

Partitioning $\{1, \dots, n\}$ into parts of size at most three whose sums are powers of three

Deep Bhattacharjee^{a,b,1,*}, Ushashi Bhattacharya^{c,2}

^aFormerly, *Electro-Gravitational Space Propulsion Laboratory (EGSPL), Bhubaneswar, 751030, Odisha, India*

^bFormerly, *University of Calcutta, Senate House, 87/1 College Street, Kolkata, 700073, West Bengal, India*

^cFormerly, *National Taiwan University, Taipei, 106319, Taiwan*

Abstract

We prove unconditionally the conjecture of Gurvich and Naumova that for every $n \geq 1$ the set $\{1, \dots, n\}$ can be partitioned into parts of at most three elements whose sums are powers of 3. The partition is given by an explicit recursion.

Keywords: set partition, powers of three, zero-sum triples, formal verification

2020 MSC: Primary 05A18, Secondary 11B75, 68V15

1. Introduction

Let $m \geq 2$ and $n \geq 1$ be integers and $[n] = \{1, \dots, n\}$. Gurvich and Naumova [1] call a partition of $[n]$ *m-good* if every part has at most m elements and the sum of the elements of every part is a power m^k , $k \geq 0$, of m . Every $[n]$ has exactly one 2-good partition [1, Proposition 1]. In an m -good partition every part other than $\{1\}$ has sum $\equiv 0 \pmod{m}$, so $n(n+1)/2 \equiv 0$ or $1 \pmod{m}$. For $n = 2(m^t + 1)$ this sum is $\equiv 3 \pmod{m}$;

*Corresponding author.

Email addresses: itsdeep@live.com (Deep Bhattacharjee),
d.bhattacharjee@erl-forschung.de (Deep Bhattacharjee),
rai.bhattacharya.in@gmail.com (Ushashi Bhattacharya)

¹ORCID: <https://orcid.org/0000-0003-0466-750X>

²ORCID: <https://orcid.org/0009-0002-2254-3914>

hence for $m \geq 4$ the set $[n]$ has no m -good partition [1, Observation 1].³ For $m = 3$ they stated:

Conjecture 1.1 ([1, Conjecture 1]). *For every $n \geq 1$ the set $[n]$ has a 3-good partition.*

They proved that a minimal counterexample is of the form $n = 3^t + 3k + 2$ with $0 \leq k < (3^{t-1} - 1)/2$ and $k \neq (3^\ell - 1)/2$ for $0 \leq \ell < t$ [1, Theorem 1], recorded further reductions found by O. Polubasoff, and reported that a computation by D. Rybin confirmed the conjecture for $n \leq 844$. We prove it.

Theorem 1.2. *For every $n \geq 1$ the set $[n]$ has a 3-good partition.*

Corollary 1.3. *Let $m \geq 2$. Every set $[n]$, $n \geq 1$, has an m -good partition if and only if $m \in \{2, 3\}$.*

The proof is a recursion on n . Let $T = 3^t \leq n < 3T$. For $n = T$, for $2n > 3T$, and for $n = T + j$ with $j \not\equiv 2 \pmod{3}$, the reductions of [1, Section 2] pass from $[n]$ to a shorter interval. For the remaining $n = T + j$, $j \equiv 2 \pmod{3}$, $2j < T$, we prove more: $[n]$ minus the triple $\{n - 1, n, T - 2j + 1\}$, whose sum is $3T$, has a 3-good partition (Proposition 4.1). This statement passes from j to n by a reflection argument (Lemma 3.1), except when j lies in the upper half of the interval $[3^u, 3^{u+1})$ that contains it. In that case a fixed-point-free involution of $\{1, \dots, 2N - 2\}$ with distinct pair sums supplies the missing triples (Lemma 3.2). Figure 1 shows the four cases, and Section 5 describes the formal verification of the proof.

2. Good and balanced sets

For integers $a \leq b$ let $[a, b] = \{x \in \mathbb{Z} : a \leq x \leq b\}$, and $[a, b] = \emptyset$ if $a > b$. For $Z \subset \mathbb{Z}$ and $c, s \in \mathbb{Z}$ write $c + sZ = \{c + sz : z \in Z\}$ and $c - Z = \{c - z : z \in Z\}$. A finite set $A \subset \mathbb{Z}$ is *admissible* if $|A| \leq 3$ and

³Partitions of an interval into small parts tied by an additive condition go back to Skolem [2], who proved that $\{1, \dots, 2n\}$ is a disjoint union of pairs $\{a_i, b_i\}$ with $b_i - a_i = i$, $1 \leq i \leq n$, if and only if $n \equiv 0$ or $1 \pmod{4}$. Such pairs give a partition of $\{1, \dots, 3n\}$ into the n triples $\{i, n + a_i, n + b_i\}$, each of the form $\{x, y, x + y\}$: the first entries run over $\{1, \dots, n\}$, the other two over $\{n + 1, \dots, 3n\}$, and $i + (n + a_i) = n + b_i$. In the problem of Gurvich and Naumova the condition is on the sum of a part rather than on a difference.

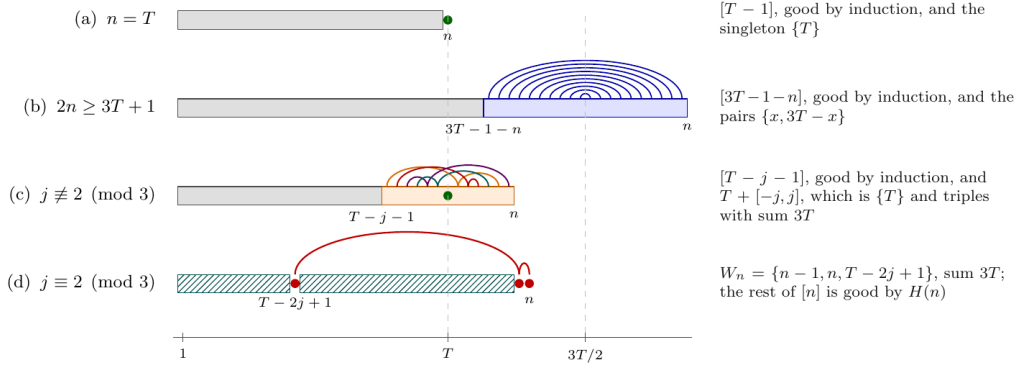


Figure 1: The four cases of the recursion for $T = 27$, drawn for $n = 27, 50, 33, 35$. In (a)–(c) the gray interval is good by induction. In (d) the three red points form W_n , and the rest of $[n]$ is good by $H(n)$, Proposition 4.1.

$\sum_{x \in A} x = 3^k$ for some $k \geq 0$. A finite set $S \subset \mathbb{Z}$ is *good* if it is a disjoint union of admissible sets;⁴ thus $[n]$ is good if and only if it has a 3-good partition. A finite set $Z \subset \mathbb{Z}$ is *balanced* if it is a disjoint union of sets each of which is $\{0\}$ or a 3-element set with sum 0. Disjoint unions of good sets are good, and disjoint unions of balanced sets are balanced.

Lemma 2.1 ([1, Lemma 1]). *If $\ell \geq 0$ and $\ell \not\equiv 2 \pmod{3}$, then $[-\ell, \ell]$ is balanced.*

Proof. Write $\ell = 3k + r$ with $r \in \{0, 1\}$. For $0 \leq i < k$ the sets

$$A_i = \{-(3k-i), 1+i, 3k-1-2i\}, \quad B_i = \{-(2k-i), -(k-i), 3k-2i\}$$

have sum 0. As i runs over $[0, k-1]$, the entries $-(3k-i)$, $-(2k-i)$, $-(k-i)$ run over $[-3k, -2k-1]$, $[-2k, -k-1]$, $[-k, -1]$, the entries $1+i$ run over $[1, k]$, and the entries $3k-1-2i$ and $3k-2i$ run over the numbers in $[k+1, 3k]$ of the parity of $3k-1$ and of $3k$. So the $2k$ sets A_i, B_i partition $[-3k, 3k] \setminus \{0\}$. Add $\{0\}$ if $r = 0$ and $\{-\ell, 0, \ell\}$ if $r = 1$. $\square$

In [1] the index i runs from 1; it has to start at 0, as the examples $k = 1, 2$ given there show.

⁴A partition of S is a cover of S in which every point lies in exactly one set. Covers in which every point lies in at least k of the sets, and the nerve complexes that record how the sets of a cover meet, are studied in [5].

Lemma 2.2. *Let Z be balanced, $q \geq 0$ and $s \neq 0$. Then $3^q + sZ$ is good and sZ is balanced.*

Proof. The map $x \mapsto 3^q + sx$ is injective; it sends $\{0\}$ to $\{3^q\}$ and a 3-element set with sum 0 to a 3-element set with sum 3^{q+1} . The map $x \mapsto sx$ is injective and preserves the two kinds of parts. $\square$

Lemma 2.3. *Let $Q = 3^q$ and let $X \subseteq [1, Q - 1]$ satisfy $Q - X = X$. Then X is good.*

Proof. Since Q is odd, $x \neq Q - x$, and X is the disjoint union of the pairs $\{x, Q - x\}$ with $x \in X$, $x < Q/2$; each has sum Q . $\square$

3. Reflection and the involutions σ_N

The next lemma uses the reflection $x \mapsto T - x$ of $[1, T - 1]$, an action of the group of order two that pairs off the points of a symmetric set.⁵

Lemma 3.1. *Let $T = 3^t$, $j \geq 0$, $D \subseteq [1, T - 1]$ and $W \subseteq [T + 1, T + j] \cup (T - D)$. Put*

$$F = \{d \in D : T - d \notin D\}, \quad R = \left(\{T\} \cup [T + 1, T + j] \cup (T - D) \right) \setminus W.$$

If F and R are good, then $[1, T + j] \setminus W$ is good.

Proof. Let $E = \{x \in [1, T - 1] : x \notin D, T - x \notin D\}$. Then $T - E = E$, so E is good by Lemma 2.3. We show that $[1, T + j] \setminus W$ is the disjoint union of E , F and R . The set E meets neither $D \supseteq F$ nor $\{T\} \cup [T + 1, T + j] \cup (T - D) \supseteq R$; and $F \subseteq [1, T - 1]$ does not meet $T - D$, hence meets neither R nor W . Since E does not meet W either, E , F , R are disjoint subsets of $[1, T + j] \setminus W$. Conversely let $x \in [1, T + j] \setminus W$. If $x \geq T$, or if $x \leq T - 1$ and $T - x \in D$, then $x \in R$. Otherwise $x \leq T - 1$, $T - x \notin D$, and $x \in F$ or $x \in E$ according as $x \in D$ or not. $\square$

⁵For an action of a group G on a set, the orbits of G partition the set; here G has order two and the orbits are the pairs $\{x, T - x\}$, all of size two because T is odd. Equivariant classes for actions of hyperbolic groups are constructed in [6].

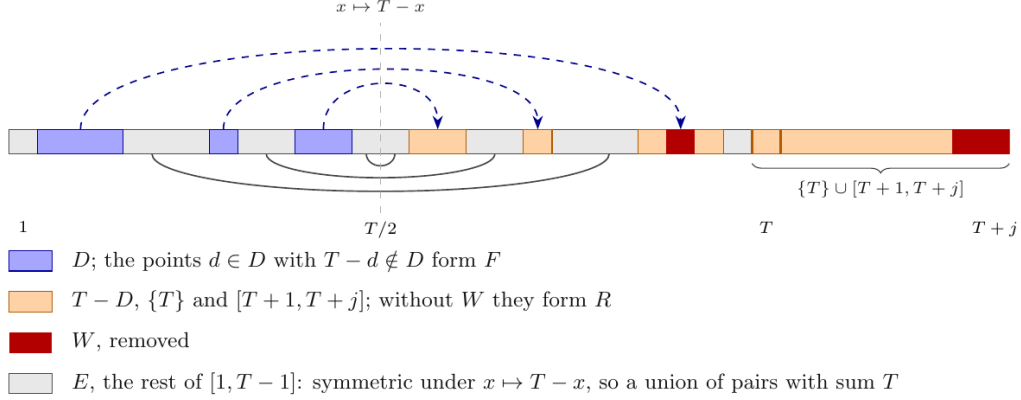


Figure 2: Lemma 3.1. The reflection $x \mapsto T - x$ maps D onto $T - D$; the rest E of $[1, T - 1]$ is symmetric and splits into pairs with sum T .

Lemma 3.2. *Let $N \geq 1$, $N \equiv 1 \pmod{3}$, and $K = 2N - 2$. There is a fixed-point-free involution⁶ σ of $[1, K]$ with the following properties, where $\Pi = \{z \in [1, K] : z < \sigma(z)\}$.*

- (i) *The sums $z + \sigma(z)$, $z \in \Pi$, are distinct.*
- (ii) *The N numbers $2N - 2$ and $z + \sigma(z) - (2N + 1)$, $z \in \Pi$, are distinct, and the set Δ_N they form is balanced.*
- (iii) *Every $\delta \in \Delta_N$ satisfies $|\delta| \leq 2N - 2$, and every positive $\delta \in \Delta_N$ is even.*

Proof. Since $N \equiv 1 \pmod{3}$, either $N = 1$, or $N = 2M$ with $M \geq 2$ and $M \equiv 2 \pmod{3}$, or $N = 2M + 1$ with $M \geq 3$ and $M \equiv 0 \pmod{3}$.

If $N = 1$, then $[1, K] = \emptyset$ and $\Delta_1 = \{0\}$.

Let $N = 2M$. Let σ exchange $z \leftrightarrow z + N - 1$ for $1 \leq z \leq N - 1$. These pairs partition $[1, K]$, since $K = 2(N - 1)$. The pair sums are $2z + N - 1$, $1 \leq z \leq N - 1$, which are distinct. Subtracting $2N + 1 = 4M + 1$ gives the numbers $2(z - M - 1)$, $1 \leq z \leq 2M - 1$, so $\Delta_N = \{4M - 2\} \cup 2[-M, M - 2]$, that is,

$$\Delta_N = \{4M - 2, -2M, -(2M - 2)\} \cup 2[-(M - 2), M - 2].$$

⁶A fixed-point-free involution is a free action of the group of order two, and its orbits are pairs. On complex surfaces the same notion gives the Enriques surfaces, which are the quotients of K3 surfaces by fixed-point-free involutions; K3 surfaces with involutions and the Enriques and Kummer surfaces obtained from them are studied in [7].

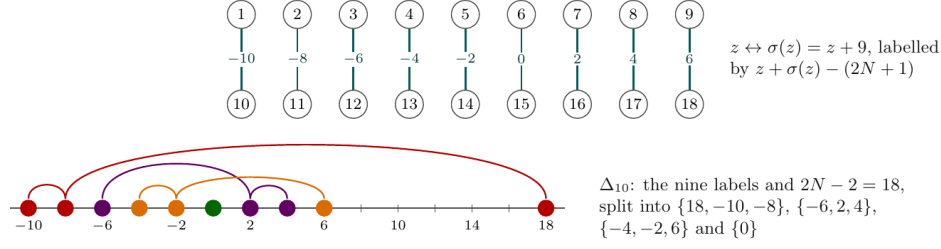


Figure 3: Lemma 3.2 for $N = 10$, $M = 5$. Each pair $\{z, \sigma(z)\}$ carries the number $z + \sigma(z) - (2N + 1)$; together with $2N - 2 = 18$ these form Δ_{10} , which splits into three triples with sum 0 and $\{0\}$.

The first set has sum 0; the second lies in $[-(2M - 4), 2M - 4]$, so it does not meet the first, and it is balanced by Lemmas 2.1 and 2.2, since $M - 2 \equiv 0 \pmod{3}$. This proves (i)–(iii), as $4M - 2 = 2N - 2$.

Let $N = 2M + 1$. Let σ exchange $1 \leftrightarrow N$, $2 \leftrightarrow N + 1$, $3 \leftrightarrow N - 1$, $4 \leftrightarrow N + 2$, $2N - 3 \leftrightarrow 2N - 2$, and $x \leftrightarrow x + N - 2$ for $5 \leq x \leq N - 2$. The last family pairs $[5, N - 2]$ with $[N + 3, 2N - 4]$, so these pairs partition $[1, K]$. The pair sums are $N + 1$, $N + 3$, $N + 2$, $N + 6$, $4N - 5$ and $2x + N - 2 \in [N + 8, 3N - 6]$, $5 \leq x \leq N - 2$, which are distinct. Subtracting $2N + 1 = 4M + 3$ gives

$$\begin{aligned} \Delta_N = \{4M, -(2M + 1), -(2M - 1)\} \cup \{4M - 4, -2M, -(2M - 4)\} \\ \cup 2[-(M - 3), M - 3]. \end{aligned}$$

The first two sets have sum 0; the third lies in $[-(2M - 6), 2M - 6]$ and is balanced since $M - 3 \equiv 0 \pmod{3}$. The three sets are disjoint, which proves (i)–(iii), as $4M = 2N - 2$. $\square$

4. Proof of Theorem 1.2

For $n \geq 1$ let t be the integer with $3^t \leq n < 3^{t+1}$. If $t \geq 1$ and $j = n - 3^t$ satisfies

$$j \equiv 2 \pmod{3}, \quad 2 \leq j, \quad 2j + 1 \leq 3^t, \quad (1)$$

put $W_n = \{n - 1, n, 3^t - 2j + 1\}$. Its elements are distinct, since $2 \leq 3^t - 2j + 1 \leq 3^t - 3$, and their sum is 3^{t+1} . Let $H(n)$ be the statement that $[n] \setminus W_n$ is good.

Proposition 4.1. *Let $n = 3^t + j$ with $t \geq 1$ and j as in (1). Suppose that $[m]$ is good for every $m < n$ and that $H(m)$ holds for every $m < n$ for which it is defined. Then $H(n)$ holds.*

Proof. Let $T = 3^t$, so $W_n = \{T+j-1, T+j, T-2j+1\}$. Since $T \geq 2j+1 \geq 5$, we have $T \geq 9$. In each case below we choose $D \subseteq [1, T-1]$ with $2j-1 \in D$, so that $W = W_n \subseteq [T+1, T+j] \cup (T-D)$, and show that the sets F and R of Lemma 3.1 are good. Lemma 3.1 then gives $H(n)$.

Case $j = 2$. Take $D = \{3\}$. Then $F = \{3\}$, as $T-3 \neq 3$, and $R = \{T, T+1, T+2, T-3\} \setminus W = \{T\}$.

For $j \geq 5$ let u be the integer with $3^u \leq j < 3^{u+1}$. Then $u \geq 1$, and $j \geq 3^u + 2$ because $j \equiv 2$ and $3^u \equiv 0 \pmod{3}$. Also $3j \neq T$, since $3j = T$ would give $j = 3^{t-1} \equiv 0 \pmod{3}$.

Cases $3j > T$, and $3j < T$ with $2j+1 \leq 3^{u+1}$. Take $D = [1, j-2] \cup \{2j-1\}$. Then $T-D = [T-j+2, T-1] \cup \{T-2j+1\}$ and

$$R = T + [-(j-2), j-2],$$

which is good by Lemmas 2.1 and 2.2, since $j-2 \equiv 0 \pmod{3}$.

If $3j > T$, then $u = t-1$. As T and $3j$ are multiples of 3, $T \leq 3j-3$, so $T-2j+1 \in [2, j-2] \subseteq D$. For $d \in [1, j-2]$ we have $T-d \geq T-j+2 > j-2$, so $T-d \in D$ only if $T-d = 2j-1$. Hence

$$F = [1, j-2] \setminus \{T-2j+1\} = [j] \setminus W_j,$$

because $3^{u+1} = T$ and $W_j = \{j-1, j, T-2j+1\}$. The number $j = 3^u + (j-3^u)$ satisfies (1) at level u , so F is good by $H(j)$.

If $3j < T$ and $2j+1 \leq 3^{u+1}$, then $3^{u+1} \leq 3j < T$ gives $T \geq 3^{u+2}$, and every $d \in D$ satisfies $T-d \geq T-2j+1 > 3^{u+2} - 3^{u+1} > 2j-1$. Hence $F = D$. Put $p = 3^{u+1} - 2j + 1$; then $2 \leq p \leq j-2$, because $2j+1 \leq 3^{u+1}$ and $j \geq 3^u + 1$. Since $W_j = \{j-1, j, p\}$,

$$F = D = ([j] \setminus W_j) \cup \{p, 2j-1\},$$

a disjoint union of a set that is good by $H(j)$ and a pair with sum 3^{u+1} .

Case $3j < T$ and $2j+1 > 3^{u+1}$. Let $P = 3^{u+1}$. As before $T \geq 3P$, so $T = 3P$ or $T \geq 9P$; and $P+1 \leq 2j \leq 2P-2$. Write $P = 2c+1$ and put

$$N = j - c, \quad e = P - 1 - j = c - N.$$

Then $N \geq 1$, $e \geq 0$, $j = e + 2N$, and $2N - 2 = 2j - P - 1 \leq P - 3$. Since $3 \mid P$ we have $c \equiv 1 \pmod{3}$, hence $N \equiv 1$ and $e \equiv 0 \pmod{3}$. Let σ , Π , Δ_N be as in Lemma 3.2, $K = 2N - 2$, and

$$D = [1, e] \cup (P + \Delta_N).$$

By Lemma 3.2(iii), $P + \Delta_N \subseteq [P - 2N + 2, P + 2N - 2] = [2e + 3, 2j - 1]$, so the union is disjoint, $D \subseteq [1, 2P - 3] \subseteq [1, T - 1]$, and $2j - 1 = P + (2N - 2) \in D$.

We show $F = D$. Let $d, d' \in D$ with $d + d' = T$. If $d \leq e$, then $d + d' \leq e + 2P - 3 < 3P \leq T$; so $d = P + \delta$ and $d' = P + \delta'$ with $\delta, \delta' \in \Delta_N$ and $\delta + \delta' = T - 2P$. If $T \geq 9P$, then $\delta + \delta' \geq 7P > 2(P - 3)$, which is impossible. If $T = 3P$, then $\delta + \delta' = P$ is odd. Since $\delta, \delta' \leq P - 3$, both are positive, hence even by Lemma 3.2(iii), a contradiction. Thus $F = D$, which is good: $[1, e] = [e]$ is good by hypothesis, and $P + \Delta_N$ by Lemmas 3.2 and 2.2.

It remains to compute R . First, $[T + 1, T + j] \setminus W = T + [1, j - 2] = T + [1, e + K]$. Second, $T - D = T + [-e, -1]$ together with $T - P - \Delta_N$; here $T - P - (2N - 2) = T - 2j + 1 \in W$, and for $z \in \Pi$, using $P - 2N - 1 = 2e$,

$$T - P - (z + \sigma(z) - (2N + 1)) = T - (2e + z + \sigma(z)).$$

By Lemma 3.2(ii) these numbers differ from $T - 2j + 1$. Hence $R = T + Z$ with

$$Z = [-e, e] \cup \bigcup_{z \in \Pi} \{e + z, e + \sigma(z), -(2e + z + \sigma(z))\}.$$

Each set $\{e + z, e + \sigma(z), -(2e + z + \sigma(z))\}$ has three elements and sum 0. As z runs over Π , the numbers $e + z$ and $e + \sigma(z)$ run over $e + [1, K]$ once each, because σ is a fixed-point-free involution; the numbers $-(2e + z + \sigma(z))$ are distinct by Lemma 3.2(i) and less than $-e$. Since $[-e, e]$ is balanced by Lemma 2.1, Z is balanced, and R is good by Lemma 2.2. $\square$

Proof of Theorem 1.2. We prove by strong induction on $n \geq 0$ that $[n]$ is good and that $H(n)$ holds whenever it is defined; $[0] = \emptyset$. The sets $[1] = \{1\}$ and $[2] = \{1, 2\}$ are admissible, and $H(1)$, $H(2)$ are not defined. Let $n \geq 3$, $T = 3^t \leq n < 3T$, $t \geq 1$. Whenever $H(n)$ is defined, it holds by Proposition 4.1 and the induction hypothesis. For $[n]$ there are four cases.

- (a) $n = T$: $[n] = [T - 1] \cup \{T\}$.
- (b) $2n \geq 3T + 1$: with $a = 3T - 1 - n < n$, $[n] = [a] \cup [a + 1, 3T - 1 - a]$, and the second set is good by Lemma 2.3 with $Q = 3T$.

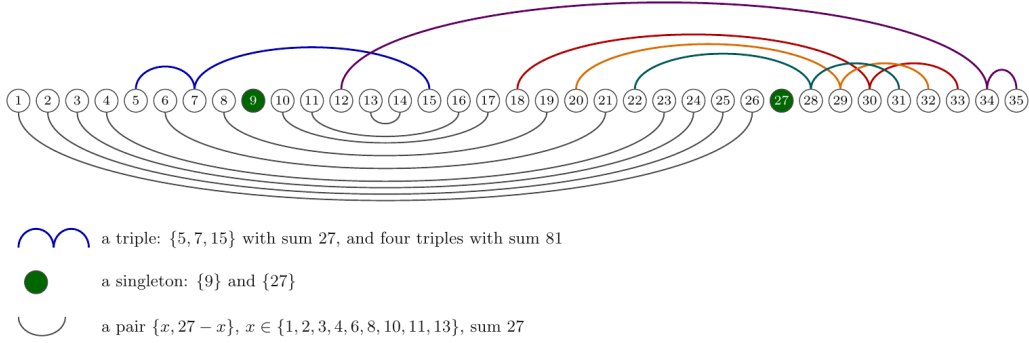


Figure 4: The 3-good partition of $[35]$ given by the recursion. Each triple is drawn as two arcs above the line, each pair as one arc below it.

- (c) $T < n$, $2n \leq 3T$ and $j = n - T \not\equiv 2 \pmod{3}$: $[n]$ is the union of $[T - j - 1]$ and $T + [-j, j]$, and the second set is good by Lemmas 2.1 and 2.2.
- (d) $T < n$, $2n \leq 3T$ and $j = n - T \equiv 2 \pmod{3}$: then j satisfies (1), and $[n] = ([n] \setminus W_n) \cup W_n$, where $[n] \setminus W_n$ is good by $H(n)$ and W_n is admissible.

In (a)–(c) the first set is good by the induction hypothesis, and in all cases the union is disjoint. $\square$

Proof of Corollary 1.3. For $m = 2$ this is [1, Proposition 1], for $m = 3$ it is Theorem 1.2, and for $m \geq 4$ the set $[2m + 2]$ has no m -good partition, as recalled in Section 1. $\square$

Example 4.2. The least n not excluded by [1, Theorem 1] is $n = 35 = 27 + 8$. Here $3j = 24 < 27$, $u = 1$ and $2j + 1 = 17 > 9$, so the last case applies with $P = 9$, $c = 4$, $N = 4$, $e = 0$. The involution σ of $[1, 6]$ is $1 \leftrightarrow 4$, $2 \leftrightarrow 5$, $3 \leftrightarrow 6$, $\Delta_4 = \{6, -4, -2, 0\}$ and $D = \{5, 7, 9, 15\}$. The construction gives the 3-good partition of $[35]$ into

$$\{5, 7, 15\}, \{9\}, \{27\}, \{18, 30, 33\}, \{20, 29, 32\}, \{22, 28, 31\}, \{12, 34, 35\}$$

and the nine pairs $\{x, 27 - x\}$, $x \in \{1, 2, 3, 4, 6, 8, 10, 11, 13\}$; see Figure 4.

The recursion computes the partition of $[n]$ in $O(n)$ arithmetic operations.

5. Formal verification

Theorem 1.2 is formalized in Lean 4 [3] with Mathlib [4], version 4.34.1, and the formal proof is checked by the Lean kernel. The formal statement says that for every natural number n there is a partition of $\{1, \dots, n\}$ into nonempty parts, each of which has at most three elements and a power of 3 as its sum. In the formal proof, good and balanced sets are the finite sets of integers built by disjoint union from the admissible sets, respectively from $\{0\}$ and the 3-element sets with sum 0, and the argument is that of Sections 2–4. The proof depends only on the axioms of Lean’s logic, namely propositional extensionality, choice and the soundness of quotients; it has no unproved step and adds no axiom.

Independent computations in exact integer arithmetic run the recursion of Section 4 and confirm that it gives a 3-good partition of $[n]$ for every $n \leq 30000$ and for further values of n up to $6.5 \cdot 10^7$, and that $H(n)$ holds for every n in these ranges for which it is defined. They also confirm Lemma 2.1 for $\ell \leq 30000$ and Lemma 3.2 for $N \leq 30001$.

Data and code availability

The L^AT_EX source of the paper, the Lean 4 formalization of Theorem 1.2 and the programs of Section 5 are archived at Zenodo under doi:10.5281/zenodo.23455972 [8]; the paper is available at <https://github.com/creelie/didactic-lamp/blob/main/preprintGurvichNaumovaConjecture/GurvichNaumovaConjecture.pdf>. Deep Bhattacharjee carried out the computations and the Lean, Julia, C, C++, Python and shell verifications, and the proofs and the text were drafted, with AI assistance (Claude, Anthropic); both authors approved the final version.

Ethical declarations

Funding: none. Competing interests: none.

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