

Selective normality and algebraic rigidity of multiplicative traces

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Abstract

A completely multiplicative sign function may be changed so that two quadratic traces remain identical at every index while a third becomes normal. We classify this freedom for the traces $n^2 + 1$, $n^2 + 2$ and $n^2 - 2$. After two traces and a finite prefix have been protected, a monic irreducible polynomial can be made normal from every prescribed source exactly when the field generated by one of its roots contains neither protected quadratic field. In the complementary case, every value of the polynomial is already fixed. A single decidable set of primes of relative density zero supports a continuous, relatively computable completion treating all admissible polynomials jointly, with uniform bounds for each fixed observation family. The proof combines weighted root counts with a sieve for moments whose order increases with the observation scale, followed by a selection using exact finite normalisers. We also give a sufficient condition on prime availability, explicit sparser reservoirs, and a divergent reservoir on which some sources have no completion with a convergent ordinary mean, despite convergence of all fixed word laws in probability. For larger protected families, the obstruction is governed by odd products of quadratic square classes. An explicit example distinguishes certificates for every numerical value from a rational norm identity in the variable.

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1 Which observations can change after two norms are protected?

Suppose that every value of a completely multiplicative sign function is prescribed on the two families

$$a^2 + 2b^2, \quad |a^2 - 2b^2|,$$

where only non-zero values are considered. These are infinitely many constraints, and they include the traces $n^2 + 2$ and $n^2 - 2$. They do not, however, determine the statistics of the remaining trace $n^2 + 1$. Starting from any source, one can preserve both families exactly and arrange that every sign word of length H on the remaining trace has frequency 2^{-H} , for every fixed H .

There is a simple reason why changes are possible. A prime $q \equiv 5 \pmod{8}$ is inert in both $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{-2})$. Its valuation in either protected norm is even. Changing the sign at q is therefore invisible to both norms. The same prime has two simple roots on $n^2 + 1$, so it can act on that trace. The difficulty is not to find one invisible change. It is to distribute enough such changes across all large observation scales, without treating the values of a multiplicative function as independent.

The algebraic boundary is already visible in two quartics. The identity

$$n^4 + 1 = (n^2 + 1)^2 - 2n^2 \tag{1.1}$$

forces every value on $n^4 + 1$ to be preserved. By contrast, the trace of

$$\Phi_5(n) = n^4 + n^3 + n^2 + n + 1$$

can be made normal by the same type of completion. Both polynomials are irreducible over $\mathbb{Q}$. The distinction is not irreducibility alone, nor degree, but the quadratic subfields of the field generated by a root.

1.1 Sources, traces and empirical laws

Write $\mathcal{M}_\pm$ for the completely multiplicative functions from $\mathbb{N}_{\geq 1}$ to $\{-1, 1\}$. Such a function takes the value one on squares and extends to positive rationals by $f(a/b) = f(a)f(b)$. Throughout,

$$\mathcal{D} = \{-1, -2, 2\}, \quad Q_d(T) = T^2 - d.$$

We choose a target $d_0 \in \mathcal{D}$ and protect the other two traces. A *compatible completion* of a source $f_0 \in \mathcal{M}_\pm$, with prefix $B \geq 1$, means a function $g \in \mathcal{M}_\pm$ satisfying

$$g(m) = f_0(m) \quad (1 \leq m \leq B), \tag{1.2}$$

$$g(Q_e(n)) = f_0(Q_e(n)) \quad (e \in \mathcal{D} \setminus \{d_0\}, n \in \mathbb{Z}, Q_e(n) > 0). \tag{1.3}$$

The source is already defined everywhere. Completion refers to realising its protected data in a class with prescribed statistics, not to filling in undefined source values.

For a finite-valued sequence $X(n)$ and a fixed starting point n_0 , set

$$\mathcal{L}_N(X) = \frac{1}{N} \sum_{n=n_0+1}^{n_0+N} \delta_{X(n)}.$$

We choose n_0 large enough that all polynomial arguments used in that law are positive. It is fixed throughout each limit. For the three quadratic traces we use $n_0 = 1$. A sign sequence

is *normal* if its words of every fixed length H have limiting law U_H , the uniform measure on $\{-1, 1\}^H$. Total variation has the convention

$$d_{\text{TV}}(\mu, \nu) = \sup_A |\mu(A) - \nu(A)| = \frac{1}{2} \sum_x |\mu\{x\} - \nu\{x\}|$$

on a finite space. All logarithms without a subscript are natural.

Here is the representative selective completion statement. The more general polynomial conclusion below will use a different, common reservoir.

Theorem 1.1 (Two norms intact, one trace normal). *Fix $d_0 \in \mathcal{D}$ and $B \geq 1$. There is a decidable set E of primes, chosen before the source, with*

$$\frac{\#(E \cap [1, x])}{\pi(x)} \longrightarrow 0,$$

and a deterministic map $f_0 \mapsto g = f_0 \omega_{f_0}$ such that $\omega_{f_0}(q) = 1$ outside E , (1.2) holds, and

$$g(|a^2 - eb^2|) = f_0(|a^2 - eb^2|) \quad (e \neq d_0, a, b \in \mathbb{Q}, a^2 - eb^2 \neq 0). \quad (1.4)$$

The map is continuous in the product topology of prime signs and computable relative to f_0 . Let

$$X_g^H(n) = (g(Q_{d_0}(n+a)))_{0 \leq a < H}, \quad Z_{f_0}^H(n) = (f_0(Q_d(n+a)))_{d \in \mathcal{D}, 0 \leq a < H}.$$

For every fixed $H \geq 1$ and $0 < \beta < \log(2)/2$,

$$\sup_{f_0 \in \mathcal{M}_{\pm}} d_{\text{TV}}(\mathcal{L}_N(X_g^H, Z_{f_0}^H), U_H \otimes \mathcal{L}_N(Z_{f_0}^H)) = O_{d_0, B, H, \beta}((\log \log N)^{-\beta}). \quad (1.5)$$

A single g has these properties for all H . If f_0 is computable, so is g . The set of primes where g differs from f_0 has infinite reciprocal sum. The disagreements have density $1/2$ both on the positive integers and along the target trace.

The true source word appears in (1.5), not a proposed limit for that word. Thus the two protected traces may have oscillating empirical laws. The new target word becomes asymptotically independent of their actual empirical data. This is an absolute estimate; it does not divide by the probability of a rare mark.

1.2 The algebraic classification

Put

$$K = \mathbb{Q}(i, \sqrt{2}), \quad K_d = \mathbb{Q}(\sqrt{d}).$$

For a monic non-constant polynomial $P \in \mathbb{Z}[T]$ irreducible over $\mathbb{Q}$, let $F_P = \mathbb{Q}(\theta)$ for a root θ . Call P *admissible* for the chosen protection if F_P contains neither of the two fields K_e with $e \neq d_0$.

Theorem 1.2 (Algebraic dichotomy and common completion). *For a monic non-constant irreducible $P \in \mathbb{Z}[T]$, the following conditions are equivalent:*

- (i) P is admissible;
- (ii) $F_P \cap K \subseteq K_{d_0}$;
- (iii) for every source f_0 and every prefix B , a compatible completion has a normal trace $g(P(n))$.

If these conditions fail, every compatible completion satisfies

$$g(|P(n)|) = f_0(|P(n)|) \quad (n \in \mathbb{Z}, P(n) \neq 0). \quad (1.6)$$

For fixed d_0, B , there is one decidable prime set E of relative density zero and one continuous, relatively computable selector $f_0 \mapsto g$ which treats all admissible polynomials jointly. Precisely, for every fixed non-empty list of distinct admissible polynomials $P_1, \dots, P_k$ ($k \geq 1$) and every fixed list of monic non-constant integer polynomials $R_1, \dots, R_l$ ($l \geq 0$), put

$$X_g(n) = (g(P_i(n)))_{i \leq k}, \quad Z_{f_0}(n) = (f_0(R_j(n)))_{j \leq l}.$$

Then, for constants $C, c > 0$ depending on the fixed lists and on d_0, B , but not on f_0 ,

$$\sup_{f_0 \in \mathcal{M}_\pm} d_{\text{TV}}(\mathcal{L}_N(X_g, Z_{f_0}), U_k \otimes \mathcal{L}_N(Z_{f_0})) \leq C \exp(-c\sqrt{\log \log N}) \quad (1.7)$$

for all sufficiently large N . The same output works for every such list. It preserves the norms (1.4) and the prefix.

The coefficients, degrees and number of observations in (1.7) are fixed before N tends to infinity. Joint words are covered when the translated polynomials actually read are distinct. Algebraic repetitions are not independent coordinates: Section 7 describes their exact finite-dimensional constraint. Monicity is part of the classification, not a dispensable convention for arbitrary leading coefficients; the example in Section 2.2 shows the constant phase that can otherwise remain free. An empty mark list means that Z_{f_0} takes its value in a singleton.

1.3 Why the reservoir is part of the theorem

The set of allowed primes is an algebraic object. A sufficiently well-placed subset can supply the statistical freedom, but infinite total harmonic mass alone does not suffice. For a set E contained in the exclusive prime class, the condition

$$\sum_{\substack{q \in E \\ \log \log x < q \leq x}} \frac{1}{q} \rightarrow \infty \quad (1.8)$$

is sufficient for normality almost surely for each fixed input, and for a uniform computable selection when E is decidable. We prove this in Section 4. It is not asserted to be necessary.

The distinction cannot be removed by a qualitative probabilistic argument. Section 8 constructs a decidable exclusive reservoir E of prime density zero and infinite harmonic mass, and a source f , such that *every* completion using only E has a target mean with oscillation at least $1/2$. Yet fair independent choices on the same E have all fixed word laws converging to uniform in probability and in every finite moment. The exceptional scales depend on the choice of signs. A statement at one size, even uniformly in the input outside the expectation, does not select a convergent realisation on the whole sequence of sizes.

Finally, protection by more than two quadratic fields can be genuinely collective. The example in Section 9 protects

$$n^2 + 2, \quad n^2 - 2, \quad n^2 - 3,$$

which is *not* the original triple. These three traces force every value of $a^2 + 3b^2$, although no pair does. The general obstruction is the closure of the protected square classes under odd products. Certificates exist for every numerical value of $n^2 + 3$, but there is no rational identity expressing $T^2 + 3$ as a product of the three original types of norms. This distinguishes arithmetic specialisation from a formula in the variable.

Route through the argument. Section 2 first identifies the exact fibre and translates the root-field criterion into a weighted supply of roots at the allowed primes. Section 3 turns that supply into moment bounds; Sections 4–5 pass to all large prefixes and then select signs by finite exact costs. The reservoirs in Sections 6–7 complete the construction. Section 8 explains why harmonic divergence alone cannot replace availability, and Sections 9–10 describe collective rigidity.

Order of choices. The uniform statements concern an algorithm with a source-dependent output, not one multiplier that works for every later source. Table 1 separates this from the probabilistic statements and from the obstruction. In every row, observation polynomials and word lengths are fixed in each limit.

Result and fixed data	Dependence and conclusion	Uniformity and effectivity
Proposition 4.2: the reservoir and a countable family of weighted tests are fixed before the fair signs.	Almost-sure convergence at all large prefixes. The full-measure event may depend on the source and tests.	The moment estimates are uniform in the fixed bounded weights. This argument alone does not select computable signs.
Theorems 1.1–1.2 and Proposition 5.1: the support and task templates precede the input.	One algorithm maps each f_0 to its own ω_{f_0} . Polynomial mark values are read from that input.	For each fixed task, the finite bound is independent of f_0 . Selection is relatively computable; no useful numerical onset is supplied.
Proposition 8.2: a divergent reservoir, a source and a fixed word test precede the signs.	Convergence in probability and every finite moment, not a conclusion about every sample path.	The supremum over sources is outside the expectation. No convergent realisation is selected by this assertion.
Theorem 8.1: E precedes $f_\star$; the base f is then constructed.	All subsets $A \subseteq E$ and finite prime changes are covered after f is fixed. Their witness sizes can differ.	The calendar is source-independent, and f is relatively computable from $f_\star$. No practical complexity bound is asserted.

Table 1: Quantifiers for completion and obstruction. A computable finite error bound and a proof that it tends to zero are distinct from a supplied usable modulus of convergence.

1.4 Relation to previous work

The precursor [16], Theorem 1.1 and Proposition 8.1, preserves polynomial traces while regularising affine observations by selective prime noise, starting from a prescribed source. Its Corollary 5.3 already protects the whole family of sums of two squares. Its Theorems 6.1–6.2 describe a polynomial factor channel under non-selective noise. The present question is which polynomial observations can be regularised *inside the exact fibre of two protected norms*. We identify that fibre, classify its frozen irreducibles, and construct a common sparse support with a relatively computable, source-dependent selection. These are static conclusions; they do not replace the precursor’s simultaneous guarantees at all positive times. The factor-parity mechanism is recalled locally in Section 7.3.

Normality and three random models. Fish [6] proves normality for independent fair prime signs extended completely multiplicatively. Najnudel [13], Proposition 6.3 in the cited arXiv edition, obtains, for each fixed word of length k in q -th roots of unity, almost-sure frequency $q^{-k} + O(N^{-1/2+\varepsilon})$ for every $\varepsilon > 0$. The case $q = 2$ is the completely multiplicative

sign model. These full-prime models do not impose our exact protection or our common sparse support, so the rates cannot be compared without their different constraints.

For polynomial observations, it is important to distinguish that sign model, sometimes called *extended Rademacher*, from two other models. A *Steinhaus* function has independent uniform unit-circle values at the primes and is extended completely multiplicatively. A *squarefree-supported Rademacher* function has independent fair prime signs, is extended multiplicatively on squarefree integers, and vanishes on the others. Our multipliers are completely multiplicative signs, random only on a specified prime set before selection; they are not squarefree-supported functions.

Klurman–Shkredov–Xu [11], Theorem 1.1 and Proposition 1.4, connect polynomial sums in the Steinhaus model with Gaussian limits and multiplicative product equations. Wang–Xu [17], Theorem 1.1 and Section 1.3, obtain asymptotics for all fixed even moments, for polynomials with at least two distinct complex roots, through equalities of products of polynomial values. Their error constants depend on the fixed moment order. Chinis–Shala [3], Theorem 1.2, prove a central limit theorem in the squarefree-supported Rademacher model for irreducible quadratics and products of at least two distinct linear factors, under the condition that no prime square divides every polynomial value. Their Section 1.4 separately discusses the completely multiplicative model and the corresponding square-product counts. The squarefree-supported theorem must not be read as a theorem for that different model.

For the latter model, de la Bretèche–Wang–Xu [4], Theorem 1.2, prove a Gaussian limit for $N^{-1/2} \sum_{n \leq N} g(P(n))$ when P is separable, has positive leading coefficient and total degree at least two, and every irreducible factor has degree at most three. Our conclusion concerns N -normalised averages and fixed word frequencies in a protected fibre. Its allowance of admissible irreducibles of arbitrary fixed degree is not an extension of that central limit theorem. Section 3 explains the different parity count and precision required here.

Sparse completion and the inherited tools. Klurman–Mangerel–Teräväinen [10], Theorem 2.1 in arXiv:2304.05344v2, give vanishing affine correlations for a completely multiplicative function negative on a prime-density-zero set of infinite harmonic mass. Thus sparsity and affine normality alone are not new conclusions. Nor does multiplication by such a fixed function give a completion for every prescribed source: if the source equals the multiplier, their product is identically one. Our quantifier is a source-dependent choice, with a uniform error estimate for the selector, rather than a single multiplier valid for all sources chosen afterwards.

The algebraic inputs are classical norm theory and the Frobenius interpretation of polynomial roots; see [12, 14, 2, 5]. The harmonic input is the fixed-extension Chebotarev–Mertens theorem of Arango-Piñeros–Keliher–Keyes [1], Theorem A. We use Selberg’s upper-bound sieve in the form recorded in [8], Section 13.2, and keep the dependence on the increasing moment order explicit. The selection uses the method of conditional expectations, with exact finite normalisers and a paid tail after every decision. Constructive Borel–Cantelli methods have a general antecedent in Galatolo–Hoyrup–Rojas [7]; the selection needed here is proved directly.

The private-prime modification used in Section 8 has a direct antecedent in Klurman [9], Lemma 2.1 and the proof of Proposition 1.1. Its role here is within a calendar that treats all exposed sign configurations, producing persistent witnesses for every subset of one reservoir. Products of norm groups also have a substantial literature: Pollio–Rapinchuk [15] provide local-global context, not a theorem contradicted by our comparison of numerical specialisations with identities over $\mathbb{Q}(T)$.

The contribution is the connection between the exact protected fibre, the weighted supply of roots, and a common completion from arbitrary sources, together with its reservoir and collective-rigidity boundaries. None of the conclusions is a new cancellation statement for an unmodified Liouville function.

2 The exact fibre and the field of a root

We first determine every prime sign that can vary under the protected data. This prevents the probabilistic construction from silently changing the fibre it is meant to occupy.

2.1 Invisible primes determine the whole fibre

Quadratic reciprocity gives the following exclusive classes:

Target d_0	Free primes $\mathcal{P}_{d_0}$	Protected absolute norms
-1	$q \equiv 5 \pmod{8}$	$a^2 + 2b^2, a^2 - 2b^2 $
-2	$q \equiv 3 \pmod{8}$	$a^2 + b^2, a^2 - 2b^2 $
2	$q \equiv 7 \pmod{8}$	$a^2 + b^2, a^2 + 2b^2$

Every prime in the indicated class has two simple roots on the target quadratic and none on either protected quadratic.

Lemma 2.1 (Exact fibre). *For $\omega \in \mathcal{M}_\pm$, the following are equivalent: ω equals one on every positive value of the two protected traces; and $\omega(q) = 1$ at every prime outside $\mathcal{P}_{d_0}$. Each such ω also equals one on every non-zero absolute rational norm $|a^2 - eb^2|$ with $e \neq d_0$. Imposing the prefix (1.2) removes exactly the free primes at most B .*

Proof. If $q \in \mathcal{P}_{d_0}$ and $e \neq d_0$, then e is not a square modulo q . For an integer pair of non-zero norm,

$$v_q(a^2 - eb^2) = 2 \min\{v_q(a), v_q(b)\}, \quad (2.1)$$

with $v_q(0) = +\infty$. After dividing the coordinates by their common power of q , further divisibility would give a square root of e modulo q . The valuation is therefore even. Clearing rational denominators multiplies the norm by a square, proving the sufficiency and the stronger norm assertion.

For necessity, the protected traces determine the sign at two: use $Q_{-1}(1) = 2$ when -1 is protected, and otherwise $Q_2(2) = 2$. If three is not exclusive, $Q_{-2}(1) = 3$ is protected. Suppose inductively that all smaller non-exclusive prime signs have been determined, and let $p \geq 5$ be non-exclusive. At least one protected quadratic has a centred root $1 \leq r \leq (p-1)/2$. For that e ,

$$0 < Q_e(r) = pc < p^2.$$

For $e = 2$, a failure of positivity could only have $r = 1$, which is not a root modulo p . Every prime factor of c is smaller than p . No exclusive prime divides $Q_e(r)$, by (2.1) with $b = 1$. Thus $\omega(c) = 1$, and the protected value gives $\omega(p) = 1$. Finally, agreement on all integers up to B is equivalent to agreement at the primes up to B . $\square$

Taking $\omega = gf_0$ identifies the complete space of compatible quotients. In particular, preserving the two traces already preserves both binary norm families; this is not an extra assumption on our constructed output.

2.2 Why a quadratic subfield freezes a polynomial

Proposition 2.2 (Norm factorisation). *Let $P \in \mathbb{Z}[T]$ be monic and irreducible. If $K_e \subseteq F_P$ for a protected e , then*

$$P(T) = A(T)^2 - eB(T)^2, \quad A, B \in \mathbb{Q}[T]. \quad (2.2)$$

Consequently every compatible completion satisfies (1.6).

Proof. Let R be the monic minimal polynomial of a root θ over K_e . Its degree is $\deg(P)/2$. If σ is the non-trivial automorphism of K_e , then RR^σ is monic, rational, has degree $\deg(P)$ and vanishes at θ . Hence $RR^\sigma = P$. Write $R = A + \sqrt{e}B$ to obtain (2.2). Lemma 2.1, applied after evaluation and clearing denominators by a square, proves the pointwise conclusion at every non-zero value, not merely away from exceptional primes. $\square$

A polynomial irreducible over $\mathbb{Q}$ becomes reducible over a quadratic field K_e exactly when $K_e \subseteq F_P$. Indeed $[F_P K_e : F_P]$ is one or two, and the tower formula computes $[F_P K_e : K_e]$. Thus admissibility can equivalently be tested by irreducibility over each protected quadratic field. The other subfields of K show that admissibility is also equivalent to $F_P \cap K \subseteq K_{d_0}$. The frozen case cannot be made normal from every input: the constant source would have a trace identically equal to one. This does not exclude a particular input whose frozen trace is already normal.

A leading coefficient can remain free. Take $d_0 = -1$, $B = 1$, and $f_0 \equiv 1$. The function $g(m) = \beta_{\{5\}}(m) := (-1)^{v_5(m)}$ is a compatible completion: the prime five lies in the free class, so both protected absolute norm families remain equal to one. The non-monic polynomial $P(T) = 5(T^2 + 2)$ is irreducible over $\mathbb{Q}$ and its root field is the protected field $\mathbb{Q}(\sqrt{-2})$. Nevertheless, $5 \nmid n^2 + 2$ for every integer n , and hence

$$g(n^2 + 2) = 1, \quad g(5(n^2 + 2)) = -1 \neq f_0(5(n^2 + 2)).$$

Thus the root field freezes the monic factor but not the free sign of the leading coefficient. This example concerns compatible completions in the full fibre; it shows why the pointwise assertion (1.6) is stated for monic polynomials.

2.3 Counting roots in the allowed prime class

For a polynomial P , write $\rho_P(q)$ for its number of distinct roots modulo the prime q . At good primes these roots are simple. The coefficient controlling the sieve is a root-weighted harmonic density, not the density of primes having at least one root. For example, when $d_0 = -1$, the free primes at which Φ_5 has roots have density $1/16$ among all primes, and each contributes four roots. The weighted coefficient is therefore $4 \times (1/16) = 1/4$. This is the quantity the fixed-point average below must recover; Section 2.4 gives the field calculation.

The Galois criteria for root existence go back to Berend–Bilu [2], Theorem 2, and are recalled in Elsholtz–Klahn–Technau [5], Theorem 2.1 and Proposition 2.2. We use the same Frobenius action, but weight each group element by its number of fixed points inside one restriction class rather than merely asking whether it has a fixed point.

Lemma 2.3 (Fixed points above an abelian quotient). *Let G be a finite group, $H \leq G$, and $\text{res} : G \twoheadrightarrow A$ a surjection onto an abelian group. For $c \in A$,*

$$\frac{1}{|G|} \sum_{\text{res}(s)=c} \text{Fix}_{G/H}(s) = \frac{\mathbf{1}_{\{c \in \text{res}(H)\}}}{|\text{res}(H)|}. \quad (2.3)$$

Proof. A coset xH is fixed by s precisely when $x^{-1}sx \in H$. Conjugation does not change the restriction to A . For each coset the number of possible s is therefore zero if $c \notin \text{res}(H)$, and otherwise $|H \cap \ker(\text{res})|$. There are $|G|/|H|$ cosets. Counting pairs (s, xH) and dividing by $|G|$ proves the formula. $\square$

Choose a finite Galois extension $M/\mathbb{Q}$ containing K and all roots of an irreducible P , and put

$$G = \text{Gal}(M/\mathbb{Q}), \quad H = \text{Gal}(M/F_P), \quad L_P = F_P \cap K.$$

The roots are the cosets G/H . The exclusive prime class corresponds in $\text{Gal}(K/\mathbb{Q})$ to the unique non-identity element c fixing K_{d_0} . Moreover $\text{res}(H) = \text{Gal}(K/L_P)$.

We use the following precise classical input. For any conjugacy class C in the Galois group of a *fixed* extension of $\mathbb{Q}$, Theorem A of [1] gives a Mertens product on its unramified primes. Taking its logarithm, with the convergent $\sum_q q^{-2}$ terms separated, gives

$$\sum_{\substack{q \leq x \\ \text{Frob}_q \in C}} \frac{1}{q} = \frac{|C|}{|G|} \log \log x + O_M(1). \quad (2.4)$$

The fixed-point interpretation of simple roots is the usual Frobenius factorisation theorem; see [12], the discussion of Galois groups over $\mathbb{Q}$. Removing discriminant and other finitely many bad primes changes only the bounded term.

Proposition 2.4 (Root-weighted coefficient). *For a monic non-constant irreducible P ,*

$$\sum_{\substack{q \leq x \\ q \in \mathcal{P}_{d_0}}} \frac{\rho_P(q)}{q} = \kappa_{d_0}(P) \log \log x + O_P(1), \quad (2.5)$$

where

$$\kappa_{d_0}(P) = \begin{cases} 1/4, & L_P = \mathbb{Q}, \\ 1/2, & L_P = K_{d_0}, \\ 0, & \text{otherwise.} \end{cases} \quad (2.6)$$

Proof. Sum (2.4) over the conjugacy classes restricting to c , weighted by their fixed-point counts on G/H . Lemma 2.3 gives coefficient $1/|\text{Gal}(K/L_P)|$ when c acts trivially on L_P , and zero otherwise. The subfields of the biquadratic field K give (2.6). $\square$

Thus every admissible polynomial has a positive supply of simple roots at allowed primes. The field and polynomial remain fixed in this argument; no estimate uniform over growing degrees or discriminants is being used. For a product of distinct monic irreducibles, the root counts add outside the finite set dividing their pairwise resultants. Its coefficient is the sum of the individual coefficients.

2.4 Examples that distinguish the criterion

For $d_0 = -1$, the forbidden fields are $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{-2})$:

$P(T)$	Information about F_P	Status
$T^2 + 1$	$F_P = \mathbb{Q}(i)$	Admissible
$T^2 - 3$	$F_P = \mathbb{Q}(\sqrt{3})$	Admissible
$T^3 - 2$	Odd degree, hence no quadratic subfield	Admissible
$\Phi_5(T)$	Only quadratic subfield $\mathbb{Q}(\sqrt{5})$	Admissible
$T^4 - 2$	Contains $\mathbb{Q}(\sqrt{2})$	Frozen
$T^4 + 2$	Contains $\mathbb{Q}(\sqrt{-2})$	Frozen
$T^4 + 1$	$F_P = \mathbb{Q}(\zeta_8) = K$	Frozen

The irreducibility assertions follow from the rational-root test in the quadratic cases and from Eisenstein, directly or after translating T by one, in the remaining examples. The familiar cyclotomic description of Φ_5 is also visible in

$$\Phi_5(T) = \left(T^2 + \frac{1+\sqrt{5}}{2}T + 1\right) \left(T^2 + \frac{1-\sqrt{5}}{2}T + 1\right).$$

Its root field intersects K in $\mathbb{Q}$, so (2.6) recovers the root-weighted coefficient announced before Lemma 2.3.

The three frozen quartics have explicit protected norm identities:

$$T^4 - 2 = (T^2)^2 - 2, \quad T^4 + 2 = (T^2)^2 + 2, \quad T^4 + 1 = (T^2 + 1)^2 - 2T^2.$$

For every irreducible of odd degree, on the other hand, the root field has no quadratic subfield, so the polynomial is admissible for each choice of d_0 .

The root field must not be replaced by the splitting field. For example, $T^3 - 3T - 10$ has no rational root and discriminant $-2592 = -2 \cdot 36^2$. Its splitting field contains $\mathbb{Q}(\sqrt{-2})$, while its cubic root field cannot contain a quadratic field. It remains admissible when $\mathbb{Q}(\sqrt{-2})$ is protected. Formula (2.3) retains exactly this distinction.

3 A moment sieve with its order visible

The algebraic calculation supplies primes at which an observation has roots. We now turn that supply into a bound on multiplicative correlations. The same finite sieve will serve both the full exclusive class and much thinner sets. Keeping the moment order explicit is essential for the latter.

The conversion of random moments into arithmetic product counts is already central to [11, 17, 3, 4]. The count needed here is different. In the Steinhaus model, orthogonality imposes equality of two products; in the full completely multiplicative sign model, it imposes even total valuation at *every* prime. With signs free only on E , it imposes parity only at primes of E . This enlarges the set of surviving tuples, so an upper bound for full square products cannot simply be substituted for the restricted-parity count below. We seek a coarse bound strong enough to make a normalised moment norm tend to zero, not Gaussian moment asymptotics. In addition, the later choice $r \asymp \log \log T$ requires visible dependence on r , not just an estimate for each fixed order. These are the two requirements addressed by the sieve.

Fix a separable polynomial $P \in \mathbb{Z}[T]$ of degree $D \geq 1$, positive on the indices $n_0 + 1, n_0 + 2, \dots$, with n_0 fixed. Let E be a set of primes. Set $\omega(q)$ to independent fair signs on E and to one elsewhere, and extend completely multiplicatively. For an even integer $r \geq 2$, define

$$C_{P,E,r}(T) = \# \left\{ (n_1, \dots, n_r) \in \{n_0 + 1, \dots, n_0 + T\}^r : v_q \left(\prod_{i=1}^r P(n_i) \right) \text{ is even for all } q \in E \right\}. \quad (3.1)$$

The dependence on the fixed starting point is suppressed. Every diagonal and repeated index is included.

Lemma 3.1 (The moment and its forbidden local event). *For deterministic complex weights $|w_n| \leq 1$,*

$$\mathbb{E} \left| \sum_{n=n_0+1}^{n_0+T} w_n \omega(P(n)) \right|^r \leq C_{P,E,r}(T). \quad (3.2)$$

Outside a finite exceptional set of primes and for $q > 4rD$, let $\rho_q = \rho_P(q)$. The proportion of tuples modulo q^2 having total valuation exactly one is

$$g_r(q) = \frac{r\rho_q}{q} \left(1 - \frac{1}{q}\right) \left(1 - \frac{\rho_q}{q}\right)^{r-1}. \quad (3.3)$$

If $\rho_q > 0$, then

$$0 < g_r(q) < \frac{1}{4}, \quad \frac{r\rho_q}{q} - \frac{(rD)^2}{q^2} \leq g_r(q) \leq \frac{rD}{q}. \quad (3.4)$$

Proof. Write $r = 2k$ and expand the product of the k -th power of the sum with the k -th power of its conjugate. The expectation of a prime-sign monomial is zero unless every free prime occurs to even total exponent, in which case it is one. Taking absolute values of the deterministic coefficients proves (3.2). There is no independence assumption on the observations $\omega(P(n))$.

At a good prime, each simple root modulo q has one lift that is still a root modulo q^2 . Thus $\rho_q(q-1)$ residue classes have valuation one, and $q^2 - \rho_q q$ have valuation zero. For total valuation one, choose the unique coordinate of valuation one in r ways; the others have valuation zero. This gives (3.3). The elementary inequality

$$1 - (1 - q^{-1})(1 - \rho_q/q)^{r-1} \leq \frac{1 + (r-1)\rho_q}{q} \leq \frac{rD}{q}$$

gives (3.4). $\square$

A tuple counted by (3.1) cannot have total valuation one at any prime in E . We exclude only this local event, rather than every odd total valuation. This relaxation enlarges the set being counted, so its direction is appropriate for an upper bound.

3.1 The congruence remainder and the Selberg weights

Discard the finitely many bad primes, the primes at most $4rD$, and those with $\rho_q = 0$. Call the remaining primes in E the sieve primes. Extend g_r multiplicatively to squarefree integers supported on them, with $g_r(1) = 1$. If t is such an integer, let A_t count tuples for which every $q \mid t$ has total valuation exactly one. The Chinese remainder theorem gives

$$A_t = T^r g_r(t) + R_t, \quad |R_t| \leq rt^2(T + t^2)^{r-1}. \quad (3.5)$$

Indeed the local conditions specify $t^{2r} g_r(t)$ residue classes in r coordinates modulo t^2 . In a fixed class, each coordinate count is $T/t^2 + \varepsilon$ with $|\varepsilon| \leq 1$. The product differs from $(T/t^2)^r$ by at most $r(T/t^2 + 1)^{r-1}$. Summing over at most t^{2r} classes proves the bound. In particular,

$$|R_t| \leq r2^{r-1}T^{r-1}t^2 \quad (t^2 \leq T). \quad (3.6)$$

The bound is unchanged by the fixed translation of the box.

Put

$$h_r(q) = \frac{g_r(q)}{1 - g_r(q)}, \quad G_r(z) = \sum_{l \leq z} h_r(l),$$

where all sums in this paragraph are over squarefree integers on the sieve primes. The classical Selberg coefficients are

$$\lambda_l = \mu(l) \prod_{q|l} (1 - g_r(q))^{-1} \frac{\sum_{\substack{t \leq z/l \\ (t,l)=1}} h_r(t)}{G_r(z)} \quad (l \leq z), \quad (3.7)$$

and zero otherwise. They satisfy

$$\lambda_1 = 1, \quad |\lambda_l| \leq 2^{\#\{q:q|l\}} \leq l, \quad \sum_{l,m \leq z} \lambda_l \lambda_m g_r([l, m]) = \frac{1}{G_r(z)}. \quad (3.8)$$

These are the weights and diagonalisation in [8], Section 13.2. To see that they apply to the present congruence events, rather than only to ordinary divisibility, one can check the finite identity directly. Set

$$y_t = \sum_{t|l} \lambda_l g_r(l).$$

Multiplicativity gives

$$\sum_{l,m} \lambda_l \lambda_m g_r([l, m]) = \sum_t \frac{y_t^2}{h_r(t)}, \quad \lambda_1 = \sum_t \mu(t) y_t.$$

Choosing $y_t = \mu(t) h_r(t) / G_r(z)$ for $t \leq z$ gives (3.8), and Möbius inversion gives (3.7). The bound on λ_l follows from $g_r(q) < 1/4$ and the fact that the numerator's sum is at most $G_r(z)$.

For a tuple $\mathbf{n}$, let $b(\mathbf{n})$ be the product of the sieve primes at most z at which its total valuation is one. On every tuple in (3.1), this product is one. Hence

$$\mathbf{1}_{\{\mathbf{n} \text{ counted in } C_{P,E,r}(T)\}} \leq \left(\sum_{l|b(\mathbf{n})} \lambda_l \right)^2.$$

Expanding the square and applying (3.6) and (3.8) proves the finite inequality

$$\boxed{\frac{C_{P,E,r}(T)}{T^r} \leq \frac{1}{G_r(z)} + r 2^{r-1} \frac{z^8}{T} \quad (z^4 \leq T).} \quad (3.9)$$

There are at most z^2 pairs of coefficients, their products have absolute value at most z^2 , and $[l, m]^2 \leq z^4$. These are the three factors in z^8 . The deliberately coarse remainder leaves every dependence on the increasing order r visible.

3.2 Keeping a useful part of the Euler product

The full Euler product for G_r is easy to estimate, but $G_r(z)$ contains only products of primes at most z in total size. We therefore retain a smaller range of primes whose products have substantial mass below z .

Let B_P bound the exceptional primes, increasing it to include any fixed primes that must not be used. Define

$$z = \lfloor T^{1/16} \rfloor, \quad x = \lfloor z^{1/(32rD)} \rfloor, \quad V = \max(B_P, 4rD), \quad (3.10)$$

$$\Delta_{P,E}(T, r) = \sum_{\substack{q \in E \\ V < q \leq x}} \frac{\rho_P(q)}{q}. \quad (3.11)$$

An empty sum is zero. Only the range $z, x \geq 2$ will be used asymptotically.

Lemma 3.2 (A bound uniform in the even moment order). *With these choices,*

$$\frac{C_{P,E,r}(T)^{1/r}}{T} \leq 2^{1/r} \exp(D - \Delta_{P,E}(T, r)) + 2r^{1/r} \exp\left(-\frac{\log T}{2r}\right). \quad (3.12)$$

Proof. We use the elementary estimate

$$\sum_{q \leq x} \frac{\log q}{q} \leq 4 \log x \quad (x \geq 2). \quad (3.13)$$

For example, the central binomial coefficient bounds the sum of $\log q$ over $n < q \leq 2n$ by $2n \log 2$. Dyadic summation gives $\vartheta(t) \leq 4t \log 2$, and partial summation gives (3.13).

Select each sieve prime $q \leq x$ independently with probability $g_r(q)$ and let R be their product. This auxiliary random product is a way to measure positive sieve weights, not a further choice of source signs. By (3.13),

$$\mathbb{E} \log R \leq rD \sum_{q \leq x} \frac{\log q}{q} \leq 4rD \log x \leq \frac{1}{8} \log z.$$

Markov's inequality shows that at least one half of its probability mass has $R \leq z$. Its law assigns to a squarefree product l a probability proportional to $h_r(l)$, so

$$G_r(z) \geq \frac{1}{2} \prod_{\substack{q \in E \\ V < q \leq x}} (1 - g_r(q))^{-1}.$$

Using (3.4), $-\log(1 - u) \geq u$ and $(rD)^2 \sum_{n > 4rD} n^{-2} \leq rD$, we obtain

$$\log G_r(z) \geq -\log 2 + r\Delta_{P,E}(T, r) - rD. \quad (3.14)$$

Take the r -th root in (3.9), use $z^8 \leq T^{1/2}$, and use $(a + b)^{1/r} \leq a^{1/r} + b^{1/r}$. This proves (3.12). $\square$

This estimate explains both cutoffs. Primes below a multiple of r cannot be used with the same local bound, while the primes above $T^{c/r}$ need not fit the truncated sieve product. It is the root-weighted mass *between* these cutoffs, not the total mass up to T , that will make the moment norm small.

4 From available primes to all large prefixes

We now pass from moment bounds at a size to a single completion valid at every large size. On the scale $T_j = 2^j$, a grid with polynomially many points in j will suffice. Taking the moment order comparable to $\log j$ lets us pay for this grid without prescribing a decay rate for the moment norm: once the ratio in Markov's inequality is below $1/8$, raising it to the chosen order gives a summable bound over all grids. The full prime class also admits a faster logarithmic rate using fixed high moments; we treat that regime separately.

4.1 A sufficient condition with no prescribed rate

For a fixed separable polynomial P , consider the root-weighted condition

$$\sum_{\substack{q \in E \\ \log \log x < q \leq x}} \frac{\rho_P(q)}{q} \longrightarrow \infty. \quad (\text{WP})$$

Finite changes to E do not affect this condition. Let

$$T_j = 2^j, \quad r_j = 2 \operatorname{bl}(j + 1), \quad \operatorname{bl}(m) = 1 + \lfloor \log_2 m \rfloor. \quad (4.1)$$

Thus r_j is even and comparable to $\log j$, or to $\log \log T_j$.

Lemma 4.1 (Vanishing moment norm). *If (WP) holds, then*

$$b_{P,j} := \frac{C_{P,E,r_j}(T_j)^{1/r_j}}{T_j} \longrightarrow 0. \quad (4.2)$$

Proof. For the cutoffs (3.10),

$$\log \log x_j = \log \log T_j - \log r_j + O_P(1), \quad V_j \ll_P \log \log x_j.$$

In particular $x_j \rightarrow \infty$. Raising the lower cutoff $y = \log \log x_j$ to at most a fixed multiple of y costs at most a constant: bound $\rho_P(q)$ by D and the reciprocal sum by the sum over all integers in that interval. Consequently

$$\Delta_{P,E}(T_j, r_j) \geq \sum_{\substack{q \in E \\ \log \log x_j < q \leq x_j}} \frac{\rho_P(q)}{q} - O_P(1) \longrightarrow \infty.$$

Both terms of (3.12) now tend to zero, since $\log T_j / r_j \rightarrow \infty$. $\square$

Proposition 4.2 (One realisation at every size). *Under (WP), for every fixed deterministic bounded sequence $|w_n| \leq 1$, almost surely*

$$\frac{1}{N} \sum_{n=n_0+1}^{n_0+N} w_n \omega(P(n)) \longrightarrow 0. \quad (4.3)$$

The conclusion holds on one event of probability one for any prescribed countable collection of these tests.

Proof. In $[T_j/2, T_j]$, take a grid $\mathcal{G}_j$ containing both endpoints, with consecutive gaps at most

$$s_j = \max \left(1, \left\lfloor \frac{T_j}{(j+1)^2} \right\rfloor \right).$$

It can be chosen with $|\mathcal{G}_j| \leq 2(j+2)^2$. For $t \in \mathcal{G}_j$, apply (3.2) at size T_j with the weights set to zero after the first t terms. Markov gives

$$\mathbb{P} \left(\left| t^{-1} \sum_{n=n_0+1}^{n_0+t} w_n \omega(P(n)) \right| > \eta \right) \leq (2b_{P,j}/\eta)^{r_j}.$$

For fixed $\eta > 0$, the ratio is at most $1/8$ eventually. Since $r_j \geq 2 \log_2(j+1)$, the bound is at most $(j+1)^{-6}$. Its sum over the grids and over j converges. Borel–Cantelli controls every sufficiently late grid point. Between neighbouring points the average of terms bounded by one changes by $O(j^{-2} + T_j^{-1})$. This gives (4.3) at every size. Intersect over rational $\eta > 0$ and then over the countable collection of tests. $\square$

The weights are fixed before the prime signs. Choosing $w_n = \omega(P(n))$ after those signs would make the average equal to one and is excluded. The probability-one event in Proposition 4.2 may depend on the input and the prescribed marks.

4.2 The exact Fourier step for marked laws

For distinct polynomial observations $R_1, \dots, R_k$, let

$$X(n) = (g(R_i(n)))_{i \leq k}, \quad Z(n) \in \mathcal{Z},$$

where $\mathcal{Z}$ is finite and Z is fixed before the random signs. Assume the products $P_J = \prod_{i \in J} R_i$, for non-empty $J \subseteq \{1, \dots, k\}$, are separable and satisfy (WP). With $g = f_0 \omega$, put

$$A_{J,v}(N) = \sum_{n=n_0+1}^{n_0+N} \mathbf{1}_{\{Z(n)=v\}} f_0(P_J(n)) \omega(P_J(n)).$$

The identity

$$\mathbf{1}_{\{X=x\}} = 2^{-k} \prod_{i=1}^k (1 + x_i X_i)$$

gives the finite, exact expansion

$$\mathcal{L}_N(X, Z)\{(x, v)\} - 2^{-k} \mathcal{L}_N(Z)\{v\} = \frac{2^{-k}}{N} \sum_{\emptyset \neq J \subseteq [k]} x_J A_{J,v}(N), \quad x_J = \prod_{i \in J} x_i. \quad (4.4)$$

Therefore

$$d_{\text{TV}}(\mathcal{L}_N(X, Z), U_k \otimes \mathcal{L}_N(Z)) \leq \frac{1}{2N} \sum_{J \neq \emptyset, v \in \mathcal{Z}} |A_{J,v}(N)|. \quad (4.5)$$

Proposition 4.2 applies to each sum, with deterministic weights $\mathbf{1}_{\{Z(n)=v\}} f_0(P_J(n))$. This proves the marked normality conclusion without any assumed convergence of the law of Z .

For the target quadratic and its distinct shifts, each product P_J has $2|J|$ simple roots at every exclusive prime outside a finite exceptional set. The condition

$$\mathfrak{a}_E(x) := \sum_{\substack{q \in E \\ \log \log x < q \leq x}} \frac{1}{q} \longrightarrow \infty \quad (\text{A})$$

thus implies (WP) for every target word moment. Combined with Lemma 2.1, it proves an almost-sure selective completion from each prescribed source. Section 5 will make a deterministic choice and strengthen the bound to uniformity over its inputs.

The criterion is stable under enlarging E inside $\mathcal{P}_{d_0}$, and under removing a subset with finite reciprocal sum. These observations follow directly from (A). They do not make the criterion necessary, or turn a support with divergent total reciprocal sum into an available support.

4.3 The faster full-class result

For comparison, suppose a fixed separable P has

$$\sum_{\substack{q \leq x \\ q \in E}} \frac{\rho_P(q)}{q} = \kappa \log \log x + O_P(1), \quad \kappa > 0. \quad (4.6)$$

This includes the full exclusive class for a product of distinct admissible irreducibles. With r fixed, (3.3) gives

$$\prod_{\substack{q \leq x \\ q \text{ a sieve prime}}} (1 - g_r(q))^{-1} \asymp_{P,r} (\log x)^{r\kappa}.$$

Select primes only up to $x = z^a$ with a sufficiently small fixed $a > 0$. The random-product argument in Lemma 3.2 retains at least half of the product mass below z . Thus $G_r(z) \gg_{P,r} (\log z)^{r\kappa}$. Taking $z = \lfloor N^{1/16} \rfloor$ in (3.9) yields

$$\mathbb{E} \left| N^{-1} \sum w_n \omega(P(n)) \right|^{2k} \ll_{P,k} (\log(2N))^{-2k\kappa} + N^{-1/2}. \quad (4.7)$$

Corollary 4.3 (Fixed-order rates). *Under (4.6), for each $0 < \beta < \kappa$ and each fixed bounded deterministic weight sequence, almost surely*

$$\sum_{n=n_0+1}^{n_0+N} w_n \omega(P(n)) = O_{\omega,P,w,\beta} (N(\log(2N))^{-\beta}).$$

Countably many fixed tests may be imposed simultaneously.

Proof. Choose an integer $A > \beta$. In $[2^j, 2^{j+1}]$, use a grid with gaps at most $2^j/j^A + 1$ and $O(j^A)$ points. Then choose k such that $2k(\kappa - \beta) > A + 2$. Markov applied to (4.7), with threshold $j^{-\beta}$, gives a summable union bound over all grid points and j . Interpolation costs $O(j^{-A})$. Rational choices of β approaching κ give all the asserted exponents. $\square$

For the target quadratic word, the smallest coefficient is $\kappa = 1/2$. Hence the full-class marked completion has rate $O((\log N)^{-\beta})$ for every $\beta < 1/2$, almost surely for each fixed source. A fixed family of positive-slope integer affine forms, pairwise non-proportional, may be adjoined to a fixed family of distinct target quadratic shifts. A non-empty moment with a affine factors and b quadratic factors has $a + 2b$ simple roots at every good exclusive prime, and coefficient $(a + 2b)/4$. The common rate is therefore $O((\log N)^{-\beta})$ for every $\beta < 1/4$ when affine coordinates are allowed. Proportional forms, for example n and $2n$, must be excluded because their signs have a fixed multiplicative relation.

All fixed progressions of origins can also be included. In $n \equiv a \pmod{M}$, substitute $n = Mt + a$; primes dividing the fixed M are the only additional exceptions. Normalise by the number of origins in that progression, and intersect the countably many resulting probability-one events. This gives no uniformity in moving moduli. Ordinary convergent word frequencies imply the corresponding logarithmic frequencies by partial summation. The logarithmic conclusion here is positive; the obstruction later concerns only ordinary means.

The full-class and sparse constructions serve different purposes. The former gives a faster almost-sure rate. Fair bits on all free primes change a prime set of relative density $1/8$ almost surely, whereas the deterministic sparse constructions below change a prime-density-zero set. Neither statement entails the other with the same cost and quantifiers.

5 Choosing the completion from finite exact costs

An almost-sure argument need not give a computable sign assignment. We instead use the method of conditional expectations, with the constructive statistical context provided by [7]. For a task a at scale j , the non-negative cost $F_{a,j}$ will penalise non-zero character sums, while an exact arithmetic count $D_{a,j} > 0$ will bound its expectation independently of the input. Thus $F_{a,j}/D_{a,j}$ has expected cost at most one. These ratios, not unknown constants from Chebotarev or the sieve, are what the algorithm computes. The asymptotic estimates enter later, to show that bounded cost forces the required empirical law.

The remaining difficulty is infinite: each prime sign must be decided without evaluating every future task. A summable positive combination of the normalised costs makes a finite rational comparison sufficient, because the entire omitted tail can still be bounded after conditioning on all earlier decisions.

5.1 Tasks and their normalisers

Assume that membership in the infinite set E is decidable. A task consists of a non-empty finite list of positive polynomial observations $R_1, \dots, R_k$ ($k \geq 1$), a fixed starting point n_a , and a finite-alphabet mark $Z_{f_0}(n)$ computable from finitely many values of the input f_0 at each n . In our applications the marks are fixed finite lists of original polynomial values. The list defining a task is fixed; its actual signs depend on f_0 . Only non-empty observation lists are enumerated. The mark list may be empty, in which case the mark alphabet is a singleton. Thus every task has at least one non-empty character J , including when no marks are requested.

Choose an effective enumeration of the tasks, indexed by $a \geq 2$. For the quadratic completion these are simply the horizons $H = a - 1$, with the target word and the original three-trace word. For the common polynomial completion they enumerate all pairs consisting of a non-empty finite observation list and a possibly empty finite mark list of monic non-constant integer polynomials. Positivity beyond a computable starting point follows from monicity, using a bound in terms of the coefficients. This enumeration need not decide which lists are admissible. Only the tasks for which the moment hypotheses hold will carry a limiting conclusion.

Use $T_j, r_j, \mathcal{G}_j$ from Section 4. For $J \neq \emptyset$ set $P_J = \prod_{i \in J} R_i$, and define

$$A_{J,v}(t) = \sum_{n=n_a+1}^{n_a+t} \mathbf{1}_{\{Z_{f_0}(n)=v\}} f_0(P_J(n)) \omega(P_J(n)), \quad (5.1)$$

$$F_{a,j}(\omega; f_0) = \sum_{t \in \mathcal{G}_j} \sum_{J \neq \emptyset} \sum_v A_{J,v}(t)^{r_j}, \quad (5.2)$$

$$D_{a,j} = |\mathcal{G}_j| \sum_{J \neq \emptyset} C_{P_J, E, r_j}(T_j). \quad (5.3)$$

Each collision count uses the task's starting point n_a . The finite parity count remains meaningful for every polynomial in the task enumeration, including non-separable products; separability is required only when a sieve estimate is subsequently applied.

For task $a = 1$, add control of the multiplier on the ordinary integers:

$$F_{1,j}(\omega) = \sum_{t \in \mathcal{G}_j} \left(\sum_{n=2}^{t+1} \omega(n) \right)^{r_j}, \quad D_{1,j} = |\mathcal{G}_j| C_{T, E, r_j}(T_j), \quad (5.4)$$

where the subscript T in C denotes the identity polynomial and the starting point is one.

All energies are non-negative because r_j is even. Their normalisers are positive, since tuples paired into equal indices contribute to each parity count. Expanding a moment and using prime-sign orthogonality gives

$$0 \leq \mathbb{E} F_{a,j}(\omega; f_0) \leq D_{a,j}. \quad (5.5)$$

There is no extra factor for the number of mark states in this inequality. For each tuple in the expansion, the sum over v of the product of the mark indicators is one if all its indices have the same mark and zero otherwise. The remaining source weights have absolute value at most one. A prefix count at $t \leq T_j$ is bounded by the full count at T_j .

Both $D_{a,j}$ and the finite polynomial $F_{a,j}$ in the free signs are exactly computable. One may factor all the finitely many polynomial values, record their free-prime parity vectors, and expand the powers. The expectation of a monomial conditional on previously fixed prime signs is zero unless every still-free sign has even exponent. The calculation uses integers and rational numbers, not an unknown constant in an asymptotic estimate. Its running time is not claimed to be practical.

One finite decision. Take the four values $n^2 + 1 = 5, 10, 17, 26$ for $n = 2, 3, 4, 5$. In this finite illustration, let $x = \omega(5)$ and $y = \omega(13)$ be independent fair signs, with every other multiplier sign equal to one. From the constant source the sum is $2x + 1 + y$. Its squared cost $F = (2x + 1 + y)^2$ has expectation 6, and

$$\mathbb{E}(F/6 \mid x = 1) = \frac{5}{3}, \quad \mathbb{E}(F/6 \mid x = -1) = \frac{1}{3}.$$

The cheaper choice is $x = -1$. From the source $f_0(m) = (-1)^{v_{17}(m)}$, the sum is instead $2x - 1 + y$ and the cheaper choice is $x = 1$, with the same normaliser. Uniform cost bounds therefore need not mean source-independent sign choices. This calculation illustrates one finite comparison, not a run of the universal selector. Its infinitely many requirements are controlled by the full potential below.

5.2 A tail estimate that survives every decision

Set

$$U_{f_0}(\omega) = \sum_{a,j \geq 1} \frac{2^{-a}}{j(j+1)} \frac{F_{a,j}(\omega; f_0)}{D_{a,j}}. \quad (5.6)$$

The two sums of weights equal one, and (5.5) gives $\mathbb{E}U_{f_0} \leq 1$, uniformly in f_0 . For the finite rectangle

$$1 \leq a \leq R, \quad 1 \leq j \leq 2^R - 1,$$

the omitted weight is

$$1 - (1 - 2^{-R})^2 \leq 2^{1-R}. \quad (5.7)$$

After fixing s signs, the cylinder has probability 2^{-s} . By positivity, the conditional expectation of the omitted part of (5.6) is therefore at most

$$2^s 2^{1-R}. \quad (5.8)$$

This estimate pays for all future tasks at once, including those not yet evaluated.

Proposition 5.1 (Irrevocable computable selection). *There is a deterministic procedure choosing every sign on E , with*

$$U_{f_0}(\omega_{f_0}) \leq \frac{5}{4}. \quad (5.9)$$

Every selected sign is obtained from finitely many oracle queries to f_0 . The map $f_0 \mapsto \omega_{f_0}$ is continuous and computable relative to the input.

Proof. Enumerate E in increasing order. At decision $s \geq 1$, use the finite rectangle with $R = 2s + 3$. Compute its two rational conditional expectations corresponding to the two possible signs at the next prime. Choose the smaller one, choosing $+1$ in a tie. After this decision, the tail bound (5.8) for each child is at most

$$\varepsilon_s = 2^{-s-2}, \quad \sum_{s \geq 1} \varepsilon_s = \frac{1}{4}.$$

Let C_{s-1} be the true conditional expectation of the whole potential before the decision. The average of the two children's true costs is C_{s-1} . The smaller finite cost is at most that average, since each omitted tail is non-negative. Adding the selected child's omitted tail gives $C_s \leq C_{s-1} + \varepsilon_s$. Thus $C_s \leq 5/4$ for every s .

Each finite partial sum of (5.6) depends on finitely many signs. Once these have all been decided, its realised value is at most the current conditional expectation of U_{f_0} , by positivity.

It is at most $5/4$. Increasing the finite partial sums proves (5.9); no probabilistic choice of an uncomputed infinite realisation is used.

The rectangle and every arithmetic computation at a decision are finite. Each output sign consequently uses finitely many input values, and its value is never revised. This proves relative computability and continuity in the product topology. For an ordinary computable input, all required oracle queries can be computed, so the output is computable. $\square$

The method also permits a finite pilot without discarding future requirements. Let $P_{f_0} \geq 0$ be a computable finite energy with $\mathbb{E}P_{f_0} \leq 1$, and fix a positive rational $\delta \leq 1$. Apply the same selection to $(1 - \delta)P_{f_0} + \delta U_{f_0}$. Its total cost is at most $5/4$, and therefore $U_{f_0} \leq 5/(4\delta)$. The pilot is always evaluated; when it alone determines an early decision, the universal part must still be omitted with the conditional tail *paid*. The limiting conclusions remain unchanged, with constants allowed to depend on δ .

5.3 A finite uniform bound for the selected output

The potential estimate gives, for each task,

$$F_{a,j}(\omega_{f_0}; f_0) \leq \frac{5}{4} 2^a j(j+1) D_{a,j}. \quad (5.10)$$

Let $L_a = (2^k - 1)|\mathcal{Z}|$ be the number of non-empty character-mark pairs. For a grid point t , Hölder's inequality and (4.5) bound the marked total variation by

$$\frac{L_a}{2t} F_{a,j}(\omega_{f_0}; f_0)^{1/r_j}.$$

At $T_j/2 \leq N \leq T_j$, compare with a grid point and also compare the two empirical mark laws. This yields the explicit finite bound

$$d_{\text{TV}}(\mathcal{L}_N(X_g, Z_{f_0}), U_k \otimes \mathcal{L}_N(Z_{f_0})) \leq \min(1, \mathcal{B}_{a,j}), \quad (5.11)$$

where

$$\mathcal{B}_{a,j} = \frac{L_a}{T_j} \left(\frac{5}{4} 2^a j(j+1) D_{a,j} \right)^{1/r_j} + \frac{8}{(j+1)^2} + \frac{8}{T_j}. \quad (5.12)$$

The two final terms bound the interpolation of the joint empirical measure and its moving product target. All quantities on the right are independent of the input f_0 . The bound is computable from finite integers; it may be larger than one at accessible sizes. Computing this bound at one scale is distinct from computing an index after which every later bound lies below a requested tolerance. The proof below establishes convergence under the moment hypotheses, but supplies no useful numerical modulus of that convergence.

If the moment norm (4.2) tends to zero for every P_J in the fixed task, then $\mathcal{B}_{a,j} \rightarrow 0$. Indeed $D_{a,j}^{1/r_j} / T_j$ is at most

$$|\mathcal{G}_j|^{1/r_j} (2^k - 1)^{1/r_j} \max_{J \neq \emptyset} b_{P_J, j},$$

and the r_j -th roots of the polynomial factors in j remain bounded because $r_j \asymp \log j$. We have therefore proved a uniform bound over all inputs of this selector, not merely an almost-sure existence statement.

In particular, if $E \subseteq \mathcal{P}_{d_0} \cap (B, \infty)$ is decidable and satisfies (A), all target quadratic tasks satisfy the moment hypotheses. Together with Lemma 2.1, Proposition 5.1 proves the computable selective completion under (A). It works for all horizons, with each horizon fixed in its own limit. No procedure here decides (A) for an arbitrary supplied membership predicate.

5.4 What sparse changes actually cost

Let

$$A(f_0) = \{q : g(q) \neq f_0(q)\} \subseteq E.$$

Whenever the identity-polynomial task has vanishing moment norm, (5.4) and the same argument give $N^{-1} \sum_{n=2}^{N+1} \omega_{f_0}(n) \rightarrow 0$, uniformly in the input. Consequently

$$\frac{1}{N} \#\{2 \leq n \leq N+1 : g(n) \neq f_0(n)\} \rightarrow \frac{1}{2}. \quad (5.13)$$

The marked target law with $H = 1$, projected onto the old and new target signs, similarly gives disagreement density $1/2$ on the target trace.

Proposition 5.2 (Infinite harmonic cost of the actual changes). *For each output having (5.13),*

$$\sum_{q \in A(f_0)} \frac{1}{q} = \infty.$$

If E has relative prime density zero, the actual changes still have relative prime density zero.

Proof. Suppose the reciprocal sum over $A = A(f_0)$ were finite. Truncating ω to negative signs only at $q \in A$, $q \leq Y$, changes its mean on the ordinary integers by at most $2 \sum_{q \in A, q \leq Y} 1/q$. For a finite set of primes, the Chinese remainder theorem and truncation of their valuations give mean

$$\prod_{\substack{q \in A \\ q \leq Y}} (1 - q^{-1}) \sum_{j \geq 0} (-1)^j q^{-j} = \prod_{\substack{q \in A \\ q \leq Y}} \frac{q-1}{q+1}.$$

Under the assumed summability this product tends to a strictly positive number, contradicting the zero mean of ω . Prime density zero follows simply from $A \subseteq E$. $\square$

Thus a small proportion of changed prime coordinates is not a small perturbation of the integer sequence. The compactness consequence is also precise: the continuous image of the compact space $\mathcal{M}_\pm$ is a compact family of outputs, uniformly normal for each tested observation family, and its projection onto the two protected traces covers all data compatible with a source. This concerns freedom after protecting two traces, not stable reconstruction from all three.

6 Explicit reservoirs of successively smaller harmonic mass

The availability theorem does not require a positive prime density or an asymptotic formula for the reservoir. Explicit regular examples nevertheless show how far it reaches. We first thin one exclusive class by a slowly growing gap in *prime rank*; the same construction can be iterated to any fixed finite depth.

Enumerate the exclusive primes as $q_1 < q_2 < \dots$. Define the integer-valued iterates

$$B_1(n) = \text{bl}(n), \quad B_{k+1}(n) = \text{bl}(B_k(n)).$$

For a fixed integer $m \geq 1$, set

$$b_m(n) = \prod_{k=2}^{m+1} B_k(n), \quad a_1 = 1, \quad a_{j+1} = a_j + b_m(a_j), \quad E_m = \{q_{a_j} : q_{a_j} > B\}. \quad (6.1)$$

The target class and the retained ranks are different indices. The construction uses only integer operations. At depth one the retained ranks begin

$$1, 2, 4, 6, 8, 11, 14, 17, 20, 23, \dots;$$

at depth two they begin

$$1, 2, 6, 10, 16, 22, 28, 34, 40, 46, \dots$$

Write $\ell_1(x) = \log x$ and $\ell_{k+1}(x) = \log(\ell_k(x))$ at sufficiently large arguments. Thus ℓ_k denotes an iterated natural logarithm, whereas bl is the integer bit length used in the rank construction.

Proposition 6.1 (Counting, harmonic mass and availability). *For fixed $m \geq 1$, with $c_m = (\log 2)^m/4$,*

$$\#(E_m \cap [1, x]) \sim c_m \frac{x}{\ell_1(x)\ell_2(x) \cdots \ell_{m+1}(x)}, \quad (6.2)$$

$$\sum_{\substack{q \in E_m \\ q \leq x}} \frac{1}{q} \sim c_m \ell_{m+2}(x). \quad (6.3)$$

These reservoirs are decidable, have relative prime density zero, and satisfy (A).

Proof. For $\sqrt{R} \leq n \leq R$, each fixed $B_k(n)$, $k \geq 2$, is $B_k(R)(1 + o(1))$ uniformly. The gaps $b_m(n)$ in this range are consequently $b_m(R)(1 + o(1))$. At most $\sqrt{R}$ retained ranks precede it, which is negligible compared with $R/b_m(R)$. Thus

$$\#\{j : a_j \leq R\} \sim \frac{R}{b_m(R)}, \quad b_m(R) \sim \frac{\ell_2(R) \cdots \ell_{m+1}(R)}{(\log 2)^m}.$$

The prime number theorem in the fixed residue class modulo eight gives $\#\{j : q_j \leq x\} \sim x/(4 \log x)$; this is the fixed-modulus instance of [14], equation (27.12.8), or [8], Theorem 4.12. Substituting this value for R gives (6.2), because all iterated logarithms from the second onwards are asymptotic to the corresponding ones at x .

Partial summation, using

$$\int_x^x \frac{dt}{t \ell_1(t) \cdots \ell_{m+1}(t)} = \ell_{m+2}(x) + O(1),$$

proves (6.3). The removed finite prefix does not affect either equivalent. At the lower cutoff in (A),

$$\sum_{\substack{q \in E_m \\ q \leq \log \log x}} \frac{1}{q} \sim c_m \ell_{m+4}(x) = o(\ell_{m+2}(x)).$$

Subtracting proves availability. Membership can be decided by enumerating the primes of the fixed class and the increasing sequence of retained ranks up to the queried prime. $\square$

For target-word moments P_J , the root number is $\rho = 2|J|$ outside finitely many primes. At the cutoffs of Lemma 3.2, with $r = r_j$,

$$\sum_{\substack{q \in E_m \\ V_j < q \leq x_j}} \frac{1}{q} = (c_m + o(1)) \ell_{m+2}(T_j). \quad (6.4)$$

Indeed $x_j = T_j^{c_P/r_j + o(1/r_j)}$, so $\ell_{m+2}(x_j) \sim \ell_{m+2}(T_j)$, while the mass below $V_j \asymp_P \log \log T_j$ is of order $\ell_{m+4}(T_j)$.

The constant-root version of (3.12) therefore gives

$$\frac{C_{P_J, E_m, r_j}(T_j)^{1/r_j}}{T_j} \leq (\ell_{m+1}(T_j))^{-\rho c_m + o(1)}. \quad (6.5)$$

The exponentially small second term in (3.12) is negligible. Equations (5.12) and (6.5) now yield the following quantitative form.

Corollary 6.2 (Fixed-depth sparse completion). *For every fixed m, d_0, B , the selector on E_m satisfies, for each fixed horizon H ,*

$$\sup_{f_0 \in \mathcal{M}_\pm} d_{\text{TV}}(\mathcal{L}_N(X_g^H, Z_{f_0}^H), U_H \otimes \mathcal{L}_N(Z_{f_0}^H)) = O_{m, d_0, B, H, \beta}((\ell_{m+1}(N))^{-\beta}) \quad (6.6)$$

whenever $0 < \beta < 2c_m$. The corresponding rate also holds almost surely under fair signs on E_m , for each fixed source. The ordinary mean of the multiplier tends to zero, with the same form of rate for $0 < \beta < c_m$.

Proof. For the selector, take r_j -th roots in the energy bound and use (6.5); the factors polynomial in j have bounded r_j -th roots. Every non-empty target subword has $\rho \geq 2$. For the identity polynomial in the multiplier task, $\rho = 1$.

For the random completion, apply Markov on the grid with threshold $(\ell_{m+1}(T_j))^{-\beta}$. The ratio of the moment norm to that threshold tends to zero. Its r_j -th power is eventually at most $(j+1)^{-6}$, and the proof of Proposition 4.2 applies. Interpolation is smaller than the stated rate. Countable intersections treat the fixed horizons and rational exponents approaching the endpoint. $\square$

Theorem 1.1 is Corollary 6.2 at depth one, together with Lemma 2.1 and Proposition 5.2. The first three depths make the cost–rate relation explicit:

Depth	Harmonic mass	Permitted target-word rate
1	$c_1 \ell_3(x)$	$(\ell_2(N))^{-\beta}, \beta < \log 2/2$
2	$c_2 \ell_4(x)$	$(\ell_3(N))^{-\beta}, \beta < (\log 2)^2/2$
3	$c_3 \ell_5(x)$	$(\ell_4(N))^{-\beta}, \beta < (\log 2)^3/2$

The depth is fixed before the limit. These examples do not permit a depth growing with x , nor an arbitrarily prescribed rate of harmonic divergence. The universal reservoir in the next section has a different task: it must not miss the prime root classes of *any* admissible polynomial.

7 One sparse reservoir for all admissible polynomials

Thinning the exclusive primes without considering root counts could remove almost every prime at which a new polynomial has a root. We avoid that loss by balancing the thinning separately in finitely many root-count classes. The number of classes increases with the block, while the blocks start sufficiently far apart. No uniform prime theorem for a growing family of fields is needed.

7.1 A root-balanced construction

Enumerate all monic non-constant integer polynomials without repetition as $P_1, P_2, \dots$. The enumeration is effective and may start with

$$T, T^3 - 2, \Phi_5(T), T^2 + 1, T^2 - 3, T^4 - 2, T^4 + 1, T^4 + 2.$$

Let

$$C_j = \prod_{i \leq j} (\deg P_i + 1), \quad D_j = \max_{i \leq j} \deg P_i, \quad \pi_*(y) = \#(\mathcal{P}_{d_0} \cap [1, y]).$$

The three thresholds below have separate roles. The first keeps the block index small relative to $\sqrt{\log \log x}$, so thinning by that index still leaves a useful root mass. The second makes the weighted discrepancies in Lemma 7.1 summable. The third controls accumulated class-count errors through a finite prime count; it is retained even though the one-sided density bound below can use the prefix-floor convention directly.

Starting with $Y_0 = B$, choose Y_j to be the smallest power of two strictly larger than Y_{j-1} and satisfying

$$Y_j \geq 2^{2^{j^2}}, \quad Y_j \geq 2^{j+2} C_j D_j, \quad \pi_*(Y_j) \geq j^2 \sum_{i \leq j} C_i. \quad (7.1)$$

Each search terminates because the fixed exclusive progression contains infinitely many primes.

In the block $(Y_j, Y_{j+1}]$, partition the exclusive primes according to

$$(\rho_{P_1}(q), \dots, \rho_{P_j}(q)). \quad (7.2)$$

There are at most C_j classes. In each class, retain every j -th prime, counting from the beginning of the block. Let E be the union of the retained primes, with no primes at most Y_1 . Membership is decidable by finding the relevant block, counting primes in its finite root classes, and comparing their ranks. Deciding admissibility or computing a limiting density is not part of this algorithm.

Lemma 7.1 (Root-weighted thinning discrepancy). *If $i \leq j$, then on every subinterval I of $(Y_j, Y_{j+1}]$,*

$$\left| \sum_{q \in E \cap I} \frac{\rho_{P_i}(q)}{q} - \frac{1}{j} \sum_{q \in \mathcal{P}_{d_0} \cap I} \frac{\rho_{P_i}(q)}{q} \right| \leq \frac{2(\deg P_i) C_j}{Y_j}. \quad (7.3)$$

For fixed i , the bounds on the right are summable over $j \geq i$.

Proof. Within one root class, the retained prefix count is the floor of $1/j$ times the full count. Its discrepancy lies in $[-1, 0]$. Partial summation with the decreasing weight $1/q$ bounds the reciprocal discrepancy of a prefix by $1/Y_j$. A subinterval is the difference of two prefixes. Since ρ_{P_i} is constant within a root class and at most $\deg P_i$, summing over the at most C_j classes proves (7.3). The second bound in (7.1) makes its right-hand side at most 2^{-j-1} for $i \leq j$. $\square$

Proposition 7.2 (Properties of the common support). *The set E is decidable and has relative density zero among the primes. For every admissible monic irreducible P , there are $c_P > 0$ and x_P such that*

$$\sum_{\substack{q \in E \\ \log \log x < q \leq x}} \frac{\rho_P(q)}{q} \geq c_P \sqrt{\log \log x} \quad (x \geq x_P). \quad (7.4)$$

In particular, E has infinite reciprocal sum and satisfies (WP) for each admissible polynomial.

Proof. Fix j_0 . In any block with $j \geq j_0$, and in a prefix of that block, the retained count in each root class is at most $1/j_0$ times its full count. Summing gives

$$\#(E \cap [1, x]) \leq O_{j_0}(1) + \pi_*(x)/j_0.$$

Divide by $\pi(x)$, let $x \rightarrow \infty$, then let $j_0 \rightarrow \infty$. This proves relative density zero. The third threshold in (7.1) also bounds accumulated class-count errors, though the prefix-floor convention already gives the displayed one-sided bound.

Now write $P = P_i$ and put $y = \log \log x$. Let J be the block containing x . The first threshold in (7.1) implies

$$J \ll \sqrt{\log \log x}. \quad (7.5)$$

For large enough x , every block met by $(y, x]$ has index at least i . On each such block its retention factor $1/j$ is at least $1/J$. Sum (7.3); the errors are bounded in total. Positivity of the root weights gives

$$\sum_{\substack{q \in E \\ y < q \leq x}} \frac{\rho_P(q)}{q} \geq \frac{1}{J} \sum_{\substack{q \in \mathcal{P}_{d_0} \\ y < q \leq x}} \frac{\rho_P(q)}{q} - O_P(1).$$

By Proposition 2.4, the full weighted sum on the right is

$$\kappa_{d_0}(P)(\log \log x - \log \log y) + O_P(1).$$

Its leading coefficient is positive, and $\log \log y = o(\log \log x)$. Together with (7.5), this proves (7.4). The identity polynomial $P(T) = T$ is admissible and has one root at every prime, so its case proves infinite harmonic mass. $\square$

This proof uses the prime theorem only for each fixed polynomial after its index has been reached. The thresholds themselves use finite prime and root counts. The absence of an effective uniform Chebotarev threshold in growing extensions is therefore not an unmentioned input.

7.2 Proof of the common completion theorem

Take the selector of Section 5 on the support just constructed, with all non-empty finite polynomial observation lists and all possibly empty finite mark lists, as specified there. For a fixed list of distinct admissible monic irreducibles, every non-empty product P_J is separable. Outside finitely many primes its root count is the sum of those of its factors. Proposition 7.2 therefore supplies (WP) for every such product.

More quantitatively, at the cutoffs (3.10), the same comparison as in Lemma 4.1 gives

$$\Delta_{P_J, E}(T_j, r_j) \geq c_J \sqrt{\log \log T_j}$$

for all sufficiently large j , after reducing $c_J > 0$ if necessary. Indeed $\log \log x_j = \log \log T_j - O_{P_J}(\log \log \log T_j)$, and raising the lower cutoff costs only a bounded amount. Lemma 3.2 gives

$$\frac{C_{P_J, E, r_j}(T_j)^{1/r_j}}{T_j} \ll_J \exp(-c_J \sqrt{\log \log T_j}). \quad (7.6)$$

The second term of that lemma is smaller than the displayed quantity.

Apply (5.12) to the fixed task. The finite list of J 's permits a common positive c , and the other r_j -th roots remain bounded. The interpolation terms are smaller as well. This proves (1.7), uniformly in f_0 . The potential covers all the tasks, so the same output works for all lists. Its support lies above B in $\mathcal{P}_{d_0}$, giving the prefix and both norms by Lemma 2.1. The continuity and computability statements are Proposition 5.1.

Distinct translates of a monic non-constant polynomial are distinct polynomials: their coefficients of next-highest degree differ. Translating preserves irreducibility and the root field. Thus the result for all finite distinct lists gives normality of the trace of each admissible irreducible. This proves the implication (i) $\Rightarrow$ (iii) of Theorem 1.2. Proposition 2.2 gives the converse and the frozen assertion, and Section 2 proved (i) $\Leftrightarrow$ (ii). The theorem is complete.

For this common selector, the identity task and the mark given by each original admissible trace also show disagreement density $1/2$ on the integers and on each such trace. Proposition 5.2 applies to its actual changed primes. The deterministic selector is not expected to choose exactly half of the reserved prime signs; only the inclusion of its changed set in E is needed for prime density zero.

7.3 Products retain their square relations

The factor-parity representation and the corresponding sign-subgroup channel occur in [16], Theorems 6.1–6.2, for non-selective noise. Here the algebraic classification removes the frozen irreducibles from the free coordinates, and the common selector supplies the remaining factor signs inside the protected fibre. The finite Fourier mechanism is inherited; the selection and the classification of active factors provide its selective form.

The restriction to distinct irreducible coordinates explains the law of general monic observations. Fix monic non-constant $R_1, \dots, R_k \in \mathbb{Z}[T]$ and factor them into monic irreducibles. List the distinct admissible factors as $P_1, \dots, P_s$, and let

$$B = (b_{ia}) \in \mathbb{F}_2^{k \times s}$$

record their multiplicities modulo two. The other irreducible factors are frozen, and even exponents contribute one to a sign. Define the sign subgroup

$$V_B = \left\{ \left((-1)^{(Bv)_i} \right)_{i \leq k} : v \in \mathbb{F}_2^s \right\} \subseteq \{-1, 1\}^k.$$

If $s = 0$, this is the singleton consisting of the all-positive vector.

Corollary 7.3 (The finite polynomial channel). *For the common selector, the empirical vector of multiplier signs*

$$(\omega_{f_0}(R_i(n)))_{i \leq k}$$

becomes uniform on V_B , asymptotically independent of any fixed finite collection of original polynomial marks. The comparison is uniform over f_0 . Equivalently, the output vector $(g(R_i(n)))_i$ is asymptotically obtained by multiplying its original source vector coordinatewise by a uniform element of V_B , while retaining the empirical original marks.

Proof. If $s = 0$, every factor is frozen or has even exponent, so the multiplier vector is identically the all-positive vector and the comparison is exact. Assume henceforth that $s \geq 1$. Include the original signs $f_0(P_a(n))$ among the marks in Theorem 1.2. Multiplication of the new independent uniform factor signs by these marked source signs leaves a uniform cube, so the multiplier factor vector is uniform and independent of the marks. Apply the group homomorphism represented by B . A uniform vector on a finite group maps to the uniform vector on its image, because all fibres have the same cardinality. Total variation contracts under this common measurable map. Frozen factors and squares make no contribution to the multiplier. Including the original R_i signs among the marks gives the output formulation. $\square$

For example, if P, Q are distinct admissible irreducibles, the three multiplier observations at P, Q, PQ have four equiprobable states, not eight: their product is always one. The original empirical vector need not itself converge, so its randomised coset need not have a fixed limiting law. Monic factorisation avoids an additional rational leading-coefficient phase, as illustrated in Section 2.2. No classification for arbitrary non-monic observation families is being substituted for the stated one.

8 Divergent reservoirs with no convergent completion

The preceding constructions have more than infinite harmonic mass: they keep supplying usable primes at the scales where high moments are needed. We now show that this distinction changes the existence of a fixed completion, not just its available convergence rate.

For a prime set A , let $\beta_A \in \mathcal{M}_\pm$ be negative exactly at the primes in A . Write

$$d_1(f, h) = \sum_{q: f(q) \neq h(q)} \frac{1}{q}, \quad M_{d,f}(N) = \frac{1}{N} \sum_{n=2}^{N+1} f(Q_d(n)),$$

and $\text{osc}_N M = \limsup_N M(N) - \liminf_N M(N)$ for a bounded real sequence.

Theorem 8.1 (A reservoir without a convergent completion). *Fix $d_0 \in \mathcal{D}$, an integer $B \geq 1$, and a positive rational ε . There exists a decidable set $E \subseteq \mathcal{P}_{d_0} \cap (B, \infty)$, independent of a reference source, with*

$$\#(E \cap [1, x]) \ll \frac{x}{\log x \log \log x}, \quad \sum_{q \in E} \frac{1}{q} = \infty, \quad (8.1)$$

such that for every $f_\star \in \mathcal{M}_\pm$ there is $f \in \mathcal{M}_\pm$ satisfying

$$d_1(f, f_\star) < \varepsilon, \quad f(m) = f_\star(m) \ (m \leq B), \quad f(q) = f_\star(q) \ (q \in E), \quad (8.2)$$

and, simultaneously for every $A \subseteq E$ and every finite set of primes F ,

$$\text{osc}_N M_{d_0, f\beta_A\beta_F}(N) \geq \frac{1}{2}. \quad (8.3)$$

The map $f_\star \mapsto f$ can be chosen continuous and relatively computable. For $F = \emptyset$, all the completions $f\beta_A$ share the prefix and both protected norms of f .

The protected norms in the last sentence belong to f , not necessarily to $f_\star$. Constructing the obstructive base is a separate modification from subsequently completing it on E . The theorem does not exclude normal completion using the *whole* exclusive class.

8.1 Nevertheless, every fixed word law converges in probability

Proposition 8.2 (What harmonic divergence alone gives). *Let $E \subseteq \mathcal{P}_{d_0}$ have infinite reciprocal sum, with no regularity or computability assumption. For fair independent prime signs on E , every fixed horizon H satisfies*

$$\lim_{N \rightarrow \infty} \sup_{f \in \mathcal{M}_\pm} \mathbb{E} d_{\text{TV}}(\mathcal{L}_N(X_{f_\omega}^H), U_H)^2 = 0. \quad (8.4)$$

The convergence also holds in every finite positive moment and in probability. For any deterministic mark sequence $Z(n)$ in a fixed finite alphabet, the analogous conclusion holds for $\mathcal{L}_N(X_{f_\omega}^H, Z)$ compared with $U_H \otimes \mathcal{L}_N(Z)$.

Proof. Let P be a fixed product of distinct translates of Q_{d_0} . At all but finitely many primes in E , it has a fixed positive number ρ of simple roots. The second-moment version of (3.2) bounds the squared norm of a weighted average by $C_{P,E,2}(N)/N^2$.

Fix a finite set $F \subseteq E$ of good primes, and put $m_F = \prod_{q \in F} q^2$. A pair counted in $C_{P,E,2}$ must avoid total valuation one at all primes in F . The forbidden local density is

$$g_2(q) = \frac{2\rho}{q}(1 - q^{-1})(1 - \rho/q) = \frac{2\rho}{q} + O_P(q^{-2}).$$

On the full period modulo m_F , the proportion of surviving pairs is $\prod_{q \in F} (1 - g_2(q))$. The two-coordinate residue count gives

$$\frac{C_{P,E,2}(N)}{N^2} \leq \prod_{q \in F} (1 - g_2(q)) + \frac{2m_F}{N} \left(1 + \frac{m_F}{N}\right). \quad (8.5)$$

First let $N \rightarrow \infty$ with F fixed, and only then let F increase. Infinite harmonic mass makes the product tend to zero. The bound was independent of the deterministic weights.

For a measure μ on the sign cube, Fourier inversion and Cauchy–Schwarz give

$$d_{\text{TV}}(\mu, U_H)^2 \leq \frac{1}{4} \sum_{\emptyset \neq J \subseteq [H]} |\widehat{\mu}(J)|^2.$$

Apply the second-moment estimate to the finitely many products P_J in these Fourier coefficients. This proves (8.4). For marks, use the weights $f(P_J(n))\mathbf{1}_{\{Z(n)=v\}}$ in (4.5), then Cauchy–Schwarz over the finitely many pairs (J, v) . All total variations lie between zero and one, so convergence in the second moment implies convergence in every finite positive moment and in probability. $\square$

The supremum over f in (8.4) is outside the expectation. It does not permit an input chosen after the signs. More importantly, convergence in probability on the full sequence of sizes does not imply a fixed realisation converging on that full sequence.

An elementary model isolates the issue. At level j , list all words $v \in \{0, 1\}^j$ and use the indicator that an infinite bit sequence begins with v . Each indicator has probability 2^{-j} , so the resulting ordered sequence tends to zero in probability. Every bit sequence nevertheless meets an indicator equal to one at every level, as well as zeros. The construction below realises this moving-exception mechanism inside a fibre of multiplicative completions.

8.2 Private primes can create a lasting discrepancy

We recall the arithmetic resource, including its cost. The method has an antecedent in [9], Lemma 2.1; the following argument records the precise range used here.

Lemma 8.3 (A supply of private large prime factors). *Fix $\tau \geq 4$, $R \geq 2$ and $d \in \mathcal{D}$. Then*

$$\#\{R + 2 \leq n \leq N + 1 : P^+(Q_d(n)) > \tau N\} \geq (1/2 - o_{\tau, R}(1))N. \quad (8.6)$$

For all sufficiently large N , each indicated value has exactly one prime factor exceeding τN , that prime has valuation one, and it divides no other value of the same trace at indices $2, \dots, N + 1$.

Proof. Put $V_N = (N + 1)^2 + 2$. Summing the logarithms of all values gives

$$\sum_{n=2}^{N+1} \log Q_d(n) = 2N \log N + O(N).$$

At an odd split prime, the two roots are simple and lift to every prime power. Thus

$$\sum_{n=2}^{N+1} v_q(Q_d(n)) \leq \frac{2N}{q-1} + \frac{2 \log V_N}{\log q}.$$

Odd inert primes contribute nothing, and the prime two contributes $O(N)$. The split primes form one half of the reduced classes modulo four or eight. The fixed-modulus prime number theorem and partial summation (equivalently [14], equation (27.11.11)) give

$$\sum_{\substack{q \leq x \\ (d/q)=1}} \frac{\log q}{q-1} = \left(\frac{1}{2} + o(1)\right) \log x.$$

Consequently the logarithmic contribution of primes at most τN is at most

$$(1 + o_\tau(1))N \log N.$$

Here the rounding contribution is $2\pi(\tau N) \log V_N = O_\tau(N)$, not an additional term of order $N \log N$. At least $(1 - o_\tau(1))N \log N$ remains above the cutoff, and each value contributes at most $\log V_N = (2 + o(1)) \log N$. At least $N/2 - o_\tau(N)$ indices have a prime above the cutoff. Removing the first R indices proves (8.6).

For large N , $\tau N > 2(N + 1)$ and $(\tau N)^2 > V_N$. The latter inequalities give uniqueness and valuation one. If the same prime divided $Q_d(n)$ and $Q_d(m)$ at distinct indices in the prefix, it would divide $(n - m)(n + m)$, whose two non-zero factors have absolute value below the prime. This is impossible. $\square$

Let $\mathcal{R}$ denote the set in (8.6), and let q_n be its private prime. For any source h , changing only the signs at these primes can set all $h(Q_d(n))$, $n \in \mathcal{R}$, to $+1$, or all to -1 . Every other value of this trace in the prefix is unchanged. The two new means differ by exactly $2|\mathcal{R}|/N$. Given an earlier mean a , at least one branch differs from a by at least $|\mathcal{R}|/N$. The harmonic cost is less than

$$\sum_{n \in \mathcal{R}} \frac{1}{q_n} < \frac{|\mathcal{R}|}{\tau N} \leq \frac{1}{\tau}. \quad (8.7)$$

No privacy between different traces is used. These primes construct the base f ; the separate exclusive reservoir will protect the other norms among its completions.

8.3 A calendar fixed before the input

Take the depth-one reservoir $S = E_1$ of Section 6. It is decidable, has the counting bound in (8.1), and has infinite reciprocal sum. Choose an increasing computable sequence of integers $\tau_\ell \geq 4$ with

$$\sum_{\ell \geq 1} \tau_\ell^{-1} < \varepsilon.$$

For example $\tau_\ell = \max(4, \lceil 2^{\ell+2}/\varepsilon \rceil)$ suffices. Start with $R_0 \geq \max(B, 2)$ and $V_0 > (R_0 + 1)^2 + 2$.

At stage $j \geq 1$, first take a finite block F_j of successive primes of S above V_{j-1} until its reciprocal sum is at least one. Let

$$E_j = F_1 \cup \dots \cup F_j, \quad D_j = E_j \cup \{q : q \text{ prime}, q \leq j\}.$$

Enumerate all sign configurations on the finite set D_j . For each configuration, with a new global task number ℓ , choose by finite search an integer N_ℓ larger than the currently protected number R of origins, large enough that

$$\tau_\ell N_\ell > \max(E_j \cup \{B, j\}), \quad |\mathcal{R}_\ell| \geq \left(\frac{1}{2} - \frac{1}{j+4}\right) N_\ell. \quad (8.8)$$

Here $\mathcal{R}_\ell$ is the private-prime set of Lemma 8.3 after the R previously protected origins. The search may additionally require $N_\ell \geq \tau_\ell$ and the elementary size conditions in that lemma. It terminates by (8.6). Replace R by N_ℓ after each task. When all configurations have been treated, set $V_j = (R + 1)^2 + 2$ and start the next stage.

This calendar uses prime factors, finite counts and the *number* of configurations, but not the source values or any chosen signs. It defines

$$E = \bigcup_{j \geq 1} F_j$$

before $f_\star$ is supplied. The blocks each have reciprocal mass at least one, while $E \subseteq S$, proving (8.1). To decide membership of a prime, run the finite calendar until a cutoff V_j exceeds it. The long gaps between free blocks are essential: they allow every configuration already exposed to receive a discrepancy witness before any further free prime can affect that witness.

Calendar procedure (no source queries). The following finite-stage procedure summarises the construction. It emits instructions that can subsequently be replayed for any f_* ; it need not store an infinite calendar.

Initialise $R \leftarrow R_0$, $V \leftarrow V_0$, $E^{\text{cur}} \leftarrow \emptyset$, $\ell \leftarrow 0$.

For $j = 1, 2, \dots$:

1. Take successive primes of S above V until their reciprocal sum is at least one; call this finite block F_j . Set $E^{\text{cur}} \leftarrow E^{\text{cur}} \cup F_j$ and $D_j \leftarrow E^{\text{cur}} \cup \{q \text{ prime} : q \leq j\}$.
2. **For each** configuration $\eta \in \{-1, 1\}^{D_j}$ in a fixed order:
 - a. Set $\ell \leftarrow \ell + 1$. Search for $N_\ell > R$ satisfying (8.8), $N_\ell \geq \tau_\ell$, $\tau_\ell N_\ell > 2(N_\ell + 1)$ and $(\tau_\ell N_\ell)^2 > (N_\ell + 1)^2 + 2$.
 - b. Compute the private indices $\mathcal{R}_\ell$ and their primes q_n . Emit the instruction

$$(j, \ell, D_j, \eta, R, N_\ell, (q_n)_{n \in \mathcal{R}_\ell}).$$

No value of η or of a source is used in the search for N_ℓ .

c. Set $R \leftarrow N_\ell$.

3. After all configurations are emitted, set $V \leftarrow (R + 1)^2 + 2$. This is V_j . Only then begin the next free block.

Reservoir: $E = \bigcup_{j \geq 1} F_j$.

8.4 One base treats every completion

Now begin from $f^{(0)} = f_*$ and follow the fixed calendar. At the task for a configuration η on D_j , extend η by one at all other primes and put $h = f^{(\ell-1)}\eta$. Its mean at the previously protected size R is the reference $a = M_{d_0, h}(R)$.

Apply the two private-prime prescriptions to h on $\mathcal{R}_\ell$ and retain the branch farther from a , with a fixed rule for ties. Make the corresponding changes to $f^{(\ell-1)}$, obtaining $f^{(\ell)}$. The old reference mean is unchanged, and

$$\left| M_{d_0, f^{(\ell)}\eta}(N_\ell) - M_{d_0, f^{(\ell)}\eta}(R) \right| \geq \frac{1}{2} - \frac{1}{j+4}. \quad (8.9)$$

Three invariants make the limiting source legitimate. A prime used at an earlier task divides an already protected value, so it cannot later be private to a new index in a larger prefix. No sign is changed twice. A used prime is larger than every prime in E_j , while every future free block lies beyond all values of the current stage; hence a used prime never belongs to E . Finally, later private modifications preserve all old values of the target trace, and later free primes are too large to divide them. Thus every discrepancy witness remains valid for all extensions of the current configuration on E .

Source-dependent replay. For an emitted instruction, let $h = f^{(\ell-1)}\eta$ and compute $a = M_{d_0, h}(R)$. For each $\sigma \in \{-1, 1\}$, compute the finite set of flips at the recorded private primes that makes $h(Q_{d_0}(n)) = \sigma$ for all $n \in \mathcal{R}_\ell$. Compare the two resulting means at N_ℓ to a , choose the farther one (choose $\sigma = +1$ in a tie), and apply its flips to $f^{(\ell-1)}$. These are the source queries and sign decisions; they do not change the emitted calendar.

Invariants of the calendar and replay.

1. Every size and prime block is chosen independently of the source.
2. Each recorded pivot is private in the entire new target-trace prefix, not only among the new indices.
3. Used pivots lie outside the current free set and every future free block; hence they lie outside E and are never used twice.
4. Later base changes preserve all earlier target values. Later free primes also miss these values, so each witness survives every extension of its exposed signs.
5. All configurations on D_j receive their witnesses before the next free block is allowed to enter.

Define f by the stabilised prime signs. The total cost is strictly below $\sum_\ell 1/\tau_\ell < \varepsilon$. Primes at most B and all primes of E were unchanged, proving (8.2). To compute $f(m)$, advance until the increasing thresholds $\tau_\ell N_\ell$ exceed m ; no later change affects a prime factor of m . All computations use finitely many values of $f_\star$. This also proves continuity and relative computability.

Fix, *after the construction*, any $A \subseteq E$ and any finite prime set F . For every sufficiently large j , all primes of F lie at most j . The restriction of $\beta_A \beta_F$ to D_j is one of the configurations treated at that stage. Free primes outside E_j lie beyond its two observation prefixes. The persistent witness (8.9) therefore applies to the final source $f\beta_A \beta_F$. The two sizes go to infinity and the lower bound tends to $1/2$, proving (8.3). The argument covers all subsets, including non-computable ones, without taking an uncountable intersection of probability-one events. Lemma 2.1 gives the protected norms for $F = \emptyset$. This completes the proof of Theorem 8.1.

8.5 Resistance to summable auxiliary changes and the exact boundary

The finite-set robustness has a useful extension.

Corollary 8.4. *For the source in Theorem 8.1, (8.3) remains true when F is replaced by any prime set C with $\sum_{q \in C} q^{-1/2} < \infty$. If $C \subseteq \mathcal{P}_{d_0}$, the two protected norms remain unchanged as well.*

Proof. A prime dividing some $Q_{d_0}(n)$ in the prefix is at most $(N+1)^2 + 2 \leq 3N^2$ for $N \geq 2$. It meets at most $2N/q + 2$ indices, so the proportion affected is at most $6q^{-1/2}$. For $C_Y = C \cap [2, Y]$, the union bound gives, uniformly in A and $N \geq 2$,

$$|M_{d_0, f\beta_A \beta_C}(N) - M_{d_0, f\beta_A \beta_{C_Y}}(N)| \leq 12 \sum_{\substack{q \in C \\ q > Y}} q^{-1/2}.$$

Oscillations of two bounded sequences at uniform distance at most δ differ by at most 2δ . Apply the finite-set theorem and let $Y \rightarrow \infty$. $\square$

Proposition 8.2 applies to the very reservoir and base of Theorem 8.1. Thus their fixed word laws converge in probability and in every finite moment, while

$$\{\omega : M_{d_0, f\omega}(N) \text{ converges as } N \rightarrow \infty\} = \emptyset.$$

There is no contradiction with almost-sure convergence along a suitable subsequence extracted from convergence in probability. The obstruction concerns the full sequence of prefixes; its discrepancy scales depend on the sign assignment.

Nor is this a negative statement about logarithmic means, which the construction does not control. The reservoir violates (A), by the positive availability theorem, but we have not proved that every reservoir violating (A) supports an obstructive source. The exact polynomial classification of Theorem 1.2 and the incomplete classification of arbitrary subreservoirs are different results.

9 Collective rigidity and certificates for numerical values

Two protected quadratic fields create no additional odd product class beyond themselves. With three independent classes, a fourth norm can become compulsory. This section concerns the different triple

$$\mathcal{D}_0 = \{-2, 2, 3\}, \quad n^2 + 2, \quad n^2 - 2, \quad n^2 - 3. \quad (9.1)$$

The third trace is $n^2 - 3$, not $n^2 + 1$. No prefix hypothesis is needed for the exact statement about these three traces.

9.1 The fourth norm is forced by all three traces

Proposition 9.1 (A collective norm constraint). *A quotient $u \in \mathcal{M}_\pm$ is one on all positive values of the traces in (9.1) if and only if its prime signs can differ from one only on*

$$S = \{q : q \text{ prime, } q \equiv 5 \pmod{24}\}.$$

Every such quotient also satisfies

$$u(a^2 + 3b^2) = 1 \quad (a, b \in \mathbb{Q}, (a, b) \neq (0, 0)). \quad (9.2)$$

Proof. For primes $q > 3$, all three symbols $(-2/q), (2/q), (3/q)$ are -1 exactly in the class $5 \pmod{24}$. Such a prime also has $(-3/q) = -1$, because

$$[-3] = [-2][2][3] \quad \text{in } \mathbb{Q}^\times / \mathbb{Q}^{\times 2}.$$

The inert valuation formula (2.1) proves sufficiency for the three norms and for (9.2).

For necessity, the traces give $2 = Q_2(2)$ and $3 = Q_{-2}(1)$. A prime $p \geq 5$ outside S has a centred root $1 \leq r \leq (p-1)/2$ in some Q_e , $e \in \mathcal{D}_0$, with

$$0 < r^2 - e = pc, \quad 1 \leq c < p/2. \quad (9.3)$$

For $e = 3$, a non-positive value would require $r = 1$, which cannot be a root modulo $p \geq 5$; the other cases are the same as in Lemma 2.1. The upper bound follows from the centred representative. All prime factors of c are smaller than p , while the free-prime valuations in c are even. Induction from two and three determines every prime outside S . $\square$

There is one small value for which each of the three traces is indispensable:

$$m = 72^2 + 3 = 5187 = 3 \cdot 7 \cdot 13 \cdot 19.$$

With no additional prefix constraint, removing any one of the traces permits a quotient negative at m :

Trace removed	Quotient	Norms still protected
$n^2 + 2$	$\beta_{\{19\}}$	$ a^2 - 2b^2 , a^2 - 3b^2 $
$n^2 - 2$	$\beta_{\{7\}}$	$a^2 + 2b^2, a^2 - 3b^2 $
$n^2 - 3$	$\beta_{\{13\}}$	$a^2 + 2b^2, a^2 - 2b^2 $

In each row the indicated prime is inert in both retained fields and has valuation one in m . These quotients work around any source, not just the constant source.

The joint determination of m can be checked without any unknown auxiliary sign:

$$\begin{aligned} & (72^2 + 3)(1^2 + 2)(3^2 - 2)(4^2 - 3)(6^2 + 2)(2^2 - 2) \\ & = 5187 \cdot 3 \cdot 7 \cdot 13 \cdot 38 \cdot 2 = 10374^2. \end{aligned} \quad (9.4)$$

Thus every source obeys $f(5187) = f(3)f(7)f(13)f(38)f(2)$, and all five factors are readings of the protected traces at indices at most six. Grouping norm factors also gives

$$4(72^2 + 3) = (4^2 + 2 \cdot 7^2)(4^2 - 2)(4^2 - 3) = 114 \cdot 14 \cdot 13. \quad (9.5)$$

For example, $(1 + \sqrt{-2})(6 + \sqrt{-2}) = 4 + 7\sqrt{-2}$ explains the first compressed norm. These are identities at a numerical value. Their distinction from an identity in T will matter shortly.

9.2 A short adaptive certificate for every permitted integer

Theorem 9.2 (Descent certificates). *Let $m \geq 1$ be an integer whose valuations at the primes 5 (mod 24) are even. There is a computable finite set $\mathcal{C}(m)$ of distinct labels (r, e) , with $e \in \mathcal{D}_0$, $r \geq 1$ and $r^2 - e > 0$, such that*

$$m \prod_{(r,e) \in \mathcal{C}(m)} (r^2 - e) = z_m^2, \quad z_m \in \mathbb{N}_{\geq 1}, \quad (9.6)$$

and

$$|\mathcal{C}(m)| \leq \lfloor \log_2 m \rfloor, \quad r \leq \max\{2, (m-1)/2\} \quad ((r, e) \in \mathcal{C}(m)). \quad (9.7)$$

Here $\log_2$ is the base-two logarithm. For $m = 1$ the set is empty. No such certificate exists when m has an odd valuation at a prime 5 (mod 24).

Proof. Set $\mathcal{C}(1) = \emptyset$, $\mathcal{C}(2) = \{(2, 2)\}$ and $\mathcal{C}(3) = \{(1, -2)\}$. For a prime $p \geq 5$ outside S , choose the first form in the order $-2, 2, 3$ having a root modulo p , and then its smallest centred positive root r . Put $c = (r^2 - e)/p < p/2$ and define

$$\mathcal{C}(p) = \{(r, e)\} \triangle \mathcal{C}(c), \quad \mathcal{C}(m) = \bigtriangleup_{v_p(m) \text{ odd}} \mathcal{C}(p). \quad (9.8)$$

Only primes outside S occur in the second formula. The free-prime valuations of c are even, so the recursive input is permitted. All arguments decrease, and factoring a finite integer gives an effective procedure.

In the square-class group, $[p] = [r^2 - e][c]$. Symmetric difference cancels repeated labels, whose product would be a square. This proves (9.6). The product is a positive integer with even prime valuations, so its square root z_m is an integer and can also be computed.

Strong induction gives, for $p \geq 5$,

$$|\mathcal{C}(p)| \leq 1 + \log_2 c < \log_2 p.$$

The bases two and three obey the non-strict bound. Summing over the prime factors of m of odd valuation gives at most $\log_2 m$, hence the integer bound in (9.7). Every root chosen is at most $(p-1)/2$, and recursive primes are smaller; the two base labels have indices at most two. This proves the height bound. Conversely, every protected observation has even valuation at every prime of S , so a certificate is impossible if m has an odd valuation there. $\square$

Let $\mathcal{H}_0$ be the subgroup of $\mathbb{Q}_{>0}^\times$ generated by non-zero absolute values of the three rational norms. The descent and inert valuation formula give

$$\mathcal{H}_0 = \{t \in \mathbb{Q}_{>0} : v_q(t) \text{ is even for every } q \in S\}. \quad (9.9)$$

For rational t , apply the integer statement to its coprime numerator and denominator. A square is itself a norm; inverses and products of norms are norms. Grouping the factors of (9.6) by field therefore represents every permitted positive integer, and then every member of (9.9), as a product of three positive rational norms, one from each field. The positivity follows by grouping the positive trace factors and their inverses.

In particular $t^2 + 3 \in \mathcal{H}_0$ for every $t \in \mathbb{Q}$. For $m = n^2 + 3$, however, the certificate permits indices $O(n^2)$, not indices bounded by the original n . Its logarithmic number of readings is not a statement about their statistical distribution or their combined loads across different values.

9.3 Every specialisation, but no rational norm identity

Over the rational function field, define

$$\mathcal{H}_T = (\mathbb{Q}(T)^\times)^2 \prod_{e \in \mathcal{D}_0} N_{\mathbb{Q}(\sqrt{e}(T))/\mathbb{Q}(T)}(\mathbb{Q}(\sqrt{e}(T))^\times). \quad (9.10)$$

Products of these norm groups mean finite products; repeated factors from one field can be grouped.

Theorem 9.3 (A functional obstruction). *The polynomial $T^2 + 3$ does not belong to $\mathcal{H}_T$. In fact there is no identity*

$$T^2 + 3 = c C(T)^2 \prod_{j=1}^s (A_j(T)^2 - e_j B_j(T)^2) \quad (9.11)$$

with $c \in \mathbb{Q}^\times$, $C, A_j, B_j \in \mathbb{Q}(T)$, $e_j \in \mathcal{D}_0$, and non-zero multiplicative factors.

Proof. Use the discrete valuation v_P at the irreducible polynomial $P = T^2 + 3$. Its residue field is $\mathbb{Q}(\sqrt{-3})$. None of $-2, 2, 3$ is a square in that field: if $(a + b\sqrt{-3})^2$ is rational, then $ab = 0$, so a rational square class becoming square there is either 1 or -3 .

For $e \in \mathcal{D}_0$ and $A, B \in \mathbb{Q}(T)$ with non-zero norm, it follows that

$$v_P(A^2 - eB^2) = 2 \min\{v_P(A), v_P(B)\}. \quad (9.12)$$

If the two valuations agree, cancellation of the leading residues would make e a square in the residue field; if they differ, the term of smaller valuation cannot cancel. Set $v_P(0) = +\infty$ for a zero coordinate. This argument includes poles. Every factor on the right of (9.11) has even v_P , and the rational constant has valuation zero. The left side has valuation one, a contradiction. $\square$

A fixed rational recipe of the form (9.11) cannot be valid at infinitely many integers where it is defined: that would make its difference from $T^2 + 3$ a rational function with infinitely many zeros, hence an identity. Nor can finitely many such recipes cover infinitely many integers. This is compatible with the computable descent, whose choices adapt to the factorisation of each input. It is not a lower bound on all possible algorithms.

Corollary 9.4 (Infinitely many independent functional classes). *The classes of $(T + j)^2 + 3$, $j \geq 0$, are linearly independent over $\mathbb{F}_2$ in $\mathbb{Q}(T)^\times / \mathcal{H}_T$. Every rational specialisation of each polynomial nevertheless belongs to $\mathcal{H}_0$.*

Proof. The polynomials are distinct and irreducible. A non-empty product has valuation one at any factor occurring in it, whose residue field is again $\mathbb{Q}(\sqrt{-3})$. Formula (9.12) rules out membership in $\mathcal{H}_T$. The specialisation statement follows from (9.9) and (9.2). $\square$

The contrast is between a norm group over $\mathbb{Q}(T)$ and the groups of its numerical specialisations, not a failure of a local-global principle over a fixed number field. Results on multinorm principles, such as [15], concern a different comparison. The valuation proof above is self-contained and does not invoke such a principle.

10 Odd-product closure for a finite protected family

The collective example admits a general description. It also identifies exactly which part of the two-field criterion must change when more traces are protected. In this section let

$$\mathcal{D}' = \{d_1, \dots, d_s\}$$

be a non-empty finite family of distinct squarefree integers, none a rational square. Protect all positive trace values $n^2 - d_i$, and protect the prime signs up to

$$B \geq B_0 := \max \left(3, 4 \max_i |d_i| \right). \quad (10.1)$$

This finite prefix is an assumption of the general result. It was not needed for the concrete triple of Section 9.

10.1 The free primes and the consistent sign pattern

Lemma 10.1 (Exact general fibre). *The quotient of two sources with the protected data can be chosen freely exactly at the primes*

$$S_{\mathcal{D}', B} = \{q > B : (d_i/q) = -1 \text{ for all } i\}. \quad (10.2)$$

It is one at all other primes. Every such quotient preserves all non-zero absolute rational norms $|a^2 - d_i b^2|$.

Proof. Inert valuations prove sufficiency. For $p > B$ outside (10.2), one of the forms has a centred root r . Since $|d_i| < p$ and d_i is not a square, the non-zero multiple $r^2 - d_i$ of p is positive. The bound (10.1) gives

$$r^2 - d_i = pc, \quad 1 \leq c < p/2.$$

Every prime factor of c is smaller than p ; its free-prime valuations are even. Induction, starting from the protected prefix, determines the remaining signs, exactly as in Lemma 2.1. $\square$

Let V be the $\mathbb{F}_2$ -space generated by the rational square classes $[d_i]$, of dimension r . The requirement that every Legendre symbol be negative amounts to a linear functional

$$\lambda : V \longrightarrow \mathbb{F}_2, \quad \lambda([d_i]) = 1 \quad (1 \leq i \leq s). \quad (10.3)$$

It is consistent exactly when every relation making a product of the d_i a rational square has an even number of factors. If an odd relation exists, (10.2) is empty. The exact fibre is then trivial: the traces and prefix determine the entire source.

When (10.3) is consistent, it is unique because the classes $[d_i]$ span V . Define

$$\mathcal{O}(\mathcal{D}') = \{v \in V : \lambda(v) = 1\}. \quad (10.4)$$

These are precisely the square classes representable by an odd product of protected classes. The set is an affine coset of $\ker \lambda$, has 2^{r-1} elements, and does not contain the trivial square class. In particular it is not itself a subgroup.

Proposition 10.2 (All norms in the odd closure are protected). *Under the consistency assumption, every compatible quotient is one on all non-zero absolute rational norms of $\mathbb{Q}(\sqrt{d})$ whenever $[d] \in \mathcal{O}(\mathcal{D}')$.*

Proof. At every free prime the Legendre symbol of an odd product is -1 . Its valuation on that norm is even. Outside the free primes the quotient equals one by Lemma 10.1. Clearing rational denominators completes the argument. $\square$

For $\mathcal{D}_0 = \{-2, 2, 3\}$, the three independent square classes have odd closure $\{-2, 2, 3, -3\}$ modulo squares. For the original triple $\{-1, -2, 2\}$, their product is a square and the relation has odd length. Protecting *all three* therefore removes the free prime class. This is an exact reconstruction assertion, not a quantitative statement about small average errors.

10.2 The root field detects the full closure

Assume consistency from now on, and set

$$K = \mathbb{Q}(\sqrt{d_1}, \dots, \sqrt{d_s}).$$

Its degree is 2^r . The functional (10.3) corresponds to the automorphism $c \in \text{Gal}(K/\mathbb{Q})$ which changes the sign of every protected square root. The free primes have Frobenius c in this multiquadratic extension. In particular they form a positive-density class after the finite prefix is removed.

Theorem 10.3 (Polynomial dichotomy for a finite family). *Let $P \in \mathbb{Z}[T]$ be monic, non-constant and irreducible, with root field F_P and $L_P = F_P \cap K$. The following conditions are equivalent:*

- (i) F_P contains no quadratic field with square class in $\mathcal{O}(\mathcal{D}')$;
- (ii) c acts trivially on L_P ;
- (iii) from every source, a compatible completion respecting the prefix (10.1) can have a normal trace at P .

If these fail, every compatible completion preserves every non-zero absolute value $|P(n)|$ pointwise. In the admissible case, fixed distinct admissible polynomial observations have jointly uniform empirical laws almost surely under fair signs on the full free reservoir, for each source fixed in advance. Fixed finite-alphabet marks may be retained.

Proof. The subextension $L_P/\mathbb{Q}$ is multiquadratic. Its automorphism c is non-trivial exactly when it is non-trivial on one of its quadratic subfields. These are precisely the contained fields whose square classes have $\lambda = 1$. This proves (i) $\Leftrightarrow$ (ii). If such a field is contained in F_P , Proposition 2.2, with Proposition 10.2 supplying protection, freezes all values of P . The constant source excludes normal completion from every input.

For the other direction, take a finite Galois extension containing K and the roots of P , and apply Lemma 2.3 to the restriction onto $\text{Gal}(K/\mathbb{Q})$. The same fixed-extension Chebotarev–Mertens input gives

$$\sum_{\substack{q \leq x \\ q \in \mathcal{S}_{\mathcal{D}', B}}} \frac{\rho_P(q)}{q} = \kappa_{\mathcal{D}'}(P) \log \log x + O_{P, \mathcal{D}', B}(1), \quad (10.5)$$

where

$$\kappa_{\mathcal{D}'}(P) = \begin{cases} [L_P : \mathbb{Q}]/2^r, & c|_{L_P} = 1, \\ 0, & c|_{L_P} \neq 1. \end{cases} \quad (10.6)$$

Indeed the coefficient is $1/|\text{Gal}(K/L_P)|$ in the first case and zero in the second.

For a fixed finite list of distinct admissible irreducibles, the root counts of a non-empty product add outside the finitely many resultant primes. Every such non-empty product has

a positive root-weighted coefficient. Lemmas 3.1–3.2, or the fixed-order estimate (4.7), apply to this full reservoir without changing their hypotheses. The marked Fourier step proves joint uniformity for each fixed source. A countable intersection covers all fixed polynomial families and all prescribed countably many marks; distinct translates give normality of each trace. This proves (iii). $\square$

The full-class proof also retains a rate. For a fixed list $P_1, \dots, P_k$, every non-empty moment has coefficient at least $\min_i \kappa_{\mathcal{D}'}(P_i)$. Corollary 4.3 and finite Fourier inversion give marked total-variation error

$$O_{\omega, f_0, \text{task}, \beta}((\log N)^{-\beta}) \quad \text{for every } 0 < \beta < \min_i \kappa_{\mathcal{D}'}(P_i),$$

almost surely. As always, the source and marks are fixed before the bits; no single multiplier is asserted to work for sources chosen afterwards.

In the collective example, $K = \mathbb{Q}(i, \sqrt{2}, \sqrt{3})$ and the fixed field of c is $\mathbb{Q}(i, \sqrt{6})$. The four forbidden quadratic classes are $-2, 2, 3, -3$. The polynomials $T^2 + 1$, $T^2 - 6$ and $T^2 + 6$ are admissible with coefficient $1/4$. Irreducibles of odd degree have coefficient $1/8$, and Φ_5 also has coefficient $1/8$, since its only quadratic subfield is $\mathbb{Q}(\sqrt{5})$. Thus protecting the fourth norm still leaves many normalisable polynomial observations.

There are two distinct algebraic boundaries. In the arithmetic fibre, the forbidden quadratic fields are those of the entire odd closure. In the norm group over $\mathbb{Q}(T)$, an irreducible polynomial can be a product of norms from the *initial* fields only if its root field contains at least one initial quadratic field: otherwise the valuation at that polynomial is even on every such norm. Containment is also sufficient by polynomial norm factorisation. This accounts for the specialisation contrast in Theorem 9.3.

The general-family result here uses fair signs on the full free reservoir. The common primedensity-zero, relatively computable selector of Theorem 1.2 was proved for the original two-field protection. No further uniform algorithmic extension to arbitrary protected families is required or asserted in Theorem 10.3.

11 Scope and remaining reservoir questions

The classification is exact for monic irreducible observations in the stated protected fibres. Failure of admissibility means pointwise preservation, not simply failure of a particular normalisation method. In the admissible case the output is selected from the fibre of a prescribed source; neither normality nor any cancellation is inferred for that source on observations that were not protected. Applied to Liouville, the construction keeps the two specified norm families at their Liouville values and changes the function elsewhere.

The reservoir problem has a different status. Infinite harmonic mass is enough for convergence of fixed word laws in probability. Availability between the cutoffs in (A) is sufficient for almost-sure normality from each fixed input, and for a uniform computable selector on a decidable reservoir. Theorem 8.1 separates these statements, but does not make (A) necessary. A sharper criterion would need to distinguish random convergence, existence of a fixed completion for every source, and selection with uniform control over the inputs. Optimising an iterated-logarithm example alone does not classify arbitrary lacunary reservoirs.

Computability is used in its exact sense: every requested sign is determined after finitely many operations and finitely many input queries. The root-balanced blocks and the rectangles in the positive potential can be extremely large. The proofs give no practical complexity bound or useful numerical onset of normality. Nor do they claim estimates for growing polynomial

degrees, growing word lengths, arbitrary moving origins, temporal paths, or a finite harmonic budget for the positive constructions.

The collective certificates have a separate value. They identify exactly what the protected observations determine and give an explicit finite witness for each permitted integer, while their height and functional obstruction show why this is not a uniform rational formula or a density estimate for reconstruction errors. This distinction persists even when all numerical specialisations of an observation lie in the generated norm group.

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Diagnostic material. The exact diagnostic archive is supplied inside the source package of the published Zenodo edition, available at <https://doi.org/10.5281/zenodo.23318326>. It contains scripts, expected outputs, tested ranges, software information and replay commands; its README distinguishes inherited suites from additional finite checks. The source package identifies the diagnostic archive by its SHA-256 checksum.

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