

Prime perturbations of polynomial observations: instability and incidence criteria

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Abstract

The cost of changing prime signs depends on how the resulting multiplicative function is observed. On a fixed prime support of finite harmonic sum, we prove that existence of a polynomial mean for every sign assignment is equivalent to uniform prime truncation and uniform convergence to local products. The criterion applies separately to ordinary and logarithmic means. For independent activations with finite expected harmonic cost, the almost-sure truncation defect is the averaged large-prime parity probability, not a sum of incidences.

These criteria separate averaging methods and observation geometries. One common support is logarithmically stable for every fixed polynomial and every assignment, yet a fixed perturbation of $(-1)^{v_2(n)}$ on that support has divergent ordinary means on $n^2 + 1, n^2 + 2, n^2 - 2$. Collective upper sieves give stability at critical counting exponents without critical summability, and harmonic stability on rectangles whose side lengths have asymptotically equal logarithms. For widths X^α , Gaussian prime estimates give the matching universal budget threshold when $119/500 < \alpha < 1$; the negative result concerns only the sum of two squares. Around every completely multiplicative sign function, fixed perturbations of arbitrarily small $\sum p^{-\sigma}$ cost can have divergent logarithmic means on $n^2 + 1$ for $125/164 < \sigma \leq 1$. A positive oscillation bound is common to this range, although the starting scales may depend on σ . Window laws further distinguish all origins from almost all origins.

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1 Introduction and principal statements

Changing the sign assigned to a large prime is inexpensive on the integers, but need not be inexpensive on a polynomial sequence. Among the ten values $n^2 + 1$, $1 \leq n \leq 10$, the prime 101 occurs only in $10^2 + 1$. Reversing its sign costs $1/101$ in the harmonic prime budget and changes a proportion $1/10$ of this sample. The example does not prove asymptotic instability. It identifies the question: *which small changes at the primes remain small after the observation is chosen?*

Three questions organize the answer. Does every choice of signs on one fixed prime support have a mean? Do ordinary and logarithmic averaging detect the same perturbations? What changes when the shape or origin of the sample changes? Section 3 already answers the first question by a self-contained argument, without the large-prime lower bounds of Section 8 or the Gaussian prime input of Section 15.

Let $\mathcal{M}_\pm$ denote the completely multiplicative functions from $\mathbb{N}_{\geq 1}$ to $\{-1, 1\}$. For $\sigma > 0$ and a set A of primes, write

$$d_\sigma(f, g) = \sum_{p: f(p) \neq g(p)} p^{-\sigma}, \quad h_A(n) = (-1)^{\sum_{p \in A} v_p(n)}. \quad (1.1)$$

Every quotient g/f is some h_A . On the integers a union of multiples gives

$$\frac{1}{X} \#\{n \leq X : f(n) \neq g(n)\} \leq \sum_{p: f(p) \neq g(p)} \frac{\lfloor X/p \rfloor}{X} \leq d_1(f, g). \quad (1.2)$$

For sign functions, the usual pretentious distance satisfies $\mathbb{D}(f, g; \infty)^2 = 2d_1(f, g)$, with the same equality at finite cutoffs; see [1]. The budget is thus an additive form of a familiar distance. Polynomial sampling changes the incidence assigned to its prime coordinates.

For a nonzero integer polynomial P , put

$$M_{P,f}^c(N) = \frac{1}{N} \sum_{\substack{n \leq N \\ P(n) \neq 0}} f(|P(n)|), \quad M_{P,f}^\ell(N) = \frac{1}{\mathcal{H}_N} \sum_{\substack{n \leq N \\ P(n) \neq 0}} \frac{f(|P(n)|)}{n}, \quad \mathcal{H}_N = \sum_{n \leq N} \frac{1}{n}. \quad (1.3)$$

Zeros are omitted without changing the denominators. There are only finitely many, and replacing finitely many observations does not change any limiting assertion. Constants are allowed when stated; the random criterion below is formulated for positive-valued polynomials of positive degree. All logarithms are natural unless a base is displayed. For $m > 1$, $P^+(m)$ denotes its largest prime factor; we set $P^+(1) = 1$.

1.1 A support criterion and a random criterion

A prime set can be studied before any of its signs are chosen. Fix a support Γ with

$$\sum_{p \in \Gamma} \frac{1}{p} < \infty. \quad (1.4)$$

For each fixed polynomial and either chosen averaging method, convergence for every assignment on this same support will force a uniform conclusion: all prime truncations are uniformly accurate and all limits are local products. This is a constructive converse, not a compactness principle. If tail coverage persisted, we could assemble one divergent assignment. The following definitions make that equivalence precise.

For $A \subseteq \Gamma$, truncate by $h_{A,y} = h_{A \cap [2,y]}$. Let $\Delta_{P,N}^c(A; y)$ and $\Delta_{P,N}^\ell(A; y)$ be the ordinary and logarithmic proportions of disagreement between $h_A(|P(n)|)$ and $h_{A,y}(|P(n)|)$. The relevant sign-free quantity is the *coverage* of the sample by odd valuations in the tail:

$$B_{P,N}(\Gamma; y) = \frac{1}{N} \#\{n \leq N : P(n) \neq 0, \exists p \in \Gamma, p > y, v_p(|P(n)|) \text{ odd}\}. \quad (1.5)$$

Set $u_P(N) = B_{P,N}(\Gamma; N)$ and, for a bounded sequence u , define its positive Abel average by

$$(\mathcal{A}u)(N) = \frac{u(N) + \sum_{k \leq N} u(k)/(k+1)}{\mathcal{H}_N}. \quad (1.6)$$

The target mean is the local product

$$\ell_P(A) = \prod_{p \in A} \ell_{P,p}, \quad \ell_{P,p} = \int_{\mathbb{Z}_p} (-1)^{v_P(P(t))} dt. \quad (1.7)$$

Haar measure is normalized to one; values at the finite zero set of P are immaterial. The products converge uniformly over $A \subseteq \Gamma$, since $|1 - \ell_{P,p}| \leq 2 \deg(P)/p$ outside finitely many primes.

Theorem 1.1 (All assignments on a fixed support). *Assume (1.4) and fix $P \in \mathbb{Z}[T] \setminus \{0\}$. Choose one averaging method $\star \in \{c, \ell\}$. The following are equivalent:*

- (i) $u_P(N) \rightarrow 0$ if $\star = c$, and $(\mathcal{A}u_P)(N) \rightarrow 0$ if $\star = \ell$;
- (ii) $\lim_{y \rightarrow \infty} \sup_{A \subseteq \Gamma} \sup_{N \geq 1} \Delta_{P,N}^\star(A; y) = 0$;
- (iii) $\sup_{A \subseteq \Gamma} |M_{P,h_A}^\star(N) - \ell_P(A)| \rightarrow 0$;
- (iv) $M_{P,h_A}^\star(N)$ converges for every $A \subseteq \Gamma$.

In (iv) no uniformity or value of the limit is assumed. The conclusion in (iii) supplies both.

The quantifier over *every subset of the same support* is essential. This is not a criterion for the existence of the mean of one specified function. Its finite mechanism is

$$\frac{1}{2} B_{P,N}^\star(\Gamma; y) \leq \sup_{A \subseteq \Gamma} \Delta_{P,N}^\star(A; y) \leq B_{P,N}^\star(\Gamma; y).$$

Fair signs on a finite tail make every covered observation disagree with probability $1/2$, even when observations share primes. If coverage persists, the converse constructs one fixed assignment with alternating excursions and recoveries. No private-prime assumption is needed for that argument.

For a prescribed random law, coverage is no longer the exact quantity: two activated sign changes can cancel. If two relevant primes have activation probabilities a, b , the reversal probability is $a + b - 2ab$, not $a + b$. Fix deterministic $0 \leq \theta_p \leq 1$, choose independent $\xi_p \sim \text{Bernoulli}(\theta_p)$, and put

$$E = \{p : \xi_p = 1\}, \quad h_E(n) = (-1)^{\sum_{p \in E} v_P(n)}, \quad h_{E,y} = h_{E \cap [2,y]}. \quad (1.8)$$

For a fixed positive-valued $P \in \mathbb{Z}[T]$ of positive degree, write

$$\Delta_{P,N}(E; y) = \frac{1}{N} \#\{n \leq N : h_E(P(n)) \neq h_{E,y}(P(n))\}, \quad (1.9)$$

$$\eta_P(N) = \frac{1}{2N} \sum_{n \leq N} \left(1 - \prod_{\substack{p > \sqrt{N} \\ v_P(P(n)) \text{ odd}}} (1 - 2\theta_p) \right). \quad (1.10)$$

Empty products are one. The prime signs are independent; the observations $P(n)$ are not assumed independent.

Theorem 1.2 (Exact random prime-truncation defect). *Assume*

$$\sum_p \frac{\theta_p}{p} < \infty. \quad (1.11)$$

Almost surely, simultaneously for every fixed polynomial as above,

$$\lim_{y \rightarrow \infty} \limsup_{N \rightarrow \infty} \Delta_{P,N}(E; y) = \limsup_{N \rightarrow \infty} \eta_P(N). \quad (1.12)$$

The same identity holds with the two inner limit superiors replaced by limit inferiors. For logarithmic disagreement the right-hand side is the limit superior, or inferior, of

$$\bar{\eta}_P(N) = (\mathcal{A}\eta_P)(N). \quad (1.13)$$

Thus natural prime truncation is valid in ordinary mean exactly when $\eta_P(N) \rightarrow 0$, and in logarithmic mean exactly when $\bar{\eta}_P(N) \rightarrow 0$.

The two criteria answer different questions. Theorem 1.1 quantifies over all assignments on a fixed support. Theorem 1.2 identifies the almost-sure defect under a specified independent law. Neither says that a multiplier creates missing means for an arbitrary source $f(P(n))$. Under $\theta_p \rightarrow 0$, the random defect vanishes and gives local joint laws conditional on the realized E ; the same p -adic coordinate is used for every polynomial at a given prime.

1.2 Fixed-source instability and separation of the two averages

For $P(n) = n^2 + 1$ abbreviate

$$A_f(N) = M_{P,f}^c(N), \quad L_f(N) = M_{P,f}^\ell(N), \quad \text{osc}_N a_N = \limsup_N a_N - \liminf_N a_N. \quad (1.14)$$

The harmonic weight is on the index n , not the height $n^2 + 1$.

Theorem 1.3 (Instability around an arbitrary fixed source). *For every $f \in \mathcal{M}_\pm$, every $\varepsilon > 0$ and every integer $M_0 \geq 1$:*

- (i) *There is a single $g \in \mathcal{M}_\pm$ with $d_1(f, g) < \varepsilon$, $g(n) = f(n)$ for $n \leq M_0$, and $\text{osc}_N A_g(N) \geq 1/2$.*
- (ii) *There is an absolute constant $c_{\log} > 0$ such that, for every fixed $125/164 < \sigma \leq 1$, there is a single $g \in \mathcal{M}_\pm$ with $d_\sigma(f, g) < \varepsilon$, $g(n) = f(n)$ for $n \leq M_0$, and $\text{osc}_N L_g(N) \geq c_{\log}$. The constant is independent of $\sigma, f, \varepsilon, M_0$. For $128/153 < \sigma \leq 1$, a separate construction gives the numerical bound $\text{osc}_N L_g(N) \geq 1/2000$.*

The functions of ordinary oscillation at least $1/2$ form a dense G_δ in every finite- d_1 class. Those of logarithmic oscillation at least $c_{\log}$ form a dense G_δ in every finite- d_σ class for $125/164 < \sigma \leq 1$; the bound $1/2000$ has the same category conclusion for $128/153 < \sigma \leq 1$. These assertions also hold with a compatible finite prefix fixed. The constructions preserve vanishing of the full family of fixed non-proportional affine correlations whenever it holds for f .

The two averaging assertions are separate constructions. Logarithmic divergence implies ordinary divergence by Abel summation, but does not transfer the ordinary oscillation constant to the same function. Section 8 retains three analytic routes: Merikoski for the value $1/2000$, Pascadi for a larger range, and Grimmelt–Merikoski combined with Merikoski’s sieve calculation for $\sigma > 125/164$. Only the positive amplitude $c_{\log}$ is uniform in σ : the source, starting scales and decay of the prime cost may depend on it. The strict endpoint $125/164$ is not included; no numerical value for $c_{\log}$ is asserted.

Two further constructions distinguish the roles of the signs and of the averaging method. Independent fair activations can remain logarithmically unstable on a deterministic support of arbitrarily small d_σ cost for $125/164 < \sigma \leq 1$ (Theorem 9.1). In the other direction, Theorem 10.1 gives a support Γ of arbitrarily small harmonic cost which is logarithmically stable for *every* fixed polynomial and every $A \subseteq \Gamma$, but admits one fixed function $g(n) = (-1)^{v_2(n)} h_A(n)$ for which all three ordinary means on $n^2 + 1, n^2 + 2, n^2 - 2$ oscillate by at least $1/8$. The three logarithmic means of that same function are zero in the limit.

There is also a distinction between probability and category. In the finite- $d_{9/10}$ class around 1, a full-support probability measure gives simultaneous polynomial stability with probability one, although logarithmic quadratic divergence is residual. Section 5.2 constructs this measure explicitly; its typical supports have divergent half-power budget.

1.3 Counting support primes and changing the observation geometry

A coordinatewise bound gives the useful sufficient estimate

$$\sup_N \frac{1}{N} \#\{n \leq N : f(n^2 + 1) \neq g(n^2 + 1)\} \leq 5d_{1/2}(f, g). \quad (1.15)$$

It is not necessary to sum this individual cost. A collective upper sieve treats the values with a nearly maximal prime factor together, allowing stability even when the critical series diverges.

For a prime set Γ , define its counting exponent by

$$\delta(\Gamma) = \limsup_{t \rightarrow \infty} \frac{\log(1 + \#(\Gamma \cap [2, t]))}{\log t}. \quad (1.16)$$

This is an exponent of growth, not a geometric dimension. Let $\mathcal{D} = \{-1, -2, 2\}$, $P_d(n) = n^2 - d$, and $Q_d(a, b) = a^2 - db^2$. Curve averages omit the nonpositive value $P_2(1)$ and remain normalized by N ; this changes (1.3) by at most $1/N$ or $1/\mathcal{H}_N$. Rectangle averages are

$$A_{d,f}(X, H) = \frac{1}{XH} \sum_{a \leq X, b \leq H} f(|Q_d(a, b)|).$$

The following statements refer to a fixed support, uniformly over all its assignments.

Theorem 1.4 (Critical support classes and a rectangular threshold).

- (i) If $\delta(\Gamma) \leq 1/2$, all three curve observations have uniformly negligible prime tails and their ordinary and logarithmic means converge uniformly over $A \subseteq \Gamma$ to

$$c_d(A) = \mathbf{1}_{\{2 \notin A\}} \prod_{\substack{p \in A, p \text{ odd} \\ (d/p)=1}} \frac{p-3}{p+1}. \quad (1.17)$$

- (ii) If $0 < \alpha < 1$, $H = \lfloor X^\alpha \rfloor$, and $\delta(\Gamma) \leq s_\alpha = (1 + \alpha)/2$, the analogous uniform tail and mean conclusions on all three rectangles hold with limits

$$C_d(A) = 3^{-\mathbf{1}_{\{2 \in A\}}} \prod_{\substack{p \in A, p \text{ odd} \\ (d/p)=1}} \left(\frac{p-1}{p+1} \right)^2. \quad (1.18)$$

- (iii) If $\sum_{p \in \Gamma} 1/p < \infty$ and $\log H(X)/\log X \rightarrow 1$, the rectangular conclusions in (ii) still hold, even when $H(X)/X \rightarrow 0$.

- (iv) For $119/500 < \alpha < 1$ and every $s_\alpha < \sigma \leq 1$, every d_σ ball around every source contains a fixed perturbation, preserving any prescribed compatible prefix, whose $a^2 + b^2$ mean on $H = \lfloor X^\alpha \rfloor$ oscillates by at least a positive constant depending only on α, σ .

All positive tail conclusions also control multiplication of an arbitrary base, without creating its missing means. The negative assertion in (iv) concerns the sum of two squares, not simultaneous instability of the other two norms.

The equality in (i)–(ii) is included without summability at that exponent. For instance, if p_j is the j th prime, $\Gamma = \{p_j : j \geq 1\}$ has $\delta(\Gamma) = 1/2$ and $\sum_{p \in \Gamma} p^{-1/2} = \infty$. Its assignment negative at every support prime has all three curve means zero. On power rectangles, (ii) and (iv) identify a universal budget and counting-exponent threshold in the stated angular range, not an individual classification of every support above it. The elementary estimate in Section 12 already proves the positive result under $\sum_{\Gamma} p^{-s_\alpha} < \infty$; Section 14 removes that critical summability assumption.

There is a separate geometric contrast at much thinner widths. If $H \rightarrow \infty$ and $H = o(\log X)$, one arbitrarily small harmonic perturbation can make *all three* rectangle means oscillate by at least $1/2$ (Theorem 12.1). The construction uses a prime private to a primitive direction, not to each point on that direction. The power-width counterexample instead obtains enough primitive directions from Gaussian primes in narrow sectors and a harmonic sum over cofactors.

Appendix A keeps the curve variable unchanged and moves the origin. If $\delta(\Gamma) \leq \alpha/2$, windows of length at least $X^{\alpha-o(1)}$ at distance comparable to X have the local laws at every origin. Under $\delta(\Gamma) \leq 1/2$, every prescribed diverging length suffices at almost every origin. Explicit extremely sparse sources with long all-positive blocks explain why these quantifiers differ.

1.4 Antecedents, contributions and reading routes

Klurman [1, Proposition 1.1, Lemma 2.1 and their proofs] already uses private large primes, summable changes and the freezing of a single function to obtain quadratic non-cancellation of size $1/2$. Finite sign prescriptions are already part of Klurman’s lemma; prefix protection alone is not a new device. The extension here combines an arbitrary infinite source, an arbitrarily small radius, actual non-convergence and residuality. Klurman’s local-factor comparisons [1, Corollary 1.2 and Theorem 1.3] also keep an explicit large-prime term. Neither those constructions nor the local-product mechanism is claimed as new.

The first criterion gains its force from the converse: convergence separately for every assignment on one harmonic support already implies uniform truncation and uniform limits. The random criterion instead identifies an exact almost-sure parity defect. The common-support separation in Theorem 10.1 distinguishes the two averaging methods arithmetically, rather than only for abstract bounded sequences. The counting-exponent results verify disappearance of the tail by combining sparse support below a moving cutoff with a collective sieve above it. The rectangular negative result joins the critical positive exponent only after an angular prime theorem has supplied a positive density of private directions.

The probabilistic input is McDiarmid’s classical bounded-differences inequality [7], in the accessible formulation of Warnke [3, Theorem 1]; no typical-differences refinement is needed. Mean periodic approximation belongs to the rationally almost periodic framework of Bergelson–Kuřaga–Przymus–Lemańczyk–Richter [8]. The periodic-multiplier argument for affine transfer is related to Klurman–Mangerel–Teräväinen [4, Section 8, especially (8.3)–(8.7)]; their Theorem 2.1 supplies the nonempty class of fully cancelling sources.

For the logarithmic lower bounds, Merikoski [2, Section 2.1 and Remark 13] supplies the numerical branch and Pascadi [10, Section 6.3, (6.20)] supplies another weighted route. The largest range used here comes from Grimmelt–Merikoski [17, Corollaries 7.1–7.2]: their unconditional Type I/II information makes available the weighted calculation in [2, Section 2.6, proof of Theorem 2]. Sec-

tion 8.5 spells out its strictly positive remaining mass before applying the population transfer. No new greatest-prime-factor estimate is claimed.

The Gaussian square mean is already known in substantially greater generality. Donoso–Le–Moreira–Sun [16, Theorem A] prove existence and an Euler product for the average of $f(a^2 + b^2)$ on squares for every bounded real completely multiplicative f . Substituting $f = h_A$ gives exactly $C_{-1}(A)$ in (1.18); harmonic summability is not needed for that existence result. Their comparison lemmas [16, Lemmas 4.1 and 4.3] also use reciprocal Gaussian-prime norms. The elementary square proof retained here prepares the explicit boundary estimate, rather than providing a new square-mean theorem. Their dilated domains arise from a fixed region [16, Definition 2.3]; this does not supply uniform control when the rectangle degenerates with $H/X \rightarrow 0$. The attribution concerns $a^2 + b^2$, not automatically the other two norms.

Shrinking angular domains also have a separate literature. Burgin–Giannitsi [18] study ergodic averages along Ω in shrinking Gaussian sectors and sector–disk comparisons for finitely generated multiplicative actions. Kuś [19, Theorems 1.7 and 1.9] treats multiplicative functions on Gaussian ideals under angular non-pretentiousness, with an additional conjugate-prime condition for short intervals. These source classes and angular or radial averages differ from uniform truncation over every subset of a prime support in a degenerating rectangle. Wang [20, Theorem 1.2 and Corollary 1.3] relates square averages over all pairs to averages over coprime pairs. This is relevant to primitive directions, but is distinct from controlling which entire direction a private prime can program.

The positive collective estimates are direct specializations of the Selberg upper sieve [11, Section 13.2]; only fixed progressions modulo four and eight enter the Mertens estimates [12]. Harman–Lewis [13, Theorem 2] supplies the angular lower bound for the negative power-rectangle branch. Frantzikinakis–Klurman–Moreira [5, Lemma 4.15] is an antecedent for square-incidence estimates, and Nair [14] for polynomial sums in short intervals. Neither is used to infer a uniform bound for a varying domain without the corresponding proof.

A neighboring probabilistic problem is studied by de la Bretèche–Wang–Xu [9], who prove central limit theorems for polynomial sums of random multiplicative functions, including the completely multiplicative model. Fair prime signs at every prime do not satisfy (1.11). Their distributional limits and our almost-sure or whole-support truncation questions are distinct.

Reading routes. For the support criterion, start with the finite-prime laws in Section 2 and the constructive converse in Section 3; Sections 4–5 give the independent-law counterpart. Sections 6–10 develop the stable and unstable realizations, with affine transfer in Section 11. For the critical exponents, the key assembly is (14.3): a support count below a moving cutoff is combined with the collective sieve above it. Section 13 supplies that estimate. For the rectangular counterexample, follow the cofactor logarithm in Section 15 through its removal and injectivity steps; Section 12 gives the elementary direction geometry first. Appendix A separates the all-origin and typical-origin conclusions. No inverse theorem for the unrestricted three curves, no cancellation for unmodified Liouville, and no Poisson limit is deduced.

2 Finite-prime approximation and deterministic control

For E a prime set, let h_E have the meaning in (1.8), without randomness. Every quotient g/f in $\mathcal{M}_\pm$ is of this form. We first state the elementary facts required later.

2.1 Metrics and finite-coordinate continuity

For a fixed $f \in \mathcal{M}_\pm$ and $\sigma > 0$, the class

$$C_\sigma(f) = \{g \in \mathcal{M}_\pm : d_\sigma(f, g) < \infty\}$$

is complete. Indeed, every prime coordinate in a Cauchy sequence is eventually constant, since changing the coordinate at p costs $p^{-\sigma}$. Let g be the coordinatewise limit. Fatou's lemma applied to the nonnegative coordinate distances gives convergence to g in d_σ and membership in the class. Truncating the changes to finitely many primes proves separability and density of finite changes. A condition on finitely many prime signs is both open and closed. In particular, each finite average in (1.14) is continuous, and preserving $g(n) = f(n)$ for $n \leq M_0$ is equivalent to preserving the prime signs at $p \leq M_0$.

2.2 Finite-prime laws are limits of periodic laws

For a finite prime set F and $r \geq 1$, the function

$$h_{F,r}(m) = (-1)^{\sum_{p \in F} \min(v_p(m), r)} \quad (2.1)$$

is periodic. The untruncated function h_F need not be periodic. For a fixed nonzero integer polynomial P , omit its integer zeros. Disagreement between $h_F(|P(n)|)$ and $h_{F,r}(|P(n)|)$ requires $P(n) \equiv 0 \pmod{p^r}$ for some $p \in F$. For a fixed p , the Haar measure of this set in $\mathbb{Z}_p$ tends to zero as $r \rightarrow \infty$: the sets decrease to the zero set of P in $\mathbb{Z}_p$, which is finite. The Chinese remainder theorem therefore proves that every finite-prime polynomial observation, and every fixed finite vector of such observations, has a mean or empirical law obtained from independent Haar coordinates at the primes in F . This argument allows repeated factors and primes of bad reduction.

For $P(n) = n^2 + 1$, the factors can be computed explicitly. If $p \equiv 1 \pmod{4}$,

$$\mathbb{P}_{u \in \mathbb{Z}_p} \{v_p(u^2 + 1) = k\} = \frac{2(p-1)}{p^{k+1}} \quad (k \geq 1).$$

Hence the probability of an odd valuation is $2/(p+1)$ and the mean sign is $1 - 4/(p+1)$. A prime $p \equiv 3 \pmod{4}$ never divides the value. At 2, the valuation is one for odd n and zero for even n . Thus

$$\lim_{N \rightarrow \infty} A_{h_F}(N) = \begin{cases} 0, & 2 \in F, \\ \prod_{\substack{p \in F \\ p \equiv 1 \pmod{4}}} \left(1 - \frac{4}{p+1}\right), & 2 \notin F. \end{cases} \quad (2.2)$$

The logarithmic mean has the same limit.

Lemma 2.1 (Abel identity). *If $b_1, b_2, \dots$ is bounded and $B(N) = N^{-1} \sum_{n \leq N} b_n$, then*

$$\frac{1}{\mathcal{H}_N} \sum_{n \leq N} \frac{b_n}{n} = \frac{B(N) + \sum_{k < N} B(k)/(k+1)}{\mathcal{H}_N}. \quad (2.3)$$

The coefficients on the right are nonnegative and sum to one. In particular, $B(N) \rightarrow b$ implies convergence of the logarithmic mean to b .

Proof. Write $b_n = nB(n) - (n-1)B(n-1)$ and telescope after division by n . The sum of the coefficients in the numerator is $1 + \sum_{k < N} (k+1)^{-1} = \mathcal{H}_N$. Every fixed initial segment has coefficient tending to zero. $\square$

Proposition 2.2 (A sufficient deterministic budget). *For all $f, g \in \mathcal{M}_\pm$ and $N \geq 1$, the ordinary and logarithmic proportions of disagreement on $n^2 + 1$ are at most $5d_{1/2}(f, g)$. If $d_{1/2}(1, h_E) < \infty$, both quadratic means converge to the infinite-prime version of (2.2).*

Proof. For an odd prime which occurs in the sample, $p \leq N^2 + 1$ and the two-root bound gives

$$\frac{1}{N} \#\{n \leq N : p \mid n^2 + 1\} \leq \frac{2}{p} + \frac{2}{N} \leq \frac{2 + 2\sqrt{2}}{\sqrt{p}} < \frac{5}{\sqrt{p}}.$$

For $p = 2$ the last bound is immediate. If $f(n^2 + 1) \neq g(n^2 + 1)$, at least one changed prime divides $n^2 + 1$. A union bound proves the ordinary estimate. Applying Lemma 2.1 to the nonnegative disagreement indicators proves the logarithmic estimate.

For a finite $F \subset E$, the means of h_F along the quadratic converge by (2.2). The mean absolute difference from h_E is at most $10 \sum_{p \in E \setminus F} p^{-1/2}$, uniformly in N , and this tends to zero as F increases to E . Finite-prime limits are consequently Cauchy and give the asserted limit. Abel gives equality of ordinary and logarithmic limits. $\square$

This is a sufficient continuity statement, not an optimal one. In particular, convergence for h_E is not asserted for an arbitrary multiplier $f(P(n))$ without further hypotheses on its periodic weighted averages.

3 A deterministic criterion for an entire prime support

The random criterion fixes probabilities and controls almost every resulting assignment. We first take a different order of quantifiers: the support is fixed, and every assignment on it must work. Coverage is then sufficient information. Parity cancellations can reduce the error of one assignment, but they cannot reduce the largest error over all assignments by more than a factor two.

Throughout this section $P \in \mathbb{Z}[T] \setminus \{0\}$ and Γ satisfies (1.4). Write $B_y(N) = B_{P,N}(\Gamma; y)$ and let $B_y^\ell(N)$ be the same coverage with weights $1/(n\mathcal{H}_N)$. At omitted zeros the indicator is zero. Put $D = \max(1, \deg P)$ and use $\star = c, \ell$ to specify the averaging method.

3.1 From all parities to coverage

Lemma 3.1 (A finite coverage comparison). *For every N, y and either averaging method,*

$$\frac{1}{2} B_y^\star(N) \leq \sup_{A \subseteq \Gamma} \Delta_{P,N}^\star(A; y) \leq B_y^\star(N). \quad (3.1)$$

Proof. A disagreement needs at least one tail prime with odd valuation, proving the upper bound. Only finitely many tail primes occur before N . Give these coordinates independent fair bits temporarily. A nonempty parity sum of fair bits is fair, so its expected disagreement is $1/2$; an empty sum gives zero. Averaging with the prescribed nonnegative weights gives expected disagreement $B_y^\star(N)/2$. Some assignment attains at least that expectation. It extends to a subset of Γ by leaving all other coordinates positive. $\square$

There is no assertion of independence between the polynomial values in this proof. Repeated rows, shared primes and nested parity supports are allowed. For general binary incidence systems the factor $1/2$ cannot be increased uniformly: with the $2^r - 1$ nonzero rows of $\mathbb{F}_2^r$, equally weighted, the largest disagreement is $2^{r-1}/(2^r - 1)$.

Lemma 3.2 (Moving the cutoff to the sample length). *With $u_P(N) = B_N(N)$,*

$$\lim_{y \rightarrow \infty} \limsup_{N \rightarrow \infty} B_y(N) = \limsup_{N \rightarrow \infty} u_P(N), \quad (3.2)$$

$$\lim_{y \rightarrow \infty} \limsup_{N \rightarrow \infty} B_y^\ell(N) = \limsup_{N \rightarrow \infty} (\mathcal{A}u_P)(N). \quad (3.3)$$

Replacing the inner limit superior on either left-hand side by $\sup_{N \geq 1}$ leaves the outer limit unchanged.

Proof. Take y beyond the primes dividing the content of P . For $N \geq y$, any extra coverage comes from a support prime between y and N . Counting at most D root classes gives

$$0 \leq B_y(N) - u_P(N) \leq D \sum_{p \in \Gamma; y < p \leq N} \frac{1}{p} + D \frac{\#(\Gamma \cap [2, N])}{N}. \quad (3.4)$$

Harmonic summability implies $\#(\Gamma \cap [2, N]) = o(N)$: for a fixed z , split off the finitely many primes at most z and bound the remaining quotient by $\sum_{p \in \Gamma, p > z} 1/p$. Taking the limits in that order proves (3.2).

Lemma 2.1 gives the exact identity $B_y^\ell = \mathcal{A}(B_y)$ for each *fixed* y . Apply this positive averaging operator to (3.4). Terms before $N = y$ have asymptotically zero total weight, the counting error tends to zero after averaging, and the remaining bound is the harmonic tail. This proves (3.3).

Finally $B_y^\star(N)$ decreases as y increases. For each finite set of sample sizes it becomes zero once y exceeds the heights of their nonzero polynomial values. Choose an initial cutoff for which the limiting upper bound is close to its outer limit; control all sufficiently large sizes at that cutoff, then enlarge the cutoff to remove the remaining finite set. This proves the assertion about the supremum. $\square$

The logarithmic diagonal quantity is $\mathcal{A}u_P$, not the logarithmic coverage obtained by imposing $p > N$ throughout a sample of size N . The two have different finite formulas: for $P(n) = n^2 + 1$, $\Gamma = \{5\}$ and $N = 10$, the latter is zero whereas $(\mathcal{A}u_P)(10) = 1092/7381$. Each $u_P(k)$ in the Abel average uses its own cutoff $p > k$.

3.2 Uniform truncation gives the local products

For finite F , the finite-prime approximation of Section 2 gives

$$\lim_N M_{P, h_F}^c(N) = \lim_N M_{P, h_F}^\ell(N) = \prod_{p \in F} \ell_{P, p}. \quad (3.5)$$

The same argument works for $|P|$ and omitted zeros: truncate valuations, apply the Chinese remainder theorem, and then use that the p -adic sets $\{p^r \mid P(t)\}$ decrease to a finite zero set. A nonzero constant is immediate. Outside the finite content-prime set,

$$|1 - \ell_{P, p}| \leq 2\mathbb{P}_{t \in \mathbb{Z}_p} \{p \mid P(t)\} \leq 2D/p.$$

Consequently the products (1.7) converge uniformly over $A \subseteq \Gamma$; any zero factor is retained.

To prove (ii) $\Rightarrow$ (iii) of Theorem 1.1, put

$$t(y) = \sup_{A \subseteq \Gamma} \sup_N \Delta_{P, N}^\star(A; y).$$

The difference of the signed means of h_A and $h_{A, y}$ is at most $2t(y)$. At fixed y , only finitely many subsets of $\Gamma \cap [2, y]$ occur, and their means in (3.5) therefore converge uniformly. Their local products approach $\ell_P(A)$ uniformly as $y \rightarrow \infty$. The triangle inequality gives

$$\limsup_N \sup_{A \subseteq \Gamma} |M_{P, h_A}^\star(N) - \ell_P(A)| \leq 2t(y) + o_{y \rightarrow \infty}(1),$$

which proves (iii), and (iii) implies (iv). Lemmas 3.1–3.2 give (i) $\Leftrightarrow$ (ii). It remains to show why pointwise convergence for *all* assignments already forces this uniform tail control.

3.3 Persistent coverage produces one divergent assignment

The converse does not need private primes. A finite parity average supplies an assignment with a low mean, and the small harmonic cost makes every finite assignment return to a high mean. Freezing the heights alternates these two observations in a single final function.

Proposition 3.3 (A bad subset of arbitrarily small harmonic cost). *Set $\delta_\star = \lim_{y \rightarrow \infty} \limsup_N B_y^\star(N)$. If $\delta_\star > 0$, then for every $0 < \gamma < \delta_\star$, every $\varepsilon > 0$ and every M_0 , there is a subset $A \subseteq \Gamma \cap (M_0, \infty)$ such that*

$$\sum_{p \in A} \frac{1}{p} < \varepsilon, \quad \text{osc}_N M_{P, h_A}^\star(N) \geq \gamma.$$

Proof. Choose $a, b > 0$ with $a + 2b < \delta_\star - \gamma$. Take $y_0 > M_0$, beyond the exceptional reduction primes, so that

$$\sum_{p \in \Gamma, p > y_0} 1/p < \varepsilon, \quad 2D \sum_{p \in \Gamma, p > y_0} 1/p < a.$$

Every finite $F \subseteq \Gamma \cap (y_0, \infty)$ has a limiting mean at least $1 - a$. Indeed, a negative value requires a prime of F to divide $P(n)$, and the limiting density of their union is at most $D \sum_F 1/p$. Equation (3.5) supplies the existence of both means.

Suppose the negative primes already chosen form a finite set F . Choose z beyond all coordinates already used and all heights of samples to be protected. There are arbitrarily large N with $B_z^\star(N) > \delta_\star - b$. Give fair bits temporarily to the finitely many primes of $\Gamma \cap (z, \infty)$ occurring to odd order in this sample. Every covered value then has expected sign zero, regardless of its old sign; the uncovered values are at most one. Some deterministic choice, extending F to a finite F' , consequently has mean at most $1 - \delta_\star + b$ at N . Since F' is finite, choose a recovery size $M > N$ where its mean is at least $1 - a - b$.

Repeat with the next cutoff beyond every $|P(n)|$, $n \leq M$, and beyond all previously used primes. Both the low sample and its recovery remain unchanged at every later stage. The increasing union of the negative sets defines one subset A , supported beyond y_0 , with the required cost and prefix. Its low and high subsequences are separated by at least $\delta_\star - a - 2b > \gamma$. Both sequences of indices tend to infinity. This proves the assertion for either weighting. $\square$

The proposition proves the contrapositive of (iv) $\Rightarrow$ (i), completing Theorem 1.1. It constructs a bad subset of the support, not divergence for each individual assignment having a nonzero truncation defect.

The harmonic hypothesis cannot simply be removed. For $P(T) = T$ and Γ equal to all primes, every real bounded multiplicative sign function has an ordinary mean by the classical theorem of Wirsing recalled in [1, Introduction]. Yet, for fixed y , an integer outside the coverage has squarefree part supported on the finitely many primes at most y . There are at most $2^{\pi(y)} \sqrt{N}$ such integers up to N , so $B_y(N) \rightarrow 1$. This example is used only to delimit the hypothesis of the support criterion.

4 The exact random truncation defect

We prove Theorem 1.2. Throughout this section $P \in \mathbb{Z}[T]$ has fixed degree $D \geq 1$ and is positive on $\mathbb{N}_{\geq 1}$. A polynomial that is only eventually positive can be treated after deleting or translating a finite prefix. Assumption (1.11) and Tonelli's theorem give

$$\sum_{p \in E} \frac{1}{p} < \infty \quad \text{almost surely.} \tag{4.1}$$

The proof separates three tasks. First, the degree bounds the total large-prime incidence, while a root count bounds the influence of each prime; multiplying the two gives a small concentration budget.

Second, the expectation of the large-prime disagreement is exactly $\eta_P(N)$, so bounded differences and Borel–Cantelli identify its realized value for all large N . Third, the intermediate primes are paid for by the tail of the *realized* budget in (4.1). Taking $N \rightarrow \infty$ before sending the fixed truncation threshold to infinity completes the comparison.

4.1 Large coordinates have small individual influence

Define

$$A_p(N) = \#\{n \leq N : v_p(P(n)) \text{ is odd}\}, \quad a_p(N) = \frac{A_p(N)}{N}.$$

Except at primes dividing the content of P , reduction modulo p gives

$$A_p(N) \leq D(N/p + 1). \quad (4.2)$$

For all sufficiently large N , depending on P , the values $P(n)$ with $n \leq N$ are at most N^{D+1} . A value therefore has at most $2(D+1)$ distinct prime factors exceeding $\sqrt{N}$. Thus, once the content primes lie below $\sqrt{N}$,

$$\sum_{p > \sqrt{N}} a_p(N) \leq 2(D+1), \quad \max_{p > \sqrt{N}} a_p(N) \leq \frac{2D}{\sqrt{N}}, \quad (4.3)$$

$$\sum_{p > \sqrt{N}} a_p(N)^2 \leq \frac{4D(D+1)}{\sqrt{N}}. \quad (4.4)$$

The last line comes from

$$\sum_{p > \sqrt{N}} a_p(N)^2 \leq \left(\max_{p > \sqrt{N}} a_p(N) \right) \sum_{p > \sqrt{N}} a_p(N).$$

Thus the concentration budget is small without summing a separate uniform error over all primes up to the polynomial height.

Let

$$Z_N = \frac{1}{N} \sum_{n \leq N} \mathbf{1} \left\{ \sum_{p > \sqrt{N}} \xi_p v_p(P(n)) \text{ is odd} \right\}.$$

Independence of the prime choices gives $\mathbb{E}Z_N = \eta_P(N)$ exactly. Changing ξ_p can change Z_N by at most $a_p(N)$. Only finitely many coordinates occur for each N . McDiarmid’s classical bounded-differences inequality [7], in the form of [3, Theorem 1], applied with (4.4), yields

$$\mathbb{P}\{|Z_N - \eta_P(N)| > t\} \leq 2 \exp \left(-\frac{t^2 \sqrt{N}}{2D(D+1)} \right). \quad (4.5)$$

For $t_N = (6D(D+1) \log N / \sqrt{N})^{1/2}$ the right side is $2N^{-3}$. Borel–Cantelli, without any independence requirement between different sample sizes, proves

$$Z_N = \eta_P(N) + O_D(N^{-1/4} \sqrt{\log N}) \quad \text{almost surely for all sufficiently large } N. \quad (4.6)$$

The starting index may depend on P and the realization.

4.2 The intermediate primes are paid by the realized budget

The comparison must establish the outer limits rather than appeal to monotonicity. For example, $8^2 + 1 = 65 = 5 \cdot 13$ and $E = \{5, 13\}$ give $h_E(65) = 1$, whereas

$$h_{E,y}(65) = \begin{cases} 1, & y < 5, \\ -1, & 5 \leq y < 13, \\ 1, & y \geq 13. \end{cases}$$

The disagreement at this value is not monotone in y . This finite example explains the issue, not an asymptotic obstruction.

Take y larger than all content primes and N large enough that $y \leq \sqrt{N}$. The parity defining $\Delta_{P,N}(E; y)$ can differ from that defining Z_N only through a changed prime between y and $\sqrt{N}$. By (4.2),

$$|\Delta_{P,N}(E; y) - Z_N| \leq D \sum_{\substack{p \in E \\ p > y}} \frac{1}{p} + \frac{D}{\sqrt{N}}. \quad (4.7)$$

This is a deterministic union bound; it does not assume independence of the effects on the sampled integers.

Combine (4.6) and (4.7). For almost every realization,

$$\left| \limsup_N \Delta_{P,N}(E; y) - \limsup_N \eta_P(N) \right| \leq D \sum_{\substack{p \in E \\ p > y}} \frac{1}{p}, \quad (4.8)$$

and the same holds for limit inferiors. Letting $y \rightarrow \infty$ proves (1.12). In particular the outer limits exist, although the individual disagreement proportions need not be monotone in y . A countable intersection of the probability-one events deals simultaneously with every integer polynomial and every integer truncation threshold.

4.3 Logarithmic truncation

Write $\Delta_{P,N}^{\log}(E; y)$ for the disagreement proportion with weights $1/(n\mathcal{H}_N)$. Lemma 2.1 gives the exact relation

$$\Delta_{P,N}^{\log}(E; y) = \frac{\Delta_{P,N}(E; y) + \sum_{k < N} \Delta_{P,k}(E; y)/(k+1)}{\mathcal{H}_N}. \quad (4.9)$$

Apply this positive averaging operator to the comparison with $\eta_P(k)$. For fixed y , the finitely many terms with $k < y^2$, or before (4.6) holds, vanish after division by $\mathcal{H}_N$. The averaged concentration error and $D/\sqrt{k}$ tend to zero. The remaining error is at most $D \sum_{p \in E, p > y} 1/p$ in limit superior. Therefore

$$\lim_{y \rightarrow \infty} \limsup_N \Delta_{P,N}^{\log}(E; y) = \limsup_N \bar{\eta}_P(N),$$

with the analogous identity for limit inferiors. All quantities are nonnegative, so the zero-defect characterizations in Theorem 1.2 follow.

Remark 4.1 (Cutoffs and the object characterized). The cutoff $\sqrt{N}$ is convenient, not essential. With N^a , $0 < a < 1$, the maximal influence is $O_D(N^{-a})$, total incidence is $O_{D,a}(1)$ and the intermediate counting error is $O_D(N^{a-1})$. The same proof gives the criterion with this cutoff.

More precisely, let $\eta_P^{(a)}(N)$ denote (1.10) with N^a in place of $\sqrt{N}$. For fixed $0 < a < b < 1$, once all content primes lie below N^a ,

$$|\eta_P^{(a)}(N) - \eta_P^{(b)}(N)| \leq D \sum_{p > N^a} \frac{\theta_p}{p} + DN^{b-1} = o(1). \quad (4.10)$$

Indeed, couple the two parities using the same prime choices. A difference requires an activated incident prime between N^a and N^b . The union bound and (4.2) give at most $D \sum_{N^a < p \leq N^b} \theta_p/p + (D/N) \sum_{p \leq N^b} \theta_p$. The first term tends to zero by (1.11); the second is at most DN^{b-1} . Thus the finite defect formulas depend on the cutoff, but their limit superiors and inferiors do not. Lemma 2.1 gives the same equivalence after Abel averaging. The logarithmic criterion still concerns that average, not an arbitrary substitute pointwise condition.

5 Stable perturbations and their local laws

Corollary 5.1 (Vanishing activation probabilities). *Assume (1.11) and $\theta_p \rightarrow 0$. Almost surely, for every fixed polynomial P as above,*

$$\lim_{y \rightarrow \infty} \limsup_N \Delta_{P,N}(E; y) = 0.$$

The same conclusion holds for logarithmic disagreement.

Proof. Put $q_N = \sup_{p > \sqrt{N}} \theta_p$. At any sampled value, an odd activated parity requires at least one activation among the relevant primes. By (4.3),

$$0 \leq \eta_P(N) \leq q_N \sum_{p > \sqrt{N}} a_p(N) \leq 2(D+1)q_N \rightarrow 0.$$

Theorem 1.2 and its Abel version apply. $\square$

Vanishing of θ_p is sufficient, not necessary. For $P(n) = n^2 + 1$, any prime support contained in $p \equiv 3 \pmod{4}$ is invisible, regardless of its activation probabilities, provided the harmonic hypothesis of the theorem is satisfied.

5.1 A joint law conditional on the selected support

Fix a realization E for which Corollary 5.1 and (4.1) hold. On a separate probability space let U_p be independent Haar-uniform elements of $\mathbb{Z}_p$. For fixed polynomials $P_1, \dots, P_m$, positive on positive integers, define

$$Y_{E,j} = \prod_{p \in E} (-1)^{v_p(P_j(U_p))}, \quad 1 \leq j \leq m. \quad (5.1)$$

At a given prime the *same* coordinate U_p is used for all polynomials. Thus shared factors and local dependencies are retained. A zero of a polynomial has Haar measure zero. Outside finitely many primes, a nontrivial factor occurs with probability $O_{P_1, \dots, P_m}(1/p)$. By (4.1) and Borel–Cantelli, only finitely many nontrivial factors occur, almost surely in the U_p ’s, so (5.1) is well-defined.

For probability measures on the finite alphabet $\mathcal{A} = \{-1, 1\}^m$, we use the convention

$$d_{\text{TV}}(\mu, \nu) = \sup_{B \subset \mathcal{A}} |\mu(B) - \nu(B)| = \frac{1}{2} \sum_{z \in \mathcal{A}} |\mu(\{z\}) - \nu(\{z\})|.$$

There are two distinct approximations: cutting the prime set, then truncating the valuations. Only the second makes the finite-prime observation periodic; a finite prime set alone does not do so.

Theorem 5.2 (Quenched empirical local law). *Under the assumptions of Corollary 5.1, almost surely in E and simultaneously for all fixed polynomial families,*

$$\frac{1}{N} \sum_{n \leq N} \delta_{(h_E(P_1(n)), \dots, h_E(P_m(n)))} \xrightarrow[N \rightarrow \infty]{\text{TV}} \mathcal{L}((Y_{E,j})_{j=1}^m). \quad (5.2)$$

The logarithmically weighted empirical measures have the same limit. Each sequence $h_E(P(n))$ is approximable in mean absolute value by periodic sign sequences, hence is rationally almost periodic in the finite-alphabet sense of [8].

Proof. For $E \cap [2, y]$ finite, truncate valuations at level r . The resulting vector is periodic, and the Chinese remainder theorem identifies its limiting distribution with independent uniform residue classes at the finitely many primes. Sending $r \rightarrow \infty$ is legitimate by the zero-set argument of Section 2. It gives the finite-prime law with complete valuations.

The total-variation discrepancy between the true empirical vector and its finite-prime truncation is at most $\sum_{j=1}^m \Delta_{P_j, N}(E; y)$. Its iterated limit is zero by Corollary 5.1. On the target side, a union bound gives an error $O_{P_1, \dots, P_m}(\sum_{p \in E, p > y} 1/p)$, once y contains the exceptional primes. Let $N \rightarrow \infty$ and then $y \rightarrow \infty$. On the finite alphabet $\{-1, 1\}^m$, convergence of all atom masses is total-variation convergence. The mean periodic approximation follows from the same two truncations. Applying Lemma 2.1 to each atom gives the logarithmic assertion. Countability gives simultaneity. $\square$

For example, almost surely,

$$\lim_N A_{h_E}(N) = \lim_N L_{h_E}(N) = \begin{cases} 0, & 2 \in E, \\ \prod_{\substack{p \in E \\ p \equiv 1 \pmod{4}}} \frac{p-3}{p+1}, & 2 \notin E. \end{cases} \quad (5.3)$$

The limit depends on the realized E . It is not the average of the product over the Bernoulli choices. Taking the family $P(n), P(n+1), \dots$ shows existence of every fixed word frequency. These frequencies need not be those of independent fair signs; normality is not asserted. The family and word length are fixed: no claim about growing words or an independent field of all positions is made.

5.2 Stability beyond the deterministic half-power budget

Protect a prescribed prefix by setting $\theta_p = 0$ for $p \leq M_0$, and set

$$\theta_p = \frac{c}{\lceil p^{1/4} \rceil} \quad (p > M_0), \quad 0 < c \leq \frac{1}{4}. \quad (5.4)$$

Then $\theta_p \rightarrow 0$, and

$$\mathbb{E} \sum_{p \in E} p^{-9/10} \leq c \sum_{n \geq 2} n^{-23/20} \leq \frac{20c}{3} < \infty. \quad (5.5)$$

However,

$$\sum_{p \in E} p^{-1/2} = \infty \quad \text{almost surely.} \quad (5.6)$$

Indeed, its expected partial sums are comparable to those of $\sum_p p^{-3/4}$ and diverge. The centered independent series has summed variance at most $\sum_p \theta_p/p < \infty$, so its partial sums converge almost surely. This proves (5.6). The stable local laws are therefore not merely instances of Proposition 2.2.

The law of h_E has full support in the subspace of $C_{9/10}(1)$ with the protected prefix. To see this, take a center h_F in that space, with F possibly infinite, and a ball of radius $r > 0$. Choose y so that both the center's $d_{9/10}$ tail and the expected random tail are small compared to r . The prescribed choices below y have positive probability. Independently of those choices, Markov's inequality gives positive probability that the random tail is within the remaining budget. The resulting function lies in the ball.

Thus a full-support probability measure can give probability one to simultaneous polynomial stability, while logarithmic quadratic divergence is residual in the same metric space by Theorem 1.3. Full support means that every nonempty open set has positive measure; it does not prevent a residual set

from being null. These are different notions of genericity. Moreover, choosing $c \leq \min(1/4, 3\varepsilon/40)$ gives

$$\mathbb{P}\{d_{9/10}(1, h_E) < \varepsilon\} \geq \frac{1}{2}.$$

Conditioning on this ball retains the probability-one conclusions and full support within the ball.

5.3 What is transmitted to a prescribed source

For every $f \in \mathcal{M}_\pm$, cancellation of its signs gives the deterministic identity

$$\mathbf{1}\{fh_E(P(n)) \neq fh_{E,y}(P(n))\} = \mathbf{1}\{h_E(P(n)) \neq h_{E,y}(P(n))\}. \quad (5.7)$$

The error bound is therefore simultaneous for all f . This does not determine the averages of $f(P(n))h_E(P(n))$. One valid transfer is the following. Its periodic-multiplier and residue-class argument is related to [4, Section 8, especially (8.3)–(8.7)]; the polynomial truncation input is supplied by Theorem 5.2.

Corollary 5.3 (Transfer of progressionwise cancellation). *Fix a regular realization E as in Theorem 5.2 and fixed polynomials $P_1, \dots, P_m$. For $f \in \mathcal{M}_\pm$, suppose that*

$$\frac{1}{N} \sum_{n \leq N} \prod_{j=1}^m f(P_j(qn + r)) \longrightarrow 0$$

for every fixed positive integer q and every integer r , with nonpositive arguments in a finite prefix omitted. Then the same assertion holds with f replaced by fh_E . The implication is reversible.

Proof. For any fixed progression, approximate the quotient factors by finitely many prime signs and then bounded valuations. Their product is periodic in n . Its correlation with the product of the f -values vanishes by subdividing into residue classes, each of which is another fixed progression. Telescope the product of quotient factors. The error tends to zero by the simultaneous truncation result for the composed polynomials, followed by finite-prime valuation approximation. Since $h_E^2 = 1$, the same argument proves the converse. $\square$

6 Stable supports with no stronger power budget

A support can be large in every stronger power budget and nevertheless have small influence on each fixed polynomial observation. The construction delays the first occurrence of every new prime. A counting argument on a product of integer values leaves enough primes to do this simultaneously for an increasing finite list of polynomials.

6.1 What an individual coordinate costs

For a nonzero integer polynomial define

$$c_P(p) = \sup_{N \geq 1} \frac{1}{N} \#\{n \leq N : P(n) \neq 0, v_p(|P(n)|) \text{ odd}\}. \quad (6.1)$$

For a nonempty finite family $\mathcal{F}$, put $c_{\mathcal{F}}(p) = \max_{P \in \mathcal{F}} c_P(p)$, and let $\Delta_{\mathcal{F}}(f, g)$ be the maximum over $P \in \mathcal{F}$ of the supremum over N of the ordinary disagreement proportion.

Proposition 6.1 (The exact coordinatewise requirement). *For all $f, g \in \mathcal{M}_\pm$,*

$$\Delta_{\mathcal{F}}(f, g) \leq \sum_{p: f(p) \neq g(p)} c_{\mathcal{F}}(p).$$

Given weights $w(p) \geq 0$, the inequality $\Delta_{\mathcal{F}}(f, g) \leq K \sum_{p: f(p) \neq g(p)} w(p)$ for every pair f, g holds if and only if $c_{\mathcal{F}}(p) \leq Kw(p)$ for every prime.

Proof. A disagreement requires a changed prime with odd valuation, so a union bound proves sufficiency. A pair differing at just one prime p has exactly the disagreement indicator in (6.1); taking the suprema proves necessity. $\square$

This optimality is limited to additive, single-coordinate upper bounds uniform over all pairs. It does not characterize convergence of means. Abel summation transfers each ordinary bound, including its tails, to logarithmic disagreement.

Let $\rho_P(p)$ be the number of roots modulo p , and let $r_P(p)$ be the first positive index where $P(n) \neq 0$ and $p \mid P(n)$, or $+\infty$ if there is none. Before this first index there is no incidence, and afterwards root counting gives

$$c_P(p) \leq \min \left(1, \frac{\rho_P(p)}{p} + \frac{\rho_P(p)}{r_P(p)} \right). \quad (6.2)$$

If $\rho_P(p) = 0$, the influence is zero. A nonzero polynomial cannot vanish identically on an infinite integer progression, so a modular root has a nonzero occurrence. For nonzero constants all relevant primes are confined to a finite set.

6.2 Choosing the support before any signs

Theorem 6.2 (Rich supports with summable observation influences). *Fix $0 < s \leq 1$, $\varepsilon > 0$, a prefix M_0 , and a nonempty finite family $\mathcal{F}_*$ of nonzero integer polynomials. There is an infinite deterministic prime set $\Gamma \subset (M_0, \infty)$ with*

$$\sum_{p \in \Gamma} p^{-s} < \varepsilon, \quad \sum_{p \in \Gamma} p^{-t} = \infty \quad (0 < t < s), \quad (6.3)$$

$$\sum_{p \in \Gamma} c_{\mathcal{F}_*}(p) < \varepsilon, \quad \sum_{p \in \Gamma} c_P(p) < \infty \quad (P \in \mathbb{Z}[T] \setminus \{0\}). \quad (6.4)$$

Proof. Enumerate the nonzero integer polynomials and let $\mathcal{F}_j$ be increasing finite families containing $\mathcal{F}_*$ and exhausting that enumeration. Put

$$a_j = \sum_{P \in \mathcal{F}_j} (\deg P + 1), \quad d_j = \max(1, \max_{P \in \mathcal{F}_j} \deg P).$$

For a large X , beyond the coefficient sums of the active polynomials, set $R = \lfloor X/(32a_j \log X) \rfloor$. Remove from $(X, 2X]$ every prime dividing a nonzero value $P(n)$ with $P \in \mathcal{F}_j$ and $n \leq R$. The logarithm of the product of these absolute values is at most $a_j R \log X$. Every removed prime exceeds X , so there are at most $a_j R \leq X/(32 \log X)$ of them. The prime number theorem gives at least $X/(2 \log X)$ primes in the interval for all sufficiently large X [12]. At least $X/(4 \log X)$ admissible primes remain.

For an admissible q , each active polynomial has $r_P(q) > R$ or no root; also $\rho_P(q) \leq \deg P$. Thus (6.2) gives $\max_{P \in \mathcal{F}_j} c_P(q) \leq 2d_j/R$. No equidistribution theorem for polynomial roots is used.

Let $\tau_j = s(1 - 1/(j+1))$. Choose successive $X_j > \max(2X_{j-1}, M_0)$ large enough for the preceding bounds and, with $k_j = \lceil (2X_j)^{\tau_j} \rceil$, for

$$k_j \leq \frac{X_j}{4 \log X_j}, \quad k_j X_j^{-s} < \varepsilon 2^{-j-2}, \quad \frac{2d_j k_j}{R_j} < \varepsilon 2^{-j-2}. \quad (6.5)$$

These choices are possible since $\tau_j < s$ and $\tau_j < 1$. Let Γ_j consist of the k_j least admissible primes and put $\Gamma = \bigcup_j \Gamma_j$.

Summing (6.5) proves the finite budgets in (6.3)–(6.4). A fixed polynomial is active from some stage onward; the earlier blocks contain finitely many primes. For $0 < t < s$, eventually $\tau_j \geq t$, and

$$\sum_{p \in \Gamma_j} p^{-t} \geq k_j (2X_j)^{-t} \geq 1.$$

Disjointness of the blocks proves divergence. Every selection is specified by finite tests and searches, so the support may be defined without randomness; no practical bound on its first scales is asserted. $\square$

Corollary 6.3 (The full cube of assignments is stable). *For the support in Theorem 6.2, for each fixed nonzero P ,*

$$\sup_{A \subseteq \Gamma} \sup_N \Delta_{P,N}^c(A; y) \leq \sum_{p \in \Gamma, p > y} c_P(p) \longrightarrow 0. \quad (6.6)$$

The same holds logarithmically. Both means converge uniformly over $A \subseteq \Gamma$ to $\ell_P(A)$, and every fixed finite vector of polynomial observations has a limiting local law.

Proof. Proposition 6.1 gives the tail bound, and Lemma 2.1 its logarithmic counterpart. The finite-prime laws and uniform product approximation prove uniform convergence, or one may use Theorem 1.1. For a finite vector, approximate all its coordinates at once; alternatively expand each atom indicator and apply the scalar statement to products of the fixed polynomials. Both routes retain shared local coordinates. $\square$

No distribution on the signs is required: choices can be arbitrarily correlated or depend on the sample size. For a moving choice $A = A_N$, uniformity means that the error relative to $\ell_P(A_N)$ tends to zero; it does not assert convergence to a single scalar if those products vary. A common limit follows when a local factor forces all products to agree, as in the next corollary.

Corollary 6.4 (Three balanced traces with a prescribed negative sign). *Take $M_0 \geq 2$ and include $T^2 + 1, T^2 + 2, T^2 - 2$ in $\mathcal{F}_*$. Put $h_0(n) = (-1)^{v_2(n)}$ and $g_A = h_0 h_A$. Then all three ordinary and logarithmic curve means of g_A tend to zero uniformly over $A \subseteq \Gamma$.*

Proof. All support primes are odd. For $n^2 + 1$ the valuation at two is one at odd indices and zero at even indices; for $n^2 \pm 2$ the roles are reversed. The local factor at two is therefore zero. The Chinese remainder theorem preserves this factor after any finite choice of odd prime signs. Multiplication by h_0 cancels in the disagreement, so (6.6) passes the zero finite-prime limits uniformly to every A . $\square$

For $s = 1$ and $A = \Gamma$, all stronger power budgets diverge, yet every trace has each sign with density $1/2$. The support remains harmonically summable. It is its delayed incidence, not a cancellation hypothesis on an unmodified source, that gives this conclusion.

7 Ordinary quadratic instability

The construction starts with a finite reservoir of sampled values having distinct private prime factors. The required reservoir is elementary apart from the prime number theorem in the fixed progression $1 \pmod{4}$. The logarithmic-product device is classical; the private-prime programming is directly related to Klurman's construction [1, Lemma 2.1].

7.1 An elementary reservoir above the sample length

The factor $1/2$ comes from a logarithmic mass balance: the product of the sampled values carries about $2N \log N$, small primes account for at most $(1 + o(1))N \log N$, and one large prime can carry at most $(2 + o(1)) \log N$ per value. The remaining mass therefore supplies almost half the sample. Its signs can be programmed at a cost tending to zero.

Choose

$$b_N = 3 + \left\lfloor \sqrt{\lfloor \log_2 N \rfloor} \right\rfloor, \quad Y_N = Nb_N, \quad I_N = \{1 \leq n \leq N : P^+(n^2 + 1) > Y_N\}. \quad (7.1)$$

Here $P^+(m)$ denotes the largest prime factor of $m > 1$, and we use the convention $P^+(1) = 1$ throughout. In particular, values $|Q_2(a, b)| = 1$ in Section 12 do not belong to a large-prime reservoir. We have $b_N \rightarrow \infty$, $b_N = o(\log N)$ and $Y_N > 2N$.

Lemma 7.1 (Half of the sample has an affordable private pivot). *As $N \rightarrow \infty$,*

$$|I_N| \geq \left(\frac{1}{2} - o(1) \right) N. \quad (7.2)$$

For each $n \in I_N$, the prime $q_n = P^+(n^2 + 1)$ has valuation one and divides no other $m^2 + 1$ with $1 \leq m \leq N$.

Proof. Set $M = N^2 + 1$. Factoring the product of all values gives

$$\sum_{n \leq N} \log(n^2 + 1) = 2 \log(N!) + O(1) = 2N \log N + O(N). \quad (7.3)$$

For an odd prime $p \equiv 1 \pmod{4}$, there are two roots of $T^2 + 1$ modulo each p^k . Summing the bound $2N/p^k + 2$ over all $k \leq \log M / \log p$, the contribution of primes at most $Y = Y_N$ is at most

$$2N \sum_{\substack{p \leq Y \\ p \equiv 1 \pmod{4}}} \frac{\log p}{p-1} + 2\pi(Y) \log M + O(N). \quad (7.4)$$

Primes $3 \pmod{4}$ do not occur, and the contribution at 2 is $O(N)$. The prime number theorem in a fixed arithmetic progression and partial summation give

$$\sum_{\substack{p \leq Y \\ p \equiv 1 \pmod{4}}} \frac{\log p}{p-1} = \left(\frac{1}{2} + o(1) \right) \log Y. \quad (7.5)$$

For this classical input one may use [6]. Since $\log Y \sim \log N$ and $\pi(Y) \ll Y / \log Y$, the second term of (7.4) is $O(Nb_N) = o(N \log N)$. The total small-prime contribution is at most $(1 + o(1))N \log N$. By (7.3), the primes above Y_N contribute at least $(1 - o(1))N \log N$.

A value $n^2 + 1 \leq M$ has at most one such prime, at exponent one, since $Y_N^2 > M$. Its contribution is at most $\log M = (2 + o(1)) \log N$. Division proves (7.2).

If $q_n \mid m^2 + 1$ as well, then $q_n \mid (n - m)(n + m)$. Since $q_n > 2N$, both factors have absolute value less than q_n , except that $n - m$ may be zero. Hence $m = n$. Distinct sites therefore have distinct pivots. $\square$

Lemma 7.2 (Weighted programming by private primes). *Let $J \subset [1, N]$ have primes $q_n \mid n^2 + 1$, $n \in J$, each with odd valuation and private to its site among the first N quadratic values. Given $f \in \mathcal{M}_\pm$ and prescribed signs s_n , changing only the q_n can realize*

$$g(n^2 + 1) = s_n \quad (n \in J)$$

without changing the other sampled values. The cost is at most $\sum_{n \in J} q_n^{-\sigma}$. For nonnegative weights w_n summing to one, the all-positive and all-negative prescriptions give two functions with weighted-average difference

$$\sum_{n \leq N} w_n g_+(n^2 + 1) - \sum_{n \leq N} w_n g_-(n^2 + 1) = 2 \sum_{n \in J} w_n. \quad (7.6)$$

All integer values smaller than the least changed prime remain fixed.

Proof. Set $g(q_n) = f(q_n)s_n/f(n^2 + 1)$, leaving all other prime signs unchanged. An odd valuation makes this change produce exactly the prescribed sign. Privacy ensures that distinct prescriptions do not interfere, and that the complement of J is unchanged. Some pivots may need no change, so the displayed sum is an upper bound on the actual cost. The weighted identity follows term by term. $\square$

For (7.1), this cost is at most

$$\sum_{n \in I_N} \frac{1}{q_n} \leq \frac{|I_N|}{Y_N} \leq \frac{1}{b_N} \rightarrow 0. \quad (7.7)$$

An individual pivot has observed incidence $1/N$, whereas its harmonic cost is less than $1/(Nb_N)$. This growing ratio is the elementary source of the instability.

7.2 Freezing a single function across infinitely many scales

At each step, keep an earlier experiment fixed and create two possible outputs at a larger scale. Since these outputs are separated by twice the controlled mass, one lies far from the earlier mean. Choose that output, then protect it from every future change. For an arbitrary source this does not force averages near both $+1$ and -1 ; separation is enough. Protection must extend to the polynomial height $N^2 + 1$, not merely to the number N of sampled indices.

We prove Theorem 1.3(i). Start with $f_0 = f$ and $N_0 \geq \max(2, M_0)$. At stage $j \geq 1$, choose $N_j > N_{j-1}$ so large that

$$Y_{N_j} > N_{j-1}^2 + 1, \quad b_{N_j}^{-1} < \varepsilon 2^{-j-1}, \quad \frac{|I_{N_j}|}{N_j} \geq \frac{1}{2} - \frac{1}{j+2}. \quad (7.8)$$

From the current f_{j-1} apply the two constant prescriptions on I_{N_j} . They have means at N_j separated by $2|I_{N_j}|/N_j$ and both preserve the mean at N_{j-1} . At least one is at distance $|I_{N_j}|/N_j$ or more from that earlier mean. Choose it as f_j .

Every later pivot exceeds the heights of all quadratic values at previously chosen scales. It also exceeds every prime sign previously changed. The prime coordinates consequently stabilize and define one function $g \in \mathcal{M}_\pm$, with $d_1(f, g) < \varepsilon$ and unchanged prefix. For this fixed g ,

$$|A_g(N_j) - A_g(N_{j-1})| \geq \frac{1}{2} - \frac{1}{j+2}.$$

For a bounded sequence, its eventual range contains both members of every sufficiently late pair. Letting $j \rightarrow \infty$ proves $\text{osc}_N A_g(N) \geq 1/2$.

7.3 Residual instability

For integers $m \geq 3$ and $k \geq 1$, consider

$$\mathcal{O}_{m,k} = \bigcup_{U, V \geq k} \left\{ g : |A_g(U) - A_g(V)| > \frac{1}{2} - \frac{1}{m} \right\}.$$

This set is open by finite-coordinate continuity. It is dense in any $C_1(f)$: within a given ball choose an anchor $R \geq k$, then choose N with $Y_N > R^2 + 1$, cost below the radius and reservoir fraction greater than $1/2 - 1/m$. One of the two branches supplied by Lemma 7.2 lies in the ball and the indicated open set. The same choice preserves any compatible finite prefix.

By completeness and Baire's theorem, $\bigcap_{m,k} O_{m,k}$ is a dense G_δ . The intersection is exactly the set with oscillation at least $1/2$. This proves the residual assertion of part (i). This argument concerns Baire category, not independent random prime signs.

7.4 Recovery near one and excursions down to zero

Proposition 7.3 (A quantitative near-constant example). *For every $0 < \varepsilon < 1/4$ and prefix M_0 , there exists $h = h_E \in \mathcal{M}_\pm$ with*

$$E \subset \{p : p \equiv 1 \pmod{4}\}, \quad \sum_{p \in E} \frac{1}{p} < \varepsilon, \quad h(n) = 1 \ (n \leq M_0),$$

such that

$$\limsup_N A_h(N) \geq 1 - 4\varepsilon, \quad \liminf_N A_h(N) \leq 0. \quad (7.9)$$

The cluster set of $A_h(N)$ contains $[0, 1 - 4\varepsilon]$, whereas

$$\lim_{X \rightarrow \infty} \frac{1}{X} \sum_{n \leq X} h(n) = \prod_{p \in E} \frac{p-1}{p+1} \geq 1 - 2\varepsilon. \quad (7.10)$$

Proof. Starting from $F_0 = \emptyset$, suppose the finite changed-prime set F_{j-1} is fixed. By (2.2), choose a recovery index R_j beyond the previous stage with

$$\left| A_{h_{F_{j-1}}}(R_j) - C_{F_{j-1}} \right| < 2^{-j}, \quad C_F = \prod_{p \in F} \left(1 - \frac{4}{p+1} \right).$$

Now choose $N_j > R_j$ so that the new pivots exceed $R_j^2 + 1$ and M_0 , the cost is less than $\varepsilon 2^{-j-1}$, and the reservoir fraction is at least $1/2 - 1/(j+2)$. Force all reservoir signs to be negative. Each pivot is new and originally positive; add to F_{j-1} exactly the pivots that need to be reversed, giving $F_j \supset F_{j-1}$. The new mean is at most

$$A_{h_{F_j}}(N_j) \leq 1 - 2|I_{N_j}|/N_j \leq \frac{2}{j+2}.$$

Subsequent stages preserve both the recovery and excursion prefixes. For $E = \bigcup_j F_j$, the total cost is less than ε , and

$$C_E = \prod_{p \in E} \left(1 - \frac{4}{p+1} \right) \geq 1 - 4 \sum_{p \in E} \frac{1}{p} > 1 - 4\varepsilon.$$

The recovery subsequence tends to C_E , and the excursion subsequence has limit inferior at most zero. This proves (7.9), without identifying either extreme exactly. Any Cesàro average of signs satisfies $|A(N+1) - A(N)| \leq 2/(N+1)$; successive crossings therefore show that its cluster set is the interval between its limit inferior and superior.

On the ordinary integers, a finite prime change at p has local mean $(p-1)/(p+1)$. The Chinese remainder theorem gives the finite product, and (1.2) controls the infinite tail uniformly. This proves (7.10). The corresponding quadratic product gives a recovery subsequential limit, not the mean of the full quadratic observation. $\square$

8 Logarithmic instability with a stronger prime budget

A single dyadic block $(N/2, N]$, even with a positive proportion of available indices, has normalized logarithmic mass only $O(1/\log N)$. To retain positive mass, we need a number of dyadic blocks proportional to $\log N$, between N^β and N . The analytic input must therefore provide a positive population in every sufficiently large dyadic interval, not merely infinitely many large factors.

The parameters have separate jobs. If the available pivots satisfy $q_n > n^\alpha$ for $N^\beta < n \leq N$, then, for large N ,

$$\begin{aligned} \alpha\beta > 1 &\implies q_n > N^{\alpha\beta} > 2N && \text{(privacy),} \\ \alpha\sigma > 1 &\implies \sum_{n \in J_N} q_n^{-\sigma} \ll N^{-\beta(\alpha\sigma-1)} \rightarrow 0 && \text{(cost).} \end{aligned}$$

Here J_N denotes the set of available indices. Its population density and $1 - \beta$ control the logarithmic mass. The same transfer and construction take three different analytic inputs, whose outputs are compared in Table 1. The widest range does not replace the numerical amplitude furnished by the narrower one.

Analytic input	Budget range	Oscillation bound
Merikoski, quantified branch	$128/153 < \sigma \leq 1$	At least $1/2000$.
Pascadi, cutoff 1.30008	$12500/16251 < \sigma \leq 1$	A common positive amplitude; not calibrated here.
Grimmelt–Merikoski with Merikoski’s weighted calculation	$125/164 < \sigma \leq 1$	A common $c_{\log} > 0$; not calibrated here.

Table 1: Three inputs to one logarithmic construction.

The lower endpoints in the table are excluded. Within each row the amplitude can be fixed independently of σ , but the starting scales and the rate at which the cost vanishes may depend on it. Each route below retains its weighted input and its passage to a dyadic population.

8.1 A quantified input from Merikoski

Fix $w \in C_c^\infty((1, 2))$ with $0 \leq w \leq 1$ and $\int w \geq 1/2$. Set

$$b_x(n) = w(n/x), \quad W_x = \int_{\mathbb{R}} b_x(t) dt \geq x/2, \quad |\mathcal{A}_d| = \sum_{d|n^2+1} b_x(n).$$

In the notation of Merikoski [2, Section 2.1], Chebyshev’s device and a nonnegative smooth dyadic partition give

$$S(x) = \sum_{p>x} |\mathcal{A}_p| \log p = W_x \log x + O_w(x) = \sum_{\substack{P \geq x \\ P=2^j x}} S(x, P) + O_w(x), \quad (8.1)$$

Here the blocks use the smooth dyadic partition of unity from that reference:

$$S(x, P) = \sum_{P \leq p \leq 4P} \psi_P(p) |\mathcal{A}_p| \log p, \quad \psi_P \geq 0, \quad \text{supp } \psi_P \subset [P, 4P].$$

The partition weights sum to one away from the lower-end transition, whose contribution is included in $O_w(x)$ in (8.1). Each subfamily of weights therefore has sum at most one there. In particular, for a fixed $a > 1$ and large x ,

$$\sum_{\substack{P > x^a \\ P=2^j x}} S(x, P) \leq \sum_{p > x^a} |\mathcal{A}_p| \log p.$$

This is the positivity comparison used below; it is not an assumption of disjoint dyadic supports.

Published input 8.1 (Merikoski’s linear-sieve bound). *Let $\vartheta = 7/64$ and $a = 153/128$. The integral upper bound in [2, Remark 13, published p. 1268; arXiv v3, p. 15] gives*

$$\sum_{\substack{x \leq P \leq x^a \\ P=2^j x}} S(x, P) \leq (\kappa + o_w(1)) W_x \log x, \quad \kappa = 4(1 - 2\vartheta) \int_1^a \frac{v}{1 - 2\vartheta v} dv < \frac{9}{10}. \quad (8.2)$$

This is an unconditional published analytic estimate, not a hypothesis on the source and not a numerical experiment. The underlying distribution and sieve estimates are imported, not reproved here. The bound “infinitely often $P^+(n^2 + 1) > n^{1.279}$ ” by itself would not supply the positive population in every sufficiently large dyadic interval used below.

For clarity, the integral actually used is

$$\kappa = \frac{25}{8} \left(-\frac{25}{28} + \frac{1024}{49} \log \frac{128}{121} \right) = 0.882619359861 \dots \quad (8.3)$$

The decimal 0.8213 printed after the integral in Remark 13, both in arXiv:1908.08816v3 and on p. 1268 of the published article, is not used. Its displayed integral, evaluated at $\vartheta = 7/64$, gives (8.3). Only the strict margin below 9/10 is needed. A rational enclosure follows from

$$\log \frac{128}{121} = 2 \sum_{j \geq 0} \frac{(7/249)^{2j+1}}{2j+1},$$

whose positive tail is bounded by a geometric series.

Lemma 8.2 (A parameterized dyadic population). *Fix $1 < \alpha < 153/128$. For every sufficiently large x , with starting point allowed to depend on α ,*

$$\#\{x < n \leq 2x : P^+(n^2 + 1) > n^\alpha\} \geq x/100. \quad (8.4)$$

Proof. Subtract (8.2) from (8.1). For large x , the blocks with $P > x^a$, $a = 153/128$, have total logarithmic mass at least $(2/25)W_x \log x$, with the partition error absorbed. The positivity comparison above bounds that mass by the actual mass at primes $p > x^a$. A value $n^2 + 1 \leq 4x^2 + 1$ contains at most one such prime, to exponent one. Since $w \leq 1$, its contribution is at most $3 \log x$ for large x . As $W_x \geq x/2$, there are at least $x/75$ such indices, and in particular at least $x/100$. Finally, $a > \alpha$ implies $x^a > (2x)^\alpha$ for every sufficiently large x depending on α . This proves (8.4). $\square$

8.2 From a dyadic population to a logarithmic reservoir

Fix $\alpha > 1$ and $1/\alpha < \beta < 1$. Define

$$J_N = \{N^\beta < n \leq N : P^+(n^2 + 1) > n^\alpha\}, \quad q_n = P^+(n^2 + 1), \quad m_N = \frac{1}{\mathcal{H}_N} \sum_{n \in J_N} \frac{1}{n}. \quad (8.5)$$

Lemma 8.3 (Parameterized mass–privacy–budget transfer). *Suppose a fixed $\nu > 0$ satisfies*

$$\#\{x < n \leq 2x : P^+(n^2 + 1) > n^\alpha\} \geq \nu x$$

for every sufficiently large x . Then, for all sufficiently large N , the primes q_n , $n \in J_N$, are distinct, have exponent one, and are private throughout the sample $(r^2 + 1)_{r \leq N}$. For every fixed $\sigma > 1/\alpha$,

$$\begin{aligned} \liminf_N m_N &\geq \frac{\nu(1 - \beta)}{2 \log 2}, \\ \sum_{n \in J_N} q_n^{-\sigma} &\ll_{\alpha, \beta, \sigma} N^{-\beta(\alpha\sigma - 1)} \longrightarrow 0. \end{aligned} \quad (8.6)$$

Proof. Since $\alpha\beta > 1$, we have $q_n > N^{\alpha\beta} > 2N$ for large N . The argument of Lemma 7.1 proves privacy and distinctness; also $q_n^2 > N^2 + 1$ for large N , which proves exponent one.

The interval $(N^\beta, N]$ contains $(1-\beta) \log N / \log 2 + O(1)$ complete dyadic blocks. For all sufficiently large N , every block lies beyond the starting point of the assumed population bound. Each block $(x, 2x]$ contributes at least νx indices, hence reciprocal mass at least $\nu/2$. Sum over these blocks and divide by $\mathcal{H}_N \sim \log N$. This gives the first bound in (8.6). For the cost, $\alpha\sigma > 1$ gives

$$\sum_{n \in J_N} q_n^{-\sigma} \leq \sum_{n > N^\beta} n^{-\alpha\sigma} \ll_{\alpha, \beta, \sigma} N^{-\beta(\alpha\sigma-1)}.$$

Thus $\alpha\beta > 1$ ensures privacy, while $\alpha\sigma > 1$ makes the cost tend to zero. No arithmetic information other than the displayed population is used in this transfer. $\square$

For $128/153 < \sigma \leq 1$, Lemma 8.2 supplies $\nu = 1/100$. Fix once and for all

$$\alpha = \frac{1}{2} \left(\frac{153}{128} + \max \left\{ \frac{7}{6}, \frac{1}{\sigma} \right\} \right), \quad \beta = \frac{9}{10}. \quad (8.7)$$

Then $\max\{7/6, 1/\sigma\} < \alpha < 153/128$, $\alpha\beta > 21/20 > 1$, and $\alpha\sigma > 1$. In particular,

$$\liminf_N m_N \geq \frac{1}{2000 \log 2} > \frac{1}{2000}.$$

This gives the numerical amplitude in the first row of Table 1. The later rows cover $\sigma = 128/153$ with an uncalibrated positive amplitude, not with the number $1/2000$.

8.3 A common construction and its category conclusion

Fix $0 < \sigma \leq 1$ and a reservoir supplied by Lemma 8.3, with $\alpha\sigma > 1$. Let $c > 0$ be strictly smaller than its limiting lower mass. The construction only needs this mass, vanishing cost, and protection of earlier heights; it does not depend on which analytic estimate supplied the population.

Start from $f_0 = f$ and $N_0 \geq \max(2, M_0)$. At stage $j \geq 1$, choose $N_j > N_{j-1}$ with

$$\min_{n \in J_{N_j}} q_n > \max(M_0, N_{j-1}^2 + 1), \quad \sum_{n \in J_{N_j}} q_n^{-\sigma} < \varepsilon 2^{-j-1}, \quad m_{N_j} > c.$$

Use Lemma 7.2 with weights $w_n = 1/(n\mathcal{H}_{N_j})$. Its two branches have logarithmic means separated by $2m_{N_j}$ and preserve the preceding mean. Choose the branch whose mean is at distance at least m_{N_j} from that preceding value. Every later changed prime exceeds the full height of the preceding sample. Thus all previously selected values are final, the prime signs stabilize to one g , and

$$d_\sigma(f, g) < \varepsilon, \quad |L_g(N_j) - L_g(N_{j-1})| > c.$$

Boundedness gives $\text{osc}_N L_g(N) \geq c$. For integers $r, k \geq 1$, the sets

$$\bigcup_{U, V \geq k} \{g : |L_g(U) - L_g(V)| > c - 1/r\}$$

are open by finite-coordinate continuity. They are dense: in any ball choose an anchor beyond k , and then a reservoir with cost less than the radius and with pivots beyond the anchor's height. One of its two branches differs from that frozen anchor by more than c . The same choices preserve any compatible finite prefix. Completeness and Baire's theorem show that their intersection, exactly the set of logarithmic oscillation at least c , is a dense G_δ .

Applying this construction with (8.7) and $c = 1/2000$ proves the quantitative part of Theorem 1.3(ii), including its category assertion. The next two routes use the identical construction with a different population and amplitude.

Because $d_1(f, g) \leq d_\sigma(f, g)$ for $\sigma \leq 1$, ordinary integer disagreement stays below ε at every prefix. For $\sigma < 1$, splitting the weighted prime sum at a fixed threshold also gives

$$\#\{p \leq x : f(p) \neq g(p)\} = o(x^\sigma).$$

Thus the changed primes have relative prime density zero in that case. This last conclusion is not deduced from harmonic summability alone at $\sigma = 1$.

8.4 Pascadi's weighted estimate without rounding its cutoff

Published input 8.4 (Weighted mass beyond Pascadi's cutoff). *Fix a nonzero $\Phi \in C_c^\infty((1, 2))$ with $0 \leq \Phi \leq 1$, and set $\omega_P = 16251/12500 = 1.30008$. The weighted estimates in Pascadi [10, Sections 6.1–6.3, especially (6.20) and the final calculation] imply that, for some $c_\Phi^P > 0$ and all sufficiently large x ,*

$$\sum_n \Phi(n/x) \sum_{\substack{p|n^2+1 \\ p > x^{\omega_P}}} \log p \geq c_\Phi^P x \log x. \quad (8.8)$$

The inner sum is over primes. Its constant is not evaluated here.

In that reference the weighted main term is $X = x \int \Phi$, not the integer sample length used elsewhere here. The total large-prime mass is $X \log x + O_\Phi(x)$. Its positive dyadic blocks up to x^{ω_P} satisfy (6.20), with a strict upper margin below $X \log x$ in the concluding calculation at 1.30008. Hence the remaining blocks have positive mass. Their sum is at most the actual mass at primes beyond that cutoff, by positivity of the partition. Absorbing $O_\Phi(x)$ proves (8.8). This uses the weighted proof, not just an infinitely-often greatest-prime-factor statement. The spectral estimates, Harman sieve and numerical integrals behind it are imported, not recomputed here.

Lemma 8.5 (A positive population at every large dyadic scale). *There is $\nu_P > 0$, independent of α , such that, for each fixed $1 < \alpha < \omega_P$ and every sufficiently large x depending on α ,*

$$\#\{x < n \leq 2x : P^+(n^2 + 1) > n^\alpha\} \geq \nu_P x.$$

Proof. For large x , a value at most $4x^2 + 1$ has at most one prime factor above x^{ω_P} , at exponent one. As $\Phi \leq 1$, its contribution to (8.8) is at most $3 \log x$. There are at least $(c_\Phi^P/3)x$ contributing indices. Since $\alpha < \omega_P$, eventually $x^{\omega_P} > (2x)^\alpha$. Take $\nu_P = c_\Phi^P/3$, or any smaller fixed positive constant. $\square$

For every fixed $12500/16251 < \sigma \leq 1$, take

$$\alpha = \frac{1}{2} \left(\omega_P + \max \left\{ \frac{7}{6}, \frac{1}{\sigma} \right\} \right), \quad \beta = \frac{9}{10}. \quad (8.9)$$

Then $\alpha < \omega_P$, $\alpha\beta > 21/20$ and $\alpha\sigma > 1$. Lemma 8.3 yields

$$\liminf_N m_N \geq \frac{\nu_P}{20 \log 2}, \quad \sum_{n \in J_N} q_n^{-\sigma} \ll_{\alpha, \sigma} N^{-(9/10)(\alpha\sigma-1)} \rightarrow 0. \quad (8.10)$$

Section 8.3 therefore supplies both a fixed perturbation and the dense G_δ conclusion with the common amplitude $c_P = \nu_P/(40 \log 2)$, as in the second row of Table 1.

The simpler cutoff $13/10$ gives a valid but weaker consequence of this route. In particular $\sigma = 10/13$ is covered: choosing $\alpha = 32501/25000$ and $\beta = 9/10$ gives $\alpha < \omega_P$, $\alpha\beta > 1$, $\alpha\sigma = 1 + 1/32500$, and cost $O(N^{-9/325000})$. The strict endpoint of this route is $12500/16251$, not $10/13$.

At $\sigma = 4/5$, the earlier choice $\alpha = 51/40$, $\beta = 9/10$ remains useful:

$$\alpha\beta = 459/400 > 1, \quad \alpha\sigma = 51/50 > 1, \quad \sum_{n \in J_N} q_n^{-4/5} \ll N^{-9/500}.$$

These negative powers are cost estimates, not practical starting scales. At the same budget, Theorem 6.2 and Corollary 6.4 give supports on which every assignment after multiplication by h_0 has all three means zero. Applying the present construction to h_0 instead gives logarithmic divergence on $n^2 + 1$, preserving its negative sign at two. Placement of the support, not the budget alone, separates these examples.

8.5 The Grimmelt–Merikoski input and the full range

The largest range used here comes from a different analytic input. Grimmelt–Merikoski supply unconditionally the Type I and Type II ranges used in Merikoski’s formerly conditional calculation. The following statement specifies the weighted consequence consumed by the construction.

Published input 8.6 (Weighted mass from Grimmelt–Merikoski and Merikoski). *Fix Φ as above and set $\omega_G = 164/125 = 1.312$. There is $c_\Phi^G > 0$ such that, for every sufficiently large x ,*

$$\sum_n \Phi(n/x) \sum_{\substack{p|n^2+1 \\ p > x^{\omega_G}}} \log p \geq c_\Phi^G x \log x. \quad (8.11)$$

The analytic inputs are [17, Section 7, Corollaries 7.1–7.2] and the weighted sieve calculation of [2, Section 2.6, pp. 15–16, proof of Theorem 2], with the fixed quadratic $n^2 + 1$.

Matching the arithmetic information. Set the coefficient and shift denoted by a, h in Grimmelt–Merikoski equal to one. Their squarefreeness and coprimality hypotheses then hold. Their Corollary 7.1 provides Type I information to level $D \leq x^{1/2-\eta}$. With $MN = x^u$, Corollary 7.2 provides Type II information in

$$x^{u-1+2\eta} \leq N \leq x^{(2-u)/3-4\eta/3},$$

with divisor-bounded coefficients and the second coefficient supported on squarefree integers. The smooth weights required there cover the present fixed weight: replace their scale by $2x$ and their argument weight by $t \mapsto \Phi(2t)$, and subdivide modulus weights into smooth dyadic pieces. Fixed scale factors are absorbed by the strict exponent margins. The power savings require only a spectral gap, not Selberg’s eigenvalue conjecture. As the paragraph following their Corollary 7.2 explicitly states, these are precisely the Type I/II ranges used in Merikoski’s proof of Theorem 2. The sieve dimension for $n^2 + 1$ is one: apart from two, it has two roots at primes $1 \pmod{4}$ and none at primes $3 \pmod{4}$.

A strict margin before taking limits. Use $W_x = x \int \Phi$ and the positive blocks $S_\Phi(x, P)$ defined as in (8.1). Merikoski’s calculation gives an upper constant 0.679914 for the mass up to $x^{5/4}$, followed by the linear-sieve term $4 \int_{5/4}^{\omega_G} v \, dv$. The bounds on the preceding Buchstab integrals are imported from that calculation. The final elementary integral has the exact evaluation

$$\begin{aligned} \kappa_G &= \frac{679914}{10^6} + 2 \left(\left(\frac{164}{125} \right)^2 - \left(\frac{5}{4} \right)^2 \right) = \frac{498801}{500000} = 0.997602, \\ 1 - \kappa_G &= \frac{1199}{500000} > 0. \end{aligned} \quad (8.12)$$

Inserting the unconditional arithmetic information into that sieve recipe gives, for each sufficiently small fixed $\zeta > 0$ after choosing its auxiliary margins,

$$\sum_{\substack{x \leq P \leq x^{\omega_G} \\ P=2^j x}} S_{\Phi}(x, P) \leq (\kappa_G + \zeta + o_{\Phi, \zeta}(1)) W_x \log x. \quad (8.13)$$

The term ζ allows the small endpoint losses in the Type I/II ranges; the same weighted recipe and its limiting integrals are those in the cited proof. Fix $0 < \zeta < (1 - \kappa_G)/4$ first, then take x large enough to absorb the little-oh term and the $O_{\Phi}(x)$ partition remainder. Subtracting from total mass $W_x \log x + O_{\Phi}(x)$ leaves a fixed positive multiple of $W_x \log x$. Positivity bounds the remaining blocks by the actual mass at primes $p > x^{\omega_G}$, proving (8.11). This is the required passage from the two cited proofs; the greatest-prime-factor theorem stated in their introductions alone would not give it. Neither their spectral estimates nor the multidimensional numerical bounds behind 0.679914 are recertified by the finite arithmetic check of (8.12).

Lemma 8.7 (Population below 1.312). *There is $\nu_G > 0$, independent of α , such that, for every fixed $1 < \alpha < 164/125$, every sufficiently large x satisfies*

$$\#\{x < n \leq 2x : P^+(n^2 + 1) > n^{\alpha}\} \geq \nu_G x.$$

The starting point may depend on α .

Proof. A value at most $4x^2 + 1$ has at most one prime factor greater than x^{ω_G} , of exponent one. Its weighted contribution in (8.11) is at most $3 \log x$. Thus $\nu_G = c_{\Phi}^G/3$, or any smaller fixed positive constant, bounds the density of contributing indices below. Finally $x^{\omega_G} > (2x)^{\alpha}$ for every sufficiently large x depending on α . $\square$

Uniform amplitude, not uniform starting scales. Fix $125/164 < \sigma \leq 1$, and put

$$\alpha_{\sigma} = \frac{1}{2} \left(\frac{164}{125} + \max \left\{ \frac{7}{6}, \frac{1}{\sigma} \right\} \right), \quad \beta = \frac{9}{10}, \quad c_{\log} = \min \left\{ \frac{1}{4000}, \frac{\nu_G}{40 \log 2} \right\} > 0. \quad (8.14)$$

Then $\alpha_{\sigma} < 164/125$, $\alpha_{\sigma}\beta > 21/20$ and $\alpha_{\sigma}\sigma > 1$. Lemma 8.3 gives $\liminf m_N \geq \nu_G/(20 \log 2) > c_{\log}$ and cost tending to zero. Section 8.3, with $c = c_{\log}$, supplies the fixed perturbation and dense G_{δ} assertion in Theorem 1.3. Here Φ and ν_G were fixed before σ , so $c_{\log}$ is common to the whole range. The strict inequality on σ is needed to choose $1/\sigma < \alpha_{\sigma} < 164/125$; this route does not cover $125/164$. No privacy on the other two curves is asserted.

The numerical value $1/2000$ remains a separate consequence of (8.7), not a value claimed for the enlarged range. The independent construction in Section 9 only consumes a positive reservoir and therefore uses (8.14) as well. The stability range on the curves and these instability ranges still leave an interval unclassified.

9 Independent choices on an unstable arithmetic support

The sufficient condition $\theta_p \rightarrow 0$ in Corollary 5.1 cannot be discarded simply because the decisions are independent. We construct a perturbation of 1 whose support is fixed before the random signs are selected. The crucial order of choices is possible because, after finitely many support blocks have been chosen, only finitely many past sign configurations are possible. A common recovery scale can be chosen for all of them, without observing the configuration eventually drawn.

Theorem 9.1 (Fair choices can retain logarithmic instability). *For every $125/164 < \sigma \leq 1$, $\varepsilon > 0$ and M_0 , there is a deterministic prime set*

$$\Gamma \subset \{p > M_0 : p \equiv 1 \pmod{4}\}, \quad \sum_{p \in \Gamma} p^{-\sigma} < \varepsilon,$$

such that independent activations $\theta_p = \frac{1}{2} \mathbf{1}_{\{p \in \Gamma\}}$ produce, almost surely, a function h_E for which $L_{h_E}(N)$ does not converge. The prefix is unchanged and the cost bound holds for every realization.

Proof. Use Lemmas 8.7 and 8.3, with the fixed parameters (8.14), and put $c_0 = c_{\log}$. In particular $\beta = 9/10$ and the reservoir mass eventually exceeds c_0 . Set $N_0 \geq \max(2, M_0)$. Fix $0 < \varepsilon_0 < \min(\varepsilon, c_0/8)$. We recursively choose disjoint finite prime blocks Γ_j , of cost less than $\varepsilon_0 2^{-j}$, without observing any random choices.

Suppose $\Gamma_1, \dots, \Gamma_{j-1}$ are fixed. There are finitely many possible subsets F of their union. By (2.2), each $L_{h_F}(N)$ tends to

$$C_F = \prod_{p \in F} \left(1 - \frac{4}{p+1}\right) \geq 1 - 4\varepsilon_0.$$

Choose a common recovery index $R_j > N_{j-1}$ such that all these finitely many averages are at least $1 - 4\varepsilon_0 - 1/j$ at R_j . Then choose $N_j > R_j$ so that the reservoir has mass $m_{N_j} \geq c_0$, its total d_σ cost is less than $\varepsilon_0 2^{-j}$, and its least pivot exceeds $R_j^2 + 1$ and M_0 . Let Γ_j consist of all these pivots. All trace values up to R_j remain protected.

Conditional on the choices in previous blocks, the signs of $h_E(n^2 + 1)$ for $n \in J_{N_j}$ are independent fair signs: each site uses one new private pivot. The other sites do not use the new block. Thus, writing T_j for the logarithmic mean at N_j after this block is chosen,

$$\mathbb{E}(T_j \mid \text{previous blocks}) \leq 1 - m_{N_j} \leq 1 - c_0.$$

Changing one pivot affects this mean by at most $2w_n$, where $w_n = 1/(n\mathcal{H}_{N_j})$. Therefore

$$\sum_{n \in J_{N_j}} (2w_n)^2 \leq \frac{4N_j^{-9/10}}{\mathcal{H}_{N_j}},$$

and the classical bounded-differences inequality [7, 3] gives

$$\mathbb{P}\{T_j > 1 - c_0 + 1/j \mid \text{previous blocks}\} \leq \exp\left(-\frac{N_j^{9/10} \mathcal{H}_{N_j}}{2j^2}\right). \quad (9.1)$$

Enlarge N_j further so that this is at most 2^{-j} . This choice uses only the deterministic reservoir and is uniform in all preceding signs.

Every later block is beyond $N_j^2 + 1$, so both T_j and the recovery mean are frozen in the final h_E . Integrating (9.1) over the past and applying Borel–Cantelli gives

$$\limsup_N L_{h_E}(N) \geq 1 - 4\varepsilon_0, \quad \liminf_N L_{h_E}(N) \leq 1 - c_0 \quad \text{almost surely.}$$

The gap is at least $c_0 - 4\varepsilon_0 > c_0/2$. Finally $E \subset \Gamma = \bigcup_j \Gamma_j$, so the cost and prefix bounds hold for every outcome. $\square$

The arithmetic concentration is in the predetermined support Γ . Independent fair choices on that support do not erase it. There is no contradiction with Corollary 5.1: the probabilities equal $1/2$ on infinitely many support primes and do not tend to zero. This theorem uses the analytic input of Section 8; the exact criterion and the vanishing-probability corollary do not.

The proof gives oscillation greater than $c_{\log}/2$ almost surely, independent of the chosen σ within this range, while the deterministic support may depend on σ . In the subrange $128/153 < \sigma \leq 1$, the original quantified route remains available: use (8.7) and $c_0 = 1/2000$ in the same proof. Thus its numerical gap greater than $1/4000$ is retained without assigning that value to the whole enlarged range.

10 A support separating ordinary and logarithmic stability

The two criteria in Theorem 1.1 have different arithmetic realizations. We construct one support whose tail is harmless logarithmically for every fixed polynomial and every assignment, but which retains a macroscopic ordinary reservoir on each of the three quadratic curves. Its first occurrences are delayed, and the scales are so far apart that their logarithmic contribution is negligible.

In this section write $A_{d,f}^c(N)$ and $A_{d,f}^\ell(N)$ for the two means on the positive values of $P_d(n) = n^2 - d$, $d \in \mathcal{D}$, normalized by N and $\mathcal{H}_N$. Put $h_0(n) = (-1)^{v_2(n)}$. This choice supplies the common recovery value: its local factor at two is zero on each of the three curves, and finite changes at odd primes preserve that zero factor. The logarithmic limits below therefore require no conjectural cancellation of the source.

Theorem 10.1 (One support, three ordinary divergences). *For every $\varepsilon > 0$ and $M_0 \geq 2$, there is a deterministic prime support $\Gamma \subset (M_0, \infty)$ with $\sum_{p \in \Gamma} 1/p < \varepsilon$ such that:*

- (i) *for every fixed nonzero integer polynomial P , logarithmic prime truncation is uniform over $A \subseteq \Gamma$ and all N , and $M_{P,h_A}^\ell(N) \rightarrow \ell_P(A)$ uniformly in A ;*
- (ii) *$\sup_{A \subseteq \Gamma} |A_{d,h_0 h_A}^\ell(N)| \rightarrow 0$ for every $d \in \mathcal{D}$;*
- (iii) *one fixed subset $A \subseteq \Gamma$ satisfies $\text{osc}_N A_{d,h_0 h_A}^c(N) \geq 1/8$ for every $d \in \mathcal{D}$;*
- (iv) *on each curve, $\limsup_N u_{P_d}(N) \geq 1/8$ and $\sum_{p \in \Gamma} c_{P_d}(p) = \infty$.*

The support and all recovery scales are chosen before the subset in (iii). The excursions for different curves need not occur at the same scales.

10.1 First occurrences and an elementary three-curve reservoir

If $r_P(p) \leq N$, then the root classes modulo p give

$$\sum_{\substack{n \leq N \\ P(n) \neq 0, p|P(n)}} \frac{1}{n} \leq \rho_P(p) \left(\frac{\mathcal{H}_N}{p} + \frac{1}{r_P(p)} \right). \quad (10.1)$$

Indeed, the first nonzero occurrence in each class is at least $r_P(p)$; subsequent occurrences are separated by at least p . Their reciprocals after the first sum to at most $\mathcal{H}_N/p$. If the first occurrence is beyond N or absent, the sum is zero.

For each fixed $B \geq 4$ and $d \in \mathcal{D}$, the elementary product argument used in Lemma 7.1 gives

$$\#\{n \leq N : P_d(n) > 0, P^+(P_d(n)) > BN\} \geq (1/2 - o_{B,d}(1))N. \quad (10.2)$$

Here is the calculation for all three curves. The logarithm of the product of their positive values, on any one curve, is $2N \log N + O_d(N)$. With $Y = BN$ and $V = N^2 + 2$, the contribution of primes at most Y is at most

$$N \sum_{\substack{p \leq Y \\ p \text{ odd}}} \frac{\rho_d(p) \log p}{p-1} + 2\pi(Y) \log V + O(N), \quad \rho_d(p) = \#\{r \bmod p : r^2 = d\}.$$

The odd roots are simple, and the valuation at two is at most one. The split primes occupy half the reduced classes modulo four or eight. The prime number theorem in these fixed progressions makes the sum $(1 + o(1)) \log Y$, while the boundary term is $O_B(N)$. Thus at least $(1 - o(1))N \log N$ remains above BN . Each value contributes at most $(2 + o(1)) \log N$, proving (10.2). A prime $q > BN > 2N$ is simple and private on that curve: a second incidence forces $q \mid (n - m)(n + m)$ and hence $n = m$. This is the same private-prime mechanism as in [1, Lemma 2.1], with a cutoff fixed separately at each stage.

10.2 Constructing the support and common recoveries

Let $\mathcal{F}_j$ be increasing finite families of nonzero integer polynomials, containing the three P_d from the start and exhausting all such polynomials. Define

$$a_j = \sum_{P \in \mathcal{F}_j} (\deg P + 1), \quad \delta_j = (64a_j)^{-1},$$

and cycle the target d_j through $-1, -2, 2$. The symbol δ_j here is an exclusion parameter, not the counting exponent of a support. Fix, before choosing the size of stage j ,

$$B_j \geq \max(4, 2^{j+4}/\varepsilon).$$

Suppose the earlier blocks and recovery indices have been fixed. Choose N_j so large that (10.2) supplies at least $N_j/3$ candidates for B_j, d_j , that N_j exceeds the active coefficient sums, and that $T_j = \lfloor \delta_j N_j \rfloor$ satisfies

$$T_j \geq \delta_j N_j / 2, \quad T_j > T_{j-1}, \quad \log T_j \geq j \left(1 + \sum_{i \leq j} \delta_i^{-1} \right), \quad (10.3)$$

$$B_j N_j > \max(M_0, M_{j-1}^2 + 2, \max \Gamma^{(j-1)}). \quad (10.4)$$

Start with recovery index $M_0, \Gamma^{(0)} = \emptyset$, and $\max \emptyset = 1$. Each constraint involves data fixed before N_j , so all can be met.

Remove candidates whose index is at most $N_j/16$, and candidates dividing any nonzero $P(m)$ with $P \in \mathcal{F}_j, m \leq T_j$. The logarithm of the latter product is at most $a_j T_j \log N_j$. Since every candidate prime exceeds N_j and the candidates are distinct, this second removal discards at most $a_j T_j \leq N_j/64$ candidates. More than $N_j/4$ remain for large N_j . Keep the first $k_j = \lfloor N_j/8 \rfloor$ in index order and call their set Γ_j . Then

$$\Gamma_j \subset (B_j N_j, N_j^2 + 2], \quad r_P(q) > T_j \ (P \in \mathcal{F}_j, q \in \Gamma_j), \quad \sum_{q \in \Gamma_j} \frac{1}{q} < \frac{1}{8B_j}. \quad (10.5)$$

Put $\Gamma^{(j)} = \bigcup_{i \leq j} \Gamma_i$. Every finite subset $F \subseteq \Gamma^{(j)}$ has all three means of $h_0 h_F$ tending to zero, by the zero factor at two and the Chinese remainder theorem. There are finitely many such subsets. Choose a *common* $M_j > N_j$ with

$$\max_{F \subseteq \Gamma^{(j)}} \max_{d \in \mathcal{D}} |A_{d, h_0 h_F}^c(M_j)| \leq 1/j. \quad (10.6)$$

This completes the stage without choosing any eventual negative subset. The union $\Gamma = \bigcup_j \Gamma_j$ is deterministic, avoids the prefix, and has harmonic cost less than ε by (10.5).

10.3 All assignments are stable logarithmically

Fix a polynomial P , active from stage j_0 onward, and take y beyond its exceptional reduction primes. Equation (10.1) and a union give

$$B_{P,N}^\ell(\Gamma; y) \leq D \sum_{q \in \Gamma, q > y} \frac{1}{q} + \frac{D}{\mathcal{H}_N} \sum_{\substack{q \in \Gamma, q > y \\ r_P(q) \leq N}} \frac{1}{r_P(q)}, \quad D = \max(1, \deg P). \quad (10.7)$$

The finitely many blocks before j_0 contribute at most a finite constant C_P to the second numerator. For $i \geq j_0$,

$$\sum_{q \in \Gamma_i} \frac{1}{r_P(q)} \leq \frac{k_i}{T_i} \leq \frac{1}{4\delta_i},$$

with absent occurrences omitted. If $T_j \leq N < T_{j+1}$, no active block after j can yet occur. Since $\mathcal{H}_N \geq \log(N+1) \geq \log T_j$, (10.3) implies

$$\frac{1}{\mathcal{H}_N} \sum_{\substack{q \in \Gamma \\ r_P(q) \leq N}} \frac{1}{r_P(q)} \leq \frac{C_P}{\mathcal{H}_N} + \frac{\sum_{i \leq j} \delta_i^{-1}}{4 \log T_j} \leq \frac{C_P}{\mathcal{H}_N} + \frac{1}{4j} \longrightarrow 0. \quad (10.8)$$

Allowing old primes which have not yet appeared only enlarges this upper bound. Taking $N \rightarrow \infty$ and then $y \rightarrow \infty$ in (10.7) gives zero coverage defect. Theorem 1.1 proves (i), including the supremum over every sample size. The same argument can also be read directly as uniform tail control, without the converse of that theorem.

For $h_0 h_A$, the zero finite-prime limits explained at the start of this section pass through the uniform logarithmic approximation: h_0 cancels in disagreement. This proves (ii), even for a moving choice $A = A_N$.

At stage j targeting d , each of its k_j primes is larger than N_j and has odd valuation at one distinct target observation. Hence

$$u_{P_d}(N_j) \geq k_j/N_j, \quad \sum_{q \in \Gamma_j} c_{P_d}(q) \geq k_j/N_j \longrightarrow 1/8.$$

Every target recurs infinitely often and the blocks are disjoint. This proves (iv). In particular, the summable-influence condition of Theorem 6.2 does not cover this logarithmically stable support.

10.4 Choosing the single ordinarily divergent assignment

Only now choose the signs on the fixed blocks. Each $q \in \Gamma_j$ is private on the target curve up to N_j , and the cofactor of its selected value contains no other prime of the block. We may therefore make all k_j selected signs of $h_0 h_A$ positive, or make them all negative. The two target means differ by exactly $2k_j/N_j$, while their complements on that curve agree. One branch has absolute mean at least k_j/N_j ; keep it.

This step need not preserve the other two current curves. The next stage, however, has all pivots beyond $M_j^2 + 2$ by (10.4), so it preserves all three earlier samples up to M_j . Every chosen excursion and every common recovery is final. The increasing union of negative choices defines one fixed $A \subseteq \Gamma$ such that, for each d ,

$$\limsup_N |A_{d, h_0 h_A}^c(N)| \geq 1/8, \quad A_{d, h_0 h_A}^c(M_j) \longrightarrow 0.$$

A bounded real sequence with zero as a cluster value and excursions of magnitude at least $1/8$ has oscillation at least $1/8$. This proves (iii) and completes Theorem 10.1.

The result does not say that logarithmic weighting repairs every support: Section 8 already supplies supports on which it does not. Nor are the three traces almost always positive in the construction. The same fixed function has three divergent ordinary means and three zero logarithmic means; neither normality of words nor a Poisson law is asserted.

11 Preserving vanishing affine correlations

Let $\mathcal{L}_\pm \subset \mathcal{M}_\pm$ consist of the functions for which

$$\frac{1}{X} \sum_{n \leq X} \prod_{i=1}^m f(A_i n + B_i) \longrightarrow 0 \quad (11.1)$$

for every fixed $m \geq 1$, $A_i \geq 1$, $B_i \in \mathbb{Z}$, with $A_i B_j - A_j B_i \neq 0$ for $i \neq j$. A finite prefix with nonpositive arguments is omitted. These quantifiers include all fixed changes of progression; they are stronger than assuming the vanishing of one specified correlation.

The proof below is a periodic-multiplier transfer. The use of periodic approximation followed by passage to arithmetic progressions is already present in [4, Section 8, especially (8.3)–(8.7)]. We include the explicit harmonic tail estimate for the full affine family used here.

Proposition 11.1 (Harmonic changes preserve the full affine property). *If $d_1(f, g) < \infty$, then $f \in \mathcal{L}_\pm$ if and only if $g \in \mathcal{L}_\pm$.*

Proof. Write $g = fu$ with $u = h_E$ and $\sum_{p \in E} 1/p < \infty$. For a finite $F \subset E$ and $r \geq 1$, approximate u by the periodic function $u_{F,r}$ of (2.1). Include in F every prime of E dividing some A_i . For the remaining primes, $p \mid A_i n + B_i$ occupies at most $X/p + O(1)$ indices. Since summability implies $\#(E \cap [1, CX]) = o(X)$ for each fixed C , the infinitely many possible boundary errors collectively contribute $o(X)$, not a new positive density. For $p \in F$, the congruence modulo p^r occupies at most $\gcd(A_i, p^r)X/p^r + O(1) \leq A_i X/p^r + O(1)$ indices. Hence

$$\limsup_X \frac{1}{X} \sum_{n \leq X} |u(A_i n + B_i) - u_{F,r}(A_i n + B_i)| \leq 2 \sum_{p \in E \setminus F} \frac{1}{p} + 2A_i \sum_{p \in F} p^{-r}. \quad (11.2)$$

The product of the $u_{F,r}$ factors is periodic in n . On any residue class of its period, the affine forms remain non-proportional, since their determinants are multiplied by the common progression step. Under $f \in \mathcal{L}_\pm$, their product has mean zero on each class. A telescoping inequality for products bounds the error by the sum of (11.2). First send $r \rightarrow \infty$, then $F \uparrow E$. This proves $g \in \mathcal{L}_\pm$. Symmetry proves the converse. $\square$

This proves the remaining assertion of Theorem 1.3, since $d_1 \leq d_\sigma$ in its range. It is a stability of an assumed full family of cancellations, not a proof of those cancellations for a prescribed source.

Corollary 11.2 (Affine normality does not ensure a quadratic mean). *There exist fixed functions in $\mathcal{L}_\pm$ whose logarithmic mean on $n^2 + 1$ does not exist. More precisely, for each $125/164 < \sigma \leq 1$, every d_σ ball around a member of $\mathcal{L}_\pm$ contains such a function, with any prescribed compatible finite prefix. Its fixed binary word frequencies are all 2^{-m} .*

Proof. The class is nonempty by Klurman–Mangerel–Teräväinen [4, Theorem 2.1 in the arXiv version]: for a prime set E_0 of relative prime density zero and divergent reciprocal sum, $(-1)^{\Omega_{E_0}(n)}$ has every affine correlation in (11.1) equal to zero in the limit. Such sets exist, for example by independently retaining each sufficiently large prime with probability $1/\log \log p$: partial summation and the prime number theorem give divergent expected reciprocal mass; the centered reciprocal series has summable variances, while the relative prime density tends to zero almost surely. Fix a realization. The published theorem covers positive shifts; translating the index treats any finite family of integer shifts in (11.1).

Apply Theorem 1.3(ii) and Proposition 11.1. To obtain word frequencies, expand the indicator of any prescribed word as a product of $(1 + s_i f(n + i))/2$. Every nonconstant term is a correlation from (11.1), and therefore vanishes. Only the constant term 2^{-m} remains. $\square$

12 Thin rectangles and primitive directions

The one-dimensional counterexample is not removed merely by allowing a second index to tend to infinity. We now retain the ratio of the side lengths in the positive incidence estimate and construct a matching failure mechanism in a thin regime.

Let $\mathcal{D} = \{-1, -2, 2\}$ and, for integers $1 \leq H \leq X$, set

$$Q_d(a, b) = a^2 - db^2, \quad A_{d,f}(X, H) = \frac{1}{XH} \sum_{a=1}^X \sum_{b=1}^H f(|Q_d(a, b)|). \quad (12.1)$$

These forms have no zeros on the positive integer quadrant. The third may have absolute value 1; our convention $P^+(1) = 1$ excludes those points from the reservoir below, and their logarithmic contribution is zero. The three sources of observation are $a^2 + b^2$, $a^2 + 2b^2$ and $|a^2 - 2b^2|$.

Theorem 12.1 (One perturbation, three thin-rectangle instabilities). *Let $H(X)$ be a deterministic positive integer sequence with $H(X) \rightarrow \infty$ and $H(X) = o(\log X)$. For every $f \in \mathcal{M}_\pm$, every $\varepsilon > 0$ and every M_0 , there is one $g \in \mathcal{M}_\pm$ with $d_1(f, g) < \varepsilon$ and $g(n) = f(n)$ for $n \leq M_0$ such that, for every $d \in \mathcal{D}$,*

$$\text{osc}_X A_{d,g}(X, H(X)) \geq \frac{1}{2}. \quad (12.2)$$

The witnessing subsequences may depend on d , and g may depend on the prescribed width H . No one perturbation is asserted to work for all widths simultaneously.

If $0 < \varepsilon < 1/4$, a perturbation $h = h_E$ of 1 can be chosen with $2 \notin E$, the same prefix and budget, and

$$\limsup_X A_{d,h}(X, H(X)) \geq C_d > 1 - 4\varepsilon, \quad \liminf_X A_{d,h}(X, H(X)) \leq 0, \quad (12.3)$$

$$\lim_{X \rightarrow \infty} A_{d,h}(X, X) = C_d, \quad C_d = \prod_{\substack{p \in E \\ (d/p)=1}} \left(\frac{p-1}{p+1} \right)^2. \quad (12.4)$$

Its mean on ordinary integers is greater than $1 - 2\varepsilon$.

No inverse theorem for three norms is used or concluded. In particular, the theorem does not construct two sources whose three thin-rectangle disagreement proportions all tend to zero.

For $d = -1$, the square limit in (12.4) is a special case of Donoso–Le–Moreira–Sun [16, Theorem A], valid there for every bounded real completely multiplicative function. Its role in this theorem is the coexistence with thin-rectangle divergence. We retain the local proof because its explicit dependence on H is needed below.

12.1 An anisotropic upper bound

The estimate has an area term $1/p$ and a boundary term $1/(H\sqrt{p})$, arising from separation of lattice points at scale $\sqrt{p}$. On a square the latter can be absorbed into the harmonic term throughout the contributing prime range. A thin rectangle retains this extra cost; the number of variables alone does not determine stability.

Reciprocal-norm comparison on fixed dilated Gaussian domains already appears in [16, Lemmas 4.1 and 4.3]; square-incidence estimates for binary forms also appear in [5, Lemma 4.15]. The next specialized proof keeps the aspect ratio instead of absorbing its boundary term.

Lemma 12.2 (Rectangle incidence). *For $d \in \mathcal{D}$, every prime p , and $1 \leq H \leq X$,*

$$\frac{1}{XH} \#\{1 \leq a \leq X, 1 \leq b \leq H : p \mid Q_d(a, b)\} \leq 12 \left(\frac{1}{p} + \frac{1}{H\sqrt{p}} \right). \quad (12.5)$$

Consequently, for $0 \leq \gamma \leq 1$ and $H = \lceil X^\gamma \rceil$,

$$\sup_X \frac{1}{XH} \#\{(a, b) : f(|Q_d(a, b)|) \neq g(|Q_d(a, b)|)\} \leq 36d_{(1+\gamma)/2}(f, g). \quad (12.6)$$

Proof. The zero set of Q_d modulo p is the union of at most two linear subspaces, allowing the zero-dimensional subspace in the inert case. This statement includes the prime 2 by direct inspection. Their inverse images in $\mathbb{Z}^2$ are lattices. Every nonzero vector v in one of these lattices satisfies

$$0 < |Q_d(v)| \leq 3\|v\|_\infty^2, \quad p \mid Q_d(v),$$

since d is not a rational square. Distinct lattice points are therefore separated by at least $\ell = \sqrt{p/3}$ in sup norm. Squares of side ℓ centered at the points have disjoint interiors and lie in a rectangle of sides $X + \ell$ and $H + \ell$. For the union of at most two lattices, division by XH gives

$$\frac{6}{p} + \frac{2\sqrt{3}}{X\sqrt{p}} + \frac{2\sqrt{3}}{H\sqrt{p}} + \frac{2}{XH}.$$

If there is a contributing point, $p \leq X^2 + 2H^2 \leq 3X^2$, so the last term is at most $2\sqrt{3}/(H\sqrt{p})$. Use $H \leq X$ and $6\sqrt{3} < 12$ to get (12.5). When there is no point the inequality is immediate.

A disagreement requires a changed prime dividing the form. In the union bound only primes $p \leq 3X^2$ occur. For those primes,

$$\frac{1}{H\sqrt{p}} \leq X^{-\gamma} p^{-1/2} \leq 3^{\gamma/2} p^{-(1+\gamma)/2}.$$

Also $p^{-1} \leq p^{-(1+\gamma)/2}$. Since $12(1 + \sqrt{3}) < 36$, this proves (12.6). $\square$

For a fixed changed-prime set E , a finite subset F , and a prescribed width, (12.5) gives a more informative tail bound:

$$\text{disagreement proportion} \leq 12 \sum_{p \in E \setminus F} \frac{1}{p} + \frac{12}{H(X)} \sum_{\substack{p \in E \setminus F \\ p \leq 3X^2}} p^{-1/2}. \quad (12.7)$$

Under $\sum_{p \in E} 1/p < \infty$, the conditions

$$H(X) \rightarrow \infty, \quad \frac{1}{H(X)} \sum_{\substack{p \in E \\ p \leq 3X^2}} p^{-1/2} \rightarrow 0 \quad (12.8)$$

are sufficient for vanishing truncation error. They are not asserted necessary. Without using the sparsity of E , Chebyshev's estimate and partial summation give $\sum_{p \leq 3X^2} p^{-1/2} \ll X/\log X$, so $H(X) \log X/X \rightarrow \infty$ is one sufficient regime.

For $H = X$, each term in (12.5) is $O(1/p)$ on the contributing range. Thus harmonic summability alone gives uniform tail control on squares. More generally, finite $d_{(1+\gamma)/2}(1, h)$ suffices for convergence along $H = \lceil X^\gamma \rceil$ when $\gamma > 0$, because both side lengths tend to infinity and finite-prime laws converge. At $\gamma = 0$ the second coordinate is fixed; the local factors are those of a one-variable observation and need not be the binary factors in (12.4).

12.2 A logarithmic-product reservoir on growing rows

The auxiliary scale B must meet two requirements: $H/B \rightarrow 0$ makes the programming cheap, while $B = o(\log X)$ makes the small-prime boundary error negligible. The hypothesis $H = o(\log X)$ allows both; it is a sufficient regime for this construction, not a claimed optimal transition.

Write $H = H(X)$ and define

$$\begin{aligned} T &= \max(1, \lfloor \log_2 X \rfloor), & B &= 3H + \lceil \sqrt{HT} \rceil, & Y &= XB, \\ I_d &= \{(a, b) \in [1, X] \times [1, H] : P^+(|Q_d(a, b)|) > Y\}. \end{aligned} \quad (12.9)$$

Since $H = o(\log X)$, we have $B = o(\log X)$, $H/B \rightarrow 0$, $Y > 2XH$ and $Y^2 > X^2 + 2H^2$ for large X .

Lemma 12.3 (Large-prime mass on a thin rectangle). *Along the prescribed width, uniformly for the three $d \in \mathcal{D}$,*

$$|I_d| \geq \left(\frac{1}{2} - o(1)\right) XH. \quad (12.10)$$

Proof. First,

$$\sum_{a \leq X, b \leq H} \log |a^2 - db^2| = (2 + o(1)) XH \log X. \quad (12.11)$$

For $d < 0$, use $a^2 \leq a^2 - db^2 \leq 3X^2$ and $\log(X!) = X \log X + O(X)$. For $d = 2$, restrict the lower bound to $a > 2H$, where $a^2 - 2b^2 \geq a^2/2$. The omitted $O(H^2)$ terms do not affect the leading order. The upper bound is immediate.

Let p be odd and split, that is $(d/p) = 1$, and fix a row b . Put $t = v_p(b)$ and $M = X^2 + 2H^2$. For $k \leq 2t$, the condition $p^k \mid Q_d(a, b)$ is equivalent to $p^{\lceil k/2 \rceil} \mid a$. For $k > 2t$ it consists of two residue classes for a modulo p^{k-t} , by lifting the two simple roots after removing p^{2t} . Summing all exponents, with their boundary errors, yields

$$\sum_{a \leq X} v_p(|Q_d(a, b)|) \leq 2X \sum_{j=1}^t p^{-j} + 2X \sum_{j>t} p^{-j} + \frac{2 \log M}{\log p} = \frac{2X}{p-1} + \frac{2 \log M}{\log p}. \quad (12.12)$$

Thus rows divisible by p have not been discarded.

If $(d/p) = -1$, then $v_p(|Q_d(a, b)|) = 2 \min(v_p(a), v_p(b))$. The sum of the logarithmic contributions of all such primes is bounded by

$$2XH \sum_p \frac{\log p}{p^2 - 1} = O(XH).$$

At 2, for all three forms, the valuation is at most $2 \min(v_2(a), v_2(b)) + 1$, giving the same order of contribution. By (12.12), all primes at most Y contribute at most

$$2XH \sum_{\substack{p \leq Y, p \text{ odd} \\ (d/p)=1}} \frac{\log p}{p-1} + 2H\pi(Y) \log M + O(XH). \quad (12.13)$$

The prime number theorem in fixed progressions modulo 4 or 8 gives $(1/2 + o(1)) \log Y$ for the sum. Since $\log Y \sim \log X$ and $\pi(Y) \ll Y/\log Y$, the second term is $O(XHB) = o(XH \log X)$. Thus (12.13) is at most $(1 + o(1))XH \log X$.

By (12.11), at least $(1 - o(1))XH \log X$ remains above Y . Each value has at most one such prime, to exponent one, contributing at most $\log M = (2 + o(1)) \log X$. Division proves (12.10). $\square$

12.3 Private directions rather than private points

For example, in $[1, 100] \times [1, 2]$ the prime $1201 > 2 \cdot 100 \cdot 2$ divides both

$$49^2 + 1 = 2 \cdot 1201, \quad 98^2 + 2^2 = 8 \cdot 1201.$$

The two points lie on the single primitive direction $(49, 1)$. A claim of pointwise privacy would be false here.

Their multiplicative signs agree because $f(k^2) = 1$. The correct unit of programming is therefore a primitive direction, with the number of its points used as a weight. The next lemma shows that a sufficiently large pivot meets only one such direction.

Lemma 12.4 (The primitive-direction quotient). *If $p > 2XH$ divides both $Q_d(a, b)$ and $Q_d(c, e)$ for two positive points in the rectangle, then $ae = bc$. Consequently, every pivot in (12.9) meets a single primitive direction. All points on that direction have the same multiplicative sign, and their multiplicities can be treated as weights.*

Proof. The two divisibilities imply

$$p \mid a^2e^2 - b^2c^2 = (ae - bc)(ae + bc).$$

Here $|ae - bc| < p$ and $0 < ae + bc \leq 2XH < p$, hence $ae = bc$. Write the direction as (a_0, b_0) with $\gcd(a_0, b_0) = 1$. Its points are $k(a_0, b_0)$, with $k \leq H < p$. Thus the pivot divides no multiplier k , and

$$f(|Q_d(ka_0, kb_0)|) = f(k^2)f(|Q_d(a_0, b_0)|) = f(|Q_d(a_0, b_0)|).$$

The pivot has exponent one throughout the direction because it does at any reservoir point. Every point of that direction in the rectangle belongs to I_d . $\square$

Assign one sign to each direction meeting I_d and change its pivot if needed. Lemma 12.4 ensures that this programming affects no other direction, and all points outside I_d are unchanged. Each prime is counted once, so the cost is at most

$$\frac{XH}{Y} = \frac{H}{B} \longrightarrow 0. \quad (12.14)$$

The all-positive and all-negative prescriptions have means separated by $2|I_d|/(XH) \geq 1 - o(1)$. They need not preserve the other two forms at the current scale; only all three forms at earlier frozen scales must be protected.

12.4 A common source and the square–rectangle separation

We prove Theorem 12.1. Choose all scales beyond a fixed starting point where $H(X) \leq X$. Cycle through the targets $d_j = -1, -2, 2, -1, \dots$. Starting from $f_0 = f$ and a sufficiently large $X_0 \geq \max(2, M_0)$, choose $X_j > X_{j-1}$ so large that

$$Y_j > \max(M_0, 3X_{j-1}^2), \quad H(X_j)/B_j < \varepsilon 2^{-j-1}, \quad |I_{d_j}|/(X_j H(X_j)) > \frac{1}{2} - \frac{1}{j+2}.$$

The two constant-direction prescriptions both leave all three previous rectangles unchanged, since the new primes exceed every previous polynomial height. Select the branch whose current d_j -mean is furthest from that target's mean at X_{j-1} . The distance is at least $1/2 - 1/(j+2)$. The prime coordinates stabilize, the costs are summable, and the final g preserves the prefix. For each target infinitely many late pairs of frozen means are separated by $1/2 - o(1)$, proving (12.2).

For the near-constant refinement, first observe that for any finite odd prime set F , as both sides tend to infinity the finite-prime law is

$$A_{d,h_F}(X, H(X)) \longrightarrow C_{d,F} = \prod_{\substack{p \in F \\ (d/p)=1}} \left(\frac{p-1}{p+1} \right)^2. \quad (12.15)$$

At an inert prime the valuation is even. At a split odd prime the two linear factors $a - rb$ and $a + rb$ are independent Haar-uniform coordinates under the invertible linear change over $\mathbb{Z}_p$. Each has mean parity sign $(p-1)/(p+1)$. The Chinese remainder theorem and finite-prime valuation truncation prove (12.15) for arbitrary diverging side lengths.

Start from $F_0 = \emptyset$. Before stage j , choose a recovery scale

$$R_j > X_{j-1}, \quad R_j \geq M_0, \quad H(R_j) \leq R_j,$$

where the three means of $h_{F_{j-1}}$ are within 2^{-j} of their products. This is possible by (12.15), since both side lengths tend to infinity along the prescribed width. Choose the attack scale $X_j > R_j$, again with $H(X_j) \leq X_j$, so that

$$Y_j > \max(M_0, 3R_j^2), \quad \frac{H(X_j)}{B_j} < \varepsilon 2^{-j-1}, \quad \frac{|I_{d_j}|}{X_j H(X_j)} > \frac{1}{2} - \frac{1}{j+2}.$$

Force all reservoir directions of target d_j to be negative. The pivots are new and positive before the change, so this adds only the necessary primes to form $F_j \supset F_{j-1}$. Every previous recovery and attack has polynomial height at most $3R_j^2$ and is protected. In particular, $R_j \rightarrow \infty$ and later changes preserve these observations even when H is not monotone.

For $E = \bigcup_j F_j$, the products converge and

$$C_d \geq 1 - \sum_{p \in E} \frac{4p}{(p+1)^2} > 1 - 4\varepsilon.$$

The recovery subsequences tend to these products, while the attack subsequences for each target have means at most $1 - 2|I_d|/(X_j H(X_j)) = o(1)$. This gives (12.3). On squares, Lemma 12.2 bounds the infinite tail by a constant times $\sum_{p \in E \setminus F} 1/p$, uniformly in X . Therefore (12.15) extends from finite F to E , proving (12.4). On the ordinary integers, (7.10) gives a mean greater than $1 - 2\varepsilon$.

Remark 12.5 (Why the width restriction occurs). The construction needs $B \gg H$ so that its cost H/B tends to zero, and $B = o(\log X)$ so that the small-prime boundary error is negligible. These requirements explain $H = o(\log X)$ in this proof. They do not identify the optimal stability threshold. No logarithmic averaging assertion for these rectangles is made. If H is irregular, no interval description of the cluster set is asserted either.

13 Collective upper bounds for nearly maximal factors

The individual estimate (12.5) pays a boundary term for every prime. Near the height of the polynomial this can be a substantial overcount. Instead, a value with a nearly maximal prime factor has a small cofactor. We sieve all values with the same cofactor together and then sum over those cofactors. The same denominator estimate works for a curve, a translated interval of that curve, and a rectangle; only the remainder of the congruence count changes.

Four quantities have different jobs. The height of the observed values bounds the possible cofactors. The interval length, or the short side of a rectangle, determines how far the sieve can run while its remainder stays small. The root count fixes the local sieve density. Grouping by cofactor then controls the very large primes collectively, instead of summing a boundary error for each of them. We keep these roles separate in the two congruence counts below.

13.1 Root counts and the two sieve densities

For $d \in \mathcal{D}$, let $\rho_d(m) = \#\{r \bmod m : r^2 \equiv d \pmod{m}\}$. For fixed m the sieve primes are

$$\mathcal{P}_d(m) = \{\ell \geq 3 : \ell \text{ prime}, (d/\ell) = 1, \ell \nmid m\}.$$

We use either of the densities

$$g_1(\ell) = \frac{2}{\ell}, \quad g_2(\ell) = \frac{2\ell - 1}{\ell^2}. \quad (13.1)$$

The first counts roots on a line. The second counts the union of two distinct lines in the residue plane. The subscript is a dimension indicator only within this section.

For $i = 1, 2$, extend g_i and $w_i(\ell) = g_i(\ell)/(1 - g_i(\ell))$ multiplicatively to squarefree integers supported on $\mathcal{P}_d(m)$. All sums in the following formulas run only over these allowed integers. Set

$$G_{i,m}(z) = \sum_{l \leq z} w_i(l), \quad \kappa_{i,d}(m) = \prod_{\substack{\ell \mid m, \ell \geq 3 \\ (d/\ell)=1}} (1 - g_i(\ell))^{-1}.$$

The Selberg weights in this normalization are

$$\lambda_l = \mu(l) \prod_{\ell \mid l} (1 - g_i(\ell))^{-1} \frac{\sum_{\substack{t \leq z/l \\ (t,l)=1}} w_i(t)}{G_{i,m}(z)}. \quad (13.2)$$

They are zero outside the allowed integers and outside $l \leq z$. The classical diagonalization of the Selberg quadratic form gives

$$\lambda_1 = 1, \quad \sum_{l_1, l_2 \leq z} \lambda_{l_1} \lambda_{l_2} g_i([l_1, l_2]) = \frac{1}{G_{i,m}(z)}, \quad |\lambda_l| \leq l. \quad (13.3)$$

Here $[l_1, l_2]$ denotes the least common multiple. The first two identities are precisely the upper-sieve weights and their minimum quadratic form [11, Section 13.2, (13.2.3)–(13.2.4), Theorem 13.1]. The coarse last bound also follows directly from (13.2): its numerator sum is at most $G_{i,m}(z)$, and each factor $(1 - g_i(\ell))^{-1}$ is at most ℓ for $\ell \geq 3$.

Lemma 13.1 (A denominator uniform in the cofactor). *For $i = 1, 2$, uniformly in $m, z \geq 2$ and $d \in \mathcal{D}$,*

$$G_{i,m}(z) \gg \frac{\log(2z)}{\kappa_{i,d}(m)}. \quad (13.4)$$

Also, uniformly for $U \geq 1$,

$$\sum_{m \leq U} \rho_d(m) \ll U, \quad \sum_{m \leq U} \rho_d(m) \kappa_{i,d}(m) \ll U, \quad \sum_{m \leq U} \frac{\rho_d(m) \kappa_{i,d}(m)}{m} \ll \log(2U). \quad (13.5)$$

All implied constants are absolute for these two densities and these three forms.

Proof. Choose a small absolute $a > 0$ and put $x = z^a$. Form a squarefree integer R by independently selecting allowed primes $\ell \leq x$ with probabilities $g_i(\ell)$. The classical estimate $\sum_{\ell \leq x} (\log \ell)/\ell \ll \log x$ allows a to be chosen so that $\mathbb{E} \log R \leq \frac{1}{2} \log z$. Thus $\mathbb{P}(R \leq z) \geq 1/2$. Its distribution is proportional to the weights w_i , whence

$$G_{i,m}(z) \geq \frac{1}{2} \prod_{\substack{\ell \leq x \\ \ell \in \mathcal{P}_d(m)}} (1 - g_i(\ell))^{-1}.$$

The split odd primes occupy half the reduced residue classes modulo four or eight. Mertens' formula in these fixed progressions gives a product comparable to $\log x$ before removing primes dividing m [12, Section 27.11]. Indeed $g_i(\ell) = 2/\ell + O(\ell^{-2})$. The removal costs at most $\kappa_{i,d}(m)$. Bounded z are absorbed using $G_{i,m}(z) \geq 1$ and $\kappa_{i,d}(m) \geq 1$. The auxiliary randomness proves a positive sum estimate; no signs of the multiplicative source are randomized.

Let χ_d have discriminant $-4, -8, 8$ for $d = -1, -2, 2$, respectively. The function ρ_d is multiplicative. At a split odd prime it has value two on every positive prime power; at an inert prime it is zero; at two its first value is one and subsequent values are zero. Hence $0 \leq \rho_d(m) \leq (1 * \chi_d)(m)$. Since χ_d is periodic of mean zero,

$$\sum_{m \leq U} \rho_d(m) \leq \sum_{r \leq U} \chi_d(r) \lfloor U/r \rfloor = U \sum_{r \leq U} \frac{\chi_d(r)}{r} + O(U) \ll U.$$

To average the extra factor, write $\kappa_{i,d}(m) = \sum_{s|m} v_i(s)$, where v_i is nonnegative, supported on squarefree split odd integers, and $v_i(\ell) = (1 - g_i(\ell))^{-1} - 1 = O(1/\ell)$. For such s , $\rho_d(st) \leq \rho_d(s)\rho_d(t)$. Therefore

$$\sum_{m \leq U} \rho_d(m) \kappa_{i,d}(m) \ll U \sum_s \frac{v_i(s) \rho_d(s)}{s} \ll U.$$

The last Euler product converges, its factors being $1 + O(\ell^{-2})$. Partial summation proves the third estimate in (13.5). $\square$

13.2 Curves and translated intervals: the length pays the error

Proposition 13.2 (One-variable upper sieve). *Let*

$$L_d(N; Z) = \#\{n \leq N : P_d(n) > 0, P^+(P_d(n)) > Z\}, \quad K = \left\lceil \frac{N^2 + 2}{Z} \right\rceil.$$

For $N \geq 2^{12}$ and $N \leq Z \leq N^2 + 2$,

$$\frac{L_d(N; Z)}{N} \ll \frac{\log(2K)}{\log(2N)} + \frac{K}{\sqrt{N}}. \quad (13.6)$$

There is also a translated form. If $X \geq L \geq 2^{12}$, $X \leq u < 2X$, $I = (u, u + L] \cap \mathbb{Z}$, $3X \leq Z \leq 9X^2 + 2$ and $K = \lceil (9X^2 + 2)/Z \rceil$, then

$$\frac{1}{L} \#\{n \in I : P^+(P_d(n)) > Z\} \ll \frac{\log(2K)}{\log(2L)} + \frac{K}{\sqrt{L}}. \quad (13.7)$$

The constants are absolute and uniform in the origin u .

Proof. Write a counted value as $P_d(n) = mp$, where $p > Z$ is prime and $m \leq K$. The index belongs to one of the $\rho_d(m)$ classes $n \equiv r \pmod{m}$. For squarefree t supported on $\mathcal{P}_d(m)$, the Chinese remainder theorem gives, on any interval of length L ,

$$\#\{n \in I : n \equiv r \pmod{m}, t \mid P_d(n)\} = \frac{L}{m} g_1(t) + R_t, \quad |R_t| \leq 2^{\omega(t)}. \quad (13.8)$$

For prefixes replace I by $[1, N]$ and L by N . Each residue class contributes a floor error at most one, independently of its position. The finite omitted value in the prefix causes no difficulty: we may upper-bound the desired positive-value count by sieving all the residue classes.

If $P_d(n)/m$ is a prime above $Z > z$, then it avoids every sieve prime, all of which are coprime to m . The indicator of such an index is at most $(\sum_{l|P_d(n)} \lambda_l)^2$. Expanding this square and applying (13.3) and (13.8) gives, for each m, r ,

$$\#\{n \in I : n \equiv r \pmod{m}, P_d(n)/m \text{ prime} > Z\} \leq \frac{L}{mG_{1,m}(z)} + z^6. \quad (13.9)$$

There are at most z^2 pairs of weights, each product is at most z^2 , and $2^{\omega([l_1, l_2])} \leq [l_1, l_2] \leq z^2$. This deliberately coarse error is sufficient.

Sum over $m \leq K$ and its roots. Lemma 13.1 bounds the result by

$$\ll L \frac{\log(2K)}{\log(2z)} + Kz^6.$$

Take $z = \lfloor L^{1/12} \rfloor$, so $z \geq 2$, $z < Z$, and divide by L . This proves (13.7); taking $L = N$ proves (13.6). The height affects K whereas the interval length determines the sieve level. $\square$

In particular, for every fixed $0 < \vartheta < 1/4$,

$$\limsup_{N \rightarrow \infty} \max_d \frac{L_d(N; N^{2-\vartheta})}{N} \ll \vartheta, \quad (13.10)$$

with an absolute implied constant independent of ϑ . The assertion is not a zero limit at fixed ϑ . We will send ϑ to zero *after* the limit in N .

13.3 Rectangles: remove common factors before sieving

Proposition 13.3 (A rectangular upper sieve with a free cutoff). *For $X \geq H \geq 2^{16}$ and $H \leq Z \leq 3X^2$, let*

$$L_d^{(2)}(X, H; Z) = \#\{a \leq X, b \leq H : P^+(|Q_d(a, b)|) > Z\}, \quad K = \left\lceil \frac{3X^2}{Z} \right\rceil.$$

Then

$$\frac{L_d^{(2)}(X, H; Z)}{XH} \ll \frac{\log(2K)}{\log(2H)} + \frac{K}{H^{5/8}} + \frac{K^2}{XH^{1/2}}. \quad (13.11)$$

The implied constant is absolute for the three forms.

Proof. Write $(a, b) = g(u, v)$ with $(u, v) = 1$. A prime $p > Z \geq H$ cannot divide $g \leq H$. If it divides the norm, then

$$|u^2 - dv^2| = mp, \quad m \leq K/g^2, \quad u \leq X/g, \quad v \leq H/g, \quad g \leq \sqrt{K}. \quad (13.12)$$

Moreover $(v, m) = 1$: a common prime divisor would divide u as well. Thus $u \equiv rv \pmod{m}$ for a root $r^2 \equiv d \pmod{m}$. After this reduction, dropping the primitivity restriction only increases the upper bound.

Fix m, r and a box $1 \leq u \leq U$, $1 \leq v \leq V$. For allowed squarefree t , the number A_t satisfying that root-lattice condition and $t \mid u^2 - dv^2$ has

$$A_t = \frac{UV}{m} g_2(t) + R_t, \quad |R_t| \leq 2^{\omega(t)} (U + V + mt). \quad (13.13)$$

There are $m \prod_{\ell|t} (2\ell - 1)$ classes modulo mt by the Chinese remainder theorem. Each contributes $UV/(m^2 t^2)$ with error at most $U/(mt) + V/(mt) + 1$. This proves (13.13).

Use the density g_2 in (13.2). A point with $|u^2 - dv^2|/m$ prime and greater than $Z > z$ passes the sieve, so its count is at most

$$\frac{UV}{mG_{2,m}(z)} + (U + V)z^6 + mz^8. \quad (13.14)$$

The two errors follow by the same weight-pair count as in (13.9), with the additional factor $mt \leq mz^2$ in the last term.

Apply (13.14) with $U = \lfloor X/g \rfloor$, $V = \lfloor H/g \rfloor$, and sum over the roots, $m \leq K/g^2$, and $g \leq \sqrt{K}$. The three estimates in (13.5) give

$$\begin{aligned} L_d^{(2)}(X, H; Z) &\ll \sum_{g \leq \sqrt{K}} \left\{ \frac{XH \log(2K)}{g^2 \log(2z)} + \frac{K(X+H)}{g^3} z^6 + \frac{K^2}{g^4} z^8 \right\} \\ &\ll XH \frac{\log(2K)}{\log(2z)} + K(X+H)z^6 + K^2 z^8. \end{aligned}$$

Overcounting a point with several eligible prime factors is harmless. Take $z = \lfloor H^{1/16} \rfloor < Z$; then $z^6 \leq H^{3/8}$, $z^8 \leq H^{1/2}$ and $\log(2z) \asymp \log(2H)$. Divide by XH and use $X+H \leq 2X$ to obtain (13.11). $\square$

The sieve bounds are positive upper estimates. They require no theorem asserting that a quadratic polynomial takes prime values. The Gaussian-prime lower bound needed for instability will enter only in Section 15.

14 Critical counting exponents and almost-square rectangles

A convergent critical power sum is sufficient but not necessary for stability. The preceding collective bounds let us place a cutoff just below the full quadratic height. Below it, the number of support primes controls the boundary errors; above it, the sieve controls all observations, independently of the support. These estimates meet even at the critical counting exponent.

For $A_\Gamma(t) = \#(\Gamma \cap [2, t])$ and $0 < s < 1$, the definition (1.16) gives

$$\delta(\Gamma) \leq s \iff \forall \varepsilon > 0, A_\Gamma(t) \ll_{\Gamma, \varepsilon} t^{s+\varepsilon} \iff \forall \varepsilon > 0, \sum_{p \in \Gamma} p^{-s-\varepsilon} < \infty. \quad (14.1)$$

For the last equivalence, use partial summation with a smaller positive exponent in one direction, and $A_\Gamma(t) \leq t^{s+\varepsilon} \sum_\Gamma p^{-s-\varepsilon}$ in the other. In particular $\delta(\Gamma) < 1$ implies harmonic summability. No convergence at exponent s itself is asserted.

14.1 Critical tails on the three curves

Define

$$T_{\Gamma, d}(N; y) = \frac{1}{N} \# \{n \leq N : P_d(n) > 0, \exists p \in \Gamma, p > y, v_p(P_d(n)) \text{ odd}\}.$$

Theorem 14.1 (Critical counting on curves). *If $\delta(\Gamma) \leq 1/2$, then*

$$\lim_{y \rightarrow \infty} \sup_{N \geq 2} \max_{d \in \mathcal{D}} T_{\Gamma, d}(N; y) = 0. \quad (14.2)$$

Uniformly over $A \subseteq \Gamma$ and $d \in \mathcal{D}$, both curve means of h_A converge to $c_d(A)$ in (1.17). The same finite-prime approximation yields the local laws of every fixed finite list of translated curve readings.

Proof. For $y \geq 2$ and $N \leq Z \leq N^2 + 2$, split the coverage at Z . The primes below it have at most two root classes; the remaining observations are counted by L_d . Thus

$$T_{\Gamma,d}(N; y) \leq 2 \sum_{p \in \Gamma, p > y} \frac{1}{p} + \frac{2A_{\Gamma}(Z)}{N} + \frac{L_d(N; Z)}{N}. \quad (14.3)$$

Fix $0 < \vartheta < 1/4$ and take $Z = N^{2-\vartheta}$. With $\varepsilon = \vartheta/8$, the counting assumption gives

$$\frac{A_{\Gamma}(Z)}{N} \ll_{\Gamma, \vartheta} N^{(2-\vartheta)(1/2+\varepsilon)-1} \leq O_{\Gamma, \vartheta}(N^{-\vartheta/4}) \rightarrow 0.$$

By (13.10), with an absolute C independent of ϑ ,

$$\limsup_N \max_d T_{\Gamma,d}(N; y) \leq 2 \sum_{p \in \Gamma, p > y} \frac{1}{p} + C\vartheta.$$

Let $\vartheta \downarrow 0$ after the limit in N , and then let $y \rightarrow \infty$. Harmonic summability follows from (14.1). The finite set of remaining small samples is removed by increasing y beyond their heights, proving the supremum in (14.2).

For a finite prime set, at a visible odd prime the density of an odd valuation is

$$2(1 - p^{-1}) \sum_{j \geq 0} p^{-2j-1} = \frac{2}{p+1},$$

so its factor is $(p-3)/(p+1)$. An invisible odd prime contributes one. At two the parity of the valuation alternates with the parity of the index, giving factor zero when two is in A . Finite-prime valuation truncation and the Chinese remainder theorem therefore give (1.17). The deficits $4/(p+1)$ are summable on Γ , so the local products converge uniformly over subsets. Equation (14.2) bounds the mean error by twice the tail coverage. At each fixed cutoff there are only finitely many assignments. This proves uniform convergence. Abel gives the logarithmic conclusion.

For any fixed translated list, each tail is bounded by its untranslated tail on a slightly larger prefix, with an $O(1/N)$ boundary per fixed shift. A finite union controls the vector. A common periodic truncation gives its joint local law, with shared Haar coordinates at each prime. No assertion of independence of those readings is made. $\square$

The tail estimate is unchanged after multiplying h_A and its truncation by any base f : their disagreement indicators are equal. This uniformity in f is an error comparison, not existence of means for $f(P_d(n))h_A(P_d(n))$.

A quantitative envelope. If $A_{\Gamma}(t) \leq C_0 t^{1/2} (\log(2t))^B$ with $B \geq 0$, take $Z = N^2/(\log N)^{2B+4}$ in (14.3). For large N , $Z \geq N$, the counting term is $O_{B,C_0}((\log N)^{-2})$, and Proposition 13.2 gives

$$T_{\Gamma,d}(N; y) \leq 2 \sum_{p \in \Gamma, p > y} \frac{1}{p} + O_{B,C_0} \left(\frac{\log \log N}{\log N} \right), \quad y \geq 2. \quad (14.4)$$

The rate concerns the prime tail. A rate for convergence to the local product must also pay for the periods of the finite-prime and valuation truncations.

14.2 Fixed supports, common envelopes and a restricted minimum

The constants in a support-specific counting bound are not automatically common to all supports with that exponent. A useful uniform statement requires a single envelope. Let $\Phi(t) \geq 1$ satisfy

$\Phi(t) \ll_{\varepsilon} t^{1/2+\varepsilon}$ for every $\varepsilon > 0$, and consider sources whose negative-prime count is at most $\Phi(t)$ for every $t \geq 2$. The proof of (14.3) is uniform in that class. So are the harmonic tails, by partial summation with any fixed exponent strictly between $1/2$ and one. The local means therefore converge uniformly over this class of possibly moving supports and assignments.

For a fixed prime p , let $m_{p,\Phi}(N)$ be the minimum of the total number of negative readings on the three positive curves, among this class and with $h(p) = -1$. Then

$$m_{p,\Phi}(N) = (\gamma_p + o(1))N, \quad \gamma_2 = \frac{3}{2}, \quad \gamma_p = \frac{2|\{d \in \mathcal{D} : (d/p) = 1\}|}{p+1} \quad (p \text{ odd}). \quad (14.5)$$

Indeed all curve factors lie in $[0, 1]$. A negative sign at an odd visible p forces $c_d(A) \leq (p-3)/(p+1)$, hence negative density at least $2/(p+1)$ on each such trace. At two all three means are zero. The source negative only at p achieves the displayed constant and is admitted because $\Phi \geq 1$. This proves both bounds in (14.5); for example $\gamma_3 = 1/2$, $\gamma_5 = 1/3$ and $\gamma_7 = 1/4$.

This is a minimum over a restricted class. The unrestricted minimum is at most $m_{p,\Phi}(N)$, not at least it. Thus (14.5) is not a linear lower bound for arbitrary sources. Likewise a fixed quotient $h_A = f/g$ with $\delta(A) \leq 1/2$ gives curve disagreement density $(1 - c_d(A))/2$, without placing either f or g individually in the sparse negative-prime class.

14.3 Examples beyond every fixed logarithmic correction

Let p_j denote the j th prime. The support

$$\Gamma_{1/2} = \{p_{j^2} : j \geq 1\}$$

has $p_{j^2} \sim 2j^2 \log j$, hence $\delta(\Gamma_{1/2}) = 1/2$, whereas $\sum_{q \in \Gamma_{1/2}} q^{-1/2}$ diverges by comparison with $\sum_j 1/(j\sqrt{\log j})$. These are consequences of the prime number theorem. Since two belongs to this support, its all-negative assignment has zero mean on every curve, and Theorem 14.1 treats all its other assignments as well.

One can exceed every fixed logarithmic correction of a critical power sum. Fix $0 < s < 1$, put $Y_j = 2^{j^2}$, and in $(Y_j, 2Y_j]$ choose the first $\lfloor Y_j^s 2^j \rfloor$ primes congruent to one modulo eight, starting at a sufficiently large j_0 . The prime number theorem in this fixed progression supplies enough primes. Their union Γ_s satisfies

$$\delta(\Gamma_s) = s, \quad \sum_{q \in \Gamma_s} \frac{1}{q} < \infty, \quad \sum_{q \in \Gamma_s} \frac{q^{-s}}{(\log q)^B} = \infty \quad (B \geq 0). \quad (14.6)$$

To see the first assertion, the number of primes in the current and earlier blocks is at most $t^{s+o(1)}$, while the block endpoints give the reverse lower bound on the exponent. The harmonic cost per block is $O(2^{-(1-s)j^2+j})$, which is summable. Its contribution to the last series is bounded below by a constant times $2^j/j^{2B}$ and does not tend to zero. Starting farther out makes the harmonic cost arbitrarily small and avoids any fixed prefix. All the selected primes are visible on all three curves.

At $s = 1/2$ these supports are stable on curves. At $s = (1 + \alpha)/2$ they will be stable on the power rectangles below. More generally, if $L(t) \geq 1$ and $\log L(t) = o(\log t)$, then $\sum_{q \in \Gamma} q^{-1/2}/L(q) < \infty$ implies $\delta(\Gamma) \leq 1/2$. The counting condition is not tied to a particular slowly varying discount.

For comparison, the class of all individual supports with $\delta \leq 1/2$ includes every finite support. Choosing at size N the primes dividing the observations up to that size can reproduce any finite assignment. Uniformity over every such moving support does not follow. The common envelope above is a substantive additional hypothesis.

14.4 The critical exponent on power rectangles

For $1 \leq H \leq X$ put

$$T_{\Gamma,d}^{(2)}(X, H; y) = \frac{1}{XH} \#\{a \leq X, b \leq H : \exists p \in \Gamma, p > y, v_p(|Q_d(a, b)|) \text{ odd}\}.$$

Theorem 14.2 (Critical counting on power rectangles). *Fix $0 < \alpha < 1$, $H = \lfloor X^\alpha \rfloor$ and $s_\alpha = (1 + \alpha)/2$. If $\delta(\Gamma) \leq s_\alpha$, then*

$$\lim_{y \rightarrow \infty} \sup_{X \geq 2} \max_d T_{\Gamma,d}^{(2)}(X, H; y) = 0. \quad (14.7)$$

Uniformly over $A \subseteq \Gamma$ and the three d , the rectangular means of h_A converge to $C_d(A)$ in (1.18).

Proof. The incidence estimate (12.5), split at Z , gives

$$T_{\Gamma,d}^{(2)}(X, H; y) \leq 12 \sum_{p \in \Gamma, p > y} \frac{1}{p} + \frac{12}{H} \sum_{\substack{p \in \Gamma \\ p \leq Z}} p^{-1/2} + \frac{L_d^{(2)}(X, H; Z)}{XH}. \quad (14.8)$$

Choose $0 < \vartheta < \alpha/4$, $Z = X^{2-\vartheta}$, and $\varepsilon = \min(\alpha\vartheta/8, (1 - \alpha)/4)$. Partial summation in the counting bound yields

$$\sum_{p \in \Gamma, p \leq Z} p^{-1/2} \ll_{\Gamma, \alpha, \vartheta} Z^{\alpha/2+\varepsilon}, \quad H^{-1} Z^{\alpha/2+\varepsilon} \ll X^{-\alpha\vartheta/4}.$$

In (13.11), $K \ll X^\vartheta$ and $H \asymp X^\alpha$. Its two power remainders tend to zero, and the limit superior of its first term is $O(\vartheta/\alpha)$ with an absolute constant. Therefore

$$\limsup_X \max_d T_{\Gamma,d}^{(2)}(X, H; y) \leq 12 \sum_{p \in \Gamma, p > y} \frac{1}{p} + C\vartheta/\alpha.$$

Send $\vartheta \downarrow 0$, then $y \rightarrow \infty$, and finally kill the finitely many small rectangles by increasing the cutoff. This proves (14.7).

At an odd split prime, the invertible p -adic change $(a, b) \mapsto (a - rb, a + rb)$ gives two independent uniform coordinates. Each has mean parity sign $(p-1)/(p+1)$, giving its square in (1.18). At an inert prime, the norm valuation is $2 \min(v_p(a), v_p(b))$, hence even. At two, among all residue pairs, mass $1/2$ has even valuation and mass $1/4$ has odd valuation; the remaining mass $1/4$ has both coordinates even. Dividing those coordinates by two changes the norm by a square and leaves its sign unchanged. Thus the local mean obeys $C = 1/4 + C/4$, giving $C = 1/3$ for each form.

Finite valuation truncations justify these local calculations, since their exceptional p -adic sets decrease to the zero set of the form, of measure zero. On diverging rectangles every fixed periodic law converges uniformly in its origins at fixed period. For subsets of a fixed prime head there are only finitely many laws. The odd deficits $4p/(p+1)^2 \leq 4/p$ are summable on Γ , so both the local-product tail and the observation tail are uniform in A . This proves the claimed limit. $\square$

At $\alpha = 0$ the second coordinate would be fixed. The curve proof, including its boundary error $2A_\Gamma(Z)/N$, is separate; the one-variable factors cannot be obtained from the two-variable products by that substitution.

For a common envelope $A_\Gamma(t) \leq C_\varepsilon t^{s_\alpha + \varepsilon}$ for every $\varepsilon > 0$, all statements are uniform across the supported family, even if its union is not harmonically summable. With $A_\Gamma(t) \leq C_0 t^{s_\alpha} (\log(2t))^B$, the choice

$$Z = X^2 / (\log X)^{(2B+4)/\alpha}$$

in (14.8) gives a harmonic tail plus $O_{\alpha, B, C_0}(\log \log X / \log X)$, as in (14.4).

The elementary budget version remains useful. For $H = \lfloor X^\alpha \rfloor$, the bound $H \geq X^\alpha/2$ and $p \leq 3X^2$ on the contributing range give

$$\sup_X \max_d T_{\Gamma,d}^{(2)}(X, H; y) \leq 60 \sum_{p \in \Gamma, p > y} p^{-(1+\alpha)/2}. \quad (14.9)$$

The counting theorem is strictly more general at equality: it assumes summability for every exponent slightly larger than s_α , rather than summability at s_α . Example (14.6) makes this distinction concrete.

14.5 Harmonic stability when the side exponents agree

A varying rectangle is not a dilation of a fixed planar region. Thus the known Gaussian mean theorem of Donoso–Le–Moreira–Sun [16, Theorems A–B and Definition 2.3] does not by itself cover the following statement. Work on shrinking sectors [18, 19] addresses related angular questions, but with different source hypotheses and averaging domains. Here the collective bound tracks the small side H for every assignment on the same harmonic support.

Theorem 14.3 (Almost-square rectangles in exponent). *Let $1 \leq H(X) \leq X$ be integer-valued and $\log H(X)/\log X \rightarrow 1$. Every fixed support Γ with finite harmonic sum satisfies*

$$\lim_{y \rightarrow \infty} \sup_{X \geq 2} \max_d T_{\Gamma,d}^{(2)}(X, H(X); y) = 0.$$

All its assignments have rectangular means converging uniformly to $C_d(A)$ in (1.18).

Proof. Use Proposition 13.3 at $Z = H^2$. Then $K = \lceil 3X^2/H^2 \rceil = X^{o(1)}$, so its logarithmic term tends to zero and its two remainders are $X^{-5/8+o(1)}$ and $X^{-3/2+o(1)}$. For $p \leq H^2$, (12.5) is at most $24/p$. Hence

$$T_{\Gamma,d}^{(2)}(X, H; y) \leq 24 \sum_{p \in \Gamma, p > y} \frac{1}{p} + \frac{L_d^{(2)}(X, H; H^2)}{XH}. \quad (14.10)$$

The second term tends to zero independently of the support. Take the two limits and remove the finite initial rectangles. The local products follow from the same finite-prime argument as in Theorem 14.2, since both side lengths diverge. $\square$

The width $H(X) = \max(1, \lfloor X \exp(-\sqrt{\log X}) \rfloor)$ is admissible, although $H/X \rightarrow 0$ and $H \log X/X \rightarrow 0$. In (13.11) its leading geometric term is $O((\log X)^{-1/2})$. More generally $X \exp(-(\log X)^\theta)$ works for every fixed $0 < \theta < 1$.

The new criterion can apply even when the older boundary sum (12.8) diverges. Set $Y_j = 2^{j^2}$ and let Γ contain *all* primes in $(Y_j, 2Y_j]$ for $j \geq j_0$. Its harmonic cost is $\sum_j O(j^{-2})$, whereas every power cost with exponent less than one diverges. At $X_j = \lceil \sqrt{Y_j} \rceil$, the current block contributes at least $cX_j/\log X_j$ to $\sum_{p \in \Gamma, p \leq 3X_j^2} p^{-1/2}$. Dividing by the width above makes the old boundary expression at least a constant times $\exp(\sqrt{\log X_j})/\log X_j$, which tends to infinity. Nevertheless Theorem 14.3 applies. A divergent sum of individual incidence bounds is not evidence of actual instability.

For two fixed sources with harmonic disagreement support A , the same theorem gives rectangular disagreement limit $(1 - C_d(A))/2$. Each $C_d(A)$ is positive, with a common positive lower bound over subsets of a fixed harmonic support. If all three disagreement limits vanish, every prime in A would have to be invisible to all forms. No odd prime has that property, and two is visible to each; hence the sources agree. This implication assumes harmonic proximity rather than deriving it from the observed errors.

Theorems 14.1, 14.2 and 14.3 prove the positive assertions of Theorem 1.4. The next section supplies the negative assertion at fixed power width.

15 The power-rectangle threshold from Gaussian primes

To match the positive exponent $s_\alpha = (1 + \alpha)/2$, the negative construction needs many private directions whose primes are close to the full height X^2 . Prime values alone would supply only about $XH/\log X$ directions. A harmonic sum over primitive cofactors restores the missing logarithm. The external input is a lower bound for Gaussian primes in every sufficiently wide sector, with the position of the sector uniform.

Theorem 15.1 (Instability above the rectangular critical exponent). *Fix $119/500 < \alpha < 1$, $H = \lfloor X^\alpha \rfloor$, and $(1 + \alpha)/2 < \sigma \leq 1$. There is $c_{\alpha, \sigma} > 0$ such that, for every $f \in \mathcal{M}_\pm$, $\varepsilon > 0$ and M_0 , there is one fixed $g \in \mathcal{M}_\pm$ with*

$$d_\sigma(f, g) < \varepsilon, \quad g(n) = f(n) \quad (n \leq M_0), \quad \text{osc}_X A_{-1, g}(X, \lfloor X^\alpha \rfloor) \geq c_{\alpha, \sigma}.$$

The oscillation constant is independent of the radius and of the initial source. This assertion concerns $a^2 + b^2$ only.

15.1 The angular input and the required margins

Published input 15.2 (Harman–Lewis). *There is an absolute $c_0 > 0$ such that, for all sufficiently large t , every angular sector of width*

$$t^{-381/1000} \leq \delta \leq \pi/2$$

contains at least $c_0 \delta t / \log t$ Gaussian primes of norm at most t , counted modulo units by their representatives in that sector. The bound is uniform in the sector position.

This is Theorem 2 of Harman–Lewis [13], with the variable denoting the *norm* bound, not the radius. We regard a translated sector as an arc modulo $\pi/2$, and use the strict margin below to handle an arc crossing a quadrant boundary. We use a lower bound in every sector, not an asymptotic in almost every sector. Its analytic proof is imported. Stucky [15] treats joint short restrictions on norm and argument; that neighboring result is not substituted for the specific uniform bound used here.

Gaussian primes which do not have rational-prime norm are the primes of norm p^2 with p inert, apart from the ramified prime above two. There are $O(\sqrt{t})$ of them. They are negligible in this input whenever $\delta\sqrt{t}/\log t \rightarrow \infty$. We shall explicitly ensure that margin.

An elementary upper bound will be used to remove small prime norms. Uniformly in sector position and $0 < \delta \leq \pi/2$,

$$\#\{z \in \mathbb{Z}[i] \setminus \{0\} : N(z) \leq T, \arg z \text{ in the sector}\} \ll \delta T + \sqrt{T} + 1. \quad (15.1)$$

The area is $\delta T/2$ and the boundary length is $O(\sqrt{T})$; count the unit squares inside and meeting the boundary. This upper bound permits subtraction from a lower bound without assuming a lower bound on a thin annulus.

The scale of the cofactor sum explains why we can afford these margins. Write $N(w) = r^2 + s^2$ for $w = r + is$, and take one representative $r > 0$, $s \geq 0$, $(r, s) = 1$ of each primitive cofactor modulo units. The elementary count proved in (15.5) gives reciprocal mass of order $\log M$ up to norm M . Consequently the relevant scale is

$$\frac{XH}{\log X} \sum_{\substack{N(w) \leq X^\tau \\ w \text{ primitive, } r > 0, s \geq 0}} \frac{1}{N(w)} \asymp_\tau XH \quad (\tau > 0 \text{ fixed}).$$

The cofactor logarithm restores exactly the logarithm lost in counting primes. This is not yet the reservoir count: the next choices make the angular estimate uniform for those cofactors, and the following subsections remove small pivots and rule out multiple counting.

Choose fixed parameters

$$0 < \rho < \min\left(1 - \alpha, 2 - \frac{1 + \alpha}{\sigma}\right), \quad (15.2)$$

$$0 < \tau < \min\left(\rho, 2\alpha, \frac{1000\alpha - 238}{381}\right). \quad (15.3)$$

They exist under the theorem's hypotheses. Put

$$M = \lfloor X^\tau \rfloor, \quad T_0 = X^{2-\rho}, \quad \mathcal{I}_X = \left[\arctan \frac{H}{4X}, \arctan \frac{H}{2X}\right], \quad \delta_X = |\mathcal{I}_X| \sim \frac{H}{4X}. \quad (15.4)$$

Every point of norm at most X^2 with argument in $\mathcal{I}_X$ has $a \leq X$ and $0 < b \leq H/2$, hence lies in the original rectangle. The condition on ρ pays for both privacy and the prime budget. The conditions on τ make the cofactor sum compatible with the angular theorem and its discarded errors.

15.2 A logarithm of primitive cofactors

Wang's comparison of full and coprime square averages [20, Theorem 1.2 and Corollary 1.3] concerns the limiting mean on primitive pairs. The count needed here has a different purpose: each primitive cofactor produces private directions, and its reciprocal norm contributes to the total programmable mass. Let

$$\mathcal{W}(M) = \{w = r + is : r > 0, s \geq 0, (r, s) = 1, r^2 + s^2 \leq M\}.$$

This is a set of representatives modulo the four units. The elementary primitive lattice-point count gives

$$|\mathcal{W}(M)| = \frac{3}{2\pi}M + O(\sqrt{M} \log(2M)), \quad \sum_{w \in \mathcal{W}(M)} \frac{1}{N(w)} = \frac{3}{2\pi} \log M + O(1). \quad (15.5)$$

For completeness, the quarter disk has $\pi M/4 + O(\sqrt{M} + 1)$ points under these conventions. Möbius inversion for coprimality replaces the leading coefficient by $(\pi/4)/\zeta(2) = 3/(2\pi)$ and produces the displayed error. Partial summation gives the harmonic formula, since the integrated error $\int_1^\infty t^{-3/2} \log(2t) dt$ is finite. The only primitive point on the nonnegative real axis is $w = 1$.

For $w \in \mathcal{W}(M)$ of norm m , apply Published input 15.2 with

$$t = X^2/m, \quad \text{sector } \mathcal{I}_X - \arg w.$$

The angular condition holds uniformly in $m \leq M$, because (15.3) implies the strict power inequality

$$1 - \alpha < \frac{381}{1000}(2 - \tau). \quad (15.6)$$

If the translated arc crosses a quadrant boundary, cut it there and retain a component of length at least $\delta_X/2$. By the strict inequality (15.6),

$$\frac{\delta_X}{2} t^{381/1000} \gg X^{\alpha-1+(381/1000)(2-\tau)} \longrightarrow \infty$$

uniformly for $m \leq M$. Hence even that component satisfies the angular hypothesis for every sufficiently large X . The constant loss is absorbed by decreasing c_0 once; no endpoint assertion for a split sector is used. Also $\delta_X \sqrt{t} \gg X^{\alpha-\tau/2}$, which tends to infinity faster than $\log t$. Removing inert-prime norms is therefore legitimate uniformly in w . For large X , at least

$$\frac{c_0}{2} \frac{\delta_X X^2}{m \log(X^2/m)}$$

Gaussian primes of rational-prime norm remain for each cofactor. Since $\log(X^2/m) \leq 2 \log X$, summing (15.5) yields

$$\sum_{w \in \mathcal{W}(M)} \#\{\pi \text{ counted for } w\} \gg \frac{\delta_X X^2}{\log X} \sum_w \frac{1}{N(w)} \gg_{\tau} XH. \quad (15.7)$$

The factor $\log M \asymp_{\tau} \log X$ is essential: omitting the cofactors would not give a positive proportion of the rectangle.

15.3 Removing small pivots and proving injectivity

Discard the pairs with prime norm $p \leq T_0$. For each w , (15.1) bounds their number by $O(\delta_X T_0 + \sqrt{T_0} + 1)$. As $|\mathcal{W}(M)| \ll M$, the total discarded count is

$$O\left(X^{1+\alpha+\tau-\rho} + X^{1+\tau-\rho/2} + X^{\tau}\right) = o(XH). \quad (15.8)$$

The first exponent is harmless because $\tau < \rho$, and the second because $\tau - \rho/2 < \tau/2 < \alpha$.

Each surviving pair gives the unique associate of $w\pi$ with argument in $\mathcal{I}_X$. Write it $a + ib$. Its norm is $mp \leq X^2$ and $p > T_0 > M$ for large X . The point is primitive. If a rational prime $\ell \neq p$ divided both coordinates, multiplication by π would be invertible modulo ℓ , forcing ℓ to divide both coordinates of w . If $\ell = p$, then $p^2 \mid mp$, impossible because $m < p$.

Conversely, a primitive value $a^2 + b^2 = mp \leq X^2$ with $m \leq M$ and $p > T_0 > X$ has exactly one such rational prime factor. Exactly one of the two Gaussian primes above p divides $a + ib$; both would force $p \mid (a, b)$. Unique factorization in $\mathbb{Z}[i]$ recovers w and π modulo units, and the convention for $\mathcal{W}(M)$ makes w unique. Thus the pairs have not hidden a multiplicity.

It follows from (15.7)–(15.8) that the set $\mathcal{R}_X$ of primitive points satisfying

$$\arg(a + ib) \in \mathcal{I}_X, \quad a^2 + b^2 = mp \leq X^2, \quad m \leq M, \quad p > X^{2-\rho}$$

has $|\mathcal{R}_X| \geq c_* XH$ at every sufficiently large X , with $c_* > 0$ depending only on the fixed parameters. These points lie in the original rectangle. In particular $p > H^2$, since $2 - \rho > 1 + \alpha > 2\alpha$. Unlike the almost-square regime, the proportion of values with a factor exceeding H^2 is bounded below here.

15.4 Programming directions and freezing a single source

Every surviving p exceeds $2XH$ for large X . Lemma 12.4 implies that its incidences on the entire rectangle lie in one primitive direction. The points touched by the pivot at $(a, b) \in \mathcal{R}_X$ are exactly

$$(ka, kb), \quad 1 \leq k \leq \min(\lfloor X/a \rfloor, \lfloor H/b \rfloor).$$

Distinct reservoir points have distinct pivots. The valuation is one throughout each direction, and all its signs equal $f(a^2 + b^2)$ because $f(k^2) = 1$. For a target sign s , assigning $g(p) = sf(m)$ in $a^2 + b^2 = mp$ programs that direction; $m \leq M$ contains no changed pivot. All other directions remain unchanged.

There are at most XH pivots, so the total cost is at most

$$\sum_{p \text{ changed}} p^{-\sigma} \leq XHX^{-\sigma(2-\rho)} \leq X^{-\eta}, \quad \eta = \sigma(2-\rho) - 1 - \alpha > 0. \quad (15.9)$$

The all-positive and all-negative prescriptions differ in their rectangular means by at least $2|\mathcal{R}_X|/(XH) \geq 2c_*$. Multiples of a primitive direction increase the controlled mass, not the number of independent choices or the prime cost.

Starting from $f_0 = f$, select successive X_j so large that the reservoir exists,

$$X_j^{2-\rho} > \max(M_0, 3X_{j-1}^2), \quad X_j^{-\eta} < \varepsilon 2^{-j-1}.$$

One of the two branches is at distance at least c_* from the previous mean. Choose it. Every later pivot exceeds the full height of all three previous rectangles; thus all previously selected observations, not only their means, are frozen. The prime coordinates stabilize to one g , its total cost is less than ε , its prefix is preserved, and successive selected means stay at distance at least c_* . This proves Theorem 15.1, and hence part (iv) of Theorem 1.4.

For example, $\alpha = 1/2$, $\rho = 1/4$, $\tau = 1/8$ and $\sigma = 7/8$ meet every strict inequality, with $\eta = 1/32$. This verifies the compatibility of the parameters; it does not calibrate c_* or the first scale at which the analytic input applies.

A finite example makes the programming visible. At $X = 256$, $H = 16$, $M = 2$ and $\rho = 1/4$, exact enumeration of $\mathcal{R}_X$ gives 61 directions, each with one label in the rectangle. Reversing their 61 pivots from the constant source changes the mean from 1 to $1987/2048$, at harmonic cost less than $1981/10^6$. The finite diagnostic supplied with this article recomputes the directions, privacy and cost. This example checks the local operation, not a lower bound for all sufficiently large rectangles.

For the Gaussian square mean, existence and its product are already supplied by [16, Theorem A]. The next corollary uses the same fixed source to separate that known square behavior from power-rectangle divergence, while controlling its prime cost.

Corollary 15.3 (Square means can coexist with power-rectangle divergence). *In the range of Theorem 15.1, one may choose a perturbation h_E of 1, preserving a prescribed prefix, with $d_\sigma(1, h_E) < \varepsilon$, such that all three square means have limits $C_d(E) > 1 - 4\varepsilon$, whereas its sum-of-two-squares mean on $H = \lfloor X^\alpha \rfloor$ does not converge. Its oscillation is bounded below by a positive constant depending only on α, σ . Its mean on the ordinary integers is greater than $1 - 2\varepsilon$.*

Proof. Use an internal budget $\varepsilon' < \min(\varepsilon, c_*/8, 1/4)$ and never change two. Before attack j , the finite negative-prime set E_{j-1} has rectangular means tending to the three products. Choose R_j beyond the previous scale where all three means are within $1/j$ of their products. Then choose $X_j > R_j$ with new pivots beyond $\max(M_0, 3R_j^2)$, and make every reservoir direction negative. Add exactly the pivots requiring a sign change. The negative sets increase and the new mean is at most $1 - 2c_*$.

All attacks and recoveries remain frozen. For $E = \bigcup_j E_j$, the products converge and satisfy $C_d(E) > 1 - 4\varepsilon'$. The recovery means tend to $C_{-1}(E)$, so the oscillation is at least $2c_* - 4\varepsilon' > c_*$. On squares the bound (12.6) at exponent one gives a uniform tail at most $36 \sum_{p \in E, p > y} 1/p$. Finite-prime approximation therefore proves convergence of all three square means to their products. The union of multiples on the integers similarly gives the product $\prod_{p \in E} (p-1)/(p+1) > 1 - 2\varepsilon'$. Finally $d_1 \leq d_\sigma$, so all displayed lower bounds hold with the requested ε . $\square$

15.5 What the threshold classifies

The stable estimate (14.9) holds at s_α , while (15.9) can be made to decay exactly when $\sigma > s_\alpha$. Thus s_α is the universal power-budget threshold in the stated angular range. This does not mean that every support above it is unstable.

There is an equivalent boundary in counting exponent. For any $t > s_\alpha$, choose $s_\alpha < \sigma < \min(t, 1)$ and use Corollary 15.3. Its unstable support has $\delta(E) \leq \sigma < t$, because its σ -budget is finite. Theorem 14.2 forces $\delta(E) > s_\alpha$. Hence all supports with $\delta \leq s_\alpha$ are stable, and unstable supports exist with exponent arbitrarily close above it. Taking different exponents in different constructions cannot yield instability at a limiting support with $\delta \leq s_\alpha$; the positive theorem rules that out.

The restriction $\alpha > 119/500$ comes from the available uniform angular lower bound. Neither its optimality nor instability for every $0 < \alpha \leq 119/500$ is established here. No simultaneous instability of the other two norms at power width, and no logarithmic-in- X assertion for these rectangles, is claimed.

16 Scope, dependence and remaining boundaries

The article separates three kinds of control. A harmonic prime budget bounds changes on ordinary integers; coverage and parity measure their effect on a chosen observation; geometry and normalization govern whether a collective estimate can make its remaining tail negligible.

The deterministic and probabilistic criteria answer different questions. On a fixed harmonically summable support, convergence for every assignment forces uniform truncation and identifies the limits. Under an independent law, the exact parity formula identifies the almost-sure truncation defect, without characterizing existence of the mean for a single arbitrary realization. Neither criterion creates an unknown mean of a prescribed base multiplied by the perturbation. The identities cancelling that base concern disagreement errors only.

The positive counting results reach exponent $1/2$ on curves, including nonsummable critical series. The logarithmic counterexample is available separately for every requested $125/164 < \sigma \leq 1$, with one positive, uncalibrated oscillation bound independent of σ . The construction and its scales may depend on σ , and the endpoint $125/164$ is not included. The quantitative $1/2000$ branch applies for $128/153 < \sigma \leq 1$. These statements do not classify the interval between the curve stability and instability ranges. In particular they do not establish the curve endpoint $1/2$ as an optimal universal counting threshold, or settle a particular support of larger exponent.

On rectangles $H = \lfloor X^\alpha \rfloor$, the universal budget and counting threshold is $(1 + \alpha)/2$ for $119/500 < \alpha < 1$, with equality on the stable side. The negative assertion concerns $a^2 + b^2$; the positive assertion concerns all three forms and holds for every $0 < \alpha < 1$. The restriction $119/500$ belongs to the angular input, not to a demonstrated barrier of the problem. Harmonic stability when $\log H / \log X \rightarrow 1$ is compatible with failure at each fixed power width in the angular range. No uniform oscillation constant as $\alpha \rightarrow 1$ is supplied. At sublogarithmic diverging widths, the separate elementary construction still makes all three means oscillate in one fixed function.

Where the external inputs enter. The whole-support criterion uses root counts, finite-prime laws, harmonic summability, a finite parity average and Abel summation. The random criterion adds classical bounded differences. The rich stable supports, ordinary instabilities and the ordinary/logarithmic separator use only prime distribution in fixed progressions beyond their finite constructions. The collective upper bounds use the Selberg quadratic form and fixed-progression Mertens estimates; their short-interval form depends on the interval length rather than the height. The three analytic routes in Table 1 feed the same mass–privacy–cost transfer. They retain different quantitative outputs, not multiple independent certifications of the argument. Harman–Lewis enters only the power-rectangle negative branch. The existence of sources with full affine cancellation comes separately from Klurman–Mangerel–Teräväinen; preservation of that property is proved here. The neighboring probabilistic, sectorial and coprime-pair results discussed in Section 1 provide context; the proof inputs are those identified above and at their uses.

The private-prime mechanism, finite prescriptions, repeated freezing, local products, periodic approximation and sieve tools have the antecedents specified in the introduction and at their uses. In particular, the Gaussian square means are known from Donoso–Le–Moreira–Sun; the degenerating geometry, support-uniform criteria and quantified coexistence results are the distinctions used here. The contribution is in the quantified support criteria, verified tail regimes, fixed-source separations and the conversion of angular mass to affordable private directions. No claim of exhaustive bibliographic priority is made.

The conclusions remain statements about the indicated source classes or constructed perturbations. They do not yield a new cancellation theorem for unmodified Liouville, a linear lower bound for the unrestricted minimum on three curves, or a Poisson limit. In particular, the restricted minimum (14.5) is not an estimate for the unrestricted one. Appendix A similarly distinguishes every permitted origin from all but a vanishing proportion of origins, without declaring its local support threshold optimal.

A Moving the origin: all windows and almost all windows

A small prefix error cannot simply be subtracted twice to control a short interval: two errors $o(X)$ need not be $o(L)$ when $L \ll X$. The translated sieve (13.7) keeps the polynomial height $O(X^2)$ separate from the length L . It gives a statement at every origin under a stronger support condition. Mean periodic approximation then gives a different statement at almost every origin under the global critical condition.

For $u \geq 2$ and $L \geq 1$ define

$$M_{d,f}(u, L) = \frac{1}{L} \sum_{u < n \leq u+L} f(P_d(n)).$$

All three polynomials are positive on these intervals. For an integer minimum length $1 \leq \ell(X) \leq X$, let

$$\mathcal{J}_\ell(X) = \{(u, u+L] \cap \mathbb{Z} : X \leq u < 2X, \ell(X) \leq L \leq X\}.$$

The origins, lengths and X in these formulas are integers. For I in this family put

$$T_{\Gamma,d}(I; y) = \frac{1}{|I|} \#\{n \in I : \exists p \in \Gamma, p > y, v_p(P_d(n)) \text{ odd}\}.$$

A.1 Every origin under the local critical condition

Theorem A.1 (Uniform local stability). *Suppose $0 < \alpha \leq 1$, $\delta(\Gamma) \leq \alpha/2$ and*

$$\liminf_{X \rightarrow \infty} \frac{\log \ell(X)}{\log X} \geq \alpha.$$

Then

$$\limsup_{y \rightarrow \infty} \sup_{X \geq 2} \sup_{I \in \mathcal{J}_\ell(X)} \max_d T_{\Gamma,d}(I; y) = 0. \quad (\text{A.1})$$

Uniformly in $A \subseteq \Gamma$, $I = (u, u+L] \in \mathcal{J}_\ell(X)$ and $d \in \mathcal{D}$,

$$M_{d,h_A}(u, L) - c_d(A) \longrightarrow 0. \quad (\text{A.2})$$

Every fixed finite list of translated readings has its corresponding local law under the same uniformity.

Proof. For large X and a cutoff $3X \leq Z \leq 9X^2 + 2$, root counting and (13.7) give

$$T_{\Gamma,d}(I; y) \leq 2 \sum_{p \in \Gamma, p > y} \frac{1}{p} + \frac{2A_\Gamma(Z)}{L} + \frac{1}{L} \#\{n \in I : P^+(P_d(n)) > Z\}. \quad (\text{A.3})$$

Fix $0 < \vartheta < \alpha/8$, set $Z = X^{2-\vartheta}$ and $\eta = \alpha\vartheta/16$. Eventually $L \geq X^{\alpha-\eta}$, and the support condition implies

$$\frac{A_\Gamma(Z)}{L} \ll_{\Gamma, \eta} X^{(2-\vartheta)(\alpha/2+\eta) - (\alpha-\eta)} \ll_{\Gamma, \eta} X^{-\alpha\vartheta/4} \rightarrow 0.$$

Here the same η is allowed in the support exponent because that bound holds for every positive excess. In (13.7), $K \ll X^\vartheta$, so $K/\sqrt{L} \rightarrow 0$, while the logarithmic term has limit superior $O(\vartheta/\alpha)$. Uniformly over the intervals,

$$\limsup_X \sup_{I \in \mathcal{J}_\ell(X)} \max_d T_{\Gamma,d}(I; y) \leq 2 \sum_{p \in \Gamma, p > y} \frac{1}{p} + C\vartheta/\alpha.$$

Send $\vartheta \downarrow 0$ and then $y \rightarrow \infty$, and remove the finitely many initial windows by a height cutoff. This proves (A.1).

For a fixed finite prime set F , truncate valuations at level r . At each odd visible prime, the unresolved classes in an interval of length L have proportion at most $2/p^r + 2/L$; at two the curve valuations are already bounded by one. A bounded periodic sequence has its period mean in every interval of length tending to infinity, uniformly in the origin. The Chinese remainder theorem gives $c_d(F)$. Uniform prime truncation and the uniformly convergent products therefore give (A.2) over all A .

For a finite translated list, take a common periodic truncation and a union of its tail events. A fixed shift changes the tail count on a window by at most $O(1/L)$ at its ends. The same proof gives the joint law, retaining all local dependencies. $\square$

The theorem includes $\ell(X) = \lfloor X^\alpha \rfloor$ and more generally integer lengths $X^{\alpha+o(1)}$ in $[1, X]$. If the support of prime disagreements of f, g has exponent at most $\alpha/2$, then the uniform disagreement limit is $(1 - c_d(A))/2$, where $A = \{p : f(p) \neq g(p)\}$. Neither negative-prime support of f or g separately has to be sparse. If they differ at a fixed prime p , their total disagreement over the three curves in any permitted interval is at least $(\gamma_p - o(1))L$, with γ_p as in (14.5). This retains a hypothesis on the disagreement support, not a conclusion about the unrestricted minimum.

The estimates are also uniform over support families with a common envelope $A_\Gamma(t) \leq \Phi(t) \ll_\varepsilon t^{\alpha/2+\varepsilon}$ for every $\varepsilon > 0$. For the more explicit bounds $A_\Gamma(t) \leq C_0 t^{\alpha/2} (\log(2t))^B$ and $L \geq X^\alpha / (\log X)^A$, with fixed $A, B \geq 0$, choose

$$Z = X^2 / (\log X)^{(2A+2B+4)/\alpha}$$

in (A.3). It gives

$$T_{\Gamma,d}(I; y) \leq 2 \sum_{p \in \Gamma, p > y} \frac{1}{p} + O_{\alpha,A,B,C_0} \left(\frac{\log \log X}{\log X} \right). \quad (\text{A.4})$$

This is still a tail bound, not the full rate of convergence to the product.

For a concrete critical example, $\Gamma = \{p_{j^4} : j \geq 1\}$ has counting exponent $1/4$ but divergent series $\sum_\Gamma p^{-1/4}$: the terms are asymptotic to a constant times $1/(j(\log j)^{1/4})$. Since two belongs to it, the all-negative assignment has all three means tending to zero in every window of length $\lfloor \sqrt{X} \rfloor$, uniformly for $X \leq u < 2X$. The critical series is not the resource paying for this local conclusion.

A.2 Almost every origin at any diverging length

Theorem A.2 (Typical windows uniformly over assignments). *If $\delta(\Gamma) \leq 1/2$ and $1 \leq H(X) \leq X$ is integer-valued with $H(X) \rightarrow \infty$, then*

$$\frac{1}{X} \sum_{u=X}^{2X-1} \sup_{A \subseteq \Gamma} \max_{d \in \mathcal{D}} |M_{d,h_A}(u, H(X)) - c_d(A)| \rightarrow 0. \quad (\text{A.5})$$

For any fixed accuracy, outside $o(X)$ origins the conclusion holds simultaneously for every assignment. The analogous assertion holds in total variation for the local laws of a fixed finite list of readings.

Proof. Let $b_{d,y}(n)$ be the indicator of a tail prime at odd valuation in $P_d(n)$, with nonpositive values omitted. Theorem 14.1 gives

$$\lim_{y \rightarrow \infty} \limsup_X \max_d \frac{1}{3X} \sum_{2 \leq n \leq 3X} b_{d,y}(n) = 0.$$

Each integer lies in at most $H = H(X)$ of the windows being averaged. Consequently

$$\frac{1}{X} \sum_{u=X}^{2X-1} \frac{1}{H} \sum_{u < n \leq u+H} b_{d,y}(n) \leq \frac{1}{X} \sum_{2 \leq n \leq 3X} b_{d,y}(n). \quad (\text{A.6})$$

For every A the absolute pointwise error from prime truncation is at most $2b_{d,y}(n)$. Thus the supremum over A can be taken before the average over origins. Sum over the three d and use (A.6).

At fixed y there are only finitely many prime assignments. Valuation truncation and periodicity make their interval means converge to the finite products uniformly in the origin as $H \rightarrow \infty$. The finite products approach $c_d(A)$ uniformly. Taking $X \rightarrow \infty$ and then $y \rightarrow \infty$ proves (A.5); Markov's inequality gives the exceptional-set statement. For a finite vector, pay for its finitely many translated tails and use the disagreement bound on total variation. This proves the last assertion as well. $\square$

The typical-window argument is a consequence of mean periodic approximation, in the classical setting discussed in [8]. It does not treat $h_A(n^2 - d)$ as a multiplicative function of n , and it provides no uniform rate for arbitrarily slowly diverging lengths.

A.3 Extremely sparse supports can still have exceptional windows

The word “almost” cannot be discarded merely because the permitted length tends to infinity. The following construction is a local counter-test, using prescribed blocks rather than a density of blocks.

Theorem A.3 (A computable source with long joint positive blocks). *For any finite multiplicative prescription compatible with $h(2) = -1$, there is a computable $h = h_\Gamma$ preserving that prescription, with*

$$A_\Gamma(t) = O(\log(2t)), \quad \sum_{q \in \Gamma} q^{-s} < \infty \quad \text{for every } s > 0,$$

and a strictly increasing sequence n_j such that

$$h((n_j + a)^2 - d) = 1 \quad (0 \leq a < j, d \in \mathcal{D}). \quad (\text{A.7})$$

Nevertheless for every fixed $0 < \alpha \leq 1$, all windows of length $\lfloor X^\alpha \rfloor$ at origins comparable to X have each sign with proportion $1/2 + o(1)$ on each curve.

Proof. Extend the finite prescription initially by positive signs at all other primes. At stage j , its negative support is finite. Let B_j exceed all previously protected values and all prescribed or processed prime coordinates. For each label (a, d) , $0 \leq a < j$, choose a new distinct prime

$$q_{a,d} \equiv 1 \pmod{8}, \quad q_{a,d} > \max\{B_j, 2^{j^2}, (j^2 + 4)^2\}.$$

Such primes exist by the theorem on primes in this fixed progression. The polynomials $P_{a,d}(T) = (T + a)^2 - d$ are distinct and irreducible over $\mathbb{Q}$. For two distinct labels, writing $v = a - b$, their resultant is

$$v^4 - 2(d + e)v^2 + (d - e)^2. \quad (\text{A.8})$$

It is nonzero and has absolute value at most $(j^2 + 4)^2$. Thus their root sets are disjoint modulo every new prime. Since $q_{a,d} \equiv 1 \pmod{8}$, the target polynomial has a simple root; choose a class modulo $q_{a,d}^2$ which lifts that root but has valuation exactly one.

The Chinese remainder theorem combines all these classes, together with a parity condition. Select a representative n_j larger than the previous origin, at least two, and at least 2^{j^2} . For each label, its assigned prime occurs once and none of the other new primes divides its value. The sign of the current finite-support source on that value can be calculated by division by its finitely many negative primes. Set the new sign at $q_{a,d}$ equal to that current sign. The resulting sign of the target value is its square, hence one. Some of these new prime signs can remain positive. All old values are preserved because the new primes exceed B_j .

At most $3j$ negative primes are added at stage j , all above 2^{j^2} . Hence

$$A_\Gamma(t) \leq C_{\text{prefix}} + \sum_{j \leq \sqrt{\log_2 t}} 3j = O_{\text{prefix}}(\log(2t)).$$

All coordinates stabilize, all blocks survive, and every positive-power sum converges. The procedure is computable: to determine $h(m)$ it suffices to carry out stages until $2^{j^2} > m$; all later changes lie above m . Each prime search, root search and Chinese-remainder calculation is finite. No claim of practical efficiency is made.

The final support has exponent zero. Applying Theorem A.1 for any fixed positive α , and using the negative sign at two, gives the balanced power-window assertion. $\square$

For an explicit finite illustration of the sign mechanism, the source negative at $\{2, 11, 31\}$ makes the three values at index eight positive:

$$65 = 5 \cdot 13, \quad 66 = 2 \cdot 3 \cdot 11, \quad 62 = 2 \cdot 31.$$

This is one block, not the infinite construction. In the proof each new pivot is private only within its newly prescribed block. It may affect other future values, but never an already frozen block.

Define a nondecreasing width equal to j between $n_j - 1$ and $n_{j+1} - 1$, with a harmless finite initial definition. It tends to infinity and is at most X there, but at $X = n_j - 1$ the window beginning at that origin has all means equal to one. Uniformity over all origins therefore fails for that width. Since $n_j \geq 2^{j^2}$, these block lengths are subpolynomial in their origins and do not contradict the power-window theorem.

Polynomial observations in short intervals have earlier treatments, including Nair [14]. The present proof uses the explicit translated Selberg estimate above rather than importing a general short-interval theorem. Prescribed blocks by congruences and freezing are also an established construction principle. The distinction established here is the combination of a critical support condition for every power window, an assignment-uniform typical-window law, and exceptions on a support of counting exponent zero. It is not an optimality result for the threshold $\alpha/2$ at a fixed power length.

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The companion source-and-diagnostics archive, distributed with this PDF, contains the sources and the finite diagnostics for this edition.

References

- [1] O. Klurman, *Correlations of multiplicative functions and applications*, Compositio Mathematica **153** (2017), no. 8, 1622–1657. doi:10.1112/S0010437X17007163. arXiv:1603.08453v1, 28 March 2016. The proposition, corollary, theorem and lemma numbers cited here refer to that version.
- [2] J. Merikoski, *On the largest prime factor of $n^2 + 1$* , Journal of the European Mathematical Society **25** (2023), no. 4, 1253–1284. doi:10.4171/JEMS/1216. The analytic locators refer to arXiv:1908.08816v3: Section 2.1, Remark 13 (p. 15; published p. 1268), and Section 2.6, pp. 15–16, proof of Theorem 2. The last calculation is used together with Grimmelt–Merikoski.
- [3] L. Warnke, *On the method of typical bounded differences*, Combinatorics, Probability and Computing **25** (2016), no. 2, 269–299. doi:10.1017/S0963548315000103. arXiv:1212.5796. Theorem 1 in the arXiv version records the classical bounded-differences inequality used here; the typical-differences refinement is not required.

- [4] O. Klurman, A. P. Mangerel and J. Teräväinen, *On Elliott’s conjecture and applications*, Proceedings of the London Mathematical Society **133** (2026), no. 3, e70205. doi:10.1112/plms.70205. Theorem numbering and references to Section 8, especially equations (8.3)–(8.7), refer to arXiv:2304.05344v2, 26 May 2023.
- [5] N. Frantzikinakis, O. Klurman and J. Moreira, *Partition regularity of generalized Pythagorean pairs*, arXiv:2407.08360v3, 27 July 2026. Lemma 4.15: prime-incidence estimates for binary forms.
- [6] T. Tao, *The prime number theorem in arithmetic progressions, and dueling conspiracies*, expository notes, 24 September 2009. terrytao.wordpress.com. Used only for the classical prime number theorem in fixed progressions and its partial-summation consequences.
- [7] C. McDiarmid, *On the method of bounded differences*, in J. Siemons (ed.), *Surveys in Combinatorics, 1989*, London Mathematical Society Lecture Note Series 141, Cambridge University Press, 1989, 148–188. doi:10.1017/CBO9781107359949.008.
- [8] V. Bergelson, J. Kułaga-Przymus, M. Lemańczyk and F. K. Richter, *Rationally almost periodic sequences, polynomial multiple recurrence and symbolic dynamics*, Ergodic Theory and Dynamical Systems **39** (2019), no. 9, 2332–2383. doi:10.1017/etds.2017.130. arXiv:1611.08392.
- [9] R. de la Bretèche, V. Y. Wang and M. W. Xu, *Random Multiplicative Functions and Making Squares from Polynomial Values*, arXiv:2607.06398v1, 7 July 2026. Context for polynomial observations of random multiplicative functions.
- [10] A. Pascadi, *Large sieve inequalities for exceptional Maass forms and the greatest prime factor of $n^2 + 1$* , Forum of Mathematics, Pi **14** (2026), e8. doi:10.1017/fmp.2026.10025. The weighted input and its locators refer to arXiv:2404.04239v3, 15 January 2026, Notations 20 and 23, Section 6.3, equation (6.20) and its final verification at 1.30008.
- [11] K. S. Kedlaya, *Notes on analytic number theory*, Chapter 13, The Selberg sieve, Section 13.2, equations (13.2.3)–(13.2.4) and Theorem 13.1. kskedlaya.org/ant/chap-selberg.html. Author’s notes, accessed 27 September 2026. Used for the upper-sieve weights and diagonalization.
- [12] NIST, *Digital Library of Mathematical Functions*, Sections 27.11–27.12. dlmf.nist.gov/27.11 and dlmf.nist.gov/27.12. Accessed 27 September 2026. In particular (27.11.9)–(27.11.11) and the prime number theorem in fixed progressions; only moduli four and eight are needed here.
- [13] G. Harman and P. Lewis, *Gaussian primes in narrow sectors*, Mathematika **48** (2001), nos. 1–2, 119–135. doi:10.1112/S0025579300014388. Theorem 2, p. 119, supplies the uniform sector lower bound at width $t^{-0.381}$ for norm bound t .
- [14] M. Nair, *Multiplicative functions of polynomial values in short intervals*, Acta Arithmetica **62** (1992), no. 3, 257–269. doi:10.4064/aa-62-3-257-269. Antecedent for the short-interval setting.
- [15] J. Stucky, *Gaussian primes in narrow sectors*, The Quarterly Journal of Mathematics **72** (2021), no. 4, 1357–1377. doi:10.1093/qmath/haab009. arXiv:2008.11325v2. Context for simultaneous restrictions on norm and argument.
- [16] S. Donoso, A. N. Le, J. Moreira and W. Sun, *Averages of completely multiplicative functions over the Gaussian integers—a dynamical approach*, Transactions of the American Mathematical Society **377** (2024), no. 10, 7081–7115. doi:10.1090/tran/9184. Locators refer to arXiv:2309.07249v2, 6 March 2024: Theorem A, Definition 2.3 and Lemmas 4.1, 4.3.
- [17] L. Grimmelt and J. Merikoski, *On the greatest prime factor and uniform equidistribution of quadratic polynomials*, preprint, arXiv:2505.00493v2, 30 May 2025. Section 7, Corollaries 7.1–7.2 and the paragraph following them; these supply the unconditional Type I/II ranges used with Merikoski’s weighted sieve calculation.
- [18] A. Burgin and C. Giannitsi, *Ergodic Averages over Shrinking Sectors of $\mathbb{Z}[i]$* , preprint, arXiv:2609.25255v1, 21 September 2026. Context for shrinking sectors and finitely generated multiplicative actions.
- [19] J. Kuś, *Halász theorems for Gaussian ideals in sectors and short intervals*, preprint, arXiv:2605.19105v1, 18 May 2026. Theorems 1.7 and 1.9: sectorial and short-interval context under non-pretentiousness hypotheses.
- [20] B. Wang, *On averages of completely multiplicative functions over co-prime integer pairs*, Proceedings of the Royal Society of Edinburgh Section A: Mathematics, published online 19 December 2025, 1–14. doi:10.1017/prm.2025.10118. Theorem 1.2 and Corollary 1.3 are cited in arXiv:2406.09243v2, 16 December 2025.