

Arithmetic structure in a complete atlas of power-residue runs below 2^{32}

Rabung profiles, Kummer densities, norm dependencies, and local codes

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Abstract

We study the arithmetic behind long runs in a complete census of power-residue colourings below 2^{32} , covering every admissible order $2 \leq r \leq 9$. The dataset contains 203 280 221 primes and 677 591 249 profiles, with exact quadratic four-window counts at every odd prime. A bounded study of its deep tail identifies algebraic windows with fewer independent character conditions than their lengths suggest. We certify their power-class groups and conjugate-root overlaps, obtaining exact Kummer densities for fixed templates, including eight additional order-four models. In a quadratic family, eleven constant positions force twenty-six neutral positions at odd character orders. In two further families, norm identities generate all remaining higher-root relations. A length-53 cubic realizes a three-wise independent binary code whose monochromatic probability is 4096 times that of independent signs; its local run law is computed exactly. A separate stratified million-profile ternary sample is analysed with the same finite catalogue filter applied to the data and to an independent-colour comparator. Finite census statements, selected explanations and asymptotic template laws concern distinct populations. No universal law of global maxima or finite-range error estimate for the template densities is asserted.

Keywords. Multiplicative characters; residue runs; van der Waerden numbers; Rabung construction; Kummer theory; Chebotarev density; binary codes; exact computation.

1 Introduction and scope

A long monochromatic residue run may resemble a rare string of independent colours. Some extreme runs instead lie on algebraic windows whose character values satisfy many exact relations. One window of length 53 gives a sharp contrast: every projection onto three positions is uniform, yet its monochromatic probability is 2^{-40} rather than the independent-sign value 2^{-52} . Section 9 proves this statement for all prime/rational-root pairs of one fixed cubic template in the limit over primes. A prime with three rational roots contributes three pairs. It is not a finite-range frequency estimate for the canonical witnesses selected from the atlas.

Three questions organize the paper: what occurs in the complete finite data; which exact identities explain selected exceptional windows; and what the resulting template laws and finite diagnostics allow us to conclude about run statistics. Rabung's criterion [4] and earlier large searches, including Monroe's [5], supply the construction context. The atlas covers all primes below 2^{32} and every admissible order from 2 to 9. The scanner method is developed in [3]; its frozen implementation and data commitments are described in Section 15.

The classification of its 11 557 deep-tail profiles is selective: 23 have no retained model after the higher-degree extensions. A template suggested by a selected witness can subsequently have

*Prepared with artificial-intelligence assistance. The scope of the assistance and the status of the manuscript are stated in Section 15.

a proved density over varying primes, but that selection is not an independent test of the density. Certified Kummer groups, norm-generated root dependencies and the exact local code law provide the mathematical explanations. The 496 order/length pairs and the bounded residual qualification are machine-readable. A separate million-profile ternary supplement supplies finite continuation diagnostics with a same-filter comparator. The short quadratic elliptic formula is background to the census.

Some ingredients are classical: the study of consecutive power residues [7], Kummer theory [8], Chebotarev’s theorem [9], and the MacWilliams identity [10]. Character-pattern counts have established connections with curves and their Jacobians [18]. Prescribed residue patterns on finite sets of rational integers were studied by Wright [11, 12] and by Karthick Babu and collaborators [13, 14]. The Kummer–Chebotarev framework is classical; our certifications concern the relations of selected algebraic-root windows and their interaction across conjugate roots. Dual distance, orthogonal arrays and limited independence are standard coding theory [15, 16]. Section 9 gives a specific arithmetic realization and its exact local run distribution, not a new general independence principle. No worldwide priority claim is made for the examples.

Relation to succinct certificates. The published certificates paper [2] proves the equivalence of the implemented boundary test with Rabung’s criterion and owns the recurrence-closed comparison of lower bounds. Here the subject is the structure visible in a complete finite population: its distributions, selected algebraic windows and their exact relations. Pointwise construction certificates, complete census coverage and limiting prime densities have different verification requirements. We retain a short Rabung specialization for readability and identify the frozen scanner and finite audit artifacts in the reproduction section.

2 The atlas and the Rabung contract

Four populations used below.

Population	Scope of the conclusion
All primes below 2^{32} and admissible orders 2–9	Complete finite census in that domain.
Canonical witnesses selected in the deep tail	Descriptive classification under stated thresholds and catalogues.
Prime/rational-root pairs of a fixed template	Asymptotic laws from its certified power-class groups and Chebotarev.
One million ternary profiles stratified on $[2^{33}, 2^{34})$	Finite comparisons of four equally sized lists, rather than a natural prime average over the band.

These populations have different weights and selection rules. In particular, neither the selected witnesses nor the equally weighted ternary strata are substituted for the prime/root population of a template law.

2.1 Definitions and conventions

For a prime p and an order $r \mid p - 1$, let

$$\chi_{p,r}(x) = x^{(p-1)/r} \in \mu_r(\mathbb{F}_p), \quad x \in \mathbb{F}_p^\times.$$

Its r values are the colours. The neutral colour is 1. Relabelling the non-neutral colours does not change any constant-run length. All maxima below concern the ordinary word at positions $1, \dots, p - 1$: the word is not joined across zero or interpreted as a circular word. Define

$$M_r(p) = \max\{m : \chi_{p,r}(w) = \dots = \chi_{p,r}(w + m - 1), \ 1 \leq w \leq p - m\},$$

and let $L_r(p)$ be the largest ℓ for which $\chi_{p,r}(1) = \cdots = \chi_{p,r}(\ell) = 1$. The canonical witness $w_r(p)$ is the least starting position attaining $M_r(p)$. Put $\sigma = \chi_{p,r}(-1)$. As $\sigma^2 = 1$, it is 1 or -1 , with $\sigma = 1$ for odd r . The atlas also records

$$K_r(p) = \begin{cases} \max(M_r(p) + 1, 2L_r(p) + 2), & \sigma = 1, \\ \max(M_r(p) + 1, L_r(p) + 2), & \sigma = -1. \end{cases} \quad (1)$$

We use *colour-first* notation: $W(r, k)$ is the least n such that every r -colouring of $[n]$ contains a monochromatic progression of length k , as in the published certificates paper [2]. Historical files retain their original schema: in particular, `W_length_first_lower_bound` in the catalogue stores the weak integer bound $W(r, k) \geq (k - 1)p + 2$. Its name is not a convention for the revised article. A strict bound $W(r, k) > B$ means $W(r, k) \geq B + 1$.

2.2 Internal and boundary conditions

For $p > k$, the atlas calls $M_r(p) < k$ criterion (a), and $K_r(p) \leq k$ the complete criterion (a)+(b). The latter has a constructive meaning.

Proposition 2.1. *If $p > k$ and $K_r(p) \leq k$, the power-residue colouring extends to an r -colouring of $[0, (k - 1)p]$ with no monochromatic k -term arithmetic progression, by colouring its k multiples of p non-constantly. Consequently $W(r, k) > (k - 1)p + 1$.*

Proof. Consider a progression of common difference d . If $p \mid d$, the only possible k -term progression in this interval is precisely the k multiples of p , which have been coloured non-constantly. Otherwise multiply its residues by d^{-1} . If none is zero, a constant progression becomes a constant ordinary consecutive segment of length k , contradicting $M < k$. If a zero is present, every nonzero term on a given side of it is a scalar multiple of $1, 2, \dots, a$. When $\sigma = 1$, at least one side has $\lceil (k - 1)/2 \rceil$ terms; a constant progression would require that many neutral prefix values. Equation (1) excludes this. When $\sigma = -1$, nonempty sides have different colours, so a monochromatic progression would have zero at an end and require $k - 1$ neutral prefix values, also excluded. $\square$

The proof explains why a good ordinary maximum alone does not certify the construction. This is the census specialization of Rabung’s theorem; the full boundary-test equivalence and the separately checked certificates are developed in the published certificates paper [2].

2.3 Coverage, exactness, and sealing

The census visits every prime $p < 2^{32}$ and every $r \in [2, 9]$ dividing $p - 1$. There are 203 280 221 primes and 677 591 249 admissible profiles. The 203 280 220 odd primes have exact quadratic quadruplet counts; the prime 2 has its specified zero slot. The 49 901 committed blocks form a contiguous partition of the domain. Independent audits comprise 3 118 periodic ordinary scans, 3 119 elliptic-count certificates, and 324 final record checks. These audit counts describe the campaign’s checks, not independent samples for inference about the tails.

The scanner implements rational coverage and exact completion. Under the coverage and implementation hypotheses of the published method, a successfully completed scan returns exact maxima and least canonical witnesses [3, Theorems 3.1 and 4.3]. The stored profile contract is stated in Section 2; the frozen implementation and retained audit certificates are identified in Section 15. Rejected batches supply no accepted partial output. Execution-policy details and campaign times are recorded in Appendix D, separate from the mathematical coverage statement.

The final manifest covers 203 files and 4 893 897 425 bytes. Its SHA-256 is

762dc0aad9beaa8a6bf2cb1a82604ad0c3d
fcbb9f67c1a852ca7a2bb4468c20.

All covered files were rehashed for this manuscript. The four preceding theoretical deliveries were also checked against their manifests. The new study uses a separate output directory.

3 Descriptive results and their limits

3.1 Marginal profiles

Table 1 summarizes the entire admissible population for each order. The extreme maxima are far beyond the central quantiles. For instance, the quadratic maximum is 100 while its 99th percentile is 36. The mean values pool small and large primes, so they do not represent a stationary distribution at a fixed prime size.

Table 1: Exact marginal profiles on the entire atlas. Quantiles refer to M_r .

r	Primes	Mean	Median	95%	99%	$M_{\max}$	$L_{\max}$	$K_{\max}$
2	203 280 220	29.84	30	34	36	100	96	166
3	101 638 091	19.27	19	22	23	60	58	118
4	101 638 380	15.45	15	17	19	42	40	82
5	50 819 996	13.42	13	15	16	36	30	62
6	101 638 091	12.13	12	14	15	40	28	58
7	33 879 793	11.22	11	13	13	30	28	58
8	50 817 342	10.54	11	12	13	30	22	46
9	33 879 336	10.01	10	11	12	26	22	46

3.2 Finite construction catalogues

The delivered catalogue records both criteria for every $2 \leq r \leq 9$ and $3 \leq k \leq 64$. Table 2 prints the thirteen previously fixed target pairs. All counts are exact within the atlas. The two best-prime columns happen to agree for these targets, but the counts do not. At $(r, k) = (2, 25)$, for example, 52 770 primes pass (a) and fail the boundary condition.

Table 2: The thirteen fixed targets. The largest prime passing (a) and the largest passing (a)+(b) coincide in these rows.

r	k	(a) count	(a)+(b) count	Largest prime
2	25	3 849 263	3 796 493	1 138 900 957
2	26	7 384 892	7 338 954	2 125 065 391
2	27	14 219 871	14 130 427	4 195 501 393
2	28	27 413 188	27 239 679	4 294 720 151
3	17	3 572 197	3 532 623	2 072 886 931
3	18	10 093 790	9 980 386	4 291 179 427
3	19	28 016 242	27 692 568	4 294 961 809
3	20	59 199 151	58 505 882	4 294 967 197
3	21	83 599 844	82 608 585	4 294 967 197
3	22	94 976 489	94 611 815	4 294 967 197
3	23	99 303 996	98 909 842	4 294 967 197
3	24	100 820 773	100 417 036	4 294 967 197
3	25	101 336 725	100 929 271	4 294 967 197

The best admissible prime for $(2, 25)$ is 1 138 900 957, giving

$$W(2, 25) > 27\,333\,622\,969.$$

At $(3, 17)$ the prime is 2 072 886 931, giving $W(3, 17) > 33\,166\,190\,897$. Relative to the fixed direct-construction entries in Table 2 of Monroe's version 7 [6], these primes increase by 18.82%

and 113.84%. They are the two improved entries among the 29 fixed comparison points. This is a comparison with that identified historical table, not a claim of current world records. The (2, 25) certificate is already in the first published edition [1], which credits Monroe's phase-2 project. It is not a new construction here. The (3, 17) value increases that edition's published strict bound 31 385 622 833 by 5.67%. Separate full ordinary CPU scans are retained in `reproducibility/bounds_update/results/`, with their source identities and commands. They also check the larger primes 5 899 373 413, 14 413 721 599 and 17 177 303 971 at lengths 18, 19 and 20. Their consequences exceed the corresponding bounded-atlas rows and retain their own pointwise checks; they are not added to the census population. The current pointwise bounds and recurrence-closed comparisons are reported in the second edition [2]. Large k values in Table 2 also approach the upper boundary of the search domain, which censors any conclusion about the largest possible construction prime.

3.3 Prime size, refinement, and apparent dependence

Maxima for different orders are correlated when all prime sizes are pooled. For (3, 6) the raw correlation is 0.376413; after restricting to $p \geq 2^{28}$ it is 0.203170. Pooling deviations from the means in the fifteen remaining bands of width 2^{28} gives 0.006293. Corresponding within-band values are 0.002830 for (2, 3), 0.004642 for (2, 4), 0.003503 for (3, 9), and 0.009308 for (4, 8). These transformations describe size adjustment and are not proofs of independence.

There is also a deterministic relation. If $r \mid s \mid p - 1$, then $\chi_{p,r} = \chi_{p,s}^{s/r}$, so a constant fine-colour run is constant after projection. Thus

$$M_s(p) \leq M_r(p), \quad L_s(p) \leq L_r(p), \quad K_s(p) \leq K_r(p).$$

The last inequality follows from the two boundary cases in (1); $\sigma_r = -1$ implies $\sigma_s = -1$. These inequalities hold throughout the common admissible populations. Small correlations in the centre do not remove dependence among extreme nested events.

For a fine run already selected, a local independent-colour comparator must condition on the fine boundary. If the fine order is s , the coarse order is $r \mid s$, and the neighbouring fine colour differs from the run colour, its probability of having the same coarse colour is $(s - r)/(r(s - 1))$. Subsequent unconditioned coarse matches have probability $1/r$. The earlier exploratory study uses the equivalent parameterization with s/r fine colours per coarse colour. Such conditioned comparisons are more appropriate than multiplying two marginal rare-tail probabilities; they remain local heuristic models.

3.4 Four-term counts and an elliptic trace

Let $A_4(p)$ count ordinary starts $1 \leq x \leq p - 4$ at which all four quadratic characters of $x, x + 1, x + 2, x + 3$ are equal. Both constant signs are included. For $p > 3$ write $\chi = \chi_{p,2}$ and define

$$B_p = 4\chi(2) + 2\chi(6) + 2\chi(3) + 2\chi(-1) + 2\chi(-2).$$

For the fixed elliptic curve

$$E_9 : y^2 = x(x - 1)(x - 9), \quad t_p = p + 1 - \#E_9(\mathbb{F}_p),$$

the exact identity used by the CPU calculation is

$$A_4(p) = \frac{p - 11 - B_p - t_p}{8}. \quad (2)$$

Here is a derivation. Away from the four zeros, the indicator of four equal signs is

$$\frac{1}{8} \left(1 + \sum_{0 \leq i < j \leq 3} \chi((x + i)(x + j)) + \chi(x(x + 1)(x + 2)(x + 3)) \right).$$

Over all $x \in \mathbb{F}_p$, each of the six pair sums equals -1 . The genus-one double cover $y^2 = x(x+1)(x+2)(x+3)$ has two rational points at infinity. This length-four quartic and the residue-pattern interpretation are treated in [18, Section 2.3]. We include the specialization and boundary count needed for the atlas column. Put $u = 2x + 3$ and $v = 4y$. Its equation becomes $v^2 = (u^2 - 1)(u^2 - 9)$, and the map

$$X = u^2, \quad Y = uv$$

maps it to E_9 . With an infinity point chosen as origin this is a degree-two isogeny for $p > 3$. Isogenous elliptic curves over $\mathbb{F}_p$ have the same Frobenius trace. Thus its full quartic character sum is $-t_p - 1$: the affine count is p plus that sum, and the two infinity points give the projective count $p + 1 - t_p$. At each excluded zero the quartic term vanishes; the total constant contribution is 4, and the remaining pair contributions sum to B_p . Subtracting these four positions gives (2).

The census sum is

$$\sum_{\substack{p < 2^{32} \\ p \text{ odd}}} A_4(p) = 53\,206\,216\,723\,199\,456.$$

All integer Hasse bounds are satisfied. Since $|B_p| \leq 12$,

$$\left| A_4(p) - \frac{p}{8} \right| \leq \frac{\sqrt{p}}{4} + \frac{23}{8}.$$

The j -invariant of E_9 is $1\,556\,068/81$. Its non-integrality excludes complex multiplication. With $Z_p = t_p/(2\sqrt{p})$, the fixed Sato–Tate comparator is $(2/\pi)\sqrt{1 - Z^2} dZ$ on $[-1, 1]$ [17]. Across $203\,280\,219$ primes $p > 3$, the empirical mean is 1.09435×10^{-5} and the second moment is 0.2500016788 , compared with limiting values 0 and $1/4$. Figure 1 uses this fixed density, without fitting a parameter.

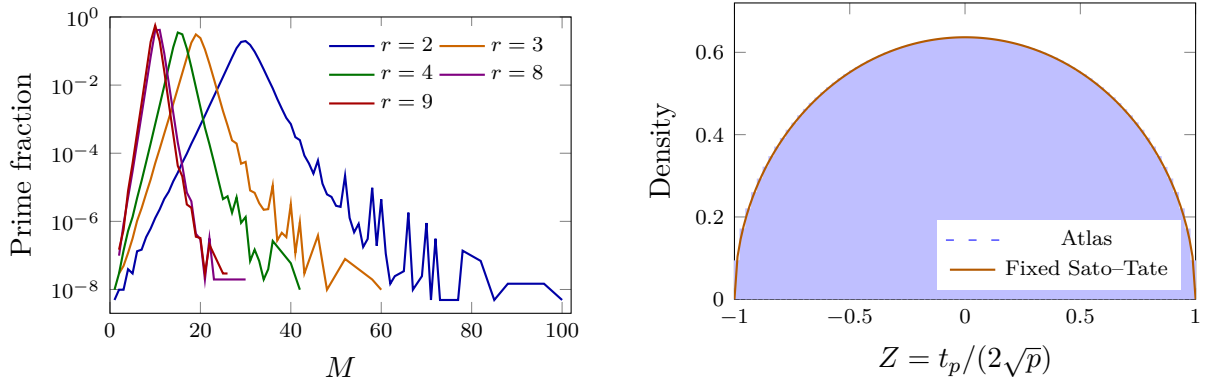


Figure 1: Marginal maxima for selected orders and normalized elliptic traces in the full atlas. The trace comparator is fixed and has no fitted parameter.

The raw correlation between A_4 and M_2 is 0.531581. Replacing the quadruplet count by its normalized elliptic trace gives -0.002831 , or -0.003086 within the fifteen large-prime bands. This removes the dominant prime-size trend; it does not prove independence. There is a more basic obstruction to recovering a global maximum from a short-pattern count. If h_ℓ counts maximal constant runs of length ℓ , then

$$A_k = \sum_{\ell \geq k} (\ell - k + 1) h_\ell.$$

A single weighted sum such as A_4 does not determine the farthest nonzero entry of h_ℓ .

4 The fixed million ternary supplement

4.1 A separate population and three different counts

The repository also contains a fixed sample of 1 000 000 ternary profiles on $[2^{33}, 2^{34})$. The separately frozen catalogue collector in `src/ternary_catalogue/` produced these profiles and fixed-threshold counts; its historical qualification has domain $p < 2^{34}$ and is documented in `data/ternary-million/README.md`. It has a different domain and contract from the worker described for $p < 2^{33}$. This supplement does not extend the complete census below 2^{32} . There are four strata of 250 000 primes each: the two bands $[2^{33}, 3 \cdot 2^{32})$ and $[3 \cdot 2^{32}, 2^{34})$, each divided according to whether two is a cube. The equal sample weights differ from the asymptotic natural relative cube-class density $1/3$. No average with these four equal weights is presented as a uniform prime average over the whole band.

For a prime p in the supplement put $h = (p - 1)/2$. Reflection $x \mapsto p - x$ preserves the cubic character. An ordinary canonical run is a maximal constant interval strictly inside this half-period; it excludes the initial run and the merged central run. Let $N_{p,k}$ count these ordinary runs of length at least k , and let C_p be the full length of the central run. Then the predecessor start count on the unfolded interval is

$$Z_{p,k} = 2N_{p,k} + \mathbf{1}_{L_p \geq k} + \mathbf{1}_{C_p \geq k}. \quad (3)$$

Here the predecessor ranges over $1, \dots, p - k - 1$. The initial run at one has no predecessor, whereas its reflected copy does; a central run has one start. Overlapping k -windows inside a longer run are a third observable and must not be substituted for $N_{p,k}$.

The stored thresholds are 18 through 27. The supplement contains 33 257 058 canonical maximal runs of length at least 18, and 1 235 473 retained canonical positions at length at least 21. Exactly 1 234 808 of the latter are ordinary. The difference is accounted for by the prefix and centre conventions. Full-million positions of lengths 18 through 20 were not retained. The stored central field is zero below length 18; it does not assert that the central run is absent.

At threshold 21, 299 150 profiles satisfy the interior criterion $M < 21$, while 293 872 satisfy both Rabung criteria $K \leq 21$. For these primes $K = \max(M + 1, 2L + 2)$, consistent with Section 2. Supplement membership includes earlier diagnostic primes and is not an independent validation of the analyses subsequently made on it. The fixed selection and its scientific commitments are documented in `data/ternary-million/`.

4.2 Intensity, continuation, and a fixed comparator

For an independent word of h uniform r -colours, the expected number of strictly interior maximal runs of length at least k is, for $k \leq h - 2$,

$$\mu_{h,k} = \frac{(r - 1)(h - k - 1) - 1}{r^k} + r^{1-h}. \quad (4)$$

For proof, an interval of exact length ℓ has probability $(r - 1)^2/r^{\ell+1}$ of being constant and different from its two neighbours. There are $h - \ell - 1$ interior placements; summing over $k \leq \ell \leq h - 2$ gives (4). The formula retains the finite-word endpoints. The terminal term is bounded explicitly below; equal population sizes alone do not cancel it when their word lengths differ.

Let S_c and S_n be the two sets of 500 000 profiles for which *two* is respectively a cube and a noncube. For any per-prime statistic U_p , write $\delta U = \sum_{p \in S_c} U_p - \sum_{p \in S_n} U_p$. Define the count contrast and its iid-centred version by

$$\delta N_k = \delta(N_{p,k}), \quad \Delta_k = \delta(N_{p,k} - \mu_{h,k}). \quad (5)$$

For $a < b$ put

$$D_{a,b}^N = \delta N_b - 3^{a-b} \delta N_a, \quad \Delta_{a,b}^N = \Delta_b - 3^{a-b} \Delta_a = D_{a,b}^N - \varepsilon_{a,b}^N,$$

where $\varepsilon_{a,b}^N = \delta(\mu_{h,b} - 3^{a-b} \mu_{h,a})$. Equation (4) gives exactly

$$\mu_{h,b} - r^{a-b} \mu_{h,a} = -\frac{(r-1)(b-a)}{r^b} + (1 - r^{a-b})r^{1-h}.$$

Only the constant term cancels between equal-sized groups. Here $h \geq 2^{32}$, so $|\varepsilon_{a,b}^N| \leq 10^6(1 - 3^{a-b})3^{1-2^{32}} < 3^{-100}$. This bound justifies every printed continuation decimal below; it does not replace the exact contrast by an identity.

Observed counts, their continuation from one threshold to another, and the number of profiles with zero count describe different features. For thresholds $a < b$, the independent reference has geometric continuation 3^{a-b} , but the actual terminal and length corrections must still be respected. A raised continuation rate can decrease a void probability, while concentration of the same total on fewer primes can increase it. An unconditioned void deficit alone is therefore not a proof of repulsion.

Conditioning a thinning model on its observed total also changes its meaning. If A initial objects are allocated in groups and exactly B survive, the conditional count in a group of size n is hypergeometric:

$$\Pr(T = t \mid B) = \frac{\binom{n}{t} \binom{A-n}{B-t}}{\binom{A}{B}}.$$

This describes an allocation model with the total learned from the same sample. Even a close fit of that conditional comparator does not estimate the unconditional first intensity or prove a Poisson law. We retain this distinction in reporting the arithmetic contrasts below.

4.3 A bounded arithmetic catalogue

For each stored ordinary interval $[x, x + \ell - 1]$, the wide quadratic catalogue asks whether it contains a position y satisfying

$$(2y + A)^2 \equiv D \pmod{p}, \quad |A| \leq 128, \quad 0 < |D| \leq 4096, \quad D \notin \mathbb{Q}^2, \quad D \equiv A^2 \pmod{4}.$$

This is a root of the integer polynomial $X^2 + AX + (A^2 - D)/4$ anywhere in the run, not necessarily at its first position. The rational catalogue asks whether the complete interval contains a reduced cut ap/q with $2 \leq q \leq 128$. The cut is nonintegral at these primes, so the two neighbouring integer positions must both belong to the interval. These parameter bounds are fixed catalogue limits, not an assertion that all possible arithmetic structures were found.

The CPU classification of all 1 234 808 ordinary runs gives 1 994 wide quadratic hits and 306 rational hits. Their union has 2300 runs on 2187 primes. The two categories happen to be disjoint in this extraction; the implementation computes their union and does not rely on that being a general property. Among the quadratic hits, internal cubic identities force two to be a cube for 1856 runs, force the constant phase to be neutral for 1935, and prove both conditions for 1855. These are necessary-condition certificates, not exact Kummer density calculations for each detected model.

Let $T_{p,k}$ count catalogue-marked ordinary runs, and put $\delta T_k = \delta(T_{p,k})$ and $D_{a,b}^T = \delta T_b - 3^{a-b} \delta T_a$. The retained counts are

k	$\sum_{S_c} N_{p,k}$	$\sum_{S_n} N_{p,k}$	$\sum_{S_c} T_{p,k}$	$\sum_{S_n} T_{p,k}$
21	619086	615722	2195	105
24	23941	22726	663	19
27	1087	825	224	4

Removing this union changes the prime-size-corrected count contrast by $\delta T_k/\Delta_k$: 60.53% at threshold 21, 52.86% at 24, and 83.93% at 27. These percentages precede subtraction of the filter's own iid background. The raw continuation innovations are

$$D_{24,27}^N = 262 - \frac{1215}{27} = 217, \quad D_{24,27}^T = 220 - \frac{644}{27} = \frac{5296}{27}, \quad D_{24,27}^N - D_{24,27}^T = \frac{563}{27}.$$

The last value is about 20.852 in cumulative half-million count units. This is a finite descriptive decomposition after model selection. It is not a causal percentage explained or a prospective test.

4.4 Applying the same filter to independent colours

Removing an arithmetic catalogue from the observed word requires evaluating that very same filter in its comparator. Here the prime and its coordinate mask are kept fixed and only its colours are replaced by independent uniform colours. The mask marks a complete maximal run if it meets any allowed quadratic position or contains any allowed rational cut; it is not truncated to its first k letters.

There is a useful exact finite geometry formula. Remove the marked vertices and cut edges from the interior positions $[2, h-1]$, and let $L_1, \dots, L_s$ be the remaining component lengths. For $L \geq k$ define

$$F_{k,r}(L) = \frac{(r-1)(L-k+1)-1}{r^k} + r^{-L-1}, \quad F_{k,r}(L) = 0 \quad (L < k).$$

If $T_{p,k}$ is the count marked by the filter, then

$$\mathbb{E}_{\text{iid}} T_{p,k} = F_{k,r}(h-2) - \sum_{j=1}^s F_{k,r}(L_j). \quad (6)$$

Indeed, for every exact length ℓ the number of marked interior placements is $(h-\ell-1) - \sum_j (L_j - \ell + 1)_+$. Multiplying by $(r-1)^2/r^{\ell+1}$ and summing proves (6). Components are used to count placements; the observed word is never shortened or concatenated.

For the wide mask there are 4032 nonsquare discriminants. In one half-period, each splitting discriminant gives at most one interval of 128 or 129 marked vertices, since its allowed A values advance by two. For $p > 20996$ these intervals avoid the word endpoints. Their total length is at most 518112, and there are 2510 rational cuts. Thus the number of intervals plus cuts is at most 6542. In general, if total marked length is at most B and this number is at most J , coverage of an exact ℓ -interval is bounded by $B + (\ell-1)J$. Summing gives the uniform bound

$$\mathbb{E}_{\text{iid}} T_{p,k} \leq \frac{(r-1)(B + (k-1)J) + J}{r^k}.$$

Using the same word lengths for all sampled primes therefore gives:

k	Observed marked runs	Upper bound for iid expected total
21	2300	124.703820
24	682	4.757640
27	228	0.181357

The decimal upper bounds are rounded upwards. They compare counts with bounded expectations; they are not p -values.

The continuation correction admits a sharper signed bound. Adding one letter to the length of a test interval increases its covered placements by a number between zero and J , apart from the final finite-word cutoff. Summing these increments gives, per profile,

$$-\frac{(r-1)(b-a)}{r^b} \leq \mathbb{E}_{\text{iid}} T_{p,b} - r^{a-b} \mathbb{E}_{\text{iid}} T_{p,a} \leq \frac{J(r-1)(b-a)}{r^b}.$$

The same bounds applied to the two equal-sized cube/noncube populations give an absolute iid contrast below 0.002575 for 24 to 27. In this calculation the background is $\varepsilon_{a,b}^T = \delta(\mathbb{E}_{\text{iid}} T_{p,b} - 3^{a-b} \mathbb{E}_{\text{iid}} T_{p,a})$, and the fully centred catalogue fraction is

$$\frac{D_{a,b}^T - \varepsilon_{a,b}^T}{D_{a,b}^N - \varepsilon_{a,b}^N}.$$

For 24 to 27, $|\varepsilon_{24,27}^T| \leq 727000000/282429536481 + 3^{-100}$; the rational part is approximately 0.002574094 and the final term bounds the finite-word remainder. Retaining also the bound on ε^N , the marked catalogue accounts for a fraction between 90.389665% and 90.392038% of the centred initial contrast $217 - \varepsilon_{24,27}^N$; the centred residual lies in $[20.849277, 20.854427]$ cumulative occurrences. These are outward-rounded rigorous enclosures for the fixed comparator, not bounds for the error of a universal maximum law.

5 Structural classification of extreme witnesses

5.1 Boundary structure and rational anchors

The neutral prefix is constrained by multiplicativity. In particular, $L_r(p) + 1$ must be prime: if it were composite, all its prime factors would already be neutral, giving a contradiction. A related identity describes half-word suffixes. Let $h = (p-1)/2$, and extend χ by $\chi(0) = 0$ when needed. Let q be the least odd prime whose character is not 1. Then

$$\chi(h-j) = \sigma \chi(2)^{-1} \chi(2j+1).$$

Every positive odd integer below q is neutral, so the suffix of $1, \dots, h$ has length $(q-1)/2$. If $\sigma = 1$, reflection joins the two half-word suffixes into a central run of length $q-1$. If $\chi(2) = 1$, the ordinary neutral prefix ends at $q-1$; otherwise it has length 1. This explains long prefix and centre events without a random-sign hypothesis. The atlas has $L_8 \geq 2$ at approximately 24.9983% of admissible primes, instead of the naive 1/8. For $p \equiv 1 \pmod{8}$, quadratic reciprocity gives $(2/p) = 1$, hence $\chi_{p,8}(2)^4 = 1$. The possible values already lie in a subgroup of size four. This arithmetic restriction illustrates why an independent-colour prefix model can fail even before examining extreme runs.

A rational anchor w can similarly place small rational multiples of consecutive offsets close to fractions of p . The resulting signs are controlled by a few small integers. An algebraic anchor instead satisfies $P(w) = 0 \pmod{p}$ for a low-degree polynomial P ; exact identities in $\mathbb{Q}[X]/(P)$ then transfer to the residue window. A polynomial congruence alone is not a compression explanation. It must produce a smaller character-class group or another proved constraint. The distinction is visible in a retained quartic at $p = 2\,654\,052\,683$, whose constant-window rank saves no condition.

Non-monic templates also enlarge the search class. For $P = aX^2 + bX + c$ and $\beta = a\alpha$,

$$\beta^2 + b\beta + ac = 0, \quad \alpha + j = (\beta + aj)/a.$$

The monic integral equation now has step a rather than step one. Scaling all positions preserves a constant colour but can change whether that colour is neutral. A search restricted to integral monic polynomials with unit-step windows need not include these non-monic families.

5.2 The selected deep tail

The exploratory classification uses order-specific cutoffs

$$(52, 36, 28, 26, 26, 22, 20, 20) \quad \text{for } r = 2, \dots, 9.$$

Table 3: Initial deep-tail classification, before the higher-degree extensions. The final unmodelled list is smaller, as described in the text.

r	Cutoff	Prefix/centre	Rational	Quadratic	Cubic	Remaining
2	52	7675	430	531	6	44
3	36	1538	26	170	1	0
4	28	573	24	203	1	9
5	26	118	1	58	0	0
6	26	43	0	4	0	0
7	22	36	1	9	0	0
8	20	12	1	22	0	0
9	20	7	1	13	0	0

It selects 11 557 profiles and their canonical witnesses. The first classification, before the higher-degree extensions, is printed in Table 3. Categories are assigned in order; they are a reproducible bookkeeping scheme, not mutually exclusive arithmetic possibilities in nature.

The first stage left 53 profiles: 44 at order two and nine at order four. Subsequent bounded searches retained models for fifteen and then seven additional binary profiles, and for eight order-four profiles. The remaining list has 23 entries: 22 binary profiles and one order-four profile. This count includes the distinction just made between a retained algebraic model and actual compression; it should not be read as a theorem that all other profiles have one unique explanation.

The 56 deepest order-eight and order-nine profiles consist of 19 prefix/centre cases, two rational cases, and 35 quadratic cases. The latter use square classes $-47, -31, -23, 33, 73, 97, 145, 148$ with respective multiplicities $1, 4, 2, 1, 10, 14, 2, 1$. After boundary and rational cases are removed, the neutral shares rise sharply in the extreme tails: at order eight they are $612/1767$ for $M \geq 16$, $97/117$ for $M \geq 18$, and $22/22$ for $M \geq 20$; at order nine they are $202/340$, $52/53$, and $13/13$. After the retained quadratic families are also removed at $M \geq 16$, the shares become $181/1327$ and $14/152$. These contrasts suggest a mixture of ordinary bulk behaviour and special arithmetic families. Because the families were found from these data, the contrasts are descriptive evidence and not significance tests.

6 Exact character-class certification

6.1 Neutral and constant windows

Let F contain μ_r , let α be a specified algebraic root, and set $a_j = \alpha + j$ for $0 \leq j < m$, assuming $a_j \neq 0$. Define

$$H_n = \langle [a_0], \dots, [a_{m-1}] \rangle \subset F^\times / F^{\times r}, \quad H_c = \left\langle \left[\frac{a_j}{a_0} \right] : 1 \leq j < m \right\rangle.$$

The neutral event requires all characters to equal 1; the constant event only requires the ratios to be neutral. Kummer theory gives the respective extension degrees $|H_n|$ and $|H_c|$. The quotient H_n/H_c is cyclic, generated by $[a_0]$, and its order is the number of compatible constant phases. It can be 1, 2, or 4 in an order-four family. For several roots, the ratio construction is made *separately at every root*; different roots need not have the same constant phase.

At order four, cardinalities must be computed over $\mathbb{Z}/4\mathbb{Z}$, not inferred from a binary rank. An image generated by multiplication by 2 has order 2, although its reduction modulo 2 is zero. We record groups as $C_4^a \times C_2^b$, with order 2^{2a+b} .

6.2 Matching upper and lower bounds

Two certification paths are used. The earlier quadratic templates use exact multiplicative decompositions in certified number fields, with PARI's unconditional `bnfcertify(bn)` call. The higher templates do not require a complete class group or a full basis of S -units. They use the following elementary principle.

Lemma 6.1. *Let H be generated by specified classes in $F^\times/F^{\times r}$. Suppose explicit identities give a finite presentation mapping onto H , of order at most U . Character homomorphisms at suitable finite places give an image of order L . If $L = U$, then $|H| = L = U$ and the presentation is exact.*

Proof. The first map gives $|H| \leq U$, and the second gives $L \leq |H|$. If a product of the radicands, all units at the place, is an r th power, its root has valuation zero and its reduction is an r th power. Hence the finite-place characters respect every actual relation and do descend to H . $\square$

In the new order-four certificates, finite character matrices produce a candidate kernel over $\mathbb{Z}/4\mathbb{Z}$. Every generator of that kernel is then proved to be a fourth power in the specified field, with its fourth root displayed in a rational power basis. This proves the upper bound. Recomputing the character matrix proves the matching lower bound. The auxiliary primes are homomorphism certificates, not samples from which a rare-event frequency is estimated.

Irreducibility and Galois groups are separately established by factorizations at unramified small primes. For cubics, cycle types 3 and $(1, 2)$ force S_3 . For the quartics considered here, types 4 and $(1, 3)$ force S_4 . The full roots, an explicit square root of -1 , and the appropriate degree prove the splitting field composita used for the order-four calculations. A pair field also carries an inverse of the difference of its two marked roots. All identities are checked using rational quotient-field arithmetic independent of PARI's power-search routines.

The finite computations can be large even when a density is tiny. The largest new fields have degree 48. Every PARI call had a 30-second deadline. A first batch containing several fourth-power searches timed out and was not accepted. Balancing the exponents in $\{-2, -1, 0, 1\}$ and certifying one generator per call gave complete certificates. No approximation enters this change of representation.

7 Prime densities through root orbits

7.1 A general inclusion–exclusion formula

Let the polynomial defining the family belong to $\mathbb{Q}[X]$, and put $C = \mathbb{Q}(\zeta_r)$. Let N/C be its splitting field with Galois group Γ , acting on roots $\alpha_1, \dots, \alpha_d$. For a nonempty root subset S , define

$$F_S = C(\alpha_i : i \in S), \quad E_S = F_S(\sqrt[r]{\alpha_i + j} : i \in S, 0 \leq j < m), \quad h_S = [E_S : F_S].$$

For constant windows replace each marked-root family by its own ratios. All primes in this section are good primes, excluding the finite ramification and non-unit exceptions.

Proposition 7.1. *Among rational primes $p \equiv 1 \pmod{r}$, the limiting density that at least one rational root has the required window is*

$$\delta = \sum_{S \neq \emptyset} \frac{(-1)^{|S|+1}}{[F_S : C] h_S}. \quad (7)$$

When subsets are grouped into Γ -orbits, an orbit represented by S has positive coefficient

$$\frac{|\text{Orb}(S)|}{[F_S : C]} = \frac{|\text{Stab}_{\text{point}}(S)|}{|\text{Stab}_{\text{set}}(S)|}.$$

If the indicated action persists over C , the density among all rational primes is $\delta/\varphi(r)$.

Proof. Take a common Galois extension containing all roots and radicals. Fixing every root in S and making its required characters neutral is exactly the condition that Frobenius fix E_S pointwise. This subgroup has proportion $1/([F_S : C]h_S)$. Inclusion–exclusion of the root events gives the stated group average, which is a central function. Chebotarev gives its limiting prime average. The orbit coefficient follows by orbit–stabilizer. Pointwise stabilization, not setwise stabilization, supplies the root-field degree. $\square$

Thus an existence probability is not the probability at one marked root, and it is not generally the sum of the root probabilities. Changing from neutral to constant windows requires different groups; it cannot always be achieved by multiplying a neutral density by a number of phases.

7.2 Symmetric actions and root-count laws

If $\Gamma = S_d$ and all size- k subsets have a common group order h_k , then

$$\delta_{S_d} = \sum_{k=1}^d \frac{(-1)^{k+1}}{k!h_k}. \quad (8)$$

Indeed $[F_S : C] = d!/(d-k)!$ and there are $\binom{d}{k}$ such subsets. For $d = 2, 3, 4$, respectively,

$$\begin{aligned} \delta_2 &= \frac{1}{g} - \frac{1}{2h_2}, & \delta_3 &= \frac{1}{g} - \frac{1}{2h_2} + \frac{1}{6h_3}, \\ \delta_4 &= \frac{1}{g} - \frac{1}{2h_2} + \frac{1}{6h_3} - \frac{1}{24h_4}. \end{aligned}$$

If Z is the number of realizing rational roots, its binomial moments are $s_k = \mathbb{E}\binom{Z}{k} = 1/(k!h_k)$, and

$$\Pr(Z = j) = \sum_{k=j}^d (-1)^{k-j} \binom{k}{j} s_k, \quad s_0 = 1. \quad (9)$$

For all rational primes, divide the positive moments by $\varphi(r)$ while retaining $s_0 = 1$. This also gives the mass at zero for primes outside the admissible congruence class.

The field in which h_k is calculated is part of the statement. For S_4 , a pair field has degree 12 over C , while the normal field has degree 24. Extending a field can turn a nontrivial radicand class into a power. Using its group order in the larger field without checking this loss can give an incorrect density. Our new quartic h_2 values are certified in the pair fields themselves; h_3 and h_4 use the normal fields.

8 Quadratic propagation and conjugate overlap

Consider

$$Q(X) = X^2 + 31X + 216, \quad A_j = \alpha + j \text{ in } \mathbb{Q}[\alpha]/(Q).$$

It was motivated by an order-nine maximum at $p = 569\,704\,789$ with witness $w = 77\,806\,457$.

Theorem 8.1. *Let $p > 512$, $Q(w) = 0 \pmod{p}$, and let χ be a character with $\chi(-1) = 1$. If $\chi(w + j)$ has one constant value for $0 \leq j \leq 10$, that value and the characters of 2, 3, 11 are 1. There is t with $t^2 = 1$ such that the 26 signs are*

$$\chi(w + j) = \begin{cases} 1, & 0 \leq j \leq 10 \text{ or } 21 \leq j \leq 25, \\ t, & 11 \leq j \leq 20. \end{cases}$$

For every odd character order, all 26 values are therefore neutral.

Proof. Every identity in Table 4 is an exact polynomial identity after reduction modulo Q . Write $u_j = \chi(w + j)$, $c = u_0 = \dots = u_{10}$, $a = \chi(2)$, and $b = \chi(3)$. The first six identities give

$$u_{22} = 1, \quad c = a^4, \quad u_{24} = a, \quad u_{21} = a^2/c, \quad u_{23} = a, \quad b = a.$$

The additional identity $A_9A_{22} = -18$ gives $c = ab^2 = a^3$. Hence $a = b = c = 1$. The eighth identity gives $\chi(11) = 1$. The first eight also determine $u_{21}, \dots, u_{25} = 1$. The ninth gives $u_{12}^2 = 1$, and identities ten to eighteen give the same value $t = u_{12}$ at all positions $11, \dots, 20$. For an odd order, $t^2 = t^r = 1$ implies $t = 1$. Finally $0 < |Q(-j)| \leq 216 < p$ for $0 \leq j \leq 31$. If $w + j = 0$ modulo p for any such j , then $p \mid Q(-j)$, a contradiction. Choose w in $1, \dots, p-1$. The interval $w, \dots, w + 31$ cannot cross p , since then it would contain zero modulo p . It therefore lies inside the ordinary word, and all its values are nonzero. No lower bound of 32 on w is asserted or needed. $\square$

Table 4: Exact propagation identities in $\mathbb{Q}[X]/(X^2 + 31X + 216)$. Add $A_9A_{22} = -18$ to force the constant phase.

Step	Identity
1	$A_{10}A_{22} = A_4$
2	$A_7A_8 = -16A_{10}$
3	$A_9A_{24} = 2A_0$
4	$A_8A_{21} = -2A_{24}$
5	$A_{10}A_{23} = 2A_7$
6	$A_7A_{21} = -3A_{23}$
7	$A_9A_{25} = 3A_3$
8	$A_3A_6 = -22A_9$
9	$A_{12}^2A_{21} = -2A_0$
10	$A_{12}A_{18} = -A_0$
11	$A_{12}A_{20} = A_{24}$
12	$A_{10}A_{20} = -A_{16}$
13	$A_{11}A_{18} = -2A_9$
14	$A_{11}A_{19} = -A_7$
15	$A_{11}A_{21} = A_{15}$
16	$A_{10}A_{19} = -2A_{13}$
17	$A_{12}A_{17} = -2A_6$
18	$A_{14}A_{15} = -2A_3$

The parity qualification is necessary. At $p = 98\,641$, with $r = 2$ and $w = 28\,104$, one obtains the pattern $+^{11}-^{10}+^5$, although $\chi(-1) = 1$. At order nine, the ten positions $\{0, 1, 2, 3, 4, 5, 6, 7, 9, 10\}$ suffice to enforce neutrality of all 26. The independent exact group is C_9^{10} , so ten is also minimal among tests supplied by a generating set of individual radicand classes. This does not bound arbitrary algorithms using other information. The earlier determinant certificate has absolute determinant 10 for these ten positions and is invertible modulo 9; the constant-phase system has absolute determinant 2 and explains the order-two obstruction in Theorem 8.1.

Three further identities are

$$A_{27}A_4 = -108, \quad A_{28}A_3 = -132, \quad A_{31}A_0 = -216.$$

They make positions 27, 28, 31 neutral in the odd-order case. If position 26 is neutral, the local run has at least 29 positions; stops at lengths 27 or 28 are impossible for this template. None of these assertions identifies a prime's global maximum.

The conjugate root is $w' = -31 - w$. For odd orders, $\chi(w' + j) = \chi(w + 31 - j)$. Thus the two neutral windows of length 26 cover positions $0, \dots, 31$ at either root. Conversely, a

neutral 32-window yields both 26-windows. With $r = 9$, the exact group orders are $g_{26} = 9^{10}$ and $h_{26} = 9^{13}$. Consequently, among admissible primes,

$$\delta_{26} = \frac{1457}{2 \cdot 9^{13}}, \quad \delta_{29} = \frac{161}{2 \cdot 9^{13}}, \quad \delta_{32} = \frac{1}{2 \cdot 9^{13}}.$$

Therefore $\Pr(A_{32} \mid A_{26}) = 1/1457$ and $\Pr(A_{32} \mid A_{29}) = 1/161$. These count primes with at least one realization. The continuation probability from 26 to 29 at a *marked root* is instead $1/9$; it answers a different question. Appendix A gives all seventeen certified quadratic density parameters.

9 A length-53 cubic as a binary code

9.1 Complete relations and the arithmetic population

Let

$$f(X) = X^3 + 163X^2 + 8010X + 121176, \quad \alpha \in \mathbb{Q}[X]/(f), \quad A_j = \alpha + j.$$

This family arose at $p = 3836617441$ in the quadratic atlas. The group generated by its 53 radicands has dimension 41 over $\mathbb{F}_2$. Its ratio group has dimension 40, so both constant signs are possible. Twelve independent square identities give an upper bound; finite-place character rows of rank 41 give the matching lower bound. The relations can be chosen with four supports of weight 4 and eight of weight 6, total weight 64. All 4096 linear combinations have the weight distribution in Table 5. The four weight-4 relations are independent, and weights at most 6 span the complete relation space. Hence 64 is the least possible total weight of a relation basis, a statement about this criterion rather than universal computational optimality.

Table 5: Weight distribution of all 4096 relations in the length-53 cubic.

Weight	0	4	6	8	10	12	14	16	18	20	22
Count	1	4	33	183	514	1003	1118	832	346	57	5

Let R be a matrix of these twelve supports. Encode a sign as $(-1)^{x_j}$ and define

$$\mathcal{C} = \{x \in \mathbb{F}_2^{53} : Rx = 0\}.$$

This is a binary code of dimension 41.

Proposition 9.1. *In the population of all pairs (p, w) with $p \leq X$ a good prime and $f(w) = 0$ in $\mathbb{F}_p$, the relative frequency of each sign vector in $\mathcal{C}$ tends to 2^{-41} as $X \rightarrow \infty$.*

Proof. Take a Galois extension containing the roots and the square roots of every conjugate family. The stabilizer of one root projects onto the Kummer group of its 41 independent classes. Every compatible sign vector has the same number of lifts in that stabilizer. For a specified vector, sum the number of marked rational roots carrying it over the conjugate roots. This is a central function of Frobenius, and Chebotarev gives its group average. Transitivity makes the mean number of rational roots equal to one; equivalently, sum the sizes of the point stabilizers. Normalizing by all prime/root pairs yields the stated uniform law. $\square$

A prime with three rational roots contributes three pairs. This population is neither a canonical-root sample nor a record-selected sample. Additional congruence conditioning can also change the law. No error term for $X = 2^{32}$ is asserted.

9.2 Three-wise independence and exceptional tails

The dual code has no nonzero vector of weight 1, 2, or 3. The implication from this dual distance to an orthogonal array of strength three is classical [15, 16]. Every projection of $\mathcal{C}$ onto at most three coordinates is therefore uniform. These signs are independent up to triples in the limiting population of Proposition 9.1. They are not independent at all orders. The positions 22, 30, 33, 37, for example, have a product identically equal to +1. Both constant vectors belong to $\mathcal{C}$, so

$$\Pr(\text{all 53 signs constant}) = \frac{2}{2^{41}} = 2^{-40}.$$

For 53 independent fair signs this probability is 2^{-52} . Thus three-wise independence is compatible with an enhancement factor $2^{12} = 4096$ at this extreme. This ratio follows from the code dimension and the presence of its two constant vectors; the result here is the certified arithmetic code and its run law.

The full weight law follows from MacWilliams' identity. If B_w is the number of dual words of weight w , the number of code words of weight k is

$$A_k = 2^{-12} \sum_w B_w \sum_j (-1)^j \binom{w}{j} \binom{53-w}{k-j}. \quad (10)$$

It was independently checked by literal enumeration of the 2^{15} compatible words on the 27 active coordinates and convolution with the 26 free coordinates. If V counts negative signs, its first four central moments are

$$0, \quad \frac{53}{4}, \quad 0, \quad \frac{8417}{16}.$$

The independent-sign fourth moment is $8321/16$, differing by 6.

9.3 The exact local longest-run law

Weight counts alone do not determine run arrangement. The longest constant segment inside this fixed 53-window was computed by two independent integer methods. A syndrome automaton retains the last sign, current run length, and a 12-bit relation syndrome. A Fourier method uses

$$\mathbf{1}_{Rx=0} = 2^{-12} \sum_{u \in \mathcal{C}^\perp} (-1)^{u \cdot x}$$

and a signed last-sign/run-length recurrence for each dual word. Both produce the same counts for all 54 strict maximum thresholds. They were also checked against literal enumeration on 21 small test codes. Figure 2 and Table 6 show the difference from independent signs.

Table 6: Exact local longest-run tails in the fixed cubic window. Decimal columns approximate the exact counts.

k	Certified probability	Decimal	Independent signs	Ratio
10	$95\,288\,989\,062/2^{41}$	0.0433324	0.0433481	0.9996
26	$2\,064\,382/2^{41}$	$9.38772e-07$	$4.32134e-07$	2.172
40	$896/2^{41}$	$4.07454e-10$	$1.36424e-11$	29.87
52	$2/2^{41}$	$9.09495e-13$	$6.66134e-16$	1365
53	$2/2^{41}$	$9.09495e-13$	$2.22045e-16$	4096

At threshold 10 the two probabilities differ very little; at 26 the enhancement is about 2.17, and at 40 about 29.87. The tail becomes especially rigid at the end. The relations on $\{36, 37, 46, 52\}$ and $\{0, 2, 17, 30, 33, 45\}$ force the missing endpoint whenever 52 consecutive signs are constant. Hence a local maximum of exactly 52 is impossible: every constant 52-segment

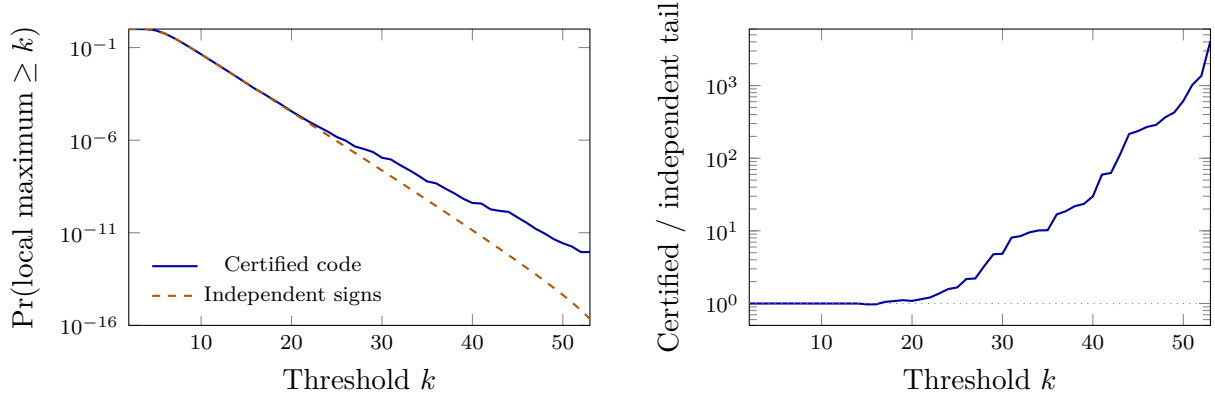


Figure 2: Exact limiting local tails in the length-53 cubic code. The law concerns prime/root pairs for this fixed window, not canonical global maxima.

completes to all 53. This is a theorem about a fixed algebraic window, not a gap theorem for the global maxima of arbitrary primes.

At the level of prime existence the same family has

$$\delta_n = 2^{-41} - 2^{-79} + \frac{1}{6 \cdot 2^{109}}, \quad \delta_c = 2^{-40} - 2^{-78} + \frac{1}{6 \cdot 2^{109}}.$$

Thus $2\delta_n - \delta_c = 1/(6 \cdot 2^{109})$. The constant density is very close to twice the neutral density, but it is not equal to it. Conjugate-root overlap supplies the correction.

10 Higher-root dependencies explained by norms

One relation makes the mechanism visible before the dimension tables. For the quartic's roots,

$$\prod_{i=1}^4 (\alpha_i + 47) = 288^2.$$

This square norm couples all four root blocks without giving a relation confined to three. The ranks below measure the remaining dependence after internal relations and common rational classes have been removed; the norm generators then identify all of it, not merely its dimension.

10.1 The two families and their dimensions

The two newest binary families are

$$\begin{aligned} P_3(X) &= X^3 - 197X^2 - 9264X - 90252, & m &= 53, \\ P_4(X) &= X^4 + 100X^3 + 3258X^2 + 34844X + 26309, & m &= 52. \end{aligned}$$

Their originating primes are 1 779 238 511 and 3 173 249 399; both observed phases are neutral. Their Galois groups are S_3 and S_4 . Work in their normal fields, and write H_i for the square classes generated by $\alpha_i + j$. For P_3 the common rational subgroup is

$$C_3 = \langle [2], [3], [5], [11], [17], [19], [59], [763] \rangle, \quad \dim C_3 = 8.$$

Each H_i has dimension 51, any pair has dimension 94, and the total has dimension 121. After quotienting by C_3 , each block has dimension 43, any two have dimension 86, and the sum has dimension 113 rather than 129.

For P_4 , the common subgroup is

$$C_4 = \langle [2], [3], [14] \rangle, \quad \dim C_4 = 3.$$

The one-, two-, three-, and four-root dimensions are respectively 44, 85, 126, 159. A block modulo C_4 has dimension 41; any three sum directly to dimension 123, but all four have dimension 156 rather than 164.

10.2 A norm description of every remaining relation

For either monic polynomial, define the rational shifted norm

$$\nu_j = \prod_{i=1}^d (\alpha_i + j) = (-1)^d P(-j).$$

Let K be the space of binary relations among all dm factors. Let K_0 be generated by relations internal to one block and by identifying the common rational classes between blocks.

Theorem 10.1. *For P_3 , K/K_0 has dimension 16; for P_4 it has dimension 8. In both cases this quotient is generated entirely by diagonal norm constraints: products of shifted norms belonging to the common rational subgroup. The Galois action on K/K_0 is trivial.*

Proof. The earlier exact square identities and matching character ranks certify the total relation spaces. Explicit square roots certify the displayed rational classes as elements of each H_i . The new verifier checks the dimensions of every proper subset of the quotient blocks. The shifted norms span dimension 27 modulo C_3 in the cubic case and dimension 33 modulo C_4 in the quartic case. Thus their kernels modulo the single-block relation spaces have dimensions $43 - 27 = 16$ and $41 - 33 = 8$. Each kernel vector supplies the same factor support in every block, followed by a common-class correction in the first block. Its actual square relation is checked as a combination of the archived exact square relations. These vectors are independent modulo K_0 and reach the full respective deficits. They therefore generate K/K_0 . A root permutation preserves a diagonal norm support; moving the common correction to another block changes it only by a generator of K_0 . Hence the quotient action is trivial. $\square$

This is a description by exact generators and dimensions, not just an observed correlation of residue signs. A short generating basis is particularly transparent. For the cubic there are seven single-position norm constraints, eight pairs, and one triple:

$$\begin{aligned} &\{7\}, \{10\}, \{12\}, \{13\}, \{26\}, \{27\}, \{28\}, \\ &\{3, 8\}, \{3, 9\}, \{17, 18\}, \{17, 19\}, \{17, 20\}, \{15, 22\}, \{15, 24\}, \{33, 48\}, \{0, 17, 23\}. \end{aligned}$$

For the quartic the basis is

$$\{22\}, \{29\}, \{31\}, \{37\}, \{38\}, \{41\}, \{47\}, \{25, 39\}.$$

For example, the cubic has $\nu_7 = 35400$ and $354\nu_7 = 3540^2$, with $354 = 2 \cdot 3 \cdot 59$ in its common subgroup. The quartic has

$$\nu_{47} = 82944 = 288^2, \quad \prod_{i=1}^4 (\alpha_i + 47) = 288^2.$$

This is a pure four-block relation. All basis relations are recorded as integer-square identities, and none of these selected bases needs a discriminant-class correction.

10.3 Probabilistic consequences under an explicit conditioning

Condition on the normal field splitting and on every common class being neutral, and choose a place over the prime uniformly to label the roots. The remaining Kummer character is uniform on the dual of the quotient class group. In the cubic case any pair of block sign vectors is independent, but all three are not. Their individual neutral probabilities are 2^{-43} , while simultaneous neutrality is 2^{-113} instead of 2^{-129} : a factor 2^{16} . In the quartic case any three block vectors are independent, while all four neutral probabilities combine to 2^{-156} instead of 2^{-164} : a factor $2^8 = 256$.

These conclusions concern the stated splitting and common-neutral population. They do not assert unconditioned pairwise independence among the original atlas maxima. The full prime-existence densities among all primes are

$$\delta(P_3) = 2^{-51} - 2^{-95} + \frac{1}{3 \cdot 2^{122}},$$

$$\delta(P_4) = 2^{-44} - 2^{-86} + \frac{1}{3 \cdot 2^{127}} - \frac{1}{3 \cdot 2^{162}}.$$

Neutral and constant modes coincide in these two families. For the cubic this can also be seen from

$$A_3 A_{12} A_{43} = 255 A_{15} A_{24}, \quad A_{18} A_{27} A_{43} = 285 A_{15} A_{26}, \quad A_{37} A_{43} A_{46} = 323 A_{22} A_{23}.$$

Each identity constrains a constant phase to the character of its coefficient. Since $255 \cdot 285 \cdot 323 = 4845^2$, the common phase must be neutral. This compresses the phase argument without discarding the full square-rank certificate.

10.4 A parity normal form and the remaining information

These dependencies admit the following linear description. It is compatible with, but logically weaker than, their explicit generation by norms.

Lemma 10.2. *Let $V_1, \dots, V_n$ be binary subspaces of dimension d , such that every sum of $n - 1$ of them is direct and $\dim(V_1 + \dots + V_n) = nd - e$. There are bases in the individual spaces for which compatible dual character coordinates satisfy exactly*

$$z_{1,j} + \dots + z_{n,j} = 0, \quad 1 \leq j \leq e.$$

The other $n(d - e)$ coordinates are free. Under the uniform law on the compatible characters, knowledge of $n - 1$ blocks leaves exactly $d - e$ free bits in the last block.

Proof. The kernel R of the sum map $\bigoplus_i V_i \rightarrow \sum_i V_i$ has dimension e . Its projection into every V_i is injective: a kernel vector with a zero i th component would be a relation among the other $n - 1$ spaces. Choose a basis $r_1, \dots, r_e$ of R . Its projected components are independent in each block; complete them to a basis there. Annihilating the r_j gives the displayed e parities, with no further relation. Each fixes one coordinate of the final block once the other blocks are known. $\square$

Apply the lemma to $V_i = H_i/C$ in the population specified above. The cubic has 16 three-block parities and 27 bits left after two block motifs are given. The quartic has 8 four-block parities and 33 bits left after three motifs are given. These are full admissible block motifs, not 43 or 41 independent original positions within a window. Explicit basis changes derived from the certified character groups are recorded in `VERIFICATION_REPORT4.json`. The linear normal form alone does not determine a Galois action; the trivial action in these two examples is established separately by Theorem 10.1 and its norm generators.

If Z counts neutral blocks in this conditioned population, its whole limiting law follows by the inversion in (9) from

$$\mathbb{E}\binom{Z}{k} = \binom{n}{k} 2^{-kd} \quad (1 \leq k < n), \quad \mathbb{E}\binom{Z}{n} = 2^{-nd+e}.$$

The supplied exact fractions distinguish this law from the unconditioned prime-existence densities above.

11 All eight new order-four density laws

The preceding residual study retained three quadratic, three cubic, and two S_4 quartic models at order four. Their exact single-root groups were already certified. The new calculation completes their pair and higher-root groups and therefore their prime densities. All relevant splitting actions are disjoint from $\mathbb{Q}(i)$, so the action remains C_2 , S_3 , or S_4 over the cyclotomic base. The largest normal composita have degree 48 over $\mathbb{Q}$; the quartic pair composita have degree 24.

Table 8 gives every group cardinality needed for (8). The numbers are base-two logarithms, so a row 48, 93, 138, 174 means $g = 2^{48}$, $h_2 = 2^{93}$, $h_3 = 2^{138}$, and $h_4 = 2^{174}$. Every row is certified by explicit fourth powers and matching characters. The neutral and constant rows are separate, and the positive moments in (9) must be divided by two to count all primes.

Table 7: The eight newly completed order-four templates. Coefficient vectors are in descending order.

Originating prime	m	Polynomial coefficients
1 065 972 469	28	(4, 168, 2485, 15006, 30001)
1 583 868 241	28	(2, 58, 411)
2 400 748 661	28	(5, 117, 640)
2 453 400 749	28	(5, 256, 3707, 16560)
2 789 197 129	29	(5, 255, 2818, 1848)
3 318 080 401	29	(2, 49, 288)
3 832 210 429	28	(9, 493, 8612, 47628)
4 129 372 337	28	(1, 41, 424, 164, -8320)

For example, the neutral density of the first quartic is exactly

$$\frac{1}{2} \left(2^{-48} - 2^{-94} + \frac{1}{6 \cdot 2^{138}} - \frac{1}{24 \cdot 2^{174}} \right).$$

The table is an exact specification of all sixteen densities, rather than decimal approximations to tiny numbers. The delivered JSON also gives all binomial moments and the full laws for the number of realizing roots, in both populations and both modes.

11.1 Short phase identities and a 30-term continuation

The quadratic phase coupling admits short identities, verified directly in the exact fields. Put $A_j = \alpha + j$ and $B_j = \bar{\alpha} + j$ for the two roots of each quadratic. For $2X^2 + 58X + 411$ the identities are

$$A_9 A_{11} = -A_{12} A_{26}, \quad A_{12} B_3 = A_9 A_{11}.$$

The first shows that even *one* constant 28-window forces $\chi_{p,4}(-1) = 1$, hence $p \equiv 1 \pmod{8}$. The second forces two constant windows to have the same phase. All four diagonal phase pairs $(1, 1), (i, i), (-1, -1), (-i, -i)$ are permitted by the exact joint group. The congruence restriction is already included in the density, so no additional factor of $1/2$ is applied.

Table 8: Exact base-two logarithms of subset group orders for every new order-four model. The pair fields are used for h_2 .

Originating prime	m	Mode	$\log_2 g$	$\log_2 h_2$	$\log_2 h_3$	$\log_2 h_4$
1 065 972 469	28	neutral	48	93	138	174
1 065 972 469	28	constant	47	91	135	171
1 583 868 241	28	neutral	38	42	—	—
1 583 868 241	28	constant	36	40	—	—
2 400 748 661	28	neutral	34	52	—	—
2 400 748 661	28	constant	32	52	—	—
2 453 400 749	28	neutral	48	90	118	—
2 453 400 749	28	constant	46	88	118	—
2 789 197 129	29	neutral	53	101	134	—
2 789 197 129	29	constant	51	98	132	—
3 318 080 401	29	neutral	38	54	—	—
3 318 080 401	29	constant	37	54	—	—
3 832 210 429	28	neutral	54	104	140	—
3 832 210 429	28	constant	52	102	140	—
4 129 372 337	28	neutral	53	102	151	183
4 129 372 337	28	constant	51	100	149	183

Here $\bar{\alpha} = -29 - \alpha$. The two constant 28-windows reflect to the intervals $0, \dots, 27$ and $2, \dots, 29$ at α . They therefore join to a constant 30-window. Conversely, a constant 30-window supplies both 28-windows. The statement holds for neutral windows as well. For $p > 1126$ the coefficient and quadratic discriminant exceptions are absent, and $0 < |P(-j)| \leq 411$ for $0 \leq j < 30$ excludes a zero or an ordinary-boundary crossing. The joined neutral group is C_4^{21} and the joined ratio group is C_4^{20} , exactly the two-root groups. The first identity puts the class of -1 in the original neutral group, so the reflected generators supply the last two positions. For the ratio groups, $B_j/B_0 = A_{29-j}/A_{29}$ and $A_{29}/A_0 = (A_{26}/A_0)/(B_3/B_0)$, which identifies the ratio groups. Among all rational primes the densities of this particular 30-window event are

$$\delta_{30,n} = 2^{-44}, \quad \delta_{30,c} = 2^{-42}, \quad \Pr(A_{30,c} \mid A_{28,c}) = \frac{1}{31}.$$

Among admissible primes the first two values double. A single 28-window need not continue: the extension is equivalent to the two-root event, not forced by one root alone.

For $5X^2 + 117X + 640$ and $2X^2 + 49X + 288$, respectively, the identities are

$$A_0 B_{16} = 16 A_8, \quad A_{10} B_{11} = A_9.$$

Since $16 = 2^4$, two constant phases ϕ, ψ satisfy $\phi\psi = \phi$, and hence $\psi = 1$; conjugation gives $\phi = 1$. These identities explain why the two joint ratio groups equal their neutral counterparts even though an isolated constant window may be non-neutral. The exact densities among all primes remain

Polynomial	Neutral density	Constant density
$2X^2 + 58X + 411$	$31/2^{44}$	$31/2^{42}$
$5X^2 + 117X + 640$	$524287/2^{54}$	$2097151/2^{54}$
$2X^2 + 49X + 288$	$131071/2^{56}$	$262143/2^{56}$

The conditional probabilities of two realizing roots given at least one constant realization are respectively $1/31$, $1/2097151$, and $1/262143$. In these three quadratic families a non-neutral constant root cannot coexist with a neutral constant root. Thus subtracting neutral from constant existence densities does count existence of a non-neutral realization, with all-prime values

$$\delta_{\text{nonneutral}} = \frac{93}{2^{44}}, \quad \frac{3}{2^{35}}, \quad \frac{1}{2^{39}},$$

in the same order. Such a subtraction is not valid for arbitrary families, where mixed neutral and non-neutral realizations may coexist. These remain limiting probabilities for fixed templates; the originating record does not establish convergence below 2^{32} .

11.2 Sixteen comparisons suffice for the discriminant-889 model

For $5X^2 + 117X + 640$, the individual ratio group is C_4^{16} . It is already generated by the ratios at

$$T = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 13, 19, 21, 24, 26, 27\}.$$

Restricting the certified character matrix to these columns gives image order 4^{16} , equal to the exact full group order. For $p > 1126$, $p \equiv 1 \pmod{4}$, and a root w of this polynomial, the sixteen comparisons

$$\chi_{p,4}(w + j) = \chi_{p,4}(w), \quad j \in T,$$

therefore force all 28 positions to have the same colour. There are seventeen distinct positions including the reference; they are not consecutive. Sixteen is minimal among generating sets made of individual ratios, since C_4^{16} needs at least sixteen generators. This is not an algorithmic complexity lower bound or a measured scan acceleration.

12 The dihedral quartic and shifted 63-windows

Consider the earlier binary quartic

$$P_D(X) = X^4 + 80X^3 + 2281X^2 + 27240X + 116640.$$

The shift $Y = X + 20$ gives $Y^4 - 119Y^2 + 4240$. Its roots are $a - 20, -a - 20, b - 20, -b - 20$, with $a^2 + b^2 = 119$ and $a^2b^2 = 4240$. The quartic field has degree 4, and its normal closure is obtained by adjoining $\sqrt{265}$, giving degree 8 and group D_4 . This action has two pair orbits: opposite pairs and adjacent pairs. Put h_o for the opposite-pair group in its quartic field, h_a for an adjacent pair in the normal field, and h_3, h_4 for three and four roots in that normal field. Equation (7) yields

$$\delta_D = \frac{1}{g} - \frac{1}{2h_o} - \frac{1}{2h_a} + \frac{1}{2h_3} - \frac{1}{8h_4}. \quad (11)$$

The exact orders for its 52-windows are

$$g = 2^{43}, \quad h_o = 2^{53}, \quad h_a = 2^{79}, \quad h_3 = 2^{89}, \quad h_4 = 2^{95}.$$

Opposite pairs are certified in the quartic field itself. Extending that field to the normal field can kill the class of 265, which is why it cannot be silently substituted in the opposite-pair term. The density is

$$\delta_D = 2^{-43} - 2^{-54} - 2^{-80} + 2^{-90} - 2^{-98}.$$

Constant and neutral events coincide. The odd-weight identity

$$9A_0A_5A_{12}A_{24}A_{27} = [\alpha(\alpha^2 + 43\alpha + 402)]^2$$

forces a constant phase to be neutral; another identity,

$$-9A_{13}A_{16}A_{28}A_{35}A_{40} = (\alpha^3 + 77\alpha^2 + 1762\alpha + 11280)^2,$$

forces $\chi(-1) = 1$ in this event.

For an opposite root $\alpha' = -40 - \alpha$, its positions $0, \dots, 51$ reflect to positions $-11, \dots, 40$ at α . With $\chi(-1) = 1$, two neutral 52-windows therefore give one neutral window on $\alpha - 11, \dots, \alpha + 51$, of length 63. Conversely that shifted 63-window gives the opposite pair. For $p > 813402$, the

exact nonzero values of $P_D(-j)$ on $-11 \leq j \leq 51$ exclude ordinary-boundary crossings; their largest absolute value is 813402.

Let B be the event that at least one opposite pair is neutral. Each pair has density $1/(4 \cdot 2^{53})$, and their intersection is the four-root event of density $1/(8 \cdot 2^{95})$. Thus

$$\delta(B) = 2^{-54} - 2^{-98}.$$

Writing A_{52} for existence of a neutral 52-window at a root,

$$\Pr(B \mid A_{52}) = \frac{5864062014805}{12003734944219221} \simeq 0.000488519785.$$

This differs from the 2^{-10} conditional continuation at a marked neutral root. The two quantities have different multiplicities. A realization implies $M_2 \geq 63$ but does not specify the global maximum or announce a new atlas occurrence.

13 The last order-four profile and bounded negative evidence

The sole order-four profile still without a retained explanatory model is

$$p = 2\,940\,141\,229, \quad M_4 = 28, \quad w = 390\,499\,731, \quad \text{phase} = -1.$$

Its immediate quadratic projection is a neutral run of exactly 29 positions, starting at $w - 1 = 390\,499\,730$. Both local ends were checked by exact modular exponentiation. Also $p \equiv 5 \pmod{8}$, so $\chi_{p,4}(-1) = -1$; reflection produces an ordinary neutral 28-window starting at 2549641471. Thus the canonical non-neutral phase does not preclude a neutral realization of the same maximum at this prime.

Reflection alone cannot improve a candidate's constant-window compression. Indeed, for $\beta = -\alpha - (m - 1)$, $b_j = \beta + j = -(\alpha + m - 1 - j)$, and

$$\langle [b_j/b_0] : 1 \leq j < m \rangle = \langle [a_j/a_0] : 1 \leq j < m \rangle.$$

Changing the common denominator from a_0 to a_{m-1} and reversing the indices does not change the ratio subgroup. This proves that reflecting each of the candidates below will not evade its proved absence of compression. It does not exclude other polynomial families. The new study searches low-height polynomials vanishing at this witness by weighted lattice reduction. It uses degrees 2 through 6, centres 0 and 14, and weights 1, 4, 14, 28: 40 specified configurations. Coefficients in $[-2, 2]$ combine each reduced basis, up to overall sign. There are 390 480 probes and 303 842 distinct primitive polynomials. Forty lowest-window-norm candidates per degree are factored, retaining their irreducible root factors.

The qualified set consists of 40 quadratic, 20 cubic, 20 quartic, 10 quintic, and 10 sextic candidates. For each, the defining polynomial is bound to its character certificate and proved irreducible over $\mathbb{Q}$ and $\mathbb{Q}(i)$. The portable verifier uses degree-preserving reductions modulo primes q , with $q \equiv 1 \pmod{4}$ for the Gaussian field. For a degree- d reduction $\bar{f}$ it checks

$$X^{q^d} \equiv X \pmod{\bar{f}}, \quad \gcd(\bar{f}, X^{q^{d/\ell}} - X) = 1 \quad \text{for every prime } \ell \mid d.$$

At a split Gaussian prime the residue field is $\mathbb{F}_q$, so the same irreducible reduction proves irreducibility over $\mathbb{Q}(i)$. The 200 stored witnesses use primes at most 59 for $\mathbb{Q}$ and 181 for $\mathbb{Q}(i)$. This identifies the single root field in which 192 finite-place character rows certify the full possible neutral group C_4^{28} and ratio group C_4^{27} . Thus none of these 100 candidates compresses the 28-window. For orientation, a quadratic candidate is $19X^2 + 1555X + 15087$ and a cubic candidate is $X^3 + 78X^2 + 891X - 5289$. Their small coefficients do not create useful fourth-power dependencies.

This is stronger than a failed heuristic search at these *qualified candidates*: each has a maximal character image, so the absence of compression for that polynomial is proved. It is weaker than an exhaustive exclusion of low-degree algebraic structure: only the stated coefficient combinations and selected factors were qualified. It also says nothing about the 22 remaining binary profiles. A wider template search, a different anchor, or several interacting windows may still explain them. There is no justification here for calling the residual profile random or for inventing a universal exceptional mechanism.

14 Consequences and research directions

The evidence supports a layered view of residue runs. Short global patterns have geometric descriptions such as (2). Prime-size variation explains much of the raw dependence of bulk maxima. Boundary prefixes and rational anchors then contribute deterministic events. Some deep tails lie on algebraic templates with far fewer character conditions than their lengths suggest. Finally, conjugate template families interact through norms, including dependencies invisible at lower orders of marginals. These statements have different scopes and should not be combined into one fitted extreme-value law.

For clarity, one critical conjecture pursued by this programme chooses a prime uniformly from $\mathcal{P}_3(X) = \{X \leq p < 2X : p \equiv 1 \pmod{3}\}$ and an integer threshold $k = \log_3 X + O(1)$. The independent-colour half-period intensity is $\lambda_{p,k} = (p-1)/3^k$. Its proposed void target is

$$\Pr_{\mathcal{P}_3(X)}(M_3(p) < k) - \mathbb{E}_{\mathcal{P}_3(X)} e^{-\lambda_{p,k}} \longrightarrow 0.$$

This is a research conjecture, not a theorem of this article. Prime-size variation is retained inside the mixture; $\mathbb{E}e^{-\lambda}$ is not replaced by $e^{-\mathbb{E}\lambda}$. Reflection also prevents interpreting the full-word start count as a simple Poisson variable merely by doubling a half-word mean. First intensity, dependence of occurrences, and a vanishing count probability each require an argument. Fixed templates, local codes, and short trace moments do not supply these estimates uniformly at growing length. The bounded catalogue removal above leaves a measured residual, and supplies no automatic asymptotic estimate for it.

Several focused questions now have concrete starting points. For a general S_d family, one can ask when the quotient of the relation space by internal and common-class identifications is generated by diagonal norms. Theorem 10.1 answers this for two families, not for every low-degree polynomial. A criterion in terms of the norm map and Galois modules could predict the first order at which joint dependence appears and identify possible nontrivial quotient representations.

Another question is to classify phase coupling for order-four families. The quadratic examples show identical phases, forced joint neutrality, and single-root non-neutral phases within the same basic framework. Smith normal forms of the combined class groups give a finite algebraic formulation; a conceptual account of when the coupling is forced would be more informative than a list of matrices. The completed cubic and quartic density laws now provide further exact examples for this question.

The local code viewpoint also offers a controlled way to study extremes. Weight enumerators describe sign imbalance, while syndrome and Fourier automata describe run placement. The next step is not to treat all atlas positions as copies of a fixed code, but to understand how many template windows exist, how they overlap, and how they compete with ordinary bulk runs. Certified finite-range errors for the rare prime-density terms would be needed before comparing them quantitatively to the observed deep tail. No such error bound is supplied here.

Finally, the residual search suggests a stopping criterion for computation. A low-degree modular fit is plentiful; a proved reduction in the size of the power-class group generated by the window is the meaningful filter. The 100 maximal-image candidates do not justify another

broad census, but they do motivate more selective searches using several product identities or multi-window geometry. No extension to 2^{33} , new GPU scan, or calculation estimated to take several hours was launched for this manuscript.

15 Reproducibility, provenance, and manuscript status

The publication repository contains the standalone `paper/PAPER_EN.tex`, its PDF, the complete `data/derived/ATLAS_CATALOGUE_K3_K64.csv`, and `data/derived/ORDER_FOUR_DENSITIES.csv`. Scientific figures are supplied separately in SVG and PNG. Their numerical coordinates are also embedded directly in the LaTeX source, which has no external figure or bibliography dependency.

`analysis/` contains the preceding descriptive analysis and exploration, including marginal histograms, prime-size bands, boundary losses, A4 summaries, and selected tail catalogues. `reproducibility/anteriorities/` contains the mathematical certificates and portable verifiers from the preceding theoretical work. The arguments and exact identities used here are presented in the manuscript and its reproducibility package. Host logs, local absolute paths, credentials, execution histories, and installed third-party libraries are excluded.

The complete $[1, 2^{32})$ atlas is supplied as pinned release assets: 70 scientific packs are byte-identical to the sealed originals, and two SQLite indices are derived for publication. The derived indices retain every scientific block extent, commitment, count, aggregate, and audit certificate, but omit execution metadata. `data/atlas/manifest.json` lists asset lengths and SHA256 hashes, the original seal identity, and the derivation of the indices. `scripts/download_data.py` retrieves the assets; `scripts/verify_atlas.py` checks their hashes, coverage, chains, aggregates, and sample frames. The two committed sample blocks are deliberately non-contiguous and are not a smaller census. The original C++ analysis in `analysis/` can decode every frame and regenerate the descriptive aggregates through `scripts/reproduce_analysis.py`. Frozen GPU sources and exact CPU workers are in `src/census/`; bounded CPU regeneration is demonstrated by `scripts/recompute_small.py`. A full GPU census is an explicit, separate operation, not a default verification step.

`reproducibility/evidence/new/` contains exact fourth-power roots, finite character matrices, common-class square identities, short norm bases, root-count laws, local code counts, and residual negative certificates. `reproducibility/proofs/verify_synthesis.py` checks the new algebra using only the standard Python library; it requires no PARI, network, GPU, or access to the raw atlas. Its checks include 445 fourth-power identities, 808 064 finite character values, the norm-generator spans, and the local weight law. It also checks the 200 irreducibility witnesses and the polynomial binding for all 100 residual models. They can be replayed separately with `reproducibility/proofs/residual_irreducibility.py`; their small primes and Frobenius remainders are stored in `reproducibility/evidence/new/RESIDUAL_IRREDUCIBILITY.json`. The C++ automaton source and bounded PARI scripts are included for regeneration rather than required for reading a certificate. This verification is an exact arithmetic audit, not a formal proof assistant development. The additional `reproducibility/proofs/verify_report4.py` checks the short quadratic identities, the sixteen comparison generators, and explicit parity normal forms.

The separate ternary supplement retains its one million profile records, counts and long-run positions in `data/ternary-million/`. Its `README.md` identifies the $p < 2^{34}$ collector in `src/ternary_catalogue/`, its 58 frozen source files, configuration and historical qualification. Run `scripts/verify_ternary.py` to check the fixed lists and retained records. This collector is not a wider-input parser for the distinct $p < 2^{33}$ worker.

The additional curated package `reproducibility/ternary_diagnostics/` contains the fixed-coordinate diagnostics and the same-filter finite comparator. `scripts/verify_editorial.py`, with `-scope atlas`, replays them; its explicit `-catalogue` option reclassifies all retained

ordinary positions on CPU and compares the archived hit lists. It does not rescan the full character words. The manuscript source compiles independently. The editorial provenance and new-file hashes are recorded separately from the original publication manifest in `docs/EDITORIAL_PROVENANCE.json` and `docs/EDITORIAL_SHA256.json`. The original complete-atlas and million-profile scientific files, their seals, and the earlier primary tests are unchanged.

The manuscript was prepared with AI assistance for computation, proof development, literature retrieval, and English writing. Prior primary statistical tests remain closed; no selected template is relabelled as a prospective test, and no universal growth law is claimed. The mathematical examples have explicit proofs or certificates. Brice Pouly is the human author and assumes responsibility for the manuscript, code, data, references and conclusions. Exact arithmetic checks do not replace specialist review of the arguments. The accompanying article archive provides the files, commands and commitments needed for bounded reproduction; full atlas assets are a separate data product identified by the manifest. Repository deposit does not constitute peer review or establish a public disclosure date.

A The seventeen quadratic density templates

For $Q = X^2 + AX + B$ with discriminant $D = A^2 - 4B$, the following fixed templates use group orders g at one root and h at both. Their neutral density among $p \equiv 1 \pmod{r}$ is $1/g - 1/(2h)$, and among all primes it is divided by $\varphi(r)$. Here the quadratic and cyclotomic fields are disjoint. Entries list orders; they need not have free C_r invariant factors. For example an order-eight group of order 8^9 can be $C_8^8 \times C_4 \times C_2$.

Table 9: Certified neutral quadratic templates. $B = (A^2 - D)/4$.

r	A	B	D	m	g	h
8	19	102	-47	20	8^8	8^8
8	19	98	-31	20	8^8	8^8
8	19	96	-23	20	8^8	8^8
8	19	72	73	20	8^7	8^7
8	19	66	97	20	8^7	8^7
8	19	54	145	20	8^8	8^8
8	17	64	33	21	8^9	8^{11}
8	19	98	-31	22	8^9	8^{10}
8	23	114	73	22	8^8	8^9
8	19	66	97	24	8^9	8^{11}
9	19	98	-31	20	9^8	9^8
9	19	96	-23	20	9^8	9^8
9	19	72	73	20	9^7	9^7
9	19	66	97	20	9^7	9^7
9	19	54	145	20	9^8	9^8
9	26	132	148	25	9^{10}	9^{11}
9	31	216	97	26	9^{10}	9^{13}

B The fifteen previously completed cubic templates

Each row has Galois group S_3 and a cyclotomically disjoint normal field. In Table 11, (a, b, c) means $(g, h_2, h_3) = (r^a, r^b, r^c)$; both modes are shown. The full polynomials are listed as coefficient vectors, and exact fractions and upper and lower certificates are in the preserved `DENSITES_CUBIQUES_EXACTES.json`. A phase-order entry 1 forces neutrality; entry 2 permits both binary constant phases and does not mean the prime densities are exactly in a ratio of two.

Table 10: The fifteen cubic coefficient vectors, in descending order.

Originating prime	Polynomial coefficients
496 108 447	(1, 65, 1342, 9000)
830 419 297	(1, 54, 815, 3906)
1 195 052 273	(1, 37, 390, 1143)
1 227 020 759	(1, 25, 66, -1008)
2 889 743 819	(1, 60, 1107, 7366)
2 936 508 541	(1, 74, 1633, 10314)
3 594 614 377	(1, 43, 586, 2592)
4 105 332 313	(1, 70, 1416, 8568)
910 400 021	(1, 94, 1944, 11620)
1 363 287 157	(1, 159, 6578, 82500)
1 478 917 823	(1, -20, -2317, -26488)
1 862 482 781	(1, -15, -3136, -45996)
2 080 022 621	(1, 187, 9104, 132770)
3 836 617 441	(1, 163, 8010, 121176)
3 920 044 319	(1, 28, -1061, -27288)

Table 11: Exact cubic group exponents in both modes: $(g, h_2, h_3) = (r^a, r^b, r^c)$.

Originating prime	r	m	Neutral (a, b, c)	Constant (a, b, c)	Phase order
496 108 447	3	37	(22, 42, 58)	(22, 42, 58)	1
830 419 297	2	54	(36, 67, 90)	(36, 67, 90)	1
1 195 052 273	2	55	(48, 94, 132)	(48, 94, 132)	1
1 227 020 759	2	53	(38, 73, 102)	(38, 73, 102)	1
2 889 743 819	2	52	(47, 87, 115)	(47, 87, 115)	1
2 936 508 541	2	52	(48, 89, 117)	(47, 88, 117)	2
3 594 614 377	4	30	(18, 33, 42)	(18, 33, 42)	1
4 105 332 313	2	54	(40, 75, 102)	(40, 75, 102)	1
910 400 021	2	53	(51, 96, 131)	(50, 95, 131)	2
1 363 287 157	2	52	(45, 84, 110)	(44, 83, 109)	2
1 478 917 823	2	53	(47, 89, 120)	(47, 89, 120)	1
1 862 482 781	2	53	(52, 101, 138)	(52, 101, 138)	1
2 080 022 621	2	52	(51, 99, 137)	(50, 98, 136)	2
3 836 617 441	2	53	(41, 78, 109)	(40, 77, 109)	2
3 920 044 319	2	54	(51, 97, 131)	(50, 96, 131)	2

C Additional single-root models

Table 12 records the twenty models from the first higher-degree extension and twelve from the next extension, including alternative fits at a common witness. All use $r = 2$. The coefficient vector is in descending order; n and c are the neutral and constant square-class ranks. The maximum possible constant rank is $m - 1$, so $m - 1 - c$ is the saving in binary conditions. A zero saving is displayed rather than counted as compression. Full product identities, power roots, and finite characters are in the corresponding archived JSON files. These are single-root certificates; a density of existence over primes requires conjugate-root groups as well. Densities are claimed only where those have been completed in the article or its cubic appendix.

Table 12: All 32 models from the two higher-degree binary extensions, including alternatives at the same witness.

Originating prime	Descending coefficient vector	m	n	c
910 400 021	(1, 94, 1944, 11620)	53	51	50
910 400 021	(1, 131, 6295, 130905, 987740)	53	51	50
1 363 287 157	(1, 108, 4376, 79104, 541056)	52	47	47
1 363 287 157	(1, 159, 6578, 82500)	52	45	44
1 428 586 013	(1, 80, 2281, 27240, 116640)	52	43	43
1 428 586 013	(1, 120, 5214, 95741, 607836)	52	46	46
1 440 133 553	(1, 114, 4753, 85452, 558016)	52	43	42
1 478 917 823	(1, -20, -2317, -26488)	53	47	47
1 532 710 759	(1, 147, 8004, 191188, 1687968)	52	45	44
1 862 482 781	(1, -15, -3136, -45996)	53	52	52
1 862 482 781	(1, 111, 4196, 63396, 359640)	53	50	50
2 080 022 621	(1, 187, 9104, 132770)	52	51	50
2 194 988 227	(1, 105, 3819, 56903, 297012)	52	48	48
2 194 988 227	(1, 108, 4223, 69528, 392772)	52	49	49
2 197 584 787	(1, 54, 940, 5466, 6883)	53	47	47
3 354 019 703	(1, 136, 6527, 130196, 906876)	55	48	47
3 732 899 699	(1, 106, 3883, 58938, 309456)	52	43	43
3 836 617 441	(1, 163, 8010, 121176)	53	41	40
3 920 044 319	(1, 28, -1061, -27288)	54	51	50
4 283 876 587	(1, 76, 2041, 21726, 73920)	52	48	47
500 177 861	(1, 107, 4112, 66262, 379356)	52	47	46
500 177 861	(1, 115, 4313, 57861, 224350)	52	48	47
500 177 861	(1, 115, 4827, 86717, 531860)	52	52	51
1 657 464 163	(1, 96, 3191, 41056, 144240)	53	52	51
1 657 464 163	(1, 130, 5957, 110909, 669309)	53	53	52
1 779 238 511	(1, -197, -9264, -90252)	53	51	51
1 779 238 511	(1, 118, 4543, 59466, 94680)	53	53	52
2 088 598 739	(1, 92, 3435, 63104, 432976)	52	50	49
2 088 598 739	(1, 98, 3463, 51510, 280488)	52	52	51
2 654 052 683	(1, 73, 1773, 14931, 9240)	52	52	51
3 173 249 399	(1, 100, 3258, 34844, 26309)	52	44	44
3 696 971 783	(1, 126, 5593, 102996, 657004)	55	54	53

D Execution provenance

The frozen scanner’s GPU batches contain at most 128 primes; CPU batches and commits contain at most 4096, with bounded worker deadlines. Two interrupted executions were followed by documented, human-authorized execution-policy changes. A rejected GPU batch was replayed

completely at a smaller denominator epoch. Partial output was not accepted. These changes did not alter the scientific engine, the domain, or already committed blocks. The scan took approximately 35.7 cumulative hours; the CPU quadratic quadruplet computation took approximately 4.4 hours. These are campaign times, not comparative performance measurements. The controlled comparisons in [3, Section 7] have separate inputs and timing boundaries.

The release `atlas32-v1.0.0`, frozen scanner, original data manifest, certificates and reproduction commands are identified in Section 15. Manuscript revisions and the 200 additional irreducibility witnesses have a separate editorial manifest. The finite computational controls verify their stated finite identities; they do not prove the proposed critical maximum law.

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