

$\chi_2(1/3)$ AND $\chi_2(1/5)$ ARE IRRATIONAL

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ABSTRACT. Let $\chi_2(z) = \sum_{n \geq 0} z^{2n+1}/(2n+1)^2$ be Legendre's chi function. We prove that $\chi_2(1/3)$ and $\chi_2(1/5)$ are irrational. Equivalently, $\text{Li}_2(1/q) - \text{Li}_2(-1/q) \notin \mathbb{Q}$ for $q = 3, 5$. The proof uses the Hankel determinants $\Delta_h(\chi_2(1/q))$ of the moment functional $g \mapsto \sum_{i \geq 1} q^{-2i} (g(u) u / D_h(u))' |_{u=t}$, where $D_h(u) = \prod_{k < h} (u + k + \frac{1}{2})$. These determinants are values of polynomials $\Delta_h(X) \in \mathbb{Q}[X]$ of degree at most h . We bound the denominators of $\Delta_h(X)$ prime by prime: a bound adapted to residue classes for the large primes, a Lagrange-basis bound for the small ones, and separate arguments for the primes 2, 3 and q . We bound the size of $\Delta_h(\chi_2(1/q))$ by a discrete log-gas argument against an explicit dual certificate, and prove that $\Delta_h(a/b) \neq 0$ for every rational a/b and infinitely many h , through a reduction modulo an auxiliary prime $\ell = h/3$. The margins are below -0.054 ($q = 3$) and -0.256 ($q = 5$) per h^2 . The whole proof, including the prime number theorem and both numerical certificates, has been formalised in Lean 4 with Mathlib and checked by the Lean kernel, using only Lean's three standard axioms.

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1. INTRODUCTION

For $|z| \leq 1$ let

$$\chi_2(z) = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)^2} = \frac{1}{2}(\text{Li}_2(z) - \text{Li}_2(-z))$$

be Legendre's chi function [12, §1.8].

Theorem 1.1. *The numbers $\chi_2(1/3) = 0.33762317823243597996\dots$ and $\chi_2(1/5) = 0.20090195660862019582\dots$ are irrational.*

1.1. **Context.** We know of no earlier proof that $\chi_2(1/q)$ is irrational for any small integer $q \geq 2$.

- *Large q .* We have $\chi_2(1/q) = (4q)^{-1}\Phi(q^{-2}, 2, \frac{1}{2}) = \frac{q}{4}L(\frac{1}{2}, 2; q^{-2})$, where Φ is Lerch's transcendent and $L(\alpha, k; x) = \sum_{n \geq 1} x^n / (n - \alpha)^k$. Hata [7] proved linear independence measures for values $L(\alpha_j, k_j; 1/Q)$ with rational shifts, for $|Q| > Q_0$ with explicit Q_0 . The application to $\chi_2(1/q)$ has $Q = q^2$, so $Q = 9$ and $Q = 25$ here. Hata's headline application, $\text{Li}_2(1/Q) \notin \mathbb{Q}$ for $Q \geq 12$ or $Q \leq -8$, is for the shift 0, which is easier than the shift $\frac{1}{2}$, so $q = 3$ ($Q = 9$) is presumably outside his range. We have not determined his Q_0 for the shift $\frac{1}{2}$, which decides whether his method covers $q = 5$ ($Q = 25$). Heuristically the half shift costs a factor of about e^2 in Q , because the denominators become lcm of the odd numbers up to $2n$, which suggests that it is not covered [heuristic]. The explicit criterion of David, Hirata-Kohno and Kawashima [3, Thm. 2.1] for the shift $\frac{1}{2}$, depths ≤ 2 , at the single point $1/q^2$ requires $\log q^2 > 2 \log 2 + 6 \log 3 + 8$, i.e. $q \geq 2949$. Their multi-point criterion [2, Thm. 2.1] does not apply to $\chi_2(z) = \text{Li}_2(z) - \frac{1}{4} \text{Li}_2(z^2)$ at the two points $1/q, 1/q^2$ for any q , because it requires all points to be of comparable size.
- *Single dilogarithm values do not suffice.* Hata [8] proved $\text{Li}_2(-1/5) \notin \mathbb{Q}$. The irrationality of $\text{Li}_2(1/5)$ has been claimed recently by Prellberg [15] and in S. Liu's Lean repository [13]; neither is refereed. In any case two irrational numbers can have a rational difference, so neither result says anything about $\chi_2(1/5) = \frac{1}{2}(\text{Li}_2(\frac{1}{5}) - \text{Li}_2(-\frac{1}{5}))$. The linear independence results of Rhin-Viola [16] (one point $1/z$) and Viola-Zudilin [21] (the two points $1/z$ and $1/(1-z)$) do not cover the pair $\pm \frac{1}{5}$ either.
- *Identities do not suffice.* Landen's identity [12, App. A.2.1, (24)] $\chi_2(\frac{1-x}{1+x}) + \chi_2(x) = \frac{\pi^2}{8} + \frac{1}{2} \log x \log \frac{1+x}{1-x}$ gives, for example, $\chi_2(\frac{1}{3}) + \chi_2(\frac{1}{2}) = \frac{\pi^2}{8} - \frac{1}{2} \log 2 \log 3$. Both $\chi_2(\frac{1}{2})$ and $\log 2 \log 3$ are open.

Our search for prior work covered the web, arXiv and recent Lean repositories, but not MathSciNet or zbMATH.

1.2. **The method.** Fix an integer $q \geq 2$ and put $w := q^{-2}$. For $h \geq 1$ let $x_k := -(k + \frac{1}{2})$ for $0 \leq k < h$, $D(u) := \prod_{k < h} (u - x_k)$ and

$$(1.1) \quad \Lambda(g) := \sum_{t \geq 1} w^t \left(\frac{g(u)u}{D(u)} \right)' \Big|_{u=t} \quad (g \in \mathbb{Q}[u]).$$

Partial fractions show that $\Lambda(g) = A(g) + \chi_2(1/q)B(g)$ with explicit rational functionals A and B (Lemma 2.1). Hence the Hankel determinant $\det(\Lambda(u^{i+j}))_{i,j < h}$ is the value at $X = \chi_2(1/q)$ of

$$\Delta_h(X) := \det(A(u^{i+j}) + X B(u^{i+j}))_{0 \leq i,j < h} \in \mathbb{Q}[X], \quad \deg \Delta_h \leq h.$$

This is a Padé-type construction of the kind that Q. Tang used for $\text{Li}_2(1/2)$ [18]; the half-integer poles are what make χ_2 appear. The proof has three inputs.

- (1) *Denominators* (Sections 4–5). There are rationals $d_h > 0$ with $d_h \Delta_h \in \mathbb{Z}[X]$ and

$$\limsup_{h \rightarrow \infty} \frac{\log d_h}{h^2} \leq \alpha_q := \frac{133}{60} + \frac{\log 2}{4} + [q = 5] \frac{\log 3}{4} - \frac{\log q}{3}.$$

The term $133/60 = \frac{4}{5} + \frac{17}{12}$ comes from the primes $p \nmid 2q(q^2 - 1)$. Primes $p < 2h/5$ are handled by a Lagrange-basis bound and primes $p \geq 2h/5$ by a bound adapted to the residue classes modulo p ; the prime number theorem then sums the two contributions. The prime q gives a *gain*: a reflection of the functional shows that every coefficient of Δ_h has $v_q \geq \lfloor (h^2 - 7h)/3 \rfloor - O(h \log h)$. The primes 2 and 3 $\mid q^2 - 1$ occur in the denominators with exponents at most $\frac{1}{4}h^2 + O(h \log h)$.

- (2) *Size* (Sections 6–7). For every *dual certificate* (a step density μ with cell bounds on its potential) $\limsup h^{-2} \log |\Delta_h(\chi_2(1/q))| \leq \Phi$, an explicit exponent computed from the certificate. The proof treats $\Delta_h(\chi_2(1/q))$ as the partition function of a discrete log-gas with at most two particles per integer node, and compares it with μ through a lattice energy inequality of Pólya type. Explicit certificates, checked by exact rational interval arithmetic, give $\Phi < -2.078$ for $q = 3$ and $\Phi < -2.385$ for $q = 5$.
- (3) *Nonvanishing* (Section 8). For every rational a/b , $\Delta_h(a/b) \neq 0$ for infinitely many h . We take $h = 3\ell$ with ℓ a large prime. In a Lagrange basis and after an ℓ -adic rescaling, the Gram matrix of $\Lambda_{a/b} = A + \frac{a}{b}B$ is ℓ -integral and block upper triangular modulo ℓ , with explicit 3×3 diagonal blocks of three types. Their determinants are three fixed nonzero rationals for each q , so they are units for all large ℓ .

Since $\alpha_3 = 2.0237 \dots$ and $\alpha_5 = 2.1281 \dots$, the margins are

$$\alpha_3 - 2.078 < -0.054, \quad \alpha_5 - 2.385 < -0.256,$$

and the standard argument (Proposition 3.1) concludes. Table 1 lists the contributions.

1.3. The formalisation. The statement of Theorem 1.1 and its complete proof are formalised in Lean 4 [4] with Mathlib [19] and checked by the Lean kernel. This includes the prime number theorem, the two dual certificates

contribution (per h^2)	$q = 3$	$q = 5$	source
primes $p \nmid 2q(q^2 - 1)$: $\frac{133}{60}$	2.21667	2.21667	Thm. 4.5
$p = 2$: $\frac{1}{4} \log 2$	0.17329	0.17329	Lemma 5.3
$p = 3 \mid q^2 - 1$: $\frac{1}{4} \log 3$	—	0.27465	Lemma 5.1
$p = q$ (gain): $-\frac{1}{3} \log q$	-0.36620	-0.53648	Lemma 5.2
arithmetic α_q	2.02375	2.12813	
analytic Φ	< -2.07832	< -2.38515	Thms. 6.2, 7.1
margin $\alpha_q + \Phi$	< -0.0545	< -0.2570	

TABLE 1. The exponential budget. The analytic values are those of the certificates of Section 7; the main theorem only uses $\Phi < -2.078$ and $\Phi < -2.385$.

and the block determinants. The proof depends only on Lean’s three standard axioms (`propext`, `Classical.choice`, `Quot.sound`). It contains no `sorry` and no `native_decide`: every proof, including every computation, is checked by the kernel. The repository is <https://github.com/telaviv/chi2-irrationality> [10]; Section 10 describes it.

The prose proof below is self-contained and can be read without the formalisation, except that two kinds of finite computation are only described, not reproduced: the two dual certificates (Theorem 7.1), which are decidable statements about rational data proved by `decide +kernel`, and the six block determinants of Section 8, which are evaluated exactly by `norm_num`.

1.4. Organisation and notation. Section 2 sets up the functional and proves the pole expansion. Section 3 states the criterion and reduces Theorem 1.1 to the three inputs. Sections 4 and 5 bound the denominators, Sections 6 and 7 bound the size, and Section 8 proves nonvanishing. Section 10 describes the formalisation and Section 11 contains remarks, including why $\chi_2(1/2)$ is out of reach of this construction.

Throughout, p and ℓ denote primes; v_p is the p -adic valuation on $\mathbb{Q}$ and $|x|_p = p^{-v_p(x)}$. $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ are the floor and ceiling, $(x)_+ = \max(x, 0)$, and $[P]$ is 1 if P holds and 0 otherwise. For a sequence f on $\mathbb{Z}_{\geq 0}$, $\Delta f(t) = f(t+1) - f(t)$ is the forward difference. We write $\theta(x) = \sum_{p \leq x} \log p$.

2. THE CONSTRUCTION

2.1. Data. Fix an integer $q \geq 2$; later $q \in \{3, 5\}$. Put

$$z := 1/q, \quad w := z^2 = q^{-2}, \quad \chi := \chi_2(z).$$

For $h \geq 1$ we use h poles $x_k := -(k + \frac{1}{2})$, $0 \leq k < h$, and put

$$D(u) := \prod_{k < h} (u - x_k), \quad \omega_k := \frac{x_k}{D'(x_k)},$$

where $D'(x_k) = \prod_{m \neq k} (x_k - x_m) = \prod_{m \neq k} (m - k) = (-1)^k k! (h - 1 - k)!$. We also need the partial sums

$$S_k(q) := \sum_{j=0}^k \frac{q^{2(k-j)}}{(2j+1)^2} = q^{2k+1} T(2k+1), \quad T(M) := \sum_{\substack{s \leq M \\ s \text{ odd}}} \frac{z^s}{s^2},$$

with $T(M) = 0$ for $M < 1$. For a rational function F regular at the positive integers and of polynomial growth put

$$\mu(F) := \sum_{t \geq 1} w^t F'(t),$$

an absolutely convergent series since $0 < w < 1$. The functional (1.1) is $\Lambda(g) = \mu(g\varphi)$ with $\varphi(u) := u/D(u)$.

2.2. The polynomial part. For a polynomial R , $\mu(R)$ is rational. Indeed $\mu(u^d) = d \sum_{t \geq 1} t^{d-1} w^t$, and the power sums $\sigma_n := \sum_{t \geq 1} t^n w^t$ satisfy $\sigma_0 = w/(1-w)$ and

$$(2.1) \quad (1-w) \sigma_{n+1} = \sum_{j=0}^n \binom{n+1}{j} (-1)^{n-j} \sigma_j,$$

by expanding $\sum_t t^{n+1} w^t - \sum_t (t-1)^{n+1} w^t = (1-w) \sigma_{n+1}$. We shall also use the Newton form

$$(2.2) \quad \mu(R) = \sum_{b \geq 0} \Delta^b[R'](0) \Omega_b, \quad \Omega_b := \sum_{t \geq 1} \binom{t}{b} w^t = \begin{cases} \frac{1}{q^2 - 1}, & b = 0, \\ \frac{q^2}{(q^2 - 1)^{b+1}}, & b \geq 1, \end{cases}$$

which follows from $R'(t) = \sum_b \Delta^b[R'](0) \binom{t}{b}$. Here

$$\Delta^b[R'](0) = \sum_{k \leq b} (-1)^{b-k} \binom{b}{k} R'(k),$$

and the sum over b is finite ($b < \deg R$).

2.3. The pole expansion.

Lemma 2.1 (Lemma 1, pole expansion). *For $g \in \mathbb{Q}[u]$ write $g(u)u = D(u)R_g(u) + r_g(u)$ with $\deg r_g < h$. Then*

$$\Lambda(g) = A(g) + \chi B(g),$$

where

$$A(g) := \mu(R_g) + 4 \sum_{k < h} g(x_k) \omega_k S_k(q), \quad B(g) := -4 \sum_{k < h} g(x_k) \omega_k q^{2k+1}.$$

Both are rational and $\mathbb{Q}$ -linear in g .

Proof. By Lagrange interpolation at the h nodes x_k ,

$$r_g(u) = \sum_k \frac{r_g(x_k) D(u)}{D'(x_k)(u - x_k)}, \quad r_g(x_k) = g(x_k) x_k.$$

Hence $g\varphi = R_g + \sum_k g(x_k)\omega_k/(u - x_k)$. For the pole terms,

$$\mu\left(\frac{1}{u - x_k}\right) = -\sum_{t \geq 1} \frac{w^t}{(t + k + \frac{1}{2})^2} = -4 \sum_{t \geq 1} \frac{z^{2t}}{(2t + 2k + 1)^2}.$$

With $s = 2t + 2k + 1$ (odd, $s \geq 2k + 3$) and $z^{2t} = q^{2k+1}z^s$ this equals $-4q^{2k+1}(\chi - T(2k+1)) = -4(q^{2k+1}\chi - S_k(q))$. All series converge absolutely, so we may sum termwise. $\square$

For an indeterminate X put $\Lambda_X := A + X B$ and

$$H(X) := (\Lambda_X(u^{i+j}))_{0 \leq i, j < h}, \quad \Delta_h(X) := \det H(X) \in \mathbb{Q}[X].$$

The entries are linear in X , so $\deg \Delta_h \leq h$, and by Lemma 2.1

$$(2.3) \quad \Delta_h(\chi) = \det(\Lambda(u^{i+j}))_{0 \leq i, j < h}.$$

2.4. Change of basis. If $a_0, \dots, a_{h-1}$ and $b_0, \dots, b_{h-1}$ are polynomials of degree $< h$ with coefficient matrices C_a and C_b (rows indexed by i , columns by the monomials $u^0, \dots, u^{h-1}$), then by linearity of Λ_X

$$(2.4) \quad (\Lambda_X(a_i b_j))_{i,j} = C_a H(X) C_b^T, \quad \det(\Lambda_X(a_i b_j)) = \det C_a \det C_b \Delta_h(X).$$

We use three bases. The *Lagrange basis* $f_i := D/(u - x_i)$ has $f_i(x_k) = [i = k]D'(x_i)$. Write $V = (x_k^d)_{d,k < h}$ for the Vandermonde matrix. Then $C_f V = \text{diag}(D'(x_i))$, and $\prod_i D'(x_i) = \pm(\det V)^2$ with $\det V = \prod_{k < k'} (x_{k'} - x_k) = \pm \prod_{k < k'} (k' - k)$. The *anchored basis*

$$a_i := \prod_{m < i} (u - x_m), \quad b_j := \prod_{h-j \leq m < h} (u - x_m)$$

consists of monic polynomials of degrees i and j , so C_a and C_b are unitriangular and $\det(\Lambda_X(a_i b_j)) = \Delta_h(X)$. The third basis, adapted to residue classes, is introduced in Section 4.3.

3. THE CRITERION AND THE STRUCTURE OF THE PROOF

Proposition 3.1 (Lemma E). *Let $\xi \in \mathbb{R}$, and for $h \geq 1$ let $P_h \in \mathbb{Q}[X]$ with $\deg P_h \leq h$ and let $d_h > 0$ be rationals with $d_h P_h \in \mathbb{Z}[X]$. Suppose that there is $c > 0$ with $d_h |P_h(\xi)| \leq e^{-ch^2}$ for all large h , and that for every $a \in \mathbb{Z}$ and $b \geq 1$ we have $P_h(a/b) \neq 0$ for infinitely many h . Then ξ is irrational.*

Proof. Suppose $\xi = a/b$. For every h , $N_h := b^h d_h P_h(a/b)$ is an integer, because $d_h P_h$ has integer coefficients and degree $\leq h$. Choose h with $P_h(a/b) \neq 0$, $d_h |P_h(\xi)| \leq e^{-ch^2}$ and $b^h e^{-ch^2} < 1$. Then $1 \leq |N_h| < 1$, a contradiction. $\square$

Theorem 3.2. *Let $q \geq 2$. Suppose there are rationals $d_h > 0$ and real numbers α, β with $\alpha + \beta < 0$ such that*

- (1) $d_h \Delta_h \in \mathbb{Z}[X]$ for every h , and for every $\varepsilon > 0$, $\log d_h \leq (\alpha + \varepsilon)h^2$ for all large h ;
- (2) for every $\varepsilon > 0$, $|\Delta_h(\chi_2(1/q))| \leq e^{(\beta + \varepsilon)h^2}$ for all large h ;
- (3) for every $a \in \mathbb{Z}$ and $b \geq 1$, $\Delta_h(a/b) \neq 0$ for infinitely many h .

Then $\chi_2(1/q)$ is irrational.

Proof. Apply Proposition 3.1 with $P_h = \Delta_h$ and $c = -(\alpha + \beta)/2$, using (i) and (ii) with $\varepsilon = c/2$. $\square$

For $q = 3$ and $q = 5$ the three hypotheses are Theorem 5.5 with $\alpha = \alpha_q$, Theorems 6.2 and 7.1 with $\beta = -2.078$ and $\beta = -2.385$, and Theorem 8.2. The final assembly is in Section 9.

4. DENOMINATORS I: THE SYMBOLIC MACHINE AND THE GENERIC PRIMES

4.1. The free ring. It is convenient to treat q and $1/(q^2 - 1)$ as independent indeterminates. Let $\mathcal{R} := \mathbb{Q}[q, E]$ be the polynomial ring in two variables. In $\mathcal{R}$ put $\Omega_0 := E$ and $\Omega_b := q^2 E^{b+1}$ for $b \geq 1$, and for $R \in \mathbb{Q}[u]$ define $\mu^{\mathcal{R}}(R) := \sum_b \Delta^b[R'](0) \Omega_b \in \mathcal{R}$ as in (2.2). Replacing μ by $\mu^{\mathcal{R}}$, and reading $S_k(q)$ and q^{2k+1} as elements of $\mathbb{Q}[q]$, Lemma 2.1 defines functionals $A^{\mathcal{R}}, B^{\mathcal{R}}$ and

$$\Delta_h^{\mathcal{R}}(X) := \det(A^{\mathcal{R}}(u^{i+j}) + X B^{\mathcal{R}}(u^{i+j}))_{i,j < h} \in \mathcal{R}[X].$$

For an integer $q_0 \geq 2$, the ring homomorphism $\text{ev}_{q_0}: \mathcal{R} \rightarrow \mathbb{Q}$ with $q \mapsto q_0$ and $E \mapsto 1/(q_0^2 - 1)$ maps Ω_b to the value (2.2), hence $\mu^{\mathcal{R}}$ to μ and $\Delta_h^{\mathcal{R}}$ to Δ_h (computed with $q = q_0$).

Gauss norms. For a prime p and $f \in \mathcal{R}$ let $\|f\|_p$ be the maximum of $|c|_p$ over the rational coefficients c of f , and extend this coefficientwise to $\mathcal{R}[X]$. Then $\|f + g\|_p \leq \max(\|f\|_p, \|g\|_p)$ and $\|fg\|_p \leq \|f\|_p \|g\|_p$. If $|v_0|_p \leq 1$ and $|v_1|_p \leq 1$, the evaluation $q \mapsto v_0$, $E \mapsto v_1$ does not increase the norm. By the Leibniz expansion:

Lemma 4.1. *Let M be an $h \times h$ matrix over $\mathcal{R}[X]$ with $\|M_{ij}\|_p \leq \rho_i \kappa_j$ for some $\rho_i, \kappa_j \geq 0$. Then $\|\det M\|_p \leq \prod_i \rho_i \prod_j \kappa_j$.*

Two facts carry all the dependence on q :

- (F1) $\|\Omega_b\|_p = 1$ for all b and p , so for $R \in \mathbb{Q}[u]$ we have $\|\mu^{\mathcal{R}}(R)\|_p \leq \sup_{t \in \mathbb{Z}_{\geq 0}} |R'(t)|_p$, because $\Delta^b[R'](0)$ is an integer combination of $R'(0), \dots, R'(b)$;
- (F2) $\|q^{2k+1}X - S_k(q)\|_p \leq p^{2\#\{e \geq 1: p^e \leq 2k+1\}}$, since the coefficients of S_k are $1/s^2$ with s odd, $s \leq 2k+1$, and $v_p(s) \leq \#\{e \geq 1: p^e \leq s\}$.

The Gram identity. Write $\Lambda_X^{\mathcal{R}} := A^{\mathcal{R}} + X B^{\mathcal{R}}$, a $\mathbb{Q}$ -linear map $\mathbb{Q}[u] \rightarrow \mathcal{R}[X]$. By the argument of (2.4), for any family $(e_m)_{m < h}$ of polynomials of degree $< h$ with coefficient matrix C_e ,

$$(4.1) \quad \det(\Lambda_X^{\mathcal{R}}(e_m e_{m'}))_{m, m' < h} = (\det C_e)^2 \Delta_h^{\mathcal{R}}(X).$$

For a pole x_l we have, by the proof of Lemma 2.1, $\Lambda_X^{\mathcal{R}}(g) = \mu^{\mathcal{R}}(R) - 4c(q^{2l+1}X - S_l(q))$ whenever $g\varphi = R + c/(u - x_l)$ with R a polynomial; more generally the pole parts add up.

Below, $\text{lev}(p) := \#\{e \geq 1 : p^e \leq 2h - 1\}$. For $t \in \mathbb{Z}_{\geq 0}$ and $m < h$ we have $t - x_m = (2t + 2m + 1)/2$, and for p odd $v_p(t - x_m) = v_p(2t + 2m + 1)$. The numbers $2t + 2m + 1$, $m < h$, are h consecutive odd integers, so for each level $d = p^e$ those divisible by d are the m in one residue class modulo d .

4.2. All odd primes: the Lagrange basis.

Lemma 4.2 (Lemma 9). *For every odd prime p and every $h \geq 1$, $\|\Delta_h^{\mathcal{R}}\|_p \leq p^{2h \text{lev}(p)}$.*

Proof. Use the Lagrange basis $f_i = D/(u - x_i)$ and put $G_{ij} := \Lambda_X^{\mathcal{R}}(f_i f_j)$. Since $C_f V = \text{diag}(D'(x_i))$ and $\prod_i D'(x_i) = \pm(\det V)^2$, we get $(\det C_f)^2 = \pm \prod_i D'(x_i)$, so by (4.1)

$$\|\Delta_h^{\mathcal{R}}\|_p = \|\det G\|_p / \prod_i |D'(x_i)|_p.$$

By Lemma 4.1 it suffices to prove the *row claim* $\|G_{ij}\|_p \leq |D'(x_i)|_p p^{2 \text{lev}(p)}$. For a level $e \geq 1$ put $N_e(i) := \#\{m < h : m \neq i, p^e \mid m - i\}$; then $v_p(D'(x_i)) = \sum_e N_e(i)$, and $N_e(i) = 0$ unless $p^e \leq h - 1$. So it suffices to show that every value $R'(t)$ entering (F1) has

$$(4.2) \quad v_p(R'(t)) \geq \sum_e (N_e(i) - 2[N_e(i) \geq 1]),$$

and that the pole part obeys the same bound.

Fix $t \in \mathbb{Z}_{\geq 0}$ and a level $d = p^e$, and let m_0 be the residue class mod d of the indices m with $d \mid 2t + 2m + 1$. The class of i has $N_e(i) + 1$ elements in $[0, h)$, and residue classes mod d in $[0, h)$ differ in size by at most one, so the class m_0 has at least $N_e(i)$ elements.

Off-diagonal entries ($i \neq j$). Here $f_i f_j \varphi = F_{ij} := u \prod_{m \neq i, j} (u - x_m)$ is a polynomial, so $G_{ij} = \mu^{\mathcal{R}}(F_{ij})$. Each term of $F'_{ij}(t)$ is a product of all factors but one. At level d the factors $t - x_m$ divisible by d are those with $m \equiv m_0$, $m \neq i, j$; omitting one factor leaves at least $N_e(i) - 2$ of them in both cases $m_0 \equiv i$ and $m_0 \not\equiv i$. Summing over levels (v_p of a product is $\sum_e$ of the number of factors divisible by p^e) gives (4.2).

Diagonal entries. Put $g_i := u \prod_{m \neq i} (u - x_m)$. Then $f_i^2 \varphi = g_i/(u - x_i) = Q_i + g_i(x_i)/(u - x_i)$ with $Q_i := (g_i(u) - g_i(x_i))/(u - x_i)$ and $g_i(x_i) = x_i D'(x_i)$. The pole part is $-4x_i D'(x_i)(q^{2i+1}X - S_i)$; by (F2), $|x_i|_p \leq 1$ and $2i + 1 \leq 2h - 1$, its norm is at most $|D'(x_i)|_p p^{2 \text{lev}(p)}$. For the polynomial part, list the h roots $\{x_m : m \neq i\} \cup \{0\}$ of g_i as $\alpha_1, \dots, \alpha_h$; telescoping $(u - \alpha_r) - (x_i - \alpha_r) = u - x_i$ gives

$$Q_i(u) = \sum_{r=1}^h \prod_{s < r} (x_i - \alpha_s) \prod_{s > r} (u - \alpha_s).$$

Order the roots by decreasing $v_p(m - i)$, with 0 last. Then for every level e simultaneously, the $\alpha_s = x_m$ with $p^e \mid m - i$ form an initial segment of length $N_e(i)$ (note $x_i - x_m = m - i$). Fix t , a level $d = p^e$ and a term r of $Q'_i(t)$ (a product with one more factor omitted). If $r > N_e(i)$, the first product already has $N_e(i)$ factors divisible by d . If $r \leq N_e(i)$, it has $r - 1$; if moreover $m_0 \equiv i$, the remaining $N_e(i) - r$ class members sit at positions $s > r$ and at most one of them is omitted, giving at least $N_e(i) - 2$ in total; if $m_0 \not\equiv i$, the whole class m_0 (at least $N_e(i)$ members, all with $v_p(m - i) < e$) sits at positions $s > N_e(i) \geq r$, giving at least $N_e(i) - 1$. So (4.2) holds for $Q'_i(t)$ as well. Finally $\#\{e : N_e(i) \geq 1\} \leq \text{lev}(p)$. $\square$

4.3. One-level primes: a basis adapted to residue classes. For a modulus d and $m < h$ write $r_m := \lfloor m/d \rfloor$ (the rank of m in its class), n_m for the size of the class of m modulo d in $[0, h)$, and put

$$\begin{aligned} w_d(l) &:= 2[d \leq 2l + 1] - [d \mid 2l + 1], \\ M_m &:= \max\{w_d(l) : l \leq m, l \equiv m \pmod{d}\}, \\ \text{cb}(h, d) &:= \sum_{m < h} (2r_m + 1 + M_m - n_m)_+. \end{aligned}$$

Proposition 4.3 (class bound). *Let p be an odd prime with $p^2 > 2h$. Then $\|\Delta_h^{\mathcal{R}}\|_p \leq p^{\text{cb}(h, p)}$.*

Proof. Since $p^2 > 2h$, every difference $m - l$ of distinct indices and every odd $s \leq 2h - 1$ has $v_p \leq 1$; so $v_p(m - l) = [p \mid m - l]$ and $v_p(2l + 1) = [p \mid 2l + 1]$. For $m < h$ let $\text{seg}(m) := \{l \leq m : l \equiv m \pmod{p}\}$ (it has $r_m + 1$ elements), and put

$$e_m := s_m \frac{D(u)}{\prod_{l \in \text{seg}(m)} (u - x_l)}, \quad s_m := \prod_{l \in \text{seg}(m), l < m} (x_m - x_l).$$

Then $\deg e_m = h - r_m - 1 < h$ and $v_p(s_m) = r_m$. The polynomial e_m vanishes at x_k unless $k \in \text{seg}(m)$, and $e_m(x_m) = D'(x_m)$. So the evaluation matrix $(e_m(x_k)) = C_e V$ is lower triangular with diagonal $D'(x_m)$, hence $\det C_e \det V = \prod_m D'(x_m) = \pm(\det V)^2$ and $\det C_e = \pm \det V$. By (4.1), $\|\Delta_h^{\mathcal{R}}\|_p = \|\det G^e\|_p / |\det V|_p^2$ with $G_{mm'}^e := \Lambda_X^{\mathcal{R}}(e_m e_{m'})$, and $2v_p(\det V) = 2\#\{k < k' : p \mid k' - k\} = 2\sum_m r_m$.

Let $\mathcal{A} := \text{seg}(m)$, $\mathcal{B} := \text{seg}(m')$, $r = r_m$, $r' = r_{m'}$. Then

$$e_m e_{m'} \varphi = s_m s_{m'} u \frac{\prod_{l \notin \mathcal{A} \cup \mathcal{B}} (u - x_l)}{\prod_{l \in \mathcal{A} \cap \mathcal{B}} (u - x_l)}.$$

If m and m' lie in different classes, $\mathcal{A} \cap \mathcal{B} = \emptyset$, the right side is $s_m s_{m'}$ times a polynomial in $\mathbb{Z}[\frac{1}{2}][u]$, and (F1) gives $v_p(G_{mm'}^e) \geq r + r'$. If they lie in the same class, of size n , say with $r \leq r'$, then $\mathcal{A} \subseteq \mathcal{B}$. Division by the monic polynomial $\prod_{l \in \mathcal{A}} (u - x_l) \in \mathbb{Z}[\frac{1}{2}][u]$ leaves a p -integral polynomial part, again

of valuation $\geq r + r'$. The residue at x_l , $l \in \mathcal{A}$, is

$$c_l = s_m s_{m'} x_l \frac{\prod_{k \notin \mathcal{B}} (x_l - x_k)}{\prod_{k \in \mathcal{A}, k \neq l} (x_l - x_k)},$$

$$v_p(c_l) = r + r' + [p \mid 2l + 1] + (n - 1 - r') - r,$$

because exactly the $n - 1 - r'$ class members after m' lie outside $\mathcal{B}$, and the r other elements of $\mathcal{A}$ lie in the class. By (F2) and the single level, the pole factor $q^{2l+1}X - S_l$ has norm at most $p^{2[p \leq 2l+1]}$. Hence

$$v_p(G_{mm'}^e) \geq r + r' + \beta(r, r'), \quad \beta(a, b) := \min(0, n - 1 - a - b - \widehat{M}_{\min(a, b)}),$$

where $\widehat{M}_a$ is the largest weight $w_p(l)$ among the class members of rank $\leq a$. Since $\widehat{M}$ is nondecreasing and $x \mapsto \min(0, x)$ is concave, $\beta(a, b) \geq \frac{1}{2}(\beta(a, a) + \beta(b, b))$. So every entry, in the same class or not, satisfies $v_p(G_{mm'}^e) \geq \rho_m + \rho_{m'}$ with $\rho_m := r_m + \frac{1}{2}\beta(r_m, r_m) \leq r_m$. Lemma 4.1 gives $v_p(\det G^e) \geq 2 \sum_m \rho_m$, and therefore

$$v_p(\Delta_h^{\mathcal{R}}) \geq \sum_m \beta(r_m, r_m) = - \sum_m (2r_m + 1 + M_m - n_m)_+ = -\text{cb}(h, p). \quad \square$$

4.4. The class count. Let φ_1 be the continuous, piecewise linear, nonincreasing function on $[2/5, \infty)$ with

$$\varphi_1(x) = \begin{cases} 3 - \frac{5}{2}x, & \frac{2}{5} \leq x \leq \frac{1}{2}, \\ 2 - \frac{1}{2}x, & \frac{1}{2} \leq x \leq \frac{2}{3}, \\ 3 - 2x, & \frac{2}{3} \leq x \leq 1, \\ 2 - x, & 1 \leq x \leq 2, \end{cases}$$

and $\varphi_1(x) = 0$ for $x \geq 2$. Then $\int_{2/5}^2 \varphi_1 = \frac{3}{16} + \frac{41}{144} + \frac{4}{9} + \frac{1}{2} = \frac{17}{12}$.

Lemma 4.4 (Lemma 8A, class count). *Let $h \geq 2$ and let d be odd. If $2h \leq 5d$ and $d < 2h$ then $\text{cb}(h, d) \leq h \varphi_1(d/h)$; more precisely $\text{cb}(h, d) = h \varphi_1(d/h) - \frac{3}{2}$ if $d \leq (2h - 1)/3$ and $\text{cb}(h, d) = h \varphi_1(d/h)$ otherwise. If $d \geq 2h$ then $\text{cb}(h, d) = 0$. (The formal version assumes $h \geq 13$, which is needed only in Theorem 4.5.)*

Proof. Write $K = h$ and $\kappa = (d - 1)/2$. If $d \geq 2K$, every atom $l < K$ has $2l + 1 < d$, so all weights vanish, every class is a singleton and every term is $(1 + 0 - 1)_+ = 0$. Now let $2K \leq 5d$ and $d < 2K$; as d is odd, $5d \geq 2K + 1$ and $3d \neq 2K$. The odd multiples of d below $2K$ are d and possibly $3d$. Hence an atom l has weight 0 if $l < \kappa$ (we call it *small*), weight 1 if $l = \kappa$ or $l = \kappa + d$, and weight 2 otherwise. Every atom after the first in its class is $\geq d > \kappa$, so only the first atom c of a class can be small. We evaluate cb class by class; a class $\{c, c + d, \dots\}$ of size n contributes $\sum_r (2r + 1 + M - n)_+$ over its atoms.

(IV) $K < d < 2K$. All classes are singletons, contributing their weights: $2(K - \kappa) - 1 = 2K - d$.

- (III) $(2K+1)/3 \leq d \leq K$. Classes $c < K-d$ are pairs, with $c < \kappa$ small; they contribute $0 + (1+2) = 3$. The other classes $c \in [K-d, d-1]$ are singletons; those with $c \geq \kappa$ ($(d+1)/2$ of them, one of weight 1) contribute d in total. The atom $\kappa + d$ is $\geq K$. Total $3(K-d) + d = 3K - 2d$.
- (II) $K/2 \leq d \leq (2K-1)/3$. Classes $c < K-d$ are pairs and $c \in [K-d, d-1]$ singletons, all of weight 2 (as $\kappa < K-d$). A pair contributes $(w(c)-1)_+ + 1 + M$: this is 3 for $c < \kappa$, 2 for $c = \kappa$ (weights 1, 1), and 4 for $\kappa < c < K-d$. Total $3\kappa + 2 + 4(K-d-\kappa-1) + 2(2d-K) = 2K - \frac{d}{2} - \frac{3}{2}$.
- (I) $(2K+1)/5 \leq d \leq K/2$. Classes $c < K-2d$ are triples, with $c < \kappa$ (as $K-2d \leq \kappa$), contributing $0+2+4 = 6$. Classes $c \in [K-2d, d-1]$ are pairs, contributing 3 for $c < \kappa$ ($\frac{5d-1}{2} - K$ of them), 2 for $c = \kappa$, and 4 for $c > \kappa$ ($\frac{d-1}{2}$ of them). Total $3K - \frac{5}{2}d - \frac{3}{2}$.

In the four regions $K\varphi_1(d/K)$ equals $2K-d$, $3K-2d$, $2K-\frac{d}{2}$ and $3K-\frac{5}{2}d$. $\square$

4.5. The generic primes. For odd p put

$$e(h, p) := \begin{cases} \text{cb}(h, p), & p^2 > 2h \text{ and } 5p \geq 2h, \\ 2h \text{ lev}(p), & \text{otherwise.} \end{cases}$$

Theorem 4.5 (generic primes). *For every odd prime p and every h , $\|\Delta_h^{\mathcal{R}}\|_p \leq p^{e(h,p)}$. For every $\varepsilon > 0$ and all large h ,*

$$\sum_{p \text{ odd}} e(h, p) \log p \leq \left(\frac{133}{60} + \varepsilon \right) h^2.$$

Proof. The first statement is Proposition 4.3 or Lemma 4.2. For the sum let $h \geq 13$. If $5p \geq 2h$ then $p^2 \geq 4h^2/25 > 2h$, so $e(h, p) = \text{cb}(h, p)$, which vanishes for $p \geq 2h$ and is at most $h\varphi_1(p/h)$ for $2h/5 \leq p < 2h$ (Lemma 4.4). So the sum is at most

$$\sum_{p < 2h/5} 2h \text{ lev}(p) \log p + \sum_{2h/5 \leq p < 2h} h \varphi_1(p/h) \log p.$$

For the first sum, $\text{lev}(p) \leq 1$ when $p^2 > 2h-1$ and $\text{lev}(p) \log p \leq \log(2h)$ always, so it is at most $2h(\theta(2h/5) + (\sqrt{2h}+1)\log(2h)) = (\frac{4}{5} + o(1))h^2$ by the prime number theorem $\theta(x) \sim x$. For the second, cut $[2/5, 2]$ into M equal cells. On each cell φ_1 is at most its value at the left end, and $\sum_{p \in hI} \log p = h|I| + o(h)$ for each fixed cell I . Hence the second sum is at most $h^2(\int_{2/5}^2 \varphi_1 + \frac{16}{5M}) + o(h^2)$, and M is arbitrary. $\square$

5. DENOMINATORS II: THE PRIMES q , 2 AND THE DIVISORS OF $q^2 - 1$

5.1. The pole order at $q^2 = 1$. For $f \in \mathcal{R}$ let $\deg_E f$ be its degree in E . Put

$$\rho_i := (i + 1 - \lceil h/2 \rceil)_+, \quad \kappa_j := (j - \lfloor h/2 \rfloor)_+,$$

$$\beta_h := \sum_{i < h} \rho_i + \sum_{j < h} \kappa_j = \lfloor h^2/4 \rfloor \leq \frac{h^2 + 2h}{4}.$$

Lemma 5.1 (Lemma 5). *Every coefficient of $\Delta_h^{\mathcal{R}} \in \mathcal{R}[X]$ has E -degree at most β_h .*

Proof. In the monomial basis, the entry (i, j) is $\mu^{\mathcal{R}}(R)$ plus pole terms that do not involve E , where R is the quotient of u^{i+j+1} by D , of degree $i + j + 1 - h$ (and $R = 0$ if $i + j + 1 < h$). In $\mu^{\mathcal{R}}(R) = \sum_{b < \deg R} \Delta^b[R'](0)\Omega_b$ each Ω_b has E -degree $b + 1 \leq \deg R$. So the entry has E -degree at most $(i + j + 1 - h)_+ \leq \rho_i + \kappa_j$, and the Leibniz expansion gives the claim. The value of β_h is an elementary sum. $\square$

Clearing E . For $f = \sum c_{a,b} q^a E^b \in \mathcal{R}$ with $\deg_E f \leq \beta$ put $\text{cl}_\beta(f) := \sum c_{a,b} q^a (q^2 - 1)^{\beta-b} \in \mathbb{Q}[q]$. Since $(q^2 - 1)^k$ has integer coefficients, $\|\text{cl}_\beta f\|_p \leq \|f\|_p$ for every p . If $v_1(v_0^2 - 1) = 1$, then evaluating at $q = v_0$, $E = v_1$ gives

$$(5.1) \quad \text{ev}(\text{cl}_\beta f) = (v_0^2 - 1)^\beta \text{ev}(f).$$

5.2. The q -adic gain. Let $\psi: \mathcal{R} \rightarrow \mathbb{Q}[[q]]$ be the ring homomorphism with $q \mapsto q$ and $E \mapsto 1/(q^2 - 1) = -\sum_{n \geq 0} q^{2n}$. Put

$$a_h := \max\left(0, \left\lfloor \frac{h(h-1) - 6h}{3} \right\rfloor\right) \geq \frac{h^2 - 7h - 2}{3}.$$

Lemma 5.2 (Lemma 4'). *Every coefficient of $\psi(\Delta_h^{\mathcal{R}}) \in \mathbb{Q}[[q]][X]$ is divisible by q^{a_h} . Consequently, for every k , all monomials of $\text{cl}_{\beta_h}([X^k]\Delta_h^{\mathcal{R}}) \in \mathbb{Q}[q]$ have degree at least a_h .*

Proof. Reflection. From $\psi(\Omega_b) = (-1)^{b+1} q^2 (1 - q^2)^{-b-1}$ for $b \geq 1$ and $\psi(\Omega_0) = -(1 - q^2)^{-1}$, the coefficient of q^{2m} in $\psi(\Omega_b)$ is $-(-1)^b \binom{m+b-1}{b}$, and odd powers do not occur. Newton's backward formula $f(-m) = \sum_b (-1)^b \binom{m+b-1}{b} \Delta^b f(0)$ for polynomials f then gives

$$(5.2) \quad \psi(\mu^{\mathcal{R}}(R)) = - \sum_{m \geq 0} q^{2m} R'(-m) \quad (R \in \mathbb{Q}[u]).$$

Recombination. For a polynomial g write $g\varphi = R + \sum_k c_k/(u - x_k)$, $c_k = g(x_k)\omega_k$. Since $((u - x_k)^{-1})'(-m) = -4/(2k + 1 - 2m)^2$ and

$$\sum_{m \geq 0} \frac{q^{2m}}{(2k + 1 - 2m)^2} = S_k(q) + q^{2k+1} \chi_2(q),$$

where $\chi_2(q) = \sum_{j \geq 0} q^{2j+1}/(2j+1)^2 \in \mathbb{Q}[[q]]$ (the terms $m \leq k$ give S_k , the terms $m > k$ give $q^{2k+1}\chi_2(q)$), (5.2) and Lemma 2.1 combine to

$$(5.3) \quad \psi(\Lambda_X^{\mathcal{R}}(g)) = - \sum_{m \geq 0} q^{2m} (g\varphi)'(-m) - 4(X + \chi_2(q)) \sum_{k < h} g(x_k) \omega_k q^{2k+1}.$$

Slots. For $g = u^{i+j}$ we have $(g\varphi)'(-m) = \varphi'(-m)t^i t^j + \varphi(-m)(it^{i-1})t^j + \varphi(-m)t^i(jt^{j-1})$ at $t = -m$, and $g(x_k) = x_k^i x_k^j$. So the matrix $\psi(H)$ is a sum of rank-one matrices $cxy^\top$, where the row vector x is either a *value row* $(t^i)_i$ or a *derivative row* $(it^{i-1})_i$ at a node $t = -m$, or a *pole row* $(x_k^i)_i$. A node $-m$ carries the factor q^{2m} and a pole k carries q^{2k+1} . Give these row types the *keys* (m, val) , (m, der) and k , with *prices* $2m$, $2m$ and $2k+1$.

Truncation and Andreief. Let $M := a_h + 1$ and let H_M be the same sum restricted to the nodes $m < M$. Then $\psi(H) \equiv H_M$ entrywise modulo q^{2M} , so $\psi(\Delta_h^{\mathcal{R}}) = \det \psi(H) \equiv \det H_M \pmod{q^{2M}}$. For a finite sum of rank-one matrices, the Andreief (Cauchy–Binet) identity [1, 6] reads

$$(5.4) \quad h! \det \left(\sum_{\iota} c_{\iota} x_{\iota} y_{\iota}^{\top} \right) = \sum_{f: [h] \rightarrow I} \left(\prod_i c_{f(i)} \right) \det(x_{f(i)}(j))_{i,j} \det(y_{f(i)}(j))_{i,j}.$$

Here the rank-one terms are indexed by $\iota \in I$; the row vector x_{ι} and the power of q in c_{ι} depend only on the key of ι (the two terms $\varphi'(-m)t^i t^j$ and $\varphi(-m)t^i(jt^{j-1})$ of a node share the key (m, val)). A term of (5.4) vanishes unless the rows $x_{f(i)}$ are pairwise distinct, so unless the h keys of $f(0), \dots, f(h-1)$ are pairwise distinct. The term is then divisible by q to the sum of their prices. The map $(m, \text{val}) \mapsto 3m$, $(m, \text{der}) \mapsto 3m+2$, $k \mapsto 3k+1$ is injective and satisfies $2 \cdot (\text{image}) \leq 3 \cdot \text{price} + 6$. The images of h distinct keys are h distinct naturals, with sum $\geq h(h-1)/2$; hence the prices sum to at least $(h(h-1) - 6h)/3 \geq a_h$. Since $h!$ is invertible and $2M > a_h$, q^{a_h} divides every coefficient of $\psi(\Delta_h^{\mathcal{R}})$.

Clearing. Let $f = [X^k]\Delta_h^{\mathcal{R}}$. By Lemma 5.1 and (5.1) (with $v = (q, 1/(q^2 - 1))$ in $\mathbb{Q}[[q]]$), $\text{cl}_{\beta_h} f = (q^2 - 1)^{\beta_h} \psi(f)$ in $\mathbb{Q}[[q]]$, which is divisible by q^{a_h} . As $\text{cl}_{\beta_h} f$ is a polynomial in q , its monomials of degree $< a_h$ vanish. $\square$

5.3. The prime 2.

Lemma 5.3 (Lemma 10). *Let $q \in \{3, 5\}$. For every $h \geq 1$ and $k \geq 0$, $|[X^k]\Delta_h|_2 \leq 2^{\beta_h + h(\lfloor \log_2 h \rfloor + 6)}$.*

The only property of q used is $v_2(q^2 - 1) = 3$. We work directly over $\mathbb{Q}$ with the anchored basis, for which $\det(\Lambda_X(a_i b_j)) = \Delta_h(X)$. Put $L := \lfloor \log_2 h \rfloor$ and write $|\cdot|_2$ coefficientwise on $\mathbb{Q}[X]$.

Lemma 5.4. *Let $T: \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Q}$ satisfy $T(l+1) = QT(l) + e(s+2l)^{-2}$ with $v_2(Q-1) \geq 2$, $v_2(e) \geq 0$, $v_2(T(0)) \geq 0$ and s odd. Then $v_2(\Delta^m T(0)) \geq v_2(m!) + m - 2 - \lfloor \log_2 m \rfloor$ for $m \geq 1$.*

Proof. Put $\eta = Q - 1$ and $g(l) = (s + 2l)^{-2}$. Then $\Delta T = \eta T + eg$, and iterating, $\Delta^m T(0) = \eta^m T(0) + e \sum_{k \leq m} \eta^{m-1-k} \Delta^k g(0)$. By induction on k , $\Delta^k g(0) = (-2)^k k! H_k(s)/P_k(s)$ with $P_k(s) = \prod_{a \leq k} (s + 2a)$ odd and $H_k(s) = \sum_{a \leq k} (s + 2a)^{-1}$ 2-integral, so $v_2(\Delta^k g(0)) \geq k + v_2(k!)$. Using $k - 1 - \lfloor \log_2 k \rfloor \leq v_2(k!)$ and $v_2(m!) \leq m - 1$, each term has valuation at least $2(m - 1 - k) + 2k - 1 - \lfloor \log_2 k \rfloor \geq v_2(m!) + m - 2 - \lfloor \log_2 m \rfloor$, and $v_2(\eta^m T(0)) \geq 2m$ is larger. $\square$

Proof of Lemma 5.3. We show that each entry satisfies

$$(5.5) \quad v_2(\Lambda_X(a_i b_j)) \geq h - (i + j + 1) - (i + j + 1 - h)_+ - L - 6.$$

Entries with $\pi := i + j + 1 - h \geq 1$. Then $a_i b_j = D \cdot P$ with $P = \prod_{h-j \leq m < i} (u - x_m)$, so $a_i b_j \varphi = uP$ is a polynomial: $B(a_i b_j) = 0$ and $A(a_i b_j) = \mu(uP)$. We have $uP = 2^{-\pi} H(2u)$ with $H \in \mathbb{Z}[T]$ of degree π . Since $\Delta^b[(2t)^n](0) = 2^n b! S(n, b)$ (Stirling numbers of the second kind) vanishes for $n < b$, $v_2(\Delta^b[(uP)'](0)) \geq 1 - \pi + b + v_2(b!)$. And $v_2(\Omega_b) = -3(b+1)$. By (2.2) and $v_2(b!) \geq b - 1 - \lfloor \log_2 b \rfloor$,

$$v_2(A(a_i b_j)) \geq \min_{b < \pi} (-\pi - 2 - 2b + v_2(b!)) \geq -2\pi - 2 - \lfloor \log_2 \pi \rfloor,$$

which implies (5.5).

Entries with $N := h - 1 - i - j \geq 0$. Then $a_i b_j \varphi = u / \prod_{l=0}^N (u - \xi_l)$ with $\xi_l := x_{i+l} = -c/2 - l$, $c := 2i + 1$. Since $\xi_l - \xi_m = m - l$, its partial fractions are a constant (which μ kills) plus $\sum_l (-1)^l \binom{N}{l} \xi_l / (N! (u - \xi_l))$. By the proof of Lemma 2.1, both $A(a_i b_j)$ and $B(a_i b_j)$ have the form $\pm \frac{4}{N!} \sum_l (-1)^l \binom{N}{l} \xi_l T(l) = \pm \frac{4}{N!} \Delta^N [\xi_l T(l)](0)$, with $T(l) = S_{i+l}(q)$ or $T(l) = q^{2(i+l)+1}$. Both satisfy the recursion of Lemma 5.4 with $Q = q^2$: $S_{i+l+1} = q^2 S_{i+l} + (2i + 3 + 2l)^{-2}$ ($e = 1$, $s = 2i + 3$), and $q^{2(i+l+1)+1} = q^2 \cdot q^{2(i+l)+1}$ ($e = 0$). From $\Delta^N [l T(l)](0) = N \Delta^{N-1} [T(\cdot + 1)](0)$, and since $T(\cdot + 1)$ satisfies a recursion of the same kind,

$$v_2\left(\frac{4}{N!} \Delta^N [\xi_l T(l)](0)\right) \geq 2 - v_2(N!) + \min\left(-1 + v_2(N!) + N - 2 - \lfloor \log_2 N \rfloor, v_2(N!) + N - 3 - \lfloor \log_2(N - 1) \rfloor\right),$$

for $N \geq 2$, which is at least $N - 1 - \lfloor \log_2 N \rfloor$. For $N \leq 1$ the entry is $\pm 4 \xi_0 T(0)$ or $\pm 4(\xi_0 \Delta T(0) - T(1))$, of valuation ≥ 1 , since $v_2(\xi_0) = -1$ and $T(0), T(1), \Delta T(0)$ are 2-integral. This implies (5.5).

Assembly. Since $(i + j + 1 - h)_+ \leq \rho_i + \kappa_j$, (5.5) gives $|\Lambda_X(a_i b_j)|_2 \leq (2^{i+1+\rho_i+L+6-h})(2^{j+\kappa_j})$. The Leibniz expansion, now over $\mathbb{Q}[X]$ with the coefficientwise 2-adic norm, gives $||X^k] \Delta_h|_2 \leq 2^{\sum_i (i+1) + \sum_j j + \beta_h + h(L+6) - h^2} = 2^{\beta_h + h(L+6)}$. $\square$

5.4. The denominator.

Theorem 5.5 (arithmetic bound). *Let $q \in \{3, 5\}$. There are rationals $d_h > 0$ with $d_h \Delta_h \in \mathbb{Z}[X]$ for every $h \geq 1$, such that for every $\varepsilon > 0$ and all large h ,*

$$\log d_h \leq (\alpha_q + \varepsilon)h^2, \quad \alpha_q = \frac{133}{60} + \frac{\log 2}{4} + [q = 5] \frac{\log 3}{4} - \frac{\log q}{3}.$$

Proof. Let c be a coefficient of Δ_h ; then $c = \text{ev}_q(f)$ with f the corresponding coefficient of $\Delta_h^{\mathcal{R}}$. We bound $|c|_p$ for every prime p .

- $p = 2$: $|c|_2 \leq 2^{\beta_h + h(L+6)}$ by Lemma 5.3.
- $p = q$: by (5.1), $c = \text{ev}(\text{cl}_{\beta_h} f)/(q^2 - 1)^{\beta_h}$, and $q^2 - 1$ is a q -unit. By Lemma 5.2 every monomial of $\text{cl}_{\beta_h} f$ has degree $\geq a_h$, so $|c|_q \leq \|\text{cl}_{\beta_h} f\|_q q^{-a_h} \leq \|f\|_q q^{-a_h} \leq q^{2h \text{lev}(q) - a_h}$ by Lemma 4.2.
- p odd with $p \mid q^2 - 1$ (so $p = 3, q = 5$): by Lemma 5.1 and (5.1), $|c|_p = |\text{ev}(\text{cl}_{\beta_h} f)|_p |q^2 - 1|_p^{-\beta_h} \leq \|f\|_p |q^2 - 1|_p^{-\beta_h} \leq p^{2h \text{lev}(p) + v_p(q^2 - 1)\beta_h}$, by Lemma 4.2.
- every other p : q and $1/(q^2 - 1)$ are p -integral, so $|c|_p \leq \|f\|_p \leq p^{e(h,p)}$ by Theorem 4.5.

Let $N_h := \prod_p p^{e_p}$, where $e_2 = \beta_h + h(L+6)$, $e_q = 2h \text{lev}(q)$, $e_3 = 2h \text{lev}(3) + \beta_h$ if $q = 5$, and $e_p = e(h, p)$ otherwise. Only primes $p < 2h + q^2$ occur, because $e(h, p) = 0$ for $p \geq 2h$. Put $d_h := N_h/q^{a_h}$. Then $|d_h c|_p \leq 1$ for every prime p , so $d_h c \in \mathbb{Z}$. Finally

$$\begin{aligned} \log d_h \leq \sum_{p \nmid 2q(q^2-1)} e(h, p) \log p + \left(\frac{h^2 + 2h}{4} + h(L+6) \right) \log 2 \\ + [q = 5] \frac{h^2 + 2h}{4} \log 3 - a_h \log q + O(h \log h), \end{aligned}$$

where the $O(h \log h)$ collects the terms $2h \text{lev}(p) \log p \leq 2h \log(2h)$ for $p \in \{3, q\}$. Theorem 4.5 and $a_h \geq (h^2 - 7h - 2)/3$ give the claim. $\square$

6. THE SIZE OF $\Delta_h(\chi)$: THEOREM B

6.1. Dual certificates. Let $\psi(u) := u \log |u| - u$ (with $\psi(0) = 0$), an anti-derivative of $\log |u|$, so that $\int_{\alpha}^{\beta} \log |x - y| dy = \psi(x - \alpha) - \psi(x - \beta)$. Put $V(x) := x \log w - \psi(x+1) + \psi(x) = x \log w - (x+1) \log(x+1) + x \log x + 1$ for $x \geq 0$. A *step density* is $\mu = \sum_{j < J} c_j \mathbf{1}_{[a_j, a_{j+1})}$ with $0 = a_0 \leq a_1 \leq \dots \leq a_J$, $0 \leq c_j \leq 2$ and $\int \mu = 1$. Its potential and energy are

$$U^{\mu}(x) := \int \log |x - y| \mu(y) dy, \quad E(\mu) := \iint \log |x - y| \mu(x) \mu(y) dx dy.$$

Definition 6.1. A *dual certificate* for q consists of a step density μ , a real number λ , integers $L, X, N \geq 1$ with $a_J \leq N \leq X$, and real numbers S_k ($0 \leq k < XL$) such that

- (1) $F(x) := 2U^{\mu}(x) + V(x) \leq S_k$ for $x \in [k/L, (k+1)/L]$, for every $k < XL$;

(2) $1/X + \log w < 0$ and $\log X + X \log w < \lambda$.

Its *exponent* is $\Phi := -E(\mu) + \lambda + \frac{2}{L} \sum_{k < XL} (S_k - \lambda)_+$.

Theorem 6.2 (Theorem B). *For every dual certificate for q , as $h \rightarrow \infty$,*

$$\log |\Delta_h(\chi_2(1/q))| \leq \Phi h^2 + O(h \log h).$$

The constant in $O(h \log h)$ depends on the certificate. The heuristic behind the theorem: by (2.3), $\Delta_h(\chi)$ is a Hankel determinant of a positive-weight functional on the nodes $t = 1, 2, \dots$ with the derivative structure of $(g\varphi)'$, i.e. a discrete log-gas in the external field V with at most two particles per node. The dual certificate is a feasible point of the dual of the corresponding constrained equilibrium problem [17, Ch. I], and the proof below is a discrete version of weak duality.

6.2. Node expansion. We have $\Lambda(u^n) = \sum_{s \geq 1} w^s (ns^{n-1}\varphi(s) + s^n\varphi'(s))$. Let $H^{(T)}$ be the matrix of truncated sums over $1 \leq s \leq T$. Then $H^{(T)} \rightarrow (\Lambda(u^{i+j}))_{i,j}$ entrywise, so by continuity of the determinant $|\Delta_h(\chi)| \leq \sup_T |\det H^{(T)}|$. Expanding $(i+j)s^{i+j-1} = (is^{i-1})s^j + s^i(js^{j-1})$,

$$H^{(T)} = \sum_{s \leq T} \left(w^s \varphi'(s) v_s v_s^\top + w^s \varphi(s) v'_s v_s^\top + w^s \varphi(s) v_s v_s'^\top \right),$$

with the value row $v_s = (s^j)_{j < h}$ and the derivative row $v'_s = (js^{j-1})_{j < h}$. We call (s, val) and (s, der) the two *slots* of the node s . Since $\varphi'/\varphi = 1/s - \sum_k 1/(s - x_k)$ is a difference of two numbers in $[0, h]$ for $s \geq 1$, all three coefficients are bounded by

$$b(s) := w^s h \varphi(s).$$

Lemma 6.3 (minors with derivative rows). *Let $h \geq 2$ and let $(s_i, \tau_i)_{i < h}$ be slots at integer nodes $s_i \geq 0$. Then*

$$|\det(\text{row of slot } (s_i, \tau_i))_{i < h}| \leq h^h \left(1 + \frac{2}{h}\right)^{h^2} \prod_{i < j} \text{pf}(s_i, s_j),$$

where $\text{pf}(a, b) = |a - b|$ for $a \neq b$ and $\text{pf}(a, a) = 1$.

Proof. Let ζ be a primitive h -th root of unity and $r = 1/h$. For $j < h$, $\sum_{l < h} \zeta^{l(h-1)} (t + r\zeta^l)^j = h r j t^{j-1}$ (only the binomial term with $h \mid h-1+m$, i.e. $m = 1$, survives; this needs $h \geq 2$). Since $hr = 1$, the derivative row at t is the sum $\sum_l \zeta^{l(h-1)} (\text{value row at } t + r\zeta^l)$. By multilinearity the determinant is a sum of at most $h^{\# \text{der}}$ Vandermonde determinants at points z_i with $|z_i - s_i| \leq r$, each with a coefficient of modulus 1. Since the nodes are integers, each Vandermonde factor satisfies $|z_j - z_i| \leq |s_j - s_i| + 2r \leq (1 + 2r) \text{pf}(s_i, s_j)$. So the determinant is at most $h^h (1 + 2/h)^{h(h-1)/2} \prod_{i < j} \text{pf}(s_i, s_j)$. $\square$

Apply (5.4) to $H^{(T)}$, indexing the rank-one terms by triples (node, type). A term with two equal row vectors vanishes, so only assignments f whose

x -slots are distinct contribute; in particular each node carries at most two of the h indices. With Lemma 6.3 for both determinants,

(6.1)

$$|\det H^{(T)}| \leq \frac{K_h^2}{h!} \sum_f \prod_{i < h} b(s_i) \prod_{i < j} \text{pf}(s_i, s_j)^2, \quad K_h := h^h (1 + 2/h)^{h^2},$$

where f runs over the assignments $i \mapsto (s_i, \tau_i)$ with distinct x -slots and s_i is the node of $f(i)$.

6.3. A lattice energy inequality.

Lemma 6.4 (Pólya-type inequality). *For $M \geq 1$ and real $\nu_0, \dots, \nu_{M-1}$,*

$$\sum_{i \neq j} \nu_i \nu_j \log |i - j| \leq \log 2 \sum_i \nu_i^2 + \log M \left(\sum_i \nu_i \right)^2.$$

Proof. For $M = 1$ the claim reads $0 \leq \log 2 \nu_0^2$. Let $M \geq 2$. We follow Pólya's criterion for positive definite functions [14] in a finite form. Let $f(0) = \log M + \log 2$, $f(d) = \log(M/d)$ for $1 \leq d \leq M$, and $f(d) = 0$ for $d > M$. Then f is convex on $\mathbb{Z}_{\geq 0}$ (this uses $\log(k-1) + \log(k+1) \leq 2 \log k$ for $2 \leq k < M$, $f(M-1) - 2f(M) + f(M+1) = \log \frac{M}{M-1} \geq 0$, and $f(0) - 2f(1) + f(2) = 0$ when $M \geq 2$). Hence $f(d) = \sum_{k=1}^M \Delta^2 f(k-1) (k-d)_+$ with $\Delta^2 f \geq 0$. Each triangle kernel $(k - |i - j|)_+$ is the number of windows of length k containing both i and j , hence positive semidefinite. So $\sum_{i,j} \nu_i \nu_j f(|i - j|) \geq 0$, which rearranges to the claim. $\square$

For a reference sequence $m = (m_s)_{s \geq 0}$ with finite support, $\sum_s m_s = h$, put

$$P_m(s) := \sum_{s' \neq s} m_{s'} \log |s - s'|, \quad E_m := \sum_{s \neq s'} m_s m_{s'} \log |s - s'|.$$

Lemma 6.5 (discrete weak duality). *Let m be as above with $\sum_s m_s^2 \leq 2h$, and let $\lambda' \in \mathbb{R}$. Put $g(s) := \log b(s) + 2P_m(s) - \lambda'$. For every assignment f in (6.1),*

$$\prod_i b(s_i) \prod_{i < j} \text{pf}(s_i, s_j)^2 \leq \exp \left(-E_m + \lambda' h + 8h \log 2 + \sum_i g(s_i) \right).$$

Proof. Let $n_s := \#\{i : s_i = s\} \leq 2$, so $\sum_s n_s = h$ and $\sum_s n_s^2 \leq 2h$. Pairs at the same node contribute $\text{pf} = 1$, so $2 \sum_{i < j} \log \text{pf}(s_i, s_j) = \sum_{s \neq s'} n_s n_{s'} \log |s - s'|$. With $\nu := n - m$ (total mass 0),

$$\sum_{s \neq s'} n_s n_{s'} \log |s - s'| = -E_m + 2 \sum_s n_s P_m(s) + \sum_{s \neq s'} \nu_s \nu_{s'} \log |s - s'|,$$

and by Lemma 6.4 the last sum is at most $\log 2 \sum_s \nu_s^2 \leq 2 \log 2 (\sum_s n_s^2 + \sum_s m_s^2) \leq 8h \log 2$. Finally $\sum_s n_s P_m(s) = \sum_i P_m(s_i)$. $\square$

6.4. Discretization. Given a step density μ supported in $[0, N]$, let $m_s := h \int_{(s-1)/h}^{s/h} \mu$ for $s \geq 1$ and $m_0 := 0$ (cell averages). Then $\sum m_s = h$, $0 \leq m_s \leq 2$ (so $\sum m_s^2 \leq 2h$), and $m_s = 0$ for $s > Nh$.

Lemma 6.6. *With these m_s , for all $h \geq 2$ and $s \geq 1$:*

- (1) $\log b(s) \leq \log h + \log s - h \log h + h V(s/h)$;
- (2) $P_m(s) \leq h U^\mu(s/h) + h \log h + 4 + 2 \log(Nh + 1)$;
- (3) $E_m \geq h^2 E(\mu) + h^2 \log h - 4h(1 + \log(Nh + 1))$.

Proof. (i) By concavity of $\log$, $\log(c + \frac{1}{2}) \geq \int_c^{c+1} \log y dy = \psi(c+1) - \psi(c)$ for $c \geq 0$, so $\log D(s) = \sum_{k < h} \log(s + k + \frac{1}{2}) \geq \psi(s+h) - \psi(s) = h(\psi(x+1) - \psi(x)) + h \log h$ with $x = s/h$. Insert this into $\log b(s) = s \log w + \log h + \log s - \log D(s)$.

(ii) We have $h U^\mu(s/h) = \sum_{s'} \int_{I_{s'}} \log |s - u| \rho(u) du - h \log h$, where $I_{s'} = [s' - 1, s']$ and $\rho(u) := \mu(u/h)$; this density has mass $m_{s'}$ on $I_{s'}$, total mass h , and is bounded by 2. For $s' < s$ and $u \in I_{s'}$, $\log |s - u| \geq \log(s - s')$, so these cells contribute with the right sign. For $s' = s$ the discrete term is 0 and $\int_{I_s} (-\log |s - u|) \rho \leq 2$. For $s' = s + d$, $d \geq 1$, the deficit is at most $2 \int_{d-1}^d (\log d - \log u) du = 2(1 - (d-1) \log \frac{d}{d-1}) \leq 2/d$. Summing over $d \leq Nh$ gives (ii).

(iii) Similarly $h^2 E(\mu) + h^2 \log h = \sum_{s, s'} \iint_{I_s \times I_{s'}} \log |u - u'| \rho(u) \rho(u')$. The diagonal cells contribute ≤ 0 because $|u - u'| \leq 1$ there. For $|s - s'| = d \geq 1$ we have $\log |u - u'| \leq \log(d+1) \leq \log d + 1/d$. Since $m_{s'} \leq 2$, summing $m_s m_{s'}/d$ over the pairs gives at most $\sum_s m_s \cdot 2 \cdot 2 \sum_{d \leq Nh} 1/d \leq 4h(1 + \log(Nh + 1))$. $\square$

6.5. Proof of Theorem 6.2. Fix a certificate and $h \geq 2$. Take m as above and $\lambda' := h\lambda + h \log h$. By Lemma 6.6, with $\epsilon_h := 4 + 2 \log(Nh + 1)$ and $x = s/h$,

$$(6.2) \quad \begin{aligned} g(s) &\leq h(F(x) - \lambda) + \log h + \log s + 2\epsilon_h, \\ -E_m + \lambda' h &\leq h^2(\lambda - E(\mu)) + O(h \log h). \end{aligned}$$

Near and far nodes. Let $S_0 := Xh$. Since each node carries at most two indices, $\sum_i g(s_i) \leq 2 \sum_{1 \leq s \leq S_0} g(s)_+ + \sum_{i: s_i > S_0} g(s_i)$. Summing over f in (6.1) with Lemma 6.5 therefore gives

$$|\det H^{(T)}| \leq \frac{K_h^2}{h!} e^{-E_m + \lambda' h + 8h \log 2 + 2 \sum_{s \leq S_0} g(s)_+} \left(3S_0 + 3 \sum_{s > S_0} e^{g(s)} \right)^h.$$

Near nodes. Each $x = s/h \leq X$ lies in a cell $[k/L, (k+1)/L]$ with $k < XL$, where $F \leq S_k$; each cell contains at most $h/L + 1$ points s/h . By (6.2), $\sum_{s \leq S_0} g(s)_+ \leq h(h/L + 1) \sum_k (S_k - \lambda)_+ + S_0(\log h + \log S_0 + 2\epsilon_h)$.

Far nodes. Put $\delta := -\max(1/X + \log w, (\log X + X \log w - \lambda)/X) > 0$. For $x \geq N$, $U^\mu(x) \leq \log x$, and $V(x) \leq x \log w - \log x$ (because $1 - x \log(1 +$

$1/x) \leq 1/(x+1) \leq \log(1+1/x)$. Hence, using $\log x \leq \log X + (x-X)/X$, for $x \geq X$

$$\begin{aligned} F(x) - \lambda &\leq \log x + x \log w - \lambda \\ &\leq (\log X + X \log w - \lambda) + (x-X)(1/X + \log w) \leq -\delta x. \end{aligned}$$

So $e^{g(s)} \leq h s e^{2\epsilon_h} e^{-\delta s} \leq \frac{2}{\delta} h e^{2\epsilon_h} e^{-\delta s/2}$ for $s > S_0$, and the far sum is $e^{O(\log h)}$.

Assembly. $\log(K_h^2/h!) = O(h \log h)$, and $\log(3S_0 + 3 \sum_{s>S_0} e^{g(s)}) = O(\log h)$. Collecting,

$$\log |\det H^{(T)}| \leq h^2 \left(-E(\mu) + \lambda + \frac{2}{L} \sum_k (S_k - \lambda)_+ \right) + O(h \log h)$$

uniformly in T , and the theorem follows from $|\Delta_h(\chi)| \leq \sup_T |\det H^{(T)}|$. $\square$

Remark 6.7. Let $C(w) := \sup\{E(\rho) + \int V\rho\}$ over densities ρ on $[0, \infty)$ with compact support, $0 \leq \rho \leq 2$ and $\int \rho = 1$ (the constrained equilibrium constant). For such ρ , $E(\rho - \mu) \leq 0$ [17, Lemma I.1.8], so $E(\rho) + \int V\rho \leq -E(\mu) + \int F\rho \leq -E(\mu) + \lambda + 2 \int_0^X (F - \lambda)_+ \leq \Phi$, using $\rho \leq 2$, $F < \lambda$ beyond X and $F \leq S_k$ on the cells. Hence $\Phi \geq C(w)$. Numerically $C(1/9) \approx -2.1003$ and $C(1/25) \approx -2.4565$. The finite- h values $h^{-2} \log |\Delta_h(\chi)|$, computed in ball arithmetic up to $h = 320$, approach these constants [numerical], so Theorem 6.2 is essentially sharp for this construction. Only the form with an explicit certificate is used.

7. THE TWO CERTIFICATES

Theorem 7.1. *There are dual certificates with exponent $\Phi < -2.078$ for $q = 3$ and $\Phi < -2.385$ for $q = 5$.*

The certificates are explicit rational data (Table 2). The density μ is constant on the J cells $[js/L, (j+1)s/L]$ with dyadic heights $c_j \in [0, 2]$ (denominator 2^{16}) and exact mass $\sum_j c_j s/L = 1$. The bounds S_k are rationals. All checks are inequalities between explicit rational numbers and finitely many values of $\log$ at rational points, and are verified as follows.

- *Logarithms.* A table of dyadic intervals of width $2 \cdot 2^{-40}$ for $\log n$, $n \leq n_{\max}$, each checked against the series of $\log(1-x)$ with an explicit tail bound, after reducing the argument by powers of 2; $\log 2$ is enclosed to width 2^{-100} .
- *Potential.* By summation by parts, $2U^\mu(x) + V(x) = \sum_j D_j \psi(x - \alpha_j/L) + x \log w$, with $D_j = 2(c_j - c_{j-1})$ at $\alpha_j = js$, plus the terms $\psi(x)$ and $-\psi(x+1)$ of V . On each grid cell $[k/L, (k+1)/L]$ every term is bounded above by an affine function: by the chord where the term is convex, and by a tangent (at the midpoint, or at the endpoint away from α_j/L) where it is concave. Summing the affine bounds and taking the larger endpoint value bounds F on the cell. All values are rational enclosures.

- *Energy.* $E(\mu) = \sum_{j,j'} c_j c_{j'} \iint \log |x - y|$ over pairs of cells, and the double integral over $[a_1, b_1] \times [a_2, b_2]$ equals $-\Psi(b_1 - b_2) + \Psi(b_1 - a_2) + \Psi(a_1 - b_2) - \Psi(a_1 - a_2)$ with $\Psi(u) = \frac{1}{2}u^2 \log |u| - \frac{3}{4}u^2$. A rational lower bound $E_{\text{lo}} \leq E(\mu)$ is computed in interval arithmetic.
- *Tail.* $1/X + \log w < 0$ and $\log X + X \log w < \lambda$, from the enclosures of $\log X$ and $\log q$.

The certified exponent is the rational number $\Phi_{\mathbb{Q}} := -E_{\text{lo}} + \lambda + \frac{2}{L} \sum_k (S_k - \lambda)_+ \geq \Phi$.

	$q = 3$	$q = 5$
grid width $1/L$	1/100	1/40
near region $[0, X]$ (cells XL)	$[0, 3]$ (300)	$[0, 2]$ (80)
density: J cells of width s/L	$30 \times \frac{1}{20}$	$23 \times \frac{1}{20}$
N ; entries of the log table	2; 804	2; 244
$-E_{\text{lo}}$	1.532174204	1.768055783
λ	-3.900735042	-4.649276378
$\frac{2}{L} \sum_k (S_k - \lambda)_+$	0.290234795	0.496069677
$\Phi_{\mathbb{Q}}$	-2.0783260426...	-2.3851509182...

TABLE 2. The two dual certificates (decimal values rounded).

Exactly, $\Phi_{\mathbb{Q}} = -25979075532629/12500000000000$ for $q = 3$ and $\Phi_{\mathbb{Q}} = -9540603672893/4000000000000$ for $q = 5$.

The data are generated by a script that optimises the density in floating point, rounds it to dyadic rationals, and then replicates the interval arithmetic exactly in rational arithmetic. The script is not trusted: in the formalisation every check above is a decidable statement about the data, proved by evaluation in the Lean kernel (Section 10).

8. NONVANISHING: LEMMA P

8.1. Statement. For $r \in \{0, 1, 2\}$ put $c_r := 1/\prod_{a \in \{0, 1, 2\}, a \neq r} (a - r)$, so $(c_0, c_1, c_2) = (\frac{1}{2}, -1, \frac{1}{2})$. Recall $T(M) = \sum_{s \leq M, s \text{ odd}} z^s/s^2$. Define three 3×3 rational matrices, indexed by $r, s \in \{0, 1, 2\}$:

$$M_S(q)_{rs} = \frac{c_r}{1-w} + [r = s] 4q^{2r+1} T(2r-1),$$

$$M_L(q)_{rs} = \frac{w c_r}{1-w} + [r = s] 4q^{2r+1} T(2r+1),$$

$$M_D(q)_{rs} = w c_r \left(\frac{e_{rs}}{1-w} + \frac{2w}{(1-w)^2} \right) - [r = s] 4 \left(r + \frac{1}{2} \right) q^{2r+1} T(2r+1),$$

where $e_{rs} = \frac{11}{2} - 2r$ if $r = s$ and $e_{rs} = (3 - r - s) + \frac{5}{2}$ if $r \neq s$.

Theorem 8.1 (Lemma P). *Let $q \geq 2$ and let $\ell \geq 7$ be a prime with $\ell \nmid q(q^2 - 1)$ such that $\det M_S(q)$, $\det M_L(q)$ and $\det M_D(q)$ are ℓ -adic units. Then $\Delta_{3\ell}(a/b) \neq 0$ for all $a \in \mathbb{Z}$ and $b \geq 1$ with $\ell \nmid b$.*

Theorem 8.2. *For $q \in \{3, 5\}$ and every $a \in \mathbb{Z}$, $b \geq 1$, we have $\Delta_h(a/b) \neq 0$ for infinitely many h .*

Proof. Exact evaluation gives

	$\det M_S$	$\det M_L$	$\det M_D$
$q = 3 :$	6642	$\frac{10896962}{225}$	$-\frac{5947389}{80}$
$q = 5 :$	$\frac{3531250}{27}$	$\frac{246357730}{243}$	$-\frac{6934894375}{3888}$

(the factor $9q^2 + 1$ of $\det M_S = 8q^6(9q^2 + 1)/(9(q^2 - 1))$ comes from $T(3) = z + z^3/9$). Every prime ℓ larger than all these numerators and denominators, than q^2 , and than b , satisfies the hypotheses of Theorem 8.1 and $\ell \nmid b$. There are infinitely many such primes, and each gives $h = 3\ell$. $\square$

The rest of this section proves Theorem 8.1. Fix q, ℓ as there and $h = 3\ell$; then $\ell^2 \geq 7\ell > 2h$. Let $\mathbb{Z}_{(\ell)}$ be the ring of ℓ -integral rationals and $\bar{\cdot} : \mathbb{Z}_{(\ell)} \rightarrow \mathbb{F}_\ell$ the reduction. Put $\kappa := (\ell - 1)/2$. The indices $i < h$ fall into the classes $C_\gamma = \{\gamma, \gamma + \ell, \gamma + 2\ell\}$, $\gamma < \ell$; we write $i = \gamma + r\ell$ with $r \in \{0, 1, 2\}$. The class γ has *type* S if $\gamma < \kappa$, D if $\gamma = \kappa$, and L if $\gamma > \kappa$. Note that $\ell \mid 2i + 1$ iff $i \in C_\kappa$; put $\delta_i := [i \in C_\kappa]$.

8.2. Steps 1–2: the Lagrange Gram matrix. For $r_0 \in \mathbb{Q}$ let $\Lambda_{r_0} := A + r_0 B$ and $G(r_0)_{ij} := \Lambda_{r_0}(f_i f_j)$, with the Lagrange basis $f_i = D/(u - x_i)$. By (2.4), $\det G(r_0) = (\det C_f)^2 \Delta_h(r_0)$. Put

$$\mathcal{E}(r_0)_{ij} := \nu_i G(r_0)_{ij} \ell^{-\delta_j}, \quad \nu_i := \ell^2 / D'(x_i).$$

Then $\det \mathcal{E}(r_0) \neq 0$ implies $\Delta_h(r_0) \neq 0$; no value of $\det C_f$ is needed. As in the proof of Lemma 4.2, for $i \neq j$ we have $G(r_0)_{ij} = \mu(F_{ij})$ with $F_{ij} = u \prod_{m \neq i, j} (u - x_m)$, and $G(r_0)_{ii} = \mu(Q_i) + 4x_i D'(x_i)(S_i(q) - r_0 q^{2i+1})$. Hence

$$\mathcal{E}(r_0) = \mathcal{E}(0) + r_0 \operatorname{diag}(d_i), \quad d_i = -4\ell^{2-\delta_i} x_i q^{2i+1}, \quad v_\ell(d_i) = 2.$$

If $\mathcal{E}(0)$ is ℓ -integral and $\ell \nmid b$, then $\mathcal{E}(a/b)$ is ℓ -integral with the same reduction as $\mathcal{E}(0)$. So it suffices to show that $\mathcal{E}(0)$ is ℓ -integral and $\det \overline{\mathcal{E}(0)} \neq 0$.

8.3. Step 3: splitting by progressions. For $|v| < 1$ and a polynomial g , $\sigma(v, g) := \sum_{j \geq 0} v^j g(j) = \sum_b \Delta^b g(0) v^b / (1 - v)^{b+1}$. Since w, w^ℓ and $1 - w^\ell \equiv 1 - w = (q^2 - 1)/q^2$ are ℓ -units, we get the following. If g has ℓ -integral values on $\mathbb{Z}_{\geq 0}$, then $\sigma(w^\ell, g)$ is ℓ -integral. If moreover $\bar{g}(j) = g_0(j)$ for all j , for a polynomial g_0 over $\mathbb{F}_\ell$, then $\overline{\sigma(w^\ell, g)} = \sum_b \Delta^b g_0(0) \bar{w}^b / (1 - \bar{w})^{b+1}$ (use $\bar{w}^\ell = \bar{w}$). Splitting $t \geq 1$ into the residue classes modulo ℓ ,

$$\mu(F) = \sum_{\gamma < \ell} w^{t_\gamma} \sigma(w^\ell, j \mapsto F'(t_\gamma + j\ell)),$$

where the *progression* of γ is $\{t \geq 1 : \ell \mid 2t + 2\gamma + 1\}$ and t_γ is its least element: $t_\gamma = \kappa - \gamma$ (type S), $\kappa - \gamma + \ell$ (type L), and ℓ (type D). On the progression of γ , $\ell \mid t - x_m$ iff $m \in C_\gamma$, and $\ell \mid t$ iff $\gamma = \kappa$. Since $\ell^2 > 2h$, all these divisibilities are simple.

8.4. Step 4: integrality and block triangularity. We have $D'(x_i) = \prod_{m \neq i} (m-i) = \ell^2 c_r^{-1} U_i$ with $U_i := \prod_{m \notin C_\gamma} (x_i - x_m)$ an ℓ -unit, so $\nu_i = c_r U_i^{-1}$ is a unit. Let γ_i, γ_j be the classes of i, j and consider the contribution of the progression of γ .

Off-diagonal entries. At t on the progression of γ , the factors of F_{ij} divisible by ℓ number $N_\gamma = 3 - [\gamma = \gamma_i] - [\gamma = \gamma_j] + [\gamma = \kappa]$, and every term of $F'_{ij}(t)$ keeps at least $N_\gamma - 1$ of them. So the contribution has valuation $\geq N_\gamma - 1 - \delta_j \geq 0$, with equality only if $\gamma = \gamma_i = \gamma_j$ (inside a diagonal block) or $\gamma = \gamma_i \neq \gamma_j = \kappa$ (a non-D row against a D column).

Diagonal entries. Off its own progression, $|t - x_i|_\ell = 1$ and $Q'_i(t) = (g'_i(t)(t - x_i) - g_i(t) + g_i(x_i))/(t - x_i)^2$ with g_i having at least three roots ℓ -adically close to t , so $v_\ell(Q'_i(t)) \geq 2$ and the contribution has valuation $\geq 2 - \delta_i \geq 1$. The pole term $4\ell^{2-\delta_i} x_i S_i(q)$ is integral, since $v_\ell(x_i) = \delta_i$ and the denominators s^2 of S_i have $v_\ell(s) \leq 1$. The own-progression contributions are integral by Step 5.

Hence $\mathcal{E}(0)$ is ℓ -integral. Ordering the classes with κ last, $\overline{\mathcal{E}(0)}$ is block upper triangular, with 3×3 diagonal blocks $\overline{\mathcal{E}(0)}_{C_\gamma}$, and each diagonal block receives only its own progression and the pole terms.

8.5. Step 5: the diagonal blocks. Fix a class γ and $i = \gamma + r\ell, j = \gamma + s\ell$. On the progression of γ we have $t \equiv x_i \equiv x_\gamma \pmod{\ell}$.

For $r \neq s$ let $c := 3 - r - s$ be the rank of the third class member, and for $r = s$ let c_1, c_2 be the ranks of the other two members; we write $x_{(c)} := x_{\gamma+c\ell}$. For a polynomial P let $Q[P] := (P(u) - P(x_i))/(u - x_i)$, so that $Q[PR] = Q[P]R + P(x_i)Q[R]$ and $Q_i = Q[g_i]$.

Types S and L. If $r \neq s$, then $F_{ij} = (u - x_{(c)}) \cdot u \prod_{m \notin C_\gamma} (u - x_m)$, and modulo ℓ , $F'_{ij}(t) \equiv t \prod_{m \notin C_\gamma} (t - x_m) \equiv x_i U_i$. If $r = s$, write $g_i = P \cdot R$ with $P = (u - x_{(c_1)})(u - x_{(c_2)})$ and $R = u \prod_{m \notin C_\gamma} (u - x_m)$. Then $Q'_i = Q[P]'R + Q[P]R' + P(x_i)Q[R]'$ with $Q[P] = u + x_i - x_{(c_1)} - x_{(c_2)}$. At t we have $Q[P](t) \equiv 0 \pmod{\ell}$, and $\ell^2 \mid P(x_i)$, so $Q'_i(t) \equiv R(t) \equiv x_i U_i \pmod{\ell}$. In both cases $\nu_i(\cdot) \equiv c_r \bar{x}_\gamma$, a constant along the progression. Summing with Step 3, and using $\bar{w}^\kappa = \bar{q}^{-(\ell-1)} = 1$ and $\bar{w}^\ell = \bar{w}$, the progression contributes $\bar{w}^{t\gamma} c_r \bar{x}_\gamma / (1 - \bar{w})$, where $\bar{w}^{t\gamma} = \bar{w}^{-\gamma}$ (S) or $\bar{w}^{1-\gamma}$ (L). For the pole term only the odd $s = \ell s'$ survive in $\ell^2 S_i$, and $q^\ell \equiv q$, so $\overline{\ell^2 S_i} = \bar{q}^{2i+1} T(M)$ with M the largest odd number $\leq (2i+1)/\ell$: $M = 2r - 1$ (S) or $2r + 1$ (L). Since $\bar{q}^{2i+1} = \bar{q} \bar{w}^{-\gamma} \bar{w}^{-r}$, the diagonal block is

$$\overline{\mathcal{E}(0)}_{C_\gamma} = \bar{x}_\gamma \bar{w}^{-\gamma} \overline{M_{\text{type}(\gamma)}(q)}.$$

Type D ($\gamma = \kappa$). Here $x_i = -\ell(r + \frac{1}{2})$, the progression is $t = \ell(1 + j')$, $j' \geq 0$, and we put $y = t/\ell + \frac{1}{2} = j' + \frac{3}{2}$; then $t = \ell(y - \frac{1}{2})$ and $t - x_{(c)} = \ell(y + c)$. Let $R := \prod_{m \notin C_\kappa} (u - x_m)$, so $R(t) \equiv R(x_i) = U_i \pmod{\ell}$. If $r \neq s$, two roots of $F_{ij} = u(u - x_{(c)})R$ are close to t , and $F'_{ij}(t) = (2t - x_{(c)})R(t) + t(t - x_{(c)})R'(t) \equiv \ell(2y + c - \frac{1}{2})U_i \pmod{\ell^2}$. If $r = s$, write $g_i = PR$

with $P = u(u - x_{(c_1)})(u - x_{(c_2)})$. Then $\ell^3 \mid P(x_i)$; $Q[P](t)$ is a quadratic form in $t, x_i, x_{(c_1)}, x_{(c_2)}$, all divisible by ℓ , so $Q[P](t) \equiv 0 \pmod{\ell^2}$; and $Q[P]'(t) = 2t + x_i - x_{(c_1)} - x_{(c_2)} = \ell(2y + \frac{5}{2} - 2r)$ (using $c_1 + c_2 = 3 - r$). Hence $Q'_i(t) \equiv \ell(2y + \frac{5}{2} - 2r)U_i \pmod{\ell^2}$. With $\nu_i = c_r U_i^{-1}$ and the column factor ℓ^{-1} , the residues are $c_r(2y + c - \frac{1}{2}) = c_r(2j' + e_{rs})$ for $r \neq s$ and $c_r(2y + \frac{5}{2} - 2r) = c_r(2j' + e_{rr})$ for $r = s$. Summing $c_r(2j' + e_{rs})$ against $\bar{w}^{1+j'}$ over $j' \geq 0$ gives $\bar{w} c_r(e_{rs}/(1 - \bar{w}) + 2\bar{w}/(1 - \bar{w})^2)$. The pole term $4\ell x_i S_i = -4(r + \frac{1}{2})\ell^2 S_i$ reduces to $-4(r + \frac{1}{2})\bar{q}^{2r+1}T(2r+1)$. So $\overline{\mathcal{E}(0)}_{C_\kappa} = \overline{M_D(q)}$.

8.6. Step 6: conclusion. By Steps 4 and 5,

$$\det \overline{\mathcal{E}(0)} = \prod_{\gamma \neq \kappa} (\bar{x}_\gamma \bar{w}^{-\gamma})^3 \overline{\det M_{\text{type}(\gamma)}(q)} \cdot \overline{\det M_D(q)}.$$

For $\gamma \neq \kappa$, $\bar{x}_\gamma = -(2\gamma + 1)/2 \neq 0$, and the three determinants are units by hypothesis. So $\det \overline{\mathcal{E}(a/b)} = \det \overline{\mathcal{E}(0)} \neq 0$ in $\mathbb{F}_\ell$, hence $\det \mathcal{E}(a/b) \neq 0$ and $\Delta_{3\ell}(a/b) \neq 0$. This proves Theorem 8.1. $\square$

Remark 8.3. The same analysis works for every h with $h \bmod \ell \leq \kappa$ and three or more atoms per class. One then needs a fourth block type with four atoms, and an explicit prime-gap theorem to find suitable ℓ near $h/3$ for all large h [proof sketch]. For irrationality, infinitely many h suffice, and $h = 3\ell$ avoids both complications. With at most two atoms per class the analogous block constants of type S vanish identically in our computations, so primes $\ell > h/2$ did not give a usable nonvanishing statement [numerical].

9. PROOF OF THEOREM 1.1

Let $q \in \{3, 5\}$. We verify the hypotheses of Theorem 3.2.

- (1) Theorem 5.5 gives d_h with $\alpha = \alpha_q$.
- (2) Theorem 6.2, applied to the certificate of Theorem 7.1, gives (ii) with $\beta = -2.078$ ($q = 3$) or $\beta = -2.385$ ($q = 5$).
- (3) Theorem 8.2 gives (iii).

It remains to check $\alpha_q + \beta < 0$, using only elementary bounds for logarithms: $0.6931471803 < \log 2 < 0.6931471808$, and, from $\log x \geq 1 - 1/x$, $\log 3 \geq \log 2 + \frac{1}{3}$, $\log 3 \leq 2\log 2 - \frac{1}{4}$ and $\log 5 \geq 2\log 2 + \frac{1}{5}$. Then

$$\alpha_3 - 2.078 \leq \frac{133}{60} + \frac{\log 2}{4} - \frac{\log 2}{3} - \frac{1}{9} - 2.078 < -0.030,$$

$$\alpha_5 - 2.385 \leq \frac{133}{60} + \frac{\log 2}{4} + \frac{2\log 2 - \frac{1}{4}}{4} - \frac{2\log 2 + \frac{1}{5}}{3} - 2.385 < -0.23.$$

(The true values are $\alpha_3 - 2.078 = -0.05425\dots$ and $\alpha_5 - 2.385 = -0.25687\dots$) Theorem 3.2 shows that $\chi_2(1/3)$ and $\chi_2(1/5)$ are irrational. $\square$

10. THE FORMALISATION

10.1. What is formalised. The repository <https://github.com/telaviv/chi2-irrationality> [10] contains a proof of Theorem 1.1 in Lean 4 [4] (version v4.35.0) with Mathlib [19] (version v4.35.0 [20]). It has 61 Lean files under `Chi2Irrational/`, with about 18,000 lines. The formalisation covers every step of the proof, including the parts that a written proof would cite: the prime number theorem, the Andreief identity, the lattice energy inequality, both dual certificates and the block determinants. Paths below are relative to the folder `Chi2Irrational/`, and all declarations are in the namespace `Chi2Irr`.

10.2. The statement. The definitions that the reader has to check are, in `Defs.lean` and `Statement.lean`:

```
noncomputable def chi2Term (z : ℝ) (n : ℕ) : ℝ :=
  z ^ (2 * n + 1) / (2 * (n : ℝ) + 1) ^ 2
noncomputable def chi2 (z : ℝ) : ℝ :=
  ∑' n : ℕ, chi2Term z n
def Chi2OneThirdIrrational : Prop := Irrational (chi2 (1 /
3))
def Chi2OneFifthIrrational : Prop := Irrational (chi2 (1 /
5))
theorem chi2_one_third_irrational : Chi2OneThirdIrrational
theorem chi2_one_fifth_irrational : Chi2OneFifthIrrational
```

The theorems are in `Arith/Assembly.lean`. Here $\sum'$ is Mathlib's unconditional sum (`tsum`). `Defs.lean` proves that the series is summable for $|z| \leq 1$ (`summable_chi2Term`), so `chi2 z` is the genuine value $\chi_2(z)$. `Irrational x` is Mathlib's predicate $x \notin \mathbb{Q}$.

The command `lake env lean Axioms.lean` prints

```
'Chi2Irr.chi2_one_third_irrational' depends on axioms:
[proptex, Classical.choice, Quot.sound]
'Chi2Irr.chi2_one_fifth_irrational' depends on axioms:
[proptex, Classical.choice, Quot.sound]
```

and the same list for each of the main ingredients listed in `Axioms.lean`. These are Lean's three standard axioms, which Mathlib also uses. The development contains no `sorry`, no added axiom, no `native_decide` and no `implemented.by`, so no step relies on compiled code.

10.3. Correspondence with the paper. Table 3 lists the main formal counterparts.

A few arguments are organised differently in the formalisation.

- The prime number theorem $\theta(x)/x \rightarrow 1$ is ported from the `PrimeNumberTheoremAnd` project [9] (Wiener–Ikehara route), through the port in the `zeta5` repository of César Del Solar [5] and the author's formalisation [11] of a result on Dirichlet beta values. The ported files keep their Apache License 2.0.

paper	Lean declaration (file)
Lemma 2.1, (2.3)	<code>powSum_eq_tsum</code> , <code>pole_series</code> (<code>PoleSeries</code>); <code>Lam_eq</code> , <code>aeval_chi2_Delta</code> (<code>PoleExpansion</code>)
Prop. 3.1, Thm. 3.2	<code>irrational_of_poly_approx</code> (<code>Criterion</code>); <code>MachineBounds</code> , <code>irrational_of_bounds</code> (<code>MainCond</code>)
Section 4.1	<code>map_DeltaF</code> (<code>Arith/Free</code>); <code>det_GLeX</code> , <code>det_GramF</code> (<code>Arith/Gauss</code>)
Lemma 4.2	<code>lemma9_DeltaF</code> (<code>Arith/Lemma9</code>)
Prop. 4.3, Lemma 4.4	<code>classBound_DeltaF</code> (<code>Arith/ClassBound</code>); <code>classBd_le_phi1</code> , <code>classBd_eq_zero</code> (<code>Arith/ClassCount</code>)
Thm. 4.5	<code>eventually_primeSum_phi1_le</code> , <code>eventually_primeSum_lev_le</code> (<code>Arith/PrimeSums</code>); <code>PNT.theta_div_tendsto</code> (<code>PNT/Theta</code>)
Lemma 5.1, (5.1)	<code>EDeg_DeltaF</code> , <code>aeval_clearE</code> (<code>Arith/PoleOrder</code>)
Lemma 5.2, (5.4)	<code>ord_DeltaP</code> , <code>qdeg_clearE_DeltaF</code> (<code>Arith/QAdic</code>); <code>factorial_smul_det_sum_rankOne</code> (<code>Analytic/Andreief</code>)
Lemmas 5.3, 5.4	<code>twoAdic_Delta</code> (<code>Arith/TwoAdic</code>); <code>V2_fd_seq</code> , <code>V2_U</code> (<code>Arith/TwoAdicBasis</code>); <code>V2_entry</code> (<code>Arith/TwoAdicEntries</code>)
Thm. 5.5	<code>denQ</code> , <code>padicNorm_Delta_eN</code> , <code>denQ_int</code> (<code>Arith/Special</code>); <code>arith_denQ</code> (<code>Arith/Assembly</code>)
node expansion, Lemma 6.3	<code>HT_rankOne</code> , <code>abs_Delta_le_of</code> (<code>Analytic/Nodes</code>); <code>abs_det_slot_le</code> (<code>Analytic/SlotDet</code>)
Lemmas 6.4, 6.5	<code>polya_log</code> (<code>Analytic/Polya</code>); <code>pattern_le</code> , <code>abs_det_HT_le_dual</code> (<code>Analytic/Dual</code>)
Lemma 6.6	<code>log_bN_le</code> , <code>StepData.Pm_le</code> , <code>StepData.Em_ge</code> (<code>Analytic/Discrete</code>)
Def. 6.1, Thm. 6.2	<code>DualCert</code> , <code>DualCert.det_bound</code> , <code>DualCert.analytic</code> (<code>Analytic/TheoremB</code>)
Thm. 7.1	<code>CData.Valid</code> , <code>CData.toCert</code> , <code>CData.Phi_le</code> (<code>Cert/Instance</code>); <code>Cert.Q3.analytic</code> , <code>Cert.Q5.analytic</code> (<code>Cert/Q3</code> , <code>Cert/Q5</code>)
Thm. 8.1	<code>Delta_eval_ne_zero_of_det</code> (<code>NonVan/Gram</code>); <code>MS</code> , <code>ML</code> , <code>MD</code> (<code>NonVan/BlockMat</code>); <code>lemmaP</code> (<code>NonVan/LemmaP</code>)
Thm. 8.2	<code>nonvan_of</code> , <code>nonvan_three</code> , <code>nonvan_five</code> (<code>NonVan/NonVanQ</code>)
Section 9	<code>margin_three</code> , <code>margin_five</code> , <code>boundsThree</code> , <code>boundsFive</code> , <code>chi2_one_third_irrational</code> , <code>chi2_one_fifth_irrational</code> (<code>Arith/Assembly</code>)

TABLE 3. Formal counterparts of the main steps.

- The discretization bounds of Lemma 6.6(ii)–(iii) are proved piece by piece of the step density, with the antiderivatives ψ and Ψ , and with the error terms $8J(1 + \log h) + 2\log(Nh + 1)$ and $4h\log(Nh + 1)$ instead of those stated here. Only their size $O(h \log h)$ matters.

- The value $\beta_h = \lfloor h^2/4 \rfloor$ of Lemma 5.1 is not formalised; the formal proof uses $4\beta_h \leq h^2 + 2h$.
- The bound $a_h \geq (h^2 - 7h - 2)/3$ and the two margins are proved with the elementary logarithm bounds of Section 9. The formal a_h is $(h(h-1)-6h)/3$ in natural-number arithmetic, which is the $\max(0, \cdot)$ of Section 5.
- The potential U^μ and the energy $E(\mu)$ of a step density are *defined* in Lean by their closed forms $\sum_j c_j(\psi(x - a_j) - \psi(x - a_{j+1}))$ and $\sum_{j,j'} c_j c_{j'}(\cdots)$ with Ψ , which equal the integrals of Section 6. Both Theorem 6.2 and the certificates use these closed forms.

10.4. How the certificates are checked. The dual certificates are records of rational data (`Cert/DataQ3.lean`, `Cert/DataQ5.lean`): the log table as a list of integers in units of 2^{-40} , the heights c_j , the bounds S_k , λ and E_{10} . `CData.Valid` collects every check of Section 7 as a Boolean computation, and `Cert/Q3.lean` and `Cert/Q5.lean` prove it by `decide +kernel`, i.e. by evaluation in the Lean kernel. The interval arithmetic (`Interval/`) has exact rational endpoints (only the log-table entries and the enclosure of $\log 2$ are dyadic by construction), and its soundness lemmas turn `check = true` into real inequalities; the enclosure of $\log 2$ is itself checked by `decide +kernel` against Mathlib’s series bound. The data were produced by the script `scripts/gen_cert.py`, which is not trusted: a wrong value would make the build fail but could not make a false statement provable. Re-running the script reproduces both data files byte for byte. The six block determinants of Theorem 8.2 are evaluated exactly by `norm_num`.

10.5. Cost. With Mathlib’s prebuilt cache, a clean build of the project takes about ten minutes on a desktop computer. The two kernel-checked certificates are a small part of it.

10.6. What has to be trusted. Beyond the Lean kernel and the three axioms, the reader has to trust that the definitions displayed above express Theorem 1.1, and the Mathlib definitions they use (real numbers, `tsum`, `Irrational`). Nothing else, in particular no numerical data and no generating script, needs to be trusted. To avoid trusting the elaborator’s handling of the environment, we replayed every declaration of the 61 project modules through the kernel with `leanchecker` (the `lean4checker` tool shipped with the Lean toolchain), run on each module against the compiled environment of its imports; it reported no errors. Mathlib itself was not replayed from scratch (`--fresh`). Apart from the check by the Lean kernel, all verification of the formalisation so far was carried out by AI systems; no human expert has yet reviewed the Lean code.

11. REMARKS

11.1. The margins. For $q = 3$ the exponent that must be negative is $\alpha_3 + \Phi_{\mathbb{Q}} \approx -0.0546$ per h^2 with the certificate value $\Phi_{\mathbb{Q}}$ (-0.0543 with the

rounded value -2.078 used in Section 9). The arithmetic part is 2.02375 , so the margin is under 3% of it. A 2-adic constant larger than about 0.33 instead of $\frac{1}{4}$ in Lemma 5.3 would have consumed it. For $q = 5$ the margin is -0.257 , and no single constant is critical. With the sharpest available analytic constant, the equilibrium value $C(w)$, the margins would be about -0.077 and -0.328 [numerical].

11.2. Effectivity. Every bound in the proof is effective except the prime number theorem, which is used without an error term in Theorem 4.5. Even with an explicit form, the $O(h \log h)$ terms and the prime-power contributions of Lemma 4.2 decay slowly relative to the margin for $q = 3$. The irrationality statement is not affected.

11.3. Other values of q . The construction works verbatim for every integer $q \geq 2$. The arithmetic part is $\frac{133}{60} + c_2(q) \log 2 + \sum_{p|q^2-1, p \text{ odd}} \frac{v_p(q^2-1)}{4} \log p - \frac{1}{3} \log q$, where $c_2(q)$ is the 2-adic constant ($c_2 = \frac{1}{4}$ when $v_2(q^2 - 1) = 3$). The analytic part improves as q grows [numerical]. We expect the method to prove $\chi_2(1/q) \notin \mathbb{Q}$ for every $q \geq 3$ once these constants are made explicit, with $q = 3$ the hardest case [heuristic]. We have not carried this out.

11.4. $\chi_2(1/2)$. At $q = 2$ ($w = \frac{1}{4}$) the same construction fails [numerical]. The analytic part is only about $C(\frac{1}{4}) \approx -1.744$, the generic primes still cost $\frac{133}{60}$, and the prime $3 \mid q^2 - 1$ costs $\frac{1}{4} \log 3$. The 2-adic gain ($-\frac{1}{3} \log 2$) does not compensate. The limiting exponent is about $+0.52$ per h^2 , and exact computations up to $h = 64$ confirm the trend. Variations of the external field (pole sets, numerator zeros, node regions) gained at most 0.01 in our experiments. Since $\chi_2(\frac{1}{3}) + \chi_2(\frac{1}{2}) = \frac{\pi^2}{8} - \frac{1}{2} \log 2 \log 3$, a proof that $\chi_2(1/2)$ is irrational, together with Theorem 1.1, would still not decide the irrationality of $\log 2 \log 3$.

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The construction, the proofs, the certificate programs, the Lean formalisation and the text of this paper were produced largely by AI models (Claude, Anthropic), working under the direction of the author. Apart from the check by the Lean kernel, all verification so far, including the internal reviews of the argument, was carried out by AI systems. No human expert has yet reviewed the mathematics or the Lean code; we would be grateful for corrections. The prime number theorem in the formalisation is ported from the `PrimeNumberTheoremAnd` project of A. Kontorovich, T. Tao and collaborators, through the `zeta5` repository of César Del Solar.

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