DURFEE INEQUALITIES FOR COMPLETE INTERSECTIONS

QIYUE TANG

Abstract. We prove that the Milnor number μ and geometric genus p_g of every singular isolated complete intersection surface satisfy $\mu > 4p_g$, with a stronger bound in each fixed embedding codimension. We obtain $\mu > 6p_g$ for hypersurfaces and $\mu > 5p_g$ in codimension two. For singular positive-weight quasihomogeneous isolated complete intersections of minimal embedding codimension two and dimension $n \geq 3$, we prove $\mu > C_{n,2}p_g$, where $C_{n,2} = (n+1)(n+2)!/(2^{n+2}-2)$.

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1. Introduction

Let $(X, 0)$ be an isolated complete intersection singularity (ICIS) over $\mathbb{C}$ of dimension $n \geq 2$. Write $\operatorname{edim}(X)$ and $\operatorname{mult}(X)$ for its embedding dimension and Hilbert–Samuel multiplicity, respectively. Let $r = \operatorname{edim}(X) - n$ be its minimal embedding codimension, and put $N = n + r$. Its Milnor number is

$$\mu(X) = \dim_{\mathbb{C}} \widetilde{H}^n(F; \mathbb{C}),$$

where F is a Milnor fibre. If $\pi : Y \to X$ is a resolution, its geometric genus is

$$p_g(X) = \dim_{\mathbb{C}}(R^{n-1}\pi_*\mathcal{O}_Y)_0.$$

These invariants compare the topology of a smoothing with the cohomology of a resolution.

Durfee's work on the signature of surface smoothings [2] gives the relation

$$2p_g = \mu_+ + \mu_0, \qquad \mu = \mu_+ + \mu_0 + \mu_-, \qquad 4p_g = \mu + \sigma + \mu_0 \tag{1.1}$$

for complete intersection surfaces. Here μ_+, μ_0, μ_- are the positive, zero, and negative indices of the Milnor intersection form, and $\sigma = \mu_+ - \mu_-$. The weak signature conjecture asks for $\sigma \leq 0$, equivalently $4p_g \leq \mu + \mu_0$. Durfee also proposed $6p_g \leq \mu$ for complete intersection surfaces, with equality only in the smooth case [2, Conjecture 5.3]. The distinction between a signature bound and a bound involving μ and p_g alone is governed by the nullity μ_0.

Several early results established stronger estimates for special classes of hypersurfaces. For weighted homogeneous surface hypersurfaces, Xu and Yau proved

$$6p_g \leq \mu - \operatorname{mult}(X) + 1, \tag{1.2}$$

with equality precisely for germs analytically equivalent to homogeneous ones [25, Theorem A]. Tomari proved $8p_g < \mu$ for isolated hypersurface double points [24], and Melle–Hernández proved the Durfee inequality for absolutely isolated surface hypersurfaces [18]. In higher dimensions, Yau and Zhang proved

$$(n+1)!p_g < \mu \tag{1.3}$$

for every singular weighted homogeneous hypersurface of dimension $n \geq 2$ [26, Theorem 1.2]. Their proof uses an upper bound for the number of positive lattice points in a real simplex.

Complete intersections of higher codimension require different constants. Kerner and Némethi constructed homogeneous complete intersection surfaces for which $6p_g > \mu$ [12]. Their

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subsequent work introduced the constants

$$C_{n,r} = \frac{\binom{n+r-1}{n}(n+r)!}{r!\,S(n+r,r)}, \tag{1.4}$$

where $S(k,r)$ is the second Stirling number, the number of partitions of a k-element set into r nonempty subsets. They proved $C_{n,r}p_g \le \mu$ for homogeneous complete intersections of arbitrary multidegree when $n > 2$ [13, Theorem 3.1]. For fixed n, these constants decrease from $C_{n,1} = (n+1)!$ to 2^n as r increases. For complete intersections of equal degree tending to infinity, the quotient μ/p_g tends to $C_{n,r}$; this explains the change from the hypersurface coefficient. Kerner and Némethi later proved the strict corrected inequality for Newton-nondegenerate complete intersections whose Newton diagrams are convenient and sufficiently large, for $r = 1$, $n \ge 2$, and for $r > 1$, $n > 2$ [14, Theorem 1.1].

For surfaces, Kerner and Némethi proposed the strict geometric inequality

$$4p_g < \mu \tag{1.5}$$

in every codimension, and proved it for homogeneous complete intersections [13, Introduction and §5]. Their homogeneous formulas also show that the formal surface coefficient $C_{2,r}$ can fail when $r > 1$. Section 15 gives an intersection of two quintics with $\mu = 1024$ and $p_g = 200$, for which $\mu < (36/7)p_g = C_{2,2}p_g$.

The weak signature problem has a different history. Kollár and Némethi proved it for arbitrary isolated hypersurface surfaces [15]. Enokizono used slope inequalities for curve fibrations to obtain bounds with an exceptional-set correction for hypersurfaces [5] and complete intersections [4]. In the latter case, let $E_{\min}$ be the reduced exceptional set of the minimal resolution and let χ_{top} denote topological Euler characteristic. The bound is

$$\frac{12(N-1)}{3N-5}p_g(X) \le \mu(X) + 1 - \chi_{\mathrm{top}}(E_{\min}), \qquad N = \operatorname{edim}(X). \tag{1.6}$$

This proves the weak signature conjecture for all isolated complete intersection surfaces. The correction in (1.6) depends on the topology of the exceptional set. We control it by applying the inequality to finite covers of a fixed germ.

Recent work of Fu, Yau, and Zuo [6] concerns the Yau multiplicity refinement for weighted homogeneous hypersurfaces. Liu [16] has announced a proof of the strict inequality $\mu > 6p_g$ for arbitrary isolated hypersurface surfaces. The present paper gives general surface bounds and incorporates the positive-weight codimension-two argument of [23] in higher dimensions.

Theorem A. *Let $(X,0)$ be a singular isolated complete intersection surface over $\mathbb{C}$, of minimal embedding codimension $r \ge 1$. Then*

$$\mu(X) > \left(4 + \frac{16}{3(r+1)^2 - 4}\right) p_g(X). \tag{1.7}$$

In particular, $\mu(X) > 4p_g(X)$ for every r, and $\mu(X) > 6p_g(X)$ when $r = 1$.

Theorem B. *Let $(X,0)$ be a singular isolated complete intersection surface over $\mathbb{C}$ of minimal embedding codimension two. Then*

$$\mu(X) > 5p_g(X). \tag{1.8}$$

Theorems A and B apply to arbitrary complex analytic complete intersection surfaces. Their inequalities are strict also when $p_g = 0$. For the higher-dimensional statement, the codimension-two constant is

$$C_{n,2} = \frac{(n+1)(n+2)!}{2^{n+2} - 2}, \qquad n \ge 3. \tag{1.9}$$

In particular, $C_{3,2} = 16$ and $C_{4,2} = 1800/31$.

Theorem C. *Let $(X,0)$ be a singular positive-weight quasihomogeneous isolated complete intersection over $\mathbb{C}$, of dimension $n \geq 3$ and minimal embedding codimension two. Then*

$$\mu(X) > C_{n,2}p_g(X). \tag{1.10}$$

For $n = 3$, one has $\mu(X) \geq 16p_g(X) + 1$.

The surface proof combines two comparisons. A canonical filtration records the extension of weighted canonical forms to a resolution. Its exact difference under a general linear section permits the calculation of the geometric genera of coordinate root covers. The first Betti numbers of their links grow more slowly than their covering degrees, so (1.6) yields a numerical comparison involving the Milnor number of a general curve section. A second comparison comes from the directional Jacobian. Principalizing its maximal-minor ideal and applying Hilbert–Burch and Riemann–Roch expresses a polar length as an intersection number. Negative definiteness then gives a quadratic bound in the same sectional data. Combining these bounds proves Theorems A and B.

For Theorem C, a finite graded Artin quotient and its Gorenstein symmetry bound the geometric genus by a convex integral. The Greuel–Hamm formula gives the Milnor number as a divided difference. When the larger equation degree is at least twice the largest weight, the comparison reduces to a product in the weights and a shifted uniform moment. In the complementary range, isolation supplies two coefficient coordinates, and a midpoint identity controls a simplex count. Separate concavity gives the endpoint estimates needed in dimensions three and four.

Section 2 gives the canonical quotient formula used in both arguments. Sections 3–7 prove the surface estimates. Sections 8–14 prove Theorem C, retaining the stronger bounds in the two degree ranges. The last section computes the quintic example.

2. The canonical quotient

We work on small Stein representatives of complex analytic germs. Resolutions and principalizations are chosen projective over the representative and are isomorphisms away from the singular point. An isolated complete intersection of dimension $n \geq 2$ is Cohen–Macaulay and regular in codimension one, hence normal. Its canonical module is invertible. For a normal Cohen–Macaulay germ, write ω_X for its canonical module; on a smooth resolution Y, the sheaf ω_Y consists of top differential forms. Choose a canonical divisor K_Y with $\mathcal{O}_Y(K_Y) \simeq \omega_Y$. The notation length means module length, equal here to complex vector-space dimension for finite-length modules. Floors and fractional parts of divisors are taken coefficientwise. For a formal series $G(t)$, write $[t^k]G(t)$ for its coefficient of t^k. An empty sum is zero and an empty product is one; we take $\binom{m}{j} = 0$ for integers $j > m \geq 0$. We will use the following formula in both parts of the proof.

Lemma 2.1. *Let $\pi : Y \to X$ be a projective resolution of a normal Cohen–Macaulay isolated complex singularity of dimension $n \geq 2$, isomorphic away from the singular point. Then*

$$\dim_{\mathbb{C}}(\omega_X/\pi_*\omega_Y)_0 = p_g.$$

Proof. Normality gives $\pi_*\mathcal{O}_Y = \mathcal{O}_X$. Write $\mathbf{R}\pi_*$ for the derived direct image, $\mathbf{R}\mathcal{H}om$ for derived sheaf Hom, and $\mathcal{H}^i$ for the ith cohomology sheaf of a complex. Let T be the mapping cone of the natural morphism $\mathcal{O}_X \to \mathbf{R}\pi_*\mathcal{O}_Y$, so that this morphism extends to a distinguished triangle with third object T. Its cohomology sheaves are

$$\mathcal{H}^i(T) = \mathrm{R}^i\pi_*\mathcal{O}_Y \quad (1 \leq i \leq n-1),$$

and are supported at the singular point. The upper bound on i follows from the bound $n-1$ on the dimension of a resolution fibre.

Let $\mathbb{D}$ denote Matlis duality on finite-length modules over the Cohen–Macaulay local ring. Local duality gives, for such a module V,

$$\mathrm{Ext}^j(V,\omega_X) = 0 \quad (j \neq n), \qquad \mathrm{Ext}^n(V,\omega_X) = \mathbb{D}(V).$$

Matlis duality preserves finite length. The spectral sequence computing the derived Hom of T therefore has only one nonzero row. Its terms in degrees zero and one are

$$\mathcal{H}^0\mathbf{R}\mathcal{H}om(T,\omega_X)=0,\qquad \mathcal{H}^1\mathbf{R}\mathcal{H}om(T,\omega_X)=\mathbb{D}(\mathrm{R}^{n-1}\pi_*\mathcal{O}_Y).$$

Our shift convention is $\mathcal{H}^i(C[n])=\mathcal{H}^{i+n}(C)$ for a complex C. Proper Grothendieck duality, with dualizing complexes $\omega_X[n]$ and $\omega_Y[n]$, yields

$$\mathbf{R}\mathcal{H}om(\mathbf{R}\pi_*\mathcal{O}_Y,\omega_X)\simeq\mathbf{R}\pi_*\omega_Y.$$

Grauert–Riemenschneider vanishing makes the positive-degree cohomology on the right zero. Applying $\mathbf{R}\mathcal{H}om(-,\omega_X)$ to the defining triangle for T gives

$$0\longrightarrow\pi_*\omega_Y\longrightarrow\omega_X\longrightarrow\mathbb{D}(\mathrm{R}^{n-1}\pi_*\mathcal{O}_Y)\longrightarrow 0.$$

Taking lengths proves the assertion. The duality inputs are as in [22, 11]; the vanishing input is Grauert–Riemenschneider vanishing for a projective resolution in characteristic zero [8]. □

3. The canonical filtration and a general section

In Sections 3–7, take $n=2$, so $N=r+2$, and let X be a singular isolated complete intersection surface. The exceptional fibre of a resolution is a proper curve.

3.1. **Conventions and standard inputs.** A general linear section is a reduced complete intersection curve, and two general linear sections form a parameter ideal of colength $e=\mathrm{mult}(X)$.

We use three standard results, in the following forms. The Lê–Greuel formula [9, Corollary 5.5] says that if $A=(F=0)$ is a positive-dimensional ICIS and h cuts an ICIS of one lower dimension, then

$$\mu(A)+\mu(A\cap(h=0))=\text{length}\,\frac{\mathcal{O}_A}{I_{q+1}(\mathrm{Jac}(F,h))},\tag{3.1}$$

where q is the codimension of A and I_{q+1} denotes the ideal of maximal minors. For the zero-dimensional section we use the convention $\mu(A\cap(h=0))=\text{length}\,\mathcal{O}_{A\cap(h=0)}-1$.

Grauert–Riemenschneider vanishing gives $R^i\pi_*\omega_Y=0$ for $i>0$ [8]. We also use relative Kawamata–Viehweg vanishing for a projective morphism of complex analytic spaces [21, Theorem 3.7]. In the form needed here, if B is a simple normal crossing real divisor with coefficients in $[0,1)$, L is a Cartier divisor, and $L-(K_Y+B)$ is relatively nef and big, then

$$R^i\pi_*\mathcal{O}_Y(L)=0\quad(i>0).\tag{3.2}$$

It is enough below to apply this statement to rational coefficients; on each bounded parameter interval, the rounded divisors are locally constant between finitely many rational walls.

3.2. **Canonical colengths.** Let $\mathfrak{m}_X$ be the ideal sheaf of the origin. Choose a resolution $\pi:Y\to X$, isomorphic outside 0, such that

$$\mathfrak{m}_X\mathcal{O}_Y=\mathcal{O}_Y(-D),$$

where D is an effective exceptional divisor with simple normal crossing support. The line bundle $\mathcal{O}_Y(-D)$ is generated by the pullbacks of generators of $\mathfrak{m}_X$. In particular, $-D$ is π-nef. For $t\geq 0$, put

$$\mathcal{J}_X(t)=\pi_*\omega_Y(-\lfloor tD\rfloor)\subseteq\omega_X,\qquad H_X(t)=\text{length}(\omega_X/\mathcal{J}_X(t)).\tag{3.3}$$

The quotient has finite length because D is exceptional and X is smooth away from the origin. The function H_X is nondecreasing and is a bounded step function on every compact interval.

Lemma 3.1. *One has $H_X(0)=p_g(X)$. Moreover, $\mathcal{J}_X(t)$ and $H_X(t)$ are independent of the chosen principalizing resolution.*

Proof. At $t = 0$, Lemma 2.1 gives $H_X(0) = p_g(X)$.

For independence, take generators $g_1, \dots, g_s$ of $\mathfrak{m}_X$ and write $|\mathfrak{m}_X|^2 = \sum |g_i|^2$ on $X \setminus \{0\}$. A meromorphic canonical form $\eta \in \omega_X$ belongs to $\mathcal{J}_X(t)$ exactly when

$$\int_{X\setminus\{0\}} \eta \wedge \overline{\eta}\, |\mathfrak{m}_X|^{-2t} < \infty \tag{3.4}$$

locally, with the harmless constant making the integrand positive understood. Indeed, on a principalizing chart the squared norm $|\mathfrak{m}_X|^2$ is comparable to $|z_1|^{2m_1}|z_2|^{2m_2}$, where $D = m_1(z_1 = 0) + m_2(z_2 = 0)$. A coefficient of order $k_i \in \mathbb{Z}$ in the z_i direction is integrable precisely when $k_i - tm_i > -1$, equivalently $k_i \geq \lfloor tm_i \rfloor$. These are exactly the conditions defining $\omega_Y(-\lfloor tD \rfloor)$. The criterion is intrinsic and is unchanged on a further resolution. □

Define

$$V = \int_0^1 H_X(v)\, \mathrm{d}v. \tag{3.5}$$

Lemma 3.1 and monotonicity give

$$V \geq H_X(0) = p_g(X). \tag{3.6}$$

Lemma 3.2. *For every $t \geq 0$,*

$$R^i\pi_*\omega_Y(-\lfloor tD \rfloor) = 0 \qquad (i > 0).$$

Proof. For rational $t > 0$, use (3.2) with $L = K_Y - \lfloor tD \rfloor$ and $B = \{tD\}$. Then

$$L - (K_Y + B) = -tD.$$

This divisor is relatively nef. It is relatively big because π is birational: its restriction to a sufficiently general, zero-dimensional fibre is big. All coefficients of B are less than one and its support has simple normal crossings. At $t = 0$ use Grauert–Riemenschneider vanishing. For real t, choose a nonnegative rational number with the same coefficientwise floor on the finitely many components of D. □

3.3. **The exact difference formula.** Let l be a sufficiently general linear form, let $C = X \cap (l = 0)$, put $\mu_1 = \mu(C)$, and let C' be its strict transform on Y. After shrinking, C' is smooth, its map $\nu : C' \to C$ is the normalization, and it meets the exceptional divisor transversely at smooth points of its support. Moreover,

$$\operatorname{div}(l \circ \pi) = C' + D. \tag{3.7}$$

These assertions follow from Bertini applied to the pullbacks of linear forms, which generate the line bundle $\mathcal{O}_Y(-D)$. We may impose them together with any finite set of further general-position conditions.

For this reduced complete intersection curve set

$$\mathcal{J}_C(t) = \nu_*\omega_{C'}(-\lfloor tD|_{C'} \rfloor), \qquad H_C(t) = \operatorname{length}(\omega_C/\mathcal{J}_C(t)). \tag{3.8}$$

Transversality ensures that restricting $\lfloor tD \rfloor$ to C' gives $\lfloor tD|_{C'} \rfloor$.

Proposition 3.3. *For every $t \geq 0$,*

$$H_X(t+1) - H_X(t) = H_C(t). \tag{3.9}$$

Proof. Write $B_t = \lfloor tD \rfloor$. The residue sequence on Y is

$$0 \longrightarrow \omega_Y(-B_t) \longrightarrow \omega_Y(C' - B_t) \longrightarrow \omega_{C'}(-B_t|_{C'}) \longrightarrow 0.$$

By (3.7), multiplication by l identifies the middle sheaf with $\omega_Y(-D - B_t)$. Since $D + B_t = \lfloor (t+1)D \rfloor$, Lemma 3.2 gives the exact sequence

$$0 \longrightarrow \mathcal{J}_X(t) \xrightarrow{\cdot l} \mathcal{J}_X(t+1) \longrightarrow \mathcal{J}_C(t) \longrightarrow 0. \tag{3.10}$$

The residue morphism agrees on the smooth locus with ordinary Gorenstein adjunction. Thus (3.10) maps into

$$0 \longrightarrow \omega_X \xrightarrow{\cdot l} \omega_X \longrightarrow \omega_C \longrightarrow 0.$$

All three vertical maps are the natural injections. The snake lemma therefore gives

$$0 \longrightarrow \omega_X/\mathcal{J}_X(t) \xrightarrow{\cdot l} \omega_X/\mathcal{J}_X(t+1) \longrightarrow \omega_C/\mathcal{J}_C(t) \longrightarrow 0.$$

Taking lengths proves (3.9). □

Set $\delta_C = \mathrm{length}(\nu_*\mathcal{O}_{C'}/\mathcal{O}_C)$. Let ρ be the number of normalization branches of C, and let $b_j \geq 1$ be the order of $\mathfrak{m}_X$ on branch j. Then

$$\sum_{j=1}^{\rho} b_j = e, \qquad \mu_1 = 2\delta_C - \rho + 1, \qquad J = \sum_{j=1}^{\rho} \frac{1}{b_j} > 0. \tag{3.11}$$

The first equality follows by intersecting C with a second general linear form; its colength is e and equals the sum of the branch orders. The second is the Milnor formula for reduced complete intersection curves [1].

Lemma 3.4. *For every $t \geq 0$,*

$$H_C(t) = \delta_C + \sum_{j=1}^{\rho} \lfloor b_j t \rfloor . \tag{3.12}$$

Consequently, $H_C(t+1) - H_C(t) = e$.

Proof. Apply $\mathrm{Hom}_{\mathcal{O}_C}(-, \omega_C)$ to the normalization sequence

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \nu_*\mathcal{O}_{C'} \longrightarrow \nu_*\mathcal{O}_{C'}/\mathcal{O}_C \longrightarrow 0.$$

The finite-length quotient has zero Hom into the depth-one module ω_C. The module $\nu_*\mathcal{O}_{C'}$ is maximal Cohen–Macaulay, so its first Ext into ω_C vanishes. Finite duality and the resulting long exact sequence give

$$0 \longrightarrow \nu_*\omega_{C'} \longrightarrow \omega_C \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_C}(\nu_*\mathcal{O}_{C'}/\mathcal{O}_C, \omega_C) \longrightarrow 0.$$

One-dimensional local duality identifies the last module with the Matlis dual of $\nu_*\mathcal{O}_{C'}/\mathcal{O}_C$, of length δ_C. On a smooth branch, twisting the canonical module down by $\lfloor b_j t \rfloor$ times its closed point adds colength $\lfloor b_j t \rfloor$. Sum over the branches to obtain (3.12). The final assertion follows because each b_j is an integer. □

Combining Proposition 3.3 and Lemma 3.4, for $k \in \mathbb{Z}_{\geq 0}$ and $0 \leq v < 1$ we obtain

$$H_X(k+v) = H_X(v) + kH_C(v) + \frac{e}{2}k(k-1). \tag{3.13}$$

This equality includes all integral endpoints through the floor convention in (3.3).

4. Coordinate root covers

Choose jointly general independent linear coordinates $l_1, \ldots, l_N$ in a minimal embedding $(X, 0) \subset (\mathbb{C}^N, 0)$. All their single sections have Milnor number μ_1, and all pairs have colength e. On a fixed principalizing resolution the divisors of the l_i have the form $D + C_i'$. We choose the strict transforms C_i' smooth and transverse to the exceptional divisor and to one another, with their points on the exceptional fibre pairwise distinct. These are finitely many open Bertini conditions. In a sufficiently small representative, two distinct branch sections do not meet away from 0.

For an integer $d \geq 2$ define

$$X_d = \{F(x) = 0,\ u_i^d = l_i(x)\ (1 \leq i \leq N)\}, \qquad p_d : X_d \longrightarrow X, \tag{4.1}$$

where $F = (F_1, \ldots, F_r)$ is a minimal defining regular sequence for X.

Lemma 4.1. *The germ X_d is an isolated complete intersection surface of embedding dimension N. The map p_d is finite flat of degree d^N, and its group of deck transformations off the branch locus is $G_d = (\mathbb{Z}/d\mathbb{Z})^N$.*

Proof. Successively adjoining roots of the monic equations $u_i^d - l_i$ gives a free $\mathcal{O}_X$-module with basis

$$u_1^{a_1} \cdots u_N^{a_N}, \qquad 0 \le a_i < d.$$

Thus the degree and flatness assertions hold. Since the l_i form coordinates, eliminating the original variables realizes X_d in $\mathbb{C}_u^N$ with the r equations $F(l^{-1}(u_1^d, \dots, u_N^d))$. They are a regular sequence: the dimension remains two under the finite extension, and there are $r = N - 2$ equations in a regular local ring. They have no linear terms, so this is a minimal embedding.

Off the origin the base is smooth. At an unramified point the cover is locally a disjoint union of smooth sheets. At a smooth point of a single branch curve the local map is $(u, v) \mapsto (u^d, v)$, up to unramified factors. The chosen branch curves have no other intersections on the punctured representative. Hence X_d is smooth away from its closed point. It is generically reduced and Cohen–Macaulay, so it is reduced; regularity in codimension one then implies normality. A normal local ring is a domain. The independent root multiplications therefore give the asserted full deck group. The same argument applies to partial covers and their general curve sections, with reducedness of the latter in place of normality. □

4.1. **The exact Milnor formula.**

Lemma 4.2. *Let $A = (F = 0)$ be a positive-dimensional ICIS, and suppose that $B = A \cap (h = 0)$ is an ICIS of the expected dimension. If $U = (F = 0, h - u^d = 0)$ is an ICIS, then*

$$\mu(U) = d\mu(A) + (d-1)\mu(B). \tag{4.2}$$

For zero-dimensional A, the length of its root extension is $d \operatorname{length} \mathcal{O}_A$.

Proof. The section $u = 0$ of U is B. Apply (3.1) to (U, u). Expand the maximal minors of $\operatorname{Jac}(F, h - u^d, u)$ along the last row. The nonzero minors are, up to sign, exactly the pullbacks of the maximal minors of $\operatorname{Jac}(F, h)$. The corresponding quotient length on U is therefore $d(\mu(A) + \mu(B))$, because $\mathcal{O}_U$ is free of rank d over $\mathcal{O}_A$. Consequently, $\mu(U) + \mu(B) = d(\mu(A) + \mu(B))$, which proves (4.2). The zero-dimensional assertion is the same finite-freeness statement applied to length. □

For a positive-dimensional ICIS A write $\chi_A = 1 + (-1)^{\dim A}\mu(A)$, the Euler characteristic of its Milnor fibre. In dimension zero put $\chi_A = \operatorname{length} \mathcal{O}_A$. Lemma 4.2 becomes

$$\chi_U = d\chi_A - (d-1)\chi_B$$

in positive dimension. Iterating first on curve sections gives, for s roots,

$$\chi_{C,s} = d^s(1 - \mu_1) - s(d-1)d^{s-1}e.$$

Here a final zero-dimensional section is counted with its full scheme length, multiplied by d for each previous root. Iterating on the surface and using this curve formula gives

$$\chi_{X,s} = d^s(1 + \mu) - s(d-1)d^{s-1}(1 - \mu_1) + \binom{s}{2}(d-1)^2 d^{s-2}e. \tag{4.3}$$

This also follows by induction on s: the new branch is the cover of a general curve section with $s - 1$ previous roots, and its final section is the zero-dimensional complete intersection.

At $s = N$, write $x_d = 1 - 1/d$. We obtain the exact identity

$$\frac{\mu(X_d)}{d^N} = \mu + 1 + Nx_d(\mu_1 - 1) + \binom{N}{2}x_d^2 e - d^{-N}. \tag{4.4}$$

In particular,

$$\lim_{d\to\infty}\frac{\mu(X_d)}{d^N}=M:=\mu+1+N(\mu_1-1)+\binom{N}{2}e. \tag{4.5}$$

4.2. **Canonical characters.**

Proposition 4.3. *The geometric genus of X_d is*

$$p_g(X_d)=\sum_{0\le j_1,\dots,j_N<d}H_X\left(\frac{j_1+\cdots+j_N}{d}\right). \tag{4.6}$$

Write $\mathbb{E}$ for expectation. If $U_1,\dots,U_N$ are independent uniform random variables on $[0,1]$ and $T=U_1+\cdots+U_N$, then

$$\lim_{d\to\infty}\frac{p_g(X_d)}{d^N}=\mathbb{E}H_X(T). \tag{4.7}$$

Proof. Choose a generator η of the Gorenstein module ω_X. Finite complete intersection duality, or the residue formula for the monic root equations, gives a generator of ω_{X_d} on the unramified locus proportional to

$$\frac{p_d^*\eta}{u_1^{d-1}\cdots u_N^{d-1}}.$$

Multiplication by the finite-free monomial basis gives the character decomposition of $p_{d*}\omega_{X_d}$, with terms

$$h\,p_d^*\eta\prod_{i=1}^N u_i^{a_i-d+1},\qquad h\in\mathcal{O}_X,\quad 0\le a_i<d. \tag{4.8}$$

The scalar factor d^{-N} is a unit.

A canonical form on the smooth part of a normal surface extends holomorphically to a resolution if and only if its squared norm is locally integrable. This can be checked in coordinates on the resolution: an integer pole of order at least one is not square integrable, and a form with no divisorial pole extends on the smooth resolution. Integration on the finite unramified locus in (4.8) is, up to a positive constant, integration of $h\eta$ on X with weight

$$\prod_{i=1}^N|l_i|^{-2\alpha_i},\qquad \alpha_i=\frac{d-1-a_i}{d}\in[0,1). \tag{4.9}$$

The branch sets have measure zero, so the integrability test is unchanged by deleting them for the change of variables.

On the chosen resolution, $\operatorname{div}(l_i)=D+C_i'$. At a strict branch component its coefficient in (4.9) is $\alpha_i<1$. The pullback of $h\eta$ is holomorphic there, since π is an isomorphism off the exceptional fibre. Such a component imposes no further integer vanishing condition. Along an exceptional component E, the weight is $(\alpha_1+\cdots+\alpha_N)\operatorname{ord}_E(D)$. The simple normal crossing condition makes the tests in the local coordinates separate. The same calculation as in (3.4) therefore says that the character form extends exactly when

$$h\eta\in\mathcal{J}_X(\alpha_1+\cdots+\alpha_N).$$

In particular the floor, including its value at an integer, is the one in (3.3).

The extending forms form a G_d-stable submodule. Character projection preserves extension, so the quotient of canonical modules is the direct sum of the quotients just computed. By Lemma 3.1 applied to the normal Gorenstein surface X_d, its total length is $p_g(X_d)$. Reindex $j_i=d-1-a_i$ to obtain (4.6). The function H_X has finitely many jumps on $[0,N]$, so this sum divided by d^N is a convergent Riemann sum on $[0,1]^N$. This proves (4.7). □

4.3. **The first Betti number of the link.** Let L denote the link of X, and let $B \subset L$ be the union of the links of the reduced curves $l_i = 0$. Let K be the exterior obtained by deleting open tubular neighbourhoods of these link components. The root cover induces an unramified G_d-cover $K_d \to K$. Completing it over the branch components fills its boundary tori with solid tori and gives a link L_d of X_d. For precision, choose a representative on which every sufficiently small positive level of the squared Euclidean radius is transverse to $X \setminus \{0\}$ and to each punctured branch curve. Off the branch locus, p_d is a local biholomorphism. At a branch point its local model is $(u, v) \mapsto (u^d, v)$; the image of its differential contains the branch tangent line, on which the differential of the radius is nonzero. Thus every such base level pulls back to a regular level for every d. The pulled-back radius is a real analytic rug function on X_d, meaning a nonnegative real analytic function with isolated zero at the origin. Its sufficiently small levels are links of X_d, and the absence of critical levels identifies them with the preimage of the fixed base level. Rug-function invariance identifies these with the ordinary spherical link [17, Chapter 2, §A]. Deleting fixed tubular neighbourhoods downstairs therefore gives the same exterior K for every d.

Lemma 4.4. *For the fixed source and the fixed general coordinate tuple,*

$$b_1(L_d) = O(d^{N-1}). \tag{4.10}$$

Proof. The maps $l_i/|l_i|$ define a homomorphism

$$\varphi : \pi_1(K) \longrightarrow \mathbb{Z}^N.$$

A small meridian about a component belonging to $l_i = 0$ maps to the ith standard basis vector, because the curve is reduced and the other l_j are nonzero there. In particular, the homomorphism is surjective, and reduction modulo d describes $K_d \to K$.

The compact manifold K has a finite CW structure. Its cellular complex with rank-one coefficients induced by φ has matrices over $\mathbb{C}[z_1^{\pm 1}, \dots, z_N^{\pm 1}]$. The local system $\mathbb{C}_z$ has monodromy $\prod_i z_i^{\varphi_i(\gamma)}$ along a loop γ. Hence

$$\Sigma = \{z \in (\mathbb{C}^*)^N : H^1(K; \mathbb{C}_z) \neq 0\}$$

is a Zariski closed algebraic subset, defined by rank conditions on two consecutive matrices. Also $\dim H^1(K; \mathbb{C}_z)$ is bounded by the fixed number of one-cells.

We verify that Σ is proper. The reduced function $g = l_1 \cdots l_N$ on X defines a Milnor open book on its link exterior. At a point of $l_i = 0$ away from the origin, the other l_j are nonzero and $dg = (\prod_{j \neq i} l_j)\, dl_i$ is nonzero on the tangent space of X. If critical points of g on the smooth locus accumulated at the origin, complex analytic curve selection in the rank-deficiency locus of $d(F, g)$ would give a nonconstant analytic arc γ of critical points tending to the origin. Then $(g \circ \gamma)' = 0$, so $g \circ \gamma = 0$, contradicting the preceding nonvanishing. Thus the Milnor fibration theorem for a function on an ICIS applies; its fibre has finite CW type [17]. The coefficient system with $z_1 = \cdots = z_N = \lambda$ is the pullback of the scalar system on the base circle of this fibration. Its Wang sequence shows that H^1 vanishes when λ avoids the finitely many eigenvalues, or their inverses according to convention, of monodromy on H^0 and H^1 of the fibre. Therefore a generic point of this diagonal does not lie in Σ.

Choose a nonzero Laurent polynomial vanishing on Σ and multiply it by a monomial to obtain a nonzero polynomial Q. If S_d is the set of dth roots of unity, elementary polynomial root counting gives

$$\#\{z \in S_d^N : Q(z) = 0\} \leq (\deg Q) d^{N-1}. \tag{4.11}$$

For completeness, this follows by induction on the number of variables: outside the zero set of the leading nonzero coefficient in the last variable, there are at most its degree many roots; the exceptional choices are bounded by the induction hypothesis. The bound also holds when the right side exceeds the grid size.

Finite abelian character decomposition over $\mathbb{C}$ gives

$$b_1(K_d) = \sum_{z \in S_d^N} \dim H^1(K; \mathbb{C}_z).$$

Only the points of Σ contribute. The uniform bound on each summand and (4.11) prove $b_1(K_d) = O(d^{N-1})$. Finally, filling a torus by a solid torus cannot increase the first Betti number: the solid torus's first homology is generated by the image of its boundary, as the Mayer–Vietoris sequence shows. Applying this to all boundary tori gives $b_1(L_d) \le b_1(K_d)$ and proves the lemma. □

Remark 4.5. For any normal surface singularity, the first Betti number of its link equals that of the reduced exceptional curve on a good resolution. Let W be a regular neighbourhood of the exceptional divisor. It retracts onto that divisor, so Poincaré–Lefschetz duality gives

$$H_1(W, \partial W; \mathbb{Q}) \simeq H^3(W; \mathbb{Q}) = 0.$$

The map $H_2(W; \mathbb{Q}) \to H_2(W, \partial W; \mathbb{Q})$ is given by the nonsingular negative definite intersection matrix and is an isomorphism. The relative homology sequence therefore identifies $H_1(\partial W; \mathbb{Q})$ with $H_1(W; \mathbb{Q})$, which is the first rational homology of the divisor. Blowups leave this first Betti number unchanged. If s is the number of exceptional components on the minimal resolution, then its connected reduced exceptional set has

$$\chi_{\mathrm{top}}(E_{\min}) = 1 - b_1(L) + s.$$

Consequently Enokizono's theorem implies

$$\frac{12(N-1)}{3N-5} p_g(X) \le \mu(X) + b_1(L) - s \le \mu(X) + b_1(L). \tag{4.12}$$

Lemma 4.4 controls the Betti term in this inequality.

5. Averaging the canonical filtration

Apply (4.12) to X_d. By Lemma 4.1, its embedding dimension is the fixed number N. Divide by d^N and use (4.5), Proposition 4.3, and Lemma 4.4. The result is

$$\frac{12(N-1)}{3N-5} \mathbb{E} H_X(T) \le M, \qquad T = U_1 + \cdots + U_N. \tag{5.1}$$

We now evaluate the left side using only the one-dimensional sectional data.

Lemma 5.1. *Let $N \ge 3$, let T be the sum of N independent uniform variables on $[0,1]$, and let $v = \{T\}$. For every bounded integrable function ϕ on $[0,1]$ and $j = 0, 1, 2$,*

$$\mathbb{E}[T^j \phi(v)] = \mathbb{E}[T^j] \int_0^1 \phi(v)\, \mathrm{d}v. \tag{5.2}$$

Moreover,

$$\mathbb{E}T = \frac{N}{2}, \qquad \mathbb{E}T^2 = \frac{N^2}{4} + \frac{N}{12}.$$

Proof. The characteristic function is

$$\mathbb{E} \exp(i\xi T) = \left(\frac{\exp(i\xi) - 1}{i\xi} \right)^N.$$

At $\xi = 2\pi k$ for every nonzero integer k, it has a zero of order N. Differentiating at most twice gives $\mathbb{E}[T^j \exp(2\pi i k T)] = 0$ for $j = 0, 1, 2$. For each j, the finite measure on the circle obtained by weighting the distribution of v by T^j therefore has all nonconstant Fourier coefficients equal to zero. Uniqueness of Fourier coefficients for finite measures identifies it with $\mathbb{E}[T^j]$ times Lebesgue measure, proving (5.2). The two moments follow by adding the means $1/2$ and variances $1/12$ of the independent summands. □

Proposition 5.2. *With V, e, J, μ_1 as above,*

$$\mathbb{E}H_X(T) = V + \frac{N-1}{4}(\mu_1 - 1) + \frac{(N-1)(3N-2)}{24}e + \frac{J}{12}. \tag{5.3}$$

Proof. Write $k = T - v$. The unit-cell formula (3.13) and Lemma 5.1 give

$$\mathbb{E}H_X(T) = V + \int_0^1 \left(\frac{N}{2} - v\right) H_C(v)\,\mathrm{d}v + \frac{e}{2}\mathbb{E}[(T-v)(T-v-1)]. \tag{5.4}$$

The same moment identities yield

$$\begin{aligned}\mathbb{E}[(T-v)(T-v-1)] &= \mathbb{E}T^2 - 2\mathbb{E}(Tv) - \mathbb{E}T + \mathbb{E}(v^2) + \mathbb{E}v \\ &= \frac{N^2}{4} - \frac{11N}{12} + \frac{5}{6} = \frac{(3N-5)(N-2)}{12}.\end{aligned}$$

For every positive integer b, direct integration over the b intervals of length $1/b$ gives

$$\int_0^1 \lfloor bv \rfloor\ \mathrm{d}v = \frac{b-1}{2}, \qquad \int_0^1 v\,\lfloor bv \rfloor\ \mathrm{d}v = \frac{b}{3} - \frac{1}{4} - \frac{1}{12b}. \tag{5.5}$$

Use (3.12) to compute

$$\begin{aligned}\int_0^1 \left(\frac{N}{2} - v\right) H_C(v)\,\mathrm{d}v &= \frac{N-1}{2}\delta_C + \frac{3N-4}{12}e - \frac{N-1}{4}\rho + \frac{J}{12} \\ &= \frac{N-1}{4}(\mu_1 - 1) + \frac{3N-4}{12}e + \frac{J}{12}.\end{aligned}$$

The second line uses $2\delta_C = \mu_1 + \rho - 1$. Substitution into (5.4) gives (5.3), since

$$2(3N-4) + (3N-5)(N-2) = (N-1)(3N-2).$$

□

Proposition 5.3. *Every singular isolated complete intersection surface of minimal embedding codimension r satisfies*

$$(3r+1)\mu \geq 12(r+1)V - (r-1)\mu_1 + (r+1)(e+J-2). \tag{5.6}$$

Proof. Insert (4.5) and (5.3) into (5.1). After multiplication by $3N-5$ and collection of terms, the inequality is

$$(3N-5)\mu \geq 12(N-1)V - (N-3)(\mu_1 - 1) + (N-1)(e+J) - (3N-5).$$

Its constant terms combine to $-2(N-1)$. Since $N = r+2$, this is (5.6). □

Corollary 5.4. *For a singular isolated hypersurface surface,*

$$\mu \geq 6V + \frac{e+J}{2} - 1 > 6p_g. \tag{5.7}$$

Proof. Set $r = 1$ in (5.6). The sectional Milnor term vanishes. Since $e \geq 2$, $J > 0$, and $V \geq p_g$, the correction $(e+J)/2 - 1$ is strictly positive. □

6. A sectional Jacobian inequality

Let $A = \mathcal{O}_{X,0}$ and let $F_1, \ldots, F_r \in \mathbb{C}\{x_1, \ldots, x_{r+2}\}$ be a minimal defining regular sequence. Each F_i has order at least two. Write

$$\mathsf{A} = \left(\frac{\partial F_i}{\partial x_j}\right) : \mathcal{O}_X^{r+2} \longrightarrow \mathcal{O}_X^r.$$

The matrix has rank r off the origin. Put

$$L_2 = \mu + \mu_1, \qquad L_1 = \mu_1 + e - 1. \tag{6.1}$$

Theorem 6.1. *With this notation,*

$$2L_2 \geq \frac{r+1}{r}\frac{L_1^2}{e} \geq (r+1)L_1. \tag{6.2}$$

We prove the theorem on one resolution. In addition to principalizing $\mathfrak{m}_X$, principalize the ideal of maximal minors of A. The zero sets of both ideals are $\{0\}$. Further modifications will also be isomorphisms outside the exceptional fibre.

6.1. **The image bundle and a target flag.**

Lemma 6.2. *On this resolution the actual image of the pulled-back Jacobian is a locally free sheaf $\mathcal{E}$ of rank r, generated by its original $r+2$ columns. It injects into $\mathcal{F} = \mathcal{O}_Y^r$, equals $\mathcal{F}$ off the exceptional fibre, and satisfies*

$$\mathcal{E} \subseteq \mathcal{F}(-D), \qquad \det \mathcal{E} = \mathcal{O}_Y(-Z)$$

for an effective exceptional divisor Z.

Proof. On a chart where the ideal generated by the pulled-back maximal minors is principal, some original minor generates it near each point: the quotients of the original minors by a local generator generate the unit ideal. Choose the corresponding r columns. Cramer's rule expresses each remaining column as their linear combination, whose coefficients are ratios of original maximal minors to the chosen one. These ratios are regular on the chart. The chosen columns are linearly independent over the meromorphic function field, so their image is free. These images glue to the actual image sheaf $\mathcal{E}$.

The determinant of its inclusion in $\mathcal{F}$ defines Z. Every Jacobian entry lies in $\mathfrak{m}_X$ by minimality of the embedding, giving $\mathcal{E} \subseteq \mathcal{F}(-D)$. A subsequent pullback preserves these properties: the pulled-back map of locally free sheaves is generically injective, and its kernel, being torsion in a locally free sheaf, is zero. Its image is generated by the same pulled-back columns. □

Lemma 6.3. *After further principalizations there are effective exceptional divisors $Z_1, \dots, Z_r$ such that*

$$Z = Z_1 + \cdots + Z_r, \qquad Z_i \geq D, \tag{6.3}$$

and

$$[\mathcal{E}^*/\mathcal{F}^*] = \sum_{i=1}^{r} [\mathcal{O}_{Z_i}(Z_i)] \tag{6.4}$$

in the Grothendieck group of coherent sheaves supported on the exceptional fibre.

Proof. Fix a constant flag in the trivial target $\mathcal{F}$. Project $\mathcal{E}$ to the last target line. Its image is an ideal sheaf equal to $\mathcal{O}_Y$ off the exceptional fibre. Principalize this image and resolve so that it is the invertible ideal $\mathcal{O}_Y(-Z_r)$. Its kernel $\mathcal{E}'$ is locally free and lies in the remaining trivial target $\mathcal{F}'$. Thus

$$0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_Y(-Z_r) \longrightarrow 0$$

is locally split. Repeat with the next constant target line. Every previously constructed sequence stays exact under later pullbacks because its quotient is locally free. All residual base ideals are principalized before the next step, so there is no residual zero-dimensional quotient at the end of this procedure.

At each step the kernel is contained in the corresponding $\mathcal{F}'(-D)$. Its projection image is therefore contained in $\mathcal{O}_Y(-D)$, which gives $Z_i \geq D$. Taking determinants of the successive locally split sequences gives $Z = \sum_i Z_i$.

For one step, dualize the displayed sequence and compare it with the dual sequence of the constant target flag. The vertical maps are injections, and the snake lemma gives

$$0 \longrightarrow \mathcal{O}_Y(Z_r)/\mathcal{O}_Y \longrightarrow \mathcal{E}^*/\mathcal{F}^* \longrightarrow (\mathcal{E}')^*/(\mathcal{F}')^* \longrightarrow 0.$$

The first quotient is $\mathcal{O}_{Z_r}(Z_r)$. Induction proves (6.4). □

Resolve the union of the exceptional divisors once more if necessary. All previous locally split sequences and divisor equalities remain valid. We use this common resolution for the following argument.

6.2. **A general directional Jacobian.**

Lemma 6.4. *For a general linear form l on $\mathbb{C}^{r+2}$, the subspace $W = \ker l$, of dimension $r+1$, generates $\mathcal{E}$ in a neighbourhood of the exceptional fibre.*

Proof. At each point y of the exceptional fibre, write $\mathcal{E}_y$ for the vector-bundle fibre $\mathcal{E} \otimes \mathbb{C}(y)$. Evaluation is a surjection $\mathbb{C}^{r+2} \to \mathcal{E}_y$ with a two-dimensional kernel K_y. The restriction to W fails to be onto exactly when $K_y \subset W$. This is a codimension-two condition in the projective space of hyperplanes in $\mathbb{C}^{r+2}$. The incidence variety of bad pairs (y, W) has dimension at most $1+(r+1-2) = r$, while the parameter space of W has dimension $r+1$. Its projection is closed by properness of the exceptional fibre. A general W therefore avoids all bad pairs. The coherent cokernel of $W \otimes \mathcal{O}_Y \to \mathcal{E}$ vanishes on the fibre; after shrinking the representative it vanishes near the fibre as well. □

Choose l also generally for the curve section and for Bertini on this resolution, and put $C = X \cap (l = 0)$. All resolutions and target-flag principalizations above can be fixed before choosing the linear forms. We impose these finite general-position requirements together with those in Section 4, and take $l = l_1$ there. Thus the curve in the intersection calculation is the same curve used to define μ_1, the b_j, and J in the covering calculation. Let A_W be the $r \times (r+1)$ matrix of the Jacobian in a constant basis of W. Expansion along the last row dl in (3.1) gives

$$\operatorname{length} A/I_r(\mathsf{A}_W) = L_2. \tag{6.5}$$

In particular, the maximal-minor ideal is primary to the maximal ideal of A and has grade two in this Cohen–Macaulay ring.

6.3. **Hilbert–Burch and the intersection calculation.** The Hilbert–Burch theorem gives a perfect free resolution

$$0 \longrightarrow A^r \xrightarrow{\mathsf{A}_W^{\mathsf{t}}} A^{r+1} \xrightarrow{\text{signed minors}} A \longrightarrow A/I_r(\mathsf{A}_W) \longrightarrow 0. \tag{6.6}$$

The grade-two Hilbert–Burch theorem applies over the Cohen–Macaulay ring A [3, Theorem 20.15(b)].

Lemma 6.5. *On the resolution above,*

$$2L_2 = -Z^2 - \sum_{i=1}^{r} Z_i^2. \tag{6.7}$$

Proof. Pull back the free complex in (6.6), putting the three terms in homological degrees $2, 1, 0$. By Lemma 6.4, there is an exact sequence

$$0 \longrightarrow \mathcal{O}_Y(Z) \longrightarrow \mathcal{O}_Y^{r+1} \longrightarrow \mathcal{E} \longrightarrow 0.$$

The first line bundle is $\det(\mathcal{E})^{-1}$ by determinants. Dualizing gives

$$0 \longrightarrow \mathcal{E}^* \longrightarrow \mathcal{O}_Y^{r+1} \longrightarrow \mathcal{O}_Y(-Z) \longrightarrow 0.$$

Composing the last map with $\mathcal{O}_Y(-Z) \hookrightarrow \mathcal{O}_Y$ gives the signed-minor map of the pulled-back complex, up to the fixed choice of orientation of the free source. Its first map is the inclusion of $\mathcal{F}^*$ in $\mathcal{E}^*$ followed by $\mathcal{E}^* \hookrightarrow \mathcal{O}_Y^{r+1}$. Thus its only nonzero homology sheaves are

$$\mathcal{H}_0 = \mathcal{O}_Z, \qquad \mathcal{H}_1 = \mathcal{E}^*/\mathcal{F}^*, \qquad \mathcal{H}_2 = 0. \tag{6.8}$$

For a coherent sheaf $\mathcal{G}$ with proper exceptional support, write

$$\chi(\mathcal{G}) = \sum_i (-1)^i \dim_{\mathbb{C}} H^i(Y, \mathcal{G}).$$

We compute the Euler difference of these homology sheaves. Let $\mathcal{K}$ be the downstairs perfect complex. Derived projection gives

$$R\pi_*(L\pi^*\mathcal{K}) \simeq \mathcal{K} \otimes^L R\pi_*\mathcal{O}_Y.$$

The sheaf $\pi_*\mathcal{O}_Y$ is $\mathcal{O}_X$ by normality. The higher direct images are of finite length at the origin. For any finite-length module Q, the Euler characteristic of $\mathcal{K} \otimes Q$ is

$$\big(1 - (r+1) + r\big) \operatorname{length} Q = 0.$$

Therefore these higher direct images make no contribution to the Euler characteristic of the displayed derived tensor product. By (6.5) and (6.8),

$$L_2 = \chi(\mathcal{O}_Z) - \chi(\mathcal{E}^*/\mathcal{F}^*). \tag{6.9}$$

All sheaves on the right have proper support.

For an effective exceptional cycle B on a smooth surface, Riemann–Roch on its components gives

$$\chi(\mathcal{O}_B) = -\frac{B^2 + K_Y B}{2}, \qquad \chi(\mathcal{O}_B(B)) = \frac{B^2 - K_Y B}{2}. \tag{6.10}$$

These formulas also hold for nonreduced cycles. To check the first one directly, remove a smooth component E once and use

$$0 \longrightarrow \mathcal{O}_E(-(B-E)) \longrightarrow \mathcal{O}_B \longrightarrow \mathcal{O}_{B-E} \longrightarrow 0.$$

Riemann–Roch on E, together with $2g(E) - 2 = (K_Y + E)E$, proves the formula by induction on the total multiplicity of B. Applying the same induction with a line bundle twist shows $\chi(\mathcal{O}_B(A)) = \chi(\mathcal{O}_B) + AB$ for any divisor A, which gives the second formula at $A = B$.

Finally insert (6.4) into (6.9) and use (6.10). Since $Z = \sum_i Z_i$, all terms involving K_Y cancel. The remaining equality is (6.7). □

6.4. **The curve section and Cauchy–Schwarz.**

Lemma 6.6. *For jointly general l, l',*

$$L_1 = -ZD, \qquad e = -D^2. \tag{6.11}$$

Proof. Fix l as above. Its strict transform C' is smooth near the exceptional fibre and meets it in finitely many transverse points, one for each normalization branch counted at its point over the origin. The space $W = \ker l$ generates $\mathcal{E}$ at these points. After conditioning on l, choose l' so that the r-plane $W' = \ker l \cap \ker l'$ maps isomorphically to $\mathcal{E}_y$ at all these finitely many points. Such a choice is possible because failure at each point is a proper closed condition on the hyperplanes of W. Write $\mathsf{A}_{W'}$ for the $r \times r$ Jacobian matrix in a constant basis of W'.

On each branch of C, the determinant of the r directional columns from W' therefore has precisely the vanishing order of Z; the determinant of their map to $\mathcal{E}$ is a unit at the point over the origin. Apply (3.1) to the complete intersection curve C in the hyperplane $l = 0$ and its function l'. The zero-dimensional section has length e and Milnor number $e-1$. Expansion of the Jacobian determinant along dl' gives

$$L_1 = \operatorname{length} \mathcal{O}_C/(\det \mathsf{A}_{W'}).$$

For a nonzerodivisor on a reduced Cohen–Macaulay curve, its colength equals the sum of its orders on the normalization branches. This follows by comparing $\mathcal{O}_C \subset \nu_*\mathcal{O}_{C'}$: multiplication on the finite-length cokernel has equal kernel and cokernel lengths, so it contributes zero to the length difference. Consequently $L_1 = ZC'$. Since $\operatorname{div}(l \circ \pi) = C' + D$ is principal and Z is compact, $ZC' = -ZD$.

Similarly, a second general linear form has, on the normalization of C, the branch orders b_j. Its colength is $e = \sum b_j = DC'$. As $D(C' + D) = 0$, we get $e = -D^2$. □

Proof of Theorem 6.1. The exceptional intersection matrix of a resolution of a normal complex surface singularity is negative definite [7, 20]. Thus

$$(A, B) = -A \cdot B$$

is a positive definite inner product on the real vector space of exceptional cycles. By (6.3), (6.7), and Cauchy–Schwarz,

$$\begin{aligned} 2L_2 &= (Z,Z) + \sum_i (Z_i, Z_i) \\ &\geq \left(1 + \frac{1}{r}\right)(Z,Z) \geq \frac{r+1}{r}\frac{(Z,D)^2}{(D,D)} = \frac{r+1}{r}\frac{L_1^2}{e}. \end{aligned} \tag{6.12}$$

The first inequality is $\sum_i \|Z_i\|^2 \geq \|\sum_i Z_i\|^2/r$. For the linear comparison, $Z - rD$ is effective by (6.3). Since $-D$ is relatively nef,

$$L_1 = -ZD \geq -rD^2 = re.$$

Substitution in the preceding quadratic bound proves the second inequality in (6.2). □

7. The surface inequalities

The surface estimates follow from Proposition 5.3 and Theorem 6.1. We give the numerical steps, including the small multiplicity cases responsible for strictness.

Proposition 7.1. *Let*

$$\mathcal{P} = 2(\mu + \mu_1) - (r+1)(\mu_1 + e - 1) \geq 0.$$

Then

$$\mu \geq 4V + \frac{2e + J - 3}{3} + \frac{\mathcal{P}}{3(r+1)}. \tag{7.1}$$

In particular, $\mu > 4p_g$.

Proof. The definition of $\mathcal{P}$ gives

$$-(r-1)\mu_1 = \mathcal{P} - 2\mu + (r+1)(e-1).$$

Insert this equality into (5.6), collect $3(r+1)\mu$ on the left, and divide by $3(r+1)$. This proves (7.1). A singular surface has $e \geq 2$, and $J > 0$, so $2e + J - 3 > 0$. Use $V \geq p_g$ to obtain the strict inequality. □

Lemma 7.2. *A minimally embedded ICIS surface of codimension* r *has* $e \geq r+1$. *In codimension two it has* $e \geq 4$.

Proof. Quotient by two general independent linear forms. Since X is Cohen–Macaulay, the resulting Artinian ring has length e. Its embedding dimension is r: in a minimal presentation it is a quotient of $\mathbb{C}\{z_1, \ldots, z_r\}$ by r equations of order at least two. Its degree-zero and degree-one associated graded pieces therefore have dimensions 1 and r, giving $e \geq r+1$.

When $r = 2$, the space of quadratic monomials has dimension three. The quadratic parts of the two equations span a space of dimension at most two, so the degree-two piece of the quotient has dimension at least one. Hence its first three Hilbert function values are at least $1, 2, 1$, and $e \geq 4$. □

Proof of Theorem A. Set

$$x = \frac{L_1}{e} \geq r, \qquad P_r(x) = \frac{r+1}{2r}x^2 - x + 1. \tag{7.2}$$

Since $\mu_1 = e(x-1) + 1$, the quadratic inequality in Theorem 6.1 is equivalent to

$$\mu + 1 \geq eP_r(x). \tag{7.3}$$

Rewriting (5.6) in these variables gives

$$(3r+1)(\mu+1) \geq 12(r+1)V + e\big(-(r-1)x + 2r\big) + (r+1)J. \tag{7.4}$$

Put $\theta = 2(r-1)/(r+1) \geq 0$ and add θ times (7.3) to (7.4). Then

$$(3r + 1 + \theta)(\mu + 1) \geq 12(r+1)V + ef_r(x) + (r+1)J, \tag{7.5}$$

where

$$f_r(x) = \theta P_r(x) - (r-1)x + 2r.$$

Direct differentiation gives

$$f_r(r) = \frac{2(3r-1)}{r+1}, \qquad f_r'(r) = \frac{(r-1)^2}{r+1}, \qquad f_r''(x) \geq 0.$$

Thus $f_r(x) \geq 2(3r-1)/(r+1)$ for $x \geq r$.

When $r = 1$, one has $\theta = 0$, $f_r = 2$, and $ef_r \geq 4 = 3r + 1 + \theta$ because $e \geq 2$. When $r \geq 2$, Lemma 7.2 gives

$$ef_r(x) \geq 2(3r-1) \geq 3r + 1 + \theta,$$

the second difference being $(r-1)(3r+1)/(r+1) \geq 0$. Since $(r+1)J > 0$, (7.5) now implies

$$(3r + 1 + \theta)\mu > 12(r+1)V.$$

Finally,

$$\frac{12(r+1)}{3r + 1 + 2(r-1)/(r+1)} = 4 + \frac{16}{3(r+1)^2 - 4}.$$

Together with $V \geq p_g$, this is (1.7). It also proves the stronger inequality with V in place of p_g. □

Proof of Theorem B. Take $r = 2$ in (7.3) and (7.4), and use $\theta = 1/5$ instead of $2/3$. The relevant expression is

$$\frac{1}{5}P_2(x) - x + 4 = \frac{3}{20}(x-4)^2 + \frac{9}{5} \geq \frac{9}{5}.$$

By Lemma 7.2, $e \geq 4$, so $e(9/5) \geq 36/5 = 7 + 1/5$. The same addition as in (7.5), now with the strictly positive term $3J$, yields

$$\frac{36}{5}\mu > 36V.$$

Hence $\mu > 5V \geq 5p_g$, as asserted. □

7.1. **Ordinary cones.** Suppose that X is the cone over a smooth projective complete intersection curve of multidegree $(d_1, \ldots, d_r)$, where $d_i \geq 2$. Blowing up the vertex resolves X; its exceptional divisor has coefficient one in D. Therefore $H_X(v) = p_g$ for $0 \leq v < 1$ and $V = p_g$. A general section consists of $e = \prod_i d_i$ smooth branches of order one, so $J = e$.

For an ordinary hypersurface cone of degree d,

$$\mu = (d-1)^3, \qquad p_g = \binom{d}{3}, \qquad e = J = d.$$

It follows that $\mu - 6p_g = d - 1$, which is equality in (5.7). For an intersection of r general quadrics,

$$e = 2^r, \qquad \mu = e\frac{r^2 - r + 2}{2} - 1, \qquad p_g = \frac{e\,r(r-1)}{8}, \qquad \mu_1 = (r-1)e + 1.$$

The graded coordinate ring has Hilbert series $(1+t)^r/(1-t)^2$. The Koszul resolution of r quadratic equations in $r+2$ variables gives its canonical shift $2r - (r+2) = r - 2$. Morales' formula [19, Theorem 4.1] and Lemma 2.1 therefore give

$$p_g = \sum_{k=0}^{r-2} [t^k] \frac{(1+t)^r}{(1-t)^2} = [t^{r-2}] \frac{(1+t)^r}{(1-t)^3}.$$

Expanding the numerator yields

$$p_g = \sum_{j=0}^{r-2} \binom{r}{j} \binom{r-j}{2} = \binom{r}{2} 2^{r-2}.$$

The sum is zero for $r = 1$. The last identity counts a chosen pair inside a subset of an r-element set: choose the pair first, then any subset of its $r-2$ remaining elements. The homogeneous Milnor formula [13, Proposition 2.1] gives

$$\begin{aligned}\mu + 1 &= 2^r[z^2]\frac{(1-z)^{r+2}}{(1-2z)^r} \\ &= 2^r\left(2r(r+1) - 2r(r+2) + \binom{r+2}{2}\right) = 2^r\frac{r^2-r+2}{2}.\end{aligned}$$

The corresponding curve formula is

$$\mu_1 - 1 = 2^r[z]\frac{(1-z)^{r+1}}{(1-2z)^r} = 2^r(r-1).$$

These values give $\mu = 4p_g + e - 1$, so equality is attained in (7.1), with $\mathcal{P} = 0$. For $r \geq 2$,

$$\frac{\mu}{p_g} = 4 + \frac{8(1-2^{-r})}{r(r-1)} \longrightarrow 4.$$

Thus 4 is the optimal constant independent of r.

8. Weighted complete intersections of codimension two

A positive-weight quasihomogeneous germ has a defining regular sequence homogeneous for one positive grading. Positive weights make each homogeneous defining series a polynomial. The weights can be chosen rational: the finitely many equal-weight conditions form a linear system with integer coefficients, whose rational solutions are dense in its real solution space. Positivity is open in that space. An equation with a nonzero linear part can be taken as a homogeneous coordinate and eliminated. A nonlinear monomial of the same positive weight involves only variables of smaller weight, so this coordinate change has a polynomial inverse. Repeating this process gives a minimal homogeneous presentation.

In Sections 8–15, take $r = 2$, so $N = n + 2$, with $n \geq 2$. Give the polynomial ring $\mathbb{C}[x_1, \ldots, x_N]$ positive integer weights $w_1, \ldots, w_N$: the weighted degree of $x_1^{i_1} \cdots x_N^{i_N}$ is $\sum_j w_j i_j$. We write $\operatorname{wt}(x_i) = w_i$ for the weight of a coordinate. Positive rational weights can be multiplied, together with all equation degrees, by a common positive integer to obtain this grading. Let f, g be weighted homogeneous polynomials of weighted degrees d_1, d_2, respectively, ordered so that $d_1 \leq d_2$. Assume they form a regular sequence. Thus f is a nonzerodivisor and the class of g is a nonzerodivisor modulo (f). Set

$$R = \mathbb{C}[x_1, \ldots, x_N]/(f, g), \qquad X = \operatorname{Spec} R.$$

Assume that $(X, 0)$ is singular at the origin and smooth on a punctured neighbourhood of the origin. Write $\mathfrak{m} = (x_1, \ldots, x_N)$ and assume $f, g \in \mathfrak{m}^2$, so the embedding is minimal. For a singular ICIS, the Milnor number is positive [17].

Order the weights and define their product P and the grading cutoff a by

$$\begin{aligned} &0 < w_1 \leq \cdots \leq w_N, \qquad p = w_1, \qquad h = w_N, \\ &P = \prod_{i=1}^{N} w_i, \qquad a = d_1 + d_2 - \sum_{i=1}^{N} w_i. \end{aligned} \tag{8.1}$$

A positive-weight scaling contracts every nonzero point towards the origin. Thus isolation at the origin gives smoothness on $X \setminus \{0\}$. The complete intersection ring is Cohen–Macaulay and regular in codimension one, hence normal.

8.1. **Hilbert series and geometric genus.** Write R_k for the weighted-degree-k component of R. For $x \in \mathbb{R}$, put $x_+ = \max(x, 0)$.

The Hilbert series and the geometric genus are related by

$$H_R(t) = \sum_{k\geq 0} (\dim_{\mathbb{C}} R_k) t^k = \frac{(1-t^{d_1})(1-t^{d_2})}{\prod_i (1-t^{w_i})}, \qquad p_g = \sum_{0\leq k\leq a} [t^k] H_R(t). \tag{8.2}$$

The Hilbert product follows by applying the multiplication exact sequence successively to f and g. For a graded module V and an integer b, the graded shift is defined by $V(b)_k = V_{k+b}$. The canonical module of R is $R(a)$. Morales' first L^2-plurigenus is the length of $\omega_X/\pi_*\omega_Y$ [19, Section 0.5.4]; his graded formula [19, Theorem 4.1], together with Lemma 2.1, gives the genus sum in (8.2). Its cutoff is closed: $a = 0$ gives $p_g = 1$, and $a < 0$ gives $p_g = 0$.

Using that the Milnor fibre has Euler characteristic $1 + (-1)^n \mu$, the Greuel–Hamm formula gives

$$P\mu = d_1 d_2 [z^{N-2}] \frac{\prod_i (1 - w_i z)}{(1 - d_1 z)(1 - d_2 z)} + (-1)^{N-1} P; \tag{8.3}$$

see [10, Corollaries 2.2 and 3.8(b)].

8.2. **The Milnor divided difference.** Define

$$Q(t) = \prod_{i=1}^{N} (t - w_i).$$

Lemma 8.1. *If $d_1 < d_2$, then*

$$P\mu = \frac{d_1 Q(d_2) - d_2 Q(d_1)}{d_2 - d_1}. \tag{8.4}$$

At $d_1 = d_2$, the continuous value is $d_1 Q'(d_1) - Q(d_1)$.

Proof. For $0 \leq k \leq N$, let $e_k(w_1, \ldots, w_N)$ be the sum of products of k distinct weights, with $e_0(w_1, \ldots, w_N) = 1$. Then

$$Q(t) = \sum_{k=0}^{N} (-1)^k e_k(w_1, \ldots, w_N) t^{N-k}.$$

For an integer $j \geq 2$,

$$\frac{d_1 {d_2}^j - d_2 {d_1}^j}{d_2 - d_1} = d_1 d_2 \sum_{i=0}^{j-2} {d_1}^i {d_2}^{j-2-i}.$$

The linear term of Q contributes zero to (8.4), while the constant term contributes $-(-1)^N P = (-1)^{N-1} P$. Expanding the two geometric series in (8.3) gives these same terms. At $d_1 = d_2$, differentiating the numerator with respect to d_2 gives $d_1 Q'(d_1) - Q(d_1)$, while the derivative of the denominator is one. □

9. A convex bound from a finite Artin quotient

For $c > 0$, let U_c denote a random variable uniformly distributed on $[0, c]$. Random variables appearing together in a sum below are independent. We write Pr for probability on their product probability space. For $\eta \geq 0$ define $\mathcal{R}_N(\eta; d_1, d_2)$ by the expression below; when it is applied to our singularity, take $\eta = (\sum_{i=1}^{N} w_i - p)/2$:

$$\mathcal{R}_N(\eta; d_1, d_2) = (d_1 + d_2 - \eta)_+^N - (d_1 - \eta)_+^N - (d_2 - \eta)_+^N. \tag{9.1}$$

The alternating expression in (9.1) is a nonnegative integral, as the next lemma shows.

Lemma 9.1. *For every $n \geq 2$,*

$$p_g \leq \frac{d_1 d_2}{n! P} \mathbb{E}(U_{d_1} + U_{d_2} - \eta)_+^n = \frac{\mathcal{R}_N(\eta; d_1, d_2)}{N! P}. \tag{9.2}$$

Proof. Normality implies that the local ring $R_{\mathfrak{m}}$ at the vertex is a domain. The map $R \to R_{\mathfrak{m}}$ is injective: if $s \notin \mathfrak{m}$ and $su = 0$, the product of the nonzero constant term of s with the lowest nonzero homogeneous term of u cannot vanish. Thus R is a domain. Since $(f, g) \subset \mathfrak{m}^2$, the minimum-weight coordinate x_1 has a nonzero class in R and is a nonzerodivisor. Choose an integer $q > \max(0, a + p, d_1 + d_2 - \eta)$ divisible by $w_2, \dots, w_N$. On $R/(x_1)$, the degree-q linear system contains a pure power of every remaining coordinate, so it is base-point free on $\operatorname{Proj}(R/(x_1))$. Over the infinite field $\mathbb{C}$, a general section avoids the generic point of every component, including each isolated point. Each successive section therefore lowers the projective dimension by one, with a zero-dimensional scheme becoming empty. Since $\dim \operatorname{Proj}(R/(x_1)) = n - 2$, after $n - 1$ sections the common projective zero scheme is empty. Their affine quotient is finite-dimensional. Thus these sections are parameters; Cohen–Macaulayness makes them a regular sequence. Denote the quotient by these parameters and x_1 by $\mathcal{A}$.

Write $h_k = \dim_{\mathbb{C}} \mathcal{A}_k$. The Hilbert series and the total dimension are

$$H_{\mathcal{A}}(t) = \frac{(1 - t^q)^{n-1}(1 - t^{d_1})(1 - t^{d_2})}{\prod_{i=2}^{N}(1 - t^{w_i})}, \qquad \ell = \sum_k h_k = \frac{pq^{n-1}d_1 d_2}{P}.$$

The length formula follows by cancelling the first-order zeros at $t = 1$. Graded complete-intersection duality gives the socle degree, the largest integer with a nonzero graded component,

$$s_{\mathcal{A}} = a + p + (n - 1)q$$

as follows. After eliminating x_1, the polynomial ring has canonical shift $-\sum_{i=2}^{N} w_i$, and the Koszul resolution of the regular sequence of degrees $d_1, d_2, q, \dots, q$ gives the canonical module $\mathcal{A}(s_{\mathcal{A}})$. The canonical module of a finite graded algebra is its graded vector-space dual, whose degree-j part is $\operatorname{Hom}_{\mathbb{C}}(\mathcal{A}_{-j}, \mathbb{C})$. Comparing degrees gives $h_k = h_{s_{\mathcal{A}}-k}$. Define the integer-valued random variable K by $\Pr(K = k) = h_k/\ell$ for $0 \le k \le s_{\mathcal{A}}$.

The identity

$$H_R(t) = \frac{H_{\mathcal{A}}(t)}{(1 - t^p)(1 - t^q)^{n-1}}$$

and (8.2), followed by $k \mapsto s_{\mathcal{A}} - k$, give

$$p_g = \ell\, \mathbb{E}\#\{(i_1, \dots, i_n) \in \mathbb{Z}_{\ge 1}^n : pi_1 + q(i_2 + \cdots + i_n) \le K\}. \tag{9.3}$$

Indeed, the original nonnegative indices are increased by one after using the symmetry; the shift is $p + (n-1)q = s_{\mathcal{A}} - a$. Our choice of q ensures $q > a + p$. If some $i_j \ge 2$ with $j \ge 2$, the left side of the inequality inside (9.3) is at least $p + nq > s_{\mathcal{A}}$. Hence all $i_2, \dots, i_n$ equal one. The remaining count is $\lfloor (K - (n-1)q)_+/p \rfloor$, which is at most $(K - (n-1)q)_+/p$. Therefore

$$p_g \le \frac{q^{n-1}d_1 d_2}{P}\mathbb{E}(K - (n-1)q)_+. \tag{9.4}$$

For the following calculation, let z be real and close to zero. The moment-generating function of U_c is $(\exp(cz) - 1)/(cz)$, with its value at zero defined by continuity. Substituting $t = \exp(z)$ into the Hilbert product gives

$$\begin{aligned} &\frac{H_{\mathcal{A}}(\exp(z))}{\ell} \prod_{i=2}^{N} \frac{\exp(w_i z) - 1}{w_i z} \\ &\qquad = \frac{\exp(d_1 z) - 1}{d_1 z} \frac{\exp(d_2 z) - 1}{d_2 z} \left(\frac{\exp(qz) - 1}{qz} \right)^{n-1}. \end{aligned}$$

Thus there is equality of moment-generating functions in

$$K + \sum_{i=2}^{N} U_{w_i} \overset{\text{law}}{=} U_{d_1} + U_{d_2} + \sum_{j=2}^{n} U_{q,j}.$$

Here $U_{q,j}$ are independent copies of U_q. Compact support makes the moment-generating functions determine the distributions. Conditioning on K and using $\sum_{i=2}^{N} \mathbb{E}U_{w_i} = \eta$, Jensen's inequality gives

$$\begin{aligned}\mathbb{E}(K-(n-1)q)_+ &\le \mathbb{E}\left(K+\sum_{i=2}^{N} U_{w_i}-\eta-(n-1)q\right)_+ \\ &= \mathbb{E}\left(U_{d_1}+U_{d_2}+\sum_{j=2}^{n} U_{q,j}-\eta-(n-1)q\right)_+ \\ &= \mathbb{E}\left(U_{d_1}+U_{d_2}-\eta-\sum_{j=2}^{n} U_{q,j}\right)_+ .\end{aligned}$$

The last equality reflects each $U_{q,j}$ to $q-U_{q,j}$. The initial choice also ensures $q > d_1+d_2-\eta$. For $z \le d_1+d_2-\eta$, the part of the positive orthant on which $(z-u_2-\cdots-u_n)_+$ is nonzero lies in $[0,q]^{n-1}$. The identity

$$\int_0^\infty (z-t)_+^\alpha \, dt = \frac{z_+^{\alpha+1}}{\alpha+1} \qquad (\alpha > 0)$$

applied successively gives

$$\int_{[0,q]^{n-1}} (z-u_2-\cdots-u_n)_+ \, du_2\cdots du_n = \frac{z_+^n}{n!}.$$

Inserting this identity into (9.4) cancels q^{n-1} and proves the first inequality in (9.2). Integrating over U_{d_1} and U_{d_2} gives

$$d_1 d_2 \, \mathbb{E}(U_{d_1}+U_{d_2}-\eta)_+^n = \frac{\mathcal{R}_N(\eta; d_1, d_2)}{(n+1)(n+2)}.$$

The possible fourth corner term is $(-\eta)_+^{n+2} = 0$. Since $N = n+2$, this proves the second equality. □

10. The range $d_2 \ge 2h$

The comparisons below apply for $n \ge 2$; the final exponential estimate uses $n \ge 5$. Define the positive polynomial in the equation degrees

$$m_0 = d_1 d_2 \sum_{k=0}^{N-2} {d_1}^k {d_2}^{N-2-k}.$$

For $d_1 < d_2$, it is $(d_1 {d_2}^N - d_2 {d_1}^N)/(d_2-d_1)$; for $d_1 = d_2$, it is $(N-1){d_2}^N$.

Lemma 10.1. *If $d_2 \ge 2h$, then*

$$d_1 \ge p + w_{N-1}, \qquad P\mu \ge m_0 \frac{Q(d_2)}{{d_2}^N}. \tag{10.1}$$

Proof. Suppose $d_1 < p + w_{N-1}$. A monomial of f involving either of the two largest-weight variables has ordinary degree at least two, hence weighted degree at least $p + w_{N-1}$; such a monomial cannot occur. Thus f and all its first derivatives vanish on their coordinate plane. The restriction of g to this plane either vanishes identically or has a positive-dimensional zero germ. At its nonzero zeros the Jacobian of (f,g) has rank at most one, contradicting isolation. This proves the first inequality.

For $d_1 < d_2$, substituting (8.4) shows that the second inequality is equivalent to $Q(d_1) \le (d_1/d_2)^N Q(d_2)$. If $d_1 \ge h$, all factors are nonnegative and

$$\frac{d_1 - w_i}{d_2 - w_i} \le \frac{d_1}{d_2}$$

for each i, which proves it. If $d_1 < h$, the first inequality in (10.1) shows that every weight except h is strictly less than d_1. Consequently $Q(d_1) < 0$, whereas $Q(d_2) > 0$, and the conclusion follows again. At $d_1 = d_2$, Lemma 8.1 gives

$$P\mu = Q(d_2)\left(\sum_i \frac{d_2}{d_2 - w_i} - 1\right) \geq (N-1)Q(d_2) = m_0 \frac{Q(d_2)}{{d_2}^N}. \qquad \square$$

Lemma 10.2. *For positive $d_1 \leq d_2$ and $N \geq 4$,*

$$m_0 \geq \frac{N-1}{2^N - 2}\mathcal{R}_N(0; d_1, d_2). \tag{10.2}$$

Proof. Dividing (10.2) by $d_1 d_2$ and expanding $\mathcal{R}_N(0; d_1, d_2)$ reduces the assertion to

$$\sum_{k=0}^{N-2} {d_1}^k {d_2}^{N-2-k} \geq \frac{N-1}{2^N - 2} \sum_{k=0}^{N-2} \binom{N}{k+1} {d_1}^k {d_2}^{N-2-k}.$$

For $0 \leq k \leq N-2$, define the two sequences

$$a_k = \binom{N}{k+1}, \qquad b_k = \frac{{d_1}^k {d_2}^{N-2-k} + {d_1}^{N-2-k} {d_2}^k}{2}.$$

Both are symmetric about the centre. Moving from an endpoint towards the centre, a_k increases and b_k decreases, since

$$b_k - b_{k+1} = \frac{d_2 - d_1}{2}\left({d_1}^k {d_2}^{N-3-k} - {d_1}^{N-3-k} {d_2}^k\right) \geq 0 \quad \left(k \leq \frac{N-3}{2}\right).$$

Thus $(a_i - a_j)(b_i - b_j) \leq 0$ for every pair of indices. Summing and expanding gives

$$(N-1)\sum_k a_k b_k \leq \left(\sum_k a_k\right)\left(\sum_k b_k\right).$$

Symmetry gives $\sum a_k b_k = \sum a_k {d_1}^k {d_2}^{N-2-k}$ and $\sum b_k = \sum {d_1}^k {d_2}^{N-2-k}$, while $\sum a_k = 2^N - 2$. This proves the desired inequality. $\square$

For integers $N \geq 4$ and real $\tau \geq 0$, define

$$F_N(\tau) = \frac{(2-\tau)_+^N - 2(1-\tau)_+^N}{2^N - 2}. \tag{10.3}$$

This is the ratio of a shifted $(N-2)$nd moment of two independent unit uniforms to its unshifted value. In particular F_N is convex on $[0, \infty)$, is positive on $[0, 2)$, and vanishes on $[2, \infty)$.

Lemma 10.3. *For $0 < d_1 \leq d_2$ and $\eta \geq 0$,*

$$\frac{\mathcal{R}_N(\eta; d_1, d_2)}{\mathcal{R}_N(0; d_1, d_2)} \leq F_N(\eta/d_2). \tag{10.4}$$

Proof. Let φ be the density of $U_{d_1} + U_{d_2}$ and ψ that of $U_{d_2} + U'_{d_2}$, where U'_{d_2} is an independent copy of U_{d_2}. On the support of φ,

$$\frac{\psi(s)}{\varphi(s)} = \begin{cases} d_1/d_2, & 0 < s < d_1, \\ s/d_2, & d_1 \leq s \leq d_2, \\ \dfrac{d_1}{d_2}\dfrac{2d_2 - s}{d_1 + d_2 - s}, & d_2 < s < d_1 + d_2. \end{cases}$$

This ratio is nondecreasing. On the last interval its derivative is $d_1(d_2 - d_1)/(d_2(d_1 + d_2 - s)^2) \geq 0$. Any mass of ψ beyond the support of φ is at its upper end. Multiply both densities by s^{N-2} and normalize each to have integral one. Their density ratio remains nondecreasing. The difference of the normalized densities changes sign at most once, from negative to positive. For an increasing test function, subtracting its value at the crossing therefore proves that its integral against the second density is at least its integral against

the first. If the densities coincide, the assertion is immediate. Apply this observation to the increasing function $s \mapsto (1-\eta/s)_+^{N-2}$ for $s>0$. Its two expectations are the corresponding shifted-to-unshifted moment ratios. The rectangle integral in Lemma 9.1 identifies the first ratio with the left side of (10.4); scaling the equal-uniform sum by d_2 identifies the second with $F_N(\eta/d_2)$. □

Lemma 10.4. *Set $\gamma_N = N(2^{N-1}-2)/(2^N-2)$ for $N \geq 4$. For $\tau \geq 0$,*

$$F_N(\tau) \leq \exp(-\gamma_N \tau)\,. \tag{10.5}$$

Moreover $-(\log F_N)'(\tau) \geq \gamma_N$ for $0 \leq \tau < 2$.

Proof. For $0 \leq \tau < 1$, set $q = (1-\tau)/(2-\tau) \in [0, 1/2]$. Differentiation of (10.3) yields

$$-\frac{d}{d\tau}\log F_N(\tau) = \frac{N}{2-\tau}\frac{1-2q^{N-1}}{1-2q^N}.$$

The derivative of the second factor with respect to q is

$$-\frac{2q^{N-2}((N-1)-Nq+2q^N)}{(1-2q^N)^2} \leq 0,$$

because $(N-1)-Nq \geq (N-2)/2 > 0$. As τ increases, q decreases, so both positive factors in the logarithmic derivative increase. Their product is at least its value at $\tau = 0$, namely γ_N. For $1 \leq \tau < 2$, the logarithmic derivative is $N/(2-\tau) \geq \gamma_N$. Integrate from zero, using $F_N(0) = 1$. For $\tau \geq 2$, (10.5) follows from $F_N(\tau) = 0$. □

Proposition 10.5. *If $n \geq 5$ and $d_2 \geq 2h$, then*

$$\mu > \exp\left(8\sum_{i=1}^N w_i/(105 d_2)\right) C_{n,2} p_g.$$

Proof. Since $p \leq \sum_{i=1}^N w_i/N$, one has $\eta \geq (N-1)\sum_{i=1}^N w_i/(2N)$. Set $\kappa_N = (N-1)(2^{N-1}-2)/(2(2^N-2))$. Lemmas 10.3 and 10.4 give

$$\frac{\mathcal{R}_N(\eta; d_1, d_2)}{\mathcal{R}_N(0; d_1, d_2)} \leq \exp\left(-\kappa_N \sum_{i=1}^N w_i/d_2\right).$$

On the other hand, concavity of $\log(1-x)$ gives its chord bound $\log(1-x) \geq -2(\log 2)x$ for $0 \leq x \leq 1/2$. Since $w_i/d_2 \leq 1/2$,

$$\frac{Q(d_2)}{{d_2}^N} \geq \exp\left(-2(\log 2)\sum_{i=1}^N w_i/d_2\right).$$

The quantity $\kappa_N = (N-1)(1/2 - 1/(2^N-2))/2$ increases with N. For $N \geq 7$,

$$\kappa_N \geq \kappa_7 = \frac{31}{21}, \qquad 2\log 2 < \frac{7}{5}.$$

The last comparison follows from

$$\exp(7/10) > \sum_{j=0}^4 \frac{(7/10)^j}{j!} = \frac{482921}{240000} > 2.$$

Thus $\kappa_N - 2\log 2 > 8/105$. Combining Lemmas 9.1, 10.1, and 10.2, we obtain

$$C_{n,2} p_g \leq \frac{N-1}{(2^N-2)P}\mathcal{R}_N(\eta; d_1, d_2) \leq \frac{m_0}{P}\exp\left(-\kappa_N \sum_{i=1}^N w_i/d_2\right),$$

$$\mu \geq \frac{m_0}{P}\exp\left(-2(\log 2)\sum_{i=1}^N w_i/d_2\right).$$

If $p_g > 0$, these inequalities imply the claimed strict estimate. If $p_g = 0$, the last displayed lower bound proves $\mu > 0$, which gives the same conclusion. □

11. The range $d_2 < 2h$

11.1. **Coordinates forced by isolation.**

Lemma 11.1. *Suppose $n \geq 3$ and $d_2 < 2h$. There are weighted polynomial coordinates $z_0, x, y, z_1, \dots, z_m$, where $m = n-1$, such that*

$$\begin{gathered} f = z_0 x + P_0(x, y, z_1, \dots, z_m), \qquad g = z_0 y + Q_0(x, y, z_1, \dots, z_m), \\ \operatorname{wt}(z_0) = h, \quad \operatorname{wt}(x) = s, \quad \operatorname{wt}(y) = t, \quad \operatorname{wt}(z_i) = b_i, \\ (d_1, d_2) = (h+s, h+t), \qquad 0 < s \leq t < h, \qquad 0 < b_i \leq h. \end{gathered} \tag{11.1}$$

Put $u = \sum_{i=1}^m b_i$. Then $a = h - u$. If $u \leq h$, isolation also implies

$$2 \max_i b_i \leq h + t. \tag{11.2}$$

Proof. Choose a variable z_0 of weight h. Neither equation contains ${z_0}^2$, since both degrees are less than $2h$, and neither contains a linear term. Hence the z_0-axis lies in X. Write the coefficients of z_0 in f, g as g_1, g_2. Along the punctured axis, the Jacobian has rank two; its transverse rows are the linear parts of g_1, g_2, multiplied by the nonzero axis coordinate. These linear parts are therefore independent. Their positive weights are $s = d_1 - h$ and $t = d_2 - h$, with $0 < s \leq t < h$.

Take g_1, g_2 as the new coordinates x, y and complete their linear parts to bases in the respective weight spaces. A nonlinear monomial of a given positive weight involves only variables of strictly smaller weights. The resulting coordinate change is therefore triangular with respect to weight, after invertible linear changes inside each weight space. Successive solution in increasing weight gives a polynomial inverse. This proves (11.1); its equation degrees also give $a = h - u$. Assume $u \leq h$, choose $b = b_{i_0} = \max b_i$, and suppose $h + t < 2b$. Both equation degrees are strictly greater than b and less than $2b$, so both equations vanish on the z_{i_0}-axis. A nonzero first derivative of g on this axis would have to come from a product of the axis variable with another coordinate. The weight of the latter must be

$$h + t - b \geq t + \sum_{i \neq i_0} b_i.$$

Since $m \geq 2$, this is strictly larger than s, t and each other b_i, but is less than b and h. There is no such coordinate. Terms involving higher powers of z_{i_0} cannot contribute, because $d_2 < 2b$. The second Jacobian row is therefore zero on the punctured axis, contradicting isolation. This proves (11.2). □

If $u > h$, then $a < 0$ and $p_g = 0$ by (8.2); the desired inequalities follow from $\mu > 0$. We may therefore assume $u \leq h$ for the rest of this section. The least weight is still $p = \min(s, t, b_1, \dots, b_m)$.

11.2. **A simplex bound.**

Since $a = h - u < h < d_1 \leq d_2$, the Hilbert truncation contains neither the coordinate z_0 nor a relation from either equation. Consequently

$$p_g = \#\{(i, j, k_1, \dots, k_m) \in \mathbb{Z}_{\geq 0}^{m+2} : si + tj + \textstyle\sum_{\ell=1}^m b_\ell k_\ell \leq h - u\}. \tag{11.3}$$

To compare this count with the midpoint of the two equation degrees, define

$$v = \frac{s+t}{2}, \qquad \varepsilon = \frac{t-s}{2}, \qquad c = h + v, \qquad \lambda = c - \frac{u-p}{2}. \tag{11.4}$$

Here λ is the intercept in the simplex integral below. We have $0 \leq \varepsilon < v < h$ and $\lambda > 0$.

Lemma 11.2. *Under the preceding assumptions,*

$$p_g \leq \frac{\lambda^{m+2}}{(m+2)!\, st \prod_i b_i}. \tag{11.5}$$

Proof. Sum first over a coordinate of weight p in (11.3). Denote the other $m+1$ weights by $c_1, \dots, c_{m+1}$. For fixed nonnegative integer indices $j_1, \dots, j_{m+1}$, the number of possible first coordinates, when the residual bound is nonnegative, is

$$\left\lfloor \frac{h - u - \sum_\ell c_\ell j_\ell}{p} \right\rfloor + 1 \le \frac{(h - u + p - \sum_\ell c_\ell j_\ell)_+}{p}.$$

When the residual bound is negative, the count is zero and the same upper bound remains valid. The positive part of an affine function is convex. On each unit cube centred at a nonnegative integer lattice point, Jensen's inequality bounds its value at the centre by its average over the cube. These cubes fill $[-1/2, \infty)^{m+1}$. Translating this region to the positive orthant changes the constant term to

$$h - u + p + \frac{1}{2}\sum_\ell c_\ell = h + \frac{s + t - u + p}{2} = \lambda.$$

Changing each integration variable by its corresponding weight and integrating successively gives

$$\int_{\mathbb{R}^{m+1}_{\ge 0}} (\lambda - \sum_\ell c_\ell x_\ell)_+ \, dx = \frac{\lambda^{m+2}}{(m+2)! \prod_\ell c_\ell}.$$

Multiplying by $1/p$ proves (11.5). □

11.3. **A midpoint identity for the Milnor numerator.** Define the polynomial

$$S(z) = \prod_{i=1}^{m} (h + z - b_i).$$

Lemma 11.3. *In the coordinates* (11.1)*,*

$$\begin{aligned} \frac{P\mu}{h} = {} & (h^2 + 2hv + 2v^2 - 2\varepsilon^2)\frac{S(s) + S(t)}{2} \\ & + h(cv + \varepsilon^2)\frac{S(t) - S(s)}{t - s}, \end{aligned} \tag{11.6}$$

where the difference quotient is $S'(v)$ when $s = t$. Moreover,

$$\frac{P\mu}{h} > c^2 S(v). \tag{11.7}$$

Proof. From (11.1),

$$Q(h + t) = ht(h + t - s)S(t), \qquad Q(h + s) = hs(h + s - t)S(s).$$

Insert these expressions into (8.4). The numerator of $P\mu/h$ is

$$(h + s)t(h + t - s)S(t) - (h + t)s(h + s - t)S(s).$$

The sum and difference of its two scalar coefficients are

$$\begin{aligned} &(h + s)t(h + t - s) + (h + t)s(h + s - t) = 2h(cv + \varepsilon^2), \\ &(h + s)t(h + t - s) - (h + t)s(h + s - t) = (t - s)(h^2 + 2hv + 2v^2 - 2\varepsilon^2). \end{aligned}$$

Splitting the numerator into its symmetric and antisymmetric parts and dividing by $t - s$ proves (11.6). Continuity gives the case $s = t$. Because $h - b_i \ge 0$, the coefficients of S in powers of z are nonnegative. Thus S and S' are convex on $[0, \infty)$. Their midpoint estimates give

$$\frac{S(s) + S(t)}{2} \ge S(v), \qquad \frac{S(t) - S(s)}{t - s} = \frac{1}{t - s}\int_s^t S'(z)\, dz \ge S'(v)$$

when $s < t$, with the second expression interpreted by continuity if $s = t$. Also

$$\frac{S'(v)}{S(v)} = \sum_i \frac{1}{c - b_i} \ge \frac{m}{c}.$$

The two coefficients in (11.6) are positive: the first equals $h^2 + h(s+t) + 2st$, and the second is $h(cv + \varepsilon^2)$. Hence

$$\frac{P\mu}{h} \geq \left(h^2 + 2hv + 2v^2 - 2\varepsilon^2 + hmv + \frac{hm\varepsilon^2}{c} \right) S(v).$$

The expression in parentheses minus c^2 is at least

$$v^2 - 2\varepsilon^2 + hmv \geq v(hm - v) > 0,$$

since $\varepsilon < v < h$ and $m \geq 2$. This proves (11.7). □

Proposition 11.4. *If $n \geq 4$ and $d_2 < 2h$, then*

$$\mu > \exp(u/(60c))\,(n+1)!p_g > C_{n,2}p_g$$

when $p_g > 0$, and $\mu > 0$ when $p_g = 0$.

Proof. Assume $u \leq h$; the case $u > h$ has $p_g = 0$. By (11.2),

$$\frac{b_i}{c} \leq \frac{h+t}{2h+s+t} < \frac{2}{3};$$

the strict inequality follows from $t < h$. The chord bound for $\log(1-x)$ on $[0, 2/3]$ therefore gives

$$\frac{S(v)}{c^m} = \prod_i \left(1 - \frac{b_i}{c}\right) \geq \exp\left(-\frac{3\log 3}{2}\frac{u}{c}\right).$$

Since $p \leq u/m$ and $1 - x \leq \exp(-x)$,

$$\left(\frac{\lambda}{c}\right)^{m+2} \leq \exp\left(-\frac{(m+2)(m-1)}{2m}\frac{u}{c}\right).$$

For $m \geq 3$, $(m+2)(m-1)/(2m) \geq 5/3$, whereas $(3/2)\log 3 < 33/20$, because

$$\exp(11/10) > \sum_{j=0}^{5} \frac{(11/10)^j}{j!} = \frac{36015101}{12000000} > 3.$$

The difference between these two rates exceeds $5/3 - 33/20 = 1/60$. Lemma 11.3 yields

$$\frac{P\mu}{h} > c^{m+2} \exp\left(-\frac{3\log 3}{2}\frac{u}{c}\right) > \exp(u/(60c))\,\lambda^{m+2}.$$

Now $P = hst\prod_i b_i$ and $m + 2 = n + 1$, so Lemma 11.2 proves the factorial estimate.

For $N = n + 2 \geq 6$, $2^N - 2 > N(N-1)$, which is equivalent to $(n+1)! > C_{n,2}$. At $N = 6$ this is $62 > 30$. If it holds at N, then

$$2^{N+1} - 2 = 2(2^N - 2) + 2 > 2N(N-1) + 2 > N(N+1),$$

where the last inequality follows from $(N-1)(N-2) > 0$. This proves the comparison of constants and the proposition. □

12. The high-degree range in dimension four

For $N = 6$, we compare F_6 with a product in the weights.

Lemma 12.1. *Let $0 < \rho \leq y_i \leq 1/2$ for $1 \leq i \leq 6$, and suppose at least one y_i equals ρ. Then*

$$\prod_{i=1}^{6}(1 - y_i) > F_6\left(\frac{\sum_i y_i - \rho}{2}\right). \tag{12.1}$$

Proof. Fix a coordinate equal to ρ. The left side of (12.1) is affine in each other coordinate separately, and F_6 is convex by its moment representation following (10.3). Thus the left side minus the right side is separately concave in the other five coordinates. On an interval, a concave function is bounded below by its linear interpolation between the endpoints. Applying this fact successively reduces the inequality to the vertices where each of those coordinates is ρ or $1/2$.

Let k be the number of coordinates equal to $1/2$, so $0 \le k \le 5$. All other $6-k$ coordinates equal ρ. With $\tau = ((5-k)\rho + k/2)/2$, we have

$$\prod_i (1-y_i) = 2^{-k}(1-\rho)^{6-k}, \qquad 2-\tau = 2 - \frac{k}{4} - \frac{5-k}{2}\rho.$$

For $k \ge 1$, the bound $F_6(\tau) \le (2-\tau)^6/62$ shows that it suffices to prove

$$2^k \frac{\left(2 - k/4 - (5-k)\rho/2\right)^6}{(1-\rho)^{6-k}} < 62. \tag{12.2}$$

For $k = 1$, the logarithmic derivative of $2(7/4-2\rho)^6/(1-\rho)^5$ is

$$-\frac{12}{7/4-2\rho} + \frac{5}{1-\rho} = \frac{2\rho - 13/4}{(7/4-2\rho)(1-\rho)} < 0.$$

Its maximum on $[0, 1/2]$ is its value at zero:

$$2(7/4)^6 = \frac{117649}{2048} < 62.$$

For $2 \le k \le 5$, the logarithmic derivative of the expression in (12.2) is

$$\frac{(6-k)(k-2)/4 - k(5-k)(1-\rho)/2}{(1-\rho)(2-k/4-(5-k)\rho/2)}.$$

Its denominator is positive and its numerator is nondecreasing in ρ, so the maximum of the expression in (12.2) occurs at an endpoint. At $\rho = 1/2$ the expression is $64(3/4)^6 = 729/64 < 62$. At $\rho = 0$ it is $2^k(2-k/4)^6$. As a function of real $k \in [2,5]$, the latter has logarithmic derivative $\log 2 - 6/(8-k) < 0$. Its maximum is therefore $729/16 < 62$, attained at $k = 2$. It remains to treat $k = 0$, for which $\tau = 5\rho/2$. If $0 < \rho \le 1/10$, Lemma 10.4 gives $-(\log F_6)' \ge 90/31$, and hence

$$\frac{d}{d\rho} \log \frac{(1-\rho)^6}{F_6(5\rho/2)} \ge \frac{225}{31} - \frac{6}{1-\rho} \ge \frac{225}{31} - \frac{20}{3} > 0.$$

The ratio equals one at $\rho = 0$, so it is strictly greater than one in this interval. If $1/10 \le \rho \le 1/2$, then

$$\frac{2-5\rho/2}{1-\rho} \le \frac{35}{18}, \qquad 35^6 = 1838265625 < 2108757888 = 62 \cdot 18^6.$$

It follows that $F_6(5\rho/2) \le (2-5\rho/2)^6/62 < (1-\rho)^6$. Successive concavity interpolation proves (12.1) throughout the cube. □

Proposition 12.2. *If $n = 4$ and $d_2 \ge 2h$, then $\mu > (1800/31)p_g$.*

Proof. Apply Lemma 12.1 with $y_i = w_i/d_2$ and $\rho = p/d_2$. Its argument of F_6 is η/d_2. Lemmas 10.1, 10.2, and 10.3 give

$$\begin{aligned} P\mu &\ge m_0 \prod_i (1 - w_i/d_2) > m_0 F_6(\eta/d_2) \\ &\ge \frac{5}{62} \mathcal{R}_6(\eta; d_1, d_2). \end{aligned}$$

Since $C_{4,2}/6! = 5/62$, Lemma 9.1 proves the assertion. □

13. The high-degree range in dimension three

Let $\boldsymbol{v} = (p, v_2, v_3, v_4, v_5)$ be a list of five positive real weights. For $0 \le j \le 5$, let $e_j(\boldsymbol{v})$ be the sum of products of j distinct entries, with $e_0(\boldsymbol{v}) = 1$. Define

$$\begin{aligned}\mathcal{M}(d_1, d_2; \boldsymbol{v}) &= d_1 d_2 \{ {d_1}^3 + {d_1}^2 d_2 + d_1 {d_2}^2 + {d_2}^3 \\ &\quad - e_1(\boldsymbol{v})({d_1}^2 + d_1 d_2 + {d_2}^2) \\ &\quad + e_2(\boldsymbol{v})(d_1 + d_2) - e_3(\boldsymbol{v}) \} + e_5(\boldsymbol{v}), \\ \Phi(d_1, d_2; \boldsymbol{v}) &= 30\mathcal{M}(d_1, d_2; \boldsymbol{v}) \\ &\quad - 4\mathcal{R}_5\left(\frac{v_2 + v_3 + v_4 + v_5}{2}; d_1, d_2\right).\end{aligned} \tag{13.1}$$

The polynomial $\mathcal{M}$ is the dimension-three Milnor numerator in (8.3). These expressions will also be used for weight lists that satisfy only the numerical inequalities below.

Lemma 13.1. *Suppose $0 < p$, $2p \le d_1 \le d_2$, and put $b = d_2/2$ and $q = \min(b, d_1 - p)$. If*

$$p \le v_2, v_3, v_4 \le q, \qquad p \le v_5 \le b,$$

then $\Phi(d_1, d_2; \boldsymbol{v})$ is at least the minimum of its values on the five lists below. An exponent on an entry specifies its number of repetitions; the order of the last four entries is immaterial:

$$(p, p, p, p, p), \qquad (p^{k+1}, q^{3-k}, b) \quad (0 \le k \le 3).$$

Proof. The polynomial $\mathcal{M}$ is affine in each weight separately. By Lemma 9.1,

$$\mathcal{R}_5(\eta; d_1, d_2) = 20 d_1 d_2 \, \mathbb{E}(U_{d_1} + U_{d_2} - \eta)_+^3.$$

This is convex in η, since the positive part of a cube is convex. Thus Φ is concave in each of v_2, v_3, v_4, v_5 separately. Repeatedly using the chord inequality on their intervals bounds its value below by the minimum of its values at the vertices of the box. A vertex with last entry b is one of the four displayed lists. At a vertex with last entry p, all other entries are p or q. If at least one is q, vary one such entry on $[p, b]$, keeping the others fixed. Concavity bounds that value below by the minimum of the two endpoint values. By symmetry in the last four entries, the endpoint b gives a displayed list, while the endpoint p reduces the number of entries equal to q. Induction ends at the all-p list. Throughout this argument the first entry remains p, so the shift in (13.1) is affine in every varied coordinate. □

Lemma 13.2. *Under the hypotheses of Lemma 13.1, each of its five endpoint values is at least $142p^5$.*

Proof. For the all-p list, differentiation in d_2 gives

$$\begin{aligned}\frac{1}{10 d_1}\frac{\partial \Phi}{\partial d_2} &= 4(d_2 - d_1)^3 + (9d_1 + 3p)(d_2 - d_1)^2 \\ &\quad + (4{d_1}^2 + 24 d_1 p - 36p^2)(d_2 - d_1) \\ &\quad + p(22{d_1}^2 - 54 d_1 p + 34p^2).\end{aligned}$$

All terms are nonnegative for $d_2 \ge d_1 \ge 2p$: the last two coefficients can be written as

$$4(d_1 - 2p)^2 + 40p(d_1 - 2p) + 28p^2, \qquad 22(d_1 - 2p)^2 + 34p(d_1 - 2p) + 14p^2.$$

Hence the minimum in $d_2 \ge d_1$ occurs at $d_2 = d_1$. Substitution there gives

$$\begin{aligned}\Phi(d_1, d_1; (p, p, p, p, p)) &= 142p^5 + 560p^4(d_1 - 2p) + 820p^3(d_1 - 2p)^2 \\ &\quad + 520p^2(d_1 - 2p)^3 + 110p(d_1 - 2p)^4 \ge 142p^5.\end{aligned}$$

It remains to treat (p^{k+1}, q^{3-k}, b) for $k \in \{0,1,2,3\}$. In the following expansions, Φ and $\mathcal{M}$ are evaluated at $(d_1, d_2; (p^{k+1}, q^{3-k}, b))$. We use the following boundary polynomial, for $\delta \geq 0$:

$$\begin{aligned} 4\mathcal{T}_k(p,\delta) &= 568p^5 + (2560+200k)p^4\delta \\ &\quad + \{4240 + 20k(47-k)\}p^3\delta^2 \\ &\quad + \{3440 + 1200k + 20k^2(3-k)\}p^2\delta^3 \\ &\quad + \{1400 + k(630 - 10k^2 - 5k^3) + 55k^2\}p\delta^4 \\ &\quad + \{224 + k(136 - 15k^2) + 10k^2 + k^4(5-k)\}\delta^5. \end{aligned} \tag{13.2}$$

The factors in braces are positive: on $0 \leq k \leq 3$,

$$630 - 10k^2 - 5k^3 \geq 405, \qquad 136 - 15k^2 \geq 1,$$

and every other displayed factor is nonnegative. Consequently

$$\mathcal{T}_k(p,\delta) \geq 142p^5. \tag{13.3}$$

The symmetric sums are obtained from the generating product

$$\sum_{j=0}^{5} e_j(\boldsymbol{v}) z^j = (1+pz)^{k+1}(1+qz)^{3-k}(1+bz). \tag{13.4}$$

More explicitly,

$$\begin{aligned} e_1 &= (k+1)p + (3-k)q + b, \\ e_2 &= \binom{k+1}{2}p^2 + (k+1)(3-k)pq + \binom{3-k}{2}q^2 \\ &\quad + b\{(k+1)p + (3-k)q\}, \\ e_3 &= \binom{k+1}{3}p^3 + \binom{k+1}{2}(3-k)p^2q + (k+1)\binom{3-k}{2}pq^2 + \binom{3-k}{3}q^3 \\ &\quad + b\left\{\binom{k+1}{2}p^2 + (k+1)(3-k)pq + \binom{3-k}{2}q^2\right\}, \\ e_5 &= p^{k+1}q^{3-k}b. \end{aligned}$$

Here e_j means $e_j(\boldsymbol{v})$ for this displayed weight list. If $\eta = (kp + (3-k)q + b)/2$, then $\eta \leq d_2$. Formula (13.1) therefore gives

$$\Phi - 4(d_1 - \eta)_+^5 = 30\mathcal{M} - 4\{(d_1 + d_2 - \eta)^5 - (d_2 - \eta)^5\}. \tag{13.5}$$

We expand this expression in the nonnegative gaps between the endpoints. The fifth-power difference in this expansion is evaluated by

$$(d_1 + d_2 - \eta)^5 - (d_2 - \eta)^5 = \sum_{j=1}^{5} \binom{5}{j} {d_1}^j (d_2 - \eta)^{5-j}.$$

Suppose first that $d_1 \le p+b$. Then $q = d_1 - p$. Write $\delta = q - p \ge 0$ and $\zeta = b - q \ge 0$, so $d_1 = 2p + \delta$ and $d_2 = 2(p + \delta + \zeta)$. Expanding (13.4) in (13.5) gives

$$\begin{aligned}\Phi - 4(d_1-\eta)_+^5 &= \mathcal{T}_k(p,\delta)\\ &\quad + \frac{5\zeta}{4}\big[\{264 + k(106 - 10k^2 - k) + k^4\}\delta^4\\ &\qquad + \{1320 + 12k(31 - 2k - k^2)\}p\delta^3\\ &\qquad + \{2328 + 12k(31 - 3k)\}p^2\delta^2\\ &\qquad + (1704 + 72k)p^3\delta + 408p^4\big]\\ &\quad + \frac{15\zeta^2}{2}(\delta + 2p)\\ &\qquad \cdot \big[\{92 + k(18 - 5k)\}\delta^2 + (168 + 4k)p\delta + 64p^2\big]\\ &\quad + 15\zeta^3(\delta + 2p)\{(39 - k)\delta + 30p\} + \frac{555\zeta^4}{4}(\delta + 2p).\end{aligned} \tag{13.6}$$

Here (13.2) is the constant term, obtained by setting $\zeta = 0$. Every remaining term is nonnegative, because

$$106 - 10k^2 - k \ge 13, \quad 31 - 2k - k^2 \ge 16, \quad 31 - 3k \ge 22, \quad 18 - 5k \ge 3.$$

Together with (13.3), this proves the required lower bound in this range.

Suppose next that $d_1 \ge p+b$. Now $q = b$. Write $\delta = b - p \ge 0$ and $x = d_1 - p - b$. The condition $d_1 \le d_2 = 2b$ gives $0 \le x \le \delta$. The same generating product yields

$$\begin{aligned}\Phi - 4(d_1-\eta)_+^5 &= \mathcal{T}_k(p,\delta)\\ &\quad + \frac{5x}{4}\big[\{80 + k(32 - k^3)\}\delta^4\\ &\qquad + \{400 + 8k(17 - k^2)\}p\delta^3\\ &\qquad + \{816 + 24k(7 - k)\}p^2\delta^2 + (736 + 32k)p^3\delta + 224p^4\big]\\ &\quad + 5x^2\big[\{16 + k(6 - k^2)\}\delta^3 + \{84 + 6k(3 - k)\}p\delta^2\\ &\qquad + 132p^2\delta + 56p^3\big]\\ &\quad + 10x^3\big[\{8 + k(2 - k)\}\delta^2 + (26 - 2k)p\delta + 14p^2\big]\\ &\quad + 10x^4\{(4 - k)\delta + 2p\} - 4x^5.\end{aligned} \tag{13.7}$$

The coefficients of x, x^2, x^3 are nonnegative. Indeed,

$$32 - k^3 \ge 5, \qquad 17 - k^2 \ge 8,$$

$$16 + k(6 - k^2) \ge 16 - 3k \ge 7, \qquad 8 + k(2 - k) \ge 5;$$

the last inequality follows from concavity in k and its endpoint values 8 and 5. Finally, $x \le \delta$ gives

$$10x^4\{(4 - k)\delta + 2p\} - 4x^5 \ge x^4(6\delta + 20p) \ge 0.$$

Equation (13.3) completes this range. The two ranges include their common boundary $d_1 = p + b$. □

Proposition 13.3. *If $n = 3$ and $d_2 \ge 2h$, then*

$$\mu - 16p_g \ge \frac{71p^5}{15P} > 0.$$

Proof. Lemma 10.1 gives $d_1 \ge p + w_4 \ge 2p$. Put $b = d_2/2$ and $q = \min(b, d_1 - p)$. The ordered actual weights satisfy

$$p \le w_2, w_3, w_4 \le q, \qquad p \le w_5 = h \le b.$$

Lemmas 13.1 and 13.2 give

$$30P\mu - 4\mathcal{R}_5(\eta; d_1, d_2) \geq 142p^5, \qquad \eta = (\sum_{i=1}^{N} w_i - p)/2.$$

By Lemma 9.1, $p_g \leq \mathcal{R}_5(\eta; d_1, d_2)/(120P)$. Division by $30P$ proves the assertion. □

14. The low-degree range in dimension three

Suppose $n = 3$ and $d_2 < 2h$. Use the coordinates and notation of Section 11; in particular, $m = 2$, $u = b_1 + b_2$, and $c = h + (s+t)/2$. The case $u > h$ has $p_g = 0$. Otherwise Lemmas 11.2 and 11.3 give

$$\frac{P\mu}{h} > c^2(c - b_1)(c - b_2), \qquad p_g \leq \frac{\lambda^4}{24stb_1b_2}, \qquad \lambda = c - \frac{u - p}{2}. \tag{14.1}$$

Proposition 14.1. *If $n = 3$ and $d_2 < 2h$, then $\mu \geq 16p_g + 1$.*

Proof. Assume $u \leq h$. Then $0 < u < c$, (11.2) gives $b_1, b_2 < 2c/3$, and $p \leq u/2$ gives

$$(\lambda/c)^4 \leq \left(1 - \frac{u}{4c}\right)^4.$$

Since $b_1b_2 \geq 0$,

$$\frac{(c - b_1)(c - b_2)}{c^2} \geq 1 - \frac{u}{c} \qquad (0 \leq u \leq 2c/3).$$

For $2c/3 \leq u \leq c$, the inequality $(2c/3 - b_1)(2c/3 - b_2) \geq 0$ instead gives

$$\frac{(c - b_1)(c - b_2)}{c^2} \geq \frac{5}{9} - \frac{u}{3c}.$$

Subtract $(2/3)(1 - u/(4c))^4$ from each affine lower bound on its interval. Each difference has second derivative $-(1 - u/(4c))^2/(2c^2) \leq 0$, so its minimum occurs at an endpoint. At $u = 0, 2c/3, c$, respectively, these differences are

$$\frac{1}{3}, \qquad \frac{23}{1944}, \qquad \frac{13}{1152},$$

which are positive. Consequently

$$\frac{(c - b_1)(c - b_2)}{c^2} > \frac{2}{3}(\lambda/c)^4.$$

Use (14.1) and $P = hstb_1b_2$ to obtain

$$\mu > \frac{2\lambda^4}{3stb_1b_2} \geq 16p_g.$$

Both μ and $16p_g$ are integers. The case $u > h$ follows from $p_g = 0$ and $\mu > 0$. □

Proof of Theorem C. For $n = 3$, Propositions 13.3 and 14.1 give the result in the complementary ranges $d_2 \geq 2h$ and $d_2 < 2h$. For $n = 4$, these ranges are covered by Propositions 12.2 and 11.4. For $n \geq 5$, use Propositions 10.5 and 11.4. In dimension three, the integrality of μ and $16p_g$ gives $\mu \geq 16p_g + 1$. □

15. An intersection of two quintics

Choosing a nonempty proper subset of a k-element set and identifying it with its complement gives $S(k, 2) = 2^{k-1} - 1$. Thus the corrected complete-intersection constant specializes in codimension two to (1.9). At $n = 2$, its formal value $36/7$ fails for surfaces, as the following example shows. Take distinct complex numbers $\lambda_1, \ldots, \lambda_4$ and the homogeneous equations

$$f = \sum_{i=1}^{4} x_i^5, \qquad g = \sum_{i=1}^{4} \lambda_i x_i^5.$$

The finite surjective map $(x_1, \dots, x_4) \mapsto (x_1^5, \dots, x_4^5)$ pulls back two independent linear equations, so the common zero set has dimension two. The two equations therefore form a regular sequence. If their Jacobian rows were dependent at a nonzero common zero, all nonzero coordinates would have the same coefficient λ_i. Distinctness forces at most one nonzero coordinate, which is incompatible with $f = 0$. The complete intersection is thus isolated, and its embedding is minimal since the equations have no linear terms.

All weights equal one and $d_1 = d_2 = 5$. With $Q(t) = (t-1)^4$, Lemma 8.1 gives

$$\mu = 5 \cdot 4 \cdot 4^3 - 4^4 = 1024.$$

The cutoff is $a = 6$. From (8.2) and $H_R(t) = (1-t^5)^2/(1-t)^4$,

$$p_g = \binom{10}{4} - 2\binom{5}{4} = 210 - 10 = 200.$$

Hence

$$1024 - \frac{36}{7} \cdot 200 = -\frac{32}{7} < 0, \qquad 1024 > 5 \cdot 200.$$

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Zhili College, Tsinghua University, Beijing 100084, China
Email address: `tangqy24@mails.tsinghua.edu.cn`