

Finite Multiset Codes and Affine Lattice Restrictions

Daniel A. Pyle

Independent Researcher

ORCID: [0009-0005-6456-1370](https://orcid.org/0009-0005-6456-1370)

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18 October 2026

Abstract

We study constant-weight nonnegative integer vectors under half- ℓ_1 distance and compare arbitrary codes with full intersections of affine additive-subgroup cosets with the discrete simplex. A heavy-coordinate decomposition gives the exact reduction $A_q(n, n-s) = q + B_q(n, s)$ to codes whose coordinates are bounded by s . It determines the unrestricted optimum throughout $2n > (q+1)s$. For every $q \geq 3$ and $s \geq 2$, at $n = qs - 1$ and $D = (q-1)s - 1$ the unrestricted and full-coset optima are respectively $q+1$ and q ; we classify all q^2 unrestricted extremizers. We retain a complete classification of both optima for three coordinates and $n < 2D$. In the next ternary band, $2D \leq n < 3D$, eight words are attainable precisely when $3n \geq 8D$; equality yields an infinite eight-versus-seven full-coset separation. Exact finite certificates also give a six-versus-five gap at $(q, n, D) = (4, 5, 3)$ and the quinary single-deletion values $A_5(9, 2) = 143$ and $A_5(10, 2) = 201$. The proofs use intersection bounds and explicit integer affine combinations. The unrestricted decomposition extends known methods, and bibliographic novelty remains unresolved, particularly against a closely related 2024 paper not compared in full.

1 Definitions and scope

For integers $q \geq 2$ and $n \geq 1$, let

$$S(n, q) = \{x \in \mathbb{Z}_{\geq 0}^q : \sum_i x_i = n\}, \quad d(x, y) = \frac{1}{2} \sum_i |x_i - y_i| = n - \sum_i \min(x_i, y_i).$$

A distance- D code has distinct words at distance at least D , where D is an integer with $1 \leq D \leq n$. Its largest size is $A_q(n, D)$. Set

$$\Lambda_{q-1} = \{z \in \mathbb{Z}^q : \sum_i z_i = 0\}.$$

Following the full-intersection definition in Kovačević and Tan [1, Definition 2], write $A_{q,\text{lin}}(n, D)$ for the largest size among codes

$$C = (t + L) \cap S(n, q), \quad L \leq \Lambda_{q-1}, \quad \sum_i t_i = n. \quad (1)$$

Here t is integral and every subgroup rank is allowed. The intersection must be *full*: no pruning is permitted. Otherwise every code would qualify by taking the ambient affine integer lattice. Every containing coset is closed under integer affine combinations whose coefficients sum to one; the resulting points matter only when they remain in the finite simplex.

Minimum distance D corresponds to correction of up to $D - 1$ deletions, or $\lfloor (D - 1)/2 \rfloor$ substitution-only errors [1, Theorems 1–2]. The deletion parameter is therefore one less than the minimum distance.

For three coordinates only, equal coordinate sums imply

$$d(x, y) = \max_i |x_i - y_i|. \quad (2)$$

Indeed, either the positive or the negative coordinates of $x - y$ occupy just one coordinate, and their total magnitude is half its ℓ_1 norm. This identity fails for $q \geq 4$: $(1, 1, 0, 0)$ and $(0, 0, 1, 1)$ have half- ℓ_1 distance two and maximum-coordinate distance one. We use overlap, not (2), in the general-alphabet results.

2 A capped-code reduction and an exact cone

For an integer $0 \leq s < n$, let $B_q(n, s)$ be the maximum size of a subset of

$$\{x \in S(n, q) : x_i \leq s \text{ for every } i\}$$

whose distinct words have overlap $\sum_i \min(x_i, y_i) \leq s$. An empty feasible set has maximum zero.

Theorem 1. *For integers $q \geq 2$ and $0 \leq s < n$,*

$$A_q(n, n - s) = q + B_q(n, s).$$

Proof. Call a word heavy if at least one coordinate exceeds s , and light otherwise. For distinct words of a distance- $(n - s)$ code the sets of coordinates exceeding s are disjoint: a shared coordinate would already contribute more than s to their overlap. Each heavy word has a nonempty such set, so there are at most q heavy words. The light words form a capped code, giving the upper bound. Conversely, adjoin all q pure vertices ne_i to a largest capped code. The overlap of ne_i with a capped word x is $x_i \leq s$, and distinct vertices have overlap zero. Since $s < n$, none of the vertices is capped. This gives the lower bound. $\square$

This is the cap- s extension of the cap-one decomposition in Kreindel–Essayag–Zabokritskiy [2, Theorem 10]. Large-coordinate counting also appears in Chen–Ma–Zhang [3, Theorems III.1 and III.3]. The method is credited here rather than presented as a new counting principle.

Corollary 2. *Put $s = n - D$. If $2n > (q + 1)s$, then*

$$A_q(n, D) = \begin{cases} q + 1, & n \leq qs, \\ q, & n > qs. \end{cases}$$

Proof. Two light words satisfy

$$\sum_i \min(x_i, y_i) = 2n - \sum_i \max(x_i, y_i) \geq 2n - qs > s.$$

Thus $B_q(n, s) \leq 1$. A light word exists exactly when $n \leq qs$: necessity follows by summation, while sufficiency follows by distributing mass as evenly as possible among q coordinates. This argument includes $s = 0$, when no light word exists. $\square$

For example, setting $s = 2$ gives the exact formula

$$A_q(n, n-2) = \begin{cases} q+1, & q+2 \leq n \leq 2q, \\ q, & n \geq 2q+1. \end{cases} \quad (3)$$

In the deletion notation of [2], this is $S_q(n, n-3)$. It sharpens the bound $q + \binom{q}{2}$ in their Proposition 12 throughout its stated range $n \geq q+2$ for $q \geq 3$. This comparison concerns that specific proposition, not priority against all earlier literature.

3 A full-coset gap for every alphabet size

Theorem 3. *For integers $q \geq 3$ and $s \geq 2$, set $n = qs - 1$ and $D = (q-1)s - 1$. Then*

$$A_q(n, D) = q+1, \quad A_{q,\text{lin}}(n, D) = q.$$

There are exactly q^2 optimal unrestricted codes on labeled coordinates. Choose an index j and let $c_j = s-1$ and $c_i = s$ for $i \neq j$. Include c and all vertices ne_i for $i \neq j$. The remaining word is either ne_j or $(n-1)e_j + e_\ell$ for one $\ell \neq j$. These codes form two coordinate-permutation orbits of sizes q and $q(q-1)$.

Proof. The cone condition holds because $2n - (q+1)s = (q-1)s - 2 > 0$, and $n \leq qs$. Hence the unrestricted optimum is $q+1$. An optimal code has one light word c and q heavy words. Their nonempty, disjoint heavy-coordinate sets must each be singletons. Label the corresponding words a_i . The deficits $s - c_i$ are nonnegative integers summing to one; thus precisely one coordinate, say j , equals $s-1$ and all others equal s .

For $i \neq j$, the overlap of a_i with c already equals s in coordinate i . Since every coordinate of c is positive, all other coordinates of a_i must vanish; thus $a_i = ne_i$. For a_j , coordinate j contributes $s-1$, leaving overlap at most one elsewhere. Every other coordinate of c equals $s \geq 2$, so the total mass of a_j outside coordinate j is at most one. This gives exactly the listed q possibilities. They all work: their overlaps with c are $s-1$ or s , with the other vertices zero or one, and distinct forced vertices have overlap zero. The unique light word distinguishes the q choices of j , and the last word distinguishes the remaining choices. Permutations act transitively separately on the unshifted and shifted choices, proving the count and orbit statement.

Every affine subgroup coset containing c and the $q-1$ forced vertices contains

$$p = qc - \sum_{i \neq j} ne_i, \quad p_j = q(s-1), \quad p_i = 1 \ (i \neq j).$$

The coefficients sum to one. This is a nonnegative integer point of weight n , distinct from c , and

$$d(p, c) = (q-1)(s-1) = D - (q-2) < D.$$

Consequently no full coset can realize an unrestricted optimum. The q vertices are exactly $(ne_j + n\Lambda_{q-1}) \cap S(n, q)$ and attain the remaining bound q . $\square$

Both hypotheses matter. The obstruction has no strict distance gap at $q = 2$. At $s = 1$, the family gives $n = q-1$, $D = q-2$, and for $q \geq 4$ it still lies *inside* the strict cone. Nevertheless a full-coset code attains $q+1$: put $c = (0, 1, \dots, 1)$, $t = ne_0$, and $L = n\Lambda_{q-1} + \mathbb{Z}(c-t)$. In the coset, coordinate zero is zero modulo n and the other coordinates have a common residue r . Weight n allows only $r = 0$, giving the vertices, or $r = 1$, giving c ; $r \geq 2$ costs at least $2n$. Thus the full intersection consists of precisely these $q+1$ words. For $q = 3, s = 1$, all six points of $S(2, 3)$ are allowed at distance one and the cone is at equality. The general theorem does not classify the full-coset optimum everywhere in the cone.

4 The complete ternary high-distance classification

Write $A = A_3$ and $A_{\text{lin}} = A_{3,\text{lin}}$ in this and the next section.

Theorem 4. *For $1 \leq D \leq n < 2D$,*

$$A(n, D) = \begin{cases} 4, & D \leq \lfloor 2n/3 \rfloor, \\ 3, & D > \lfloor 2n/3 \rfloor, \end{cases} \quad A_{\text{lin}}(n, D) = \begin{cases} 4, & D \leq 2\lfloor n/3 \rfloor, \\ 3, & D > 2\lfloor n/3 \rfloor. \end{cases}$$

The optima differ precisely at $n = 3k + 2, D = 2k + 1, k \geq 1$; there are nine unrestricted optimal codes, in orbits of sizes three and six.

Proof. For $q = 3$ the cone condition is $n < 2D$, and $n \leq 3(n - D)$ is equivalent to $D \leq \lfloor 2n/3 \rfloor$. This proves the unrestricted formula.

For the full-coset construction let $k = \lfloor n/3 \rfloor$, $r = n - 3k$, and suppose $D \leq 2k$. Put

$$H = \{z \in \Lambda_2 : z_0 \equiv z_1 \equiv z_2 \pmod{3}\}, \quad L = kH, \quad t = (n, 0, 0).$$

The least nonzero half- ℓ_1 norm in H is two: a norm-one vector is a permutation of $(1, -1, 0)$, which fails the congruences, whereas $(1, 1, -2)$ has norm two. The full intersection therefore has minimum distance at least $2k$. It contains the four words

$$(n, 0, 0), \quad (r, 3k, 0), \quad (r, 0, 3k), \quad (r + k, k, k),$$

whose differences from t are k times vectors in H . The unrestricted upper bound forces the intersection to have exactly four words.

The only possible remaining four-word parameters satisfy $2\lfloor n/3 \rfloor < D \leq \lfloor 2n/3 \rfloor$; checking n modulo three gives exactly $n = 3k + 2, D = 2k + 1$. The strict inequality $n < 2D$ forces $k \geq 1$. Theorem 3, with $s = k + 1$, proves the full-coset upper bound and classification. The vertices always give a full-coset code of size three, completing all cases. $\square$

For instance, at $(n, D) = (5, 3)$ the vertices together with $(1, 2, 2)$ form an optimal unrestricted code. Any coset containing $(1, 2, 2)$ and the last two vertices contains $(3, 1, 1)$, at distance two from $(1, 2, 2)$. The first gap parameters are $(5, 3), (8, 5), (11, 7), (14, 9)$. The result concerns a finite boundary with distance proportional to length, not a fixed-radius asymptotic density advantage.

5 Eight words in the next ternary band

Theorem 5. *For $2D \leq n < 3D$, one has $A(n, D) \leq 8$, with equality if and only if $3n \geq 8D$. At $n = 8k, D = 3k, k \geq 1$, the only optimal codes are the three coordinate-permutation images of k times*

$$E = \{(8, 0, 0), (0, 8, 0), (0, 0, 8), (5, 3, 0), \\ (5, 0, 3), (2, 5, 1), (2, 1, 5), (0, 4, 4)\}.$$

At these parameters $A_{\text{lin}}(8k, 3k) = 7$.

5.1 Partition and upper bound

Put $r = n - 2D$, so $0 \leq r < D$. Partition words by the triple of integer parts $\lfloor x_i/D \rfloor$. Its sum is zero, one, or two. The ten cells are O (all coordinates below D), A_i (coordinate i at least $2D$), B_{ij} (coordinates i, j at least D), and C_i (coordinate i in $[D, 2D)$, the others below D). Every cell has diameter less than D by (2), and hence holds at most one word.

If $o \in O$, every $o_i > r$, since its other two coordinates total less than $2D$. In B_{ij} the two large coordinates are at most $D + r$ and the remaining one at most r . Thus o is at distance less than D from every word of every B cell. A code containing O has at most seven words.

For a word in C_i write c_{ij} for its j th coordinate. Two simple consequences of separation will be used:

$$A_i, C_i \text{ occupied} \implies \sum_{j \neq i} c_{ij} \geq D; \quad (4)$$

$$B_{ij}, C_i \text{ occupied} \implies c_{ij} \leq (B_{ij})_j - D. \quad (5)$$

In (4), separation is possible only in coordinate i , giving $c_{ii} \leq n - D$. In (5), it is possible only in coordinate j , since coordinate i is in $[D, 2D)$ for both words and the third coordinate is below D . If C_j also occurs, summing the two resulting inequalities gives $c_{ij} + c_{ji} \leq r$. If only C_i occurs, it still gives $c_{ij} \leq r$.

Nine words would occupy every A, B, C cell. Summing (4) would make the total off-diagonal mass at least $3D$, while summing the three paired consequences of (5) makes it at most $3r$, impossible. Thus eight is an upper bound.

5.2 The sharp threshold

An eight-word code occupies eight of the nine A, B, C cells. There are three types of missing cell.

If A_0 is missing, the off-diagonal masses in rows C_1, C_2 are at least D , and the mass in row C_0 exceeds r because $c_{00} < 2D$. The three B constraints bound their sum by $3r$, implying $2D + r < 3r$, impossible.

If B_{12} is missing, all A, C cells occur. The remaining B constraints and (4) yield

$$c_{10} + c_{20} \leq 2r - (c_{01} + c_{02}) \leq 2r - D.$$

Separation of C_1, C_2 is possible only in coordinate one or two. After exchanging these coordinates, suppose $c_{11} - c_{21} \geq D$. The sum of row C_1 then gives $c_{10} + c_{12} + c_{21} \leq D + r$. Since $c_{10} + c_{12} \geq D$, we obtain $c_{21} \leq r$. Since $c_{20} + c_{21} \geq D$, we obtain $c_{20} \geq D - r$. Consequently $2r - D \geq D - r$, or $3r \geq 2D$. Equality would force $c_{10} = 0$ and then $c_{12} \geq D$, contradicting its cell definition. This case requires strict inequality.

If C_0 is missing, the remaining two rows have off-diagonal mass at least D , while the three B constraints give

$$c_{10} \leq r, \quad c_{20} \leq r, \quad c_{12} + c_{21} \leq r.$$

Their sum gives $2D \leq 3r$. We have proved necessity of $3n \geq 8D$.

Conversely, if $2D/3 \leq r < D$, take the vertices and the five words

$$\begin{aligned} (D + r, D, 0), \quad (D + r, 0, D), \\ (r, D + r, D - r), \quad (r, D - r, D + r), \quad (0, 2D - r, 2r). \end{aligned}$$

Every entry is nonnegative, all sums are n , and each coordinate is at most $n - D$, ensuring separation from the vertices. The first pair differ by D ; the second pair by $2r \geq D$; every word of the first pair

differs from every word of the second by D in coordinate zero. The final word differs from the first pair by $D + r$ in coordinate zero. Against the second pair, respectively, it has coordinate differences $3r - D \geq D$ and D . This proves sufficiency.

5.3 Rigidity and the full-coset gap

At $n = 8k, D = 3k, r = 2k$, only the missing- C case is possible. Suppose C_0 is missing. Equality in all three constraints and in the two row bounds forces

$$C_1 = (2k, 5k, k), \quad C_2 = (2k, k, 5k).$$

For example, $c_{10} = c_{20} = 2k$ and $c_{12} + c_{21} = 2k$, while each remaining row sum is at least $3k$; hence $c_{12} = c_{21} = k$. The individual B constraints force

$$B_{01} = (5k, 3k, 0), \quad B_{02} = (5k, 0, 3k), \quad B_{12} = (0, 4k, 4k).$$

Indeed the relevant large coordinates have lower bounds summing to $8k$, leaving no slack. Separation of A_1 from C_1 , and of A_2 from C_2 , forces their distinguished coordinates to be $8k$. If A_0 were not the remaining vertex, separation from B_{01} would force coordinate one to vanish: coordinate zero cannot differ by $3k$, and coordinate two is below $3k$ for A_0 . Similarly B_{02} would force coordinate two to vanish, a contradiction. Thus all three A words are vertices. The three choices of the missing C cell yield exactly the stated three codes.

Any coset containing kE contains the integer affine combination

$$p = (8k, 0, 0) + (0, 8k, 0) - (5k, 3k, 0) = (3k, 5k, 0).$$

It lies in $S(8k, 3)$ and is at distance $k < 3k$ from $(2k, 5k, k)$. Permuting coordinates excludes every optimal eight-word code from being a full coset.

To attain seven, let

$$L_0 = \{z \in \Lambda_2 : 2z_0 + 3z_1 \equiv 0 \pmod{8}\}, \quad L = kL_0, \quad t = (0, 0, 8k).$$

The least nonzero norm in L_0 is three. To exclude norms one and two, enumerate $z_0 \in \{-2, -1, 0, 1, 2\}$ and $z_1 \in [-2, 2] \cap [-z_0 - 2, -z_0 + 2]$; only $(0, 0)$ satisfies the congruence. The vector $(1, 2, -3)$ attains three. Dividing coordinates by k , the full intersection is exactly k times

$$\{(0, 0, 8), (0, 8, 0), (1, 2, 5), (2, 4, 2), (4, 0, 4), (5, 2, 1), (8, 0, 0)\}.$$

This list follows by checking $x, y \geq 0$, $x + y \leq 8$, and $2x + 3y \equiv 0 \pmod{8}$. It gives the desired seven-word full-coset construction and completes the proof.

6 A finite four-coordinate certificate

Theorem 6 (Exact finite computation). *At $(q, n, D) = (4, 5, 3)$,*

$$A_4(5, 3) = 6, \quad A_{4,\text{lin}}(5, 3) = 5.$$

There are 288 unrestricted optimal codes, in 14 coordinate-permutation orbits.

The proof is an exhaustive finite certificate, rather than an all-parameter argument. The supplied dependency-free program enumerates the compatibility graph on the $\binom{8}{3} = 56$ words, with adjacency defined by half- ℓ_1 distance at least three. Exact maximum-clique search with integer greedy-color bounds gives maximum six and enumerates all 288 six-cliques. Permuting four coordinates partitions them into 14 orbits. For each representative, the certificate records its generated affine subgroup and a forbidden pair in its full simplex intersection.

For clarity, the subgroup-membership test has no numerical tolerance. Project onto the first three coordinates, choose a codeword as base, and let G contain the integral difference generators. Every representative has rank three. The index of its column lattice in $\mathbb{Z}^3$ is the positive gcd of its 3×3 minors. A vector belongs to that lattice exactly when adjoining it leaves the gcd unchanged: the enlarged lattice contains the original, and equal finite index forces equality. Testing all 56 simplex points gives the entire generated intersection. A forbidden pair there excludes every larger containing affine subgroup as well. The recorded per-orbit witnesses therefore prove the full-coset upper bound five, while the clique computation proves the unrestricted upper bound six.

For explicit lower bounds, a six-word code is

$$\{(5, 0, 0, 0), (2, 2, 1, 0), (1, 0, 2, 2), \\ (0, 5, 0, 0), (0, 1, 4, 0), (0, 2, 0, 3)\}.$$

A five-word full intersection is specified by $x_0 - 3x_1 - 5x_2 \equiv 0 \pmod{15}$ in $S(5, 4)$; its words are

$$\{(0, 0, 0, 5), (0, 0, 3, 2), (0, 5, 0, 0), \\ (1, 2, 2, 0), (3, 1, 0, 1)\}.$$

Enumeration verifies this entire intersection and its minimum distance three.

The ternary full-coset threshold has no direct four-coordinate analogue $D \leq 3\lfloor n/4 \rfloor$. At $(n, D) = (6, 4)$, the first three coordinates generated by columns of

$$\begin{pmatrix} 4 & 0 & 2 \\ 0 & 4 & 2 \\ 0 & 0 & 2 \end{pmatrix}$$

with base $(0, 0, 0, 6)$ give a full five-word code. At $(11, 7)$, the congruence $x_0 - 110x_1 - 124x_2 \equiv 0 \pmod{163}$ gives a full five-word code, despite $\max(3\lfloor 11/4 \rfloor, 2\lfloor 11/3 \rfloor) = 6 < 7$. The accompanying checker verifies both full intersections. These examples limit simple extensions; they do not claim a complete four-coordinate full-coset classification.

7 Prior work and limits of originality

Kovačević and Tan [1] provide the multiset-code framework, full lattice-intersection definition, finite bounds, and fixed-radius asymptotics. Their Corollary 16 already gives asymptotic unrestricted/linear equality for three-symbol codes at every fixed error radius. Our finite gaps have distance proportional to length and establish no infinite-lattice density improvement. Their Theorem 13 gives a Sidon-type converse under an interior-point hypothesis $t_i \geq h$, where $h = D - 1$. In our general gap family,

$$qh - n = q(q - 2)s - 2q + 1 \geq 1,$$

so no point of weight n satisfies that hypothesis. The finite affine-combination argument avoids presuming such an interior point.

The heavy-coordinate method is closely related to [2, Theorem 10] and [3, Theorems III.1 and III.3], as explained above. In [3], the number of coordinates corresponds to our q , constant weight to our n , and their ordinary ℓ_1 distance to twice our distance. Their bounded ternary-coordinate-alphabet results concern entries in $\{0, 1, 2\}$, rather than three unrestricted multiplicities. Perfect-code classification [4] addresses tilings and does not itself resolve maximum size for arbitrary finite full-coset intersections.

Chen, He, and Ge [5] is a materially close source for the unrestricted ternary statements: its constant- ℓ_1 -weight, ℓ_∞ framework agrees with ours in three coordinates. Its bibliographic record was verified, but its full text was not obtained in the public-access routes checked. Neither the high-distance formula nor the eight-word threshold and rigidity statement has been cleared against all of its contents. For $q \geq 4$ the metrics differ, so formulas cannot be transferred without an additional argument. The source remains unresolved, neither cleared nor demonstrated to duplicate these results.

Older relative-distance and homothetic-triangle packing formulations [6] are also relevant. For $T = \{(u, v) : u, v \geq 0, u + v \leq 1\}$, the norm $\max(|u|, |v|, |u + v|)$ has unit ball $T - T$. The unrestricted ternary problem is a denominator- n grid version of that finite geometric packing problem. The continuous four-point threshold is $2/3$ in this norm, or $4/3$ in the classical relative-distance normalization. No claim of geometric originality follows from the elementary proof here.

No exact all-coset obstruction or complete extremizer classification matching the displayed families was located in the sources inspected. This is a bounded negative finding, not exhaustive novelty clearance. The note is offered for mathematical and bibliographic review without a priority claim or a claim of AI-model superiority.

8 Reproducibility and verification limits

The manuscript and reproducibility bundle are available at <https://github.com/dapyle11/finite-multiset-codes>.

All-parameter conclusions in Theorems 1, 3, 4, and 5 rest on the proofs above. The finite theorem rests on exhaustive integer computation. The companion bundle contains the editable L^AT_EX source, original ternary checkers, new general-alphabet and eight-word checks, and the finite four-coordinate certificates, with a README giving exact commands and dependency requirements.

The original generic half- ℓ_1 and determinant checker independently computes both finite ternary optima for all 90 high-distance pairs through $n = 18$, examining 17,391 four-word sets, of which 324 have valid generated full intersections. General-alphabet checks enumerate all cone cases through $n = 12, 10, 7, 5$ for $q = 3, 4, 5, 6$, respectively, and compare all extremizers in six small members of the general separation family. The eight-word checker enumerates every eight-clique at $(8k, 3k)$ through $k = 8$, checking the three-code classification and exact seven-word construction. A separate cell-based audit checks all 78 ternary-band cases with $1 \leq D \leq 12$. The four-coordinate certificate lists all 14 orbits and exact closure obstructions. Appendix A supplies exact rational and integer branch certificates for both additional quinary cases. These finite checks support the proofs but cannot establish arbitrary-parameter statements by themselves.

Generated-subgroup closure is sufficient to exclude all containing cosets, because every containing coset includes that closure. This is not an enumeration of infinitely many lattices. Exploratory optimizer output is not used as a proof of an optimum. Verification was performed by separate AI checks and exact programs; no formal proof assistant, specialist human verification, or peer review is claimed.

9 Contributions and attribution

Daniel A. Pyle is an uncredentialed independent researcher who initiated the project and requested research, verification, and preparation. Lyra, an AI assistant powered by OpenAI, performed the scientific work, including mathematical analysis and proof development, literature review, code, computational analysis, separate AI checks, and drafting. The separate checks were AI-based and should not be interpreted as independent human peer review. No specialized mathematical verification by Daniel is claimed. OpenAI did not sponsor, endorse, or officially author this research. This note and its repository are prepared for review; preparation does not imply publication or acceptance.

A Two certified quinary single-deletion cases

The following finite result complements the structural full-coset theorems. In the notation of [2], put $S_5(n, 1) = A_5(n, 2)$.

Theorem 7 (Computer-assisted). *One has $S_5(9, 1) = 143$ and $S_5(10, 1) = 201$.*

This verifies the length-nine and length-ten instances of Conjecture 21 in [2, Section 5.5], whose July 2026 version reports computations through length eight. It is a finite-case extension, not a resolution of the conjecture or exhaustive priority clearance.

A.1 Valid inequalities and exact covers

Write $V_n = S(n, 5)$. The conflict graph joins two words of ordinary ℓ_1 distance two; a single-deletion code is an independent set. Two types of cliques are

$$U(a) = \{a + e_i : 0 \leq i < 5\} \quad (a \in V_{n-1}), \quad F(b) = \{b - e_i : b_i > 0\} \quad (b \in V_{n+1}).$$

We write F for the downward clique to avoid confusing it with minimum distance D . Each clique contains at most one codeword. For $a \in V_{n-k}$, the translate $T_k(a) = a + V_k$ contains at most $S_5(k, 1)$ codewords, since translation preserves distance.

Assign nonnegative rational weights w_R to finitely many such sets R , each with a valid cardinality bound b_R . If every vertex has coverage $\sum_{R \ni x} w_R \geq 1$, then every code satisfies

$$|C| \leq \sum_{x \in C} \sum_{R \ni x} w_R = \sum_R w_R |C \cap R| \leq \sum_R w_R b_R. \quad (6)$$

Orbit-symmetric weights do not presume that the unknown code is symmetric: each individual inequality remains valid and the certificate expands every orbit.

A.2 The prerequisite bounds

First, $S_5(3, 1) \leq 7$. Give weight one to each $U(a)$ with a a permutation of $(0, 0, 0, 0, 2)$, and weight $1/2$ to each $F(b)$ with b a permutation of $(0, 1, 1, 1, 1)$. A word of type $(3, 0, 0, 0, 0)$ or $(2, 1, 0, 0, 0)$ belongs to a U clique; a word of type $(1, 1, 1, 0, 0)$ belongs to two F cliques. Thus every vertex is covered, the total is $5 + 5/2 = 15/2$, and integrality gives seven.

Second, $S_5(6, 1) \leq 42$ is established by the supplied complete binary certificate tree on the 210-vertex conflict graph. Its 1,637 nodes comprise 818 branches and 819 leaves. Each internal node selects an active vertex v . The exclusion branch removes v ; the inclusion branch selects v and

removes it and its neighbors. These two cases exhaust all independent sets. At a leaf, nonnegative integer clique weights have denominator $M = 10,000,000$, and cover each remaining active vertex to total at least M . If a vertices have already been selected, the verified inequality is

$$\sum_j w_j + aM < 43M.$$

Consequently every leaf permits at most 42 total vertices. The verifier reconstructs the graph from pairwise distances, checks every clique, the complete tree, all cover inequalities, and strict integer bounds. The smallest positive scaled slack is 288; exact arithmetic, rather than a numerical tolerance, makes that slack conclusive. No solver status is a premise of this proof.

A.3 The length-ten certificate and construction

Table 1 gives the complete 28-orbit certificate. For each representative, include every distinct coordinate permutation of the corresponding base and assign it the displayed numerator divided by 1380. The bounds are one for U, F , seven for T_3 , and 42 for T_6 . Expanded, these are 835 valid inequalities. Exact verification over all $\binom{14}{4} = 1001$ vertices gives coverage at least one and total weighted right-hand side exactly 201. Equation (6) proves the upper bound.

The congruence code

$$C = \{x \in V_{10} : x_1 + 2x_2 + 3x_3 + 4x_4 \equiv 0 \pmod{5}\}$$

has 201 words by direct enumeration. Two conflicting words differ by $e_i - e_j$ for distinct indices, which changes their syndrome by the nonzero residue $i - j$. Thus the entire syndrome class is a distance-two code. The verifier also checks all 20,100 pairs directly, proving the lower bound and equality.

At length nine, a second exact binary tree proves the upper bound 143. The graph has 715 vertices and 4,950 edges. Its constraints are 1,491 cliques, 210 translates of V_3 with bound seven, and 35 translates of V_6 with bound 42. The tree has 221 nodes, comprising 110 branches and 111 leaves. At a leaf with selected set Q and active set W , a constraint R has residual bound $b_R - |Q \cap R|$. Nonnegative integer weights cover every active vertex to at least $M = 10,000,000$, and the verifier checks

$$|Q|M + \sum_R w_R(b_R - |Q \cap R|) < 144M.$$

This proves at most 143 words in every leaf. The minimum scaled slack is 6,965, and all checks use exact integers. The zero-syndrome class at length nine contains 143 words; all 10,153 pairs are checked, proving equality. The earlier rational cover giving upper bound 144 remains an optional weaker certificate, not the final result.

A.4 Necessary conditions for equality

The independently expanded certificate has coverage exactly one except on permutations of $(0, 0, 0, 1, 9)$, with coverage $55/46$, and of $(0, 1, 3, 3, 3)$, with coverage $231/230$. If $|C| = 201$, both inequalities in (6) must be equalities. Such a code therefore omits those 40 excess-coverage vertices. Every positive-weight constraint is tight; in particular, each $T_6(4e_i)$ contains exactly 42 codewords, equivalently exactly 42 words satisfy $x_i \geq 4$. These are necessary conditions, not a uniqueness or sufficiency claim.

Table 1: Length-ten cover. Each weight is its numerator divided by 1380.

Family	Representative	Numerator	Orbit size
U	(0, 0, 0, 0, 9)	564	5
U	(0, 0, 0, 1, 8)	270	20
U	(0, 0, 0, 2, 7)	72	20
U	(0, 0, 0, 3, 6)	222	20
U	(0, 0, 1, 1, 7)	24	30
U	(0, 0, 1, 2, 6)	246	60
U	(0, 0, 1, 3, 5)	222	60
U	(0, 0, 2, 3, 4)	198	60
U	(0, 0, 3, 3, 3)	396	10
U	(0, 1, 1, 2, 5)	144	60
U	(0, 1, 1, 3, 4)	100	60
U	(0, 1, 2, 2, 4)	202	60
U	(0, 1, 2, 3, 3)	304	60
U	(0, 2, 2, 2, 3)	279	20
U	(1, 1, 1, 1, 5)	64	5
U	(1, 1, 1, 3, 3)	58	10
F	(1, 1, 1, 1, 7)	492	5
F	(1, 1, 1, 2, 6)	101	20
F	(1, 1, 1, 4, 4)	242	10
F	(1, 1, 2, 3, 4)	56	60
F	(1, 2, 2, 3, 3)	64	30
T_3	(0, 0, 0, 0, 7)	396	5
T_3	(0, 0, 0, 2, 5)	222	20
T_3	(0, 0, 0, 3, 4)	48	20
T_3	(0, 0, 2, 2, 3)	24	30
T_3	(0, 1, 1, 2, 3)	1	60
T_3	(1, 1, 1, 2, 2)	138	10
T_6	(0, 0, 0, 0, 4)	420	5

A.5 Certificate verification

The bundle contains the prerequisite orbit certificate, compressed length-six and length-nine trees, final length-ten translated-simplex certificate, optional weaker length-nine cover, and two verifiers. The independent verifier imports no search code or numerical solver; it regenerates compositions, builds metric edges directly, expands every constraint orbit, and uses arbitrary-precision integers and rational fractions. Its checks remain active under Python optimization. Negative-control tests reject corrupted denominators, repeated or unreachable tree nodes, missing leaf coverage, and deleted orbit weights. Search software used floating-point optimization to discover candidate certificates, but only the checked exact data are premises of the result.

References

- [1] M. Kovačević and V. Y. F. Tan, “Codes in the Space of Multisets—Coding for Permutation Channels with Impairments,” *IEEE Transactions on Information Theory* 64(7), 5156–5169 (2018). doi:10.1109/TIT.2017.2789292. <https://arxiv.org/abs/1612.08837>.
- [2] A. Kreindel, I. B. Essayag, and A. L. Zabokritskiy (Yohananov), “Multiset Deletion Codes: Cyclic Constructions, Bounds, and Exact Results,” arXiv:2601.05636v2 (2026; revised 13 July). <https://arxiv.org/abs/2601.05636v2>.
- [3] T. Chen, Y. Ma, and X. Zhang, “Optimal Codes with Small Constant Weight in ℓ_1 -Metric,” arXiv:2003.00483v2 (2020; revised 9 October). <https://arxiv.org/abs/2003.00483v2>.
- [4] M. Kovačević and D. Vukobratović, “Perfect Codes in the Discrete Simplex,” *Designs, Codes and Cryptography* 75(1), 81–95 (2015). doi:10.1007/s10623-013-9893-5. <https://arxiv.org/abs/1307.3142>.
- [5] T. Chen, Z. He, and G. Ge, “Constant ℓ_1 -Weight Codes Under ℓ_∞ -Metric,” *IEEE Transactions on Information Theory* 70(7), 4928–4945 (2024). doi:10.1109/TIT.2024.3385419. Full-text comparison unresolved.
- [6] Z. Lángi, *Relative Distance of Points of a Convex Body*, doctoral dissertation, ELTE TTK, Mathematics Doctoral School (2003), Chapter 2 and Theorem 6. <https://www.math.u-szeged.hu/~zlangi/publications/dissertation.pdf>.