

ADELIC GALOIS REPRESENTATIONS ARE MAXIMAL FOR ALMOST ALL DRINFELD MODULES

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ABSTRACT. Let $r \geq 3$ be an integer, let q be an odd prime power, and set $A = \mathbb{F}_q[T]$ and $F = \mathbb{F}_q(T)$. We study the Galois representations associated with rank r Drinfeld modules of the form $\phi_T = T + g_1\tau + \cdots + g_r\tau^r$, where $g_i \in A$ and $g_r \neq 0$. Ordering the coefficient tuples $\vec{g} = (g_1, \dots, g_r)$ by their maximum degree, we prove that, for a density-one set of tuples, the λ -adic Galois representation is surjective for every prime λ of A , and the adelic Galois image contains $\mathrm{SL}_r(\hat{A})$. Moreover, for every tuple in this set, the index of the adelic image equals the index of the corresponding rank-one determinant representation and hence divides $q - 1$. These results extend Zywina's theorem in rank 2 to arbitrary ranks $r \geq 3$. As a further consequence, we establish the existence of rank r Drinfeld modules with surjective adelic Galois representations for every $r \geq 3$, extending previous existence results which were known for prime values of r .

1. INTRODUCTION

1.1. Background and motivation. The torsion of a Drinfeld module carries commuting actions of its coefficient ring and of the absolute Galois group. These actions give a compatible family of representations, one for each nonzero ideal of the coefficient ring. These images reflect information about the Galois groups of the division fields and the entanglement relations among them. It is expected, however not known that the adelic Galois images are maximal for many Drinfeld modules of a given rank r over a global field. These Galois representations are positive characteristic analogues of the characteristic zero representations that arise from elliptic curves and abelian varieties. More broadly, Drinfeld modules are closely related to the theory of Drinfeld shtukas, which played a central role in the proofs of the global Langlands correspondence over function fields, see [Dri77, Laf02].

In this article, we specialize to rational function fields over finite fields. Let q be an odd prime power and set $A = \mathbb{F}_q[T]$ and $F = \mathbb{F}_q(T)$. A Drinfeld module of rank r over F is determined by an expression

$$\phi_T = T + g_1\tau + \cdots + g_r\tau^r, \quad (1.1)$$

where $g_r \neq 0$ and τ acts on an element x by $x \mapsto x^q$. For a nonzero ideal $\mathfrak{a} \subset A$, the $\mathfrak{a}$ -torsion is free of rank r over $A/\mathfrak{a}$. This gives a homomorphism $\rho_{\phi, \mathfrak{a}} : G_F \rightarrow \mathrm{GL}_r(A/\mathfrak{a})$, where $G_F = \mathrm{Gal}(F^{\mathrm{sep}}/F)$. Passing to compatible inverse limits gives

$$\hat{\rho}_{\phi} : G_F \longrightarrow \mathrm{GL}_r(\hat{A}),$$

where $\hat{A} = \prod_{\lambda \in \Omega_A} A_{\lambda}$ and Ω_A is the set of nonzero prime ideals of A and A_{λ} is the completion of A at λ .

Pink and Rüttsche proved that this adelic image is open if its geometric endomorphism ring is A , cf. [PR09a]. It is natural to study the maximal possible image as one varies over all Drinfeld modules over F of a given rank r . The analogous questions for elliptic curves

2020 *Mathematics Subject Classification.* Primary 11G09, 11F80; Secondary 11R45, 11N36.

Key words and phrases. Drinfeld modules, adelic Galois representations, density results.

over number fields have been studied extensively from a statistical perspective. Given an elliptic curve E over $\mathbb{Q}$, a prime ℓ is said to be exceptional if the Galois representation

$$\rho_{E,\ell} : G_{\mathbb{Q}} \rightarrow \mathrm{GL}_2(\mathbb{F}_{\ell})$$

associated to $E[\ell]$ is not surjective. Serre's open image theorem implies that a non-CM elliptic curve has only finitely many exceptional primes. A striking result of Duke [Duk97] establishes that almost all elliptic curves over $\mathbb{Q}$, when ordered by height, have no exceptional primes. Jones [Jon10] subsequently strengthened this result by proving that almost all elliptic curves over $\mathbb{Q}$ are Serre curves, meaning that the image of their adelic Galois representation has index 2 in $\mathrm{GL}_2(\widehat{\mathbb{Z}})$. This is the smallest possible index for an elliptic curve over $\mathbb{Q}$, owing to an unavoidable relation between its division fields.

Similar questions have been investigated for higher-dimensional abelian varieties. Landesman, Swaminathan, Tao and Xu [LSTX19] proved such a result for non-isotrivial rational families of abelian varieties with big monodromy. They showed that the adelic Galois image is as large as permitted by the arithmetic and geometric constraints of the family for a density-one set of specializations. Their results apply to families of Jacobians of hyperelliptic, trigonal and plane curves, and imply, in particular, that for every $g \geq 3$ there exist infinitely many abelian varieties over $\mathbb{Q}$ whose adelic Galois image is $\mathrm{GSp}_{2g}(\widehat{\mathbb{Z}})$. Zywinia [Zyw23] subsequently generalized these results. For an abelian scheme over a nonempty open subset of projective space over a number field, he proved that the adelic Galois image of a density-one set of specializations has uniformly bounded index in the monodromy group of the family.

In the setting of Drinfeld modules, the author [Ray26] established an analogue of Duke's result for the T -adic Galois representation. More precisely, let $A = \mathbb{F}_q[T]$ and $F = \mathbb{F}_q(T)$, and consider rank r Drinfeld modules of the form

$$\phi_T = T + g_1\tau + \cdots + g_r\tau^r,$$

where $g_i \in A$ and $g_r \neq 0$. For every integer $r \geq 2$, it was shown that the T -adic Galois representation

$$\widehat{\rho}_{\phi,T} : G_F \longrightarrow \mathrm{GL}_r(\mathbb{F}_q[[T]])$$

is surjective for a density-one set of coefficient tuples $(g_1, \dots, g_r)$, ordered by their maximum degree. Thus, for almost all Drinfeld modules in this family, the Galois group of the T^n -division field is $\mathrm{GL}_r(A/(T^n))$ for every $n \geq 1$. The proof combines Hilbert irreducibility over function fields, Drinfeld–Tate uniformization, and sieve methods. However, this result concerns only the prime (T) and does not address the Galois representations at other primes of A , or the possible entanglement relations between their division fields.

More recently, Zywinia [Zyw25] obtained an analogue of the result of Jones for rank 2 Drinfeld modules. For $A = \mathbb{F}_q[T]$, he considered the family

$$\phi_T = T + g_1\tau + g_2\tau^2, \quad \text{where } g_1, g_2 \in A, \quad \text{and } g_2 \neq 0,$$

with the coefficients ordered by degree. He proved that, when $q > 2$, the index of the adelic Galois image in $\mathrm{GL}_2(\widehat{A})$ divides $q - 1$ for a density-one set of coefficient pairs. When $q = 2$, the corresponding index divides 4.

There has also been considerable interest in constructing individual Drinfeld modules with surjective adelic Galois representations. In an earlier paper, Zywinia [Zyw11] exhibited an explicit example of rank 2, namely

$$\phi_T = T + \tau - T^{q-1}\tau^2,$$

and proved that its adelic Galois representation is surjective whenever $q \geq 5$ is an odd prime power. Chen [Che25] extended this construction to prime ranks. More precisely,

for an odd prime r , he considered the Drinfeld module

$$\phi_T = T + \tau^{r-1} + T^{q-1}\tau^r$$

and established adelic surjectivity under suitable hypotheses on q , depending on r . These constructions provide examples with maximal adelic Galois image, but the restriction to prime rank in Chen's argument leaves open the corresponding existence question for arbitrary composite ranks. The results of the present paper address this question and, more generally, extend Zywin's density theorem to every rank $r \geq 3$.

1.2. The main results. Throughout the paper, unless a different hypothesis is stated, $r \geq 3$ is an integer and $q = p^e$ is odd. Write $A_{\leq n} = \{a \in A : \deg a \leq n\}$, with 0 included, and set

$$\mathcal{B}_n = A_{\leq n}^{-1} \times (A_{\leq n} \setminus \{0\}).$$

For a subset S of the coefficient family, denote by

$$\underline{\delta}(S) = \liminf_{n \rightarrow \infty} \frac{|S \cap \mathcal{B}_n|}{|\mathcal{B}_n|} \quad \text{and} \quad \bar{\delta}(S) = \limsup_{n \rightarrow \infty} \frac{|S \cap \mathcal{B}_n|}{|\mathcal{B}_n|}$$

the lower and upper densities respectively. When the two values agree, we denote their common value by $\delta(S)$.

For a coefficient tuple $\mathbf{g} = (g_1, \dots, g_r) \in A^r$ with $g_r \neq 0$, let $\phi^{\mathbf{g}}$ denote the associated Drinfeld module. For each nonzero prime ideal λ of A , the action of $G_F = \text{Gal}(F^{\text{sep}}/F)$ on the λ -adic Tate module of $\phi^{\mathbf{g}}$ gives a representation

$$\rho_{\mathbf{g}, \lambda} : G_F \longrightarrow \text{GL}_r(A_\lambda),$$

where A_λ is the completion of A at λ . Taking the product over all nonzero primes λ gives the adelic Galois representation

$$\rho_{\mathbf{g}} : G_F \longrightarrow \text{GL}_r(\hat{A}).$$

We denote its image by

$$G_{\mathbf{g}} = \rho_{\mathbf{g}}(G_F) \subseteq \text{GL}_r(\hat{A}).$$

For a nonzero polynomial $f \in A$, we write $\text{lc}(f) \in \mathbb{F}_q^\times$ for its leading coefficient.

Theorem A. *There is a set $\mathcal{M}$ of coefficient tuples of density one such that, for $\mathbf{g} \in \mathcal{M}$, the following statements hold:*

- (i) $\hat{\rho}_{\mathbf{g}, \lambda}(G_F) = \text{GL}_r(A_\lambda)$ for every $\lambda \in \Omega_A$;
- (ii) the adelic image $G_{\mathbf{g}}$ contains $\text{SL}_r(\hat{A})$, and its commutator subgroup equals $\text{SL}_r(\hat{A})$;
- (iii) if $d = \deg g_r$ and $e_r(g_r)$ is the order of $(-1)^{d+r-1} \text{lc}(g_r)$ in $\mathbb{F}_q^\times$, then

$$[\text{GL}_r(\hat{A}) : G_{\mathbf{g}}] = \gcd\left(d-1, \frac{q-1}{e_r(g_r)}\right). \quad (1.2)$$

In particular, the adelic index divides $q-1$ on $\mathcal{M}$.

For $d=0$, the greatest common divisor in (1.2) is interpreted using $\gcd(-1, b) = 1$.

Let $\mathcal{P}$ be the set of tuples for which $(-1)^{\deg g_r + r - 1} \text{lc}(g_r)$ generates $\mathbb{F}_q^\times$, and let $\mathcal{S}_{\text{ad}}$ be the set with full adelic image $\text{GL}_r(\hat{A})$.

Theorem B. *The set $\mathcal{P}$ has density $\varphi(q-1)/(q-1)$, and*

$$\lim_{n \rightarrow \infty} \frac{|\mathcal{S}_{\text{ad}} \cap \mathcal{P} \cap \mathcal{B}_n|}{|\mathcal{P} \cap \mathcal{B}_n|} = 1.$$

Consequently, $\underline{\delta}(\mathcal{S}_{\text{ad}}) \geq \varphi(q-1)/(q-1)$.

1.3. Method of proof. We begin by studying the Galois action at primes where the Drinfeld module has stable reduction of rank $r - 1$. More precisely, we consider primes $\mathfrak{p}$ of A for which

$$v_{\mathfrak{p}}(g_{r-1}) = 0 \quad \text{and} \quad v_{\mathfrak{p}}(g_r) = 1. \quad (1.3)$$

Here $v_{\mathfrak{p}}$ denotes the normalized valuation at $\mathfrak{p}$. Drinfeld–Tate uniformization gives a description of the module in terms of a Drinfeld module of smaller rank and a lattice. This allows us to study the inertia action on the division points. Under the conditions in (1.3), we show that the inertia image contains a large unipotent subgroup (Theorem 3.6).

The next step is to study the irreducibility of the residual representations. We show that the primes at which reducibility can occur have bounded degree, where the bound depends on the degrees of two auxiliary primes of good reduction (Theorem 7.3). These auxiliary primes can be chosen independently of the coefficient tuple in the families we consider. Once irreducibility is established, the unipotent subgroups obtained from inertia can be used to generate the special linear group (Proposition 7.7). Together with the determinant representation, this allows us to establish surjectivity of the individual λ -adic representations (Proposition 7.8).

The surjectivity of the individual λ -adic representations does not by itself imply that the adelic image is large. Indeed, there could be relations between the images at distinct primes, even when each projection is surjective. We address this problem by studying the inertia action simultaneously on division points of coprime orders. This allows us to show that the images at distinct primes are independent, in the sense that their joint image contains the product of the corresponding special linear groups (Proposition 8.2). Using the fact that the special linear groups are perfect, we extend this to any finite collection of primes. It follows that the adelic image contains $\mathrm{SL}_r(\hat{A})$ (Theorem 8.3). The remaining question is then to determine the determinant image, which accounts for the possible index in $\mathrm{GL}_r(\hat{A})$.

Finally, we show that the conditions needed in these arguments hold for a density-one set of coefficient tuples. The required conditions on stable reduction can be expressed in terms of congruences satisfied by the coefficients. We use sieve theoretic arguments to show that almost all tuples satisfy these conditions at sufficiently many primes (Proposition 9.3). The remaining primes are handled using a result of Zywinia from [Zyw25] (see Theorem 7.1). Combining these results, we obtain a density-one set of tuples for which the preceding arguments apply (Proposition 9.5).

Declarations.

Conflict of interest: Not applicable; there is no conflict of interest to report.

Availability of data and materials: No data was generated or analyzed in obtaining the results in this article.

Funding: No funding was received during the preparation of this article.

Use of artificial intelligence: The author used ChatGPT 6 Astra as an exploratory tool to study related papers and to assist in correcting mistakes in the article. It was not used to generate the mathematical ideas or the main results of the article.

2. BASIC NOTIONS

This section is devoted to setting up notation and discussing preliminaries. Let q be a prime power and put $A = \mathbb{F}_q[T]$ and $F = \mathbb{F}_q(T)$. A place of F is an equivalence class of valuations. These correspond to the monic irreducible polynomials in A as well as the place at ∞ . We identify a prime ideal λ with its monic generator. We then set

$$d_{\lambda} = \deg \lambda, \quad k_{\lambda} = A/\lambda, \quad Q_{\lambda} = |k_{\lambda}| = q^{d_{\lambda}} \quad \text{and} \quad A_{\lambda} = \varprojlim_n A/\lambda^n.$$

The fraction field of A_λ is F_λ , which is the completion of F at λ . Its valuation is normalized by $v_\lambda(\lambda) = 1$. We use the same symbol v_λ for the extension of this valuation to an algebraic closure $\overline{F}$, with values in $\mathbb{Q}$.

The remaining place is ∞ , and the valuation is normalized by $v_\infty(T) = -1$. The corresponding absolute value is given by $|a|_\infty = q^{\deg a}$ for $a \in A \setminus \{0\}$. For a nonzero ideal $\mathfrak{a} = (a)$ with a monic, set $\deg \mathfrak{a} = \deg a$ and $N \mathfrak{a} = q^{\deg a}$.

For a field K , let $G_K = \text{Gal}(K^{\text{sep}}/K)$. For a local field K , write K^{un} and K^t for its maximal unramified and maximal tamely ramified extensions in K^{sep} . Thus $I_K = \text{Gal}(K^{\text{sep}}/K^{\text{un}})$ is the inertia subgroup and the wild inertia group is $P_K = \text{Gal}(K^{\text{sep}}/K^t)$.

Let $q = p^e$, and let K be a field containing $\mathbb{F}_q$. A polynomial $P(X) \in K[X]$ is called *additive* if $P(X + Y) = P(X) + P(Y)$ in $K[X, Y]$. Equivalently, an additive polynomial is one of the form $P(X) = \sum_i a_i X^{p^i}$, with $a_i \in K$. An additive polynomial is called $\mathbb{F}_q$ -*linear* if $P(cX) = cP(X)$ for every $c \in \mathbb{F}_q$. Equivalently, the $\mathbb{F}_q$ -linear polynomials are precisely those of the form $P(X) = \sum_i a_i X^{q^i}$, where only finitely many coefficients are nonzero.

The ring $K\{\tau\}$ of twisted polynomials consists of finite sums $\sum_i a_i \tau^i$, with multiplication determined by $\tau c = c^q \tau$. Consequently,

$$(a\tau^i)(b\tau^j) = ab^{q^i} \tau^{i+j}.$$

The correspondence $\sum_i a_i \tau^i \mapsto \sum_i a_i X^{q^i}$ identifies multiplication with composition of additive polynomials. The coefficient a_0 equals the formal derivative of the corresponding polynomial. Thus an additive polynomial with $a_0 \neq 0$ is separable.

An A -field is a field K equipped with an $\mathbb{F}_q$ -algebra homomorphism $\gamma : A \rightarrow K$. A Drinfeld module of rank r is a homomorphism $\phi : A \rightarrow K\{\tau\}$ such that the constant coefficient of ϕ_a is $\gamma(a)$ and $\deg_\tau \phi_a = r \deg a$ for $a \neq 0$. Since A is a polynomial ring, it is enough to specify $\phi_T = \gamma(T) + g_1 \tau + \cdots + g_r \tau^r$ with $g_r \neq 0$. For $a = \sum_i c_i T^i$, one has that

$$\phi_a = \sum_i c_i \phi_T^i.$$

Its τ -degree is $r \deg a$.

A Drinfeld module is said to have generic characteristic if γ is injective. Otherwise $\ker \gamma$ is a finite prime, called its A -characteristic. Over F and its completions we always use the natural embedding $A \hookrightarrow F$. In generic characteristic every ϕ_a for $a \neq 0$ is separable.

Let ψ and ϕ be Drinfeld modules over the same A -field. A morphism $u : \phi \rightarrow \psi$ is a twisted polynomial satisfying $u\phi_a = \psi_a u$ for all $a \in A$. A nonzero morphism is an isogeny. The only units of $K\{\tau\}$ are $K^\times$. This is easy to see given that the τ -degrees of nonzero products add. Thus isomorphisms arise from scalars $c \in K^\times$. A Drinfeld module ϕ and a scalar c gives rise to an isomorphic Drinfeld module ψ , which is defined by

$$\psi_T = c^{-1} \phi_T c = T + \sum_{i=1}^r c^{q^i-1} g_i \tau^i. \quad (2.1)$$

The action $a \cdot x = \phi_a(x)$ equips K^{sep} with the structure of an A -module. If a is prime to the A -characteristic, define the a -torsion by

$$\phi[a] = \ker \phi_a \subset K^{\text{sep}}.$$

It is an $A/(a)$ -module. The number of its elements is $q^{r \deg a}$, since ϕ_a is separable of degree $q^{r \deg a}$.

Lemma 2.1. *If (a) is prime to the A -characteristic, then $\phi[a] \simeq (A/(a))^r$ as an A -module.*

Proof. This is [Pap23, Corollary 3.5.3]. $\square$

The Galois action commutes with A and gives rise to a homomorphism

$$\rho_{\phi, \mathfrak{a}} : G_K \rightarrow \mathrm{GL}_r(A/\mathfrak{a}).$$

The splitting field $K(\phi[\mathfrak{a}])$ is defined to be the fixed field of the kernel of the above homomorphism.

Let W be a finite $\mathbb{F}_q$ -subspace of a field L containing $\mathbb{F}_q$. We set

$$P_W(X) = \prod_{w \in W} (X - w) \quad \text{and} \quad E_W(X) = X \prod_{0 \neq w \in W} (1 - X/w).$$

Lemma 2.2. *The polynomials P_W and E_W are $\mathbb{F}_q$ -linear. Let $W \subseteq W'$ be finite $\mathbb{F}_q$ -subspaces. Then $E_W(W')$ is also an $\mathbb{F}_q$ -subspace and*

$$E_{W'} = E_{E_W(W')} \circ E_W. \quad (2.2)$$

Proof. We prove additivity of P_W by induction on $\dim_{\mathbb{F}_q} W$. For $W = 0$ the assertion is immediate. Write $W = W_0 \oplus \mathbb{F}_q v$ with $v \notin W_0$. By induction P_{W_0} is $\mathbb{F}_q$ -linear, and therefore

$$\begin{aligned} P_W(X) &= \prod_{c \in \mathbb{F}_q} P_{W_0}(X - cv) \\ &= \prod_{c \in \mathbb{F}_q} (P_{W_0}(X) - cP_{W_0}(v)) \\ &= P_{W_0}(X)^q - P_{W_0}(v)^{q-1} P_{W_0}(X). \end{aligned}$$

This is again $\mathbb{F}_q$ -linear. Differentiating with respect to X , we obtain

$$P'_W(X) = -P_{W_0}(v)^{q-1} P'_{W_0}(X),$$

since the derivative of $P_{W_0}(X)^q$ vanishes in characteristic p . As $v \notin W_0$, we have $P_{W_0}(v) \neq 0$. Moreover, $P'_{W_0}(X)$ is a nonzero constant by the induction hypothesis. It follows that $P'_W(X)$ is also a nonzero constant. The product formula gives $P'_W(0) = \prod_{0 \neq w \in W} (-w) \neq 0$, and $E_W = P_W/P'_W(0)$. This proves the first assertion.

For the second, $E_W(W')$ is an $\mathbb{F}_q$ -space of size $|W'|/|W|$, since the kernel of E_W on W' is W . The composition on the right of (2.2) has degree $|W'|$, derivative one, and root set exactly W' . Thus there exists a nonzero scalar c such that

$$E_{E_W(W')} \circ E_W = cE_{W'}.$$

By definition, $E_U(0) = 0$ and $E'_U(0) = 1$ for every finite $\mathbb{F}_q$ -subspace U . Differentiating the preceding identity at $X = 0$ and applying the chain rule gives

$$E'_{E_W(W')}(0)E'_W(0) = cE'_{W'}(0),$$

and hence $c = 1$. $\square$

Given a prime λ , ϕ_λ gives rise to a map $\phi[\lambda^{n+1}] \rightarrow \phi[\lambda^n]$ is ϕ_λ . It is easy to see that the map is surjective. Indeed, for $x \in \phi[\lambda^n]$, a root of $\phi_\lambda(Y) - x$ lies in K^{sep} and is killed by $\phi_{\lambda^{n+1}}$. Compatible bases may be chosen recursively by lifting a basis at each stage. Consider the Tate module

$$T_\lambda(\phi) = \varprojlim_n \phi[\lambda^n] \simeq A_\lambda^r.$$

The associated representation is denoted by $\widehat{\rho}_{\phi, \lambda} : G_K \rightarrow \mathrm{GL}_r(A_\lambda)$, and its residual representation by $\bar{\rho}_{\phi, \lambda} : G_K \rightarrow \mathrm{GL}_r(k_\lambda)$. The adelic Tate module is

$$T(\phi) := \prod_\lambda T_\lambda(\phi)$$

and $\widehat{\rho}_\phi : G_K \rightarrow \mathrm{GL}_r(\widehat{A})$ be the associated big Galois representation, where $\widehat{A} = \prod_\lambda A_\lambda$.

3. STABLE REDUCTION AND FULL TRANSLATION INERTIA

Let $K = F_{\mathfrak{p}}$, let $\mathcal{O}_K$ be its valuation ring, and write $v = v_{\mathfrak{p}}$ normalized by $v(\mathfrak{p}) = 1$. Denote by $\mathfrak{m}_K$ its maximal ideal. Let $\mathbb{C}_K = \widehat{\overline{K}}$ denote the completed algebraic closure of K and extend v to $\mathbb{C}_K$. In particular, if L/M is a finite Galois extension and M is algebraic over K , one has that

$$v(\mathrm{Norm}_{L/M}(x)) = [L : M]v(x) \quad (3.1)$$

for $x \in L^\times$.

An integral model of ϕ is an expression $\phi_T = T + \sum_i g_i \tau^i$ with $g_i \in \mathcal{O}_K$. It has stable reduction of rank $s \geq 1$ if g_s is a unit and g_i lies in the maximal ideal for $i > s$. Its reduction then has rank s . We say that ϕ has good reduction if $s = r$.

Lemma 3.1. *Every Drinfeld module over K acquires stable reduction after a tame extension whose ramification index divides $q^s - 1$ for some $1 \leq s \leq r$.*

Proof. Set

$$\mu = \min_{\substack{1 \leq i \leq r \\ g_i \neq 0}} \frac{v(g_i)}{q^i - 1},$$

and let s be the largest index for which the minimum is attained. Write $\mu = a/b$ in lowest terms, with $b \geq 1$. Since $\mu = v(g_s)/(q^s - 1)$ and $v(g_s) \in \mathbb{Z}$, we have $b \mid q^s - 1$. In particular, b is prime to p .

Let ϖ be a uniformizer of $K = F_{\mathfrak{p}}$, and choose u satisfying $u^b = \varpi$. Then $L = K(u)$ is a totally and tamely ramified extension of degree b . The element $c = u^{-a}$ satisfies $v(c) = -\mu$. The Drinfeld module $\widetilde{\phi}$ defined by $\widetilde{\phi}_a = c^{-1} \phi_a c$ for $a \in A$ is isomorphic to ϕ over L , via multiplication by c . Explicitly,

$$\widetilde{\phi}_T = T + \sum_{i=1}^r g_i c^{q^i - 1} \tau^i.$$

For every i with $g_i \neq 0$, its coefficient satisfies

$$v(g_i c^{q^i - 1}) = v(g_i) - (q^i - 1)\mu \geq 0,$$

with equality for $i = s$. Thus $\widetilde{\phi}$ has stable reduction. $\square$

Let $\psi : A \rightarrow K\{\tau\}$ be a Drinfeld module over K . Write $K^{\mathrm{sep}, \psi}$ for the additive group of K^{sep} equipped with the A -action $a \cdot x = \psi_a(x)$. This makes $K^{\mathrm{sep}, \psi}$ into an A -module. A ψ -lattice is a discrete, G_K -stable A -submodule of $K^{\mathrm{sep}, \psi}$ that is free of finite rank.

For a ψ -lattice Λ , its associated exponential $e_\Lambda : \mathbb{C}_K \rightarrow \mathbb{C}_K$ is defined by

$$e_\Lambda(X) = X \prod_{0 \neq \gamma \in \Lambda} (1 - X/\gamma).$$

Lemma 3.2. *The product $e_\Lambda(z)$ converges for every $z \in \mathbb{C}_K$ and defines an entire $\mathbb{F}_q$ -linear function $e_\Lambda : \mathbb{C}_K \rightarrow \mathbb{C}_K$ with kernel Λ . It has an expansion*

$$e_\Lambda(z) = z + \sum_{i \geq 1} c_i z^{q^i}$$

with coefficients $c_i \in K$ and is G_K -equivariant:

$$e_\Lambda(\sigma z) = \sigma(e_\Lambda(z))$$

with $\sigma \in G_K$ and $z \in \mathbb{C}_K$.

Proof. The convergence, $\mathbb{F}_q$ -linearity, and description of the kernel follow from the lattice-exponential construction in [Pap23, Proposition 5.1.3], whose argument applies over $\mathbb{C}_K$; see also [Pap23, Section 6.2].

Choose a finite Galois extension L/K containing an A -basis of Λ . Since ψ is defined over K , we have $\Lambda \subset L$, and completeness of L implies that the coefficients of e_Λ lie in L . The product formula and G_K -stability of Λ give the asserted equivariance and show that these coefficients are fixed by $\text{Gal}(L/K)$. They therefore belong to K . For further details, see [Pap23, Section 6.2]. $\square$

We use the following form of Drinfeld–Tate uniformization; see [Pap23, Section 6.2, especially Theorem 6.2.11].

Theorem 3.3 (Drinfeld–Tate uniformization). *Suppose an integral model of ϕ/K has stable reduction of rank s . There exist a Drinfeld module ψ/K of rank s with good reduction and a ψ -lattice Λ of rank $r - s$ such that*

$$e_\Lambda \circ \psi_a = \phi_a \circ e_\Lambda$$

for $a \in A$. For every nonzero $a \in A$, the exponential induces a G_K -equivariant isomorphism of $A/(a)$ -modules

$$\psi_a^{-1}(\Lambda)/\Lambda \xrightarrow{\sim} \phi[a], \quad (3.2)$$

defined by $[z] \mapsto e_\Lambda(z)$.

Proof. For further details, see for instance [Pap23, Section 6.2, especially Theorem 6.2.11]. $\square$

Lemma 3.4. *For every nonzero $a \in A$ there is a G_K -equivariant exact sequence of $A/(a)$ -modules*

$$0 \longrightarrow \psi[a] \longrightarrow \phi[a] \longrightarrow \Lambda/a\Lambda \longrightarrow 0. \quad (3.3)$$

Proof. We refer to [Pap23, Section 6.2] for this result. $\square$

Assume $s = r - 1$, $v(g_{r-1}) = 0$ and $v(g_r) = t > 0$. In this case, Λ has rank 1. Choose an A -generator γ of Λ . Since $\text{Aut}_A(\Lambda) = A^\times = \mathbb{F}_q^\times$, the Galois action has the form $\sigma\gamma = \eta(\sigma)\gamma$ for a character $\eta : G_K \rightarrow \mathbb{F}_q^\times$. Thus $K(\gamma)/K$ is Galois of degree dividing $q - 1$, and $\gamma \in K^t$.

Lemma 3.5. *One has*

$$v(\gamma) = -\frac{t}{q-1}. \quad (3.4)$$

For every nonconstant $a \in A$ and every z with $\psi_a(z) = \gamma$,

$$v(z) = -\frac{t}{(q-1)q^{(r-1)\deg a}}. \quad (3.5)$$

Proof. We first show that $v(\gamma) < 0$. By the integral Tate uniformization construction [Pap23, Section 6.2], we may choose the uniformizing exponential so that

$$e_\Lambda(X) \equiv X \pmod{\mathfrak{m}_K}$$

coefficientwise. Thus, we may write

$$e_\Lambda(X) = X + \sum_{i \geq 1} c_i X^{q^i},$$

we have $c_i \in \mathfrak{m}_K$ for every $i \geq 1$. If $v(\gamma) \geq 0$, then every nonzero higher term satisfies

$$v(c_i \gamma^{q^i}) \geq 1 + q^i v(\gamma) \geq 1 + v(\gamma).$$

Consequently, the sum of the higher terms has valuation strictly greater than $v(\gamma)$ and cannot cancel the linear term. This contradicts $e_\Lambda(\gamma) = 0$, so $v(\gamma) < 0$.

Since ψ has good reduction, it follows that for every $b \in A \setminus \{0\}$,

$$\psi_b(X) = \sum_{i=0}^{s \deg b} u_i X^{q^i},$$

where $u_i \in \mathcal{O}_K$ and $u_{s \deg b} \in \mathcal{O}_K^\times$. If $v(x) < 0$, then for every $i < s \deg b$ with $u_i \neq 0$,

$$v(u_i x^{q^i}) \geq q^i v(x) > q^{s \deg b} v(x).$$

The leading term therefore has strictly smaller valuation than every other nonzero term, and hence

$$v(\psi_b(x)) = q^{s \deg b} v(x). \quad (3.6)$$

Every nonzero $\delta \in \Lambda$ has the form $\delta = \psi_b(\gamma)$ for some $b \neq 0$. Thus

$$v(\delta) = q^{s \deg b} v(\gamma) \leq v(\gamma).$$

Choose $c \in \mathbb{F}_q$ such that $\mathfrak{p} \nmid T - c$, put $b = T - c$, and choose z_b satisfying $\psi_b(z_b) = \gamma$. Since ψ_b has integral coefficients and $v(\gamma) < 0$, we must have $v(z_b) < 0$. Applying (3.6) and using $\deg b = 1$, we obtain

$$v(z_b) = \frac{v(\gamma)}{q^s} > v(\gamma).$$

For every nonzero $\delta \in \Lambda$, it follows that

$$v(z_b/\delta) = v(z_b) - v(\delta) \geq v(z_b) - v(\gamma) > 0.$$

Each factor $1 - z_b/\delta$ in the product defining $e_\Lambda(z_b)$ therefore has valuation zero. The finite partial products have absolute value one, as does their limit. Consequently,

$$v(e_\Lambda(z_b)) = v(z_b) = \frac{v(\gamma)}{q^s} < 0.$$

In particular, $e_\Lambda(z_b) \neq 0$. We have that

$$\phi_b(e_\Lambda(z_b)) = e_\Lambda(\psi_b(z_b)) = e_\Lambda(\gamma) = 0.$$

Since $b = T - c$, we have

$$\phi_b(X) = bX + g_1 X^q + \cdots + g_s X^{q^s} + g_r X^{q^{s+1}}.$$

Here $v(b) = v(g_s) = 0$, $v(g_r) = t > 0$, and all coefficients are integral. Its Newton polygon consists of a horizontal segment from $(1, 0)$ to $(q^s, 0)$, followed by the segment from $(q^s, 0)$ to (q^{s+1}, t) . Thus every root of negative valuation has valuation

$$-\frac{t}{q^{s+1} - q^s};$$

see [Pap23, Section 2.5]. Applying this to $e_\Lambda(z_b)$ gives

$$\frac{v(\gamma)}{q^s} = -\frac{t}{q^{s+1} - q^s},$$

and therefore $v(\gamma) = -t/(q - 1)$.

Finally, let $a \in A$ be nonconstant and suppose $\psi_a(z) = \gamma$. Integrality of the coefficients again implies $v(z) < 0$, so (3.6) yields

$$q^{s \deg a} v(z) = v(\gamma) = -\frac{t}{q - 1}.$$

Since $s = r - 1$, this proves (3.5). $\square$

For a ring R , write

$$U(R) = \left\{ \begin{pmatrix} I_{r-1} & v \\ 0 & 1 \end{pmatrix} : v \in R^{r-1} \right\}.$$

Theorem 3.6. *Assume that ϕ has integral coefficients, that g_{r-1} is a unit, and that g_r has positive valuation t not divisible by p . If $\mathfrak{a}$ is prime to $\mathfrak{p}$, then, in a basis adapted to (3.3),*

$$\rho_{\phi, \mathfrak{a}}(P_K) = U(A/\mathfrak{a}).$$

In particular, for every nonzero prime $\lambda \neq \mathfrak{p}$, there is a compatible basis of $T_\lambda(\phi)$ with respect to which

$$\widehat{\rho}_{\phi, \lambda}(P_K) = U(A_\lambda).$$

Proof. Put $s = r - 1$, and let (ψ, Λ) be the Tate uniformization of ϕ , where ψ has good reduction of rank s and $\Lambda = A\gamma$ has rank one. Recall that the A -action on Λ is given by ψ , so that $a\Lambda = \psi_a(\Lambda)$. Let a be the monic generator of $\mathfrak{a}$. The finite-level assertion is trivial when $a = 1$, so assume that $\deg a > 0$; set $R = A/(a)$ and $N = q^{s \deg a}$. Since $\psi[a] \simeq R^s$, the group $\psi[a]$ has N elements.

Tate uniformization gives the G_K -equivariant exact sequence

$$0 \longrightarrow \psi[a] \xrightarrow{e_\Lambda} \phi[a] \xrightarrow{\eta_a} \Lambda/a\Lambda \longrightarrow 0;$$

see [Pap23, Section 6.2, pp. 355–356]. We first show that wild inertia fixes $\psi[a]$ and $\Lambda/a\Lambda$ pointwise. The quotient map is given by

$$\eta_a(e_\Lambda(x)) = \psi_a(x) \pmod{a\Lambda},$$

where $x \in \psi_a^{-1}(\Lambda)$. Replacing x by $x + \delta$, with $\delta \in \Lambda$, changes $\psi_a(x)$ by $\psi_a(\delta) \in a\Lambda$, so this formula is well defined.

Good reduction of ψ implies that ψ_a has integral coefficients and a unit leading coefficient. Moreover, $\psi'_a(X) = a$ is a unit because $\mathfrak{a}$ is prime to $\mathfrak{p}$. Its reduction therefore has degree N and N distinct roots over an algebraic closure of the residue field. Choose a finite unramified extension of K whose residue field contains these roots. Hensel's lemma lifts them to N distinct roots of ψ_a in that extension. Thus

$$\psi[a] \subset K^{\text{un}} \subset K^t,$$

and $P_K = \text{Gal}(K^{\text{sep}}/K^t)$ fixes $\psi[a]$ pointwise.

The action of G_K on the rank-one A -module Λ gives a continuous character

$$\chi_\Lambda : G_K \longrightarrow \text{Aut}_A(\Lambda) = A^\times = \mathbb{F}_q^\times.$$

Since P_K is a pro- p group and $q - 1$ is prime to p , the restriction of χ_Λ to P_K is trivial. Consequently, $\gamma \in K^t$, and P_K also fixes $\Lambda/a\Lambda$ pointwise.

Write $M = e_\Lambda(\psi[a]) \subset \phi[a]$. The preceding observations show that P_K fixes M pointwise and acts trivially on $\phi[a]/M$. Hence, in any basis adapted to this submodule, its image is contained in

$$U(R) = \left\{ \begin{pmatrix} I_s & u \\ 0 & 1 \end{pmatrix} : u \in R^s \right\}.$$

It remains to prove that every column $u \in R^s$ occurs.

Choose $z \in K^{\text{sep}}$ satisfying $\psi_a(z) = \gamma$, and set

$$L = K^t(z).$$

Since ψ_a is additive, its division equation has precisely the roots

$$\{z + w : w \in \psi[a]\}.$$

All these roots belong to L , because $\psi[a] \subset K^t$. The polynomial $\psi_a(X) - \gamma$ is separable, its derivative being $a \neq 0$, so L/K^t is finite Galois.

For $\sigma \in \text{Gal}(L/K^t)$, define

$$c(\sigma) = \sigma(z) - z.$$

Applying ψ_a gives

$$\psi_a(c(\sigma)) = \sigma(\gamma) - \gamma = 0,$$

so $c(\sigma) \in \psi[a]$. Since every element of $\psi[a]$ is fixed by $\text{Gal}(L/K^t)$, we have

$$\begin{aligned} c(\sigma\tau) &= \sigma(\tau(z) - z) + \sigma(z) - z \\ &= c(\tau) + c(\sigma). \end{aligned}$$

Thus c is a homomorphism to the additive group $\psi[a]$. It is injective: if $c(\sigma) = 0$, then σ fixes z and therefore fixes $L = K^t(z)$. In particular,

$$D := [L : K^t] = |\text{Gal}(L/K^t)| \leq |\psi[a]| = N.$$

We now use the valuation of z to prove the reverse inequality. By (3.5),

$$v(z) = -\frac{t}{(q-1)N}.$$

Every K -automorphism of K^{sep} preserves the extension of v , since K is complete. Therefore

$$\begin{aligned} v(\text{Norm}_{L/K^t}(z)) &= \sum_{\sigma \in \text{Gal}(L/K^t)} v(\sigma(z)) \\ &= Dv(z) = -\frac{Dt}{(q-1)N}. \end{aligned}$$

The norm belongs to $(K^t)^\times$. Every element of K^t lies in a finite tamely ramified extension of K , whose ramification index is prime to p . Since $v(K^\times) = \mathbb{Z}$, it follows that

$$v((K^t)^\times) \subset \mathbb{Z}_{(p)} := \left\{ \frac{m}{n} : m \in \mathbb{Z}, n \geq 1, p \nmid n \right\}.$$

Consequently,

$$-\frac{Dt}{(q-1)N} \in \mathbb{Z}_{(p)}.$$

Now N is a power of p , whereas both t and $q-1$ are prime to p . Thus all the factors of p in the denominator must be cancelled by D , which forces $N \mid D$. Together with $D \leq N$, this gives $D = N$. The injective map c is therefore an isomorphism

$$\text{Gal}(L/K^t) \xrightarrow{\sim} \psi[a].$$

Since restriction gives a surjection $P_K \rightarrow \text{Gal}(L/K^t)$, we conclude that

$$\{\sigma(z) - z : \sigma \in P_K\} = \psi[a].$$

To translate this statement into the action on $\phi[a]$, put

$$y = e_\Lambda(z).$$

The functional equation gives

$$\phi_a(y) = e_\Lambda(\psi_a(z)) = e_\Lambda(\gamma) = 0,$$

and the definition of η_a gives

$$\eta_a(y) = \gamma \pmod{a\Lambda}.$$

Thus y lifts a generator of the free rank-one R -module $\Lambda/a\Lambda$.

Choose an R -basis $w_1, \dots, w_s$ of $\psi[a]$ and put $v_i = e_\Lambda(w_i)$. The restriction of e_Λ to $\psi[a]$ is injective: its kernel is $\psi[a] \cap \Lambda$, which is zero because Λ is torsion-free as an A -module. Hence $v_1, \dots, v_s, y$ is a basis of M , and $v_1, \dots, v_s, y$ is an R -basis of $\phi[a]$. Indeed, its images generate the quotient, and any relation among these elements first has zero coefficient of y after applying η_a , and then has all remaining coefficients zero by independence in M .

For every $\sigma \in P_K$, equivariance and additivity of e_Λ give

$$\sigma(y) - y = e_\Lambda(\sigma(z)) - e_\Lambda(z) = e_\Lambda(\sigma(z) - z).$$

As σ varies, these differences range over all of M . Since σ fixes each v_i , its matrix in the chosen basis is

$$\begin{pmatrix} I_s & u_\sigma \\ 0 & 1 \end{pmatrix},$$

where u_σ is the coordinate column of $\sigma(y) - y$ in the basis $v_1, \dots, v_s$. These columns range over all of R^s , proving

$$\rho_{\phi, \mathfrak{a}}(P_K) = U(R).$$

Finally, fix a nonzero prime $\lambda \neq \mathfrak{p}$, and let $\ell \in A$ be its monic generator. At the levels $a = \ell^n$, the transition maps in the torsion sequences are ψ_ℓ on $\psi[\ell^{n+1}]$, ϕ_ℓ on $\phi[\ell^{n+1}]$, and natural reduction on $\Lambda/\ell^{n+1}\Lambda$. These maps are compatible with the injections and quotient maps. They therefore induce a map

$$\eta : T_\lambda(\phi) \longrightarrow \varprojlim_n \Lambda/\ell^n \Lambda = \Lambda \otimes_A A_\lambda$$

whose kernel is $T_\lambda(\psi)$.

We can exhibit a compatible lift of the generator of this quotient. Set $z_0 = \gamma$ and choose recursively $z_{n+1} \in K^{\text{sep}}$ such that $\psi_\ell(z_{n+1}) = z_n$. Then $\psi_{\ell^n}(z_n) = \gamma$. For $n \geq 1$, put $y_n = e_\Lambda(z_n)$. Using the identity $\phi_a \circ e_\Lambda = e_\Lambda \circ \psi_a$ and $y_n = e_\Lambda(z_n)$, we obtain

$$\begin{aligned} \phi_{\ell^n}(y_n) &= e_\Lambda(\psi_{\ell^n}(z_n)) = e_\Lambda(\gamma) = 0, \\ \phi_\ell(y_{n+1}) &= e_\Lambda(\psi_\ell(z_{n+1})) = e_\Lambda(z_n) = y_n. \end{aligned}$$

Here we have used $\psi_{\ell^n}(z_n) = \gamma$, $\psi_\ell(z_{n+1}) = z_n$, and $\gamma \in \Lambda = \ker(e_\Lambda)$. Thus $y = (y_n)_{n \geq 1}$ belongs to $T_\lambda(\phi)$, and

$$\eta(y) = (\gamma \bmod \ell^n \Lambda)_{n \geq 1} = \gamma \otimes 1.$$

This element generates $\Lambda \otimes_A A_\lambda$, so η is surjective and we obtain the exact sequence

$$0 \longrightarrow T_\lambda(\psi) \longrightarrow T_\lambda(\phi) \longrightarrow \Lambda \otimes_A A_\lambda \longrightarrow 0.$$

Adjoining y to an A_λ -basis of the free rank- s module $T_\lambda(\psi)$ gives an adapted basis of $T_\lambda(\phi)$. Its reductions give compatible adapted bases at every finite level.

In this basis, let

$$H = \widehat{\rho}_{\phi, \lambda}(P_K) \subset U(A_\lambda).$$

The group H is closed, being the continuous image of the compact group P_K in the Hausdorff group $U(A_\lambda)$. The finite-level assertion shows that its reduction modulo λ^n is $U(A/\lambda^n)$ for every n . Given $u \in U(A_\lambda)$, choose $h_n \in H$ such that $h_n \equiv u \pmod{\lambda^n}$. Then $h_n \rightarrow u$, and closedness of H implies $u \in H$. Hence

$$\widehat{\rho}_{\phi, \lambda}(P_K) = U(A_\lambda),$$

as required. $\square$

4. INERTIA AT THE RESIDUAL PRIME

In this section, we study the ramification of the determinant characters associated with a non-zero G_F -stable subspace $W \subsetneq \phi[\lambda]$ and its quotient $\phi[\lambda]/W$. Let λ be a nonzero prime of A , let ℓ be its monic generator, and set $d = \deg \ell$ and $Q = q^d$. We work over $K = F_\lambda$, with valuation normalized by $v(\ell) = 1$. Let $\psi : A \rightarrow \mathcal{O}_K\{\tau\}$ be a Drinfeld module with good reduction of rank s . Its λ -division polynomial can be expressed as follows:

$$\psi_\ell(X) = \ell X + \sum_{i=1}^{sd} b_i X^{q^i},$$

where $b_i \in \mathcal{O}_K$ and $b_{sd} \in \mathcal{O}_K^\times$. Let m be the least index for which b_m is a unit. We first check that $d \mid m$. For every $a \in A$, consider the identity $\psi_\ell \circ \psi_a = \psi_a \circ \psi_\ell$. Comparing the reduced coefficients of X^{q^m} gives

$$\bar{b}_m \bar{a}^{q^m} = \bar{a} \bar{b}_m.$$

Since $\bar{b}_m \neq 0$ and the residues of elements of A range over $\mathbb{F}_{q^d}$, every element of $\mathbb{F}_{q^d}$ is fixed by the q^m -power map. Thus $d \mid m$. Write $m = hd$; the integer h , which satisfies $1 \leq h \leq s$, is the height of the reduction.

Define

$$C = \{0\} \cup \{x \in \psi[\lambda] : v(x) > 0\}.$$

The k_λ -action on $\psi[\lambda]$ is induced by $a \cdot x = \psi_a(x)$.

Lemma 4.1. *The space C is an I_λ -stable k_λ -subspace of dimension h . The action of I_λ on C is irreducible, and its action on $\psi[\lambda]/C$ is trivial.*

Proof. The space C consists of the λ -torsion points that reduce to zero. It is therefore the subspace denoted by $\psi[\lambda]^\circ(K^{\text{sep}})$ in [PR09b, Section 2]. Since ψ has good reduction and $K = F_\lambda$, Proposition 2.7 of that reference applies. Together with the discussion preceding it, this gives $\dim_{k_\lambda} C = h$ and shows that inertia acts trivially on $\psi[\lambda]/C$. It also gives a k_λ -linear isomorphism

$$\iota : C \xrightarrow{\sim} \mathbb{F}_{Q^h}$$

under which the inertia action is multiplication by a surjective character

$$\chi : I_\lambda \longrightarrow \mathbb{F}_{Q^h}^\times.$$

Thus

$$\iota(\sigma x) = \chi(\sigma) \iota(x)$$

for $\sigma \in I_\lambda$ and $x \in C$. To prove irreducibility, let $D \subseteq C$ be a nonzero I_λ -stable k_λ -subspace, and choose $0 \neq x \in D$. For any $0 \neq y \in C$, the surjectivity of χ gives an element $\sigma \in I_\lambda$ satisfying

$$\chi(\sigma) = \frac{\iota(y)}{\iota(x)}.$$

Consequently $\iota(\sigma x) = \iota(y)$, and hence $\sigma x = y$. Since D is stable under inertia, this implies $y \in D$. Therefore $D = C$, proving that the inertia action on C is irreducible. $\square$

Set

$$L_r = \prod_{i=1}^r (q^i - 1) \quad \text{and} \quad N_r = L_r^2. \quad (4.1)$$

For a finite-dimensional k_λ -representation V and a stable subspace W , write $\det W$ and $\det(V/W)$ for the characters

$$\sigma \longmapsto \det(\sigma|_W) \quad \text{and} \quad \sigma \longmapsto \det(\sigma|_{V/W}),$$

respectively.

Lemma 4.2. *Let $1 \leq t \leq r$, and let $M \in \text{GL}_t(A)$. Assume that M has finite order. Then every eigenvalue of M , and every eigenvalue of its reduction modulo a nonzero prime of A , has order dividing L_r .*

Proof. Choose $n \geq 1$ such that $M^n = I_t$. Every eigenvalue ξ of M satisfies $\xi^n = 1$, so $\xi \in \overline{\mathbb{F}_q}$. The coefficients of the characteristic polynomial of M are elementary symmetric functions of its eigenvalues. They consequently lie in

$$A \cap \overline{\mathbb{F}_q} = \mathbb{F}_q.$$

The minimal polynomial of ξ over $\mathbb{F}_q$ divides this characteristic polynomial. Its degree is therefore some $i \leq t$, so $\xi \in \mathbb{F}_{q^i}^\times$ and its order divides $q^i - 1$, hence L_r .

Since the characteristic polynomial lies in $\mathbb{F}_q[X]$, reduction modulo any nonzero prime of A leaves it unchanged. Its roots after reduction therefore satisfy the same order bound. $\square$

Proposition 4.3. *Let ϕ/F have rank r and let W be a G_F -stable subspace of $V = \phi[\lambda]$. For every finite prime $\mathfrak{p} \neq \lambda$, both $(\det W)^{N_r}$ and $(\det(V/W))^{N_r}$ are trivial on $I_{\mathfrak{p}}$.*

Proof. Set $K = F_{\mathfrak{p}}$ and choose a uniformizer ϖ . The construction in Lemma 3.1 gives a totally tamely ramified extension $K' = K(\varpi^{1/b})$ where $b \mid L_r$, over which ϕ is isomorphic to an integral module with stable reduction. This isomorphism is defined over K' , so it identifies the torsion representations of $G_{K'}$. We set $E = K'K^{\text{un}}$ and $J = \text{Gal}(K^{\text{sep}}/E)$. Since K^{un} contains the b -th roots of unity, E/K^{un} is cyclic of degree b . Thus J is a normal subgroup of $I_{\mathfrak{p}}$ of index b . In particular, we have that $\sigma^b \in J$ for all $\sigma \in I_{\mathfrak{p}}$.

By Tate uniformization over K' there is a Drinfeld module ψ with good reduction and a ψ -lattice Λ of rank at most $r - 1$ for which there is the associated short exact sequence

$$0 \longrightarrow \psi[\lambda] \longrightarrow V \longrightarrow \Lambda/\lambda\Lambda \longrightarrow 0.$$

Since $\mathfrak{p} \neq \lambda$ and ψ has good reduction, $\psi[\lambda]$ is unramified over K' . The group $J \subseteq I_{K'}$ therefore fixes $\psi[\lambda]$ pointwise.

Choose an A -basis $\gamma_1, \dots, \gamma_t$ of Λ . Since these elements belong to K^{sep} , there is a finite Galois extension L/K' containing all of them. Every element of Λ has the form

$$\psi_{a_1}(\gamma_1) + \dots + \psi_{a_t}(\gamma_t),$$

where $a_1, \dots, a_t \in A$. The coefficients of each ψ_{a_i} lie in K' , so every element of Λ belongs to L . Consequently, $J \cap \text{Gal}(K^{\text{sep}}/L)$ fixes Λ pointwise. This subgroup has finite index in J , proving that the action of J on Λ has finite image. The intersection of these finitely many stabilizers fixes the whole lattice, because the action commutes with every ψ_a . The kernel of the lattice action is consequently open and has finite index. Thus the action of J on Λ has finite image.

For $\sigma \in J$, its matrix on Λ therefore has finite order. Lemma 4.2 shows that every eigenvalue on $\Lambda/\lambda\Lambda$ has order dividing L_r . The other eigenvalues on V are all one, since J fixes $\psi[\lambda]$. This conclusion also holds when $\Lambda = 0$.

Because W is stable, a basis of W extended to a basis of V gives a block upper triangular matrix for σ . Its characteristic polynomial factors as

$$\det(XI_V - \sigma|_V) = \det(XI_W - \sigma|_W) \det(XI_{V/W} - \sigma|_{V/W}).$$

Each determinant character is therefore a product of eigenvalues whose orders divide L_r . Hence for $\sigma \in J$, we have that

$$(\det W)(\sigma)^{L_r} = (\det(V/W))(\sigma)^{L_r} = 1.$$

Let χ denote either of these determinant characters. For arbitrary $\sigma \in I_{\mathfrak{p}}$, we have

$$\chi(\sigma)^{bL_r} = \chi(\sigma^b)^{L_r} = 1.$$

Since $b \mid L_r$, the integer bL_r divides N_r . Thus χ^{N_r} is trivial on $I_{\mathfrak{p}}$, as claimed. $\square$

Proposition 4.4. *Suppose that ϕ admits an integral model over F_λ with good reduction or stable reduction of rank $r - 1$. For every G_F -stable subspace $0 \neq W \neq V = \phi[\lambda]$, at least one of $(\det W)^{N_r}$ and $(\det(V/W))^{N_r}$ is unramified at λ .*

Proof. All the local representations in this proof are taken over $K = F_\lambda$, with $v_\lambda(\ell) = 1$. Replacing ϕ by the given integral model preserves its I_λ -representation on λ -torsion.

Suppose first that ϕ has good reduction. Apply Lemma 4.1 to ϕ , and let $C \subseteq V$ be its positive-valuation torsion. Since C is irreducible under inertia, its stable subspace $W \cap C$ is either zero or C .

If $W \cap C = 0$, the map $W \rightarrow V/C$ is injective. Inertia acts trivially on V/C , so it also acts trivially on W . Thus $\det W$ is unramified. If $C \subseteq W$, the natural map $V \rightarrow V/W$ factors through V/C . Hence inertia acts trivially on V/W , and $\det(V/W)$ is unramified. This proves the assertion in the good-reduction case.

Now suppose that the integral model has stable reduction of rank $r - 1$. Tate uniformization over K gives a good-reduction module ψ/K of rank $r - 1$ and a rank-one lattice Λ , with exact sequence

$$0 \longrightarrow \psi[\lambda] \xrightarrow{e_\Lambda} V \longrightarrow \Lambda/\lambda\Lambda \longrightarrow 0.$$

Here the sequence is applied at the residual prime λ ; this is permitted because K has generic A -characteristic. Identify $\psi[\lambda]$ with its image in V , and let C be the image of its positive-valuation subspace. By Lemma 4.1, C is irreducible under I_λ , and inertia acts trivially on $B = \psi[\lambda]/C$.

Put $D = \Lambda/\lambda\Lambda$. The action on the rank-one lattice takes values in $\text{Aut}_A(\Lambda) = \mathbb{F}_q^\times$. Thus the action on the one-dimensional k_λ -space D is given by a character

$$\eta : I_\lambda \longrightarrow \mathbb{F}_q^\times \subseteq k_\lambda^\times, \quad \eta^{q-1} = 1.$$

Quotienting the torsion sequence by C gives

$$0 \longrightarrow B \longrightarrow V/C \longrightarrow D \longrightarrow 0.$$

We again have $W \cap C = 0$ or C .

If $W \cap C = 0$, let W' be the image of W in V/C , so that $W \simeq W'$ as inertia modules. Let D_W be the image of W' in D . There is an exact sequence

$$0 \longrightarrow W' \cap B \longrightarrow W' \longrightarrow D_W \longrightarrow 0.$$

Inertia acts trivially on $W' \cap B$, while D_W is either zero or all of D . Multiplicativity of determinants therefore gives

$$(\det W)|_{I_\lambda} = 1 \quad \text{or} \quad (\det W)|_{I_\lambda} = \eta.$$

In either case, $(\det W)^{q-1}$ is unramified at λ .

If $C \subseteq W$, put $Z = V/W$. The quotient map $V/C \rightarrow Z$ is surjective. Let B_Z be the image of B in Z . Inertia acts trivially on B_Z , and Z/B_Z is a quotient of D , hence is either zero or isomorphic to D . The exact sequence

$$0 \longrightarrow B_Z \longrightarrow Z \longrightarrow Z/B_Z \longrightarrow 0$$

then gives

$$(\det(V/W))|_{I_\lambda} = 1 \quad \text{or} \quad (\det(V/W))|_{I_\lambda} = \eta.$$

Thus $(\det(V/W))^{q-1}$ is unramified at λ in this case. Since $q - 1 \mid N_r$, both cases give the asserted alternative. $\square$

5. FROBENIUS POLYNOMIALS AND THEIR EXTERIOR POWERS

In this section we prove estimates for Frobenius polynomials which are used to bound the degrees of primes at which the residual representation is reducible. These bounds will be applied in Section 7.

Let $\mathfrak{p}$ be a prime of good reduction for ϕ , and set $d_{\mathfrak{p}} = \deg \mathfrak{p}$. As always, we denote by $\mathfrak{p}$ its monic generator, and normalize the infinite absolute value by $|T|_\infty = q$.

Theorem 5.1. *There is a monic polynomial*

$$P_{\phi, \mathfrak{p}}(X) = X^r + a_1 X^{r-1} + \cdots + a_r \in A[X]$$

with the following properties.

- (i) *For every nonzero ideal $\mathfrak{a}$ prime to $\mathfrak{p}$, $\rho_{\phi, \mathfrak{a}}$ is unramified at $\mathfrak{p}$, and*

$$P_{\phi, \mathfrak{p}}(X) \equiv \det(XI - \rho_{\phi, \mathfrak{a}}(\text{Frob}_{\mathfrak{p}})) \pmod{\mathfrak{a}},$$

where $\text{Frob}_{\mathfrak{p}}$ denotes arithmetic Frobenius.

- (ii) *The constant term is $a_r = \epsilon_{\mathfrak{p}} \mathfrak{p}$ for some $\epsilon_{\mathfrak{p}} \in \mathbb{F}_q^\times$.*

- (iii) *Every root α satisfies $|\alpha|_\infty = q^{d_{\mathfrak{p}}/r}$ at every extension of $|\cdot|_\infty$ to a field containing α .*

Proof. The existence of $P_{\phi, \mathfrak{p}} \in A[X]$ and part (i) are recalled in [Zyw25, Section 5 and Proposition 5.1]. The constant-term formula and the coefficient bounds

$$|a_i|_\infty \leq q^{id_{\mathfrak{p}}/r}$$

for $1 \leq i \leq r$ are given in [Che25, Propositions 2.9 and 2.10].

To deduce part (iii), fix any extension of $|\cdot|_\infty$ to a splitting field and put $R = q^{d_{\mathfrak{p}}/r}$. If a root α satisfied $|\alpha|_\infty > R$, then

$$|a_i \alpha^{r-i}|_\infty \leq R^i |\alpha|_\infty^{r-i} < |\alpha|_\infty^r$$

for $1 \leq i \leq r$. The leading term could therefore not cancel in $P_{\phi, \mathfrak{p}}(\alpha) = 0$. Thus all r roots, counted with multiplicity, have absolute value at most R . Their product has absolute value $|a_r|_\infty = q^{d_{\mathfrak{p}}} = R^r$, so every root has absolute value exactly R . $\square$

Lemma 5.2. *Let L/F be a finite extension containing the roots of $P_{\phi, \mathfrak{p}}$, and let β be a product of a nonempty subset of these roots, counted with multiplicity. Then β is integral over A , its valuation is zero at every finite place of L not lying over $\mathfrak{p}$, and its valuation is positive at some place above $\mathfrak{p}$.*

Proof. Write the roots as $\alpha_1, \dots, \alpha_r$. They are integral over A because $P_{\phi, \mathfrak{p}}$ is monic. Consequently, at every finite place w of L , all $v_w(\alpha_i)$ are nonnegative. Moreover,

$$\sum_{i=1}^r v_w(\alpha_i) = v_w((-1)^r a_r) = v_w(\mathfrak{p}).$$

If w does not lie above $\mathfrak{p}$, this sum is zero, so every summand is zero. This proves integrality and the asserted vanishing for β .

Suppose that β is a product of $j > 0$ roots. By Theorem 5.1, its absolute value at every place above infinity is $q^{jd_{\mathfrak{p}}/r} > 1$, when the absolute value extends our normalization on F . Its valuation is therefore negative at every such place. If all its finite valuations were zero, its principal divisor would have negative degree, contradicting the fact that principal divisors have degree zero. Some finite valuation is therefore positive, and the preceding paragraph shows that it must lie above $\mathfrak{p}$. $\square$

For $1 \leq j \leq r-1$, put $m_j = \binom{r}{j}$ and define

$$P_{\phi, \mathfrak{p}, j}(X) = \prod_{\substack{I \subseteq \{1, \dots, r\} \\ |I|=j}} \left(X - \prod_{i \in I} \alpha_i \right).$$

For any matrix M over a field, the eigenvalues of $\bigwedge^j M$ are the products of j eigenvalues of M , counted with multiplicity. Indeed, triangularizing M gives a triangular matrix on the exterior power with precisely these diagonal entries.

Let $M_{\mathfrak{p}}$ be the companion matrix of $P_{\phi, \mathfrak{p}}$. The preceding observation gives

$$P_{\phi, \mathfrak{p}, j}(X) = \det(XI - \bigwedge^j M_{\mathfrak{p}}).$$

The entries of $\bigwedge^j M_{\mathfrak{p}}$ are minors of $M_{\mathfrak{p}}$ and hence belong to A . Thus $P_{\phi, \mathfrak{p}, j} \in A[X]$, and this construction commutes with reduction modulo an ideal.

For a monic polynomial $P(X) = \prod_{u=1}^m (X - \beta_u)$ and an integer $N \geq 1$, set

$$P^{[N]}(X) = \prod_{u=1}^m (X - \beta_u^N) = \text{Res}_Y(P(Y), X - Y^N).$$

The resultant expression shows that $P^{[N]} \in A[X]$ whenever $P \in A[X]$, and that powering the roots also commutes with reduction. Triangularization shows that $P^{[N]}$ is the characteristic polynomial of M^N whenever P is the characteristic polynomial of M . Therefore, for $\lambda \neq \mathfrak{p}$,

$$P_{\phi, \mathfrak{p}, j}^{[N]}(X) \equiv \det\left(XI - \left(\bigwedge^j \rho_{\phi, \lambda}(\text{Frob}_{\mathfrak{p}})\right)^N\right) \pmod{\lambda}.$$

Here is how this identity detects determinant characters. Let $V = \phi[\lambda]$, and let $W \subset V$ be a nonzero proper G_F -stable subspace. If $j = \dim_{k_\lambda} W$, then $\bigwedge^j W$ is a stable line in $\bigwedge^j V$, with character $\theta = \det(W)$. If instead $j = \dim_{k_\lambda}(V/W)$, then $\bigwedge^j(V/W)$ is a one-dimensional quotient of $\bigwedge^j V$, with character $\theta = \det(V/W)$. In either case, a basis adapted to this line or quotient makes the Frobenius matrix block triangular. Its characteristic polynomial has the factor $X - \theta(\text{Frob}_{\mathfrak{p}})$. Consequently,

$$P_{\phi, \mathfrak{p}, j}^{[N]}(\theta(\text{Frob}_{\mathfrak{p}})^N) \equiv 0 \pmod{\lambda}.$$

Lemma 5.3. *Let $\mathfrak{p}_1 \neq \mathfrak{p}_2$ be primes of good reduction. For $1 \leq j_1, j_2 \leq r-1$ and $N \geq 1$, the polynomials $P_{\phi, \mathfrak{p}_1, j_1}^{[N]}$ and $P_{\phi, \mathfrak{p}_2, j_2}^{[N]}$ have no common root. Their resultant is therefore a nonzero element of A .*

Proof. Work in a finite extension containing all the Frobenius roots at $\mathfrak{p}_1$ and $\mathfrak{p}_2$. A common root would give $\beta_1^N = \beta_2^N$, where β_i is a product of j_i Frobenius roots at $\mathfrak{p}_i$. By Lemma 5.2, there is a place w above $\mathfrak{p}_1$ for which $v_w(\beta_1) > 0$. Since $\mathfrak{p}_1 \neq \mathfrak{p}_2$, the same lemma gives $v_w(\beta_2) = 0$. Taking valuations in the proposed equality would give $Nv_w(\beta_1) = Nv_w(\beta_2)$, a contradiction. $\square$

Lemma 5.4. *Let $\mathfrak{p}_1 \neq \mathfrak{p}_2$ be primes of good reduction of the same degree d . For $1 \leq j_1, j_2 \leq r-1$ and $N \geq 1$,*

$$\begin{aligned} \deg_T \text{Res}(P_{\phi, \mathfrak{p}_1, j_1}^{[N]}, P_{\phi, \mathfrak{p}_2, j_2}^{[N]}) &\leq \frac{Nd}{r} \max(j_1, j_2) m_{j_1} m_{j_2} \\ &\leq Nd \max_{1 \leq j \leq r-1} m_j^2. \end{aligned}$$

Proof. Denote the resultant by R ; the preceding lemma gives $R \in A \setminus \{0\}$. Let $\beta_1, \dots, \beta_{m_{j_1}}$ be the roots of $P_{\phi, \mathfrak{p}_1, j_1}$, and let $\gamma_1, \dots, \gamma_{m_{j_2}}$ be the roots of $P_{\phi, \mathfrak{p}_2, j_2}$. Fix an extension of $|\cdot|_\infty$ to a field containing these roots. Theorem 5.1 gives

$$|\beta_u^N|_\infty = q^{Nj_1 d/r} \quad \text{and} \quad |\gamma_v^N|_\infty = q^{Nj_2 d/r}.$$

Since both polynomials are monic, the product formula for the resultant and the ultrametric inequality give

$$|R|_\infty = \prod_{u=1}^{m_{j_1}} \prod_{v=1}^{m_{j_2}} |\beta_u^N - \gamma_v^N|_\infty \leq q^{(Nd/r) \max(j_1, j_2) m_{j_1} m_{j_2}}.$$

Now $|R|_\infty = q^{\deg_T R}$, which proves the first bound. The second follows from $\max(j_1, j_2) \leq r$ and $m_{j_1} m_{j_2} \leq \max_j m_j^2$. $\square$

6. THE DETERMINANT REPRESENTATION

Once the adelic image contains $\mathrm{SL}_r(\widehat{A})$, its index is determined by its determinant. We identify that character with a rank-one Drinfeld representation and apply Gekeler's image formula. We also record the local surjectivity criterion and the elementary count needed for Theorems A and B.

For $\phi_T = T + g_1\tau + \cdots + g_r\tau^r$, define

$$\Delta = (-1)^{r-1}g_r \quad \text{and} \quad \psi_T = T + \Delta\tau. \quad (6.1)$$

Thus ψ is a rank-one Drinfeld module over F . The Weil pairing theorem [Pap23, Theorem 3.7.1] gives compatible G_F -equivariant isomorphisms

$$\bigwedge_{A/\mathfrak{a}}^r \phi[\mathfrak{a}] \simeq \psi[\mathfrak{a}] \quad \text{and} \quad \bigwedge_{A_\lambda}^r T_\lambda(\phi) \simeq T_\lambda(\psi). \quad (6.2)$$

Since the action on the top exterior power is the determinant,

$$\det \rho_{\phi, \mathfrak{a}} = \rho_{\psi, \mathfrak{a}} \quad \text{and} \quad \det \widehat{\rho}_\phi = \widehat{\rho}_\psi.$$

We use Gekeler's rank-one image theorem in the form stated in [Zyw25, Theorem 6.3]; see also [Gek16, Theorem 3.13].

Theorem 6.1 (Gekeler). *Let $\psi_T = T + \Delta\tau$ with $0 \neq \Delta \in A$. Put $d = \deg \Delta$, and let e be the order of $(-1)^d \mathrm{lc}(\Delta)$ in $\mathbb{F}_q^\times$. Then*

$$[\widehat{A}^\times : \widehat{\rho}_\psi(G_F)] = \gcd\left(d-1, \frac{q-1}{e}\right), \quad (6.3)$$

$$[(A/\mathfrak{a})^\times : \rho_{\psi, \mathfrak{a}}(G_F)] = \gcd\left(d-1, \frac{q-1}{e}, \{v_{\mathfrak{p}}(\Delta) : \mathfrak{p} \nmid \mathfrak{a}\}\right) \quad (6.4)$$

for every nonzero ideal $\mathfrak{a}$. In the second formula, $\mathfrak{p}$ runs over the nonzero prime ideals of A not dividing $\mathfrak{a}$.

Corollary 6.2. *Suppose $v_{\mathfrak{p}}(g_r) = 1$. For every level $\mathfrak{a}$ prime to $\mathfrak{p}$,*

$$\det \rho_{g, \mathfrak{a}}(G_F) = (A/\mathfrak{a})^\times.$$

For every prime $\lambda \neq \mathfrak{p}$,

$$\det \widehat{\rho}_{g, \lambda}(G_F) = A_\lambda^\times.$$

For every coefficient tuple, without a valuation hypothesis, the adelic determinant index divides $q-1$.

Proof. For the determinant module in (6.1), we have $v_{\mathfrak{p}}(\Delta) = v_{\mathfrak{p}}(g_r) = 1$. If $\mathfrak{p} \nmid \mathfrak{a}$, the greatest common divisor in (6.4) therefore contains an entry equal to one. The finite-level index is one, proving the first assertion.

If $\lambda \neq \mathfrak{p}$, apply this assertion to $\mathfrak{a} = \lambda^n$ for every $n \geq 1$. The image of the continuous character $\det \widehat{\rho}_{g, \lambda}$ is compact and hence closed in $A_\lambda^\times$. Surjectivity modulo every λ^n makes it dense in $A_\lambda^\times$, so it is the whole group. Finally, (6.3) shows that the adelic determinant index divides $(q-1)/e$, and thus $q-1$, for every tuple. $\square$

Recall that $\mathcal{P}$ consists of the tuples for which $(-1)^{\deg g_r + r-1} \mathrm{lc}(g_r)$ generates $\mathbb{F}_q^\times$.

Lemma 6.3. *For every $n \geq 0$,*

$$\frac{|\mathcal{P} \cap \mathcal{B}_n|}{|\mathcal{B}_n|} = \frac{\varphi(q-1)}{q-1}.$$

In particular, $\delta(\mathcal{P}) = \varphi(q-1)/(q-1)$.

Proof. Fix $0 \leq d \leq n$. For each $c \in \mathbb{F}_q^\times$, there are exactly q^d polynomials g_r of degree d with leading coefficient c . Multiplication by $(-1)^{d+r-1}$ permutes $\mathbb{F}_q^\times$, which has exactly $\varphi(q-1)$ generators. Thus, among the $(q-1)q^d$ possibilities for g_r of degree d , precisely $\varphi(q-1)q^d$ satisfy the defining condition of $\mathcal{P}$.

The remaining $r-1$ coefficients can be chosen freely in $A_{\leq n}$, giving $q^{(r-1)(n+1)}$ possibilities. Hence

$$|\mathcal{P} \cap \mathcal{B}_n| = q^{(r-1)(n+1)} \varphi(q-1) \sum_{d=0}^n q^d,$$

$$|\mathcal{B}_n| = q^{(r-1)(n+1)} (q-1) \sum_{d=0}^n q^d.$$

Taking their ratio proves the assertion. $\square$

7. A UNIFORM BOUNDS AND BIG IMAGE CRITERIA

The following theorem of Zywna will be of central importance in our density estimates. Recall that

$$A_{\leq n} = \{f \in \mathbb{F}_q[T] : \deg f \leq n\} \quad \text{and} \quad \mathcal{B}_n = A_{\leq n}^{r-1} \times (A_{\leq n} \setminus \{0\}),$$

where 0 is included in $A_{\leq n}$. Thus $\mathcal{B}_n$ consists of coefficient tuples $\mathbf{g} = (g_1, \dots, g_r)$ with each coefficient of degree at most n and $g_r \neq 0$, defining rank- r Drinfeld modules

$$\phi_T = T + g_1\tau + \dots + g_r\tau^r.$$

Theorem 7.1. *For every fixed nonzero ideal $\mathfrak{a} \subset A$,*

$$\lim_{n \rightarrow \infty} \frac{|\{\mathbf{g} \in \mathcal{B}_n : \rho_{\mathbf{g}, \mathfrak{a}}(G_F) = \mathrm{GL}_r(A/\mathfrak{a})\}|}{|\mathcal{B}_n|} = 1.$$

Proof. This is [Zyw25, Theorem 8.1], which is stated for arbitrary rank $r \geq 2$. The density there is computed in $A_{\leq n}^r$. Since

$$\frac{|\mathcal{B}_n|}{|A_{\leq n}^r|} = 1 - q^{-(n+1)} \longrightarrow 1,$$

the same conclusion holds with our normalization. $\square$

We combine the inertia-character results of Section 4 with the resultant bounds of Section 5. The argument uses a determinant character of an invariant subspace or quotient, so it applies to invariant subspaces of every dimension.

Remark 7.2. *Let $\chi : G_F \rightarrow k_\lambda^\times$ be a continuous character unramified at every finite prime. Then χ factors through $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$. In particular, there is $\zeta \in k_\lambda^\times$ such that*

$$\chi(\mathrm{Frob}_{\mathfrak{p}}) = \zeta^{\deg \mathfrak{p}}$$

for every finite prime $\mathfrak{p}$.

Let N_r be defined as in (4.1), and set

$$C_r = N_r \max_{1 \leq j \leq r-1} \binom{r}{j}^2. \quad (7.1)$$

Theorem 7.3. *Suppose ϕ/F has good reduction at distinct primes $\mathfrak{p}_1, \mathfrak{p}_2$ of the same degree d . Let λ be a prime at which ϕ has good reduction or stable reduction of rank $r - 1$. If $\bar{\rho}_{\phi, \lambda}$ is reducible, then $\deg \lambda \leq C_r d$.*

Proof. If $\lambda \in \{\mathfrak{p}_1, \mathfrak{p}_2\}$, then $\deg \lambda = d \leq C_r d$. We may therefore assume that λ differs from both auxiliary primes.

Choose a nonzero proper G_F -stable subspace $W \subset V = \phi[\lambda]$. By Proposition 4.3, both $(\det W)^{N_r}$ and $(\det(V/W))^{N_r}$ are unramified at every finite prime other than λ . Proposition 4.4 shows that at least one of these two characters is also unramified at λ . Choose one of the two representations W and V/W with this property, denote it by V_0 , and put $\theta = \det V_0$ and $j = \dim_{k_\lambda} V_0$. Thus $1 \leq j \leq r - 1$, and $\chi = \theta^{N_r}$ is unramified at every finite prime.

Remark 7.2 supplies $\zeta \in k_\lambda^\times$ with $\theta(\text{Frob}_\mathfrak{p})^{N_r} = \zeta^{\deg \mathfrak{p}}$. The exterior-power construction in Section 5 therefore gives, at every good prime $\mathfrak{p} \neq \lambda$,

$$P_{\phi, \mathfrak{p}, j}^{[N_r]}(\zeta^{\deg \mathfrak{p}}) \equiv 0 \pmod{\lambda}. \quad (7.2)$$

In particular, the two monic polynomials $P_{\phi, \mathfrak{p}_1, j}^{[N_r]}$ and $P_{\phi, \mathfrak{p}_2, j}^{[N_r]}$ have the common root ζ^d after reduction modulo λ .

Their resultant

$$R = \text{Res}(P_{\phi, \mathfrak{p}_1, j}^{[N_r]}, P_{\phi, \mathfrak{p}_2, j}^{[N_r]}) \in A$$

is consequently divisible by λ .

Lemma 5.3 implies that $R \neq 0$, and Lemma 5.4 asserts that $\deg R \leq C_r d$. Since the nonzero polynomial R is divisible by λ , we obtain $\deg \lambda \leq \deg R \leq C_r d$. $\square$

Suppose $\mathfrak{p} \neq \lambda$ is a prime at which ϕ has integral coefficients and

$$v_\mathfrak{p}(g_{r-1}) = 0 \quad \text{and} \quad v_\mathfrak{p}(g_r) = 1.$$

By Theorem 3.6, there is a basis of $T_\lambda(\phi)$ adapted to the uniformization sequence in which

$$\widehat{\rho}_{\phi, \lambda}(P_{F_\mathfrak{p}}) = U(A_\lambda) = \left\{ \begin{pmatrix} I_{r-1} & u \\ 0 & 1 \end{pmatrix} : u \in A_\lambda^{r-1} \right\}.$$

Thus the global image contains every matrix in this translation group. We show that residual irreducibility then forces it to contain $\text{SL}_r(A_\lambda)$.

For $i \neq j$ and a in a commutative ring R , write $E_{ij}(a) = I + ae_{ij}$, where e_{ij} is the corresponding matrix unit. We use the notation $[x, y] = xyx^{-1}y^{-1}$.

Lemma 7.4. *For every $r \geq 2$, there is an integer $B(r)$ such that, for every commutative local ring R , every element of $\text{SL}_r(R)$ is a product of at most $B(r)$ elementary matrices.*

Proof. By [VSS12, Theorem 1 and Section 1], every element of $\text{SL}_r(R)$ is a product of four matrices, each upper or lower triangular with all diagonal entries equal to one. Each such factor is a product of at most $r(r-1)/2$ elementary matrices. This gives a bound depending only on r . $\square$

Lemma 7.5. *For a commutative local ring R and $r \geq 3$, the group $\text{SL}_r(R)$ is perfect. Moreover, $\text{SL}_r(\widehat{A})$ is perfect.*

Proof. Since $r \geq 3$, for every pair $i \neq j$ we can choose an index k distinct from both. We find that

$$[E_{ik}(a), E_{kj}(b)] = E_{ij}(ab). \quad (7.3)$$

Taking $b = 1$ shows that every elementary matrix belongs to the commutator subgroup of $\text{SL}_r(R)$. By Lemma 7.4, elementary matrices generate $\text{SL}_r(R)$, so this group is perfect.

We now prove the assertion for $\widehat{A} = \prod_{\lambda} A_{\lambda}$. Let $M = (M_{\lambda})_{\lambda} \in \mathrm{SL}_r(\widehat{A})$, and put $B = B(r)$. Each M_{λ} is a product of at most B elementary matrices. List the pairs (i, j) with $i \neq j$ as

$$(i_1, j_1), \dots, (i_m, j_m), \quad \text{where } m = r(r-1),$$

and write

$$\mathcal{E}(x_1, \dots, x_m) = E_{i_1 j_1}(x_1) \cdots E_{i_m j_m}(x_m).$$

Every elementary matrix can be written in this form by taking its coefficient in the corresponding position and setting all other parameters equal to zero. The identity matrix is obtained by setting every parameter equal to zero.

Consequently, after adding identity factors when necessary, we can write each local component as

$$M_{\lambda} = \prod_{t=1}^B \mathcal{E}(a_{\lambda,t,1}, \dots, a_{\lambda,t,m}), \quad a_{\lambda,t,s} \in A_{\lambda}.$$

For each fixed t and s , form the element

$$a_{t,s} = (a_{\lambda,t,s})_{\lambda} \in \widehat{A}.$$

We then have

$$M = \prod_{t=1}^B \mathcal{E}(a_{t,1}, \dots, a_{t,m}).$$

Indeed, projection to the λ -coordinate gives exactly the displayed factorization of M_{λ} .

Thus M is a product of at most Bm elementary matrices over $\widehat{A}$. The bound is independent of λ , so this is a finite product even though $\widehat{A}$ has infinitely many local factors. Applying (7.3) with $b = 1$ over $\widehat{A}$ expresses each factor as a commutator in $\mathrm{SL}_r(\widehat{A})$. Every element of $\mathrm{SL}_r(\widehat{A})$ is therefore a finite product of commutators, proving perfectness. $\square$

Before stating the next result, we introduce some further notation. Let R be a commutative local ring and put $V = R^r$. An R -linear functional $f \in V^* = \mathrm{Hom}_R(V, R)$ is called primitive if it is surjective.

For a primitive functional f and a vector $v \in \ker f$, write vf for the R -linear endomorphism of V defined by

$$(vf)(x) = f(x)v.$$

Thus $I + vf$ sends x to $x + f(x)v$, where I denotes the identity map on V . Define

$$U_f = \{I + vf : v \in \ker f\}.$$

For $v, w \in \ker f$, the relation $f(w) = 0$ gives $(vf)(wf) = 0$. Consequently,

$$(I + vf)(I + wf) = I + (v + w)f \quad \text{and} \quad (I + vf)^{-1} = I - vf.$$

These identities show that U_f is a subgroup of $\mathrm{GL}_r(R)$. Since f is surjective, the map

$$v \mapsto I + vf$$

is injective. This identifies the additive group $\ker f$ with U_f .

We describe this group in an adapted basis. Let $e_1, \dots, e_r$ be the standard basis of V , and write

$$f(x_1, \dots, x_r) = a_1 x_1 + \cdots + a_r x_r.$$

Surjectivity means that the coefficients a_i generate R . Since R is local, at least one a_i is a unit. After reordering the basis, assume $a_r \in R^{\times}$, and set $b_i = e_i - a_i a_r^{-1} e_r$ for $1 \leq i < r$ and set $b_r = a_r^{-1} e_r$. The change-of-basis matrix has determinant a_r^{-1} , so $b_1, \dots, b_r$ form a basis of V . Moreover, $f(b_i) = 0$ for $1 \leq i < r$ and $f(b_r) = 1$. Thus $b_1, \dots, b_{r-1}$ form a

basis of $\ker f$. If $v = \sum_{i=1}^{r-1} u_i b_i$, then $I + vf$ fixes $b_1, \dots, b_{r-1}$ and sends b_r to $b_r + v$. Its matrix in this basis is therefore

$$\begin{pmatrix} I_{r-1} & u \\ 0 & 1 \end{pmatrix}$$

where $u = (u_1, \dots, u_{r-1})^\top$. Consequently, in this basis U_f is exactly the group $U(R)$ appearing in Theorem 3.6. In particular, $U_f \subseteq \mathrm{SL}_r(R)$.

Finally, let $g \in \mathrm{GL}_r(R)$. For every $x \in V$,

$$g(I + vf)g^{-1}(x) = x + f(g^{-1}x)gv.$$

Hence

$$g(I + vf)g^{-1} = I + (gv)(f \circ g^{-1}).$$

The functional $f \circ g^{-1}$ is primitive, and g maps $\ker f$ bijectively onto $\ker(f \circ g^{-1})$. Letting v range over $\ker f$ therefore gives

$$gU_f g^{-1} = U_{f \circ g^{-1}}. \quad (7.4)$$

Lemma 7.6. *If $f_1, \dots, f_r$ form an R -basis of V^* , then the subgroups $U_{f_1}, \dots, U_{f_r}$ generate $\mathrm{SL}_r(R)$.*

Proof. Let $e_1, \dots, e_r$ be the dual basis of V . For $i \neq j$ and $a \in R$, the vector ae_j lies in $\ker f_i$, so $I + ae_j f_i \in U_{f_i}$. These are all elementary matrices in the basis $e_1, \dots, e_r$. Apply Lemma 7.4. $\square$

Proposition 7.7. *Let R be a commutative local ring with residue field k , and let $G \subseteq \mathrm{GL}_r(R)$. Suppose the reduction of G acts irreducibly on k^r and G contains U_f for a primitive functional f . Then $G \supseteq \mathrm{SL}_r(R)$. If also $\det G = R^\times$, then $G = \mathrm{GL}_r(R)$.*

Proof. The dual residual representation is irreducible. Indeed, the annihilator of a proper nonzero invariant subspace of $(k^r)^*$ would be a proper nonzero invariant subspace of k^r . Since f is primitive, its reduction $\bar{f}$ is nonzero. The span of its orbit is therefore the whole dual space. Choose $g_1, \dots, g_r \in G$ such that the reductions of $f_i = f \circ g_i^{-1}$ form a basis of $(k^r)^*$.

The matrix with rows $f_1, \dots, f_r$ has nonzero determinant modulo the maximal ideal. Its determinant is a unit in R , so these functionals form an R -basis of V^* . By (7.4), G contains all the groups U_{f_i} . Lemma 7.6 now implies that $\mathrm{SL}_r(R) \subseteq G$.

If $\det G = R^\times$, take any $M \in \mathrm{GL}_r(R)$ and choose $g \in G$ with $\det g = \det M$. Then $Mg^{-1} \in \mathrm{SL}_r(R) \subseteq G$, whence $M \in G$. $\square$

Proposition 7.8. *Suppose $\bar{\rho}_{\phi, \lambda}$ is irreducible and there is a prime $\mathfrak{p} \neq \lambda$ with*

$$v_{\mathfrak{p}}(g_{r-1}) = 0 \quad \text{and} \quad v_{\mathfrak{p}}(g_r) = 1.$$

Then $\widehat{\rho}_{\phi, \lambda}(G_F) = \mathrm{GL}_r(A_\lambda)$.

Proof. Theorem 3.6 implies that the full translation group U_f lies inside the image over A_λ . Proposition 7.7 therefore asserts $\mathrm{SL}_r(A_\lambda)$ is contained inside that image. Corollary 6.2 gives surjectivity for the determinant $A_\lambda^\times$, so the last assertion of Proposition 7.7 proves the claim. $\square$

8. NON-ENTANGLEMENT OF THE λ -ADIC IMAGES

We now study possible entanglement between the λ -adic images $\widehat{\rho}_{\phi, \lambda}(G_F) \subseteq \mathrm{GL}_r(A_\lambda)$. For distinct primes λ_1, λ_2 , write

$$G_{12} = (\widehat{\rho}_{\phi, \lambda_1}, \widehat{\rho}_{\phi, \lambda_2})(G_F) \subseteq \mathrm{GL}_r(A_{\lambda_1}) \times \mathrm{GL}_r(A_{\lambda_2})$$

for the joint image. We show that, if both λ -adic images are full and a prime $\mathfrak{p} \notin \{\lambda_1, \lambda_2\}$ satisfies (1.3), then

$$G_{12} \supseteq \mathrm{SL}_r(A_{\lambda_1}) \times \mathrm{SL}_r(A_{\lambda_2}).$$

The proof uses Theorem 3.6 as well as [Zyw25, Lemma 3.3].

Lemma 8.1. *Suppose $\mathfrak{p}$ satisfies (1.3), and let λ_1, λ_2 be distinct primes different from $\mathfrak{p}$. In compatible bases adapted to uniformization, the image of $P_{F_{\mathfrak{p}}}$ in the two local factors is*

$$U(A_{\lambda_1}) \times U(A_{\lambda_2}).$$

Proof. For $i = 1, 2$, the uniformization sequence is

$$0 \longrightarrow T_{\lambda_i}(\psi) \longrightarrow T_{\lambda_i}(\phi) \longrightarrow \Lambda \otimes_A A_{\lambda_i} \longrightarrow 0.$$

Choose a basis of $T_{\lambda_i}(\psi)$ and extend it to a basis of $T_{\lambda_i}(\phi)$ by lifting a generator of the rank-one quotient. Reducing these bases modulo powers of λ_i gives compatible adapted bases at every finite level.

Since ψ has good reduction at $\mathfrak{p}$ and $\lambda_i \neq \mathfrak{p}$, inertia acts trivially on $T_{\lambda_i}(\psi)$. Moreover, the action on the rank-one lattice Λ takes values in $A^\times = \mathbb{F}_q^\times$. This group has order prime to p , so the pro- p group $P_{F_{\mathfrak{p}}}$ acts trivially on Λ and hence on $\Lambda \otimes_A A_{\lambda_i}$. Thus, in the chosen bases, the joint image

$$H = (\widehat{\rho}_{\phi, \lambda_1}, \widehat{\rho}_{\phi, \lambda_2})(P_{F_{\mathfrak{p}}})$$

satisfies

$$H \subseteq U(A_{\lambda_1}) \times U(A_{\lambda_2}).$$

We prove that this inclusion is an equality by considering both factors simultaneously at finite level. Fix $n, m \geq 1$ and put $\mathfrak{a} = \lambda_1^n \lambda_2^m$. The Chinese remainder theorem gives a Galois-equivariant decomposition

$$\phi[\mathfrak{a}] \simeq \phi[\lambda_1^n] \oplus \phi[\lambda_2^m],$$

compatible with the submodules arising from ψ . The two reduced adapted bases therefore determine an adapted basis of $\phi[\mathfrak{a}]$ over $A/\mathfrak{a}$.

Since $\mathfrak{a}$ is prime to $\mathfrak{p}$, Theorem 3.6 identifies the wild inertia image at this level with $U(A/\mathfrak{a})$. This group is

$$U(A/\mathfrak{a}) \simeq U(A/\lambda_1^n) \times U(A/\lambda_2^m).$$

Consequently, reduction induces a surjection

$$H \longrightarrow U(A/\lambda_1^n) \times U(A/\lambda_2^m).$$

In particular, any prescribed pair of translation matrices modulo λ_1^n and λ_2^m is realized by the same element of $P_{F_{\mathfrak{p}}}$.

As n and m vary, the kernels of these reduction maps form a neighbourhood basis of the identity in $U(A_{\lambda_1}) \times U(A_{\lambda_2})$. The preceding surjectivity therefore shows that H is dense in this product. On the other hand, $P_{F_{\mathfrak{p}}}$ is compact and the representations are continuous, so H is compact and hence closed in the Hausdorff target. It follows that

$$H = U(A_{\lambda_1}) \times U(A_{\lambda_2}),$$

as required. $\square$

Proposition 8.2. *Let λ_1, λ_2 be distinct primes such that*

$$\widehat{\rho}_{\phi, \lambda_i}(G_F) = \mathrm{GL}_r(A_{\lambda_i}) \quad (i = 1, 2).$$

Suppose there is a prime $\mathfrak{p} \notin \{\lambda_1, \lambda_2\}$ satisfying (1.3). Then the joint image

$$G_{12} = (\widehat{\rho}_{\phi, \lambda_1}, \widehat{\rho}_{\phi, \lambda_2})(G_F)$$

contains $\mathrm{SL}_r(A_{\lambda_1}) \times \mathrm{SL}_r(A_{\lambda_2})$.

Proof. Put $R_i = A_{\lambda_i}$ and $S_i = \mathrm{SL}_r(R_i)$. By Lemma 8.1, there are primitive functionals $f_i : (R_i)^r \rightarrow R_i$ such that

$$U_{f_1} \times U_{f_2} \subseteq G_{12}.$$

In particular,

$$U_{f_1} \times \{I\} \subseteq G_{12}, \quad \{I\} \times U_{f_2} \subseteq G_{12}.$$

We first show that the first of these inclusions implies $S_1 \times \{I\} \subseteq G_{12}$.

Let $\epsilon_1, \dots, \epsilon_r$ be the coordinate functionals on R_1^r . Since f_1 is primitive, there is a basis in which it is a coordinate functional. Permuting this basis shows that, for every j , there is a matrix $h_j \in \mathrm{GL}_r(R_1)$ such that

$$f_1 \circ h_j^{-1} = \epsilon_j.$$

The first projection of G_{12} is $\mathrm{GL}_r(R_1)$ by hypothesis. We can therefore choose

$$g_j = (h_j, k_j) \in G_{12}$$

for some $k_j \in \mathrm{GL}_r(R_2)$.

Conjugating by g_j and using (7.4), we obtain

$$g_j(U_{f_1} \times \{I\})g_j^{-1} = U_{\epsilon_j} \times \{I\} \subseteq G_{12}.$$

The second component remains the identity because $k_j I k_j^{-1} = I$. Thus G_{12} contains all the groups $U_{\epsilon_j} \times \{I\}$, for $1 \leq j \leq r$. Since the ϵ_j form a basis of $(R_1^r)^*$, Lemma 7.6 shows that these groups generate $S_1 \times \{I\}$.

Applying the same argument to the second projection gives $\{I\} \times S_2 \subseteq G_{12}$. Finally, for arbitrary $s_i \in S_i$,

$$(s_1, s_2) = (s_1, I)(I, s_2) \in G_{12}.$$

Hence we conclude that $S_1 \times S_2 \subseteq G_{12}$. $\square$

Theorem 8.3. *Suppose that*

$$\widehat{\rho}_{\phi, \lambda}(G_F) = \mathrm{GL}_r(A_\lambda)$$

for every nonzero prime ideal $\lambda \subset A$, and that at least three distinct primes satisfy (1.3). Let $G = \widehat{\rho}_\phi(G_F)$. Then one has that

$$[G, G] = \mathrm{SL}_r(\widehat{A}), \quad G = (\det)^{-1}(\det G),$$

where $[G, G]$ denotes the commutator subgroup of G .

Proof. Set

$$S_\lambda = \mathrm{SL}_r(A_\lambda), \quad S = \prod_{\lambda} S_\lambda = \mathrm{SL}_r(\widehat{A}), \quad \text{and} \quad H = \overline{[G, G]}.$$

The closure is taken in $\mathrm{GL}_r(\widehat{A})$. Since G is the continuous image of the compact group G_F , it is compact and hence closed. Therefore $H \subseteq G$. Every commutator has determinant one, and S is closed, so $H \subseteq S$.

We first determine the projection of H onto any two factors. Fix distinct primes λ_1, λ_2 , and write

$$\pi_{12} : G \longrightarrow G_{12}$$

for the projection onto the corresponding joint image. Of the three primes satisfying (1.3), at least one differs from both λ_1 and λ_2 . Proposition 8.2 therefore gives

$$S_{\lambda_1} \times S_{\lambda_2} \subseteq G_{12}.$$

Each S_{λ_i} is perfect by Lemma 7.5, so their product is also perfect. It follows that

$$\begin{aligned} S_{\lambda_1} \times S_{\lambda_2} &= [S_{\lambda_1} \times S_{\lambda_2}, S_{\lambda_1} \times S_{\lambda_2}] \\ &\subseteq [G_{12}, G_{12}] \subseteq S_{\lambda_1} \times S_{\lambda_2}. \end{aligned}$$

Thus

$$[G_{12}, G_{12}] = S_{\lambda_1} \times S_{\lambda_2}.$$

Since π_{12} is surjective,

$$\pi_{12}([G, G]) = [G_{12}, G_{12}].$$

Indeed, a commutator $[x, y]$ in G_{12} is the image of $[\tilde{x}, \tilde{y}]$ for any lifts $\tilde{x}, \tilde{y} \in G$, and the same observation applies to finite products of commutators. Since $[G, G] \subseteq H \subseteq S$, we conclude that

$$\pi_{12}(H) = S_{\lambda_1} \times S_{\lambda_2}.$$

Now let $\mathcal{L}$ be a finite set of at least two primes, and let $H_{\mathcal{L}}$ be the image of H in $\prod_{\lambda \in \mathcal{L}} S_{\lambda}$. The group $H_{\mathcal{L}}$ is compact and hence closed. Its projection onto every pair of factors is surjective by the preceding argument. Each factor is perfect, so [Zyw25, Lemma 3.3] gives

$$H_{\mathcal{L}} = \prod_{\lambda \in \mathcal{L}} S_{\lambda}.$$

Consequently, the projection of H onto every finite collection of factors of S is surjective. To see that this makes H dense in S , take any $s = (s_{\lambda})_{\lambda} \in S$ and any basic open neighbourhood of s . Such a neighbourhood imposes conditions on only finitely many coordinates. Surjectivity onto those coordinates supplies an element of H agreeing with s there, and hence lying in the neighbourhood. Since H is closed, we obtain

$$H = S = \mathrm{SL}_r(\hat{A}).$$

In particular, $S \subseteq G$. We now use the perfectness of S , established in Lemma 7.5:

$$S = [S, S] \subseteq [G, G] \subseteq S.$$

Thus $[G, G] = S$.

It remains to determine G from its determinant image. The inclusion $G \subseteq (\det)^{-1}(\det G)$ follows from the definition. Conversely, take $M \in \mathrm{GL}_r(\hat{A})$ with $\det M \in \det G$. Choose $g \in G$ such that $\det g = \det M$. Then

$$\det(Mg^{-1}) = 1,$$

so $Mg^{-1} \in S \subseteq G$, and hence $M \in G$. This proves $G = (\det)^{-1}(\det G)$. $\square$

9. CONGRUENCE CONDITIONS AND DENSITY ESTIMATES

We begin by recalling that $\mathcal{B} = A^{r-1} \times (A \setminus \{0\})$ and that $\mathcal{B}_n = A_{\leq n}^{r-1} \times (A_{\leq n} \setminus \{0\})$. In particular,

$$|\mathcal{B}_n| = q^{(r-1)(n+1)}(q^{n+1} - 1) = q^{r(n+1)}(1 - q^{-(n+1)}).$$

Let $\mathfrak{b}$ be a fixed nonzero ideal of degree e . For $n+1 \geq e$, reduction gives a surjective $\mathbb{F}_q$ -linear map $A_{\leq n} \rightarrow A/\mathfrak{b}$. Its source and target have dimensions $n+1$ and e , respectively as $\mathbb{F}_q$ -vector spaces. Every fibre therefore has q^{n+1-e} elements. Consequently, for any subset $\Omega \subseteq (A/\mathfrak{b})^r$, the proportion of tuples in $A_{\leq n}^r$ reducing into Ω is exactly $|\Omega|/|A/\mathfrak{b}|^r$ once $n+1 \geq e$.

The tuples with $g_r = 0$ is a set with density $q^{-(n+1)}$ in $A_{\leq n}^r$. Removing them does not change the limiting values. Thus we find that

$$\lim_{n \rightarrow \infty} \frac{|\{g \in \mathcal{B}_n : g \bmod \mathfrak{b} \in \Omega\}|}{|\mathcal{B}_n|} = \frac{|\Omega|}{|A/\mathfrak{b}|^r}.$$

Definition 9.1. For a nonzero prime $\mathfrak{p} \subset A$, let

$$E_{\mathfrak{p}} = \{g \in \mathcal{B} : v_{\mathfrak{p}}(g_{r-1}) = 0, \quad \text{and} \quad v_{\mathfrak{p}}(g_r) = 1\}.$$

These conditions depend only on the tuple modulo $\mathfrak{p}^2$. Let $\overline{E}_{\mathfrak{p}} \subseteq (A/\mathfrak{p}^2)^r$ be the set of residue tuples satisfying them, and define

$$c_{\mathfrak{p}} = \frac{|\overline{E}_{\mathfrak{p}}|}{|A/\mathfrak{p}^2|^r}.$$

In other words, $c_{\mathfrak{p}}$ is the probability that the conditions defining $E_{\mathfrak{p}}$ hold when the coefficients are chosen uniformly modulo $\mathfrak{p}^2$.

Lemma 9.2. *For every nonzero prime $\mathfrak{p}$, the set $E_{\mathfrak{p}}$ has density $c_{\mathfrak{p}}$. Writing $Q_{\mathfrak{p}} = N\mathfrak{p} = q^{\deg \mathfrak{p}}$, we have*

$$c_{\mathfrak{p}} = (1 - Q_{\mathfrak{p}}^{-1}) (Q_{\mathfrak{p}}^{-1} - Q_{\mathfrak{p}}^{-2}) = Q_{\mathfrak{p}}^{-1} (1 - Q_{\mathfrak{p}}^{-1})^2. \quad (9.1)$$

Moreover,

$$\sum_{\mathfrak{p}} c_{\mathfrak{p}} = \infty.$$

There are also at least two primes of every sufficiently large degree.

Proof. Fix $\mathfrak{p}$, and put $Q = Q_{\mathfrak{p}}$. We first compute the proportion $c_{\mathfrak{p}}$. There are Q^2 residue classes modulo $\mathfrak{p}^2$, of which Q are divisible by $\mathfrak{p}$. Hence there are $Q^2 - Q$ choices for g_{r-1} satisfying $v_{\mathfrak{p}}(g_{r-1}) = 0$.

For g_r , we require divisibility by $\mathfrak{p}$ but not by $\mathfrak{p}^2$. Its residue class must therefore be a nonzero element of $\mathfrak{p}/\mathfrak{p}^2$, giving $Q - 1$ choices. The other $r - 2$ coefficients are unrestricted, with Q^2 choices each. Thus

$$|\overline{E}_{\mathfrak{p}}| = (Q^2 - Q)(Q - 1)Q^{2(r-2)}.$$

Dividing by the total number Q^{2r} of residue tuples gives

$$c_{\mathfrak{p}} = \frac{Q^2 - Q}{Q^2} \frac{Q - 1}{Q^2} = Q^{-1}(1 - Q^{-1})^2.$$

For all sufficiently large n , the tuples in $A_{\leq n}^r$ are uniformly distributed modulo $\mathfrak{p}^2$. Exactly a proportion $c_{\mathfrak{p}}$ satisfies the required conditions. Every such tuple has $g_r \neq 0$, so it already belongs to $\mathcal{B}_n$. Consequently,

$$\frac{|E_{\mathfrak{p}} \cap \mathcal{B}_n|}{|\mathcal{B}_n|} = \frac{c_{\mathfrak{p}} q^{r(n+1)}}{q^{r(n+1)}(1 - q^{-(n+1)})} \longrightarrow c_{\mathfrak{p}}.$$

This proves $\delta(E_{\mathfrak{p}}) = c_{\mathfrak{p}}$.

We now prove that $\sum_{\mathfrak{p}} c_{\mathfrak{p}}$ diverges. The formula just obtained shows that a prime of degree d contributes

$$q^{-d}(1 - q^{-d})^2.$$

To determine the total contribution from degree d , we must therefore count how many primes have that degree. Let $\pi_q(d)$ denote this number, or equivalently the number of monic irreducible polynomials of degree d . Then

$$\sum_{\deg \mathfrak{p}=d} c_{\mathfrak{p}} = \pi_q(d) q^{-d} (1 - q^{-d})^2.$$

We will show that this quantity is asymptotic to $1/d$.

An irreducible polynomial of degree d over $\mathbb{F}_q$ divides $T^{q^n} - T$ precisely when $d \mid n$. Moreover, $T^{q^n} - T$ has no repeated factors, since its derivative is -1 . Comparing degrees in its factorization therefore gives

$$q^n = \sum_{d \mid n} d \pi_q(d).$$

By Möbius inversion,

$$d \pi_q(d) = \sum_{a \mid d} \mu(a) q^{d/a},$$

where μ is the Möbius function. The term with $a = 1$ is q^d . For every other divisor a , we have $d/a \leq d/2$, so

$$\left| \sum_{\substack{a \mid d \\ a > 1}} \mu(a) q^{d/a} \right| \leq \sum_{j=1}^{\lfloor d/2 \rfloor} q^j = O_q(q^{d/2}).$$

It follows that

$$\pi_q(d) = \frac{q^d}{d} + O_q\left(\frac{q^{d/2}}{d}\right).$$

Substituting this estimate into the contribution from degree d , we obtain

$$\sum_{\deg \mathfrak{p} = d} c_{\mathfrak{p}} = \pi_q(d) q^{-d} (1 - q^{-d})^2 = \frac{1}{d} + O_q\left(\frac{q^{-d/2}}{d}\right).$$

Thus the number of primes of degree d compensates for the small contribution of each individual prime. Summing over $1 \leq d \leq D$ gives

$$\sum_{\deg \mathfrak{p} \leq D} c_{\mathfrak{p}} = \sum_{d=1}^D \frac{1}{d} + O_q(1),$$

because $\sum_{d \geq 1} q^{-d/2}/d$ converges. The harmonic sum tends to infinity, proving $\sum_{\mathfrak{p}} c_{\mathfrak{p}} = \infty$.

Finally, the estimate for $\pi_q(d)$ shows that $\pi_q(d) \rightarrow \infty$. In particular, there are at least two primes of every sufficiently large degree. $\square$

We now use the divergence of $\sum_{\mathfrak{p}} c_{\mathfrak{p}}$ to show that almost every coefficient tuple satisfies the required valuation conditions at at least three primes.

Proposition 9.3. *Let $\mathcal{R}$ be the set of tuples satisfying $E_{\mathfrak{p}}$ at at least three distinct primes. Then $\delta(\mathcal{R}) = 1$.*

Proof. Let $\varepsilon > 0$. Since $\sum_{\mathfrak{p}} c_{\mathfrak{p}} = \infty$, we can choose a finite set $\mathcal{P}$ of primes such that

$$S_{\mathcal{P}} := \sum_{\mathfrak{p} \in \mathcal{P}} c_{\mathfrak{p}} > 2 \quad \text{and} \quad \frac{S_{\mathcal{P}}}{(S_{\mathcal{P}} - 2)^2} < \varepsilon.$$

We keep this set $\mathcal{P}$ fixed throughout the following counting argument.

Define

$$\mathcal{C}_{\mathcal{P}} = \{g \in \mathcal{B} : \#\{\mathfrak{p} \in \mathcal{P} : g \in E_{\mathfrak{p}}\} \leq 2\}.$$

This set depends on $\mathcal{P}$, but it contains the fixed exceptional set $\mathcal{B} \setminus \mathcal{R}$. Indeed, a tuple satisfying the valuation conditions at fewer than three primes in total necessarily satisfies them at fewer than three primes in $\mathcal{P}$. Thus

$$\mathcal{B} \setminus \mathcal{R} \subseteq \mathcal{C}_{\mathcal{P}}.$$

We next bound the density of $\mathcal{C}_{\mathcal{P}}$. Put

$$\mathfrak{b}_{\mathcal{P}} = \prod_{\mathfrak{p} \in \mathcal{P}} \mathfrak{p}^2,$$

and choose a residue tuple uniformly in $(A/\mathfrak{b}_{\mathcal{P}})^r$. For each $\mathfrak{p} \in \mathcal{P}$, let $X_{\mathfrak{p}}$ be one if the tuple satisfies the conditions defining $E_{\mathfrak{p}}$, and zero otherwise. Set

$$X_{\mathcal{P}} = \sum_{\mathfrak{p} \in \mathcal{P}} X_{\mathfrak{p}}.$$

The Chinese remainder theorem gives

$$(A/\mathfrak{b}_{\mathcal{P}})^r \simeq \prod_{\mathfrak{p} \in \mathcal{P}} (A/\mathfrak{p}^2)^r.$$

Under the uniform measure, $X_{\mathfrak{p}}$ depends only on the $\mathfrak{p}$ -component and has mean $c_{\mathfrak{p}}$. Consequently,

$$\mathbb{E}[X_{\mathcal{P}}] = S_{\mathcal{P}} \quad \text{and} \quad \text{Var}(X_{\mathcal{P}}) = \sum_{\mathfrak{p} \in \mathcal{P}} c_{\mathfrak{p}}(1 - c_{\mathfrak{p}}) \leq S_{\mathcal{P}}.$$

If $X_{\mathcal{P}} \leq 2$, then $|X_{\mathcal{P}} - S_{\mathcal{P}}| \geq S_{\mathcal{P}} - 2$. Chebyshev's inequality therefore gives

$$\Pr(X_{\mathcal{P}} \leq 2) \leq \frac{\text{Var}(X_{\mathcal{P}})}{(S_{\mathcal{P}} - 2)^2} \leq \frac{S_{\mathcal{P}}}{(S_{\mathcal{P}} - 2)^2} < \varepsilon.$$

Membership in $\mathcal{C}_{\mathcal{P}}$ is determined modulo the fixed ideal $\mathfrak{b}_{\mathcal{P}}$. For $n + 1 \geq \deg \mathfrak{b}_{\mathcal{P}}$, reduction from $A_{\leq n}^r$ to $(A/\mathfrak{b}_{\mathcal{P}})^r$ has equal-sized fibres. The proportion of tuples in $A_{\leq n}^r$ satisfying at most two of the conditions is therefore exactly $\Pr(X_{\mathcal{P}} \leq 2)$. Removing the tuples with $g_r = 0$ does not change the limit, since their proportion is $q^{-(n+1)}$. Hence

$$\delta(\mathcal{C}_{\mathcal{P}}) = \lim_{n \rightarrow \infty} \frac{|\mathcal{C}_{\mathcal{P}} \cap \mathcal{B}_n|}{|\mathcal{B}_n|} = \Pr(X_{\mathcal{P}} \leq 2) < \varepsilon.$$

Using $\mathcal{B} \setminus \mathcal{R} \subseteq \mathcal{C}_{\mathcal{P}}$, we obtain

$$\bar{\delta}(\mathcal{B} \setminus \mathcal{R}) \leq \delta(\mathcal{C}_{\mathcal{P}}) < \varepsilon.$$

The set $\mathcal{B} \setminus \mathcal{R}$ is independent of ε and $\mathcal{P}$. Since $\varepsilon > 0$ was arbitrary, its upper density is zero. Therefore $\delta(\mathcal{R}) = 1$. $\square$

Let $\mathcal{S}_m$ be the set of tuples for which no prime of degree greater than m divides both g_{r-1} and g_r . We set $\varepsilon_m = \bar{\delta}(\mathcal{B} \setminus \mathcal{S}_m)$.

Lemma 9.4. *We have that $\varepsilon_m \rightarrow 0$ as $m \rightarrow \infty$.*

Proof. Let $\mathcal{D}_m \subseteq A^2$ consist of pairs (u, v) such that some prime of degree greater than m divides both u and v . By [Zyw25, Lemma 9.5],

$$\eta_m := \limsup_{n \rightarrow \infty} \frac{|\mathcal{D}_m \cap A_{\leq n}^2|}{q^{2(n+1)}} \rightarrow 0$$

as $m \rightarrow \infty$. A tuple in $\mathcal{B}_n \setminus \mathcal{S}_m$ has $(g_{r-1}, g_r) \in \mathcal{D}_m \cap A_{\leq n}^2$. For each such pair, there are $q^{(r-2)(n+1)}$ choices for the other coordinates. Consequently,

$$|\mathcal{B}_n \setminus \mathcal{S}_m| \leq q^{(r-2)(n+1)} |\mathcal{D}_m \cap A_{\leq n}^2|.$$

Dividing by $|\mathcal{B}_n|$ gives

$$\frac{|\mathcal{B}_n \setminus \mathcal{S}_m|}{|\mathcal{B}_n|} \leq \frac{1}{1 - q^{-(n+1)}} \frac{|\mathcal{D}_m \cap A_{\leq n}^2|}{q^{2(n+1)}}.$$

Taking the limit superior in n , with m fixed, yields $\varepsilon_m \leq \eta_m$. This completes the proof. $\square$

For all sufficiently large integers m , set

$$d(m) = \left\lfloor \frac{m}{2C_r} \right\rfloor \quad \text{and} \quad \mathfrak{a}_m = \prod_{\deg \lambda \leq m} \lambda.$$

Since $d(m) \rightarrow \infty$, we may choose two distinct primes $\mathfrak{p}_1(m), \mathfrak{p}_2(m)$ of degree $d(m)$. These primes are fixed once m is chosen and do not depend on the coefficient tuple.

Let $\mathcal{T}_m$ be the set of tuples for which

$$\mathfrak{p}_1(m) \nmid g_r \quad \text{and} \quad \mathfrak{p}_2(m) \nmid g_r.$$

For a fixed prime $\mathfrak{p}$ of degree d and $n \geq d$, there are $q^{n+1-d} - 1$ nonzero multiples of its monic generator in $A_{\leq n}$. Thus the proportion of nonzero polynomials in $A_{\leq n}$ divisible by $\mathfrak{p}$ is

$$\frac{q^{n+1-d} - 1}{q^{n+1} - 1} \rightarrow q^{-d}.$$

The other coefficients can be chosen freely, so the same limiting proportion holds in $\mathcal{B}_n$. Applying the union bound to the two chosen primes gives

$$\bar{\delta}(\mathcal{B} \setminus \mathcal{T}_m) \leq 2q^{-d(m)}. \quad (9.2)$$

Every tuple in $\mathcal{T}_m$ has good reduction at both primes: all coefficients are integral there, and the leading coefficient g_r is a unit.

Let

$$\mathcal{U}_m = \{\mathbf{g} \in \mathcal{B} : \rho_{\mathbf{g}, \mathfrak{a}_m}(G_F) = \mathrm{GL}_r(A/\mathfrak{a}_m)\}.$$

Theorem 7.1 implies that $\delta(\mathcal{U}_m) = 1$. Define

$$\mathcal{V}_m = \mathcal{R} \cap \mathcal{S}_m \cap \mathcal{T}_m \cap \mathcal{U}_m. \quad (9.3)$$

Proposition 9.5. *For all sufficiently large m ,*

$$\underline{\delta}(\mathcal{V}_m) \geq 1 - \varepsilon_m - 2q^{-d(m)}.$$

Consequently, $\underline{\delta}(\mathcal{V}_m) \rightarrow 1$ as $m \rightarrow \infty$.

Proof. Given a fixed value m , a tuple outside $\mathcal{V}_m$ fails at least one of the four conditions in (9.3). Therefore

$$\mathcal{B} \setminus \mathcal{V}_m \subseteq (\mathcal{B} \setminus \mathcal{R}) \cup (\mathcal{B} \setminus \mathcal{S}_m) \cup (\mathcal{B} \setminus \mathcal{T}_m) \cup (\mathcal{B} \setminus \mathcal{U}_m).$$

Counting in $\mathcal{B}_n$, dividing by $|\mathcal{B}_n|$, and taking the limit superior gives

$$\begin{aligned} \bar{\delta}(\mathcal{B} \setminus \mathcal{V}_m) &\leq \bar{\delta}(\mathcal{B} \setminus \mathcal{R}) + \bar{\delta}(\mathcal{B} \setminus \mathcal{S}_m) \\ &\quad + \bar{\delta}(\mathcal{B} \setminus \mathcal{T}_m) + \bar{\delta}(\mathcal{B} \setminus \mathcal{U}_m). \end{aligned}$$

The first and fourth terms are zero. The second is ε_m , and the third is at most $2q^{-d(m)}$. Since

$$\underline{\delta}(\mathcal{V}_m) = 1 - \bar{\delta}(\mathcal{B} \setminus \mathcal{V}_m),$$

we obtain the asserted lower bound. Finally, $\varepsilon_m \rightarrow 0$ by Lemma 9.4, and $q^{-d(m)} \rightarrow 0$ because $d(m) \rightarrow \infty$. $\square$

10. PROOF OF THE MAIN THEOREMS

In this section, we prove the main results of the article.

Proposition 10.1. *For every sufficiently large m and every $\mathbf{g} \in \mathcal{V}_m$,*

$$\widehat{\rho}_{\mathbf{g},\lambda}(G_F) = \mathrm{GL}_r(A_\lambda)$$

for every nonzero prime $\lambda \subset A$.

Proof. Fix $\mathbf{g} \in \mathcal{V}_m$, and let ϕ be the Drinfeld module determined by this tuple. We first prove that $\bar{\rho}_{\mathbf{g},\lambda}$ is irreducible for every λ .

Suppose first that $\deg \lambda \leq m$. Then $\lambda \mid \mathfrak{a}_m$. Since $\mathbf{g} \in \mathcal{U}_m$, the representation at level $\mathfrak{a}_m$ has image $\mathrm{GL}_r(A/\mathfrak{a}_m)$. The Chinese remainder theorem shows that the projection

$$\mathrm{GL}_r(A/\mathfrak{a}_m) \longrightarrow \mathrm{GL}_r(k_\lambda)$$

is surjective. Hence

$$\bar{\rho}_{\mathbf{g},\lambda}(G_F) = \mathrm{GL}_r(k_\lambda).$$

The natural action of this group on k_λ^r is irreducible: the orbit of any nonzero vector consists of all nonzero vectors, so a nonzero invariant subspace must be the entire space.

Now suppose that $\deg \lambda > m$. Since $\mathbf{g} \in \mathcal{S}_m$, the prime λ does not divide both g_{r-1} and g_r . There are two possibilities. If $\lambda \nmid g_r$, all coefficients are integral at λ and the leading coefficient is a unit, so ϕ has good reduction there. If $\lambda \mid g_r$, then $\lambda \nmid g_{r-1}$. The reduction of the displayed integral model then has rank $r - 1$, so ϕ has stable reduction of that rank at λ .

Membership in $\mathcal{T}_m$ supplies the two distinct good primes $\mathfrak{p}_1(m), \mathfrak{p}_2(m)$, both of degree $d(m)$. Thus the reduction hypotheses of Theorem 7.3 hold. If $\bar{\rho}_{\mathbf{g},\lambda}$ were reducible, that theorem would give

$$\deg \lambda \leq C_r d(m) = C_r \left\lfloor \frac{m}{2C_r} \right\rfloor \leq \frac{m}{2}.$$

This contradicts $\deg \lambda > m$. The residual representation is therefore irreducible in this case as well.

We have proved residual irreducibility at every prime. Furthermore, $\mathbf{g} \in \mathcal{R}$ supplies three distinct primes satisfying (1.3). For any given λ , choose one of them, say $\mathfrak{p}$, different from λ . Then

$$v_{\mathfrak{p}}(g_{r-1}) = 0, \quad v_{\mathfrak{p}}(g_r) = 1.$$

The hypotheses of Proposition 7.8 are now satisfied at λ . It follows that

$$\widehat{\rho}_{\mathbf{g},\lambda}(G_F) = \mathrm{GL}_r(A_\lambda).$$

Since λ was arbitrary, this holds for every prime. □

Proof of Theorem A. For $\mathbf{g} \in \mathcal{B}$, write

$$G_{\mathbf{g}} = \widehat{\rho}_{\mathbf{g}}(G_F).$$

Let $\mathcal{M}$ be the set of tuples for which

$$\widehat{\rho}_{\mathbf{g},\lambda}(G_F) = \mathrm{GL}_r(A_\lambda) \quad \text{for every prime } \lambda, \quad [G_{\mathbf{g}}, G_{\mathbf{g}}] = \mathrm{SL}_r(\widehat{A}).$$

We first prove that $\delta(\mathcal{M}) = 1$. Fix a sufficiently large m and take $\mathbf{g} \in \mathcal{V}_m$. Proposition 10.1 gives surjectivity of every λ -adic representation. Since $\mathbf{g} \in \mathcal{R}$, there are also three distinct primes satisfying (1.3). Theorem 8.3 therefore gives

$$[G_{\mathbf{g}}, G_{\mathbf{g}}] = \mathrm{SL}_r(\widehat{A}).$$

Thus $\mathcal{V}_m \subseteq \mathcal{M}$ for every sufficiently large m . By Proposition 9.5,

$$\underline{\delta}(\mathcal{M}) \geq \underline{\delta}(\mathcal{V}_m) \geq 1 - \varepsilon_m - 2q^{-d(m)}.$$

Letting $m \rightarrow \infty$ gives $\underline{\delta}(\mathcal{M}) \geq 1$. Since all box proportions are at most one, this proves $\delta(\mathcal{M}) = 1$.

Here the limits defining the densities are taken with m fixed. The subsequent limit in m improves the lower bound for the single set $\mathcal{M}$. This is why the fixed-level density statements suffice to obtain simultaneous surjectivity at all primes.

Now fix $\mathbf{g} \in \mathcal{M}$. Since the commutator subgroup is contained in the group itself,

$$\mathrm{SL}_r(\widehat{A}) \subseteq G_{\mathbf{g}}.$$

If $M \in \mathrm{GL}_r(\widehat{A})$ has determinant in $\det G_{\mathbf{g}}$, choose $g \in G_{\mathbf{g}}$ with $\det g = \det M$. Then $Mg^{-1} \in \mathrm{SL}_r(\widehat{A}) \subseteq G_{\mathbf{g}}$, so $M \in G_{\mathbf{g}}$. Hence

$$G_{\mathbf{g}} = (\det)^{-1}(\det G_{\mathbf{g}}).$$

The determinant map on $\mathrm{GL}_r(\widehat{A})$ is surjective, as is seen from the matrices $\mathrm{diag}(u, 1, \dots, 1)$. It therefore induces an isomorphism

$$\mathrm{GL}_r(\widehat{A})/G_{\mathbf{g}} \simeq \widehat{A}^\times / \det G_{\mathbf{g}}.$$

In particular, the adelic index is the determinant index.

The determinant representation is the adelic representation attached to the rank-one Drinfeld module

$$\psi_T = T + \Delta\tau, \quad \text{where } \Delta = (-1)^{r-1}g_r.$$

Put $d = \deg g_r = \deg \Delta$. Then

$$(-1)^d \mathrm{lc}(\Delta) = (-1)^{d+r-1} \mathrm{lc}(g_r),$$

whose order in $\mathbb{F}_q^\times$ is $e_r(g_r)$. Applying (6.3) gives

$$\begin{aligned} [\mathrm{GL}_r(\widehat{A}) : G_{\mathbf{g}}] &= [\widehat{A}^\times : \det G_{\mathbf{g}}] \\ &= \gcd\left(d-1, \frac{q-1}{e_r(g_r)}\right). \end{aligned}$$

This index divides $q-1$, and all the asserted properties hold on the density-one set $\mathcal{M}$. $\square$

Proof of Theorem B. Recall that $\mathcal{P}$ consists of the tuples for which

$$(-1)^{\deg g_r + r-1} \mathrm{lc}(g_r)$$

generates $\mathbb{F}_q^\times$. For such a tuple, $e_r(g_r) = q-1$. The determinant formula (6.3) therefore gives

$$[\widehat{A}^\times : \det G_{\mathbf{g}}] = \gcd(\deg g_r - 1, 1) = 1.$$

Thus $\det G_{\mathbf{g}} = \widehat{A}^\times$ for every $\mathbf{g} \in \mathcal{P}$.

If $\mathbf{g}$ also belongs to $\mathcal{M}$, Theorem A gives

$$G_{\mathbf{g}} = (\det)^{-1}(\det G_{\mathbf{g}}) = \mathrm{GL}_r(\widehat{A}).$$

Consequently,

$$\mathcal{M} \cap \mathcal{P} \subseteq \mathcal{S}_{\mathrm{ad}},$$

where $\mathcal{S}_{\mathrm{ad}}$ is the set of tuples with surjective adelic representation.

Put

$$c = \frac{\varphi(q-1)}{q-1} > 0.$$

By Lemma 6.3,

$$|\mathcal{P} \cap \mathcal{B}_n| = c|\mathcal{B}_n|$$

for every n . A tuple in $\mathcal{P}$ whose adelic representation is not surjective cannot belong to $\mathcal{M}$. Hence

$$(\mathcal{P} \setminus \mathcal{S}_{\text{ad}}) \cap \mathcal{B}_n \subseteq \mathcal{B}_n \setminus \mathcal{M}.$$

Dividing by $|\mathcal{P} \cap \mathcal{B}_n|$ yields

$$0 \leq 1 - \frac{|\mathcal{S}_{\text{ad}} \cap \mathcal{P} \cap \mathcal{B}_n|}{|\mathcal{P} \cap \mathcal{B}_n|} \leq \frac{|\mathcal{B}_n \setminus \mathcal{M}|}{c|\mathcal{B}_n|}.$$

On the other hand,

$$\frac{|\mathcal{B}_n \setminus \mathcal{M}|}{c|\mathcal{B}_n|} \rightarrow 0$$

since $\delta(\mathcal{M}) = 1$. This proves that adelic surjectivity has relative density one inside $\mathcal{P}$.

Note that

$$\begin{aligned} |\mathcal{S}_{\text{ad}} \cap \mathcal{B}_n| &\geq |\mathcal{M} \cap \mathcal{P} \cap \mathcal{B}_n| \\ &\geq |\mathcal{P} \cap \mathcal{B}_n| - |\mathcal{B}_n \setminus \mathcal{M}|. \end{aligned}$$

After dividing by $|\mathcal{B}_n|$ and taking the limit inferior, we obtain

$$\underline{\delta}(\mathcal{S}_{\text{ad}}) \geq c = \frac{\varphi(q-1)}{q-1}.$$

This proves the claimed lower bound. $\square$

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