

Durfee inequalities for quasihomogeneous complete intersections of codimension two

Qiyue Tang

Abstract

For positive-weight quasihomogeneous isolated complete intersections of minimal embedding codimension two, we prove $\mu \geq 4p_g + 1$ in dimension two and $\mu > C_{n,2p_g}$ in every dimension $n \geq 3$.

1 Introduction

The Milnor number of an isolated complete intersection singularity measures the middle cohomology of its Milnor fibre, whereas its geometric genus is defined by the structure-sheaf cohomology of a resolution. Durfee inequalities compare these topological and analytic invariants. For homogeneous complete intersections of dimension greater than two, Kerner and Némethi established an inequality with a constant depending on both the dimension and the codimension [3]. Their later work treats convenient Newton-nondegenerate complete intersections with sufficiently large Newton diagrams [4]. For complex dimension $n \geq 3$ and codimension two, the proposed corrected constant is

$$C_{n,2} = \frac{(n+1)(n+2)!}{2^{n+2} - 2} \quad (n \geq 3). \quad (1.1)$$

Thus $C_{3,2} = 16$ and $C_{4,2} = 1800/31$. In dimension two, the corresponding target is $\mu > 4p_g$. The distinction between these two statements is illustrated in Section 10.

We work over $\mathbb{C}$. Fix $n \geq 2$ and let $N = n + 2$ be the ambient complex dimension. Give the polynomial ring $\mathbb{C}[x_1, \dots, x_N]$ positive integer weights $w_1, \dots, w_N$: the weighted degree of $x_1^{i_1} \cdots x_N^{i_N}$ is $\sum_j w_j i_j$. We write $\text{wt}(x_i) = w_i$ for the weight of a coordinate; on a tuple of coordinates, wt lists their weights in the same order. Positive rational weights can be multiplied, together with all equation degrees, by a common positive integer to obtain this grading. Let f, g be weighted homogeneous polynomials of weighted degrees d_1, d_2 , respectively, ordered so that $d_1 \leq d_2$. Assume they form a regular sequence. Thus f is a nonzerodivisor and the class of g is a nonzerodivisor modulo (f) . Set

$$R = \mathbb{C}[x_1, \dots, x_N]/(f, g), \quad X = \text{Spec } R.$$

Assume that $(X, 0)$ is singular at the origin and smooth on a punctured neighbourhood of the origin. Write $\mathfrak{m} = (x_1, \dots, x_N)$ and assume $f, g \in \mathfrak{m}^2$, so the embedding is minimal.

Order the weights and define their product P and the grading cutoff a by

$$0 < w_1 \leq \cdots \leq w_N, \quad p = w_1, \quad h = w_N, \quad (1.2)$$

$$P = \prod_{i=1}^N w_i, \quad a = d_1 + d_2 - \sum_{i=1}^N w_i.$$

2020 *Mathematics Subject Classification*. Primary 32S05; Secondary 32S10, 32S25.

Keywords. Quasihomogeneous isolated complete intersection, Milnor number, geometric genus, Durfee inequality.

Let F be a Milnor fibre of (f, g) , and let $\tilde{H}^n(F; \mathbb{C})$ denote its reduced singular cohomology with complex coefficients. The Milnor number is $\mu = \dim_{\mathbb{C}} \tilde{H}^n(F; \mathbb{C})$. For a projective resolution $\pi : \tilde{X} \rightarrow X$ that is an isomorphism over the smooth locus, $R^{n-1}\pi_* \mathcal{O}_{\tilde{X}}$ denotes the $(n-1)$ st higher direct image of the structure sheaf. It is supported at the origin. Define the ordinary geometric genus by

$$p_g = \dim_{\mathbb{C}} (R^{n-1}\pi_* \mathcal{O}_{\tilde{X}})_0.$$

On a sufficiently small Stein representative, this is also $\dim_{\mathbb{C}} H^{n-1}(\tilde{X}, \mathcal{O}_{\tilde{X}})$. We use the standard positivity of the Milnor number of a singular isolated complete intersection: $\mu > 0$.

Theorem 1.1. *For a singularity satisfying the preceding hypotheses, the following statements hold.*

- (i) *If $n = 2$, then $\mu \geq 4p_g + 1$.*
- (ii) *If $n \geq 3$, then $\mu > C_{n,2}p_g$.*

An Artin quotient of R gives a convex integral upper bound for p_g , while the Greuel–Hamm formula expresses $P\mu$ as a divided difference. For $d_2 \geq 2h$, we compare a product in the weights with a shifted uniform moment. If $d_2 < 2h$, the largest-weight variable occurs linearly in both equations; isolation provides two coefficient coordinates. A midpoint identity then compares the Milnor numerator with a simplex count. In dimension three, separate concavity in the weights reduces the high-degree comparison to endpoint estimates.

Use of generative AI. Codex contributed to the mathematical arguments and the preparation of this manuscript, including the estimates in Sections 6, 7, and 8.

2 Numerical formulas and the ordinary geometric genus

For a formal power series $G(t) = \sum_{k \geq 0} c_k t^k$, write $[t^k]G = c_k$; an absent coefficient is zero. For example, $[t^2](1 + 3t^2) = 3$ and $[t](1 + 3t^2) = 0$. The variables t and z in generating series below are formal variables. Write R_k for the weighted-degree- k component of R . For real x , put $x_+ = \max(x, 0)$ and $x_- = \max(-x, 0)$. For integers $m, j \geq 0$, $\binom{m}{j}$ counts j -element subsets of an m -element set and is zero for $j > m$. A sum with an empty index range is zero, and $\#$ denotes the cardinality of a finite set.

The Hilbert series and the geometric genus are related by

$$H_R(t) = \sum_{k \geq 0} (\dim_{\mathbb{C}} R_k) t^k = \frac{(1 - t^{d_1})(1 - t^{d_2})}{\prod_i (1 - t^{w_i})}, \quad p_g = \sum_{0 \leq k \leq a} [t^k] H_R(t). \quad (2.1)$$

The Hilbert product follows by applying the multiplication exact sequence successively to f and g . The genus formula follows from Morales’ formula for the first L^2 -plurigenus [2, Theorem 4.1] and its identification with the ordinary geometric genus, recalled below. The cutoff in (2.1) is closed: $a = 0$ gives $p_g = 1$, and $a < 0$ gives $p_g = 0$.

The Greuel–Hamm formula is

$$P\mu = d_1 d_2 [z^{N-2}] \frac{\prod_i (1 - w_i z)}{(1 - d_1 z)(1 - d_2 z)} + (-1)^{N-1} P; \quad (2.2)$$

see [1, Corollary 3.8]. These two classical numerical formulas are the singularity-theoretic inputs to our estimates.

2.1 The duality identification

For a normal Cohen–Macaulay germ, let ω_X denote its canonical module; on a smooth resolution, $\omega_{\tilde{X}}$ is the line bundle of top differential forms. The first L^2 -plurigenus is the length of $\omega_X/\pi_*\omega_{\tilde{X}}$, by [2, Section 0.5.4]. The following argument specifies its relation to p_g .

Lemma 2.1. *For a normal Cohen–Macaulay isolated complex singularity of dimension $n \geq 2$,*

$$\dim_{\mathbb{C}}(\omega_X/\pi_*\omega_{\tilde{X}})_0 = p_g.$$

Proof. Normality gives $\pi_*\mathcal{O}_{\tilde{X}} = \mathcal{O}_X$. Write $\mathbf{R}\pi_*$ for the derived direct image, $\mathbf{R}\mathcal{H}om$ for derived sheaf Hom, and $\mathcal{H}^i$ for the i th cohomology sheaf of a complex. Let T be the mapping cone of the natural morphism $\mathcal{O}_X \rightarrow \mathbf{R}\pi_*\mathcal{O}_{\tilde{X}}$, so that this morphism extends to a distinguished triangle with third object T . Its cohomology sheaves are

$$\mathcal{H}^i(T) = \mathbf{R}^i\pi_*\mathcal{O}_{\tilde{X}} \quad (1 \leq i \leq n-1),$$

and are supported at the singular point. The upper bound on i follows from the bound $n-1$ on the dimension of a resolution fibre.

Let D denote Matlis duality on finite-length modules over the Cohen–Macaulay local ring. Local duality gives, for such a module V ,

$$\mathrm{Ext}^j(V, \omega_X) = 0 \quad (j \neq n), \quad \mathrm{Ext}^n(V, \omega_X) = D(V),$$

Matlis duality preserves finite length. The spectral sequence computing the derived Hom of T therefore has only one nonzero row. Its terms in degrees zero and one are

$$\mathcal{H}^0\mathbf{R}\mathcal{H}om(T, \omega_X) = 0, \quad \mathcal{H}^1\mathbf{R}\mathcal{H}om(T, \omega_X) = D(\mathbf{R}^{n-1}\pi_*\mathcal{O}_{\tilde{X}}).$$

For a complex C , use the cohomological shift convention $\mathcal{H}^i(C[n]) = \mathcal{H}^{i+n}(C)$. Proper Grothendieck duality, with dualizing complexes $\omega_X[n]$ and $\omega_{\tilde{X}}[n]$, yields

$$\mathbf{R}\mathcal{H}om(\mathbf{R}\pi_*\mathcal{O}_{\tilde{X}}, \omega_X) \simeq \mathbf{R}\pi_*\omega_{\tilde{X}}.$$

Grauert–Riemenschneider vanishing makes the positive-degree cohomology on the right zero. Applying $\mathbf{R}\mathcal{H}om(-, \omega_X)$ to the defining triangle for T gives

$$0 \longrightarrow \pi_*\omega_{\tilde{X}} \longrightarrow \omega_X \longrightarrow D(\mathbf{R}^{n-1}\pi_*\mathcal{O}_{\tilde{X}}) \longrightarrow 0.$$

Taking lengths proves the assertion. The duality inputs are as in [5]; the vanishing input is Grauert–Riemenschneider vanishing for a projective resolution in characteristic zero [6]. $\square$

For a graded module V and an integer b , the graded shift $V(b)$ is defined by $V(b)_k = V_{k+b}$. The canonical module of the complete intersection R is $R(a)$. Thus the integer in Morales’ formula is $a = d_1 + d_2 - \sum_{i=1}^N w_i$, and Lemma 2.1 gives (2.1).

2.2 The Milnor divided difference

Define the polynomial $Q(t) = \prod_{i=1}^N (t - w_i)$.

Lemma 2.2. *If $d_1 < d_2$, then*

$$P\mu = \frac{d_1 Q(d_2) - d_2 Q(d_1)}{d_2 - d_1}. \tag{2.3}$$

At $d_1 = d_2$, the continuous value is $d_1 Q'(d_1) - Q(d_1)$.

Proof. For $0 \leq k \leq N$, let $e_k(w_1, \dots, w_N)$ be the sum of products of k distinct weights, with $e_0(w_1, \dots, w_N) = 1$. Then

$$Q(t) = \sum_{k=0}^N (-1)^k e_k(w_1, \dots, w_N) t^{N-k}.$$

For an integer $j \geq 2$,

$$\frac{d_1 d_2^j - d_2 d_1^j}{d_2 - d_1} = d_1 d_2 \sum_{i=0}^{j-2} d_1^i d_2^{j-2-i}.$$

The linear term of Q contributes zero to (2.3), while the constant term contributes $-(-1)^N P = (-1)^{N-1} P$. Expanding the two geometric series in (2.2) gives these same terms. At $d_1 = d_2$, differentiating the numerator with respect to d_2 gives $d_1 Q'(d_1) - Q(d_1)$, while the derivative of the denominator is one. $\square$

3 A convex bound from a finite Artin quotient

For $c > 0$, let U_c denote a random variable uniformly distributed on $[0, c]$. Random variables appearing together in a sum below are independent. We write $\Pr$ for probability and $\mathbb{E}$ for expectation on their product probability space. For $\eta \geq 0$ define $\mathcal{R}_N(\eta; d_1, d_2)$ by the expression below; when it is applied to our singularity, take $\eta = (\sum_{i=1}^N w_i - p)/2$:

$$\mathcal{R}_N(\eta; d_1, d_2) = (d_1 + d_2 - \eta)_+^N - (d_1 - \eta)_+^N - (d_2 - \eta)_+^N. \quad (3.1)$$

The alternating expression in (3.1) is a nonnegative integral, as the next lemma shows.

Lemma 3.1. *For every $n \geq 2$,*

$$p_g \leq \frac{d_1 d_2}{n! P} \mathbb{E}(U_{d_1} + U_{d_2} - \eta)_+^n = \frac{\mathcal{R}_N(\eta; d_1, d_2)}{N! P}. \quad (3.2)$$

Proof. A complete intersection is Cohen–Macaulay. Since $n \geq 2$ and the singularity is isolated, it is regular in codimension one and hence normal by Serre’s criterion. Its local ring at the vertex is a domain. The positive grading makes every irreducible component pass through the vertex, so R is also a domain. Since $(f, g) \subset \mathfrak{m}^2$, the minimum-weight coordinate x_1 has a nonzero class in R and is a nonzerodivisor.

Choose an integer $q > \max(0, a + p, d_1 + d_2 - \eta)$ divisible by $w_2, \dots, w_N$. On $R/(x_1)$, the degree- q linear system contains a pure power of every remaining coordinate, so it is base-point free on $\text{Proj}(R/(x_1))$. Over the infinite field $\mathbb{C}$, a general section avoids the generic point of every component, including each isolated point. Each successive section therefore lowers the projective dimension by one, with a zero-dimensional scheme becoming empty. Since $\dim \text{Proj}(R/(x_1)) = n - 2$, after $n - 1$ sections the common projective zero scheme is empty. Their affine quotient is finite-dimensional. Thus these sections are parameters; Cohen–Macaulayness makes them a regular sequence. Denote the quotient by these parameters and x_1 by $\mathcal{A}$.

Write $h_k = \dim_{\mathbb{C}} \mathcal{A}_k$. The Hilbert series and the total dimension are

$$H_{\mathcal{A}}(t) = \frac{(1 - t^q)^{n-1} (1 - t^{d_1}) (1 - t^{d_2})}{\prod_{i=2}^N (1 - t^{w_i})}, \quad \ell = \sum_k h_k = \frac{pq^{n-1} d_1 d_2}{P}.$$

The length formula follows by cancelling the first-order zeros at $t = 1$. Graded complete-intersection duality gives the socle degree, the largest integer with a nonzero graded component,

$$s_{\mathcal{A}} = a + p + (n - 1)q$$

as follows. After eliminating x_1 , the polynomial ring has canonical shift $-\sum_{i=2}^N w_i$, and the Koszul resolution of the regular sequence of degrees $d_1, d_2, q, \dots, q$ gives the canonical module $\mathcal{A}(s_{\mathcal{A}})$. The canonical module of a finite graded algebra is its graded vector-space dual, whose degree- j part is $\text{Hom}_{\mathbb{C}}(\mathcal{A}_{-j}, \mathbb{C})$. Comparing degrees gives $h_k = h_{s_{\mathcal{A}}-k}$. Define the integer-valued random variable K by $\Pr(K = k) = h_k/\ell$ for $0 \leq k \leq s_{\mathcal{A}}$.

The identity

$$H_R(t) = \frac{H_{\mathcal{A}}(t)}{(1-t^p)(1-t^q)^{n-1}}$$

and (2.1), followed by $k \mapsto s_{\mathcal{A}} - k$, give

$$p_g = \ell \mathbb{E} \# \{i_1, \dots, i_n \geq 1 : pi_1 + q(i_2 + \dots + i_n) \leq K\}. \quad (3.3)$$

Indeed, the original nonnegative indices are increased by one after using the symmetry; the shift is $p + (n-1)q = s_{\mathcal{A}} - a$.

Our choice of q ensures $q > a + p$. If some $i_j \geq 2$ with $j \geq 2$, the left side of the inequality inside (3.3) is at least $p + nq > s_{\mathcal{A}}$. Hence all $i_2, \dots, i_n$ equal one. The remaining count is $\lfloor (K - (n-1)q)_+ / p \rfloor$, which is at most $(K - (n-1)q)_+ / p$. Therefore

$$p_g \leq \frac{q^{n-1} d_1 d_2}{P} \mathbb{E}(K - (n-1)q)_+. \quad (3.4)$$

For the following calculation, let z be real and close to zero. The moment-generating function of U_c is $(e^{cz} - 1)/(cz)$, with its value at zero defined by continuity. Substituting $t = e^z$ into the Hilbert product gives

$$\frac{H_{\mathcal{A}}(e^z)}{\ell} \prod_{i=2}^N \frac{e^{w_i z} - 1}{w_i z} = \frac{e^{d_1 z} - 1}{d_1 z} \frac{e^{d_2 z} - 1}{d_2 z} \left(\frac{e^{qz} - 1}{qz} \right)^{n-1}.$$

Thus there is equality of moment-generating functions in

$$K + \sum_{i=2}^N U_{w_i} \stackrel{\text{law}}{=} U_{d_1} + U_{d_2} + \sum_{j=2}^n U_{q,j}.$$

All variables have compact support, so these functions determine their distributions. Given K , Jensen's inequality for the convex positive-part function, together with $\sum_{i=2}^N \mathbb{E} U_{w_i} = \eta$, gives

$$\mathbb{E}(K - (n-1)q)_+ \leq \mathbb{E} \left(U_{d_1} + U_{d_2} - \eta - \sum_{j=2}^n U_{q,j} \right)_+,$$

where the q -uniform variables on the right have also been reflected by $U_{q,j} \mapsto q - U_{q,j}$.

The initial choice also ensures $q > d_1 + d_2 - \eta$. For $z \leq d_1 + d_2 - \eta$, the part of the positive orthant on which $(z - u_2 - \dots - u_n)_+$ is nonzero lies in $[0, q]^{n-1}$. The identity

$$\int_0^\infty (z - t)_+^r dt = \frac{z_+^{r+1}}{r+1} \quad (r \geq 0)$$

applied successively gives

$$\int_{[0,q]^{n-1}} (z - u_2 - \dots - u_n)_+ du_2 \dots du_n = \frac{z_+^n}{n!}.$$

Inserting this identity into (3.4) cancels q^{n-1} and proves the first inequality in (3.2). Integrating over U_{d_1} and U_{d_2} gives

$$d_1 d_2 \mathbb{E}(U_{d_1} + U_{d_2} - \eta)_+^n = \frac{\mathcal{R}_N(\eta; d_1, d_2)}{(n+1)(n+2)}.$$

The possible fourth corner term is $(-\eta)_+^{n+2} = 0$. Since $N = n+2$, this proves the second equality. $\square$

4 The range $d_2 \geq 2h$

The comparisons below apply for $n \geq 2$; the final exponential estimate uses $n \geq 5$. Define the positive polynomial in the equation degrees

$$m_0 = d_1 d_2 \sum_{k=0}^{N-2} d_1^k d_2^{N-2-k}.$$

For $d_1 < d_2$, it is $(d_1 d_2^N - d_2 d_1^N)/(d_2 - d_1)$; for $d_1 = d_2$, it is $(N-1)d_2^N$.

Lemma 4.1. *If $d_2 \geq 2h$, then*

$$d_1 \geq p + w_{N-1}, \quad P\mu \geq m_0 \frac{Q(d_2)}{d_2^N}. \quad (4.1)$$

Proof. Suppose $d_1 < p + w_{N-1}$. A monomial of f involving either of the two largest-weight variables has ordinary degree at least two, hence weighted degree at least $p + w_{N-1}$; such a monomial cannot occur. Thus f and all its first derivatives vanish on their coordinate plane. The restriction of g to this plane either vanishes identically or has a positive-dimensional zero germ. At its nonzero zeros the Jacobian of (f, g) has rank at most one, contradicting isolation. This proves the first inequality.

For $d_1 < d_2$, substituting (2.3) shows that the second inequality is equivalent to $Q(d_1) \leq (d_1/d_2)^N Q(d_2)$. If $d_1 \geq h$, all factors are nonnegative and

$$\frac{d_1 - w_i}{d_2 - w_i} \leq \frac{d_1}{d_2}$$

for each i , which proves it. If $d_1 < h$, the first inequality in (4.1) shows that every weight except h is strictly less than d_1 . Consequently $Q(d_1) < 0$, whereas $Q(d_2) > 0$, and the conclusion follows again. At $d_1 = d_2$, Lemma 2.2 gives

$$P\mu = Q(d_2) \left(\sum_i \frac{d_2}{d_2 - w_i} - 1 \right) \geq (N-1)Q(d_2) = m_0 \frac{Q(d_2)}{d_2^N}. \quad \square$$

Lemma 4.2. *For positive $d_1 \leq d_2$ and $N \geq 4$,*

$$m_0 \geq \frac{N-1}{2^N-2} \mathcal{R}_N(0; d_1, d_2). \quad (4.2)$$

Proof. Dividing (4.2) by $d_1 d_2$ and expanding $\mathcal{R}_N(0; d_1, d_2)$ reduces the assertion to

$$\sum_{k=0}^{N-2} d_1^k d_2^{N-2-k} \geq \frac{N-1}{2^N-2} \sum_{k=0}^{N-2} \binom{N}{k+1} d_1^k d_2^{N-2-k}.$$

For $0 \leq k \leq N-2$, define the two sequences

$$a_k = \binom{N}{k+1}, \quad b_k = \frac{d_1^k d_2^{N-2-k} + d_1^{N-2-k} d_2^k}{2}.$$

Both are symmetric about the centre. Moving from an endpoint towards the centre, a_k increases and b_k decreases, since

$$b_k - b_{k+1} = \frac{d_2 - d_1}{2} (d_1^k d_2^{N-3-k} - d_1^{N-3-k} d_2^k) \geq 0 \quad \left(k \leq \frac{N-3}{2} \right).$$

Thus $(a_i - a_j)(b_i - b_j) \leq 0$ for every pair of indices. Summing and expanding gives

$$(N-1) \sum_k a_k b_k \leq \left(\sum_k a_k \right) \left(\sum_k b_k \right).$$

Symmetry gives $\sum a_k b_k = \sum a_k d_1^k d_2^{N-2-k}$ and $\sum b_k = \sum d_1^k d_2^{N-2-k}$, while $\sum a_k = 2^N - 2$. This proves the desired inequality. $\square$

For integers $N \geq 4$ and real $\tau \geq 0$, define

$$F_N(\tau) = \frac{(2-\tau)_+^N - 2(1-\tau)_+^N}{2^N - 2}. \quad (4.3)$$

This is the ratio of a shifted $(N-2)$ nd moment of two independent unit uniforms to its unshifted value. In particular F_N is convex on $[0, \infty)$, is positive on $[0, 2)$, and vanishes on $[2, \infty)$.

Lemma 4.3. For $0 < d_1 \leq d_2$ and $\eta \geq 0$,

$$\frac{\mathcal{R}_N(\eta; d_1, d_2)}{\mathcal{R}_N(0; d_1, d_2)} \leq F_N(\eta/d_2). \quad (4.4)$$

Proof. Let φ be the density of $U_{d_1} + U_{d_2}$ and ψ that of $U_{d_2} + U'_{d_2}$, where U'_{d_2} is an independent copy of U_{d_2} . On the support of φ ,

$$\frac{\psi(s)}{\varphi(s)} = \begin{cases} d_1/d_2, & 0 < s < d_1, \\ s/d_2, & d_1 \leq s \leq d_2, \\ \frac{d_1}{d_2} \frac{2d_2 - s}{d_1 + d_2 - s}, & d_2 < s < d_1 + d_2. \end{cases}$$

This ratio is nondecreasing. On the last interval its derivative is $d_1(d_2 - d_1)/(d_2(d_1 + d_2 - s)^2) \geq 0$. Any mass of ψ beyond the support of φ is at its upper end.

Multiply both densities by s^{N-2} and normalize each to have integral one. Their density ratio remains nondecreasing. The difference of the normalized densities changes sign at most once, from negative to positive. For an increasing test function, subtracting its value at the crossing therefore proves that its integral against the second density is at least its integral against the first. If the densities coincide, the assertion is immediate.

Apply this observation to the increasing function $s \mapsto (1 - \eta/s)_+^{N-2}$ for $s > 0$. Its two expectations are the corresponding shifted-to-unshifted moment ratios. The rectangle integral in Lemma 3.1 identifies the first ratio with the left side of (4.4); scaling the equal-uniform sum by d_2 identifies the second with $F_N(\eta/d_2)$. $\square$

Lemma 4.4. Set $\gamma_N = N(2^{N-1} - 2)/(2^N - 2)$ for $N \geq 4$. For $\tau \geq 0$,

$$F_N(\tau) \leq e^{-\gamma_N \tau}. \quad (4.5)$$

Moreover $-(\log F_N)'(\tau) \geq \gamma_N$ for $0 \leq \tau < 2$.

Proof. For $0 \leq \tau < 1$, set $q = (1 - \tau)/(2 - \tau) \in [0, 1/2]$. Differentiation of (4.3) yields

$$-\frac{d}{d\tau} \log F_N(\tau) = \frac{N}{2 - \tau} \frac{1 - 2q^{N-1}}{1 - 2q^N}.$$

The derivative of the second factor with respect to q is

$$-\frac{2q^{N-2}((N-1) - Nq + 2q^N)}{(1 - 2q^N)^2} \leq 0,$$

because $(N-1) - Nq \geq (N-2)/2 > 0$. As τ increases, q decreases, so both positive factors in the logarithmic derivative increase. Their product is at least its value at $\tau = 0$, namely γ_N . For $1 \leq \tau < 2$, the logarithmic derivative is $N/(2-\tau) \geq \gamma_N$. Integrate from zero, using $F_N(0) = 1$. For $\tau \geq 2$, (4.5) follows from $F_N(\tau) = 0$. $\square$

Proposition 4.5. *If $n \geq 5$ and $d_2 \geq 2h$, then*

$$\mu > e^{8\sum_{i=1}^N w_i/(105d_2)} C_{n,2p_g}.$$

Proof. Since $p \leq \sum_{i=1}^N w_i/N$, one has $\eta \geq (N-1)\sum_{i=1}^N w_i/(2N)$. Set $\kappa_N = (N-1)(2^{N-1} - 2)/(2(2^N - 2))$. Lemmas 4.3 and 4.4 give

$$\frac{\mathcal{R}_N(\eta; d_1, d_2)}{\mathcal{R}_N(0; d_1, d_2)} \leq e^{-\kappa_N \sum_{i=1}^N w_i/d_2}.$$

On the other hand, concavity of $\log(1-x)$ gives its chord bound $\log(1-x) \geq -2(\log 2)x$ for $0 \leq x \leq 1/2$. Since $w_i/d_2 \leq 1/2$,

$$\frac{Q(d_2)}{d_2^N} \geq e^{-2(\log 2)\sum_{i=1}^N w_i/d_2}.$$

The quantity $\kappa_N = (N-1)(1/2 - 1/(2^N - 2))/2$ increases with N . For $N \geq 7$,

$$\kappa_N \geq \kappa_7 = \frac{31}{21}, \quad 2\log 2 < \frac{7}{5}.$$

The last comparison follows from

$$e^{7/10} > \sum_{j=0}^4 \frac{(7/10)^j}{j!} = \frac{482921}{240000} > 2.$$

Thus $\kappa_N - 2\log 2 > 8/105$. Combining Lemmas 3.1, 4.1, and 4.2, we obtain

$$\begin{aligned} C_{n,2p_g} &\leq \frac{N-1}{(2^N-2)P} \mathcal{R}_N(\eta; d_1, d_2) \leq \frac{m_0}{P} e^{-\kappa_N \sum_{i=1}^N w_i/d_2}, \\ \mu &\geq \frac{m_0}{P} e^{-2(\log 2)\sum_{i=1}^N w_i/d_2}. \end{aligned}$$

If $p_g > 0$, these inequalities imply the claimed strict estimate. If $p_g = 0$, the last displayed lower bound proves $\mu > 0$, which gives the same conclusion. $\square$

5 The range $d_2 < 2h$

5.1 Coordinates forced by isolation

Lemma 5.1. *Suppose $n \geq 3$ and $d_2 < 2h$. There are weighted polynomial coordinates $z_0, x, y, z_1, \dots, z_m$, where $m = n-1$, such that*

$$\begin{aligned} f &= z_0x + P_0(x, y, z_1, \dots, z_m), & g &= z_0y + Q_0(x, y, z_1, \dots, z_m), \\ \text{wt}(z_0) &= h, & \text{wt}(x) &= s, & \text{wt}(y) &= t, & \text{wt}(z_i) &= b_i, \\ (d_1, d_2) &= (h+s, h+t), & 0 &< s \leq t < h, & 0 &< b_i \leq h. \end{aligned} \tag{5.1}$$

Put $u = \sum_{i=1}^m b_i$. Then $a = h - u$. If $u \leq h$, isolation also implies

$$2 \max_i b_i \leq h + t. \tag{5.2}$$

Proof. Choose a variable z_0 of weight h . Neither equation contains z_0^2 , since both degrees are less than $2h$, and neither contains a linear term. Hence the z_0 -axis lies in X . Write the coefficients of z_0 in f, g as g_1, g_2 . Along the punctured axis, the Jacobian has rank two; its transverse rows are the linear parts of g_1, g_2 , multiplied by the nonzero axis coordinate. These linear parts are therefore independent. Their positive weights are $s = d_1 - h$ and $t = d_2 - h$, with $0 < s \leq t < h$.

Take g_1, g_2 as the new coordinates x, y and complete their linear parts to bases in the respective weight spaces. A nonlinear monomial of a given positive weight involves only variables of strictly smaller weights. The resulting coordinate change is therefore triangular with respect to weight, after invertible linear changes inside each weight space. Successive solution in increasing weight gives a polynomial inverse. This proves (5.1); its equation degrees also give $a = h - u$.

Assume $u \leq h$, choose $b = b_{i_0} = \max b_i$, and suppose $h + t < 2b$. Both equation degrees are strictly greater than b and less than $2b$, so both equations vanish on the z_{i_0} -axis. A nonzero first derivative of g on this axis would have to come from a product of the axis variable with another coordinate. The weight of the latter must be

$$h + t - b \geq t + \sum_{i \neq i_0} b_i.$$

Since $m \geq 2$, this is strictly larger than s, t and each other b_i , but is less than b and h . There is no such coordinate. Terms involving higher powers of z_{i_0} cannot contribute, because $d_2 < 2b$. The second Jacobian row is therefore zero on the punctured axis, contradicting isolation. This proves (5.2). $\square$

If $u > h$, then $a < 0$ and $p_g = 0$ by (2.1); the desired inequalities follow from $\mu > 0$. We may therefore assume $u \leq h$ for the rest of this section. The least weight is still $p = \min(s, t, b_1, \dots, b_m)$.

5.2 A simplex bound

Since $a = h - u < h < d_1 \leq d_2$, the Hilbert truncation contains neither the coordinate z_0 nor a relation from either equation. Consequently

$$p_g = \#\{(i, j, k_1, \dots, k_m) \in \mathbb{Z}_{\geq 0}^{m+2} : si + tj + \sum_{\ell=1}^m b_\ell k_\ell \leq h - u\}. \quad (5.3)$$

To compare this count with the midpoint of the two equation degrees, define

$$v = \frac{s+t}{2}, \quad e = \frac{t-s}{2}, \quad c = h + v, \quad \lambda = c - \frac{u-p}{2}. \quad (5.4)$$

Here v is the midpoint of s, t , e is their half-difference, $c = (d_1 + d_2)/2$ is the midpoint of the equation degrees, and λ is the intercept in the simplex integral below. These quantities satisfy $0 \leq e < v < h$ and $\lambda > 0$.

Lemma 5.2. *Under the preceding assumptions,*

$$p_g \leq \frac{\lambda^{m+2}}{(m+2)! \, st \prod_i b_i}. \quad (5.5)$$

Proof. Sum first over a coordinate of weight p in (5.3). Denote the other $m+1$ weights by $c_1, \dots, c_{m+1}$. For fixed nonnegative integer indices $j_1, \dots, j_{m+1}$, the number of possible first coordinates, when the residual bound is nonnegative, is

$$\left\lfloor \frac{h - u - \sum_{\ell} c_\ell j_\ell}{p} \right\rfloor + 1 \leq \frac{(h - u + p - \sum_{\ell} c_\ell j_\ell)_+}{p}.$$

When the residual bound is negative, the count is zero and the same upper bound remains valid.

The positive part of an affine function is convex. On each unit cube centred at a nonnegative integer lattice point, Jensen's inequality bounds its value at the centre by its average over the cube. These cubes fill $[-1/2, \infty)^{m+1}$. Translating this region to the positive orthant changes the constant term to

$$h - u + p + \frac{1}{2} \sum_{\ell} c_{\ell} = h + \frac{s + t - u + p}{2} = \lambda.$$

Changing each integration variable by its corresponding weight and integrating successively gives

$$\int_{\mathbb{R}_{\geq 0}^{m+1}} (\lambda - \sum_{\ell} c_{\ell} x_{\ell})_+ dx = \frac{\lambda^{m+2}}{(m+2)! \prod_{\ell} c_{\ell}}.$$

Multiplying by $1/p$ proves (5.5). □

5.3 A midpoint identity for the Milnor numerator

Define the polynomial

$$S(z) = \prod_{i=1}^m (h + z - b_i).$$

It records the factors of Q from the remaining m coordinates.

Lemma 5.3. *In the coordinates (5.1),*

$$\begin{aligned} \frac{P\mu}{h} &= (h^2 + 2hv + 2v^2 - 2e^2) \frac{S(s) + S(t)}{2} \\ &\quad + h(cv + e^2) \frac{S(t) - S(s)}{t - s}, \end{aligned} \tag{5.6}$$

where the difference quotient is $S'(v)$ when $s = t$. Moreover,

$$\frac{P\mu}{h} > c^2 S(v). \tag{5.7}$$

Proof. From (5.1),

$$Q(h + t) = ht(h + t - s)S(t), \quad Q(h + s) = hs(h + s - t)S(s).$$

Insert these expressions into (2.3). The numerator of $P\mu/h$ is

$$(h + s)t(h + t - s)S(t) - (h + t)s(h + s - t)S(s).$$

The sum and difference of its two scalar coefficients are

$$\begin{aligned} (h + s)t(h + t - s) + (h + t)s(h + s - t) &= 2h(cv + e^2), \\ (h + s)t(h + t - s) - (h + t)s(h + s - t) &= (t - s)(h^2 + 2hv + 2v^2 - 2e^2). \end{aligned}$$

Splitting the numerator into its symmetric and antisymmetric parts and dividing by $t - s$ proves (5.6). Continuity gives the case $s = t$.

Because $h - b_i \geq 0$, the coefficients of S in powers of z are nonnegative. Thus S and S' are convex on $[0, \infty)$. Their midpoint estimates give

$$\frac{S(s) + S(t)}{2} \geq S(v), \quad \frac{S(t) - S(s)}{t - s} = \frac{1}{t - s} \int_s^t S'(z) dz \geq S'(v)$$

when $s < t$, with the second expression interpreted by continuity if $s = t$. Also

$$\frac{S'(v)}{S(v)} = \sum_i \frac{1}{c - b_i} \geq \frac{m}{c}.$$

The two coefficients in (5.6) are positive: the first equals $h^2 + h(s + t) + 2st$, and the second is $h(cv + e^2)$. Hence

$$\frac{P\mu}{h} \geq \left(h^2 + 2hv + 2v^2 - 2e^2 + hmv + \frac{hme^2}{c} \right) S(v).$$

The expression in parentheses minus c^2 is at least

$$v^2 - 2e^2 + hmv \geq v(hm - v) > 0,$$

since $e < v < h$ and $m \geq 2$. This proves (5.7). $\square$

Proposition 5.4. *If $n \geq 4$ and $d_2 < 2h$, then*

$$\mu > e^{u/(60c)}(n+1)!p_g > C_{n,2}p_g$$

when $p_g > 0$, and $\mu > 0$ when $p_g = 0$.

Proof. Assume $u \leq h$; the case $u > h$ has $p_g = 0$. By (5.2),

$$\frac{b_i}{c} \leq \frac{h+t}{2h+s+t} < \frac{2}{3};$$

the strict inequality follows from $t < h$. The chord bound for $\log(1-x)$ on $[0, 2/3]$ therefore gives

$$\frac{S(v)}{c^m} = \prod_i \left(1 - \frac{b_i}{c} \right) \geq \exp \left(-\frac{3 \log 3}{2} \frac{u}{c} \right).$$

Since $p \leq u/m$ and $1 - x \leq e^{-x}$,

$$\left(\frac{\lambda}{c} \right)^{m+2} \leq \exp \left(-\frac{(m+2)(m-1)}{2m} \frac{u}{c} \right).$$

For $m \geq 3$, $(m+2)(m-1)/(2m) \geq 5/3$, whereas $(3/2) \log 3 < 33/20$, because

$$e^{11/10} > \sum_{j=0}^5 \frac{(11/10)^j}{j!} = \frac{36015101}{12000000} > 3.$$

The difference between these two rates exceeds $5/3 - 33/20 = 1/60$. Lemma 5.3 yields

$$\frac{P\mu}{h} > c^{m+2} \exp \left(-\frac{3 \log 3}{2} \frac{u}{c} \right) > e^{u/(60c)} \lambda^{m+2}.$$

Now $P = hst \prod_i b_i$ and $m+2 = n+1$, so Lemma 5.2 proves the factorial estimate.

For $N = n+2 \geq 6$, $2^N - 2 > N(N-1)$, which is equivalent to $(n+1)! > C_{n,2}$. At $N = 6$ this is $62 > 30$. If it holds at N , then

$$2^{N+1} - 2 = 2(2^N - 2) + 2 > 2N(N-1) + 2 > N(N+1),$$

where the last inequality follows from $(N-1)(N-2) > 0$. This proves the comparison of constants and the proposition. $\square$

6 The high-degree range in dimension four

For $N = 6$, we compare F_6 with a product in the weights.

Lemma 6.1. *Let $0 < \rho \leq y_i \leq 1/2$ for $1 \leq i \leq 6$, and suppose at least one y_i equals ρ . Then*

$$\prod_{i=1}^6 (1 - y_i) > F_6 \left(\frac{\sum_{i=1}^6 y_i - \rho}{2} \right). \quad (6.1)$$

Proof. Fix a coordinate equal to ρ . The left side of (6.1) is affine in each other coordinate separately, and F_6 is convex by its moment representation following (4.3). Thus the left side minus the right side is separately concave in the other five coordinates. On an interval, a concave function is bounded below by its linear interpolation between the endpoints. Applying this fact successively reduces the inequality to the vertices where each of those coordinates is ρ or $1/2$.

Let k be the number of coordinates equal to $1/2$, so $0 \leq k \leq 5$. All other $6 - k$ coordinates equal ρ . With $\tau = ((5 - k)\rho + k/2)/2$, we have

$$\prod_i (1 - y_i) = 2^{-k} (1 - \rho)^{6-k}, \quad 2 - \tau = 2 - \frac{k}{4} - \frac{5-k}{2} \rho.$$

For $k \geq 1$, the bound $F_6(\tau) \leq (2 - \tau)^6/62$ shows that it suffices to prove

$$2^k \frac{(2 - k/4 - (5 - k)\rho/2)^6}{(1 - \rho)^{6-k}} < 62. \quad (6.2)$$

For $k = 1$, the logarithmic derivative of $2(7/4 - 2\rho)^6/(1 - \rho)^5$ is

$$-\frac{12}{7/4 - 2\rho} + \frac{5}{1 - \rho} = \frac{2\rho - 13/4}{(7/4 - 2\rho)(1 - \rho)} < 0.$$

Its maximum on $[0, 1/2]$ is its value at zero:

$$2(7/4)^6 = \frac{117649}{2048} < 62.$$

For $2 \leq k \leq 5$, the logarithmic derivative of the expression in (6.2) is

$$\frac{(6 - k)(k - 2)/4 - k(5 - k)(1 - \rho)/2}{(1 - \rho)(2 - k/4 - (5 - k)\rho/2)}.$$

Its denominator is positive and its numerator is nondecreasing in ρ , so the maximum of the expression in (6.2) occurs at an endpoint. At $\rho = 1/2$ the expression is $64(3/4)^6 = 729/64 < 62$. At $\rho = 0$ it is $2^k(2 - k/4)^6$. As a function of real $k \in [2, 5]$, the latter has logarithmic derivative $\log 2 - 6/(8 - k) < 0$. Its maximum is therefore $729/16 < 62$, attained at $k = 2$.

It remains to treat $k = 0$, for which $\tau = 5\rho/2$. If $0 < \rho \leq 1/10$, Lemma 4.4 gives $-(\log F_6)' \geq 90/31$, and hence

$$\frac{d}{d\rho} \log \frac{(1 - \rho)^6}{F_6(5\rho/2)} \geq \frac{225}{31} - \frac{6}{1 - \rho} \geq \frac{225}{31} - \frac{20}{3} > 0.$$

The ratio equals one at $\rho = 0$, so it is strictly greater than one in this interval. If $1/10 \leq \rho \leq 1/2$, then

$$\frac{2 - 5\rho/2}{1 - \rho} \leq \frac{35}{18}, \quad 35^6 = 1838265625 < 2108757888 = 62 \cdot 18^6.$$

It follows that $F_6(5\rho/2) \leq (2 - 5\rho/2)^6/62 < (1 - \rho)^6$. Successive concavity interpolation proves (6.1) throughout the cube. $\square$

Proposition 6.2. *If $n = 4$ and $d_2 \geq 2h$, then $\mu > (1800/31)p_g$.*

Proof. Apply Lemma 6.1 with $y_i = w_i/d_2$ and $\rho = p/d_2$. Its argument of F_6 is η/d_2 . Lemmas 4.1, 4.2, and 4.3 give

$$\begin{aligned} P\mu &\geq m_0 \prod_i (1 - w_i/d_2) > m_0 F_6(\eta/d_2) \\ &\geq \frac{5}{62} \mathcal{R}_6(\eta; d_1, d_2). \end{aligned}$$

Since $C_{4,2}/6! = 5/62$, Lemma 3.1 proves the assertion. $\square$

7 The high-degree range in dimension three

Let $\mathbf{v} = (p, v_2, v_3, v_4, v_5)$ be a list of five positive real weights. For $0 \leq j \leq 5$, let $e_j(\mathbf{v})$ be the sum of products of j distinct entries, with $e_0(\mathbf{v}) = 1$. Define

$$\begin{aligned} \mathcal{M}(d_1, d_2; \mathbf{v}) &= d_1 d_2 \{d_1^3 + d_1^2 d_2 + d_1 d_2^2 + d_2^3 \\ &\quad - e_1(\mathbf{v})(d_1^2 + d_1 d_2 + d_2^2) + e_2(\mathbf{v})(d_1 + d_2) - e_3(\mathbf{v})\} + e_5(\mathbf{v}), \\ \Phi(d_1, d_2; \mathbf{v}) &= 30\mathcal{M}(d_1, d_2; \mathbf{v}) - 4\mathcal{R}_5\left(\frac{v_2 + v_3 + v_4 + v_5}{2}; d_1, d_2\right). \end{aligned} \quad (7.1)$$

The polynomial $\mathcal{M}$ is the dimension-three Milnor numerator in (2.2). These expressions will also be used for weight lists that satisfy only the numerical inequalities below.

Lemma 7.1. *Suppose $0 < p$, $2p \leq d_1 \leq d_2$, and put $b = d_2/2$ and $q = \min(b, d_1 - p)$. If*

$$p \leq v_2, v_3, v_4 \leq q, \quad p \leq v_5 \leq b,$$

then $\Phi(d_1, d_2; \mathbf{v})$ is at least the minimum of its values on the five lists below. An exponent on an entry specifies its number of repetitions; the order of the last four entries is immaterial:

$$(p, p, p, p, p), \quad (p^{k+1}, q^{3-k}, b) \quad (0 \leq k \leq 3).$$

Proof. The polynomial $\mathcal{M}$ is affine in each weight separately. By Lemma 3.1,

$$\mathcal{R}_5(\eta; d_1, d_2) = 20d_1 d_2 \mathbb{E}(U_{d_1} + U_{d_2} - \eta)_+^3.$$

This is convex in η , since the positive part of a cube is convex. Thus Φ is concave in each of v_2, v_3, v_4, v_5 separately. Repeatedly using the chord inequality on their intervals bounds its value below by the minimum of its values at the vertices of the box.

A vertex with last entry b is one of the four displayed lists. At a vertex with last entry p , all other entries are p or q . If at least one is q , vary one such entry on $[p, b]$, keeping the others fixed. Concavity bounds that value below by the minimum of the two endpoint values. The endpoint b gives a displayed list, while the endpoint p reduces the number of entries equal to q . Induction ends at the all- p list. Throughout this argument the first entry remains p , so the shift in (7.1) is affine in every varied coordinate. $\square$

Lemma 7.2. *Under the hypotheses of Lemma 7.1, each of its five endpoint values is at least $142p^5$.*

Proof. For the all- p list, differentiation in d_2 gives

$$\begin{aligned} \frac{1}{10d_1} \frac{\partial \Phi}{\partial d_2} &= 4(d_2 - d_1)^3 + (9d_1 + 3p)(d_2 - d_1)^2 \\ &\quad + (4d_1^2 + 24d_1 p - 36p^2)(d_2 - d_1) \\ &\quad + p(22d_1^2 - 54d_1 p + 34p^2). \end{aligned}$$

All terms are nonnegative for $d_2 \geq d_1 \geq 2p$: the last two coefficients can be written as

$$4(d_1 - 2p)^2 + 40p(d_1 - 2p) + 28p^2, \quad 22(d_1 - 2p)^2 + 34p(d_1 - 2p) + 14p^2.$$

Hence the minimum in $d_2 \geq d_1$ occurs at $d_2 = d_1$. Substitution there gives

$$\begin{aligned} \Phi(d_1, d_1; (p, p, p, p, p)) &= 142p^5 + 560p^4(d_1 - 2p) + 820p^3(d_1 - 2p)^2 \\ &\quad + 520p^2(d_1 - 2p)^3 + 110p(d_1 - 2p)^4 \geq 142p^5. \end{aligned}$$

It remains to treat (p^{k+1}, q^{3-k}, b) for $k \in \{0, 1, 2, 3\}$. In the following expansions, Φ and $\mathcal{M}$ are evaluated at $(d_1, d_2; (p^{k+1}, q^{3-k}, b))$. The multiplicity k lets us handle these four lists together. We use the following boundary polynomial, for $\delta \geq 0$:

$$\begin{aligned} 4\mathcal{T}_k(p, \delta) &= 568p^5 + (2560 + 200k)p^4\delta \\ &\quad + \{4240 + 20k(47 - k)\}p^3\delta^2 \\ &\quad + \{3440 + 1200k + 20k^2(3 - k)\}p^2\delta^3 \\ &\quad + \{1400 + k(630 - 10k^2 - 5k^3) + 55k^2\}p\delta^4 \\ &\quad + \{224 + k(136 - 15k^2) + 10k^2 + k^4(5 - k)\}\delta^5. \end{aligned} \tag{7.2}$$

The factors in braces are positive: on $0 \leq k \leq 3$,

$$630 - 10k^2 - 5k^3 \geq 405, \quad 136 - 15k^2 \geq 1,$$

and every other displayed factor is nonnegative. Consequently

$$\mathcal{T}_k(p, \delta) \geq 142p^5. \tag{7.3}$$

The symmetric sums are obtained from the generating product

$$\sum_{j=0}^5 e_j(\mathbf{v})z^j = (1 + pz)^{k+1}(1 + qz)^{3-k}(1 + bz). \tag{7.4}$$

More explicitly,

$$\begin{aligned} e_1 &= (k+1)p + (3-k)q + b, \\ e_2 &= \binom{k+1}{2}p^2 + (k+1)(3-k)pq + \binom{3-k}{2}q^2 \\ &\quad + b\{(k+1)p + (3-k)q\}, \\ e_3 &= \binom{k+1}{3}p^3 + \binom{k+1}{2}(3-k)p^2q + (k+1)\binom{3-k}{2}pq^2 + \binom{3-k}{3}q^3 \\ &\quad + b\left\{\binom{k+1}{2}p^2 + (k+1)(3-k)pq + \binom{3-k}{2}q^2\right\}, \\ e_5 &= p^{k+1}q^{3-k}b. \end{aligned}$$

Here e_j means $e_j(\mathbf{v})$ for this displayed weight list. If $\eta = (kp + (3-k)q + b)/2$, then $\eta \leq d_2$. Formula (7.1) therefore gives

$$\Phi - 4(d_1 - \eta)_+^5 = 30\mathcal{M} - 4\{(d_1 + d_2 - \eta)^5 - (d_2 - \eta)^5\}. \tag{7.5}$$

We expand this expression in the nonnegative gaps between the endpoints. The fifth-power difference in this expansion is evaluated by

$$(d_1 + d_2 - \eta)^5 - (d_2 - \eta)^5 = \sum_{j=1}^5 \binom{5}{j} d_1^j (d_2 - \eta)^{5-j}.$$

Suppose first that $d_1 \leq p + b$. Then $q = d_1 - p$. Write $\delta = q - p \geq 0$ and $e = b - q \geq 0$, so $d_1 = 2p + \delta$ and $d_2 = 2(p + \delta + e)$. Expanding (7.4) in (7.5) gives

$$\begin{aligned}
\Phi - 4(d_1 - \eta)_+^5 &= \mathcal{T}_k(p, \delta) \\
&+ \frac{5e}{4} [\{264 + k(106 - 10k^2 - k) + k^4\}\delta^4 \\
&\quad + \{1320 + 12k(31 - 2k - k^2)\}p\delta^3 \\
&\quad + \{2328 + 12k(31 - 3k)\}p^2\delta^2 \\
&\quad + (1704 + 72k)p^3\delta + 408p^4] \\
&+ \frac{15e^2}{2} (\delta + 2p) [\{92 + k(18 - 5k)\}\delta^2 + (168 + 4k)p\delta + 64p^2] \\
&+ 15e^3(\delta + 2p)\{(39 - k)\delta + 30p\} + \frac{555e^4}{4}(\delta + 2p).
\end{aligned} \tag{7.6}$$

Here (7.2) is the constant term, obtained by setting $e = 0$. Every remaining term is nonnegative, because

$$106 - 10k^2 - k \geq 13, \quad 31 - 2k - k^2 \geq 16, \quad 31 - 3k \geq 22, \quad 18 - 5k \geq 3.$$

Together with (7.3), this proves the required lower bound in this range.

Suppose next that $d_1 \geq p + b$. Now $q = b$. Write $\delta = b - p \geq 0$ and $x = d_1 - p - b$. The condition $d_1 \leq d_2 = 2b$ gives $0 \leq x \leq \delta$. The same generating product yields

$$\begin{aligned}
\Phi - 4(d_1 - \eta)_+^5 &= \mathcal{T}_k(p, \delta) \\
&+ \frac{5x}{4} [\{80 + k(32 - k^3)\}\delta^4 + \{400 + 8k(17 - k^2)\}p\delta^3 \\
&\quad + \{816 + 24k(7 - k)\}p^2\delta^2 + (736 + 32k)p^3\delta + 224p^4] \\
&+ 5x^2 [\{16 + k(6 - k^2)\}\delta^3 + \{84 + 6k(3 - k)\}p\delta^2 \\
&\quad + 132p^2\delta + 56p^3] \\
&+ 10x^3 [\{8 + k(2 - k)\}\delta^2 + (26 - 2k)p\delta + 14p^2] \\
&+ 10x^4 \{(4 - k)\delta + 2p\} - 4x^5.
\end{aligned} \tag{7.7}$$

The coefficients of x, x^2, x^3 are nonnegative. Indeed,

$$32 - k^3 \geq 5, \quad 17 - k^2 \geq 8,$$

$$16 + k(6 - k^2) \geq 16 - 3k \geq 7, \quad 8 + k(2 - k) \geq 5;$$

the last inequality follows from concavity in k and its endpoint values 8 and 5. Finally, $x \leq \delta$ gives

$$10x^4 \{(4 - k)\delta + 2p\} - 4x^5 \geq x^4(6\delta + 20p) \geq 0.$$

Equation (7.3) completes this range. The two ranges include their common boundary $d_1 = p + b$. $\square$

Proposition 7.3. *If $n = 3$ and $d_2 \geq 2h$, then*

$$\mu - 16p_g \geq \frac{71p^5}{15P} > 0.$$

Proof. Lemma 4.1 gives $d_1 \geq p + w_4 \geq 2p$. Put $b = d_2/2$ and $q = \min(b, d_1 - p)$. The ordered actual weights satisfy

$$p \leq w_2, w_3, w_4 \leq q, \quad p \leq w_5 = h \leq b.$$

Lemmas 7.1 and 7.2 give

$$30P\mu - 4\mathcal{R}_5(\eta; d_1, d_2) \geq 142p^5, \quad \eta = \left(\sum_{i=1}^N w_i - p\right)/2.$$

By Lemma 3.1, $p_g \leq \mathcal{R}_5(\eta; d_1, d_2)/(120P)$. Division by $30P$ proves the assertion. $\square$

8 The low-degree range in dimension three

Suppose $n = 3$ and $d_2 < 2h$. In the coordinates of Lemma 5.1, $m = 2$ and $u = b_1 + b_2$. The case $u > h$ has $p_g = 0$. Otherwise Lemmas 5.2 and 5.3 give

$$\frac{P\mu}{h} > c^2(c - b_1)(c - b_2), \quad p_g \leq \frac{\lambda^4}{24stb_1b_2}, \quad \lambda = c - \frac{u - p}{2}. \quad (8.1)$$

Proposition 8.1. *If $n = 3$ and $d_2 < 2h$, then $\mu \geq 16p_g + 1$.*

Proof. Assume $u \leq h$. Then $0 < u < c$, (5.2) gives $b_1, b_2 < 2c/3$, and $p \leq u/2$ gives

$$(\lambda/c)^4 \leq \left(1 - \frac{u}{4c}\right)^4.$$

Since $b_1b_2 \geq 0$,

$$\frac{(c - b_1)(c - b_2)}{c^2} \geq 1 - \frac{u}{c} \quad (0 \leq u \leq 2c/3).$$

For $2c/3 \leq u \leq c$, the inequality $(2c/3 - b_1)(2c/3 - b_2) \geq 0$ instead gives

$$\frac{(c - b_1)(c - b_2)}{c^2} \geq \frac{5}{9} - \frac{u}{3c}.$$

Subtract $(2/3)(1 - u/(4c))^4$ from each affine lower bound on its interval. Each difference has second derivative $-(1 - u/(4c))^2/(2c^2) \leq 0$, so its minimum occurs at an endpoint. At $u = 0, 2c/3, c$, respectively, these differences are

$$\frac{1}{3}, \quad \frac{23}{1944}, \quad \frac{13}{1152},$$

which are positive. Consequently

$$\frac{(c - b_1)(c - b_2)}{c^2} > \frac{2}{3}(\lambda/c)^4.$$

Use (8.1) and $P = hstb_1b_2$ to obtain

$$\mu > \frac{2\lambda^4}{3stb_1b_2} \geq 16p_g.$$

Both μ and $16p_g$ are integers. The case $u > h$ follows from $p_g = 0$ and $\mu > 0$. $\square$

9 The surface inequality

In this section $n = 2$. For any four weights with minimum p , retain $P = \prod_i w_i$, $a = d_1 + d_2 - \sum_i w_i$, and $\eta = (\sum_i w_i - p)/2$. Define the following numerical functions, which will be used in both degree ranges:

$$\begin{aligned} \mathcal{G}(d_1, d_2) &= \frac{d_1^2 + d_2^2}{3} - \frac{1}{2} \sum_i w_i^2 - pa - \frac{p^2}{2}, \\ \mathcal{J}(\eta, d_1) &= \eta^4 - (\eta - d_1)_+^4, \\ \mathcal{F}(d_1, d_2) &= 6d_1d_2\mathcal{G}(d_1, d_2) - 6P + \mathcal{J}(\eta, d_1). \end{aligned} \quad (9.1)$$

All weights are held fixed when differentiating these functions in d_1 or d_2 .

Lemma 9.1. *If $d_2 \geq \eta$, then*

$$\mu - 4p_g \geq \frac{\mathcal{F}(d_1, d_2)}{6P}. \quad (9.2)$$

Proof. Put $Z = U_{d_1} + U_{d_2} - \eta$. The degree-two coefficient in (2.2) is

$$d_1^2 + d_1 d_2 + d_2^2 - \sum_{i=1}^N w_i(d_1 + d_2) + \frac{1}{2} \left(\sum_{i=1}^N w_i^2 - \sum_i w_i^2 \right).$$

Using $a = d_1 + d_2 - \sum_{i=1}^N w_i$, we obtain

$$\mu + 1 = \frac{d_1 d_2}{2P} \left(a^2 + d_1^2 + d_2^2 - \sum_i w_i^2 \right).$$

Lemma 3.1 gives $p_g \leq (d_1 d_2 / (2P)) \mathbb{E} Z_+^2$. Since

$$\mathbb{E} Z^2 = \frac{d_1^2 + d_2^2}{12} + \frac{(a+p)^2}{4}, \quad Z_+^2 = Z^2 - Z_-^2,$$

the difference between the two quadratic expressions is

$$\begin{aligned} & \frac{1}{2} \left(a^2 + d_1^2 + d_2^2 - \sum_i w_i^2 \right) - 2 \left(\frac{d_1^2 + d_2^2}{12} + \frac{(a+p)^2}{4} \right) \\ &= \frac{d_1^2 + d_2^2}{3} - \frac{1}{2} \sum_i w_i^2 - pa - \frac{p^2}{2} = \mathcal{G}(d_1, d_2). \end{aligned}$$

It follows that

$$\mu - 4p_g \geq \frac{d_1 d_2}{P} \mathcal{G}(d_1, d_2) - 1 + \frac{2d_1 d_2}{P} \mathbb{E} Z_-^2.$$

The condition $d_2 \geq \eta$ allows direct integration over the rectangle:

$$\mathbb{E} Z_-^2 = \frac{\eta^4 - (\eta - d_1)_+^4}{12d_1 d_2}.$$

For completeness, integrating $(\eta - u - v)_+^2$ first in $v \in [0, d_2]$ gives $(\eta - u)_+^3/3$, since $\eta - u \leq d_2$; integrating in $u \in [0, d_1]$ then gives the stated fourth-power difference. Substitution proves (9.2). $\square$

9.1 The range $d_2 \geq 2h$

Proposition 9.2. *For $n = 2$ and $d_2 \geq 2h$, one has $\mu - 4p_g \geq 1/2$.*

Proof. Write the four ordered weights as $p \leq u \leq v \leq w$, so $h = w$, $P = puvw$, and $\eta = (u + v + w)/2$. Lemma 4.1 gives $d_1 \geq p + v$, and $d_2 \geq 2w > \eta$. We will show $\mathcal{F}(d_1, d_2) \geq 3P$.

The derivatives $\mathcal{G}_{d_1} = 2d_1/3 - p$ and $\mathcal{G}_{d_2} = 2d_2/3 - p$ are positive throughout this region. At the boundary $(d_1, d_2) = (p + v, 2w)$, expansion gives

$$\begin{aligned} 6\mathcal{G}(p + v, 2w) &= v^2 + 4pv - 4p^2 + 3(v - u)(v + u - 2p) \\ &\quad + (w - v)(5(w + v) - 6p) \\ &\geq v^2 + 4pv - 4p^2 \geq p^2 > 0. \end{aligned}$$

Every additional term is nonnegative under the ordering of the weights, and the last inequality follows from $v^2 + 4pv - 4p^2 = p^2 + (v - p)(v + 5p)$. Thus $\mathcal{G}(d_1, d_2) > 0$ on the whole region. Since $\mathcal{J}_{d_1} = 4(\eta - d_1)_+^3 \geq 0$ and $\mathcal{J}_{d_2} = 0$,

$$\mathcal{F}_{d_1} = 6d_2(\mathcal{G}(d_1, d_2) + d_1\mathcal{G}_{d_1}) + \mathcal{J}_{d_1} > 0, \quad \mathcal{F}_{d_2} = 6d_1(\mathcal{G}(d_1, d_2) + d_2\mathcal{G}_{d_2}) > 0.$$

Lowering d_1 to $p + v$ and then d_2 to $2w$ therefore gives $\mathcal{F}(d_1, d_2) \geq \mathcal{F}(p + v, 2w)$; the degree ordering is preserved because $p + v \leq 2w$.

At the boundary, $2\eta = (p + v) + (u + w - p)$ and $u + w - p \geq w \geq v$. The elementary inequality $t_+^4 \leq t^4$ gives

$$\begin{aligned} \mathcal{J}(\eta, p + v) &\geq \eta^4 - (\eta - (p + v))^4 \\ &= \frac{(p + v)(u + w - p)((p + v)^2 + (u + w - p)^2)}{2} \\ &\geq \frac{(p + v)w((p + v)^2 + v^2)}{2}. \end{aligned}$$

Combining this with the lower bound for $6\mathcal{G}(p + v, 2w)$ and $u \leq v$ yields

$$\begin{aligned} \mathcal{F}(p + v, 2w) - 3P &\geq w \left[(p + v) \left(3v^2 + 9pv - \frac{15p^2}{2} \right) - 9pv^2 \right] \\ &= \frac{3w}{2}(v - p)(2v^2 + 4pv + 5p^2) \geq 0. \end{aligned}$$

Thus $\mathcal{F}(d_1, d_2) \geq 3P$, and Lemma 9.1 proves the assertion. $\square$

9.2 The range $d_2 < 2h$

The coordinate construction in the first part of Lemma 5.1 also applies when $n = 2$. Write the resulting coordinates as x, y, z, z_0 , with

$$\begin{aligned} f &= z_0x + P_0(x, y, z), & g &= z_0y + Q_0(x, y, z), \\ \text{wt}(x, y, z, z_0) &= (\alpha, \beta, s, h), & (d_1, d_2) &= (h + \alpha, h + \beta), & 0 < \alpha \leq \beta < h. \end{aligned} \tag{9.3}$$

The independent linear parts of the coefficients of z_0 give this form by the weighted triangular coordinate change in Lemma 5.1.

Lemma 9.3. *In (9.3), $2s \leq h + \beta$.*

Proof. If both $P_0(0, 0, z)$ and $Q_0(0, 0, z)$ vanished identically, the surface would contain the plane $x = y = 0$. Its defining ideal $I = (f, g)$ is a height-two prime ideal, by normality and the irreducibility argument in Lemma 3.1. Thus $I \subset (x, y)$ is an inclusion of prime ideals of the same height and must be equality. This contradicts $I \subset \mathfrak{m}^2$. At least one restriction is therefore a nonzero pure power of z . Its exponent is at least two, and its weighted degree is d_1 or d_2 . Hence $2s \leq d_2 = h + \beta$. $\square$

Proposition 9.4. *For $n = 2$ and $d_2 < 2h$, one has $\mu - 4p_g \geq 1/2$.*

Proof. The minimum weight in (9.3) is $p = \min(\alpha, s)$, since $\alpha \leq \beta < h$. We consider its two possible locations. In both cases, use the functions in (9.1) for these four weights.

Suppose first that $p = \alpha \leq s$. Then $P = p\beta sh$ and $\eta = (\beta + s + h)/2$. Lemma 9.3 gives $p \leq s \leq (h + \beta)/2$, so $d_2 = h + \beta > \eta$. Direct expansion gives

$$6\mathcal{G}(d_1, d_2) = h^2 + (4\beta - 2p)h + 6ps - 4p^2 - \beta^2 - 3s^2.$$

This expression is nonincreasing in $s \geq p$. Substituting the upper bound $s \leq (h + \beta)/2$ therefore gives

$$6\mathcal{G}(d_1, d_2) \geq \frac{1}{4} \left(h^2 + (10\beta + 4p)h + 12p\beta - 7\beta^2 - 16p^2 \right).$$

Also $2\eta = d_1 + (\beta + s - p)$, and $\beta + s - p \geq \beta$. Using $t_+^4 \leq t^4$ as in Proposition 9.2, we obtain

$$\mathcal{J}(\eta, d_1) \geq \frac{d_1\beta(d_1^2 + \beta^2)}{2}.$$

Finally $P \leq p\beta h(h + \beta)/2 = p\beta h d_2/2$. Consequently $4(\mathcal{F}(d_1, d_2) - 3P)$ is bounded below by

$$\begin{aligned} & d_1 d_2 \left(h^2 + (10\beta + 4p)h + 12p\beta - 7\beta^2 - 16p^2 \right) \\ & + 2d_1\beta(d_1^2 + \beta^2) - 18p\beta h d_2. \end{aligned} \tag{9.4}$$

To see its sign, put $x = \beta - p \geq 0$ and $y = h - \beta \geq 0$ only in this expression. Substitution of $d_1 = h + p$ and $d_2 = h + \beta$ and expansion by total degree in p give

$$\begin{aligned} & p^3(66x + 52y) + p^2(114x^2 + 198xy + 66y^2) \\ & + p(60x^3 + 174x^2y + 126xy^2 + 22y^3) \\ & + 12x^4 + 44x^3y + 48x^2y^2 + 17xy^3 + y^4 \geq 0. \end{aligned}$$

Thus $\mathcal{F}(d_1, d_2) \geq 3P$, and (9.2) gives the required bound in this case.

Suppose next that $p = s \leq \alpha$. Now $P = p\alpha\beta h$ and $\eta = (\alpha + \beta + h)/2$. Both $d_1 = h + \alpha$ and $d_2 = h + \beta$ exceed η . Expansion gives

$$6\mathcal{G}(d_1, d_2) = h^2 + 4h(\alpha + \beta) - \alpha^2 - \beta^2 - 6ph.$$

For fixed p, h , this increases with each of $\alpha, \beta \in [p, h]$, so

$$6\mathcal{G}(d_1, d_2) \geq h^2 + 2ph - 2p^2 > 0.$$

Since $(h + \alpha)/\alpha \geq 2$ and $(h + \beta)/\beta \geq 2$,

$$\frac{d_1 d_2}{P} \geq \frac{4}{ph}, \quad \frac{d_1 d_2}{P} \mathcal{G}(d_1, d_2) \geq \frac{2}{3} \left(\frac{h}{p} + 2 - \frac{2p}{h} \right) \geq \frac{2}{3}.$$

The arithmetic–geometric mean inequality for α, β, h gives

$$\frac{\eta^4}{P} \geq \frac{81}{16} \frac{(\alpha\beta h)^{1/3}}{p} \geq \frac{81}{16}.$$

Here $\mathcal{J}(\eta, d_1) = \eta^4$, so Lemma 9.1 yields

$$\mu - 4p_g \geq \frac{2}{3} - 1 + \frac{81}{96} = \frac{49}{96} > \frac{1}{2}.$$

The two cases cover every minimum weight, including their common boundary. $\square$

Proof of Theorem 1.1. For $n = 2$, Propositions 9.2 and 9.4 give $\mu - 4p_g \geq 1/2$ in the complementary ranges $d_2 \geq 2h$ and $d_2 < 2h$. Integrality gives part (i). For $n = 4$, Proposition 6.2 treats $d_2 \geq 2h$, while Proposition 5.4 treats $d_2 < 2h$. For $n \geq 5$, these ranges are covered by Propositions 4.5 and 5.4. For $n = 3$, Propositions 7.3 and 8.1 cover the two degree ranges. This proves part (ii); in dimension three, integrality gives $\mu \geq 16p_g + 1$. $\square$

10 The surface constant

For an integer $m \geq 2$, the second Stirling number $S(m, 2)$ counts partitions of m labelled elements into two unordered nonempty subsets. Choosing a nonempty proper subset and identifying it with its complement gives $S(m, 2) = 2^{m-1} - 1$. Thus the corrected complete-intersection constant specializes in codimension two to (1.1). At $n = 2$, its formal value $36/7$ fails for surfaces, as the following example shows.

Take distinct complex numbers $\lambda_1, \dots, \lambda_4$ and the homogeneous equations

$$f = \sum_{i=1}^4 x_i^5, \quad g = \sum_{i=1}^4 \lambda_i x_i^5.$$

The finite surjective map $(x_1, \dots, x_4) \mapsto (x_1^5, \dots, x_4^5)$ pulls back two independent linear equations, so the common zero set has dimension two. The two equations therefore form a regular sequence. If their Jacobian rows were dependent at a nonzero common zero, all nonzero coordinates would have the same coefficient λ_i . Distinctness forces at most one nonzero coordinate, which is incompatible with $f = 0$. The complete intersection is thus isolated, and its embedding is minimal since the equations have no linear terms.

All weights equal one and $d_1 = d_2 = 5$. With $Q(t) = (t - 1)^4$, Lemma 2.2 gives

$$\mu = 5 \cdot 4 \cdot 4^3 - 4^4 = 1024.$$

The cutoff is $a = 6$. From (2.1) and $H_R(t) = (1 - t^5)^2/(1 - t)^4$,

$$p_g = \binom{10}{4} - 2 \binom{5}{4} = 210 - 10 = 200.$$

Hence

$$1024 - \frac{36}{7} \cdot 200 = -\frac{32}{7} < 0, \quad 1024 > 4 \cdot 200.$$

The surface conclusion in Theorem 1.1 therefore has a different coefficient from the higher-dimensional corrected inequality.

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ZHILI COLLEGE, TSINGHUA UNIVERSITY, BEIJING 100084, CHINA
 Email address: tangqy24@mails.tsinghua.edu.cn