 bisections.

Proof of Theorem A. Lemma 2.13(ii) gives (R) for $38,358,837,683 \leq x \leq 10^{19}$. The computation above gives $M(L) > 0$, hence (R), for $43.749 \leq L \leq L_A$, and $e^{43.749} < 10^{19}$, $L_A > 753.98$. $\square$

Theorem A depends on [CH25, Cor. 1.3]. By Corollary E(i), arguments of type (a)–(c) at height 3,000,175,332,800 (Section 5) cannot go beyond $\log x = 754.40$; the barrier computed in Section 5.4 lies at $\log x = 754.39478$, and Theorem A is within 0.41 of it.

4.3. Refereed inputs only.

Proposition 4.1. *Inequality (R) holds for every real x with $38,358,837,683 \leq x \leq e^{672.84}$. The proof uses neither [CH25] nor [BKLNW21, Table 8]: its inputs are Lemmas 2.3, 2.4, 2.5, 2.9 and 2.13(ii). All of them are refereed except the correction of Büthe's Theorem 1 in [Bha24] (Lemma 2.4).*

Rows from Büthe's Theorem 1. For b in the grid

$$\mathcal{B} = \{43.749\} \cup \{44, 45, \dots, 120\} \cup \{125, 130, \dots, 800\} \cup \{825, 850, \dots, 3300\}$$

(314 values), we evaluate Lemma 2.4 with $x_* = e^b$, $T = c/\varepsilon \leq 3,000,175,332,800$ (permitted by Lemma 2.5), and the best of a small set of parameters (c, T, α) . For $b < 70$ the search runs over $c \in \{28, 30, 32, 33, 34, 35\}$, over $\alpha \in \{10^{-3}, 3 \cdot 10^{-3}, 10^{-2}\}$, and over $T = 3,000,175,332,800$ and $T = 10^{12}, 3 \cdot 10^{11}, 10^{11}, 3 \cdot 10^{10}, 10^{10}, 3 \cdot 10^9, 10^9, 3 \cdot 10^8$. For $b \geq 70$ it runs over $T = 3,000,175,332,800$, over $c \in \{33.5, 34.25, 35, 35.75, 36.5\}$, and over $\alpha \in \{3 \cdot 10^{-4}, 10^{-3}, 3 \cdot 10^{-3}\}$. Any admissible choice gives a valid bound; the search only sharpens it.

The quantities in Lemma 2.4 are enclosed as follows. I_0 and I_1 are summed from their power series (400 terms), with a geometric bound for the tail. Since η_c is decreasing on $[0, 1]$ and $0 < \mu_c \leq \frac{1}{2}$ is decreasing on $[0, \alpha]$,

$$\begin{aligned} \mu_c(\alpha) &\in \left[\frac{1}{2} - \alpha\eta_c(0), \frac{1}{2} - \alpha\eta_c(\alpha)\right], \\ |\nu_c(\alpha)| &\in \left[\frac{I_1(c)}{2\sinh c} - \frac{\alpha}{2}, \frac{I_1(c)}{2\sinh c} - \alpha\mu_c(\alpha)\right]. \end{aligned}$$

In E_1 we use the upper end for $|\nu_c(\alpha)|$ and the lower end for B , and E_3 is bounded by $2(x_*^{-1/2} + x_*^{-1})^{\frac{\log^2(T/2\pi)}{4\pi}} + \frac{1.92}{x_*}$. The second condition in Lemma 2.4 holds for all parameters used here, since $\varepsilon < 10^{-3}$, $|\nu_c(\alpha)| < 1/2$ and $\mu_c(\alpha) > 1/4$. The result is a number $\delta(b)$ with

$$|\psi(x) - x| \leq \delta(b)x \quad \text{for all } x \geq e^{b+\alpha\varepsilon}.$$

In all cases $\alpha\varepsilon < 10^{-6}$, and we use $\delta(b)$ for $x \geq e^{b+10^{-6}}$. Note that $e^{43.749+10^{-6}} < 10^{19}$. Table 4 shows some of the rows.

TABLE 4. Some of the rows $\delta(b)$ (rounded up) from Lemma 2.4 with RH up to height 3,000,175,332,800. Here $T_{\text{RH}} = 3,000,175,332,800$.

b	$\delta(b)$	c	T	α
43.749	$2.2730 \cdot 10^{-8}$	28	$3 \cdot 10^9$	10^{-2}
50	$1.3051 \cdot 10^{-9}$	32	10^{11}	10^{-2}
60	$1.3621 \cdot 10^{-11}$	35	T_{RH}	10^{-2}
100	$2.1766 \cdot 10^{-12}$	35	T_{RH}	$3 \cdot 10^{-3}$
300	$1.7440 \cdot 10^{-12}$	35	T_{RH}	$3 \cdot 10^{-3}$
600	$1.6622 \cdot 10^{-12}$	35	T_{RH}	$3 \cdot 10^{-3}$
700	$1.6512 \cdot 10^{-12}$	35	T_{RH}	$3 \cdot 10^{-3}$
1000	$1.6317 \cdot 10^{-12}$	35	T_{RH}	$3 \cdot 10^{-3}$
3000	$1.6021 \cdot 10^{-12}$	35	T_{RH}	10^{-3}

For comparison, [BKLNW21, Table 8] prints $1.98894 \cdot 10^{-12}$ at $b = 600$. With $\log(3c)$ restored and (c, α) re-optimized, Büthe's method at the height used in [BKLNW21] gives at best about $2.02478 \cdot 10^{-12}$ there (Section 7.4, by quadrature). The rows of Table 4 are smaller because they use the larger height of Lemma 2.5. They are also larger than the constant $\pi/(3 \cdot 10^{12}) = 1.0472 \cdot 10^{-12}$ of Lemma 2.8(i), which is why Theorem A goes further.

Segment bounds. For integers $20 \leq k \leq 3400$ define $\varepsilon_{A'}(k)$ as follows:

- if $k \leq 42$: $\varepsilon_{A'}(k) = 0.94e^{-k/2}$ (Lemma 2.3; note $e^{43} < 10^{19}$);
- if $k = 43$: $\varepsilon_{A'}(43) = \max\{0.94e^{-43/2}, \delta(43.749)\}$;

- if $k \geq 44$: $\varepsilon_{A'}(k)$ is the least of $\varepsilon_1(k)$, $\delta(b)$ for the largest $b \in \mathcal{B}$ with $b + 10^{-6} \leq k$, and $\varepsilon_5(x'_0)$ for the largest tabulated $\log x'_0 \leq k$.

Each $\varepsilon_{A'}(k)$ bounds $|\psi(t) - t|/t$ on $[e^k, 10^{19}]$ for $k \leq 42$ and on $[e^k, \infty)$ for $k \geq 43$, by Lemmas 2.3, 2.4, 2.5 and 2.9 and the monotonicity of ε_1 . At $x = e^l$ the pointwise bound is $\varepsilon^{\text{pt}} = 0.94 e^{-l/2}$ if $l \leq 43.749$, and otherwise the least of $\varepsilon_{A'}(\lfloor l \rfloor)$, $\varepsilon_1(l)$ and $\delta(b)$ for the largest $b \in \mathcal{B}$ with $b + 10^{-6} \leq l$. The majorant $d_{A'}$ is D of Corollary 3.4 with these choices and $N = \lfloor l \rfloor$. Since all intervals used have width at most 0.25 and start at $l \geq 42.749$, the validity ranges above cover every t at which $\varepsilon_{A'}(N)$ is applied.

Verification. The script `buthe_thesis_run.py`, which runs `buthe_ramanujan.py` with E_2 and E_3 as in Lemma 2.4, evaluates M with $d = d_{A'}$ on the grid of step 0.25 starting at $L = 43.749$. It finds $M > 0$ on every interval up to $L = 672.749$, bisecting where necessary; the next interval fails. A second run with step 0.002 from 672.749 finds $M > 0$ up to $L_{A'} = 672.849\dots$, and the interval $[L_{A'}, L_{A'} + 0.002]$ fails even after 12 bisections.

Proof of Proposition 4.1. Lemma 2.13(ii) gives (R) for $38,358,837,683 \leq x \leq 10^{19}$. The computation above gives $M(L) > 0$, hence (R), for $43.749 \leq L \leq L_{A'}$, and $e^{43.749} < 10^{19}$, $L_{A'} > 672.84$. $\square$

4.4. Proof of Theorem C.

4.4.1. *The bound of Fiori, Kadiri and Swidinsky, re-implemented.* Theorem 1.1 of [FKS23] states that, for $\log x_0 > 1000$,

$$|\psi(x) - x| \leq \varepsilon(x_0)x \quad \text{for all } x \geq x_0, \quad \varepsilon(x_0) = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4,$$

where $\varepsilon_1, \dots, \varepsilon_4$ are given by [FKS23, (3.4), (3.8), (3.17), (3.22)]. These terms depend on the following data:

- the height H_0 up to which RH is verified; here $H_0 = 3 \cdot 10^{12}$, by Lemma 2.5;
- the constants $T_0 = 30,610,046,000$ and $S_0 = 39.5797647802$ of [FKS23], and Rosser's constants $b_1 = 0.137$, $b_2 = 0.443$, $b_3 = 1.588$ in the bound for the number of zeros up to height t ;
- zero-density estimates (Lemma 2.11);
- a zero-free region $\sigma \geq 1 - 1/(R \log |t|)$;
- parameters $\sigma_1 = 0.9$, σ_2 , $c = 3$, N , K , with $T = \exp(c\sqrt{\log x_0/R})$.

The zero-free region enters only through R : in ε_3 through the heights $H_\sigma = \max\{H_0, \exp(1/(R(1 - \sigma)))\}$ above which the density estimates are applied, and in ε_4 through t_0 and the factor $x^{-1/(R \log t)}$. In [FKS23], $t_0 = \max\{H_{\sigma_2}, \exp(\sqrt{\log x/R})\}$ depends on x ; with the parameters of a row fixed, the bound holds only while $\log x \leq R \log^2 t_0$. We therefore take $t_0 = \max\{H_{\sigma_2}, \exp(\sqrt{L_{\max}/R})\}$, where $L_{\max}$ is the right end of the range of $\log x$ on which the row is used, so that $t_0 \geq \exp(\sqrt{\log x/R})$ for every x at which the row is used. The density estimate enters ε_3 through the interval constants of [FKS23, Table 8] and ε_4 through a bound $\tilde{N}(\sigma_2, t) \geq N(\sigma_2, t)$; any such upper bound may be used in ε_4 , in particular the pointwise minimum of two of them.

The script `fks_eps.py` implements the four terms exactly as printed in [FKS23]; the rows for Theorems C and C* use $N = 10^4$ and $K = 10^5$. As for all tabulated constants (Section 2), the density constants of [FKS23, Tables 7 and 8] and the constants C and B of [Bel24, Table 2] are used after adding one unit in their last printed digit. For ε_4 it uses the per- σ constants of [FKS23, Table 7] at $\sigma_2 \in \{0.990, 0.991, \dots, 0.999\}$, for ε_3 the interval constants of [FKS23, Table 8], and it minimizes over σ_2 on this grid. Table 5 compares the result, with the constant $R = 5.5666305$ used in [FKS23] and the first block of [FKS23, Table 7] ($a_1 = 0.77$), against [FKS23, Table 5]. Our values are 0.4–2.8% larger, because Fiori, Kadiri and Swidinsky optimize σ_2 continuously, with density constants for each σ that they do not print; the terms ε_1 and ε_2 reproduce exactly. The evaluation is in double precision (NumPy and SciPy). A 40-digit mpmath evaluation of the same formulas (`fks_eps_mpmath.py`) agrees to a relative difference of about 10^{-15} , and every computed value is multiplied by $1 + 10^{-9}$ before it is used.

4.4.2. *The computation. Rows.* For Theorem C we use the zero-free region of Lemma 2.10(i), $R = 5.558691$, and the second block of [FKS23, Table 7] ($a_1 = 0.618$, Lemma 2.11(i)) in ε_4 , and evaluate $\varepsilon(x_0)$ at every integer $\log x_0 \in [2900, 3160]$. Each row bounds $|\psi(x) - x|/x$ for all $x \geq x_0$ at which it is used. The rows decrease from $8.98693 \cdot 10^{-15}$ at $\log x_0 = 2900$ to $1.91915 \cdot 10^{-15}$ at $\log x_0 = 3150$.

Segment bounds. The segment bounds $\varepsilon_C(k)$ ($20 \leq k \leq 10050$) and the pointwise bounds are formed as in Section 4.2, with the rows above in place of ε_{CH} , so that [CH25] is not used. For $k \geq 44$, $\varepsilon_C(k)$ is the

TABLE 5. Our evaluation of [FKS23, Thm. 1.1] (with σ_2 on the grid of [FKS23, Table 7], $a_1 = 0.77$ and $R = 5.5666305$) against [FKS23, Table 5].

$\log x_0$	ours	[FKS23, Table 5]
2000	$2.4843 \cdot 10^{-12}$	$2.4746 \cdot 10^{-12}$
2500	$1.1143 \cdot 10^{-13}$	$1.0972 \cdot 10^{-13}$
2900	$9.4623 \cdot 10^{-15}$	$9.2091 \cdot 10^{-15}$
3000	$5.1066 \cdot 10^{-15}$	$4.9678 \cdot 10^{-15}$

least of $\varepsilon_1(k)$, $\varepsilon_5(x'_0)$ for the largest tabulated $\log x'_0 \leq k$, and the row at the largest integer $\log x_0 \leq k$ in [2900, 3160].

Verification on $[L_C, 10^4]$. The script `fks_fine.R.py` scans [2920, 3200] with step 0.5 and then the last failing region with step 0.002. It finds that the last interval with $M \leq 0$ ends at $L_C = 3035.00199\dots$, and from L_C to 10^4 it verifies $M > 0$ on a grid of step 0.5, bisecting where necessary.

The tail $L \geq 10^4$. Here we use only Lemma 2.9(ii), splitting the integral in Proposition 3.3 at $t = e^{l/2}$. With $\eta(t) = \varepsilon_1(20)$ on $[e^{20}, e^{l/2}]$ and $\eta(t) = \varepsilon_1(l/2)$ on $[e^{l/2}, x]$, Lemma 3.1 (with $a = l/2$ on the second piece) gives the majorant

$$d_T(l) = \frac{\varepsilon_1(l) + R_{\text{pt}}(l)}{l} + K_0 e^{-l} + \frac{\varepsilon_1(20) e^{-l/2}}{(l/2)^2 (1 - 2/20)} + \frac{\varepsilon_1(l/2)}{l^2 (1 - 4/l)} + R_{\text{int}}(l),$$

valid for $l \geq 40$. Here $\varepsilon_1(20) \approx 18.62$.

Lemma 4.2. *For $L \geq 338$, $L^6 d_T(L)$, $\text{Err}_T(L) = L^6 \left[\frac{d_T(L-1)}{L} + 2j(L)d_T(L) + d_T(L)^2 \right]$ and $j(L)$ are nonincreasing.*

Proof. $L^6 d_T(L)$ is a sum of the following nonnegative terms, each nonincreasing on the stated range:

- $L^5 \varepsilon_1(L) = A_1 L^{13/2} e^{-c_1 \sqrt{L}}$, for $L > (13/c_1)^2 \approx 235.2$;
- $L^4 \varepsilon_1(L/2) \cdot (1 - 4/L)^{-1}$, a product of two positive nonincreasing functions for $L > 2(11/c_1)^2 \approx 336.78$;
- terms $C L^k e^{-\gamma L}$ with $k \leq 6$ and $\gamma \geq 1/2$, coming from K_0 , $\varepsilon_1(20)$, R_{pt} and R_{int} , nonincreasing for $L > k/\gamma$, hence for $L \geq 12$.

Next, $J'(x) = 1/\log x$ gives $j'(l) = 1/l - j(l) < 0$, since $j(l) > 1/l$. Finally, $\text{Err}_T(L)$ is the sum of $\frac{L^5}{(L-1)^6} \cdot (L-1)^6 d_T(L-1)$, of $2j(L) \cdot L^6 d_T(L)$, and of $L^6 d_T(L) \cdot L^{-6} \cdot L^6 d_T(L)$. Each is a product of positive nonincreasing functions for $L-1 > 336.78$. $\square$

The script checks the thresholds of Lemma 4.2 and, as a numerical sanity check, that $L^6 d_T(L)$ decreases along $L = 10^4 \cdot \{1, 2, 4, 8, 64\}$. It then evaluates at $L_1 = 10^4$:

- an enclosure of $\text{Main}(L) = N_6(u)/(1-u)^{12}$ over the whole interval $u \in [0, 10^{-4}]$, that is, over all $L \geq 10^4$, with lower end 1.0000000000 to ten decimals;
- $\text{Err}_T(10^4) \leq 9.1 \cdot 10^{-8}$;
- $|\text{Corr}(10^4)| \leq 4.8 \cdot 10^{-24}$.

By Lemma 4.2, $\text{Err}_T(L) \leq \text{Err}_T(10^4)$ for all $L \geq 10^4$. For the correction term, Proposition 3.6 gives $0 \leq \rho(l) \leq 2 \cdot 12! l^{-13}$ for $l \geq 9999$, and $P(1/L) \leq 2/L$; hence $|\text{Corr}(L)| \leq 10^{-20}$ for all $L \geq 10^4$. Therefore

$$M(L) \geq \inf_{0 \leq u \leq 10^{-4}} \text{Main} - 10^{-20} - \text{Err}_T(10^4) \geq 0.99999990 \quad (L \geq 10^4);$$

the script reports 0.9999999090 for the same bound with $|\text{Corr}(10^4)|$ in place of 10^{-20} .

Proof of Theorem C. The verification gives $M(L) > 0$ for $L_C \leq L \leq 10^4$, and the tail argument gives $M(L) \geq 0.99999990$ for $L \geq 10^4$. Since $L_C < 3035.01$, Proposition 3.5 gives (R) for all $x \geq e^{3035.01}$. $\square$

The proof uses Lemmas 2.1, 2.2, 2.3, 2.9, 2.10(i) and 2.11(i), and [FKS23, Thm. 1.1] in the re-implementation above; all of these are published. In particular it uses neither [DP15, Lemma 2.1], nor [Büt16, Thm. 1], nor [CH25].

Remark 4.3 (Variants). (a) With the first block of [FKS23, Table 7] ($a_1 = 0.77$) in place of the second, the same computation verifies $M > 0$ from $L = 3040.00199 \dots$ to 10^4 , which gives (R) for $x \geq e^{3040.01}$.

(b) If in addition the constant $R = 5.5666305$ of [FKS23] replaces Lemma 2.10(i), the computation verifies $M > 0$ from $L = 3045.00199 \dots$ to 10^4 , which gives (R) for $x \geq e^{3045.01}$.

(c) With the printed rows of [FKS23, Table 5] and [FKS23, Cor. 1.4] only (`extend_ramanujan.py`), the method gives (R) for $x \geq e^{3157.94}$, as follows. *Mode F.* For integers $20 \leq k \leq 10050$ let $\varepsilon_F(k)$ be the least of $\varepsilon_1(k)$ and $\varepsilon_5(x'_0)$ for the largest tabulated $\log x'_0 \leq k$ (Lemma 2.9). Each $\varepsilon_F(k)$ bounds $|\psi(t) - t|/t$ for all $t \geq e^k$. The script also lists the rows $b = 2500$ and $b = 3000$ of [BKLNW21, Table 8] among the candidates. On the ranges where these rows apply, ε_5 is smaller ($1.0972 \cdot 10^{-13}$ against $9.06304 \cdot 10^{-13}$, and $4.9678 \cdot 10^{-15}$ against $4.59972 \cdot 10^{-14}$), so they never attain the minimum, and this variant does not depend on them. The pointwise bound at $x = e^l$ is $\min\{\varepsilon_F(\lfloor l \rfloor), \varepsilon_1(l)\}$, and the majorant d_F is D of Corollary 3.4 with $N = \lfloor l \rfloor$. For example, $\varepsilon_F(3000) = 4.9679 \cdot 10^{-15}$ (the entry of Table 2 plus one unit) and $\varepsilon_F(3158) \approx 3.3542 \cdot 10^{-15}$, which comes from ε_1 .

Verification on $[L_F, 10^4]$. A scan of $[3000, 3200]$ with step 0.5 finds that the last interval with $M \leq 0$ ends at 3158.5. A scan of $[3155.5, 3159]$ with step 0.002 finds that the last such interval ends at $L_F = 3157.93199 \dots < 3157.932$. From L_F to 10^4 , the script verifies $M > 0$ on a grid of step 0.5, bisecting where necessary. This uses 13,692 intervals of width at most 0.5; exactly one interval of the initial grid had to be subdivided. Since $L_F < 3157.94$, the tail argument above completes the proof.

The step from $e^{3157.94}$ to $e^{3045.01}$ comes from the rows at every integer $\log x_0$, which replace the gap between the printed rows at 3000 and 4000; the step to $e^{3040.01}$ comes from the constant of [MTY23], and the step to $e^{3035.01}$ from the density constants based on [HPY24].

(d) With the preprint of Bellotti, that is, with the pointwise minimum of the two bounds of Lemma 2.11 in ε_4 , and with $R = 5.558691$, rows at every integer $\log x_0 \in [2850, 3160]$ give (R) for $x \geq e^{2992.01}$; the verification starts at $L = 2992.00199 \dots$

4.5. Proof of Theorem C*. Theorem C* depends on two preprints: on [BTY26] through Lemma 2.10(ii), and on [Bel24] through Lemma 2.11(ii). Since $\zeta(s)$ has no zeros with $|\operatorname{Im} s| < 14$, Lemma 2.10(ii) can replace Lemma 2.10(i) in [FKS23, Thm. 1.1], with $R = 4.896$. In ε_4 we use the pointwise minimum of the two bounds of Lemma 2.11. We evaluate $\varepsilon(x_0)$ with these inputs at every integer $\log x_0 \in [2400, 3000]$; the rows decrease from $2.86401 \cdot 10^{-14}$ at $\log x_0 = 2400$ to $4.11372 \cdot 10^{-16}$ at $\log x_0 = 3000$. With these rows in place of those of Section 4.4, `fxs.fine.R.py` finds that the last interval with $M \leq 0$ ends at $L_{C^*} = 2591.00199 \dots$, and verifies $M > 0$ on a grid of step 0.5 from L_{C^*} to 10^4 , bisecting where necessary.

Proof of Theorem C.* The verification gives $M(L) > 0$ for $L_{C^*} \leq L \leq 10^4$, and the tail argument of Section 4.4 gives $M(L) > 0$ for $L \geq 10^4$. Since $L_{C^*} < 2591.01$, Proposition 3.5 gives (R) for all $x \geq e^{2591.01}$. $\square$

Remark 4.4. With [BTY26] alone, that is, with $R = 4.896$ and the first block of [FKS23, Table 7] in ε_4 , rows at every integer $\log x_0 \in [2300, 3160]$ give (R) for $x \geq e^{2599.01}$; the verification starts at $L = 2599.00199 \dots$

4.6. Proof of Theorem B and Corollary F. *The bound for ψ .* Let $0 < \theta < 1$ and assume that every zero $\beta + i\gamma$ of $\zeta(s)$ satisfies $\beta \leq \theta$. Let $L = \log x \geq 110$ and $T = e^{L/2-2}$. Then $T > \max\{51, L\}$ and $T < (\sqrt{x} - 2)/2$, so Lemma 2.7 applies; we use it in the weaker form described after it. Every zero with $0 < \gamma \leq H = 3,000,175,332,800$ has $\beta = 1/2$ (Lemma 2.5), every zero has $\beta \leq \theta$, $|x^{\rho-1}/\rho| \leq x^{\beta-1}/|\gamma|$, the zeros are symmetric under $\gamma \mapsto -\gamma$, and $\sum_{0 < \gamma \leq t} 1/\gamma \leq \log^2(t/2\pi)/(4\pi)$ for $t \geq 5000$ [Büt16, Lemma 3]. Hence $|\psi(x) - x| \leq \varepsilon_\theta(L)x$, where

$$\varepsilon_\theta(L) = \frac{200L^2}{T} + \frac{2e^{-L/2} \log^2(H/2\pi)}{4\pi} + \frac{2e^{(\theta-1)L} \log^2(2.45T/2\pi)}{4\pi}.$$

The function ε_θ is decreasing in L for $L \geq \max\{110, 2/(1-\theta) + 6\}$, so $\varepsilon_\theta(a)$ bounds $|\psi(x) - x|/x$ for every $x \geq e^a$, and it is increasing in θ . The computations below add ε_θ to the bounds of Lemmas 2.3 and 2.9, formed into segment and pointwise bounds as in Section 4.2 but without ε_{CH} . They use neither [CH25] nor [FJ26].

The case $\theta = 7/8$. The script `qrh_ramanujan.py` gives $\varepsilon_{7/8}(250) \leq 6.3570 \cdot 10^{-11}$, $\varepsilon_{7/8}(300) \leq 1.7814 \cdot 10^{-13}$, $\varepsilon_{7/8}(672) \leq 5.8361 \cdot 10^{-33}$ and $\varepsilon_{7/8}(3000) \leq 4.9189 \cdot 10^{-158}$, and verifies $M > 0$ on every interval of the grid of step 0.5 on $[247.5, 3036.0]$.

The case $\theta \leq 0.945092$. The script `qrh_theta.py` verifies $M > 0$ on every interval of the grid of step 0.5 on $[672.5, 3036.0]$ for $\theta = 0.945092$. A bisection over $\theta \in [0.875, 0.99]$, with 14 steps, shows that the verification succeeds at $\theta = 0.945092$ and fails at $\theta = 0.945099$. At every failing value the first failure is on $[672.5, 673.0]$, at the lower end of the range, where Proposition 4.1 ends.

Proof of Theorem B. Let $\theta \leq 0.945092$. Under the hypothesis every zero satisfies $\beta \leq \theta$, and since $\varepsilon_\theta \leq \varepsilon_{0.945092}$, the computation gives (R) for $e^{672.5} \leq x \leq e^{3036}$. Lemma 2.13(ii) and Proposition 4.1 give (R) on $[38, 358, 837, 683, e^{672.84}]$, and Theorem C gives it on $[e^{3035.01}, \infty)$. Lemma 2.13(i) gives $N_R = 38, 358, 837, 683$. All inputs are refereed except the correction of Büthe's Theorem 1 in [Bha24], which enters through Proposition 4.1. $\square$

Proof of Corollary F. By Lemma 2.6, $\zeta(s) \neq 0$ for $\operatorname{Re} s > 7/8$. Since $7/8 < 0.945092$, Theorem B gives the claim. $\square$

Robustness of Corollary F. For $\theta = 7/8$ the computation covers $[e^{247.5}, e^{3036}]$, so Proposition 4.1 is needed only on $[10^{19}, e^{247.5}]$, and there it has a large margin: if every row of Lemma 2.4 is multiplied by 2, the verification of Section 4.3 still succeeds for $43.749 \leq L \leq 564.999$ (`robust_Aref.py`). Theorem A, which rests on the independent method of Chirre and Helfgott [CH25], also covers $[10^{19}, e^{753.98}]$. So for $\theta = 7/8$ the range $[10^{19}, e^{247.5}]$ has two independent proofs.

4.7. Proof of Theorem B*. Let $\theta_* = 0.968978$, and assume that every zero ρ of $\zeta(s)$ satisfies $\operatorname{Re} \rho \leq \theta_*$. Lemma 2.12 with $\sigma = \theta_*$ gives $|\psi(x) - x| \leq \varepsilon_{\theta_*}^{\text{FJ}}(\log x) x$ for $\log x > 5$, where

$$\varepsilon_{\theta_*}^{\text{FJ}}(l) = A l^2 e^{-l/4} + C(x_0) e^{(\theta-1)l}, \quad x_0 \text{ the largest entry of Table 3 with } \log x_0 < l.$$

The function $\varepsilon_{\theta_*}^{\text{FJ}}$ is nonincreasing for $l \geq 8$, since $C(x_0)$ is nonincreasing in x_0 . The segment bounds $\varepsilon_B(k)$ are defined as $\varepsilon_A(k)$ in Section 4.2, with $\varepsilon_{\theta_*}^{\text{FJ}}(k)$ added to the list of candidates, and the pointwise bound at $x = e^l$ is the least of that of Section 4.2 and $\varepsilon_{\theta_*}^{\text{FJ}}(l)$.

Verification. With the resulting majorant, the script verifies $M > 0$ on every interval of the grid of step 0.5 on $[752.985, 3158.5]$, bisecting where necessary.

Sharpness for the method. A bisection over $\theta \in [0.875, 0.999]$, with 16 steps, each a full verification of $M > 0$ on $[752.985, 3158.5]$, shows that the verification succeeds at $\theta = 0.968978$ and fails at $\theta = 0.968980$. At every failing value the first failure is on the grid interval $[753.985, 754.485]$, at the left end of the open range. There Lemma 2.8(i), the input of Theorem A, has just ceased to suffice, and Lemma 2.12 must take over.

Proof of Theorem B.* Assume that $\zeta(s) \neq 0$ for $\operatorname{Re} s > 0.96897$. Then every zero satisfies $\operatorname{Re} \rho \leq 0.96897 < \theta_*$, and the computation gives (R) for $e^{752.985} \leq x \leq e^{3158.5}$. Lemma 2.13(ii) and Theorem A give (R) on $[38, 358, 837, 683, e^{753.98}]$, and Theorem C gives it on $[e^{3035.01}, \infty)$. These ranges cover $[38, 358, 837, 683, \infty)$. Lemma 2.13(i) shows that (R) fails at $38, 358, 837, 682$, so $N_R = 38, 358, 837, 683$. $\square$

Remark 4.5. Under the stronger hypothesis $\operatorname{Re} \rho \leq 7/8$, Lemma 2.12 with $\sigma = 7/8$ and $x_0 = e^5$ alone suffices from $L = 137$ on. Take $\eta(t) = A \log^2 t t^{-1/4} + C t^{-1/8}$ with $C = 0.02636425$ in Proposition 3.3, and bound the integrals by $\int_{e^{20}}^x A t^{-1/4} dt \leq \frac{4}{3} A x^{3/4}$ and

$$\int_{e^{20}}^x \frac{C t^{-1/8}}{\log^2 t} dt \leq \frac{C x^{7/8}}{l^2(7/8 - 4/l)} + \frac{8C}{7 \cdot 400} x^{7/16}$$

(split at $\sqrt{x}$). The script `verify_ramanujan.py` then finds $M > 0$ on every interval of $[137, 3158.5]$: a scan of $[100, 200]$ with step 0.25 finds that the last interval with $M \leq 0$ ends at $L = 137$, and on $[200, 3158.5]$ all 5917 intervals of width 0.5 pass. The smallest lower bound there is 1.004447 , attained on $[3158, 3158.5]$; the error term is at most $1.93 \cdot 10^{-41}$ at $L \approx 950$. So under Lemma 2.6, and with the preprint [FJ26], the margin $M(L)$ is of size 1 throughout the open range.

4.8. Proof of Theorem B'. Let $R_B = 1.73730$, and assume that $\zeta(\beta + i\gamma) \neq 0$ whenever $|\gamma| > 3 \cdot 10^{12}$ and $\beta > 1 - 1/(R_B \log |\gamma|)$. Theorem 1.1 of [FKS23] uses the zero-free region only at heights at least $H_0 = 3 \cdot 10^{12}$ (through $H_\sigma \geq H_0$ and $t_0 \geq H_0$; Section 4.4.1), and below H_0 Lemma 2.5 applies. So the re-implementation of Section 4.4.1 applies with $R = R_B$.

Rows below $\log x_0 = 1000$. Fiori, Kadiri and Swidinsky state their Theorem 1.1 for $\log x_0 > 1000$. The ingredients of its proof, [FKS23, Props. 3.4, 3.6, 3.8, 3.11 and 3.14], require only $x > e^{50}$, $3 \log x < T < \sqrt{x}/3$ and $T > t_0$, which the script checks for every row. As in Section 4.4.1, t_0 is chosen so that $t_0 \geq \exp(\sqrt{\log x/R})$ for every x at which the row is used. We therefore use rows from $\log x_0 = 700$ on: at every 3 in $\log x_0$ up to 1200 and at every 25 above, with $N = 4000$ and $K = 4 \cdot 10^4$. Each row is minimized over $\sigma_2 \in \{0.95, 0.96, 0.97, 0.98, 0.985, 0.990, 0.993, 0.996\}$, with the constants of the second block of [FKS23, Table 7] at 0.990, 0.993 and 0.996 and those of [FKS23, Table 8] otherwise.

Verification. With these rows added to the candidates of Section 4.2, the script `zfr.B.py` verifies $M > 0$ on every interval of the grid of step 0.5 on $[752.985, 3035.502]$, bisecting where necessary.

Sharpness for the method. A bisection over $R \in [1, 3]$, with 12 steps, each a full verification of $M > 0$ on $[752.985, 3035.502]$, shows that the verification succeeds at $R = 1.73730$ and fails at $R = 1.73779$. At every failing value the first failure is on the grid interval $[753.985, 754.485]$, at the left end of the open range.

Proof of Theorem B'. Let $R \leq 1.7373 = R_B$. Since $1/(R \log |\gamma|) \geq 1/(R_B \log |\gamma|)$, the hypothesis of Theorem B' implies the hypothesis above, and the computation gives (R) for $e^{752.985} \leq x \leq e^{3035.502}$. Lemma 2.13(ii) and Theorem A give (R) on $[38, 358, 837, 683, e^{753.98}]$, and Theorem C gives it on $[e^{3035.01}, \infty)$. Lemma 2.13(i) gives $N_R = 38, 358, 837, 683$. $\square$

The best known constants are $R = 5.558691$ (Lemma 2.10(i)) and, in the preprint [BTY26], $R = 4.896$ (Lemma 2.10(ii)). Theorem B' needs a constant smaller by a factor of about 3.2, respectively 2.8.

4.9. Proof of Theorem D. Let $T_D = 4.16017 \cdot 10^{14}$, and assume that every zero $\beta + i\gamma$ of $\zeta(s)$ with $0 < \gamma \leq T_D$ satisfies $\beta = 1/2$. The proof combines three bounds for $|\psi(x) - x|/x$:

- (D1) Lemma 2.8(ii) at height T_D , that is, ε_{T_D} ;
- (D2) Lemma 2.8(i), that is, ε_{CH} ;
- (D3) [FKS23, Thm. 1.1], re-implemented as in Section 4.4.1 with the inputs of Theorem C ($R = 5.558691$ and the second block of [FKS23, Table 7] in ε_4), with $N = 4000$, $K = 4 \cdot 10^4$ and t_0 chosen as there, but with the verification height $H_0 = 3 \cdot 10^{12}$ replaced by T_D .

All inputs except [CH25] are published.

Validity of (D3). In the proof of [FKS23, Thm. 1.1] the height H_0 is used in exactly three places, each time only as a height below which every zero has $\beta = 1/2$:

- in ε_2 , through $B_1(H_0, T)$: the zeros with $1/2 < \beta < \sigma_1$ have $\gamma \geq H_0$;
- in $H_\sigma = \max\{H_0, \exp(1/(R(1 - \sigma)))\}$, through $N(\sigma, H_\sigma) = 0$;
- in t_0 in ε_4 , through H_{σ_2} .

Under our hypothesis each of these remains valid with T_D in place of H_0 . The density constants of [FKS23, Thm. 2.7] were computed with $H_0 = 3 \cdot 10^{12}$; since RH holds up to that height (Lemma 2.5), they remain valid for every verification height $T \geq 3 \cdot 10^{12}$. We evaluate (D3) at every 4 in $\log x_0$ on $[1700, L_C]$, where $L_C = 3035.00199 \dots$ is as in Section 4.4.

Verification. The segment and pointwise bounds are formed as in Section 4.2, with (D1) and (D3) added to the candidates. With the resulting majorant, the script `hybrid.D.py` verifies $M > 0$ on every interval of the grid of step 0.5 on $[752.985, L_C + 0.5]$, bisecting where necessary.

Sharpness for the method. A geometric bisection over $T \in [1.5 \cdot 10^{14}, 6 \cdot 10^{14}]$, with 11 steps, each a full verification of $M > 0$ on $[752.985, L_C + 0.5]$, shows that the verification succeeds at $T = 4.16017 \cdot 10^{14}$ and fails at $T = 4.15735 \cdot 10^{14}$. At the failing height the first failure is on the grid interval $[2579.985, 2580.485]$, inside the open range.

Proof of Theorem D. Since $4.1602 \cdot 10^{14} > T_D$, the hypothesis of Theorem D implies the hypothesis above, and the computation gives (R) for $e^{752.985} \leq x \leq e^{L_C + 0.5}$. Lemma 2.13(ii) and Theorem A cover $38, 358, 837, 683 \leq x \leq e^{753.98}$, and Theorem C covers $x \geq e^{3035.01}$. Lemma 2.13(i) gives $N_R = 38, 358, 837, 683$. $\square$

Remark 4.6 (With the preprints [BTY26] and [Bel24]). With the inputs of Theorem C* in (D3), that is, $R = 4.896$ and the pointwise minimum of the two bounds of Lemma 2.11 in ε_4 , with rows on $[1700, L_{C^*}]$, and with Theorem C* in place of Theorem C, the same computation succeeds at $T = 2.14878 \cdot 10^{14}$ and fails at $T = 2.14733 \cdot 10^{14}$, with the first failure on $[2187.485, 2187.985]$. So, given [BTY26] and [Bel24], RH up to height $2.1488 \cdot 10^{14}$, about 72 times 3,000,175,332,800, implies $N_R = 38,358,837,683$.

Remark 4.7 (The zeros through [CH25] only). Suppose the zeros above 3,000,175,332,800 enter only through Lemma 2.8(ii), that is, (D3) is replaced by the rows of Section 4.4, computed with $H_0 = 3 \cdot 10^{12}$. Then the method needs a larger height. The verification on $[752.985, L_C + 0.5]$ succeeds at $T = 7.96116 \cdot 10^{14}$ and fails at $T = 7.96091 \cdot 10^{14}$, with the first failure on $[3034.985, 3035.485]$; so RH up to height $7.9612 \cdot 10^{14}$ suffices. With the rows of Remark 4.3(d), which use [Bel24], and with (R) for $x \geq e^{2992.01}$ from there, the verification succeeds at $T = 7.51915 \cdot 10^{14}$ and fails at $T = 7.51891 \cdot 10^{14}$ (first failure on $[2991.985, 2992.485]$), so RH up to height $7.5192 \cdot 10^{14}$ suffices. With the rows of Section 4.5 and Theorem C*, it succeeds at $T = 4.22899 \cdot 10^{14}$ and fails at $T = 4.22886 \cdot 10^{14}$ (first failure on $[2590.985, 2591.485]$), so RH up to height $4.2290 \cdot 10^{14}$ suffices. These runs use `fxs_fine_R.py`.

With the printed rows of [FKS23] only and the version of Theorem C in Remark 4.3(c), the height is larger again. Let $T'_D = 9.33566 \cdot 10^{14}$ and assume RH up to T'_D . With $\varepsilon_{T'_D}(k)$ added to the candidates of Section 4.2 for $k \geq 43$, the script `ch_ramanujan.py` verifies $M > 0$ on every interval of the grid of step 0.5 on $[752.985, 3158.5]$; together with Theorem A and Theorem C, RH up to height $9.3357 \cdot 10^{14}$ implies $N_R = 38,358,837,683$. A geometric bisection over $T \in [3,000,175,332,800, 10^{16}]$, with 18 steps, shows that this verification succeeds at $T = 9.33566 \cdot 10^{14}$ and fails at $T = 9.33537 \cdot 10^{14}$, with the first failure on $[3157.485, 3157.985]$.

In all of these cases the binding constraint lies at the top of the range covered by the verification.

Remark 4.8 (Büthe's method). With Lemma 2.4 in place of Lemma 2.8(ii) the same approach needs a larger height. Let $T^* = 1.3277 \cdot 10^{15}$ and assume that every zero $\beta + i\gamma$ of $\zeta(s)$ with $0 < \gamma \leq T^*$ satisfies $\beta = 1/2$. Since $T^* > 3,000,175,332,800$, all rows $\delta(b)$ of Section 4.3 remain valid. For $b \in \mathcal{B}$ with $b \geq 600$, the script `confirm_theorem_D_thesis.py` recomputes the rows of Lemma 2.4 with $T = T^*$. It searches over $c \in c_0 + \{-1, 0, 1\}$ with $c_0 = 35 + \log(T^*/3,000,175,332,800)$ and $\alpha \in \{3 \cdot 10^{-4}, 10^{-3}\}$, and for each b it keeps the smaller of the new row and the row of Section 4.3. Under the hypothesis the new rows satisfy

$$\begin{aligned} \delta(600) &\leq 4.1260 \cdot 10^{-15}, & \delta(1000) &\leq 4.0311 \cdot 10^{-15}, \\ \delta(2000) &\leq 3.9627 \cdot 10^{-15}, & \delta(3000) &\leq 3.9404 \cdot 10^{-15}, \end{aligned}$$

more than two orders of magnitude below the rows of Table 4. With the majorant built from these rows exactly as $d_{A'}$ in Section 4.3, the script verifies $M > 0$ on every interval of the grid of step 0.5 on $[671.85, 3035.502]$, bisecting where necessary. Together with Proposition 4.1 and Theorem C, this gives $N_R = 38,358,837,683$ under RH up to height $1.3277 \cdot 10^{15}$, from refereed inputs and [Bha24]. A geometric bisection over $T \in [3,000,175,332,800, 10^{17}]$, with 16 steps (`buthe_thesis_run.py`), shows that this verification succeeds at $T = 1.3277 \cdot 10^{15}$ and fails at $T = 1.3275 \cdot 10^{15}$, with the first failure on the grid interval $[3035.349 \dots, 3035.502]$, at the start of Theorem C. Theorem D, which uses Lemma 2.8(ii) instead, needs a height about 3.19 times smaller.

4.10. An independent check. A separate script, which is not part of the proofs, recomputes $L^6[j(L-1)/L - j(L)^2]$ directly from mpmath's logarithmic integral, using $J(x) = \text{li}(x) - \text{li}(2) + 2/\log 2$ and no asymptotic expansion, at $L = 43, 50, 80, 103, 150, 300, 642, 1000, 3158$. Every value lies inside the enclosure of Main + Corr used in the proofs.

5. A SPECTRAL BARRIER

5.1. The classes.

Definition 5.1. Let $T > 0$. We write $\mathcal{A}_T$ for the set of real sequences $a = (a_n)_{n \geq 1}$ such that:

- (a) $a_n \geq 0$ for all n ;
- (b) $A(s) = \sum_{n \geq 1} a_n n^{-s}$ converges absolutely for $\text{Re } s > 1$ and continues meromorphically to $\mathbb{C}$;
- (c) A has a simple pole at $s = 1$ with residue 1, and no other pole with $|\text{Im } s| \leq T$.

We write $\mathcal{A}_T^{\text{RH}}$ for the larger class defined in the same way, except that in (c) A may also have poles with $|\text{Im } s| \leq T$ on the line $\text{Re } s = 1/2$ and in the half-plane $\text{Re } s < 0$.

Thus $\mathcal{A}_T \subseteq \mathcal{A}_T^{\text{RH}}$, and $\mathcal{A}_{T_1} \supseteq \mathcal{A}_{T_2}$ for $T_1 \leq T_2$. For $T = 3,000,175,332,800$ the sequence $a = \Lambda$, with $A = -\zeta'/\zeta$, lies in $\mathcal{A}_T^{\text{RH}}$: the poles of $-\zeta'/\zeta$ with $|\text{Im } s| \leq T$ are $s = 1$, with residue 1, the nontrivial zeros, which lie on $\text{Re } s = 1/2$ by Lemma 2.5, and the trivial zeros $-2, -4, \dots$. We call an argument *of type (a)–(c) at height T* if it is valid for every sequence in $\mathcal{A}_T^{\text{RH}}$. Conditions (a)–(c) impose no growth condition on A . In particular they do not include the boundedness conditions of [CH25, Thm. 1.1], which the sequences of Theorem E do not satisfy (Remark 5.3), so Theorem E says nothing about arguments that use those conditions.

For a sequence a put

$$\psi_a(y) = \sum_{n \leq y} a_n, \quad \pi_a(y) = \sum_{2 \leq n \leq y} \frac{a_n}{\log n}.$$

For $a = \Lambda$, π_a is Riemann's prime-power counting function $\Pi(y) = \sum_{k \geq 1} \pi(y^{1/k})/k$. It differs from $\pi(y)$ by $O(\sqrt{y}/\log y)$, and for $x \geq e^{700}$ replacing π by Π changes both sides of (R) by less than $10^{-100}x^2/\log^6 x$, far below the margins considered here.

5.2. The function Φ . With j as in Section 2, put

$$M_0(L) = L^6 \left(\frac{j(L-1)}{L} - j(L)^2 \right) = \text{Main}(L) + \text{Corr}(L),$$

the normalized margin (3.1) with $d = 0$. For $T' > 0$ put $v = \pi/T'$,

$$s_+ = v(\coth v + 1) - 1, \quad s_- = v(\coth v - 1) - 1,$$

and

$$\Phi(L, T') = M_0(L) + L^6 \left[\frac{s_-}{L(L-1)} - \frac{2j(L)s_+}{L} - \frac{s_+^2}{L^2} \right] + \frac{2\pi^2 L^3}{T'^2}.$$

For fixed L we also regard Φ as a function of v .

Lemma 5.2. *Let $700 \leq L \leq 10^4 + 1$ and $10^6 \leq T' \leq 10^{17}$, and put $v = \pi/T'$. Then*

- (a) $-3.01 L^4 \leq \partial\Phi/\partial v \leq -3(1-v)L^4$;
- (b) $\partial\Phi/\partial L \leq -11 v L^3$.

Consequently, for each $L \in [700, 10^4 + 1]$ there is a unique $\bar{T}(L) \in (10^6, 10^{17})$ with $\Phi(L, \bar{T}(L)) = 0$, and $\Phi(L, T') < 0$ if and only if $T' < \bar{T}(L)$. The function $\bar{T}$ is increasing, with $\frac{d}{dL} \log \bar{T}(L) \geq 3.6/L$. If $T' \leq (1-\varepsilon)\bar{T}(L)$ with $0 < \varepsilon < 1$, then $\Phi(L, T') \leq -2.99 \varepsilon L^4 v$.

Proof. We use three elementary facts. First, for $l \geq 700$,

$$(5.1) \quad 1 \leq l j(l) \leq 1 + \frac{1}{l-4} + 10^{-100}.$$

The lower bound holds because $J(x) \geq x/\log x$. For the upper bound write $j(l) = 1/l + e^{-l} \int_2^e dt/\log^2 t$, split the integral at e^{20} and $e^{l/2}$, and apply Lemma 3.1 to the two upper pieces (with $a = 20$ and $a = l/2$). Second, $j'(l) = 1/l - j(l)$, since $J'(x) = 1/\log x$. Third, $1 \leq v \coth v \leq 1 + v^2/3$ and $0 \leq f(v) \leq v$ for $v > 0$, where $f(v) = \coth v - v/\sinh^2 v$ is the derivative of $v \coth v$. Hence $v \leq s_+ \leq v + v^2/3$ and $-v \leq s_- \leq -v + v^2/3$.

- (a) Since $ds_{\pm}/dv = f(v) \pm 1$,

$$\frac{\partial\Phi}{\partial v} = -(1-f) \frac{L^5}{L-1} - 2(1+f)L^5 j(L) - 2(1+f)s_+ L^4 + 4v L^3.$$

By (5.1), $3 \leq L/(L-1) + 2Lj(L) \leq 3 + 3.02/L$. The upper bound follows from $f \geq 0$, $s_+ \geq 0$ and $f \leq v$: $\partial\Phi/\partial v \leq -3L^4 + vL^4(L/(L-1) + 4/L) \leq -3(1-v)L^4$. The lower bound follows from $f \leq v \leq \pi \cdot 10^{-6}$ and $s_+ \leq 2v$.

- (b) We have

$$\frac{\partial\Phi}{\partial L} = M'_0(L) + s_- \frac{d}{dL} \frac{L^5}{L-1} - 2s_+ \frac{d}{dL} (L^5 j(L)) - 4s_+^2 L^3 + 6v^2 L^2.$$

Here $\frac{d}{dL} \frac{L^5}{L-1} = \frac{L^4(4L-5)}{(L-1)^2} \geq 4L^3$, and $\frac{d}{dL} (L^5 j(L)) = L^4(1 - (L-5)j(L)) \geq 3.8L^3$ by (5.1). For $M_0 = \text{Main} + \text{Corr}$, put $w = 1/L$. Then $\frac{d}{dw} \text{Main} = Q(w)(1-w)^{-13}$ with $Q(w) = (1-w)N'_6(w) + 12N_6(w) = 14 + 108w + 1558w^2 + \dots \in \mathbb{Z}[w]$. The negative coefficients of Q occur only in degrees ≥ 8 , and all coefficients

of Q are less than $5 \cdot 10^{19}$ in absolute value, so $Q > 0$ on $[0, 1/700]$. Hence Main is decreasing in $L \geq 700$, and $\text{Main}(L) \geq \text{Main}(\infty) = 1$. Proposition 3.6 (with $q \leq 0.024$) and $g'(l) = l^{-13} - g(l)$ give $|\rho(l)| \leq 4.91 \cdot 10^8 l^{-13}$ and $|\rho'(l)| \leq 1.2 \cdot 10^7 l^{-13}$ for $l \geq 699$, and differentiating the formula for Corr gives $|\text{Corr}(L)| \leq 10^{-13}$ and $|\text{Corr}'(L)| \leq 10^{-14}$ for $L \geq 700$. Therefore $M'_0(L) \leq 10^{-14}$, and

$$\frac{\partial \Phi}{\partial L} \leq 10^{-14} - 4L^3 \left(v - \frac{v^2}{3} \right) - 7.6 v L^3 + 6v^2 L^2 \leq -11 v L^3,$$

because $vL^3 \geq \pi \cdot 10^{-17} \cdot 700^3 > 10^{-8}$.

For the consequences, note that $\Phi = M_0(L)$ at $v = 0$. By (a), $M_0(L) - 3.01L^4 v \leq \Phi(L, T') \leq M_0(L) - 2.99L^4 v$. Since $1 - 10^{-13} \leq M_0(L) \leq \text{Main}(700) + 10^{-13} < 1.03$, this gives $\Phi(L, 10^6) < 0 < \Phi(L, 10^{17})$, and (a) gives the existence and uniqueness of $\bar{T}(L)$ and the equivalence. With $\bar{v}(L) = \pi/\bar{T}(L)$, implicit differentiation and (a), (b) give $\bar{v}' = -\Phi_L/\Phi_v \leq -11\bar{v}L^3/(3.01L^4) \leq -3.6\bar{v}/L$, which is the bound for $\frac{d}{dL} \log \bar{T}$. Finally, if $T' \leq (1 - \varepsilon)\bar{T}(L)$ then $\bar{v} \leq (1 - \varepsilon)v$, so $\Phi(L, T') \leq \Phi(L, \bar{T}(L)) - 3(1 - v)L^4(v - \bar{v}) \leq -2.99\varepsilon L^4 v$. $\square$

5.3. Theorem E.

Theorem E. *Let $700 \leq L \leq 10^4$ and $T \geq 10^6$. Let m be an integer such that $T' = 2\pi m + 2\delta_0$, with $\delta_0 = 10^{-10}$, satisfies $T < T' \leq \min\{T + 2\pi + 1, 10^{17}\}$, and put $v = \pi/T'$. Suppose that*

$$(5.2) \quad \Phi(L', T') \leq -10^{-9} L'^4 v \quad \text{for every } L' \in [L, L + 2\pi/T'].$$

Then there are $a \in \mathcal{A}_T$ and x with $L \leq \log x < L + 2\pi/T'$ such that

$$\pi_a(x)^2 > \frac{ex}{\log x} \pi_a\left(\frac{x}{e}\right).$$

In particular, the inequality $\pi_a(x)^2 < \frac{ex}{\log x} \pi_a(x/e)$ does not hold for all $a \in \mathcal{A}_T^{\text{RH}}$ and all x with $L \leq \log x \leq L + 2\pi/T'$.

Theorem E therefore rules out, at such x , every proof of (R) that is valid for all of $\mathcal{A}_T^{\text{RH}}$, that is, every argument of type (a)–(c) at height T : every argument whose only inputs are non-negativity, analytic continuation, and the location and residues of the poles up to height T as recorded in (c).

Remark 5.3 (What Theorem E does not cover). Theorem E concerns the class defined by (a)–(c) only. Arguments that use further arithmetic structure of Λ are not covered: for instance the bound $\Lambda(n) \leq \log n$, the Euler product, or the growth of $-\zeta'/\zeta$ in the left half-plane. The sequences of Theorem E are far from arithmetic. Their first term is $a_1 = F_H(0) = H + 1 = 10^{40} + 1$, and a_n has spikes of that size wherever $T' \log n$ is close to a multiple of 2π . Nor do they satisfy the boundedness conditions of [CH25, Thm. 1.1]: by the functional equation, $|\zeta(\sigma + it)|$ grows like $\Gamma(1 - \sigma)(2\pi)^\sigma e^{\pi|t|/2}$ as $\sigma \rightarrow -\infty$, so $A(s)$ is unbounded on horizontal lines in the left half-plane. Theorem E is therefore a limitation of a class of methods, not a statement about the primes.

Proof. Step 1: the sequence. This is the construction of [CH25, Prop. 3.1], with T replaced by T' . Let $H = 10^{40}$, and let

$$F_H(t) = \sum_{|k| \leq H} w_k e^{ikt}, \quad w_k = 1 - \frac{|k|}{H+1},$$

be the Fejér kernel. Then $F_H(t) = \frac{1}{H+1} \left(\frac{\sin((H+1)t/2)}{\sin(t/2)} \right)^2 \geq 0$, $F_H(0) = H + 1$, and $F_H(t) \leq \pi^2/((H+1)t^2)$ for $0 < |t| \leq \pi$. Put $a_n = F_H(T' \log n)$. Then $a_n \geq 0$, $a_1 = H + 1$, and

$$A(s) = \sum_{|k| \leq H} w_k \zeta(s - ikT').$$

The poles of A are at $1 + ikT'$, $|k| \leq H$, with residues w_k . Since $T' > T$, the only one with $|\text{Im } s| \leq T$ is $s = 1$, with residue $w_0 = 1$. Since the sum is finite, A converges absolutely for $\text{Re } s > 1$ and continues meromorphically to $\mathbb{C}$. Hence $a \in \mathcal{A}_T$.

Step 2: ψ_a . For real τ and $y \geq 1$, Euler–Maclaurin summation gives $\sum_{n \leq y} n^{i\tau} = y^{1+i\tau}/(1+i\tau) + \mathcal{E}_\tau(y)$ with $|\mathcal{E}_\tau(y)| \leq 3 + |\tau| \log y$. Hence

$$\psi_a(y) = y S(y) + \mathcal{E}(y), \quad S(y) = \sum_{|k| \leq H} \frac{w_k y^{ikT'}}{1 + ikT'}, \quad |\mathcal{E}(y)| \leq (H+1)(3 + HT' \log y).$$

Step 3: values of S . For $y > 0$ let G_y be the 2π -periodic function with

$$G_y(t) = v(\coth v + 1) \exp(-r_y(t)/T'),$$

where $r_y(t) \in [0, 2\pi)$ is the residue of $T' \log y - t$ modulo 2π . Since $v(\coth v + 1) = (2\pi/T')/(1 - e^{-2\pi/T'})$, a direct computation gives $\frac{1}{2\pi} \int_{-\pi}^{\pi} G_y(t) e^{ikt} dt = y^{ikT'}/(1 + ikT')$, and therefore

$$S(y) = \frac{1}{2\pi} \int_{-\pi}^{\pi} G_y(t) F_H(t) dt.$$

Suppose that $T' \log y \equiv \pm \delta_0 \pmod{2\pi}$. Then the only discontinuity of G_y in $[-\pi, \pi]$ is at $t = \pm \delta_0$, so G_y is smooth on $(-\delta_0, \delta_0)$ with $|G'_y| = G_y/T' \leq 2/T'$ there, and $0 < G_y \leq v(\coth v + 1) \leq 2$ everywhere. Since $\frac{1}{2\pi} \int_{-\pi}^{\pi} F_H = 1$ and $F_H(t) \leq \min\{H + 1, \pi^2/((H + 1)t^2)\}$,

$$\begin{aligned} |S(y) - G_y(0)| &\leq \frac{1}{2\pi} \int_{|t| < \delta_0} \frac{2|t|}{T'} F_H(t) dt + \frac{1}{2\pi} \int_{\delta_0 \leq |t| \leq \pi} 4F_H(t) dt \\ &\leq \frac{1 + 2\pi^2 \log((H + 1)\delta_0)}{\pi(H + 1)T'} + \frac{4\pi}{(H + 1)\delta_0} < 10^{-28}. \end{aligned}$$

If $T' \log y = 2\pi h + \delta_0$ with $h \in \mathbb{Z}$, then $r_y(0) = \delta_0$ and $G_y(0) = v(\coth v + 1)e^{-\delta_0/T'}$. If $T' \log y = 2\pi h - \delta_0$, then $r_y(0) = 2\pi - \delta_0$ and $G_y(0) = v(\coth v + 1)e^{-2\pi/T'}e^{\delta_0/T'} = v(\coth v - 1)e^{\delta_0/T'}$. Since $\delta_0/T' = (\delta_0/\pi)v$, $v(\coth v \pm 1) \leq 1 + 2v$, and $10^{-28} < 4 \cdot 10^{-12}v$ (as $v \geq \pi \cdot 10^{-17}$), we obtain

$$(5.3) \quad |S(y) - 1 - s_{\pm}| < 4 \cdot 10^{-11} v \quad \text{if } T' \log y \equiv \pm \delta_0 \pmod{2\pi}.$$

Step 4: the choice of x . Let $h = \lceil (T'L - \delta_0)/(2\pi) \rceil$ and $x = \exp((2\pi h + \delta_0)/T')$. Then $L \leq \log x < L + 2\pi/T'$. Since $T' = 2\pi m + 2\delta_0$,

$$\log(x/e) = \frac{2\pi h + \delta_0 - T'}{T'} = \frac{2\pi(h - m) - \delta_0}{T'}.$$

So (5.3) applies with the sign $+$ at $y = x$ and with the sign $-$ at $y = x/e$.

Step 5: π_a . Partial summation gives, for $y \geq 2$,

$$\pi_a(y) = \frac{\psi_a(y)}{\log y} - \frac{a_1}{\log 2} + \int_2^y \frac{\psi_a(t)}{t \log^2 t} dt = J(y) + \frac{y(S(y) - 1)}{\log y} + \text{Osc}(y) + W(y),$$

where

$$\text{Osc}(y) = \int_2^y \frac{S(t) - 1}{\log^2 t} dt, \quad W(y) = \frac{\mathcal{E}(y)}{\log y} - \frac{a_1}{\log 2} + \int_2^y \frac{\mathcal{E}(t)}{t \log^2 t} dt.$$

For $\tau \neq 0$, integration by parts gives

$$\left| \int_2^y \frac{t^{i\tau}}{\log^2 t} dt \right| \leq \frac{1}{|1 + i\tau|} \left(\frac{y}{\log^2 y} + \frac{2}{\log^2 2} + 2 \int_2^y \frac{dt}{\log^3 t} \right).$$

Numerical quadrature gives $\int_2^{e^{20}} dt/\log^3 t < 7.3 \cdot 10^4$, and $\frac{d}{dt} \frac{t}{\log^3 t} \geq \frac{0.85}{\log^3 t}$ for $t \geq e^{20}$; hence $\int_2^y dt/\log^3 t \leq 7.3 \cdot 10^4 + 1.18 y/\log^3 y$. For $\log y \geq 600$ the integral above is therefore at most $(1.004 y/\log^2 y + 1.5 \cdot 10^5)/|\tau|$. Since $S(t) - 1 = \sum_{k \neq 0} w_k t^{ikT'}/(1 + ikT')$ and $|1 + ikT'| \geq |k|T'$,

$$|\text{Osc}(y)| \leq \sum_{k \neq 0} \frac{1}{k^2 T'^2} \left(\frac{1.004 y}{\log^2 y} + 1.5 \cdot 10^5 \right) = \frac{\pi^2}{3T'^2} \left(\frac{1.004 y}{\log^2 y} + 1.5 \cdot 10^5 \right) \quad (\log y \geq 600).$$

For the remaining term, $a_1 = H + 1$ and Step 2 give $|W(y)| \leq 4(H + 1)^2 T' \log y$, which is less than $10^{-200} y$ for $698 \leq \log y \leq 10^4 + 1$.

Step 6: the margin. Let $\lambda = \log x$, $\Delta = \frac{ex}{\lambda} \pi_a(x/e) - \pi_a(x)^2$, $\tilde{s}_+ = S(x) - 1$, $\tilde{s}_- = S(x/e) - 1$ and $\mathcal{R}(y) = \text{Osc}(y) + W(y)$. Since $J(x) = x j(\lambda)$ and $J(x/e) = (x/e) j(\lambda - 1)$, Step 5 gives

$$\frac{\lambda^6 \Delta}{x^2} = M_0(\lambda) + \lambda^6 \left[\frac{\tilde{s}_-}{\lambda(\lambda - 1)} - \frac{2j(\lambda)\tilde{s}_+}{\lambda} - \frac{\tilde{s}_+^2}{\lambda^2} \right] + \text{Rem},$$

$$\text{Rem} = \lambda^6 \left[\frac{\mathcal{R}(x/e)}{\lambda x/e} - 2 \left(j(\lambda) + \frac{\tilde{s}_+}{\lambda} \right) \frac{\mathcal{R}(x)}{x} - \frac{\mathcal{R}(x)^2}{x^2} \right].$$

By Step 5, $R^* = \max\{|\mathcal{R}(x)|/x, |\mathcal{R}(x/e)|/(x/e)\} \leq \frac{\pi^2}{3} \cdot \frac{1.004}{(\lambda-1)^2 T'^2} + 2 \cdot 10^{-200}$. Using (5.1) and $|\tilde{s}_+| < 2v$ (by (5.3)), we get $|\text{Rem}| \leq \lambda^5(3+4/\lambda)R^* \leq 1.05\pi^2\lambda^3/T'^2$. Next, by (5.3), replacing $\tilde{s}_\pm$ by $s_\pm$ in the bracket changes $\lambda^6[\dots]$ by at most

$$4 \cdot 10^{-11}v \lambda^4 \left(\frac{\lambda}{\lambda-1} + 2\lambda j(\lambda) + 2s_+ + 4 \cdot 10^{-11}v \right) < 1.3 \cdot 10^{-10}\lambda^4v.$$

Comparing with the definition of Φ , we conclude that

$$\frac{\lambda^6\Delta}{x^2} \leq \Phi(\lambda, T') + 1.3 \cdot 10^{-10}\lambda^4v - 0.95 \frac{\pi^2\lambda^3}{T'^2} < \Phi(\lambda, T') + 10^{-9}\lambda^4v \leq 0$$

by (5.2), since $\lambda \in [L, L + 2\pi/T']$. Hence $\Delta < 0$, which is the claim. $\square$

5.4. Proof of Corollary E. The script `barrier.py` evaluates Φ in 40-digit arithmetic, with $M_0(L)$ replaced by the upper end of its interval enclosure (Main + Corr, as in Section 4) and $j(L)$ by the lower end of its enclosure. Since $j(L)$ and $s_+ > 0$ enter Φ with a minus sign, the computed function is an upper bound for Φ , up to rounding errors of relative size 10^{-38} . It computes $\bar{T}(L)$ by bisection in T' ; by Lemma 5.2, the value returned is a lower bound for $\bar{T}(L)$.

Proof of Corollary E. (i) Let $T = 3,000,175,332,800$. A bisection in L locates the point $754.39478\dots$ where $\bar{T}$ crosses T : it finds $L_b < 754.3948$ with $\bar{T}(L_b) \geq T$. By Lemma 5.2, $\log(\bar{T}(L')/\bar{T}(L_b)) \geq 3.6 \log(754.40/754.3948) > 2 \cdot 10^{-5}$ for every $L' \geq 754.40$. Given $L \in [754.40, 10^4]$, choose an integer m with $T < T' = 2\pi m + 2\delta_0 \leq T + 2\pi + 1$. Then $T' \leq (1 - 10^{-9})\bar{T}(L')$ for every $L' \in [L, L + 2\pi/T']$, so $\Phi(L', T') \leq -2.99 \cdot 10^{-9}L'^4v$ by Lemma 5.2, and Theorem E applies. Since $2\pi/T' < 10^{-11}$, (i) follows.

(ii), (iii) For (ii), let $(L_0, T_0, T_1) = (3035.002, 7.9624 \cdot 10^{14}, 7.96241 \cdot 10^{14})$, and for (iii), let $(L_0, T_0, T_1) = (3157.93, 9.3344 \cdot 10^{14}, 9.33450511 \cdot 10^{14})$. The script gives $\bar{T}(3035.002) \geq 7.96241 \cdot 10^{14}$ and $\bar{T}(3157.93) \geq 9.33450511 \cdot 10^{14}$, so in both cases $\bar{T}(L_0) \geq T_1$. Since $\mathcal{A}_T \supseteq \mathcal{A}_{10^6}$ for $T < 10^6$, we may assume $10^6 \leq T \leq T_0$. Choose m with $T < T' = 2\pi m + 2\delta_0 \leq T + 2\pi + 1$. Then $T' \leq T_0 + 2\pi + 1 < (1 - 10^{-9})T_1$. As $\bar{T}$ is increasing, $T' \leq (1 - 10^{-9})\bar{T}(L')$ for every $L' \geq L_0$, so (5.2) holds with $L = L_0$, and Theorem E applies. Since $2\pi/T' < 10^{-5}$, (ii) and (iii) follow. $\square$

Theorem A reaches $\log x = 753.985$, and $754.39478 - 753.985 < 0.41$. At the top of the open range the barrier is $\bar{T}(3035.002) = 7.96241 \cdot 10^{14}$. The version of Theorem D in which the zeros above 3,000,175,332,800 enter only through [CH25] (Remark 4.7) uses $7.9612 \cdot 10^{14}$, slightly less, because near the top the rows of Section 4.4, which carry zero-free-region and zero-density information, take over. With the rows of Remark 4.3(d) that version uses $7.5192 \cdot 10^{14}$, against $\bar{T}(2992.002) \approx 7.520 \cdot 10^{14}$, and with Theorem C* it uses $4.2290 \cdot 10^{14}$, against $\bar{T}(2591.002) \approx 4.226 \cdot 10^{14}$. With the printed rows of [FKS23] it uses $9.33566 \cdot 10^{14}$, which is 1.000124 times $\bar{T}(3157.93) = 9.33450511 \cdot 10^{14}$. None of these methods is of type (a)–(c), so Theorem E does not bound them from below; the comparison shows how close to the barrier they come.

5.5. The barrier and the methods of Theorem D. Table 6 compares $\bar{T}(L)$ with the least height $T_{\text{suf}}(L)$ at which the bounds of Section 4.2 together with Lemma 2.8(ii) at height T , as in the last part of Remark 4.7, prove $M > 0$ on $[L - 0.25, L]$; `barrier.py` computes T_{suf} by a geometric bisection with 40 steps. The last column is the approximation $3\pi L^4/(1 + 14/L)$, which agrees with $\bar{T}(L)$ to within 0.11% on the table.

TABLE 6. The barrier height $\bar{T}(L)$, the height $T_{\text{suf}}(L)$ that suffices near e^L when the zeros above 3,000,175,332,800 enter only through Lemma 2.8(ii), and the approximation $3\pi L^4/(1 + 14/L)$.

L	$\bar{T}(L)$	$T_{\text{suf}}(L)$	$T_{\text{suf}}/\bar{T}$	$3\pi L^4/(1 + 14/L)$
800	$3.7978905 \cdot 10^{12}$	$3.8054345 \cdot 10^{12}$	1.001986	$3.7939942 \cdot 10^{12}$
1000	$9.3026197 \cdot 10^{12}$	$9.3173894 \cdot 10^{12}$	1.001588	$9.2946528 \cdot 10^{12}$
1500	$4.7300245 \cdot 10^{13}$	$4.7350252 \cdot 10^{13}$	1.001057	$4.7271736 \cdot 10^{13}$
2000	$1.4981773 \cdot 10^{14}$	$1.4993645 \cdot 10^{14}$	1.000792	$1.4974821 \cdot 10^{14}$
2500	$3.6624326 \cdot 10^{14}$	$3.6647536 \cdot 10^{14}$	1.000634	$3.661052 \cdot 10^{14}$
3000	$7.601022 \cdot 10^{14}$	$7.6050353 \cdot 10^{14}$	1.000528	$7.59861 \cdot 10^{14}$
3157.93	$9.3345051 \cdot 10^{14}$	$9.3207897 \cdot 10^{14}$	0.998531	$9.3316841 \cdot 10^{14}$

For $800 \leq L \leq 3000$ this method comes within a factor $1+c(L)$ of the barrier, with $5 \cdot 10^{-4} < c(L) < 2 \cdot 10^{-3}$ at the tabulated points. The method is not of type (a)–(c): it uses [CH25, Prop. 9.1] and the bounds of [FKS23], so Theorem E does not bound T_{suf} from below. At $L = 3157.93$ the ratio $T_{\text{suf}}/\bar{T}$ is $0.998531 < 1$. There the bound ε_1 of Lemma 2.9(ii) is smaller than ε_T at $T = T_{\text{suf}}(3157.93)$: $\varepsilon_1(3158) \approx 3.3542 \cdot 10^{-15}$ (Remark 4.3), while $\pi/T_{\text{suf}}(3157.93) > 3.37 \cdot 10^{-15}$. That bound rests on an explicit zero-free region, which the sequences of Theorem E violate. This is where the version of Theorem C in Remark 4.3(c) begins.

The combined method. Theorem D goes far below the barrier. It needs $T = 4.1602 \cdot 10^{14}$, about $0.52\bar{T}(3035.002)$, and $2.1488 \cdot 10^{14}$ with [BTY26] and [Bel24]. Its input (D3) uses, above height T , the zero-free region of Lemma 2.10 and zero-density estimates. These are not of type (a)–(c), and the sequences of Theorem E violate them: their poles at $1+ikT'$ lie on the line $\text{Re } s = 1$. Where it fails, at $T = 4.15735 \cdot 10^{14}$, the first failure lies near the point where $\bar{T}(L)$ reaches T , that is, where the bound of Lemma 2.8(ii) stops sufficing and (D3) is not yet strong enough.

Remark 5.4. The sequences of Theorem E have poles on the line $\text{Re } s = 1$, at the points $1+ikT'$ with $k \neq 0$. For $-\zeta'/\zeta$ such poles are excluded by the nonvanishing of ζ on $\text{Re } s = 1$, and explicit zero-free regions and zero-density estimates quantify this. Such information is not of type (a)–(c). With it, Theorems C and C* prove (R) for all $x \geq e^{3035.01}$ and all $x \geq e^{2591.01}$, respectively, although by Corollary E(i) $\mathcal{A}_T$ with $T = 3,000,175,332,800$ contains counterexamples within 10^{-11} in $\log x$ of every point with $754.40 \leq \log x \leq 10^4$.

Remark 5.5. To first order in v , $\Phi(L, T') = 0$ reads $\pi/T' \approx M_0(L)/(L^4(L/(L-1)+2Lj(L)))$, which explains the size $\bar{T}(L) \approx 3\pi L^4$. The Fejér sequence is a prime distribution that is consistent with every pole up to height T , yet it can raise $\psi_a(x)$ by about $(\pi/T)x$ just after one of its peaks and lower $\psi_a(x/e)$ by about $(\pi/T)(x/e)$ just before another. Inequality (R) weighs the first change twice, through $\pi(x)^2$, and the second once, which gives the factor 3; the margin of (R) is $x^2/\log^6 x$, which gives the factor L^4 .

6. WHAT REMAINS WITHOUT THE HALF-PLANE $\text{Re } s > 7/8$

Corollary F closes the whole range. If one relies only on the peer-reviewed literature, Theorems A and C leave the range $e^{753.98} < x < e^{3035.01}$ open; with the preprints [BTY26] and [Bel24], Theorem C* reduces it to $e^{753.98} < x < e^{2591.01}$.

Arguments of type (a)–(c) do not suffice. By Corollary E(i), for every $L \in [754.40, 10^4]$ the class $\mathcal{A}_T$ with $T = 3,000,175,332,800$ contains a sequence that violates the analogue of (R) at some x with $L \leq \log x \leq L+10^{-11}$. An argument of type (a)–(c) at height 3,000,175,332,800 (Section 5) therefore cannot prove (R) on any interval $[e^L, e^{L+10^{-11}}]$ with $754.40 \leq L < 3035.01$, and it can extend the range $\log x \leq 753.985$ verified for Theorem A by less than 0.42. The explicit methods for ψ in this range ([Büt16], [CH25], [BKLNW21, FKS23]) use, beyond pole data up to height T , only zero-free regions and zero-density estimates above T . Theorems C and C* use these, through [FKS23] and Lemmas 2.10 and 2.11. Theorem E does not cover arguments that use further arithmetic structure (Remark 5.3).

Routes. Each of the following closes the range and gives $N_R = 38,358,837,683$:

- (1) a verification of RH up to height $4.1602 \cdot 10^{14}$, about 139 times 3,000,175,332,800; with [BTY26] and [Bel24], up to height $2.1488 \cdot 10^{14}$, about 72 times 3,000,175,332,800 (Theorem D and Remark 4.6);
- (2) a zero-free region $\beta \leq 1 - 1/(R \log |\gamma|)$ for $|\gamma| > 3 \cdot 10^{12}$ with $R \leq 1.7373$, a constant about 3.2 times smaller than that of Lemma 2.10(i) and 2.8 times smaller than that of Lemma 2.10(ii) (Theorem B');
- (3) a zero-free half-plane $\text{Re } s > 0.945092$ (Theorem B, refereed inputs and [Bha24]), or $\text{Re } s > 0.96897$ with the preprints [FJ26] and [CH25] (Theorem B*). Lemma 2.6 provides $\text{Re } s > 7/8$ (Corollary F).

For Theorem B the binding constraint lies at $L = 672.5$, where Proposition 4.1 ends; for Theorems B* and B' it lies at $L \approx 754$, where Lemma 2.8(i) stops sufficing. For route (1) it lies inside the range, at $L \approx 2580$ ($L \approx 2187$ with the preprints).

Trade-off between routes (1) and (2). Both are extreme cases of the hypothesis that every zero with $0 < \gamma \leq T$ has $\beta = 1/2$ and that there are no zeros with $\beta \geq 1 - 1/(R \log |\gamma|)$ and $|\gamma| > T$. Table 7 gives, for several values of R , a height $T(R)$ for which this hypothesis implies $N_R = 38,358,837,683$. The method is that of Theorem D, with (D3) computed for the constant R and with rows from $\log x_0 = 700$ as in Section 4.8; like Theorem D it uses [CH25]. Each value comes from a geometric bisection with 12 steps (`tradeoff.py`), and the failing value just below it is recorded in the output files. For $R = 1.7373$ the

current verification height suffices, which is Theorem B'. For $R = 5.558691$ the value is close to the height in Theorem D, whose rows start at $\log x_0 = 1700$.

TABLE 7. Heights $T(R)$ such that RH for $0 < \gamma \leq T(R)$ and a zero-free region $\beta < 1 - 1/(R \log |\gamma|)$ for $|\gamma| > T(R)$ imply $N_R = 38,358,837,683$.

R	$T(R)$
1.7373	$3.00018 \cdot 10^{12}$ (current verification height)
2.0	$5.43248 \cdot 10^{12}$
2.5	$1.40572 \cdot 10^{13}$
3.0	$3.08641 \cdot 10^{13}$
3.5	$5.48834 \cdot 10^{13}$
4.0	$9.74689 \cdot 10^{13}$
4.896	$2.37951 \cdot 10^{14}$
5.558691	$4.16074 \cdot 10^{14}$

Remark 6.1 (Density sensitivity; hypothetical). If the zero-density bounds of Lemma 2.11(i) (second block of [FKS23, Table 7]) were smaller by a factor K , in both ε_3 and ε_4 , the method of Theorem C would verify (C) from $L = 3035.0$ for $K = 1$ (the control), 2548.0 for $K = 10$, 2032.0 for $K = 100$, 1464.0 for $K = 10^3$, 1238.0 for $K = 10^4$ and 1220.0 for $K = 10^6$, on a grid of step 0.5 (`sens_density.py`). So improved density estimates alone cannot reach e^{754} .

The height needed at each point. Table 6 lists across the range the barrier $\bar{T}(L)$, below which arguments of type (a)–(c) at height T cannot prove (R) near e^L , and the height $T_{\text{suf}}(L)$ at which the method of Remark 4.7, which uses [CH25, Prop. 9.1] at height T , succeeds there. Up to $L = 3000$ the two agree to within 0.2%. Both grow like $3\pi L^4$: at $L = 2000$ the barrier is $1.498 \cdot 10^{14}$, about 50 times 3,000,175,332,800, and at $L = 3035.002$ it is $7.96241 \cdot 10^{14}$, about 265 times 3,000,175,332,800. Route (1) needs only about half of the latter, because above height T it uses a zero-free region and zero-density estimates.

7. REMARKS ON THE LITERATURE

7.1. The preprint arXiv:2408.02591. De [De24b] claims (R) for all $x \geq e^{43.51}$ unconditionally, which would contain Theorems A and C. The argument fails at its final step (29). From $\pi(x) < e\pi(x/e)$ and $\pi(x) > x/\log x$ it concludes $\pi(x)^2 < \frac{ex}{\log x}\pi(x/e)$. Multiplying the first inequality by $\pi(x)$ gives only $\pi(x)^2 < e\pi(x)\pi(x/e)$. To reach (R) one needs $\pi(x) \leq x/\log x$, which is the reverse of the inequality the step itself uses, and which is false for all $x \geq 17$. The true margin in (R) is of size $x^2/\log^6 x$ (Lemma 3.7), and bounds as crude as these cannot detect it.

7.2. The preprint arXiv:2407.12052. De [De24a] (version 6) claims (R) for all $x \geq e^{547}$ unconditionally, which would cover the upper part of the range of Theorem A and the whole range left open here. It is not a proof:

- its monotonicity step (Prop. 6.1) assumes Sterneck's conjecture $|M(x)| < \frac{1}{2}\sqrt{x}$ for $x \geq 201$, where here $M(x) = \sum_{n \leq x} \mu(n)$ is the Mertens function. That conjecture is false; Neubauer [Neu63] found a counterexample near $7.8 \cdot 10^9$;
- its bounds for $\pi(x)$ carry O -terms with unspecified constants, so none of its estimates is explicit;
- its “check” at e^{547} is a numerical evaluation of the auxiliary function G of that paper, not a proven bound.

7.3. Lemma 2.1 of Dudek–Platt and Axler's threshold. Lemma 2.1 of [DP15] gives a sufficient condition for (R) in terms of the constants m_a and M_a in explicit lower and upper bounds for $\pi(x)$. IEANTN [Tao+26] (node `DudekPlatt.v3`) records a defect in its proof, reported by D. Platt: the term carrying the lower-bound constant m_a is bounded in the wrong direction when $m_a < 0$, and $m_a < 0$ in every application. IEANTN gives a repaired form of the criterion, checked in Lean, in which m_a is multiplied by $(1 + 1/\log x_1)^6$ (notation of that node).

Axler [Ax22] proves (R) for $x \geq e^{3158.442}$ by arguing “as in the proof of” [DP15, Lemma 2.1], with $m_a = -936.646$. His threshold condition $\widetilde{M}_a - \widetilde{m}_a < \log x$ holds at $\log x = 3158.442$ with a margin of only 0.003. With the repair, the condition costs about 1.78 more, and Axler’s own parameters then give about $e^{3160.3}$ instead of $e^{3158.442}$. We have not re-optimized Axler’s parameters. For (R), Theorem C settles the question, since its proof does not use [DP15, Lemma 2.1]. Other thresholds obtained through [DP15, Lemma 2.1] with $m_a < 0$ need the same check; IEANTN notes that this includes the threshold e^{3915} reached in the PrimeNumberTheoremAnd project.

7.4. Bütthe’s Theorem 1 and Table 8 of BKLNW. *Prior work.* Bhattacharjee [Bha24, §2.3] lists errors in [Büt16]. Among them:

- the factor $\log(3c)$, which is present in [Büt16, Prop. 3, (4.1)], is missing from E_2 in Theorem 1;
- in [Büt16, Lemma 2], which is taken from [FKBJ17, Lemma 2.4], an integral is evaluated as $\log(2c/2\pi)/(2\pi c)$ instead of $\log(2c/(2\pi\epsilon))/(2\pi c)$;
- E_1 lacks the summand $e^{\alpha\epsilon} - 1$;
- constant terms are missing from E_2 and E_3 ;
- the limits of integration in [Büt16, Prop. 1] and the bound (3.7) in [Büt16, Lemma 1] need correction.

Her corrected Theorem 7 is our Lemma 2.4. She recomputes Bütthe’s Table 2 at his parameters and finds his values too small by 4.9–8.9%, except at e^{55} (0.003%) and e^{60} (1.4%). The missing factor $\log(3c)$ and the discrepancy in Table 2 were thus found by Bhattacharjee. Bütthe’s own partial correction, version v4 of arXiv:1410.7015 (May 2022), adds the summand $e^{\alpha\epsilon} - 1$ to E_1 and states that the explicit bounds in Tables 1 and 2 remain valid. The factor $\log(3c)$ is absent from E_2 in all versions (v1, v3 and v4), although it is present in [Büt16, Lemma 2], in [Büt16, Prop. 3, (4.1)], from which E_2 is derived, and in the corresponding step (7.6) of the proof of [Büt16, Thm. 2].

Our contributions. They are the following.

- (1) The rows $40 \leq b \leq 2000$ of [BKLNW21, Table 8] come from Bütthe’s Theorem 1 and reproduce from the printed formula (we checked $100 \leq b \leq 2000$), so they inherit the error; [Bha24] does not discuss [BKLNW21]. After re-optimizing the parameters, the rows $b \leq 50$ change by less than 0.03%; the row $b = 40$, at $x = 10^{19}$, even stays below its printed value $1.93378 \cdot 10^{-8}$, at $1.91948 \cdot 10^{-8}$. The rows $100 \leq b \leq 2000$ rise by 1.3–2.2%. The rows $b = 20, \dots, 35$ coincide with Bütthe’s bound $0.94 e^{-b/2}$ from [Büt18] and are not affected.
- (2) The re-optimization shows that the gap persists at the best parameters (Table 8).
- (3) The consequences in the literature (Section 7.5).

In the rest of this subsection “with $\log(3c)$ ” refers to the version of Bütthe’s Theorem 1 in which only the factor $\log(3c)$ is restored, with $E_2 = 0.16(1 + x_*^{-1})(\sinh c)^{-1}e^{0.71\sqrt{c\epsilon}}\log(3c)\log(c/\epsilon)$ and E_3 as printed. The full correction of Lemma 2.4 gives slightly larger values: re-optimized, the rows $b = 100, 200, 600, 1000$ and 2000 of [BKLNW21, Table 8] rise by 1.5–2.4%.

At Bütthe’s parameters. The published tables match the formula as printed. Evaluated without the factor $\log(3c)$ at the parameters used for them ($c = 29$ for [Büt16, Table 1] and $c = 33$ for [Büt16, Table 2]), Theorem 1 reproduces Bütthe’s values; see Table 8. Both of his tables were therefore computed without the factor. With $\log(3c)$ restored at the same parameters, the values rise to those in the fourth column of Table 8.

With re-optimized parameters. At fixed c the correction looks large, but most of it can be recovered by choosing c and α again. For each row of Table 8 the script `buthe_reopt_fine.py` minimizes the corrected bound, at the row’s own height, over $c \in [20, 48]$ and $\alpha \in [10^{-5}, 0.3]$. It evaluates $\mu_c(\alpha)$ and $|\nu_c(\alpha)|$ by 40-digit quadrature, because the interval enclosures of Section 4.3 are loose for larger α and would overstate the minimum. The least corrected values (last column) exceed the printed ones by 2.54% for [Büt16, Table 1] at e^{500} , by 2.17% and 2.15% for [Büt16, Table 2] at e^{1000} and e^{4000} , and by 1.27% and 1.80% for the rows $b = 200$ and $b = 600$ of [BKLNW21, Table 8]. On this grid no (c, α) reaches a printed value, so the tables are not justified as printed. The same search with the rigorous enclosures (`buthe_reopt.py`) gives values between 1.89% and 2.62% above the printed ones; those rigorous rows are the ones used for the range at the end of this subsection.

Quadrature check. As a check on the enclosures of Section 4.3, which is not part of the proofs, we recomputed $\mu_c(\alpha)$ and $|\nu_c(\alpha)|$ by numerical quadrature (mpmath, 40 digits, with I_0 and I_1). Every quadrature

TABLE 8. Bütthe's Theorem 1 against the published tables. Columns 3 and 4: our evaluation by quadrature, without and with $\log(3c)$, at the published parameters. Column 5: the least value with $\log(3c)$ after re-optimizing (c, α) , by quadrature.

Row	printed	without $\log(3c)$	with $\log(3c)$	re-optimized
[Büt16, Table 1], $x = e^{500}$	$1.47067 \cdot 10^{-10}$	$1.47057 \cdot 10^{-10}$	$1.53869 \cdot 10^{-10}$	$1.50799 \cdot 10^{-10}$
[Büt16, Table 1], $x = e^{1000}$	$1.43770 \cdot 10^{-10}$	$1.43760 \cdot 10^{-10}$	$1.50572 \cdot 10^{-10}$	–
[Büt16, Table 1], $x = e^{3000}$	$1.41594 \cdot 10^{-10}$	$1.41584 \cdot 10^{-10}$	$1.48397 \cdot 10^{-10}$	–
[Büt16, Table 2], $x = e^{1000}$	$1.94751 \cdot 10^{-12}$	$1.94737 \cdot 10^{-12}$	$2.10026 \cdot 10^{-12}$	$1.98974 \cdot 10^{-12}$
[Büt16, Table 2], $x = e^{4000}$	$1.90866 \cdot 10^{-12}$	$1.90852 \cdot 10^{-12}$	$2.06141 \cdot 10^{-12}$	$1.94976 \cdot 10^{-12}$
[BKLNW21, Table 8], $b = 200$	$2.18154 \cdot 10^{-12}$	–	–	$2.20935 \cdot 10^{-12}$
[BKLNW21, Table 8], $b = 600$	$1.98894 \cdot 10^{-12}$	–	–	$2.02478 \cdot 10^{-12}$

value lies inside the corresponding enclosure. For $c = 35$, $\alpha = 0.001$ gives $\mu_c(\alpha) = 0.4976312704$ and $|\nu_c(\alpha)| = 0.0662056161$, and $\alpha = 0.003$ gives 0.492894138 and 0.0652150908 . In the version with only $\log(3c)$ restored, the rows at $b = 600$ and $b = 700$ computed by quadrature, $1.65775 \cdot 10^{-12}$ and $1.64671 \cdot 10^{-12}$, lie just below the interval upper bounds $1.65801 \cdot 10^{-12}$ and $1.64697 \cdot 10^{-12}$; the enclosures are the same for Lemma 2.4.

Consequences for this paper. The rows $40 \leq b \leq 2000$ of [BKLNW21, Table 8], and Bütthe's Tables 1 and 2, are not justified by the lemmas from which they are derived, although the corrected values exceed them by only 1.3–2.6% at the rows we re-optimized. These rows of [BKLNW21, Table 8] were computed with RH verified up to height 2,445,999,556,030. With rows corrected for $\log(3c)$ and re-optimized at that height, the method of this paper proves (R) for $43.749 \leq \log x \leq 640.001$, that is, up to about e^{640} (`buthe_reopt.py`). Theorem A and Proposition 4.1 avoid these rows entirely; Proposition 4.1 goes further because Lemma 2.5 provides a larger height. The printed values of [BKLNW21, Table 8] quoted here agree with the independent machine extraction of that table in IEANTN (node BKLNW.v1).

7.5. Consequences in the literature. We searched the papers that cite [Büt16], [BKLNW21] and the papers below for numerical uses of Bütthe's Theorem 1 or of the affected rows; the details and scripts are in the folder `downstream` of the data (`DOWNSTREAM_REPORT.md`). We recomputed the first three items ourselves. The others are findings of a model-based search, labelled as such, which we have not verified independently.

Johnston–Yang [JY23]. Their Theorem 1.1 states $|\psi(x) - x| \leq 9.39 x (\log x)^{1.515} \exp(-0.8274\sqrt{\log x})$ for $x \geq 2$. Their Section 3.1 covers $(e^{2000}, e^{2488}]$ by their Lemma 2.3, with $\varepsilon = 1.570 \cdot 10^{-12}$, a recomputation of [Büt16, Thm. 1] with the printed E_2 , $\alpha = 0$, $c = 33.6$ and $\varepsilon = 1.12 \cdot 10^{-11}$ in Bütthe's notation. From these parameters we obtain $1.569362 \cdot 10^{-12}$; with $\log(3c)$, $1.654295 \cdot 10^{-12}$; re-optimized, $1.604805 \cdot 10^{-12}$ (at $c = 35.156$, $\alpha = 0.0029$). With $f(L) = 9.39 L^{1.515} e^{-0.8274\sqrt{L}}$ we have $f(2488) = 1.562386 \cdot 10^{-12} < 1.570 \cdot 10^{-12}$, so even as printed Lemma 2.3 covers only $\log x \leq 2487.367$, and corrected only $\log x \leq 2484.515$. Their Theorem 1.1 itself remains true, because [FKS23, Cor. 1.4], valid for $x > 2$, implies it, and $x = 2$ is trivial.

Lichtman–Pomerance [LP18]. Their Proposition 2.1 states $|\theta(x) - x| < 2.9 \cdot 10^{-10} x$ for $x > e^{55}$. It follows from Bütthe's Table 1 value $2.885 \cdot 10^{-10}$ (printed as $2.88434 \cdot 10^{-10}$) and $\psi(x) - \theta(x) < 1.02\sqrt{x} + 3x^{1/3}$, with 0.14% to spare. With $\log(3c)$ restored, the least value of Bütthe's bound over a broad parameter scan is $2.90948 \cdot 10^{-10}$, which gives $2.92111 \cdot 10^{-10}$, 0.73% above the stated bound. This uses a model of the zero sum, which reproduces Bütthe's printed value $2.88434 \cdot 10^{-10}$ as $2.88432 \cdot 10^{-10}$. The statement remains true by later bounds [FKS23b, Table 2].

PrimeNumberTheoremAnd. In the Lean project [PNT], the file `IEANTN/BKLNW_app.lean` states [Büt16, Thm. 1] as the theorem `bklnw_thm_16` with $E_2 = 0.16 (1 + x_0^{-1})(\sinh c)^{-1} e^{0.71\sqrt{c\varepsilon}} \log(c/\varepsilon)$, without the factor $\log(3c)$; its proof is `sorry`.

Findings of the model-based search.

- Fiori, Kadiri and Swidinsky [FKS23b] recompute Bütthe's Theorem 1 with the printed E_2 . Six of the eleven range endpoints in [FKS23b, Table 7] shrink, for example from 1358.6 to 1357.4–1357.5 for the bound $x^{-1/50}$, and from 270.8 to 270.66–270.68 for $x^{-1/10}$.
- In [BKLNW21, Table 15 and Cor. 11.1], M_1 and M_2 rise by about 2%. The values $M_3 = 0.024334$ and $M_4 = 57.184$ are exceeded on the grid of step 25 used there (0.02462 and 57.24), but look recoverable on a finer grid.

- In Johnston, Leong and Tudzi [JLT25], the constant $B_4 = 75.109$ becomes 76.65.

No published theorem we examined becomes false, but several proofs need a patch or a newer reference. Of the papers we checked, 40 were unaffected, and 7 could not be checked. Results that rely on the affected rows of [BKLNW21, Table 8], or on [Büt16, Thm. 1] as printed, should be rechecked with the corrected constant.

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DATA AVAILABILITY

The Python scripts used in the proofs and their outputs are available from the author. They require Python 3 and mpmath; `fks_eps.py` and the scripts that use it also require NumPy and SciPy. The outputs for Theorems C, C*, B' and D and their variants are regenerated by `regen_final.sh` with the final scripts; it calls `fks_fine_R.py` with arguments R , the range of $\log x_0$, a tag and a density mode (KLN: first block of [FKS23, Table 7], $a_1 = 0.77$; KLN618: second block, $a_1 = 0.618$; BEL618: pointwise minimum of the second block and [Bel24]), and `hybrid_D.py` and `zfr_B.py` with the corresponding arguments.

- `final_C.PUB.txt`: Theorem C ($R = 5.558691$, KLN618) and the first case of Remark 4.7;
- `final_C.PUBBEL.txt`: Remark 4.3(d) and the corresponding case of Remark 4.7;
- `final_C.BTYBEL.txt`: Theorem C* ($R = 4.896$, BEL618) and the corresponding case of Remark 4.7;
- `final_C.MTY77.txt`, `final_C.FKSR.txt`: Remark 4.3(a) and (b);
- `final_C.BTYKLN.txt`: Remark 4.4;
- `final_D.PUB.txt`: Theorem D (`hybrid_D.py`, $R = 5.558691$, KLN618);
- `final_D.BTYBEL.txt`: Remark 4.6 ($R = 4.896$, BEL618);
- `final_Bprime.txt`: Theorem B' (`zfr_B.py`);
- `buthe_reopt_fine_output.txt`: the re-optimized values of Table 8 (`buthe_reopt_fine.py`, quadrature);
- `buthe_reopt_output.txt`: the re-optimization with rigorous enclosures and the range at height 2,445,999,556,030 in Section 7.4 (`buthe_reopt.py`);
- `qrh_ramanujan_output.txt`: the case $\theta = 7/8$ in Section 4.6 (`qrh_ramanujan.py`);
- `qrh_theta_output.txt`: Theorem B (`qrh_theta.py`);
- `buthe_thesis_output.txt`: Proposition 4.1 and the bisection of Remark 4.8 (`buthe_thesis_run.py`, with $D_{TOP}=3035.502$);
- `confirm_D_thesis_output.txt`: the verification at height $1.3277 \cdot 10^{15}$ in Remark 4.8 (`confirm_theorem_D_thesis.py`);
- `robust_Aref_K*.txt`: the robustness of Proposition 4.1 in Section 4.6 (`robust_Aref.py`);
- `tradeoff_R*.txt`: Table 7 (`tradeoff.py`);
- `sens_K*.txt`: Remark 6.1 (`sens_density.py`);
- the folder `downstream`: the report `DOWNSTREAM_REPORT.md` and the scripts and outputs for Section 7.5.

The remaining scripts and their outputs are:

- `ch_ramanujan.py`: Theorems A and B*, and the printed-rows version of Theorem D in Remark 4.7;
- `fks_eps.py`: the re-implementation of [FKS23, Thm. 1.1] and Table 5; `fks_eps_mpcheck.py`: its 40-digit check;
- `barrier.py`: Corollary E and Table 6;
- `extend_ramanujan.py`: the tail $L \geq 10^4$ and Remark 4.3(c);
- `buthe_bound.py` and `buthe_ramanujan.py`: the evaluation of Lemma 2.4, in the corrected form and in the version with only $\log(3c)$ restored;
- `verify_ramanujan.py`: the main term, the interval routines shared by all scripts, and Remark 4.5;
- `crosscheck.py`: the independent check of Section 4.10;
- `crosscheck_buthe.py`: the quadrature values in Section 7.4 and Table 8.

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