

Prescribed Eigenvalues in Power Bands of Finite Prime-Cover Graphs

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Abstract

Let G_X be the graph on the positive integers at most X , with an edge between n and np whenever p is prime and $np \leq X$. Write $\lambda_1(X) \geq \lambda_2(X) \geq \dots > 0$ for its positive adjacency eigenvalues, counted with multiplicity. We prove that, for every fixed $21/61 < \theta < 1/2$ and every integer $X^\theta / \log X \leq a \leq X^\theta$,

$$\lambda_a(X)^2 = \pi(X/a) + O_\theta\left(\frac{a}{\log X}\right),$$

where π is the prime-counting function. This identifies the eigenvalue at the prescribed rank a throughout a polynomially growing band of centres. For every $0 < \theta < 1/2$, ordered mean-square bounds give a weaker estimate outside $o(X^\theta)$ centres below X^θ . When $1/3 < \theta < 1/2$, the same $O_\theta(a/\log X)$ error holds outside $o(X^\theta)$ centres using only global prime estimates. The results extend to deletion of any fixed finite set of prime directions.

Edges labelled by primes exceeding $\sqrt{X}$ form disjoint stars. Retaining their first-exit interactions gives local comparison operators and a family of approximate eigenvectors. Bounds on their overlaps and residuals control eigenvalue counts across clusters, allowing ranks to be recovered where the global perturbation bound no longer separates neighbouring star levels. The Baker–Harman–Pintz lower bound and the Brun–Titchmarsh upper bound supply the short-interval prime counts used in the uniform every-centre theorem.

1 Introduction

For $X \geq 1$, we study the graph on the positive integers at most X in which two vertices are joined when their ratio is prime. Its edges are

$$n \longleftrightarrow np, \quad p \text{ prime, } np \leq X. \quad (1.1)$$

These are the cover relations of divisibility, so the graph is the Hasse diagram of the divisibility poset on $\{1, \dots, \lfloor X \rfloor\}$. Splitting the edges at $Y = \lfloor \sqrt{X} \rfloor$ gives an explicit reference spectrum. An integer at most X cannot contain two distinct prime factors larger than Y , so the edges labelled by $p > Y$ form disjoint stars. The star centred at $a \leq Y$ has degree

$$d_{a,X} = \pi(X/a) - \pi(Y) \quad (1.2)$$

and positive eigenvalue $\mu_a = \sqrt{d_{a,X}}$. Writing $j(a) = a$ for the arithmetic rank in the undeleted graph, this suggests the ordered profile

$$\lambda_{j(a)}(X)^2 \approx \pi(X/a). \quad (1.3)$$

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The companion paper [15] treats fixed centres. The first-order interaction of a star mode with the small-prime adjacency vanishes, allowing the remaining eigenvalue correction to be evaluated through constant order. That work also proves localization of the corresponding eigenvectors. Here the centre grows as a power of X , and we seek the eigenvalue at its prescribed global rank. Comparing the smooth adjacent-star gap μ_a/a with the global perturbation norm gives the scale $a \asymp X^{1/6}$. Beyond this scale, that comparison no longer establishes separation of neighbouring levels. We recover the ranks by comparing families of approximate eigenvectors and counting eigenvalues across spectral clusters.

We state the result with a fixed finite set S of deleted primes. Put

$$V_S(X) = \{n \leq X : p \nmid n \text{ for every } p \in S\},$$

and let $A_{S,X}$ be the adjacency matrix of (1.1) on $V_S(X)$, with only prime labels outside S . Write

$$\pi_S(t) = \#\{p \leq t : p \text{ prime, } p \notin S\},$$

and enumerate the allowed positive integers as $a_1 < a_2 < \dots$. Equivalently,

$$j_S(a) = \#\{n \leq a : n \in V_S(\infty)\} \quad (1.4)$$

is the one-based arithmetic rank of an allowed centre a . The positive eigenvalues of $A_{S,X}$, counted with multiplicity, are denoted by $\lambda_1(A_{S,X}) \geq \lambda_2(A_{S,X}) \geq \dots > 0$.

For $Y = \lfloor \sqrt{X} \rfloor$, define

$$d_{S,X}(c) = \pi_S(X/c) - \pi_S(Y) \quad (1.5)$$

when $c \leq Y$, and set it equal to zero when the corresponding star is inactive. For an active centre a , put

$$r_{q,X}(a) = \frac{d_{S,X}(aq)}{d_{S,X}(a)}, \quad s_{q,X}(a) = \frac{d_{S,X}(a/q)}{d_{S,X}(a)}, \quad (1.6)$$

where $q \leq Y$ is an allowed prime in the first expression and $q \mid a$ is prime in the second. Inactive up-stars have $r_{q,X}(a) = 0$. For the ranges below, all nonzero denominators are separated from zero once X is sufficiently large. Define the first-exit correction

$$M_{S,a,X} = 4 \sum_{\substack{q \leq Y \\ q \notin S}} \frac{r_{q,X}(a)}{1 - r_{q,X}(a)} + \sum_{\substack{q \mid a \\ q \text{ prime}}} \frac{5 - s_{q,X}(a)}{1 - s_{q,X}(a)}. \quad (1.7)$$

Our main theorem identifies every prescribed ordered eigenvalue in a terminal power band below the square-root boundary.

Theorem 1.1 – prescribed eigenvalues in a terminal power band

Fix a finite set S of primes and

$$\frac{21}{61} < \theta < \frac{1}{2}.$$

There is a constant $C_{S,\theta}$ such that, for all sufficiently large X , every allowed centre

$$\frac{X^\theta}{\log X} \leq a \leq X^\theta$$

obeys

$$\boxed{|\lambda_{j_S(a)}(A_{S,X})^2 - \pi_S(X/a) - M_{S,a,X}| \leq C_{S,\theta} \frac{a}{\log X}.} \quad (1.8)$$

Since $M_{S,a,X} \ll_{S,\theta} \log X = o(a/\log X)$ uniformly on the terminal band, the correction is smaller than the stated error. The same estimate therefore has the following equivalent form.

Corollary 1.2

Under the hypotheses of Theorem 1.1, every allowed centre in the terminal band satisfies

$$\boxed{\lambda_{j_S(a)}(A_{S,X})^2 = \pi_S(X/a) + O_{S,\theta}\left(\frac{a}{\log X}\right).} \quad (1.9)$$

Among all allowed centres $a \leq X^\theta$, the only centres not covered by (1.8)–(1.9) lie in the explicit initial segment $a < X^\theta/\log X$, whose size is

$$O_{S,\theta}\left(\frac{X^\theta}{\log X}\right).$$

Ordered mean-square estimates give a density-one result throughout $0 < \theta < 1/2$.

Theorem 1.3 – density-one estimate on sub-square-root power bands

Fix a finite set S of primes and

$$0 < \theta < \frac{1}{2}.$$

Then there is an exponent $D = D(S, \theta)$ such that, outside $o_{S,\theta}(X^\theta)$ allowed centres $a \leq X^\theta$,

$$\begin{aligned} \lambda_{j_S(a)}(A_{S,X})^2 &= \pi_S(X/a) + M_{S,a,X} \\ &+ O_{S,\theta}\left(\frac{a}{\log X} + (\log X)^D \sqrt{\frac{X}{a \log X}}\right). \end{aligned} \quad (1.10)$$

More precisely, for each fixed $R > 0$, D may be chosen depending on S, θ, R so that the terminal-band exceptions number $O_{S,\theta,R}(X^\theta/(\log X)^R)$. Counting all centres below X^θ additionally omits the initial segment $a < X^\theta/\log X$. On every retained dyadic band the error is $o(\pi_S(X/a))$.

Two perturbative estimates provide the comparison scales. The star forest and Weyl's inequality give, for every fixed $0 < \theta < 1/2$,

$$\sup_{a \leq X^\theta} \left| \frac{\lambda_{j_S(a)}(A_{S,X})^2}{\pi_S(X/a)} - 1 \right| \ll_{S,\theta} X^{-(1-2\theta)/4}. \quad (1.11)$$

This is Proposition 2.3. On a dyadic terminal band $a \asymp X^\theta$, its squared-error scale is $X^{3/4-\theta/2}/\log X$. Theorem 1.1 reduces that scale to $X^\theta/\log X$, an improvement by the factor

$$X^{3(1-2\theta)/4}.$$

A positive-star prefix and a separated-complement Schur bound give the stronger benchmark $O_{S,\theta}(\sqrt{X}/\log X)$ in Proposition 2.4. The table compares both benchmarks with the results of this paper.

Estimate	Centres covered	Squared-error scale on $a \asymp X^\theta$
Raw Weyl, Proposition 2.3	Every centre, $0 < \theta < 1/2$	$X^{3/4-\theta/2}/\log X$
Positive-star block, Proposition 2.4	Every terminal-band centre, $0 < \theta < 1/2$	$\sqrt{X}/\log X$
Theorem 1.1	Every terminal-band centre, $21/61 < \theta < 1/2$	$X^\theta/\log X$
Theorem 1.3	Density one, $0 < \theta < 1/2$	$X^\theta/\log X + (\log X)^D X^{(1-\theta)/2}/\sqrt{\log X}$
Corollary 10.3	Every terminal-band centre, $0 < \theta < 1/2$	$X^\theta/\log X + (\log X)^{C'} [X^{(1-\theta)/2} + X^{1/4+\theta/2}]/\sqrt{\log X}$

Theorem 1.3 improves the positive-star block scale by a positive power for every fixed $0 < \theta < 1/2$, as quantified below. Corollary 10.3 gives a weaker every-centre improvement on the same full range.

Corollary 1.4 – density-one improvement over the block baseline

Put

$$\delta_0 = \min \left\{ \frac{1}{2} - \theta, \quad \frac{\theta}{2} \right\} > 0.$$

Then, for every $0 < \delta < \delta_0$, outside $o_{S,\theta}(X^\theta)$ allowed centres $a \leq X^\theta$,

$$\left| \lambda_{j_S(a)}(A_{S,X})^2 - \pi_S(X/a) - M_{S,a,X} \right| \ll_{S,\theta,\delta} X^{1/2-\delta}. \quad (1.12)$$

The exponent comparison follows from Theorem 1.3 and is proved in Section 10.

Equation (1.8) identifies the eigenvalue at the specific position $j_S(a)$. We recover this rank by comparing eigenvalue counts across clusters with the ordered energies of the local comparison operators, called molecules below. Two-sided short-interval prime counts control the arithmetic ordering at the required width.

The target width also explains why this collective step is necessary. On a dyadic band $a \asymp B$, the smooth PNT profile has mean spacing between its squared targets

$$\asymp_{S,\theta} \frac{X}{B^2 \log X},$$

whereas the error in (1.8) is $B/\log X$. Thus one theorem-sized window contains approximately

$$K_B = \frac{B^3}{X}$$

model levels. For $B = X^{\theta+o(1)}$ with $\theta > 1/3$, this number tends to infinity. The error windows overlap, so we use threshold counts across clusters of K_B labels to recover the prescribed ranks. The mean spacing here is the derivative scale of the smooth PNT profile; consecutive discrete prime-counting targets may coincide.

The same argument has a counting form. Corollary 9.4 shows that a buffered interval of terminal-band centres and its corresponding arithmetic energy window contain the same number of modes up to $O(B^3/X)$ boundary labels.

The proof begins with a two-stage Schur comparison between the full graph and the one-exit compression (Section 3). This comparison holds throughout $a \leq X^{1/2-\varepsilon}$. Section 4 constructs the local molecules and evaluates their energies. Sections 5–6 control the Gram matrix and residuals of the molecule family, allowing the vectors to be used together. The proof then divides into two parts. Sections 7–9 establish the sharp every-centre theorem by eliminating spectrally distant modes and bounding the number of internal eigenvalues that can cross a cluster threshold. Short-interval prime counts supply

enough arithmetic labels on either side to determine the rank. Section 10 proves the density-one theorem directly from an ordered mean-square comparison above a separated complement.

The arithmetic input determines the lower threshold in Theorem 1.1. At the target width, the relevant prime interval has base $x \asymp X^{1-\theta}$ and length $h \asymp X^\theta$. The Brun–Titchmarsh upper bound [12] and the Baker–Harman–Pintz lower bound $\pi(x) - \pi(x - x^{21/40}) \gg x^{21/40}/\log x$ [16] give the two-sided count used in the proof. The strict inequality $\theta/(1-\theta) > 21/40$ gives $\theta > 21/61$. Using the Guth–Maynard asymptotic [11] at exponent $17/30$ would instead require $\theta > 17/47$. More generally, a positive-proportion prime lower bound in intervals of length $x^{\kappa+\varepsilon}$ gives the arithmetic condition $\theta > \kappa/(1+\kappa)$. The cluster count also requires $\theta > 1/3$, since $K_B = B^3/X$ must tend to infinity. The density-one theorem uses global prime estimates and sorting contraction, which accommodates repeated prime-counting targets.

Cover-matrix spectra of structured product posets were studied by Anđelić and da Fonseca [1]. Related divisibility graphs usually retain every comparable pair or restrict to the divisor lattice of one integer [2,3]; the present graph is the cover graph of the radial truncation $([1, X], |)$. Hub-controlled extreme spectra occur in sparse random graphs [4,5]. Alt, Ducatez, and Knowles prove Poisson statistics and exponential localization at the spectral edge of sparse Erdős–Rényi graphs [6], while Frieze and Tsourakakis obtain a star-forest decomposition in a random Apollonian model [7]. The present result is deterministic and labels a growing family of ordered modes. Helfgott and Radziwiłł [8] study a different prime-divisibility operator with additive moves and a global expansion objective. The Schur complement, min–max principle, and inertia arguments used below are standard finite-dimensional tools [9,10].

Kline’s sparse Redheffer descendant is an arithmetic matrix with $O(X)$ nonzero entries. Its determinant represents the Mertens function, and its two dominant eigenvalues have leading scale $\pm\sqrt{\pi(X)}$ with a logarithmic correction [13,14]. That operator retains a distinguished Redheffer interaction. The present paper treats the symmetric zero-diagonal cover adjacency and identifies a family of order X^θ ordered positive eigenvalues.

A companion Lean 4 formalization of Theorem 1.3 and Corollary 1.4 is registered in Palomar [17, version 1]. Its scope is described in the formal-verification section.

2 The large-prime forest and the small-prime norm

Let $L_{S,X}$ contain the edges labelled by primes $p > Y$, and put

$$A_{S,X} = L_{S,X} + H_{S,X}. \quad (2.1)$$

Figure 1 shows the split for $X = 16$: solid large-prime edges form the disjoint stars, while dashed small-prime edges comprise the remainder.

Proposition 2.1 – exact star decomposition

The graph with adjacency $L_{S,X}$ is a disjoint union of stars. The nonempty star centred at $a \leq Y$ has degree $d_{S,X}(a)$, and its two nonzero normalized eigenvectors are

$$u_a^\pm = \frac{1}{\sqrt{2}} \left(\delta_a \pm \frac{1}{\sqrt{d_{S,X}(a)}} \sum_{\substack{p > Y, p \notin S \\ ap \leq X}} \delta_{ap} \right), \quad (2.2)$$

with eigenvalues $\pm\mu_a$, where $\mu_a = \sqrt{d_{S,X}(a)}$.

Proof.

An edge labelled by $p > Y$ has lower endpoint at most Y and upper endpoint above Y . An integer at most X cannot have two distinct prime divisors larger than Y . Hence every large-prime leaf has

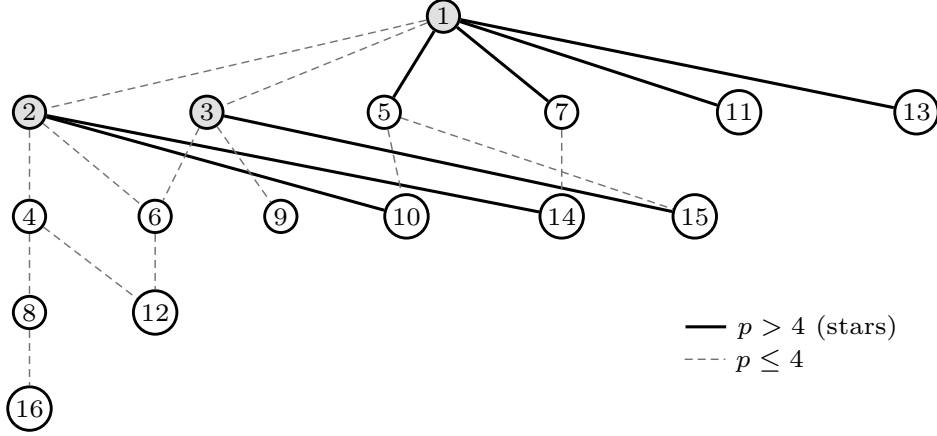


Figure 1: The prime-cover graph at $X = 16$, split at $Y = 4$. Solid edges belong to $L_{S,X}$; dashed edges belong to $H_{S,X}$.

a unique centre and cannot itself be a centre. The components are stars, and (2.2) is the standard spectrum of $K_{1,d}$. $\square$

The remaining graph has the sharp norm scale needed throughout the paper.

Proposition 2.2 – small-prime norm

For fixed S ,

$$\|H_{S,X}\| \asymp_S \frac{X^{1/4}}{\sqrt{\log X}}. \quad (2.3)$$

Proof.

The principal subgraph on the root and the allowed primes at most Y is $K_{1,\pi_S(Y)}$, which gives the lower bound, using the prime number theorem [12]. For the upper bound use the positive Schur weight $w(n) = n^{-\alpha}$. Uniformly for α near $1/2$, partial summation and Chebyshev's estimate [12] give

$$\sum_{p \leq Y} p^{-\alpha} \ll \frac{Y^{1-\alpha}}{\log Y}. \quad (2.4)$$

For $n \leq X < (Y+1)^2$, the incoming row sum is $\sum_{p|n, p \leq Y} p^{-\alpha}$. Prime divisors at most $Y^{1/2}$ contribute $o(Y^\alpha)$. For $Y \geq 4$, at most four distinct prime divisors of n exceed $Y^{1/2}$: five would have product greater than $Y^{5/2} \geq (Y+1)^2 > X$. Thus the incoming sum is $O(Y^\alpha)$. With

$$\alpha = \frac{1}{2} - \frac{\log \log Y}{2 \log Y},$$

both row contributions are $O(Y^{1/2}/\sqrt{\log Y})$. The weighted Schur test proves the upper bound. $\square$

We shall repeatedly use, uniformly for $a \leq X^{1/2-\varepsilon}$,

$$\mu_a^2 \asymp_{S,\varepsilon} \frac{X}{a \log(X/a)}, \quad \|H_{S,X}\| = o_{S,\varepsilon}(\mu_a). \quad (2.5)$$

These follow from the prime number theorem and (2.3). In particular, the min-max principle applied to the positive-star subspace shows that all indices $j_S(a)$ used below correspond to positive eigenvalues once X is sufficiently large.

Proposition 2.3 – Weyl baseline

Fix a finite set S of primes and $0 < \theta < 1/2$. Uniformly for allowed centres $a \leq X^\theta$,

$$|\lambda_{j_S(a)}(A_{S,X})^2 - \pi_S(X/a)| \ll_{S,\theta} \frac{X^{3/4}}{\sqrt{a} \log X} + \frac{X^{1/2}}{\log X}. \quad (2.6)$$

Consequently the relative estimate (1.11) holds.

Proof.

Since $a \mapsto d_{S,X}(a)$ is nonincreasing, the positive large-prime eigenvalue at arithmetic rank $j_S(a)$ is μ_a . Weyl's inequality and the exact identity

$$d_{S,X}(a) = \pi_S(X/a) - \pi_S(Y)$$

give

$$|\lambda_{j_S(a)}(A_{S,X})^2 - \pi_S(X/a)| \leq \|H_{S,X}\| (2\mu_a + \|H_{S,X}\|) + \pi_S(Y). \quad (2.7)$$

Combining (2.3), (2.5), and the PNT proves (2.6). After division by $\pi_S(X/a)$, the dominant term is $\|H_{S,X}\|/\mu_a \ll_{S,\theta} X^{-(1-2\theta)/4}$; the squared-norm and boundary terms are $O_{S,\theta}(X^{-1/2+\theta})$. This proves (1.11). $\square$

Proposition 2.4 – positive-star block baseline

Fix a finite set S of primes and $0 < \theta < 1/2$. Uniformly for allowed centres $X^\theta/\log X \leq a \leq X^\theta$,

$$|\lambda_{j_S(a)}(A_{S,X})^2 - \pi_S(X/a)| \ll_{S,\theta} \frac{\sqrt{X}}{\log X}. \quad (2.8)$$

Proof.

Choose a sufficiently large fixed $C_\star$, put $Z = \lceil C_\star a \rceil$, and let P_Z project onto the positive large-prime modes with allowed centres $c \leq Z$. All these stars are active for large X . The compression is $P_Z A_{S,X} P_Z = \Lambda_Z + W_Z$, with $\Lambda_Z = \text{diag}(\mu_c)$. Unique factorization gives

$$(W_Z)_{c,cq} = \frac{1 + \sqrt{d_{S,X}(cq)/d_{S,X}(c)}}{2} \leq 1 \quad (cq \leq Z), \quad (2.9)$$

for allowed primes q , and every other off-diagonal entry is zero. The diagonal is zero. Entrywise domination by the prime-cover adjacency on $V_S(Z)$, followed by the large/small-prime split at size Z , therefore gives

$$\|W_Z\| \leq \|A_{S,Z}\| \ll_S \sqrt{\frac{Z}{\log Z}} \ll_{S,\theta} \sqrt{\frac{a}{\log X}}. \quad (2.10)$$

Here nonnegative symmetric matrices satisfy this norm domination by the Rayleigh principle applied to coordinatewise absolute values. The bound on $A_{S,Z}$ uses $\|L_{S,Z}\| \leq \sqrt{\pi_S(Z)}$, Proposition 2.2 at size Z , and the PNT; it uses no molecule estimate.

Let τ_j be the ordered eigenvalues of $\Lambda_Z + W_Z$. Monotonicity of the star degrees and Weyl give $|\tau_{j_S(a)} - \mu_a| \leq \|W_Z\| = o(\mu_a)$. On $P_Z^\perp$, the forest has positive top at most μ_b , where b is the first allowed centre above Z ; all other sectors are nonpositive. Uniform PNT gives $\mu_b^2/\mu_a^2 \leq C_\star^{-1} + o(1)$. Consequently the full adjacency complement has top below $\tau_{j_S(a)} - c\mu_a$, after increasing $C_\star$, since $\|H_{S,X}\| = o(\mu_a)$. Its coupling to P_Z has norm at most $\|H_{S,X}\|$.

The finite-dimensional Schur comparison in Lemma 5.2 now yields

$$|\lambda_{j_S(a)}(A_{S,X}) - \mu_a| \leq \|W_Z\| + O_{S,\theta}(\|H_{S,X}\|^2/\mu_a) \ll_{S,\theta} \sqrt{\frac{a}{\log X}}. \quad (2.11)$$

Squaring costs $O_{S,\theta}(\sqrt{X}/\log X + a/\log X)$. Finally $\mu_a^2 = \pi_S(X/a) - \pi_S(Y)$ and $\pi_S(Y) \ll_S \sqrt{X}/\log X$, proving (2.8). $\square$

3 The exact one-exit compression

Let $P = P_+$ be the orthogonal projection onto the span of all positive star modes u_a^+ , and put $Q = I - P$. Define

$$C_0 = QL_{S,X}Q, \quad K = QH_{S,X}P. \quad (3.1)$$

Let $\mathcal{E}$ be the smallest reducing subspace of the self-adjoint operator C_0 containing $\text{ran } K$. The one-exit space and its compressed adjacency are

$$\mathcal{R} = P\ell^2(V_S(X)) \oplus \mathcal{E}, \quad G_{S,X} = P_{\mathcal{R}}A_{S,X}|_{\mathcal{R}}. \quad (3.2)$$

This space keeps every positive star mode and exactly the nonpositive large-prime spectral components reached in one small-prime step. It is a proper compression of the full graph: most mean-zero leaf modes of a large star are not charged by K .

We first record the linear-algebra comparison that transfers results for $G_{S,X}$ back to $A_{S,X}$.

Lemma 3.1 – two-stage Schur comparison

Let

$$\mathcal{A} = \begin{pmatrix} T & K^* & 0 \\ K & D & R^* \\ 0 & R & C \end{pmatrix}, \quad \mathcal{G} = \begin{pmatrix} T & K^* \\ K & D \end{pmatrix} \quad (3.3)$$

be Hermitian, and suppose

$$\lambda_{\max}(D), \lambda_{\max}(C), \|K\|, \|R\| \leq B.$$

If the j -th eigenvalue ρ_j of $\mathcal{G}$ satisfies $\rho_j \geq 4B$, then

$$0 \leq \lambda_j(\mathcal{A})^2 - \rho_j^2 \leq 7 \frac{B^4}{\rho_j^2}. \quad (3.4)$$

Proof.

Interlacing gives the lower bound. Put

$$\delta = 3B^4/\rho_j^3, \quad z = \rho_j + \delta, \quad s = B^2/(z - B).$$

Eliminating the last block at energy z adds to the D -block a positive operator bounded above by sI . Eliminate the D -block in this upper comparison and compare it with the Schur form of $\mathcal{G} - \rho_j$. If $s \leq \delta$, the change of self-energy is nonpositive. If $s > \delta$, the resolvent identity gives

$$\|K^*((z - s - D)^{-1} - (\rho_j - D)^{-1})K\| \leq \frac{8B^4}{3\rho_j^3} < \delta.$$

Sylvester inertia therefore shows that $\mathcal{A}$ has at most $j - 1$ eigenvalues above z , so $0 \leq \lambda_j(\mathcal{A}) - \rho_j \leq \delta$. Since $\rho_j \geq 4B$, squaring gives (3.4). $\square$

Theorem 3.2 – transfer from one exit to the full graph

Let $\rho_1 \geq \rho_2 \geq \dots > 0$ be the positive eigenvalues of $G_{S,X}$. For fixed $\varepsilon > 0$, uniformly whenever $a_j \leq X^{1/2-\varepsilon}$,

$$0 \leq \lambda_j(A_{S,X})^2 - \rho_j^2 \ll_{S,\varepsilon} \frac{a_j \log(X/a_j)}{(\log X)^2} \ll_{S,\varepsilon} \frac{a_j}{\log X}. \quad (3.5)$$

Proof.

With $\mathcal{F} = Q\ell^2 \ominus \mathcal{E}$, the decomposition $P \oplus \mathcal{E} \oplus \mathcal{F}$ has the form (3.3). The P -to- $\mathcal{F}$ block vanishes by the definition of $\mathcal{E}$, and all remaining off-diagonal blocks have norm at most $B_X = \|H_{S,X}\|$. The D - and C -blocks have top eigenvalue at most B_X , since $C_0 \leq 0$.

Interlacing with the positive-star compression gives $\rho_j \geq \mu_{a_j} - B_X$. Equations (2.3)–(2.5) imply $\rho_j \geq 4B_X$ uniformly in the stated range. Lemma 3.1 yields

$$\lambda_j^2 - \rho_j^2 \ll \frac{B_X^4}{\mu_{a_j}^2} \ll_{S,\varepsilon} \frac{a_j \log(X/a_j)}{(\log X)^2},$$

which is (3.5). $\square$

Corollary 3.3 – uniform relative one-exit comparison

Fix $\varepsilon > 0$. Uniformly over the indices whose arithmetic centres satisfy $a_j \leq X^{1/2-\varepsilon}$,

$$\sup_{a_j \leq X^{1/2-\varepsilon}} \frac{|\lambda_j(A_{S,X})^2 - \rho_j(G_{S,X})^2|}{\pi_S(X/a_j)} \longrightarrow 0. \quad (3.6)$$

In particular, (3.6) holds simultaneously for every $a_j \leq X^\theta$ when $\theta < 1/2$ is fixed.

Proof.

Theorem 3.2 and the prime number theorem give, uniformly in the stated range,

$$\frac{|\lambda_j^2 - \rho_j^2|}{\pi_S(X/a_j)} \ll_S \frac{a_j^2 \log^2(X/a_j)}{X(\log X)^2} \ll_{S,\varepsilon} X^{-2\varepsilon} = o(1).$$

This proves (3.6). $\square$

4 The first-exit correction and principal molecules

Fix an active centre a . Put $u = u_a^+$, $\mu = \mu_a$,

$$h_a = PH_{S,X}u, \quad k_a = QH_{S,X}u. \quad (4.1)$$

Every component of $h_a \oplus k_a$ lies in a large-prime star reached from the a -star by multiplying or dividing the centre by one allowed small prime. Distinct targets are orthogonal. If $q \leq Y$ is an allowed prime, the up-target is aq ; if $q \mid a$, the down-target is a/q .

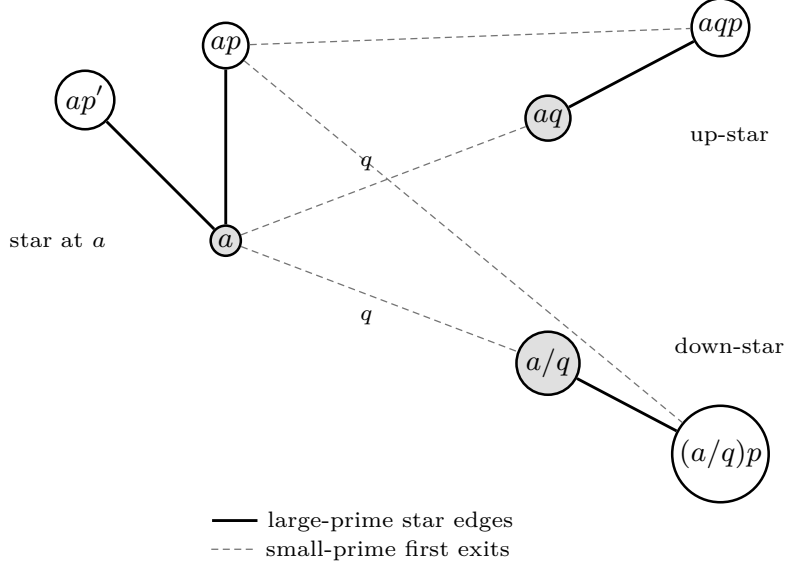


Figure 2: The two first-exit families from the star at a . Multiplication by q reaches the up-star at aq ; division by a prime $q \mid a$ reaches the down-star at a/q .

Lemma 4.1 – exact local self-energy

Let D_+ be the restriction of the positive-star energy operator to the target modes reached from the a -star, excluding u_a^+ . The vector h_a belongs to this target space. All inverses involving $\mu - D_+$ below are taken on that space, not on the full positive sector, where μ is an eigenvalue. Retain C_0 on the nonpositive sector. For the sub-square-root ranges used below the target energies are separated from μ , and

$$\boxed{2\mu \left(\langle h_a, (\mu - D_+)^{-1} h_a \rangle + \langle k_a, (\mu - C_0)^{-1} k_a \rangle \right) - \pi_S(Y) = M_{S,a,X}.} \quad (4.2)$$

Proof.

For an active up-target put $r = d_{S,X}(aq)/d_{S,X}(a)$. Its positive and negative coefficients are

$$\frac{1 + \sqrt{r}}{2}, \quad \frac{1 - \sqrt{r}}{2},$$

at energies $+\mu\sqrt{r}$ and $-\mu\sqrt{r}$. Their contribution to twice the normalized self-energy is

$$\frac{(1 + \sqrt{r})^2}{2(1 - \sqrt{r})} + \frac{(1 - \sqrt{r})^2}{2(1 + \sqrt{r})} = 1 + \frac{4r}{1 - r}. \quad (4.3)$$

An inactive up-target contributes 1. Summing these leading terms gives $\pi_S(Y)$.

For a down-target $c = a/q$, put $s = d_{S,X}(c)/d_{S,X}(a) > 1$. The positive, negative, and mean-zero leaf components on the c -star give

$$\frac{5 - s}{1 - s}. \quad (4.4)$$

This follows by inverting $\mu \begin{smallmatrix} 1 & -\sqrt{s} \\ -\sqrt{s} & 1 \end{smallmatrix}$ on the centre-constant-leaf plane and adding the kernel mass $(s - 1)/(2s\mu)$. Unique factorization makes the target spaces orthogonal, so (4.3)–(4.4) add to (4.2). $\square$

Let $\mathcal{B}_a$ be the coordinate space of the entire boundary star at a , including both modes $u_a^\pm$ and its mean-zero leaf kernel. Let $\mathcal{U}_a$ be the orthogonal sum of the entire target stars at aq , for allowed primes $q \leq Y$, and at a/q , for prime divisors $q \mid a$. An inactive up-target is the singleton $\{aq\}$, not an omitted coordinate. On a fixed sub-square-root range all these singleton vertices are at most X . Put $K_a = P_{\mathcal{U}_a} H_{S,X}|_{\mathcal{B}_a}$. The raw principal molecule is

$$\mathcal{M}_a = \begin{pmatrix} L_{S,X}|_{\mathcal{B}_a} & K_a^* \\ K_a & L_{S,X}|_{\mathcal{U}_a} \end{pmatrix} \quad (4.5)$$

on $\mathcal{B}_a \oplus \mathcal{U}_a$. This is a principal submatrix of $A_{S,X}$. Indeed, small-prime edges preserve the retained large-prime factor and reverse the parity of the remaining number of prime factors; all first-exit targets have the same such parity, so no small-prime edge joins two of them. The stars themselves are disjoint.

Let $\nu_a > 0$ be the eigenvalue continuing μ_a , and extend its unit eigenvector $\tilde{f}_a$ by zero to the full graph, with $\langle u_a^+, \tilde{f}_a \rangle > 0$. Write its boundary part as $\alpha_a^+ u_a^+ + \alpha_a^- u_a^- + \zeta_a$, where $\zeta_a \in \ker L_{S,X}|_{\mathcal{B}_a}$. The full boundary, rather than only u_a^+ , is needed for the cancellation below. An uncharged zero-mode component may lie outside $\mathcal{R}$. We therefore use the projected, normalized vector

$$z_a = (I - P_{\mathcal{R}})\tilde{f}_a, \quad f_a = \frac{P_{\mathcal{R}}\tilde{f}_a}{\|P_{\mathcal{R}}\tilde{f}_a\|}, \quad g_a = G_{S,X}f_a - \nu_a f_a. \quad (4.6)$$

Put

$$F_{S,a,X} = \pi_S(X/a) + M_{S,a,X}. \quad (4.7)$$

Proposition 4.2 – molecule energy, projection, and residual

Fix $0 < \vartheta < 1/2$. Uniformly for allowed $a \leq X^\vartheta$,

$$\nu_a^2 = F_{S,a,X} + e_{a,X}, \quad (4.8)$$

where

$$|e_{a,X}| \ll_{S,\vartheta} (\log \log X + \omega(a)) \frac{a}{\sqrt{X}} + \frac{\log X}{\sqrt{X}}. \quad (4.9)$$

The discarded projection satisfies

$$\begin{aligned} \|f_a - u_a^+\|^2 &\ll_{S,\vartheta} \frac{a}{\sqrt{X}}, & L_{S,X}z_a &= 0, \\ \|z_a\|^2 &\ll_{S,\vartheta} \frac{a(\log X)^2}{X^{3/2}} + \frac{a^3\omega(a)\log X}{X^2}, \\ \|A_{S,X}z_a\|^2 &\ll_{S,\vartheta} \frac{a\log X}{X} + \frac{a^3\omega(a)}{X^{3/2}}. \end{aligned} \quad (4.10)$$

Apart from the projection term in (4.10), g_a comes only from small-prime edges from a target space to the next unretained layer. Moreover, for every fixed $C_\star$, every $B \leq X^\vartheta$, and some fixed exponent C_S ,

$$\sum_{a \leq C_\star B} \|g_a\|^2 \ll_{S,\vartheta} B(\log X)^{C_S}. \quad (4.11)$$

Proof.

All active up-target ratios are at most $1/2 + o(1)$, and all down-target ratios are at least $2 + o(1)$; the latter follows from the fixed-ratio argument in (9.7b). Thus the diagonal forest in (4.5) has a simple eigenvalue

μ_a separated from its other eigenvalues by a fixed multiple of μ_a . Since $\|K_a\| \leq \|H_{S,X}\| = o(\mu_a)$, Weyl's inequality gives a unique molecule eigenvalue in a fixed neighborhood of μ_a .

Eliminating the target stars gives the exact boundary equation

$$(\nu_a - L_{\mathcal{B}_a} - \Sigma_a(\nu_a))v_a = 0, \quad \Sigma_a(\lambda) = K_a^*(\lambda - L_{\mathcal{U}_a})^{-1}K_a, \quad v_a = P_{\mathcal{B}_a}\tilde{f}_a. \quad (4.12)$$

First compress this equation to $\text{span}\{u_a^+, u_a^-\}$; the boundary-kernel contribution will be bounded below. Write $z = \lambda/\mu_a$, $P = \pi_S(Y)$, and $d = \mu_a^2$. Let $\mathbf{F}(z)$ have entries $\mu_a \langle K_a u_a^\sigma, (z\mu_a - L_{\mathcal{U}_a})^{-1} K_a u_a^\tau \rangle$. The two-mode equation is $\det(\text{diag}(z-1, z+1) - \mathbf{F}(z)/d) = 0$. For an up-target of ratio r , its deviation from $(2z)^{-1} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ is

$$\mathbf{R}_r(z) = \frac{r}{2z(z^2 - r)} \begin{pmatrix} (z+1)^2 & -(z^2 - 1) \\ -(z^2 - 1) & (z-1)^2 \end{pmatrix}. \quad (4.13)$$

After the boundary self-energy is divided by d in the dimensionless two-mode determinant, the universal terms sum to $\gamma(z) \frac{1}{1} \frac{1}{1}$, where $\gamma(z) = P/(2zd)$, and the corresponding determinant is exactly

$$(z-1-\gamma)(z+1-\gamma) - \gamma^2 = z^2 - 1 - P/d. \quad (4.14)$$

Thus no pure P^2/d^2 term remains. The matrices (4.13), their first derivatives, and the analogous down-target rational functions have total norm $O_{S,\vartheta}(\log \log X + \omega(a))$ on a fixed neighborhood of $z = 1$; all denominators there are separated from zero. The first local identities also give

$$2R_{++}(1) = M_{S,a,X}, \quad R_{+-}(1) = -\omega(a)/2.$$

Let $\nu_a^{(2)}$ denote the positive root of this two-mode equation. Expanding its determinant at the unique positive root of (4.14), the pure P -terms cancel and every remaining term contains at least one $R_{\sigma\tau}$ -factor. The mean-value theorem and the preceding derivative bounds therefore give

$$|z-1| \ll_{S,\vartheta} \frac{a}{\sqrt{X}}, \quad |(\nu_a^{(2)})^2 - F_{S,a,X}| \ll_{S,\vartheta} (\log \log X + \omega(a)) \frac{a}{\sqrt{X}} + \frac{\log X}{\sqrt{X}}.$$

This proves (4.8)–(4.9) for the two-mode compression, apart from the boundary-kernel response estimated below. The cancellation between the positive and negative entries of an up-star is essential; estimating them separately would leave a spurious quadratic boundary term.

Before treating the uncharged kernel, the direct-sum target resolvent gives

$$\|\tilde{f}_a - u_a^+\|^2 \ll_{S,\vartheta} \frac{\pi_S(Y) + \omega(a)}{\mu_a^2} + O_{S,\vartheta}\left(\frac{a^2}{X}\right) \ll_{S,\vartheta} \frac{a}{\sqrt{X}}.$$

Projection and normalization change this by the smaller quantity estimated below, proving the first assertion of (4.10).

We first bound the boundary kernel ζ_a , separately from the projection z_a . For a boundary leaf ap , put

$$n_a(p) = \#\{q \leq Y : q \notin S, q \text{ prime}, apq \leq X\}.$$

Let Q_0 project onto the boundary leaf kernel, and define $c_{a,\pm} = Q_0 \Sigma_a(z\mu_a)u_a^\pm$. Put

$$H_a(p; z) = \sum_{\substack{q \leq Y, q \notin S \\ q \text{ prime}, apq \leq X}} \frac{1}{z^2 - r_{q,X}(a)}, \quad \bar{H}_a(z) = \frac{1}{d_a} \sum_{Y < p \leq X/a} H_a(p; z),$$

where all leaf sums omit primes in S . The explicit up-star inverse gives the exact leaf values

$$c_{a,+}(ap) = \frac{z+1}{\sqrt{2}d_a}(H_a(p;z) - \overline{H}_a(z)), \quad c_{a,-} = \frac{1-z}{1+z}c_{a,+}.$$

The down-target values are constant on all boundary leaves and disappear under Q_0 . Each summand in $H_a(p;z)$ is uniformly bounded, so subtracting its mean and using $|H_a(p;z)| \ll n_a(p)$ gives

$$\|c_{a,+}\|^2 + \|c_{a,-}\|^2 \ll_{S,\vartheta} \frac{1}{\mu_a^2 d_{S,X}(a)} \sum_{Y < p \leq X/a} n_a(p)^2. \quad (4.15)$$

Chebyshev's estimate, after splitting p into dyadic intervals, gives

$$\sum_{Y < p \leq X/a} n_a(p)^2 \ll_{S,\vartheta} \frac{XY}{a^2 \log X \log^2(2Y/a)}. \quad (4.16)$$

Indeed on $p \asymp P$ one has $n_a(p) \ll X/(aP \log(2X/(aP)))$, while the interval contains $O(P/\log P)$ primes. Put $R = X/(aY)$ and $P \asymp 2^j Y$. The remaining dyadic sum is bounded by $\sum_{2^j \leq R} 2^{-j}/\log^2(2R/2^j) \ll 1/\log^2(2R)$. For $2^j \leq \sqrt{R}$ use the lower bound $\frac{1}{2} \log R$ for the logarithm; the other terms sum to $O(R^{-1/2})$, which is smaller. Since $\mu_a^2 = d_{S,X}(a) \asymp X/(a \log(X/a))$, (4.15)–(4.16) imply

$$\|c_{a,+}\|^2 + \|c_{a,-}\|^2 \ll_{S,\vartheta} \frac{\log X}{\sqrt{X}}. \quad (4.17)$$

On the root window, $\|Q_0 \Sigma_a(\lambda) Q_0\| \ll \|H_{S,X}\|^2/\mu_a = o(\mu_a)$. The kernel block is therefore separated from ν_a by $\gg \mu_a$. Its Schur contribution to the two-mode equation has norm $O((\|c_{a,+}\|^2 + \|c_{a,-}\|^2)/\mu_a)$, with the same bound for its dimensionless derivative. Consequently

$$|\nu_a^2 - (\nu_a^{(2)})^2| \ll_{S,\vartheta} \frac{\log X}{\sqrt{X}}, \quad \|\zeta_a\|^2 \ll_{S,\vartheta} \frac{a(\log X)^2}{X^{3/2}}, \quad |\alpha_a^-| \ll_{S,\vartheta} \frac{a}{\sqrt{X}}. \quad (4.17a)$$

The first estimate follows by the mean-value theorem for the simple two-mode root; the second follows from the boundary-kernel equation; the third follows from the negative-mode row, whose diagonal gap is $2\mu_a + o(\mu_a)$. This proves the full-molecule energy estimate. The kernel-driven target response has norm $O(\|H_{S,X}\| \|\zeta_a\|/\mu_a) = o(\|\zeta_a\|)$.

We next locate the discarded projection. If ℓ is the smallest prime outside S , the negative a -mode occurs with nonzero coefficient in $QH_{S,X}u_{\ell a}^+$. Every other down-target there has centre at most $\ell a/(\ell+1)$, and every up-target has centre at least $\ell^2 a$. Uniform PNT separates their degrees from d_a . The spectral projection of C_0 at $-\mu_a$ therefore shows $u_a^- \in \mathcal{E}$. Here $\ell a \leq Y$ for all large X .

On the negative target modes, the responses to u_a^- and to ζ_a are functions of the target degree times the corresponding spectral components of $QH_{S,X}u_a^+$. To check equal-degree targets, note that two up-stars of the same degree use the identical initial segment of boundary primes, so their partial sums of $\zeta_a(ap)$ agree. Down-targets use the whole boundary segment, whose kernel sum is zero. Up- and down-target degrees lie on opposite sides of d_a . Thus all negative target components belong to $\mathcal{E}$. Only zero modes can be discarded, proving $L_{S,X}z_a = 0$.

There is a second zero-mode source besides ζ_a . Write the zero-mode part of $H_{S,X}u_a^+$ as $C_a + D_a$, where C_a is the inactive up-target part and D_a is the down-target leaf-kernel part. For $H_{S,X}u_a^-$ the corresponding part is $C_a - D_a$, and $\|D_a\|^2 \leq \omega(a)/2$. Since $C_a + D_a \in \mathcal{E}$, the uncharged signed response is exactly $-2\alpha_a^- \nu_a^{-1}(I - P_{\mathcal{R}})D_a$. Together with the kernel-driven response, this gives

$$\|z_a\|^2 \ll_{S,\vartheta} \|\zeta_a\|^2 + \frac{|\alpha_a^-|^2 \omega(a)}{\nu_a^2} \ll_{S,\vartheta} \frac{a(\log X)^2}{X^{3/2}} + \frac{a^3 \omega(a) \log X}{X^2}.$$

Since $A_{S,X}z_a = H_{S,X}z_a$, Proposition 2.2 proves the last estimate in (4.10).

For the residual, write $\tilde{f}_a = s_a + i_a + k_a^{\ker}$, where $s_a = \alpha_a^+ u_a^+ + \alpha_a^- u_a^-$, $i_a = (\nu_a - L_{\mathcal{U}_a})^{-1} K_a s_a$, and $k_a^{\ker} = \zeta_a + (\nu_a - L_{\mathcal{U}_a})^{-1} K_a \zeta_a$. Let P_a be the coordinate projection onto the whole principal molecule. Its raw residual is exactly $(I - P_a)H_{S,X}(i_a + k_a^{\ker} - \zeta_a)$: every return to the entire boundary star is removed before taking a norm.

Write $s_a = \beta_0 \delta_a + \beta_1 \psi_a$, with $|\beta_0| + |\beta_1| = O(1)$, and put $z = \nu_a / \mu_a$, $d = d_a$. On any target star of degree D , a source $(b_0, (b_p)_p)$ has the exact resolvent values

$$v(c) = \frac{\nu_a b_0 + \sum_p b_p}{\nu_a^2 - D}, \quad v(cp) = \frac{b_p + v(c)}{\nu_a}. \quad (4.17b)$$

For an up-target this gives $v(aq) = (z\beta_0 + r\beta_1)/(\mu_a(z^2 - r))$ and $v(aqp) = (\beta_0 + z\beta_1)/(d(z^2 - r))$. For a down-target, $s = d_{a/q}/d \gg_{S,\vartheta} q$ and $v(a/q) = (z\beta_0 + \beta_1)/(\mu_a(z^2 - s))$. Thus the required absolute majorants are

Target coordinate	Range	Coefficient bound
Up centre aq	$q \leq Y$	C/μ_a
Down centre a/q	$q \mid a$	$C/(q\mu_a)$
Up leaf aqp	$aqp \leq X$	C/d_a
Down leaf $(a/q)p$	$ap \leq X$	C/d_a
Down leaf $(a/q)p$	$X/a < p \leq Xq/a$	$C/(qd_a)$

After one further small-prime edge and deletion of boundary returns, a reduced base quotient is qr , q/r with $q \neq r$, or $1/(qr)$. The two down-primes must divide the source in their stated order. Each quotient has at most two step orderings; a retained prime $p > Y$ is unique. These facts bound multiplicity before squaring.

Put $\mathcal{P}_Y = 1 + \sum_{q \leq Y} q^{-1}$. Chebyshev summation gives

$$\#\{q, r \text{ prime} : qr \leq T\} \ll \frac{T \log \log(3T)}{\log(2T)}.$$

For leaf labels sum over the retained prime first; its logarithm is $\asymp \log X$. The coefficient table yields

Residual family	Squared-mass bound, up to a fixed power of $1 + \omega(a)$
Centre, two up-steps	$C\mathcal{P}_Y$
Centre, mixed or two down-steps	$C(\pi_S(Y) + 1)/d_a$
Leaf, two up-steps	$C\mathcal{P}_Y^2/d_a$
Leaf, mixed or two down-steps	$C(1 + a)\mathcal{P}_Y/d_a$

For the last row, an inner down-leaf segment has at most $O(d_a q \mathcal{P}_Y)$ subsequent prime pairs and coefficient $O(d_a^{-1})$; use $q \leq a$. On the outer segment the same count is multiplied by q^{-2} , retaining the down-star weight. For two up-steps the leaf count is at most $O(d_a \mathcal{P}_Y^2)$. All four rows are polylogarithmic uniformly on $a \leq X^\vartheta$, because $\pi_S(Y)/d_a \ll a/\sqrt{X}$ and $a/d_a \ll a^2 \log X/X = o(1)$. The signed raw residual therefore has squared norm $O((\log X)^C)$. The kernel-driven residual has squared norm at most $C\|H_{S,X}\|^4 \|\zeta_a\|^2 / \mu_a^2$, which is smaller.

Projection contributes $P_{\mathcal{R}}H_{S,X}z_a$, followed by the bounded normalization in (4.6). Summing its squared norm gives

$$\sum_{a \leq C_* B} \|P_{\mathcal{R}}H_{S,X}z_a\|^2 \leq \|H_{S,X}\|^2 \sum_{a \leq C_* B} \|z_a\|^2 \ll_{S,\vartheta} \frac{B^2 L}{X} + \frac{B^4 L^C}{X^{3/2}} \ll_{S,\vartheta} B L^C.$$

Here $L = \log X$, and the fixed exponent absorbs the bounded word multiplicities. The raw-residual bound and this projection estimate prove (4.11). $\square$

Lemma 4.3 – size of the first-exit correction

For every fixed S and $0 < \vartheta < 1/2$, uniformly for allowed $a \leq X^\vartheta$,

$$|M_{S,a,X}| \ll_{S,\vartheta} \log X. \quad (4.18)$$

Proof.

Uniform PNT bounds on this sub-square-root range give $r_{q,X}(a) \ll_{S,\vartheta} q^{-1}$, with all nonzero up-ratios bounded away from one. Thus the up-sum is $O_S(\log \log X)$. They also give $s_{q,X}(a) \geq 1 + c_{S,\vartheta}$ for every prime divisor $q \mid a$, so every down summand is $O_{S,\vartheta}(1)$. The down-sum is $O_{S,\vartheta}(\omega(a)) = O(\log X)$. $\square$

5 Finite-dimensional tools

We now pass from individual molecules to their combined span. The following lemmas control reorthonormalization, the complementary spectrum, and ordered eigenvalue errors. Section 6 applies them to the molecule family.

Lemma 5.1 – frame commutator

Let $T = T^*$, let V have full column rank, and suppose

$$TV = VD + J, \quad D = D^*, \quad \Gamma = V^*V. \quad (5.1)$$

Then

$$D\Gamma - \Gamma D = V^*J - J^*V. \quad (5.2)$$

If $\|\Gamma - I\| \leq 1/2$, put

$$Q = V\Gamma^{-1/2}, \quad E = TQ - QD. \quad (5.3)$$

For either the operator norm or the Hilbert–Schmidt norm, denoted by $\|\cdot\|_*$,

$$\|E\|_* \ll \|J\|_*. \quad (5.4)$$

Moreover, with

$$M = Q^*TQ, \quad R = (I - QQ^*)TQ,$$

one has

$$\|M - D\|_* \leq \|E\|_*, \quad \|R\|_* \leq \|E\|_*, \quad * \in \{2, \infty\}. \quad (5.5)$$

Proof.

Self-adjointness gives

$$V^*TV = \Gamma D + V^*J = D\Gamma + J^*V,$$

which proves (5.2). On $[1/2, 3/2]$, the divided difference of $x \mapsto x^{-1/2}$ is bounded. The integral formula for negative fractional powers (equivalently, the divided-difference formula in an eigenbasis of Γ) gives, for $* \in \{2, \infty\}$,

$$\| [D, \Gamma^{-1/2}] \|_* \ll \| [D, \Gamma] \|_* \ll \| J \|_*.$$

Since

$$E = V[D, \Gamma^{-1/2}] + J\Gamma^{-1/2},$$

the frame bounds prove (5.4). Finally, $M - D = Q^*E$ and $R = (I - QQ^*)E$. $\square$

Hilbert–Schmidt refinement. Under the same hypotheses, write $\text{sym } B = (B + B^*)/2$. Then

$$\| M - D - \text{sym}(V^*J) \|_{\text{HS}} \ll \| \Gamma - I \| \| J \|_{\text{HS}}. \quad (5.5a)$$

Indeed, self-adjointness gives $V^*TV = (\Gamma D + D\Gamma)/2 + \text{sym}(V^*J)$. Conjugating the symmetrized residual by $\Gamma^{-1/2}$ changes it in Hilbert–Schmidt norm by at most $O(\| \Gamma - I \| \| J \|_{\text{HS}})$. In an eigenbasis of Γ , the remaining difference from D has entries

$$\frac{(\sqrt{\gamma_i} - \sqrt{\gamma_j})^2}{2\sqrt{\gamma_i\gamma_j}} D_{ij}.$$

They vanish when $\gamma_i = \gamma_j$. Otherwise divide their coefficient by $\gamma_j - \gamma_i$; the resulting multiplier is $O(\| \Gamma - I \|)$, uniformly for $\gamma_i, \gamma_j \in [1/2, 3/2]$. Equation (5.2) and $\| V \| \leq \sqrt{3/2}$ give (5.5a).

In particular, under the global estimates (6.3) and (6.4), the squared cost of this Gram correction is bounded by

$$(\log X)^C \left(\frac{B^2}{X} + \frac{B^5}{X^2} \right) = o\left(\frac{B^3}{X \log X} \right) \quad (5.5b)$$

on a fixed positive sub-square-root power band. This is a bound for the complete matrix before taking its local compression. The matrix that remains to be estimated is $\text{sym}(V^*J)$, whose entries involve the complete output vectors; coefficients of U^*J for an auxiliary star-mode frame U are not interchangeable with them.

Lemma 5.2 – ordered eigenvalues above a separated complement

Let

$$T = \begin{pmatrix} M & R^* \\ R & C \end{pmatrix} = T^*$$

on $\mathbb{C}^m \oplus \mathcal{K}$, and let $\tau_1 \geq \dots \geq \tau_m$ be the eigenvalues of M . If

$$\sup \sigma(C) \leq \beta < \tau_k,$$

then

$$0 \leq \lambda_k(T) - \tau_k \leq \frac{\|R\|^2}{\tau_k - \beta}. \quad (5.6)$$

Proof.

Compression interlacing gives $\lambda_k(T) \geq \tau_k$. Put

$$\delta = \frac{\|R\|^2}{\tau_k - \beta}, \quad t = \tau_k + \delta.$$

The complement block $C - tI$ is negative definite. Sylvester inertia and Schur elimination give

$$\#\{j : \lambda_j(T) > t\} = n_+(M - tI + R^*(t - C)^{-1}R).$$

Since

$$0 \leq R^*(t - C)^{-1}R \leq \frac{\|R\|^2}{t - \beta}I \leq \delta I,$$

the Schur form is bounded above by $M - \tau_k I$, which has at most $k - 1$ positive eigenvalues. Hence $\lambda_k(T) \leq t$, proving (5.6). $\square$

Lemma 5.3 – complement of an energy-weighted graph

Let $L = L^*$ reduce $\mathcal{P} \oplus \mathcal{P}^\perp$, suppose $L|_{\mathcal{P}} = \Lambda \geq 0$, and write $\beta = \sup \sigma(L|_{\mathcal{P}^\perp}) \geq 0$. Let $\mathcal{Q} = \{p + Tp : p \in \mathcal{P}\}$ be the graph of $T : \mathcal{P} \rightarrow \mathcal{P}^\perp$. For every self-adjoint H , the compression $C = P_{\mathcal{Q}^\perp}(L + H)|_{\mathcal{Q}^\perp}$ satisfies

$$\boxed{\sup \sigma(C) \leq \beta + \|T\Lambda^{1/2}\|^2 + \|H\|}. \quad (5.7)$$

Proof.

Every $x \in \mathcal{Q}^\perp$ has the unique form $x = -T^*y + y$, with $y \in \mathcal{P}^\perp$. The two summands are orthogonal, so $\|y\| \leq \|x\|$, and the reducing decomposition gives

$$\langle x, Lx \rangle = \langle y, L|_{\mathcal{P}^\perp}y \rangle + \langle T^*y, \Lambda T^*y \rangle \leq (\beta + \|T\Lambda^{1/2}\|^2)\|y\|^2.$$

For unit x , add $|\langle x, Hx \rangle| \leq \|H\|$. The coefficient $\beta + \|T\Lambda^{1/2}\|^2$ is nonnegative, so replacing $\|y\|^2$ by $\|x\|^2 = 1$ preserves the upper bound. The variational principle proves (5.7). $\square$

Lemma 5.4 – ordered residual comparison

Let $T = T^*$ on a finite-dimensional Hilbert space $\mathcal{H}$, let $Q : \mathbb{C}^m \rightarrow \mathcal{H}$ be an isometry, and let $D = \text{diag}(\nu_1, \dots, \nu_m)$ be real. Write $\hat{\nu}_1 \geq \dots \geq \hat{\nu}_m$ for the decreasing rearrangement of its entries and put

$$E = TQ - QD, \quad e = \|E\|, \quad h = \|E\|_{\text{HS}}, \quad M = Q^*TQ, \quad C = T|_{\text{ran}(Q)^\perp},$$

where the last expression denotes compression. Let ξ_j and ζ_j be the decreasing eigenvalues of M and T , respectively. Suppose $\mu > 0$, $0 < c_0 \leq C_0$, and a set $I \subset \{1, \dots, m\}$ satisfy

$$c_0\mu \leq \hat{\nu}_j \leq C_0\mu \quad (j \in I), \quad \sup \sigma(C) \leq c_0\mu/2, \quad e \leq c_0\mu/4. \quad (5.8)$$

There are numbers $\epsilon_j = |\xi_j - \hat{\nu}_j|$ with

$$\sum_{j=1}^m \epsilon_j^2 \leq h^2, \quad \max_j \epsilon_j \leq e, \quad (5.9)$$

such that, at the same ordered indices $j \in I$,

$$0 \leq \zeta_j - \xi_j \leq \frac{4e^2}{c_0\mu}, \quad |\zeta_j^2 - \hat{\nu}_j^2| \ll_{c_0, C_0} \mu \epsilon_j + e^2. \quad (5.10)$$

In particular,

$$\sum_{j \in I} |\zeta_j^2 - \hat{\nu}_j^2|^2 \ll_{c_0, C_0} \mu^2 h^2 + |I|e^4. \quad (5.11)$$

Proof.

The identities $M - D = Q^*E$ and $(I - QQ^*)TQ = (I - QQ^*)E$ bound both blocks by E in operator and Hilbert–Schmidt norm. The Hermitian Hoffman–Wielandt inequality, in decreasing order, and Weyl’s inequality give (5.9); see [9]. For $j \in I$, $\xi_j \geq 3c_0\mu/4$, so its gap above C is at least $c_0\mu/4$. Lemma 5.2 proves the first inequality in (5.10) without reindexing. Also $\xi_j \leq (C_0 + c_0/4)\mu$, and the Schur shift is at most $c_0\mu/4$, so all three roots are positive and $O_{c_0, C_0}(\mu)$. Factoring differences of squares proves the second inequality. Squaring and summing proves (5.11). $\square$

We will also use the elementary sorting contraction. If $p_1 \geq \dots \geq p_m$ and $|x_i - p_i| \leq u$ for all i , then their decreasing rearrangement satisfies

$$|\hat{x}_j - p_j| \leq u \quad (1 \leq j \leq m). \quad (5.12)$$

Indeed the first j entries of the original list are at least $p_j - u$, while all entries with index at least j are at most $p_j + u$. These two counts prove (5.12), including ties. Thus a comparison of ordered values need not bound the permutation of labels.

6 The signed molecule frame

Fix $0 < \theta < 1/2$ and a dyadic scale B satisfying

$$X^\theta / \log X \leq B \leq X^\theta, \quad (6.1)$$

and retain all allowed centres $c \leq C_\star B$, where $C_\star > 4$ is fixed. Let $V_B e_c = f_c$, $D_B = \text{diag}(\nu_c)$, and

$$J_B = G_{S,X} V_B - V_B D_B, \quad \Gamma_B = V_B^* V_B. \quad (6.2)$$

The following estimates bound the Gram matrix and the residual of this molecule family.

All occurrences of $X^{o(1)}$ in Sections 6–9 arise from bounded-word divisor multiplicities. After S and θ are fixed, only finitely many such factors are composed. Consequently, for every fixed $\xi > 0$, their complete product is $O_{S, \theta, \xi}(X^\xi)$. Only the later target-width threshold-count argument requires $\theta > 1/3$ and chooses $\xi < 3\theta - 1$. The global estimates in this section hold throughout $0 < \theta < 1/2$.

Proposition 6.1 – collision budget

For some fixed exponent C_S , uniformly in (6.1),

$$\|\Gamma_B - I\| \ll_{S, \theta} (\log X)^{C_S} \left(\sqrt{\frac{B}{X}} + \frac{B^2}{X} \right) = o(1), \quad (6.3)$$

$$\|J_B\|_{\text{HS}}^2 \ll_{S, \theta} B (\log X)^{C_S}, \quad \boxed{\|J_B\|^2 \ll_{S, \theta} (\log X)^{C_S} \left(1 + \frac{B^2}{\sqrt{X}} \right)}. \quad (6.4)$$

Proof.

We first work with the raw full-star vectors $\tilde{f}_c$ and raw residuals $\tilde{g}_c = (A_{S,X} - \nu_c)\tilde{f}_c$; projection and normalization will be estimated at the end. Their support dictionary is

$$c, \quad cp, \quad cq, \quad cqp, \quad c/q, \quad (c/q)p, \quad p > Y, \quad q \leq Y, \quad (6.4a)$$

with $q \mid c$ for a down-step. A signed raw residual has one further small-prime edge, and no coordinate on its own full principal molecule. We use the coefficients in (4.17b), keeping the outer down-leaf factor q^{-1} .

First consider the Gram matrix. If two source centres are prime-adjacent, each boundary star belongs to the other molecule. Self-adjointness gives

$$(\nu_c - \nu_d)\langle \tilde{f}_d, \tilde{f}_c \rangle = \langle \tilde{g}_d, \tilde{f}_c \rangle - \langle \tilde{f}_d, \tilde{g}_c \rangle.$$

Only first-exit interiors occur in these pairings. Their norms are $O(\sqrt{c/\sqrt{X}})$ and $O(\sqrt{d/\sqrt{X}})$, while the raw residuals are polylogarithmically bounded by Proposition 4.2. The two roots are separated by $\gg \max(\mu_c, \mu_d)$. If the centres are not prime-adjacent, their boundary stars do not intersect the other molecule. Their interiors can share at most two target stars: solve $cq^{\pm 1} = dr^{\pm 1}$. On any such target the complete response has norm $O(1/\mu_c)$, respectively $O(1/\mu_d)$, since the one-target coupling is a partial coordinate isometry and the target resolvent gap is $\gg \mu_c$. Consequently

$$|\langle \tilde{f}_c, \tilde{f}_d \rangle| \ll L^C \begin{cases} \sqrt{cd}/X^{3/4}, & c, d \text{ prime-adjacent,} \\ 1/(\mu_c \mu_d), & c, d \text{ share a target but are not adjacent,} \\ 0, & \text{otherwise,} \end{cases} \quad c \neq d. \quad (6.5)$$

For the first matrix use the Schur weight $w_c = c^{-1/2}$. An up-neighbour $d = cq \leq C_\star B$ contributes $c/X^{3/4}$ to the weighted row; there are at most $C_\star B/c$ such neighbours. Down-neighbours contribute the same weight and have at most $\omega(c)$ choices. The other matrix has ordinary row sum at most $L^C B^2/X$, using $\mu_c^{-1} \ll \sqrt{cL/X}$. Thus

$$\|\tilde{V}_B^* \tilde{V}_B - I\| \ll L^C \left(\frac{B}{X^{3/4}} + \frac{B^2}{X} \right) \ll L^C \left(\sqrt{\frac{B}{X}} + \frac{B^2}{X} \right). \quad (6.6)$$

This controls the complete raw vectors, including their boundary kernels.

For the residual operator, take two distinct sources c, d . A centre-type equality has the form

$$cu_1/u_2 = dv_1/v_2, \quad \frac{d}{c} = \frac{u_1 v_2}{u_2 v_1}, \quad (6.7)$$

where each side uses at most two prime steps. Denominator slots divide the source at the step where they are removed. There are a bounded number of orientations, and at most L^4 source-dividing choices for each. Since $c \neq d$, at least two of the four prime slots are fixed by the reduced quotient; a free prime must occupy one slot on each side. There is therefore at most one free common small prime, of range at most Y . This gives $O(YL^C)$ representations, including the common-descendant multiplicity.

For leaf-type equalities the same large prime occurs on both sides. On a source piece indexed by ℓ_c , the leaf range is $p \leq X\ell_c/c$ and the coefficient is $O((\ell_c d_c)^{-1})$: take $\ell_c = 1$ for up-leaves and inner down-leaves, and $\ell_c = q$ for an outer down-leaf segment. When no free common small prime remains, summing p gives at most $XL^C \min(\ell_c/c, \ell_d/d)$ choices. Their weighted contribution is bounded using the exact inequality

$$\frac{\min(\ell_c/c, \ell_d/d)}{\ell_c \ell_d} \leq \frac{1}{\max(c, d)}.$$

Thus the extended outer down-leaf range retains its reciprocal down-prime factor. If a common small prime also remains, an inner down-leaf segment can have a larger product cutoff. We use the safe bound XL^C for the number of prime pairs, with coefficients $O(d_c^{-1})$ and $O(d_d^{-1})$, instead of claiming the preceding smaller range. Centre and leaf residuals have disjoint large-prime signatures.

Together with the diagonal bound from Proposition 4.2, the two off-diagonal residual-Gram row bounds are

$$\sum_{d \leq C_\star B} O\left(\frac{YL^C}{\mu_c \mu_d}\right) \ll L^C \frac{B^2}{\sqrt{X}}, \quad (6.8a)$$

and

$$\sum_{d \leq C_\star B} O\left(\frac{XL^C}{d_c d_d}\right) \ll L^C \frac{B^3}{X} \leq L^C \frac{B^2}{\sqrt{X}}. \quad (6.8b)$$

The Schur test proves the signed raw-residual operator bound in (6.4). Its Hilbert–Schmidt bound is the sum of the one-source estimates.

It remains to pass to the actual projected columns. Let Z_B have columns z_c , and let N_B be diagonal with entries $(1 - \|z_c\|^2)^{-1/2}$. Exactly,

$$V_B = P_{\mathcal{R}} \tilde{V}_B N_B, \quad J_B = (P_{\mathcal{R}} \tilde{J}_B - P_{\mathcal{R}} H_{S,X} Z_B) N_B. \quad (6.8c)$$

By (4.10),

$$\|Z_B\|_{\text{HS}}^2 \ll L^C \left(\frac{B^2}{X^{3/2}} + \frac{B^4}{X^2} \right), \quad \|H_{S,X} Z_B\|_{\text{HS}}^2 \ll L^C \left(\frac{B^2}{X} + \frac{B^4}{X^{3/2}} \right). \quad (6.8d)$$

The first is $o(1)$ and is $O(L^C B^2/X)$. Consequently (6.6) survives the Gram subtraction $Z_B^* Z_B$ and the diagonal normalization, proving (6.3).

The kernel-driven raw residual has squared Hilbert–Schmidt norm at most $L^C B^3/X^{3/2}$, by (4.17a) and the extra $\|H_{S,X}\|/\mu_c$ factor in its target response. This and the second bound in (6.8d) are $O(L^C(1+B^2/\sqrt{X}))$ in operator-norm squared and $O(BL^C)$ in Hilbert–Schmidt norm squared, because $B/\sqrt{X} \rightarrow 0$. Normalization has the same or a smaller cost. These estimates complete (6.4) for the full projected residual. The $B^2/\sqrt{X}$ term retains the coherent prime-source rectangle. $\square$

Because of (6.3), the symmetric reorthonormalization

$$Q_B = V_B \Gamma_B^{-1/2} \quad (6.9)$$

is defined for large X . Put

$$E_B = G_{S,X} Q_B - Q_B D_B. \quad (6.10)$$

Corollary 6.2 – signed prefix budget

Uniformly in (6.1),

$$\|E_B\|_{\text{HS}}^2 \ll_{S,\theta} B(\log X)^{C_S}, \quad \|E_B\|^2 \ll_{S,\theta} (\log X)^{C_S} \left(1 + \frac{B^2}{\sqrt{X}} \right). \quad (6.11)$$

Proof.

Apply Lemma 5.1 to Proposition 6.1. $\square$

The Hilbert–Schmidt bound in (6.11) controls the number of bad cluster boundaries. The operator bound controls the self-energy after elimination of the separated complement, where its coherent term is divided by a root-scale gap. The factor $1 + B^2/\sqrt{X}$ is retained in that estimate.

Lemma 6.3 – the prefix as an energy-weighted graph

Let $\mathcal{P}_B$ be the span of the positive star modes with allowed centres $c \leq C_\star B$, let Π_B be its orthogonal projection, and write $\Lambda_B = L_{S,X}|_{\mathcal{P}_B}$. For all sufficiently large X , the map $W_B = \Pi_B V_B$ is invertible, and $\text{ran } Q_B = \text{ran } V_B$ is the graph of

$$T_B = (I - \Pi_B) V_B W_B^{-1} : \mathcal{P}_B \longrightarrow \mathcal{P}_B^\perp. \quad (6.12)$$

Moreover

$$\|T_B\|^2 \ll_{S,\theta} L^{C_S} \frac{B}{\sqrt{X}} = o(1), \quad \|T_B \Lambda_B^{1/2}\|^2 \ll_{S,\theta} L^{C_S} \sqrt{\frac{B}{L}} = o(\mu_B). \quad (6.13)$$

The largest forest eigenvalue on $\mathcal{P}_B^\perp$ is $\sqrt{d_{S,X}(c_B)}$, where c_B is the first allowed centre above $C_\star B$.

Proof.

Let $U_B e_c = u_c^+$, and identify Λ_B on the source space with $\hat{\Lambda}_B = \text{diag}(\mu_c)$. The coefficient table following (4.17b) gives both

$$\|V_B - U_B\|^2 \ll L^C \frac{B}{\sqrt{X}}, \quad \|(V_B - U_B)\hat{\Lambda}_B^{1/2}\|^2 \ll L^C \sqrt{\frac{B}{L}}. \quad (6.14)$$

Here are the row and column counts for the weighted estimate; the unweighted estimate follows from the same counts without the column factor $\sqrt{\mu_c}$.

- An up-centre row has at most $\omega(n) \leq L$ source representations. Its weighted column squared mass is $O(\pi_S(Y)/\mu_c) \ll \sqrt{c/L}$.
- A down-centre coefficient is $O((q\sqrt{\mu_c})^{-1})$. The matrix of weights q^{-1} has logarithmic row and column sums. Replacing μ_c by its prefix minimum bounds its squared norm by $L^C/\min \mu_c$.
- An up-leaf row has at most L source representations after its unique large prime is removed. Its weighted column squared mass is $O(\mathcal{P}_Y/\mu_c)$.
- A down-leaf row has at most $C_\star B$ sources. The inner and outer segments each have weighted column squared mass $O(L^C/\mu_c)$; on the outer segment use $d_{c/q} \ll qd_c$ with its coefficient $q^{-1}d_c^{-1}$. Their squared norms are at most $L^C B/\min \mu_c$.

The last bound is at most $L^{C'} \sqrt{B/L}$, since $B/\sqrt{X} \rightarrow 0$. The signed boundary coefficients differ from u_c^+ by $O(c/\sqrt{X})$, and their weighted squared mass is at most $L^C c^{3/2}/\sqrt{X}$. The boundary-kernel contribution has weighted Hilbert–Schmidt mass at most $L^C B^{3/2}/X$, by (4.17a); its target response has the extra factor $\|H\|/\mu_c = o(1)$. Finally projection is a contraction fixing U_B , and normalization adds only the diagonal error $O(\|z_c\|^2)$. These estimates prove (6.14).

It follows that $U_B^* W_B = I + o(1)$, proving invertibility and the graph representation. Put $Z'_B = (I - \Pi_B)V_B$. In particular

$$\mu_c \|Z'_B e_c\|^2 \ll L^C \sqrt{c/L}, \quad (6.15)$$

and the operator estimate in (6.14), rather than only this column bound, gives

$$\|Z'_B \hat{\Lambda}_B^{1/2}\|^2 \ll L^C \sqrt{B/L}. \quad (6.16)$$

To move the energy weight through W_B^{-1} , set $A_B = U_B^* W_B - I$. Its signed off-diagonal entries are nonzero only for prime-adjacent centres. After conjugation by $\hat{\Lambda}_B^{1/2}$, their bounds, with source a and output b , are

$$C \sqrt{\frac{L}{X}} \begin{cases} (ab)^{1/4}, & b = aq, \\ b^{3/4} a^{-1/4}, & b = a/q. \end{cases}$$

The Schur weight $a^{-1/2}$ gives row and column sums $L^C B^{3/4}/\sqrt{X}$. The diagonal normalization and source-amplitude terms have size $O(B/\sqrt{X})$; these terms must also be retained. The kernel-driven positive response has Hilbert–Schmidt norm at most $L^C B^{3/2}/X$, and conjugation costs at most $O(B^{1/4})$. Thus the sufficient conjugated bound is

$$\|\hat{\Lambda}_B^{-1/2} A_B \hat{\Lambda}_B^{1/2}\| \ll L^C \frac{B}{\sqrt{X}} = o(1). \quad (6.17)$$

Indeed $B^{7/4}/X \leq B/\sqrt{X}$ on this range.

The inverse conjugation is therefore uniformly bounded, and

$$\begin{aligned} \hat{\Lambda}_B^{-1/2} (I + A_B)^{-1} \hat{\Lambda}_B^{1/2} &= (I + \hat{\Lambda}_B^{-1/2} A_B \hat{\Lambda}_B^{1/2})^{-1}, \\ \|T_B \Lambda_B^{1/2}\| &\leq \|Z'_B \hat{\Lambda}_B^{1/2}\| \|\hat{\Lambda}_B^{-1/2} (I + A_B)^{-1} \hat{\Lambda}_B^{1/2}\|. \end{aligned} \quad (6.18)$$

Equations (6.14), (6.16), and (6.18) prove (6.13). Removing the positive modes with centres at most $C_\star B$ leaves forest top $\sqrt{d_{S,X}(c_B)}$; the other sectors are nonpositive. $\square$

7 Arithmetic clusters and exact complement elimination

Let

$$A = X^\theta, \quad L = \log X, \quad K_B = \left\lfloor \frac{B^3}{X} \right\rfloor, \quad (7.1)$$

and use the root scale

$$\mu_B^2 \asymp_{S,\theta} \frac{X}{BL}, \quad \Delta_B = \frac{B}{\mu_B L}. \quad (7.2)$$

Since

$$\Delta_B^2 \asymp \frac{B^3}{XL} = \frac{K_B}{L},$$

the quantity Δ_B is the root width of a cluster of K_B labels; the derivative-scale spacing between adjacent roots is $\asymp \Delta_B/K_B$.

Proposition 7.1 – target-width prime clusters

Suppose $21/61 < \theta < 1/2$. Uniformly for B in (6.1), if allowed centres $a < b$ in $[B/2, B]$ satisfy

$$c_1 K_B \leq b - a \leq c_2 K_B \quad (7.3)$$

for fixed positive c_1, c_2 , then

$$F_{S,a,X} - F_{S,b,X} \asymp_{S,\theta,c_1,c_2} \frac{B}{\log X}. \quad (7.4)$$

If the labelled list $F_{S,a,X}$ is sorted into decreasing order, each label moves by $O_S(K_B/L^4) = o(K_B)$ places. Across any fixed number of adjacent K_B -blocks, the target oscillation is $O_{S,\theta}(B/L)$.

Let $\widehat{\nu}_{B,1} \geq \cdots \geq \widehat{\nu}_{B,m_B}$ be the decreasing rearrangement of the molecule eigenvalues $\{\nu_c : c \leq C_\star B\}$. Then, uniformly for every allowed $a \in [B/2, B]$,

$$\left| \widehat{\nu}_{B,j_S(a)}^2 - F_{S,a,X} \right| \ll_{S,\theta} \frac{B}{L}. \quad (7.4a)$$

Proof.

Put

$$x = X/b, \quad h = X/a - X/b.$$

Then $x \asymp X/B$, $h \asymp B$, and

$$\frac{\log h}{\log x} = \frac{\theta}{1-\theta} + o(1) > \frac{21}{40}. \quad (7.5)$$

Two classical inputs give a two-sided short-interval count. Baker, Harman, and Pintz [16] prove, in an equivalent forward-interval form,

$$\pi(z + z^{21/40}) - \pi(z) \gg \frac{z^{21/40}}{\log z}$$

for all large z . Greedily partition $[x, x+h]$ into consecutive blocks $[z, z + z^{21/40}]$, discarding only the final incomplete block. Because $h/x \rightarrow 0$, all block lengths and logarithms are uniformly comparable to $x^{21/40}$ and $\log x$. Summing the block bounds gives the lower half of

$$\pi(x+h) - \pi(x) \asymp_{\theta} \frac{h}{\log x}, \quad (7.6)$$

uniformly whenever $h/x^{21/40} \rightarrow \infty$ and $h = o(x)$; the upper half is the Brun–Titchmarsh inequality [12], since by (7.5) the ratio $\log h / \log x$ is bounded below by a fixed θ -constant. Integer parts cost $O(1)$, and deleting the fixed set S does not change the interval count for large X . Thus the prime-count difference is $\asymp_{S,\theta} B/L$.

The square-root boundary cancels between the two star degrees. Lemma 4.3 gives $|M_{S,a,X} - M_{S,b,X}| \ll_{S,\theta} L = o(B/L)$, proving (7.4). If $b - a \geq K_B/L^4$, the interval length $h_{\text{sort}} \asymp B/L^4$ still satisfies

$$\frac{h_{\text{sort}}}{x^{21/40}} \asymp \frac{X^{(61/40)(\theta-21/61)}}{(\log X)^4} \rightarrow \infty,$$

so the lower half of (7.6) applies and

$$F_{S,a,X} - F_{S,b,X} \gg_{S,\theta} B/L^5 > 0.$$

Thus an inversion in the labelled target list can involve only centres in an interval of length K_B/L^4 . A fixed finite prime deletion changes the number of labels in such an interval by at most a fixed factor, so sorting moves each label by $O_S(K_B/L^4) = o(K_B)$ positions. The upper oscillation bound follows from the upper half of (7.6), applied across a fixed number of K_B -blocks, together with Lemma 4.3.

The sorting statement holds throughout the full retained prefix $c \leq C_* B$. Uniform PNT at fixed multiplicative scales separates labels outside an overlapping dyadic neighborhood by $\gg X/(BL)$, while the preceding short-interval argument handles the overlapping neighborhoods. Thus no distant label can cross a target in $[B/2, B]$.

Finally Proposition 4.2 gives, uniformly for $c \asymp B$,

$$|\nu_c^2 - F_{S,c,X}| = o_{S,\theta}(B/L^5).$$

This error is smaller than the separation just used, so sorting the ν_c has the same $O_S(K_B/L^4)$ label displacement. The target oscillation bound then proves (7.4a). $\square$

Lemma 7.2 – quantitative cluster-boundary transport

Let $x_1 \geq \dots \geq x_m$ and $y_1 \geq \dots \geq y_m$, and let $K \rightarrow \infty$ with $K = o(m)$. Suppose that at all but N grid boundaries $b = rK$,

$$|N_x(y_b) - b| \leq C_0 K, \quad (7.6a)$$

where $N_x(t) = \#\{i : x_i \geq t\}$. If the target oscillation across any fixed number of adjacent grid blocks is at most η , then

$$\#\{i : |x_i - y_i| > C\eta\} \ll_{C_0} K(N+1). \quad (7.6b)$$

Proof. Put $R = \lceil C_0 \rceil + 2$, and discard every block within R grid steps of a bad boundary. These blocks contain $O_{C_0}(KN)$ indices. For an index i in a remaining block, choose good grid boundaries $b_- < i < b_+$ at distances between RK and $(R+1)K$. Equation (7.6a) gives

$$N_x(y_{b_-}) < i \leq N_x(y_{b_+}),$$

and hence $y_{b_+} \leq x_i \leq y_{b_-}$. The target-oscillation hypothesis gives $|x_i - y_i| \leq C\eta$. The two outer incomplete block neighborhoods cost $O_{C_0}(K)$ further indices. $\square$

The elementary scale facts used below are

$$\frac{B}{\sqrt{X}} \rightarrow 0, \quad \frac{\|H_{S,X}\|}{\mu_B} \rightarrow 0, \quad \frac{(\log X)^C}{\mu_B} = o(\Delta_B), \quad (7.7)$$

$$K_B/(\log X)^C \rightarrow \infty, \quad K_B = o(B), \quad (7.8)$$

uniformly in (6.1). The lower limit in (7.8) uses $\theta > 21/61 > 1/3$; the other limits use $\theta < 1/2$.

Write

$$\begin{aligned} M_B &= Q_B^* G_{S,X} Q_B, & C_B &= (I - Q_B Q_B^*) G_{S,X} (I - Q_B Q_B^*), \\ R_B &= (I - Q_B Q_B^*) G_{S,X} Q_B. \end{aligned} \quad (7.9)$$

For a target centre $a_b \in [B/2, B]$, set

$$z_b = \sqrt{F_{S,a_b,X}}. \quad (7.10)$$

Proposition 7.3 – separated complement

For $C_\star$ sufficiently large and all large X ,

$$\sup \sigma(C_B) \leq z_b - c_{S,\theta} \mu_B \quad (7.11)$$

uniformly in B and b .

Proof.

Let $\mathcal{Q}_B = \text{ran } Q_B$. By Lemma 6.3 this is the graph of $T_B : \mathcal{P}_B \rightarrow \mathcal{P}_B^\perp$. The large-prime forest reduces $\mathcal{P}_B \oplus \mathcal{P}_B^\perp$, and its top on the second summand is $\sqrt{d_{S,X}(c_B)} \geq 0$, where c_B is the first allowed centre above $C_\star B$. Thus the sign hypothesis in Lemma 5.3 holds, and that lemma gives the direct Rayleigh bound

$$\sup \sigma(C_B) \leq \sqrt{d_{S,X}(c_B)} + \|T_B \Lambda_B^{1/2}\|^2 + \|H_{S,X}\|. \quad (7.14)$$

Here Lemma 5.3 is first applied to $L_{S,X} + H_{S,X} = A_{S,X}$ on the full graph of T_B . The operator C_B is the further compression of that orthogonal-complement form to $\mathcal{R} \cap \mathcal{Q}_B^\perp$; restricting the form can only lower its spectral supremum. Thus (7.14) applies to the one-exit complement used in (7.9).

Uniform PNT at the two fixed multiplicative scales gives $d_{S,X}(c_B) \leq (2/C_\star + o(1))\mu_B^2$, whereas every target root in $[B/2, B]$ is bounded below by a fixed multiple of μ_B . By (6.13) and (7.7), the last two terms in (7.14) are $o(\mu_B)$. Choose $C_\star$ large. This proves (7.11). $\square$

8 Quantitative cluster-boundary estimate

The every-centre proof below uses a threshold count. We first record the Hilbert–Schmidt cluster estimate from which that refinement starts.

Proposition 8.1 – quantitative almost-all comparison

Fix $21/61 < \theta < 1/2$, put $A = X^\theta$, and let

$$\mathcal{E}_B = \{a \in V_S(X) : B/2 \leq a \leq B, \ |\lambda_{j_S(a)}(A_{S,X})^2 - F_{S,a,X}| > C_{S,\theta} B / \log X\}.$$

For a sufficiently large fixed exponent $D = D(S, \theta)$,

$$|\mathcal{E}_B| \ll_{S,\theta} (\log X)^D \left(\frac{X}{B^2} + \frac{B^3}{X} \right). \quad (8.9)$$

Consequently the union of these exceptional sets over $A/\log X \leq B \leq A$ has the same form with $B = A$, up to a further power of $\log X$.

Proof. We first prove the result for $G_{S,X}$. Restrict to

$$A/L \leq a \leq A, \quad A = X^\theta, \quad (8.1)$$

and decompose this interval into dyadic centre bands $[B/2, B]$. Fix one such band. Let $\xi_{B,1} \geq \dots \geq \xi_{B,m_B}$ be the eigenvalues of M_B , and retain the notation $\widehat{\nu}_{B,k}$ from Proposition 7.1. Corollary 6.2 and (5.5) give

$$\sum_{k=1}^{m_B} |\xi_{B,k} - \widehat{\nu}_{B,k}|^2 \leq \|M_B - D_B\|_{\text{HS}}^2 \ll_{S,\theta} BL^{C_S}. \quad (8.2)$$

Put $K = K_B$, and list the allowed labels in the band in increasing order. At a grid boundary $b = rK$, compare the tail count of the $\xi_{B,k}$'s at the target root $z_b = \sqrt{F_{S,a_b,X}}$ with its arithmetic index. If this count is wrong by more than a fixed multiple of K , then Proposition 7.1 forces at least K ordered pairs in (8.2) to move by $\gg_{S,\theta} \Delta_B$. Choosing a disjoint subfamily of bad boundaries gives

$$N_B \ll_{S,\theta} \frac{BL^{C_S}}{K\Delta_B^2} \ll_{S,\theta} \frac{BL^{C_S+1}}{K^2}, \quad (8.3)$$

where N_B denotes the number of bad grid boundaries and $\Delta_B^2 \asymp K/L$. This is the step at which an exceptional set enters: (8.2) controls the total squared displacement, not its maximum.

It remains to transfer the good boundary counts from the retained prefix to the full signed one-exit operator. Proposition 7.3 separates the complement from every target root by $\gg_{S,\theta} \mu_B$. For z in the target window, exact Schur elimination and (6.11) give

$$\|R_B^*(z - C_B)^{-1} R_B\| \ll_{S,\theta} \frac{L^{C_S}}{\mu_B} \left(1 + \frac{B^2}{\sqrt{X}} \right). \quad (8.4)$$

Relative to the root width this is

$$\frac{\|R_B^*(z - C_B)^{-1} R_B\|}{\Delta_B} \ll_{S,\theta} L^{C_S+1} \left(\frac{1}{B} + \frac{B}{\sqrt{X}} \right) = o(1), \quad (8.5)$$

uniformly for $A/L \leq B \leq A$. Sylvester inertia therefore changes the prefix count only across a root interval of width $o(\Delta_B)$. Proposition 7.1 shows that such an interval contains $O_{S,\theta}(K)$ target roots. Hence, outside the N_B boundaries counted in (8.3),

$$|N_G(F_{S,a_{rK},X}) - rK| \ll_{S,\theta} K, \quad (8.6)$$

where $N_G(t) = \#\{j : \rho_j(G_{S,X})^2 \geq t\}$, with the fixed offset of the dyadic band understood. Lemma 7.2 and the oscillation estimate in Proposition 7.1 now give

$$\rho_{j_S(a)}(G_{S,X})^2 = F_{S,a,X} + O_{S,\theta}(a/L) \quad (8.7)$$

outside a set of size

$$\ll_{S,\theta} K(N_B + 1) \ll_{S,\theta} \frac{XL^{C_S+1}}{B^2} + \frac{B^3}{X}. \quad (8.8)$$

Finally choose a fixed ε with $0 < \varepsilon < 1/2 - \theta$. Theorem 3.2 transfers (8.7) from $G_{S,X}$ to the full adjacency $A_{S,X}$ at an additional cost $O_{S,\theta}(a/L)$. Summing (8.8) over the dyadic bands $A/L \leq B \leq A$ gives

$$\ll_{S,\theta} L^{D_{S,\theta}} \left(\frac{X}{A^2} + \frac{A^3}{X} \right)$$

exceptions. This proves (8.9). $\square$

9 Buffered threshold counts and proof of Theorem 1.1

To compare eigenvalues at the cluster root width Δ_B , we separate the coherent common-descendant interactions into couplings with spectrally distant molecules and interactions within the target band.

Put

$$L = \log X, \quad B/2 \leq a \leq B, \quad \mu_B^2 \asymp \frac{X}{BL},$$

and define the target label and root widths

$$K_B = \frac{B^3}{X}, \quad \Delta_B = \frac{B}{\mu_B L}, \quad \Delta_B^2 \asymp \frac{B^3}{XL}. \quad (9.1)$$

In the range $21/61 < \theta < 1/2$, K_B is a fixed positive power of X .

Lemma 9.1 – buffered exterior elimination

Choose fixed $0 < c_- < 1/2 < 1 < C_+ < C_*$, and retain labels in

$$\mathcal{A}_B = \{c : c_-B \leq c \leq C_+B\}. \quad (9.2)$$

For thresholds in a fixed root interval containing the targets for $B/2 \leq a \leq B$, the exterior spectrum is separated by $\gg \mu_B$. Its inertia contribution is constant, and its Schur self-energy satisfies

$$\frac{\|\Sigma_{\text{out}}(t)\|}{\Delta_B} \ll_{S,\theta} X^{o(1)} \left(\frac{1}{B} + \frac{B}{\sqrt{X}} \right) = o(1). \quad (9.3)$$

Proof.

Use the orthogonal coordinates $\text{ran } Q_B \oplus (\mathcal{R} \cap \text{ran } Q_B^\perp)$. In these coordinates G has blocks $M = D_{\text{prefix}} + Q_B^* E_B$, $R = (I - Q_B Q_B^*) E_B$, and C_B . Proposition 7.3 puts C_B below the whole target interval after C_* is chosen sufficiently large.

For labels outside the buffer, $\nu_c/\mu_c = 1 + o(1)$ and PNT at fixed multiplicative scales put $c < c_-B$ above the target interval and $c > C_+B$ below it, with a gap $\gg \mu_B$. Thus $\text{diag}(\nu_c : c \notin \mathcal{A}_B) \oplus C_B$ is gapped on both sides of every target threshold. The actual exterior differs from this reference by $O(\|E_B\|)$. By (6.11),

$$\frac{\|E_B\|^2}{\mu_B^2} \ll L^C \left(\frac{B}{X} + \frac{B^3}{X^{3/2}} \right) = o(1).$$

The homotopy multiplying this difference by a parameter in $[0, 1]$ therefore preserves the gap and its inertia. At parameter zero the above-threshold exterior count is exactly the number of allowed labels

$c < c_-B$, since C_B and the later labels are below the threshold. This identifies the constant offset r_B with the arithmetic prefix count.

The coupling from the retained buffer to the exterior is $O(\|E_B\|)$, because D_{prefix} is diagonal. Two-sided Schur elimination gives

$$\|\Sigma_{\text{out}}(t)\| \ll \frac{\|E_B\|^2}{\mu_B} \ll \frac{L^C}{\mu_B} \left(1 + \frac{B^2}{\sqrt{X}}\right).$$

Divide by $\Delta_B = B/(\mu_B L)$ to obtain (9.3). The coherent $g = 1$ prime rectangle is included in the coupling to the distant exterior, so its contribution is bounded through this Schur self-energy. $\square$

Lemma 9.2 – internal core budget

Let $D_B = \text{diag}(\nu_c : c \in \mathcal{A}_B)$. The fixed local block, before adding its exterior self-energy, has an exact decomposition

$$D_B + \mathcal{K}_{B,\text{int}} + \mathcal{R}_{B,\text{noncoh}}, \quad (9.4)$$

with

$$\boxed{\|\mathcal{K}_{B,\text{int}}\|_{\text{HS}}^2 \ll_{S,\theta} X^{o(1)} \Delta_B^2, \quad \|\mathcal{R}_{B,\text{noncoh}}\| \leq X^{o(1)} = o(\Delta_B).} \quad (9.5)$$

The first term includes the coherent entries and the small kernel, projection, and Gram corrections, including entries introduced by reorthonormalization. As usual, the combined $X^{o(1)}$ is $O_{S,\theta,\xi}(X^\xi)$ for each fixed $\xi > 0$.

Proof.

Use the raw splitting $\tilde{f}_a = s_a + i_a + k_a^{\text{ker}}$ from Proposition 4.2, and put $g_a^s = (I - P_a)H_{S,X}i_a$. On the full prefix define the complete signed-response pairing

$$(\mathcal{S}_B)_{ba} = \langle s_b + i_b, g_a^s \rangle.$$

This matrix, not the selected coefficients $\langle u_b^+, g_a \rangle$, is the starting comparison object.

Coherent entries. A prime-sibling pair has the form

$$a = gr, \quad b = gq, \quad r, q \text{ prime}, \quad r \neq q. \quad (9.6)$$

For a fixed retained large-prime signature, i_b has parity $\Omega(b) + 1$, while g_a^s has parity $\Omega(a)$. These are opposite for a sibling pair, so $\langle i_b, g_a^s \rangle = 0$ exactly. An edge from the boundary at b to a target at a reduces to $qt^\sigma = ru^\tau$. Unique factorization leaves only the common down-target g and common up-target grq . Both active and inactive up-targets are included.

For example, if $gr \leq gq$ and $grq > Y$, write $s_a = \alpha\delta_{gr} + \beta\psi_{gr}$ and put $z = \nu_{gr}/\mu_{gr}$, $s = d_g/d_{gr}$, and $t = d_{gq}/d_{gr}$. Pairing the common-core response with u_{gq}^+ gives

$$\frac{1}{\sqrt{2}\mu_{gr}} \left[\frac{\alpha}{z} + \frac{z\alpha + \beta}{z^2 - s} + \sqrt{t} \left\{ \frac{\alpha + z\beta/s}{z^2 - s} + \frac{\beta(1 - 1/s)}{z} \right\} \right]. \quad (9.7a)$$

For a denominator bound uniform in r , put $x = X/(gr)$ and $P_Y = \pi_S(Y)$. Monotonicity gives

$$s = \frac{\pi_S(rx) - P_Y}{\pi_S(x) - P_Y} \geq \frac{\pi_S(2x) - P_Y}{\pi_S(x) - P_Y} \geq \frac{\pi_S(2x)}{\pi_S(x)} = 2 + o_{S,\theta}(1). \quad (9.7b)$$

The second inequality uses $\pi_S(2x) \geq \pi_S(x) > P_Y$. Since $z^2 = 1 + o(1)$,

$$|z^2 - s| \geq \frac{1}{2} \quad (9.7c)$$

eventually. The output ratio t is not inverted; $t = 1$ is allowed.

The coefficient table (4.17b) also covers the reverse orientation and active common up-targets. A centre contribution is $O(1/\mu_a)$. For an up-target there are at most d_b output leaves, each with coefficient $O(1/d_a)$. For the down-target split the output leaves at X/a ; the inner coefficient is $O(1/d_a)$, and

the outer coefficient is $O(1/(rd_a))$. Pairing with either output-star sign therefore gives, in both orientations,

$$|(\mathcal{S}_B)_{ba}| \ll_{S,\theta} \frac{1}{\mu_a} + \frac{\mu_b}{\mu_a^2}. \quad (9.7)$$

This estimates the full signed output s_b , since its two coefficients are bounded, and its interior contribution has already vanished.

For a given pair a, b , the core is uniquely $g = \gcd(a, b)$; its two common targets are added within that entry. There are at most $O(B/g)$ possible sources and outputs, and $\mu_a, \mu_b \asymp \mu_B$ in the buffer. Hence

$$\sum_g \sum_{\substack{a=gr, b=gq \\ a, b \in \mathcal{A}_B}} |(\mathcal{S}_B)_{ba}|^2 \ll \sum_g \frac{B^2}{g^2} \frac{BL}{X} \ll \frac{B^3 L}{X} \asymp L^2 \Delta_B^2. \quad (9.8)$$

Thus the contributions of all cores are aggregated at the actual matrix entries before taking the norm.

Noncoherent entries. If the source and output have the same parity, only the boundary pairing can occur. Outside the sibling mask, its nonzero entries have $a/b = qr$ or $b/a = qr$. Since both labels lie in the fixed buffer, $qr \leq C_+/c_-$; each row has only boundedly many such partners. The coefficient bound (4.17b) makes their operator norm $O(1/\mu_B)$.

For opposite parities only $\langle i_b, g_a^s \rangle$ occurs. Equality of its labels uses one prime step at b and two at a . A free common small prime uses one slot on each side, leaving a/b or b/a equal to one prime. That prime is bounded by C_+/c_- , so there are boundedly many such partners in each row. Their centre-coordinate sum is $O(L^C Y/(\mu_a \mu_b))$. For leaf coordinates, sum a possible common small prime together with the unique large prime under their product cutoff. If the common step is upward, its prime q and the retained prime p satisfy $pq \ll X/B$, giving $O(XL^C/B)$ choices. If the common step is downward, q divides the source and has only $O(L)$ choices. The inner segment has $p \ll X/B$; the outer segment has $p \ll Xq/B$ but retains a factor q^{-1} from each outer response. Splitting inner and outer segments therefore gives the same $O(XL^C/B)$ weighted count. The remaining source/output ratio is a bounded prime, so changing its orientation changes only the fixed constant. If no free common small prime remains, all three small-prime slots are fixed by the reduced quotient, up to L^C bounded-word choices. Centre terms then cost $O(L^C/(\mu_a \mu_b))$; leaf terms satisfy the weighted-range inequality preceding (6.8a). Summing these bounds over the at most $O(B)$ partners gives, in both row and column orientation,

$$\|\text{sym}(\mathcal{S}_B)_{\text{noncoh}}\| \ll L^C \left(\mu_B^{-1} + \frac{B}{\sqrt{X}} + \frac{B^2}{X} \right) \leq L^{C'}.$$

The diagonal is zero, since g_a^s is exterior to its own molecule.

Complete vectors and the fixed local matrix. Let $\tilde{J}_B$ be the raw full residual synthesis matrix. Equation (4.17a) gives $\sum_a \|k_a^{\text{ker}}\|^2 \ll L^C B^2/X^{3/2}$. The kernel-driven residual has squared Hilbert–Schmidt norm $O(L^C B^3/X^{3/2})$. Since $\|\tilde{V}_B\| = O(1)$ and $\|\tilde{J}_B\|_{\text{HS}}^2 \ll BL^C$, it follows that

$$\|\tilde{V}_B^* \tilde{J}_B - \mathcal{S}_B\|_{\text{HS}}^2 \ll L^C B^3/X^{3/2}.$$

Next use the exact projection identities (6.8c), including Z_B , HZ_B , and N_B . Equations (6.8d) bound their total paired-matrix cost. Finally (5.5a), on the full prefix before local compression, prices the Gram correction. Combining the three costs gives

$$\|Q_B^* G_{S,X} Q_B - D_{\text{prefix}} - \text{sym}(\mathcal{S}_B)\|_{\text{HS}}^2 \ll L^C \left(\frac{B^2}{X} + \frac{B^4}{X^{3/2}} \right) = o(\Delta_B^2). \quad (9.9)$$

For projection, the terms $Z_B^* \tilde{J}_B$ and $(P_{\mathcal{R}} \tilde{V}_B)^* HZ_B$ cost, respectively, $L^C (B^3/X^{3/2} + B^5/X^2)$ and $L^C (B^2/X + B^4/X^{3/2})$; normalization is smaller. For the Gram term use (5.5b) and $B/\sqrt{X} = o(1)$. The ratio of the displayed majorant to Δ_B^2 is $O(L^{C+1}(1/B + B/\sqrt{X}))$, which tends to zero.

Restrict (9.9) to the buffer. Define $\mathcal{R}_{B,\text{noncoh}}$ to be the noncoherent part of $\text{sym}(\mathcal{S}_B)$, and $\mathcal{K}_{B,\text{int}}$ to be its coherent part plus the exact matrix difference in (9.9). This defines (9.4) on the same coordinate space and proves (9.5). The matrix difference in (9.9) accounts for the additional entries introduced by Gram conjugation. $\square$

Proposition 9.3 – threshold-count comparison

Let $N_G(t)$ count positive one-exit eigenvalues at least t , and let $N_D(t)$ count entries of D_B at least t . There is an integer r_B , independent of t in the target window, such that

$$r_B + N_D(t + 2\Delta_B) - X^{o(1)} \leq N_G(t) \leq r_B + N_D(t - 2\Delta_B) + X^{o(1)}. \quad (9.10)$$

Proof. Lemma 9.1 and Sylvester inertia reduce the full count at each fixed threshold to the local matrix (9.4), with the constant higher-energy exterior count r_B , at an operator cost $o(\Delta_B)$. Apply Hoffman–Wielandt only to the fixed matrix $\mathcal{K}_{B,\text{int}}$. A root that crosses a threshold by more than Δ_B spends at least Δ_B^2 of (9.5), so at most $X^{o(1)}$ roots can do so. The noncoherent term and the exterior Schur term consume the other root-width unit. $\square$

Proof of Theorem 1.1. Proposition 7.1 implies, uniformly for every allowed $a \in [B/2, B]$,

$$\begin{aligned} r_B + N_D(\nu_a - c\Delta_B) - j_S(a) &\gg_{S,\theta} K_B, \\ j_S(a) - r_B - N_D(\nu_a + c\Delta_B) &\gg_{S,\theta} K_B \end{aligned} \quad (9.11)$$

for a sufficiently large fixed c . To see the margin, first fix a positive integer m , then take mK_B allowed labels on either side of a . A fixed finite deletion set has bounded gaps between allowed integers, so these labels lie within $O_S(mK_B)$ in centre distance. The upper target-oscillation estimate in Proposition 7.1 places all their squared targets within $C_{S,\theta,m}B/L$ of the target at a , and hence all their molecule roots within $C'_{S,\theta,m}\Delta_B$ of ν_a . Choose $c > C'_{S,\theta,m} + 2$ after m . Both label blocks are then contained in the corresponding root window. Since $mK_B/B \rightarrow 0$, the buffered band contains these blocks even at a dyadic endpoint. Sorting moves only $o(K_B)$ labels, leaving a fixed positive multiple of K_B on each side. The offset r_B accounts for the higher-energy labels outside the local ordering.

Since $\theta > 21/61 > 1/3$,

$$X^{o(1)} = o(K_B).$$

Combining (9.10)–(9.11), with a larger fixed root-width constant, gives

$$N_G(\nu_a - C\Delta_B) \geq j_S(a), \quad N_G(\nu_a + C\Delta_B) \leq j_S(a) - 1. \quad (9.12)$$

Thus

$$|\rho_{j_S(a)}(G_{S,X}) - \nu_a| \ll_{S,\theta} \Delta_B$$

for every allowed centre in the terminal band. Squaring costs

$$O(\mu_B\Delta_B + \Delta_B^2) = O(B/L),$$

because $\Delta_B^2/(B/L) = B^2/X = o(1)$. Proposition 4.2 replaces ν_a^2 by $F_{S,a,X}$, and Theorem 3.2 transfers the result to $A_{S,X}$ at the same squared scale. Summing over the dyadic bands $X^\theta/L \leq B \leq X^\theta$ proves (1.8).

Lemma 4.3 gives $M_{S,a,X} \ll_{S,\theta} \log X = o(a/\log X)$ on the terminal band, proving Corollary 1.2. $\square$

Corollary 9.4 – terminal-band counting rigidity

Under the hypotheses of Theorem 1.1, let $[B/2, B]$ be a consumed dyadic band and put $K_B = \lfloor B^3/X \rfloor$. There are constants $C_1, C_2 > 0$, depending only on S and θ , with the following property. Let $I = [u, v]$ be an interval for which

$$[u - C_1K_B, v + C_1K_B] \subset [B/2, B],$$

and suppose that $I_S = I \cap V_S(\infty)$ is nonempty. For

$$J_I = \left[\min_{a \in I_S} F_{S,a,X} - C_2 \frac{B}{\log X}, \max_{a \in I_S} F_{S,a,X} + C_2 \frac{B}{\log X} \right],$$

one has

$$|\#\{k : \lambda_k(A_{S,X})^2 \in J_I\} - \#I_S| \ll_{S,\theta} K_B. \quad (9.13)$$

Proof.

Theorem 1.1 places the eigenvalue at every rank $j_S(a)$, $a \in I_S$, inside J_I once C_2 is sufficiently large. Proposition 7.1 shows that moving a fixed multiple of K_B allowed labels past either endpoint changes the target by a fixed multiple of $B/\log X$, while sorting the labelled targets moves each label by only $o(K_B)$ places. Choose C_1 after C_2 . Theorem 1.1 then puts the corresponding exterior boundary eigenvalues strictly above and below J_I . Since the positive eigenvalues are ordered, only $O_{S,\theta}(K_B)$ ranks adjacent to the two endpoints can enter or leave the window. This proves (9.13). $\square$

10 Ordered comparison and proof of Theorem 1.3

The density-one argument uses the full prefix and the maximum-norm contraction of sorted values. Throughout this section fix $0 < \theta < 1/2$, $L = \log X$, $A = X^\theta$, and a dyadic scale $A/L \leq B \leq A$. Put

$$\mathcal{A}_B = [B/2, B] \cap V_S(\infty), \quad \mu_B = \sqrt{\frac{X}{BL}}, \quad \mathfrak{d}_{a,X} = |\lambda_{j_S(a)}(A_{S,X})^2 - F_{S,a,X}|.$$

Theorem 10.1 – ordered mean-square and tail estimates

There are constants $C, C_0, C_1, C_2 > 0$, depending only on S, θ , such that, uniformly in B for all sufficiently large X ,

$$\sum_{a \in \mathcal{A}_B} \mathfrak{d}_{a,X}^2 \ll_{S,\theta} XL^C + \frac{B^3}{L^2}. \quad (10.1)$$

More precisely, for every $t > 0$,

$$\#\{a \in \mathcal{A}_B : \mathfrak{d}_{a,X} > C_0 B/L + C_1 t \mu_B\} \leq C_2 \frac{BL^C}{t^2}. \quad (10.2)$$

Proof.

Retain all allowed centres $c \leq C_\star B$, with $C_\star$ a sufficiently large fixed constant, and use Q_B, D_B, E_B from Section 6. Choose $\theta < \vartheta < 1/2$; then $C_\star B \leq X^\vartheta$ eventually. Proposition 4.2 and Lemma 4.3 give, uniformly on this complete prefix,

$$\nu_c > 0, \quad |\nu_c^2 - p_c| \ll_{S,\theta} L, \quad p_c = \pi_S(X/c). \quad (10.3)$$

Indeed the error in (4.9) is $o(1)$, using $\omega(c) \ll \log c$ and $C_\star B/\sqrt{X} \rightarrow 0$.

The list p_c , in arithmetic order of c , is nonincreasing. Apply (5.12) to the squared molecule energies. Since all molecule roots are positive, sorting their squares also sorts the roots. Thus

$$|\widehat{\nu}_{j_S(a)}^2 - p_a| \ll L, \quad |\widehat{\nu}_{j_S(a)}^2 - F_{S,a,X}| \ll L \quad (a \in \mathcal{A}_B). \quad (10.4)$$

The index here is the global arithmetic rank: the prefix includes every allowed centre smaller than a . No band-local reindexing or strict separation of the p_c 's is used. Global PNT gives $\widehat{\nu}_{j_S(a)} \asymp_{S,\theta} \mu_B$.

Write $e_B = \|E_B\|$ and $h_B = \|E_B\|_{\text{HS}}$. Corollary 6.2 gives

$$h_B^2 \ll BL^C, \quad e_B^2 \ll L^C(1 + B^2/\sqrt{X}), \quad \frac{e_B^2}{\mu_B^2} \ll L^{C+1} \left(\frac{B}{X} + \frac{B^3}{X^{3/2}} \right) = o(1). \quad (10.5)$$

The complementary compression is below the entire target band. To see this without an arithmetic cluster, apply Lemmas 5.3 and 6.3 directly:

$$\sup \sigma(C_B) \leq \sqrt{d_{S,X}(c_B)} + L^C \sqrt{B/L} + \|H_{S,X}\|, \quad (10.6)$$

where c_B is the first allowed centre above $C_\star B$. Relative to μ_B , the last two terms are $O(L^C B/\sqrt{X})$ and $O(\sqrt{B}/X^{1/4})$, respectively, hence are $o(1)$. By PNT the first term is $O_{S,\theta}(C_\star^{-1/2} \mu_B)$. Choose $C_\star$

large enough for the complement condition in (5.8). The constants in the lower bound for the target roots are independent of this prefix factor.

Lemma 5.4, applied to $T = G_{S,X}$, now supplies errors $\epsilon_j \geq 0$ satisfying

$$\sum_j \epsilon_j^2 \ll BL^C, \quad \max_j \epsilon_j \leq e_B, \quad |\rho_{j_S(a)}(G_{S,X})^2 - \widehat{\nu}_{j_S(a)}^2| \ll \mu_B \epsilon_{j_S(a)} + e_B^2. \quad (10.7)$$

Its ordered indices are unchanged by the complement elimination. In particular the relevant eigenvalues are positive, so they are the positive eigenvalues used in Theorem 3.2.

The deterministic Schur cost is smaller than B/L :

$$\frac{e_B^2}{B/L} \ll L^{C+1} \left(\frac{1}{B} + \frac{B}{\sqrt{X}} \right) = o(1). \quad (10.8)$$

Both limits are uniform on $A/L \leq B \leq A$, since $0 < \theta < 1/2$ is fixed. Also $L = o(B/L)$. Combining (10.4), (10.7), and the full-graph transfer (3.5) gives

$$\mathfrak{d}_{a,X} \leq C_0 B/L + C_1 \mu_B \epsilon_{j_S(a)} \quad (a \in \mathcal{A}_B). \quad (10.9)$$

Squaring and summing proves (10.1), because $\mu_B^2 BL^C = XL^{C-1}$, which is absorbed after increasing C . Markov's inequality applied to the squared errors ϵ_j^2 proves (10.2). $\square$

Global prime estimates suffice for this proof. Sorting contraction in (10.4) accommodates repeated targets, and the coherent term retained in (10.5) is bounded using the complement gap.

Proof of Theorem 1.3.

Fix $R > 0$ and take $t = L^D$ in (10.2), with $2D > C + R$. Outside $O_{S,\theta,R}(B/L^R)$ centres of each dyadic band,

$$\mathfrak{d}_{a,X} \ll_{S,\theta,R} \frac{a}{L} + L^D \sqrt{\frac{X}{aL}}. \quad (10.10)$$

Here a and B differ by at most a factor of two. Cover $[A/L, A]$ by the bands with upper endpoints $A, A/2, \dots$, stopping when the upper endpoint first lies in $[A/L, 2A/L)$. Their lengths have geometric sum $O(A)$. All estimates use the same constants, so their exceptional counts sum to $O(A/L^R)$. The omitted initial segment has $O(A/L)$ centres. This proves the stated density-one conclusion below A , as well as the sharper terminal-band exceptional count. Below A , the two contributions to the exceptional set have sizes $O(A/L^R)$ and $O(A/L)$, respectively.

For the relative conclusion, division of (10.10) by $\pi_S(X/a) \asymp X/(aL)$ leaves

$$O_{S,\theta,R} \left(\frac{a^2}{X} + L^{D+1/2} \sqrt{\frac{a}{X}} \right) = o(1).$$

Thus (1.10) holds throughout the asserted range. $\square$

Proof of Corollary 1.4.

On the retained terminal bands, the two error exponents in (1.10), apart from fixed logarithmic factors, are θ and $(1 - \theta)/2$. Their margins below the block-baseline exponent $1/2$ are $1/2 - \theta$ and $\theta/2$. Any smaller positive margin absorbs the logarithmic factors. $\square$

Corollary 10.2 – quantitative sharp density-one law

If $1/3 < \theta < 1/2$, then on each retained dyadic band the number of centres for which

$$|\lambda_{j_S(a)}^2 - F_{S,a,X}| > C_3 B/L$$

is at most

$$O_{S,\theta} \left(\frac{XL^{C+1}}{B^2} \right) = o(B). \quad (10.11)$$

Consequently the error is $O_{S,\theta}(a/L)$ outside $o(X^\theta)$ centres below X^θ .

Proof.

Set $t = B/(\mu_B L)$ in (10.2) and take $C_3 = C_0 + C_1$. Then $BL^C/t^2 = XL^{C+1}/B^2$, and its ratio to B tends to zero uniformly because $B \geq X^\theta/L$ and $3\theta > 1$. The dyadic summation and initial segment are handled as above. $\square$

Corollary 10.3 – every-centre bound on a sub-square-root band

For every fixed $0 < \theta < 1/2$, every allowed $X^\theta/L \leq a \leq X^\theta$ satisfies

$$\boxed{|\lambda_{js(a)}^2 - F_{S,a,X}| \ll_{S,\theta} \frac{a}{L} + L^{C'} \left[\sqrt{\frac{X}{aL}} + X^{1/4} \sqrt{\frac{a}{L}} \right]}. \quad (10.12)$$

This is smaller by a positive power than the block baseline $\sqrt{X}/L$.

Proof.

Use $\epsilon_j \leq e_B$ in (10.9), and take the square root of the corrected operator bound in (10.5):

$$\mu_B e_B \ll L^{C'} \left[\sqrt{\frac{X}{BL}} + X^{1/4} \sqrt{\frac{B}{L}} \right].$$

Replace B by a within its dyadic band. The three error exponents are θ , $(1 - \theta)/2$, and $1/4 + \theta/2$. Their minimum margin below $1/2$ is

$$\min\{\theta/2, (1 - 2\theta)/4\} > 0.$$

This proves the comparison after absorbing logarithmic factors. Theorem 1.1 gives the sharper error on its stated range. $\square$

11 Scope of the argument

Theorem 1.1 identifies every prescribed rank in $[X^\theta/\log X, X^\theta]$ for $21/61 < \theta < 1/2$. The proof eliminates couplings with spectrally distant modes and bounds the number of internal eigenvalues that can cross a cluster threshold. The prime-counting comparison supplies more arithmetic labels in each cluster than this bound allows to cross. The initial segment below $X^\theta/\log X$ is the theorem's only omission among centres up to X^θ .

The ordered mean-square argument gives Theorem 1.3 throughout $0 < \theta < 1/2$, using global prime estimates. Its error improves the positive-star block benchmark by a positive power of X . For $\theta > 1/3$, Corollary 10.2 gives the sharp $O(a/\log X)$ error outside a quantitative exceptional set. Corollary 10.3 gives a weaker every-centre improvement throughout the full sub-square-root range.

At $B \asymp \sqrt{X}$, the frame and squaring factor B^2/X ceases to tend to zero, the retained prefix reaches the large-prime cutoff, and $\|H_{S,X}\|/\mu_B$ is no longer small. The target-star denominator in (9.7b) remains separated from zero. Extending the results to the endpoint requires control of the full boundary block beyond the smallness estimates used here.

Formal verification

The Palomar registration [17, version 1] covers the Lean 4 formalization of Theorem 1.3 and Corollary 1.4. The selected declarations were checked by the Lean kernel and NanoDa. These two density-one statements constitute the registered verification scope. Formalization of the sharp every-centre Theorem 1.1 and Corollary 1.2 is work in progress.

Declaration of generative AI and AI-assisted technologies in the manuscript preparation process

During the preparation of this work, the author used OpenAI Codex, including AI-assisted research and coding tools, to support mathematical exploration, proof-development assistance, manuscript revision, finite diagnostic design, and Lean 4 formalization. The author reviewed and edited the output as needed and takes full responsibility for the content of this manuscript.

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