

Prime noise, polynomial observations and trace-preserving completion of multiplicative functions

Brice Pouly

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Abstract

For every prescribed completely multiplicative sign function, every integer polynomial irreducible over $\mathbb{Q}$ of degree at least two and eventually positive, and every prescribed prefix, we construct a completion preserving the polynomial trace and the prefix exactly, with vanishing correlations along all fixed admissible affine families. The changed primes have relative density zero, although the changed integers have asymptotic density one half and the changed primes have infinite harmonic mass. One noise trajectory has these properties almost surely at all positive real times. The construction combines a classical root-free prime reservoir with decreasing clock rates. Its probabilistic input is affine regularisation against arbitrary bounded weights fixed before the noise, uniformly in time after a vanishing delay. Marks retain their possibly non-convergent empirical distribution. For non-selective noise, the local channel acts through irreducible factors rather than output coordinates; square identities determine its invariant characters, and rational constants supply a time-evolving random translation. The proofs separate arithmetic moments, temporal concentration and finite Fourier analysis. The constant-source existence case already follows from work of Klurman–Mangerel–Teräväinen. The conclusions concern the constructed function, not new cancellation for the unperturbed source.

1 Preserving a trace while regularising other observations

Normality of a sign sequence concerns words of fixed length. A trace such as $n \mapsto f(n^2 + 1)$ samples a multiplicative function in a different way. Our aim is to separate these kinds of information: rather than calculate the trace of a prescribed source, we show that it can survive pointwise inside a function with vanishing affine correlations.

For a concrete example, a prime $p \equiv 3 \pmod{4}$ never divides $n^2 + 1$. Flipping signs only at these primes, beyond a prescribed prefix, therefore preserves every value of this trace. Such a prime does act on a non-constant affine form whose leading coefficient it does not divide: the form has one root modulo p . Decreasing clock rates can thus leave the polynomial trace untouched while supplying enough harmonic mass to regularise affine observations. With the Liouville source $f = \lambda$, all values $\lambda(n^2 + 1)$ survive exactly; it is the new function, not λ , whose affine correlations vanish. The trace's mean is not calculated.

1.1 The function classes and the completion theorem

Write $\mathcal{M}_{\pm}$ for the completely multiplicative functions $f : \mathbb{N}_{\geq 1} \rightarrow \{-1, 1\}$. Each extends to $\mathbb{Q}_{>0}$ by $f(a/b) = f(a)f(b)$ and takes the value one on positive rational squares.

An admissible affine family is a fixed non-empty family

$$\ell_i(n) = A_i n + B_i, \quad A_i \in \mathbb{N}_{\geq 1}, \quad B_i \in \mathbb{Z}, \quad A_i B_j - A_j B_i \neq 0 \quad (i \neq j). \quad (1.1)$$

Let $\mathcal{L}_\pm$ consist of the $g \in \mathcal{M}_\pm$ such that, for every family (1.1),

$$\frac{1}{X} \sum_{n \leq X} \prod_{i=1}^k g(\ell_i(n)) \longrightarrow 0. \quad (1.2)$$

A fixed initial segment is omitted when some arguments are not positive. This does not change the limit. Expanding word indicators over all subfamilies of consecutive shifts gives frequency 2^{-m} for each word of length m . Thus every member of $\mathcal{L}_\pm$ is normal.

Theorem 1.1 (Completion of an irreducible trace). *Let $P \in \mathbb{Z}[T]$ be irreducible over $\mathbb{Q}$, of degree at least two, and positive at all sufficiently large positive integers. For every $f \in \mathcal{M}_\pm$ and every integer $M_0 \geq 1$, there exists $g \in \mathcal{L}_\pm$ such that*

$$g(P(n)) = f(P(n)) \quad (n \in \mathbb{Z}, P(n) > 0), \quad (1.3)$$

$$g(n) = f(n) \quad (1 \leq n \leq M_0), \quad (1.4)$$

$$\frac{\#\{p \leq x : g(p) \neq f(p)\}}{\pi(x)} \longrightarrow 0. \quad (1.5)$$

More precisely, there is a process obtained from independent prime Poisson clocks, starting from f , for which these conclusions and membership in $\mathcal{L}_\pm$ hold almost surely at all real times $t > 0$ simultaneously. The preservation assertions (1.3)–(1.4) hold at every time, including $t = 0$.

The source is fixed before the clocks. No initial balance, normality or convergence of its correlations is assumed. The theorem preserves a trace *already compatible with a multiplicative function*; it does not prescribe arbitrary signs on the values of P .

With $\text{Tr}_P(f) = (f(P(n)))_{n \in \mathbb{Z}, P(n) > 0}$, we obtain

$$\text{Tr}_P(\mathcal{L}_\pm) = \text{Tr}_P(\mathcal{M}_\pm). \quad (1.6)$$

Here “completion” refers to the protected data, not to missing values of f : the source is already defined on every integer, and we realise its protected data inside the smaller class $\mathcal{L}_\pm$. Equality (1.6) does not say that every binary sequence is a compatible trace. It fails for a positive affine polynomial: the trace of the constant source $f \equiv 1$ would contradict the one-form instance of (1.2).

For probability measures on a common measurable space we use the convention

$$d_{\text{TV}}(\mu, \nu) = \sup_{B \text{ measurable}} |\mu(B) - \nu(B)|.$$

On a finite state space this equals $\frac{1}{2} \sum_x |\mu\{x\} - \nu\{x\}|$. The next statement uses this distance for the joint law of the affine signs and the protected marks.

Corollary 1.2 (Preserved marks). *In the construction of Theorem 1.1, fix an admissible affine family $(\ell_i)_{i \leq k}$ and a finite-alphabet sequence $z(n)$, fixed before the clocks and formed from source values at protected arguments. For example,*

$$z(n) = (f(P(n + b_1)), \dots, f(P(n + b_r)))$$

for fixed integer shifts. Choose an integer $n_0 \geq 0$, fixed before the clocks and large enough that all arguments under consideration are positive, and set

$$I_X = \{n_0 + 1, \dots, n_0 + X\}, \quad \nu_X = \frac{1}{X} \sum_{n \in I_X} \delta_{z(n)}.$$

Almost surely, for every $0 < u < T < \infty$,

$$\sup_{u \leq t \leq T} d_{\text{TV}} \left(\frac{1}{X} \sum_{n \in I_X} \delta_{((F_t^f(\ell_i(n)))_{i \leq k}, z(n))}, \text{Unif}(\{-1, 1\}^k) \otimes \nu_X \right) \longrightarrow 0. \quad (1.7)$$

The same assertion holds separately for $I_X = [X, 2X) \cap \mathbb{Z}$, for all sufficiently large X . No convergence of ν_X is required.

Here F_t^f denotes the constructed function at time t . For one affine form, the joint mass of either sign and a prescribed mark state z is $\frac{1}{2}\nu_X\{z\} + o(1)$, uniformly on the stated time intervals. This is an absolute error; it does not give a relative conditional estimate when $\nu_X\{z\}$ tends to zero.

The probability-one event may depend on the fixed marks; a countable collection of mark sequences can be treated on one event. These are empirical laws as the origin ranges over the specified intervals, not independence of all integer positions. In particular, no uniformity over all translates of an interval, or over arbitrary moving origins, is asserted.

1.2 Two noise regimes and two scales

Throughout, the model is

$$W_t(r) = (-1)^{\sum_p v_p(r) Q_p(t)}, \quad F_t^f(r) = f(r) W_t(r), \quad r \in \mathbb{Q}_{>0}, \quad (1.8)$$

where the Q_p are independent Poisson processes of rates $a_p \in [0, 1]$, starting at zero. Each rational uses finitely many primes, even if the sum of all rates is infinite. A clock ring flips the sign; this is not fair resampling. In particular,

$$\mathbb{E} W_t(p) = e^{-2ta_p}.$$

For full prime noise, $a_p = 1$ and the local scale is $t \log \log X$. For selective completion, the rates tend to zero and vanish outside a reservoir adapted to P ; the scale is $t\Theta_X$, where

$$\Theta_X = \sum_{p \leq X} \frac{a_p}{p} \asymp_P \log \log \log X.$$

The last scale belongs to our convenient choice of rates; it is not claimed to be an optimal or intrinsic scale for all completions. The polynomial classification in Section 6 concerns full noise and does not transfer unchanged to trace-preserving noise.

Sparse among primes, not close on the integers. For selective noise, Proposition 5.2 contrasts three costs on one probability-one event: relative density zero for the changed primes, asymptotic density $1/2$ for the changed integers, and infinite reciprocal sum over the changed primes at every positive time. The first limit is uniform on $[0, T]$, the second on positive compact intervals $[u, T]$; no uniform speed is asserted for the third. Thus this completion does not have finite harmonic prime budget. A sparse set of changed prime coordinates need not produce a small change in the integer sequence.

1.3 Prior work and the contribution of the transfer

The existence of root-free primes for an irreducible polynomial of degree at least two is classical: see Serre's account of Jordan's theorem and its Chebotarev application [13, Theorems 1 and 4, Section 4]. We use the harmonic form supplied by Arango-Piñeros–Keliher–Keyes [1, Theorem

A]. Common reservoirs for several polynomials are governed by the Galois criteria of Berend–Bilu [3, Theorem 2] and Elsholtz–Klähn–Technau [6, Theorem 2.1 and Proposition 2.2]. Neither the reservoir nor this criterion is a new ingredient here.

There is also a direct antecedent for the constant source. Theorem 2.1 of Klurman–Mangerel–Teräväinen [9], with numbering as in arXiv:2304.05344v2, gives (1.2) for

$$w(n) = (-1)^{\Omega_E(n)}$$

whenever E has relative density zero among the primes and $\sum_{p \in E} 1/p = \infty$. Here Ω_E counts prime factors in E with multiplicity. Choosing such an E inside a root-free reservoir, beyond the prescribed prefix, already gives Theorem 1.1’s existence conclusion for $f \equiv 1$. Initial translations of the index accommodate the integer constants in (1.1). A sparse set of the required type can be obtained by independently retaining each reservoir prime with probability $1/b(p)$ from Section 2.2, and fixing a realisation: the prime density follows from bounded-variable concentration, while the harmonic mass diverges because its centred series has summable variances. We do not claim this constant-source consequence as a separate novelty.

The arbitrary-source conclusion does not follow just by multiplying such a w by f : taking $f = w$ would give $fw \equiv 1$. The mechanism used here is instead Theorem 5.1, which regularises against *arbitrary bounded weights fixed before the noise*. No entropy or complexity assumption is imposed on the weights. This use of “fixed” is different from the zero-entropy meaning of “deterministic sequence” preceding Corollary 2.2 in [9]. For each prescribed source and weights, the construction yields a probability-one event; there is no single event for all sources chosen after observing the clocks.

The technical transfer is Proposition 4.2: arithmetic degree and incidence bounds are converted into a uniform-in-time estimate by revealing the small-prime trajectories, enriching a regular time mesh with their jumps, and controlling the fraction of origins affected between mesh points. Theorem 5.1 then supplies the completion, the marked law and a vanishing-delay version. The concentration inequality itself is the classical bounded-differences inequality of McDiarmid [10]; Warnke [14, Theorem 1] recalls the form used here, rather than providing an additional “typical differences” input.

For probabilistic context, Fish [7] established almost-sure normality for fair prime signs. Najnudel [11] treats consecutive values for prime laws uniform on the circle or on roots of unity; his Section 7 studies more general prime laws for one coordinate. Multiplying fair independent signs by a fixed source would not change their law, so arbitrary-source language alone does not extend that fair model. Here the source is combined with biased, non-stationary, selective noise and exact protection. Biased random multiplicative functions have also been studied by Aymone–Sidoravicius [2], in a squarefree-supported model and for partial-sum questions, rather than the completely multiplicative multivariate completion considered here.

The remaining analytic input is a root-counting Mertens formula [8]. Boolean noise and Fourier analysis [12] are classical. For full noise, the result is their quantitative transfer to a moving empirical arithmetic law, including the random translation from rational constants. All transfer arguments used below are given explicitly. No relation profile for growing words is imported.

1.4 Two routes through the proof

To follow the completion theorem, read Sections 2, 3.2, 4.2 and 5. Section 4.1 additionally describes its finite-local-time memory. For the polynomial channel under full noise, read Sections 3.1, 4, 6 and 7. The examples in Sections 8.1–8.2 illustrate that channel; Section 8.3 returns to selective protection.

Both routes use the same sequence of ideas. Identify primes acting to odd parity on an observation. Estimate their typical total rate. Count how many origins a single prime can affect. Convert these arithmetic budgets into temporal concentration, before applying finite Fourier analysis to the desired joint law.

2 Invisible primes and sparse clocks

2.1 A root-free reservoir

Lemma 2.1 (Root-free reservoir: Jordan–Chebotarev and Mertens). *For P as in Theorem 1.1, there are a prime set S , disjoint from any prescribed finite set, and a number $\delta > 0$ such that*

$$p \nmid P(n) \quad (p \in S, n \in \mathbb{Z}), \quad \sum_{\substack{p \leq x \\ p \in S}} \frac{1}{p} = \delta \log \log x + O_P(1). \quad (2.1)$$

The remainder constant may also depend on the removed finite set.

Proof. We recall the classical argument; see [13]. Let K be the splitting field and $G = \text{Gal}(K/\mathbb{Q})$ act on the $d = \deg P$ roots. This action is transitive. Burnside’s formula gives mean fixed-point count one. If every element had a fixed point, the identity, with $d > 1$ fixed points, would make this mean exceed one. Thus the set of derangements is non-empty and invariant under conjugation.

Remove the primes dividing the content, leading coefficient or discriminant, the ramified primes and the prescribed finite set. At every remaining prime, the number of roots modulo p is the number of fixed points of Frobenius on the roots. The derangement classes therefore supply a root-free Chebotarev set of density

$$\delta = \frac{\#\{\sigma \in G : \sigma \text{ fixes no root}\}}{|G|} > 0.$$

The Mertens theorem for these classes [1, Theorem A], after taking the logarithm of its product and summing the p^{-2} terms, gives (2.1). A prime without a root divides no integer value of P , proving the pointwise assertion. $\square$

The protection is exact. It uses neither a density estimate for the values of P nor a conjecture about their large prime factors.

2.2 An explicit choice of rates

The rates must serve three purposes. The condition $a_p \rightarrow 0$ makes the primes that ring on a bounded horizon sparse. Divergence of $\sum_p a_p/p$ leaves enough cumulative action for dissipation. Finiteness of $\sum_p a_p^2/p$ gives a bounded weighted second moment in Lemma 3.2. Our choice has the simple asymptotic $a_p \sim (\log 2)/\log \log p$ along S ; the integer-valued denominator below makes it exact from the first prime onwards.

For a set S satisfying (2.1), put

$$b(p) = 1 + \lfloor \log_2 \lfloor \log_2 p \rfloor \rfloor, \quad a_p = \frac{\mathbf{1}_{p \in S}}{b(p)}, \quad \Theta_X = \sum_{p \leq X} \frac{a_p}{p}. \quad (2.2)$$

This is defined from $p = 2$ onwards. It is a convenient choice, not an optimality assertion.

Lemma 2.2 (Rate budget). *With $P_X = \lceil (\log X)^4 \rceil$, we have*

$$a_p \longrightarrow 0, \quad \Theta_X = \delta(\log 2) \log \log \log X + O_S(1) \longrightarrow \infty, \quad (2.3)$$

$$\sum_p \frac{a_p^2}{p} < \infty, \quad \Theta_{P_X} = O_S(\log(2 + \Theta_X)) = o(\Theta_X). \quad (2.4)$$

Proof. We have $b(p) = \log \log p / \log 2 + O(1)$, and hence, for sufficiently large p ,

$$\frac{1}{b(p)} = \frac{\log 2}{\log \log p} + O\left(\frac{1}{(\log \log p)^2}\right).$$

Partial summation in (2.1) gives $\delta(\log 2) \log \log \log X + O_S(1)$ for the sum of the first term weighted by $1/p$. The error is summable, as is a_p^2/p . Small primes only change a constant. Substituting $P_X = (\log X)^4 + O(1)$ into the asymptotic proves the last assertion. $\square$

For Theorem 1.1, all primes $p \leq M_0$ are removed. The clock support S may have positive relative density. It is the set of primes whose signs actually change, or even whose clocks ring during a fixed bounded horizon, that will have relative density zero.

3 The two arithmetic budgets

Unless the dyadic alternative is explicitly selected, averages are over

$$I_X = \{n_0 + 1, \dots, n_0 + X\},$$

with n_0 fixed before the noise and large enough to make all arguments positive and non-zero. The same proofs hold for $[X, 2X) \cap \mathbb{Z}$. Implicit constants may depend on the fixed polynomials and on n_0 . We always take X sufficiently large for $h = \log \log X$ to be positive.

3.1 Full noise: the number of irreducible factors

A prime matters to a sign observation precisely when its valuation is odd. The typical number of such primes will be $k \log \log X$, where k counts irreducible factors, not the algebraic degree: $T^2 + 1$ has $k = 1$, whereas $T(T + 1)$ has $k = 2$. This is why the limiting channel has one clock per factor. The calculation below is not a sieve asserting that many polynomial values are squarefree. For a primitive, squarefree, non-constant and eventually positive polynomial $R \in \mathbb{Z}[T]$, let k be its number of distinct irreducible factors over $\mathbb{Q}$, and put $D = \deg R$. If $\rho_R(p)$ counts roots modulo p , the classical input is

$$\sum_{p \leq z} \frac{\rho_R(p)}{p} = k \log \log z + O_R(1). \quad (3.1)$$

This follows, for example, from Theorems 1–2 of Garcia–Lee–Suh–Yu [8]. The family is fixed: no uniformity in a varying discriminant is used. Define

$$S_R(n) = \{p : v_p(R(n)) \text{ is odd}\}, \quad K_R(n) = |S_R(n)|, \quad A_p = \#\{n \in I_X : p \in S_R(n)\}.$$

The incidence A_p is the number of origins at which prime p can change the observed sign. A flip changes a normalised average by at most $2A_p/X$, so the squared incidences in the next lemma are exactly the bounded-differences budget for large primes. The sum of incidences controls the expected fraction of origins touched in a short time interval. The centred moment ensures that few origins have too little noise. These are the three inputs of Proposition 4.2.

Lemma 3.1 (Typical degree and influences). *With $P_X = \lceil (\log X)^4 \rceil$ and $h = \log \log X$,*

$$\frac{1}{X} \sum_{n \in I_X} (K_R(n) - kh)^2 \ll_R h, \quad (3.2)$$

$$\frac{1}{X^2} \sum_{n, m \in I_X} |S_R(n) \cap S_R(m)| \ll_R 1, \quad (3.3)$$

$$\sum_{p > P_X} (A_p/X)^2 \ll_R P_X^{-1}, \quad \sum_p A_p/X \ll_R h. \quad (3.4)$$

Estimate (3.2) remains valid with $K_R(n)$ replaced by $|S_R(n) \cap (P_X, \infty)|$ and with the same centre kh .

Proof. First remove primes dividing the leading coefficient or discriminant. Their contribution to K_R is bounded. At all other primes the roots are simple, and their number is unchanged on lifting to modulus p^2 .

Put $z = X^{1/4}$ and $V(n) = \sum_{p \leq z}^* \mathbf{1}_{p|R(n)}$, where the star removes the exceptional primes. Counting congruence classes gives, for $p \neq q$,

$$\frac{1}{X} \sum_{n \in I_X} \mathbf{1}_{p|R(n)} = \frac{\rho_R(p)}{p} + O_R(X^{-1}), \quad \frac{1}{X} \sum_{n \in I_X} \mathbf{1}_{pq|R(n)} = \frac{\rho_R(p)\rho_R(q)}{pq} + O_R(X^{-1}).$$

Expanding the square and applying (3.1) yields

$$\frac{1}{X} \sum_{n \in I_X} (V(n) - kh)^2 \ll_R h + \frac{z^2 + zh}{X} \ll_R h. \quad (3.5)$$

The discrepancy between divisibility and odd valuation among these primes is bounded by

$$J(n) = \sum_{p \leq z}^* \mathbf{1}_{p^2|R(n)}.$$

Its second moment satisfies

$$\frac{1}{X} \sum_{n \in I_X} J(n)^2 \ll_R \sum_p p^{-2} + \left(\sum_p p^{-2} \right)^2 + \frac{z^2}{X} \ll_R 1.$$

Indeed, the main terms come from congruences modulo p^2 and p^2q^2 , and the $O(z^2)$ counting errors contribute $O_R(z^2/X)$. No assertion that many values $R(n)$ are squarefree is needed. Beyond z , each value $R(n) \ll_R X^D$ has only $O_R(1)$ distinct prime factors. This proves (3.2).

For the truncated version, start the sum at P_X . Its centre is

$$k \log \log z - k \log \log P_X + O_R(1) = kh + O_R(\log h).$$

The square of the displacement is $o(h)$, so the same argument applies.

Since R is primitive, it is not identically zero modulo any prime, and

$$A_p \leq D(X/p + 1). \quad (3.6)$$

For $P_X < p \leq X$, summing the squares in (3.6) gives $O_R(P_X^{-1} + \log X/X)$. For $p > X$, there are at most D relevant indices, so

$$\sum_{p > X} \frac{A_p^2}{X^2} \leq \frac{D}{X^2} \sum_p A_p \ll_R \frac{\log X}{X},$$

using the elementary bound $\omega(R(n)) \leq \log R(n)/\log 2$. This handles prime divisors as large as X^D without summing a constant error over every prime up to that size. Without the cutoff, the same estimates give $\sum_p (A_p/X)^2 \ll_R 1$. The identity

$$\sum_{n,m \in I_X} |S_R(n) \cap S_R(m)| = \sum_p A_p^2$$

proves (3.3). Finally, (3.2) gives $\sum_p A_p/X = kh + O_R(\sqrt{h})$. $\square$

Polynomial factors with even exponent disappear completely from the sign observation.

3.2 Selective noise: weighted degree of an affine product

For selective noise the same incidences are weighted by the clock rates. The typical total rate is now $k\Theta_X$; its centred second moment is bounded, thanks to $\sum_p a_p^2/p < \infty$. In the temporal argument, $\sum_p a_p A_p/X$ plays the role previously taken by the unweighted incidence sum. For an admissible family, put $R(n) = \prod_{i=1}^k \ell_i(n)$ and

$$S_R(n) = \{p : v_p(R(n)) \text{ is odd}\}, \quad d_R(n) = \sum_{p \in S_R(n)} a_p.$$

The rates now are (2.2), and in particular vanish outside S . Keep $A_p = \#\{n \in I_X : p \in S_R(n)\}$.

Lemma 3.2 (Weighted moments). *We have*

$$\frac{1}{X} \sum_{n \in I_X} (d_R(n) - k\Theta_X)^2 \ll_{\ell,S} 1, \quad (3.7)$$

$$\frac{1}{X^2} \sum_{n,m \in I_X} \sum_{p \in S_R(n) \cap S_R(m)} a_p \ll_{\ell} 1, \quad (3.8)$$

$$\sum_{p > P_X} (A_p/X)^2 \ll_{\ell} P_X^{-1}, \quad \sum_p a_p A_p/X \ll_{\ell,S} 1 + \Theta_X. \quad (3.9)$$

After removing primes $p \leq P_X$, the weighted degree still has bounded centred second moment, with centre $k(\Theta_X - \Theta_{P_X})$.

Proof. The primes dividing some A_i or some determinant $A_i B_j - A_j B_i$ form a finite set and contribute a bounded amount to the degree. At every other prime, R has exactly k simple roots modulo p and p^2 , and k^2 roots modulo pq and $p^2 q^2$ for distinct primes p, q .

With $z = X^{1/4}$ and $V(n) = \sum_{p \leq z}^* a_p \mathbf{1}_{p|R(n)}$, the same congruence counts give

$$\frac{1}{X} \sum_{n \in I_X} \left(V(n) - k \sum_{p \leq z}^* \frac{a_p}{p} \right)^2 \ll_{\ell} \sum_p \frac{a_p^2}{p} + \frac{z^2 + z(1 + \Theta_X)}{X} \ll_{\ell,S} 1.$$

The second moment of $\sum_{p \leq z}^* a_p \mathbf{1}_{p^2|R(n)}$ is bounded as above. Each value $R(n) \ll_{\ell} X^k$ has $O_{\ell}(1)$ distinct prime factors exceeding z . Moreover,

$$0 \leq \Theta_X - \Theta_z \leq \sum_{z < p \leq X} \frac{1}{p} = O(1).$$

This proves (3.7); starting at P_X proves its truncated version.

For non-exceptional primes, $A_p \leq k(X/p + 1)$, and every prime dividing one of the affine forms is at most $C_\ell X$. The exceptional primes lie below P_X for large X . The sum of squared influences beyond P_X is therefore $O_\ell(P_X^{-1})$. Double counting gives

$$\frac{1}{X^2} \sum_{n,m} \sum_{p \in S_R(n) \cap S_R(m)} a_p = \sum_p a_p (A_p/X)^2 \ll_\ell 1.$$

The last assertion in (3.9) is the mean of d_R , bounded using (3.7). $\square$

4 The common probabilistic mechanism

4.1 A finite lemma, uniform at deterministic times

Lemma 4.1 (Bias and variance of prime monomials). *Let independent Poisson clocks have rates $a_p \geq 0$, let $S_1, \dots, S_M$ be finite sets, and let $|c_i| \leq 1$ be real coefficients. Put*

$$\xi_A(t) = \prod_{p \in A} (-1)^{Q_p(t)}, \quad Z(t) = \frac{1}{M} \sum_{i=1}^M c_i \xi_{S_i}(t), \quad d_i = \sum_{p \in S_i} a_p, \quad g_{ij} = \sum_{p \in S_i \cap S_j} a_p.$$

For $\mu > 0$, define

$$b = \frac{1}{M} \#\{i : d_i < \mu/2\}, \quad D_1 = \frac{1}{M} \sum_i |d_i - \mu|, \quad G = \frac{1}{M^2} \sum_{i,j} g_{ij}, \quad \bar{c} = \frac{1}{M} \sum_i c_i.$$

Then, for every $t \geq 0$,

$$|\mathbb{E}Z(t) - e^{-2t\mu}\bar{c}| \leq b + \frac{2D_1}{\mu}, \quad \text{Var } Z(t) \leq 2b + \frac{8G}{\mu}. \quad (4.1)$$

Proof. Independence at the primes gives the exact identities

$$\mathbb{E}\xi_{S_i}(t) = e^{-2td_i}, \quad (4.2)$$

$$\text{Cov}(\xi_{S_i}(t), \xi_{S_j}(t)) = e^{-2t(d_i+d_j-2g_{ij})}(1 - e^{-4tg_{ij}}) \geq 0. \quad (4.3)$$

When $d_i \geq \mu/2$, the mean value theorem gives

$$|e^{-2td_i} - e^{-2t\mu}| \leq 2te^{-t\mu}|d_i - \mu| \leq 2|d_i - \mu|/\mu.$$

All remaining indices cost at most b , proving the bias estimate.

For the variance, the non-negative covariances allow us to bound $c_i c_j$ in absolute value. Pairs containing a small degree cost at most $2b$. Pairs with $g_{ij} > \mu/4$ have proportion at most $4G/\mu$. For the other pairs, $d_i + d_j - 2g_{ij} \geq \mu/2$ and the covariance is at most

$$4tg_{ij}e^{-t\mu} \leq 4g_{ij}/\mu.$$

Summation gives the stated bound, including the diagonal pairs. $\square$

The monomials need not be independent. A bound for the small-degree proportion and for mean overlap suffices. Together with Lemmas 3.1 and 3.2, the lemma gives L^2 errors $O(h^{-1/2})$ and $O(\Theta_X^{-1/2})$, respectively, uniform at deterministic times. Such estimates do not by themselves control a random temporal supremum.

4.2 Concentration on an entire time interval

Small primes can affect many origins at once. We therefore reveal their entire paths: they remain present as a known time-dependent mask, rather than being discarded or frozen. Large primes have small individual influence, so their squared-incidence budget gives conditional concentration at a fixed time, including a time determined only by the small-prime paths.

To pass between such times, enrich a regular mesh with every small-prime jump. Within a mesh cell, control the fraction of origins touched by a large-prime ring, not the number of those rings. An origin touched repeatedly still contributes at most a difference of two signs. This is the step that turns fixed-time concentration into a bound for a whole trajectory.

Proposition 4.2 (Temporal concentration after revealing small-prime paths). *For each integer X , let $S_X(n)$, $n \in I_X$, be finite prime supports, and put $P_X = \lceil (\log X)^4 \rceil$. Let the clock rates satisfy $0 \leq a_p \leq 1$, and let $|c_n| \leq 1$ be arbitrary real coefficients fixed before the noise. Write*

$$A_p = \#\{n : p \in S_X(n)\}, \quad d_{>P_X}(n) = \sum_{\substack{p \in S_X(n) \\ p > P_X}} a_p, \quad Z_X(t) = \frac{1}{X} \sum_{n \in I_X} c_n \xi_{S_X(n)}(t).$$

Assume $\Lambda_X \rightarrow \infty$, $\Lambda_X \leq \log X$ for all sufficiently large X , and

$$\frac{1}{X} \#\{n : d_{>P_X}(n) < \Lambda_X/2\} \leq C\Lambda_X^{-1}, \quad (4.4)$$

$$\sum_{p > P_X} (A_p/X)^2 \leq C/P_X, \quad \sum_p a_p A_p/X \leq C\Lambda_X. \quad (4.5)$$

For each $0 < T < \infty$ there exist $c, C_T > 0$ such that, for $0 \leq u \leq T$ and $\varepsilon > 0$,

$$\mathbb{P} \left\{ \sup_{u \leq t \leq T} |Z_X(t)| > e^{-u\Lambda_X} + C_T \Lambda_X^{-1} + 3\varepsilon \right\} \leq C_T (1 + P_X + \Lambda_X^2) e^{-c\varepsilon^2 P_X}. \quad (4.6)$$

Consequently, for every positive sequence u_X , fixed before the noise, with $u_X \rightarrow 0$ and $u_X \Lambda_X \rightarrow \infty$, almost surely for all sufficiently large X ,

$$\sup_{u_X \leq t \leq T} |Z_X(t)| \ll_T e^{-u_X \Lambda_X} + \Lambda_X^{-1}. \quad (4.7)$$

The constants depend on the constants in the assumptions, but not on the coefficients or on u . The coefficients may vary with X provided they are specified before the clocks.

Proof. For each X , the union $\bigcup_{n \in I_X} S_X(n)$ is finite. Only these clocks influence Z_X , so its relevant paths have finitely many jumps on a bounded horizon, almost surely. All temporal suprema below are therefore well defined; the argument is not an intersection over uncountably many time-indexed events.

Condition on the entire trajectories of the primes $p \leq P_X$. They give a known, time-dependent sign mask; their values are not frozen in time. For $t \geq u$, assumption (4.4) gives

$$|\mathbb{E}(Z_X(t) \mid \text{small-prime paths})| \leq \frac{1}{X} \sum_{n \in I_X} e^{-2ud_{>P_X}(n)} \leq e^{-u\Lambda_X} + C\Lambda_X^{-1}. \quad (4.8)$$

Changing one large-prime sign affects $Z_X(t)$ by at most $2A_p/X$. The classical bounded-differences inequality [10, 14], using (4.5), yields

$$\mathbb{P}\{|Z_X(t)| > e^{-u\Lambda_X} + C\Lambda_X^{-1} + \varepsilon \mid \text{small-prime paths}\} \leq 2e^{-c\varepsilon^2 P_X}. \quad (4.9)$$

The estimate also holds for a time determined solely by the small-prime paths, and for the mask immediately before or after one of their rings. At a time fixed by these paths, large clocks almost surely do not ring exactly at that time.

Choose the regular deterministic mesh

$$N_X = \max\{1, \lceil T\Lambda_X^2 \rceil\}, \quad t_j = \frac{jT}{N_X} \quad (0 \leq j \leq N_X).$$

It has $O_T(1 + \Lambda_X^2)$ points and step at most Λ_X^{-2} . Add u and all ring times of primes $p \leq P_X$ in $[0, T]$. The enriched mesh has expected cardinality $O_T(1 + \Lambda_X^2 + P_X)$, since the rates are at most one. Conditional on the small-prime paths, these points are fixed independently of the large clocks, so (4.9) applies at each point in $[u, T]$, recording left and right values where needed.

For one deterministic mesh cell I , let J_I be the fraction of origins whose support contains at least one large prime ringing in I . A union bound gives

$$\mathbb{E}J_I \leq |I| \sum_p a_p A_p / X \leq C\Lambda_X^{-1}.$$

The large-prime ringing indicators are independent. Changing one indicator affects J_I by at most A_p/X . A second application of bounded differences gives

$$\mathbb{P}\{J_I > C\Lambda_X^{-1} + \varepsilon\} \leq e^{-c\varepsilon^2 P_X}.$$

Between consecutive enriched mesh points, the small-prime mask is constant and any difference of two values of Z_X is bounded by $2J_I$ in the containing deterministic cell. Multiple rings at one origin do not invalidate this bound: it controls a difference of values, not the total variation of the entire path. A conditional union bound over enriched points, a union bound over cells, and integration prove (4.6). The cost is the mass of affected origins, not the number of large-prime rings.

Finally, take $\varepsilon = 1/\log X$ and $u = u_X$. Since $P_X \asymp (\log X)^4$ and $\Lambda_X \leq \log X$, the right-hand side is

$$O_T((\log X)^4 e^{-c(\log X)^2}),$$

summable over all positive integers X . Borel–Cantelli gives (4.7), since $1/\log X \leq 1/\Lambda_X$. No monotonicity of u_X is needed. $\square$

For full noise, Lemma 3.1 verifies the assumptions with $\Lambda_X = h$. For selective noise on an affine product, the truncated centre is $k(\Theta_X - \Theta_{P_X}) \sim k\Theta_X$ and the centred second moment is bounded. The proportion below $\Theta_X/2$ is therefore even $O_{\ell,S}(\Theta_X^{-2})$. Proposition 4.2 applies with $\Lambda_X = \Theta_X$.

5 Selective completion: proof and cost

5.1 Weighted cancellation and local memory

Theorem 5.1 (Affine regularisation against fixed bounded weights). *Fix S satisfying (2.1), rates (2.2), a source $f \in \mathcal{M}_\pm$ and arbitrary real weights $|v_n| \leq 1$, all before the clocks. For an admissible affine family and a positive sequence $u_X \rightarrow 0$ fixed before the noise with $u_X \Theta_X \rightarrow \infty$, almost surely,*

$$\sup_{u_X \leq t \leq T} \left| \frac{1}{X} \sum_{n \in I_X} v_n \prod_{i=1}^k F_t^f(\ell_i(n)) \right| \ll_{\ell,S,T} e^{-u_X \Theta_X} + \Theta_X^{-1} \quad (5.1)$$

for all sufficiently large X and every fixed $T < \infty$. The event may depend on f , v and u_X . No complexity or entropy assumption is made on the weights.

Proof. Write the product as $f(R(n))\xi_{S_R(n)}(t)$, where $R = \prod_i \ell_i$. Apply Proposition 4.2 with coefficients $c_n = v_n f(R(n))$, supports and estimates from Lemma 3.2, and $\Lambda_X = \Theta_X$. Intersect over positive integer horizons to obtain every finite horizon. $\square$

The information retained at finite local time is also useful. For arbitrary real coefficients $|c_n| \leq 1$ fixed before the noise, Lemmas 3.2 and 4.1, with $\mu = k\Theta_X$, give

$$\sup_{t \geq 0} \left\| \frac{1}{X} \sum_{n \in I_X} c_n W_t(R(n)) - e^{-2kt\Theta_X} \frac{1}{X} \sum_{n \in I_X} c_n \right\|_{L^2(\mathbb{P})} \ll_{\ell, S} \Theta_X^{-1/2}. \quad (5.2)$$

Indeed, $b \ll \Theta_X^{-2}$, $D_1 \ll 1$ and $G \ll 1$ in (4.1). The supremum lies outside the norm. At time $t = s/\Theta_X$, the memory multiplier is the non-zero number e^{-2ks} .

There is no conflict between this memory and regularity at every positive time. Each fixed integer prefix involves finitely many clocks, so almost surely it stays equal to its initial value for a short random interval after zero. Regularisation concerns the limit $X \rightarrow \infty$ at positive times, or after delays satisfying $u_X \Theta_X \rightarrow \infty$. These assertions do not exchange the limits in time and observation size.

5.2 Proof of the main theorem and the marked version

Proof of Theorem 1.1. Choose the reservoir in Lemma 2.1, removing all $p \leq M_0$, and use rates (2.2). No active prime divides a value of P , so (1.3) holds pointwise along every trajectory at every time. Integers $n \leq M_0$ involve only inactive primes, giving (1.4).

For large X , take $u_X = \Theta_X^{-1/2}$. Theorem 5.1 with $v_n = 1$, followed by a countable intersection over integer-coefficient affine families and integer horizons, gives $F_t^f \in \mathcal{L}_\pm$ at every real $t > 0$ on one probability-one event. Each such time eventually lies above u_X . Proposition 5.2 below gives the prime-density assertion, uniformly on bounded horizons. Fixing $t = 1$ on an admissible realisation supplies g . $\square$

Proof of Corollary 1.2. For $x \in \{-1, 1\}^k$ and a mark state z , expand

$$\mathbf{1}_{(F_t^f(\ell_i(n)))_{i=1}^k = x} \mathbf{1}_{z(n)=z} = 2^{-k} \mathbf{1}_{z(n)=z} \prod_{i=1}^k (1 + x_i F_t^f(\ell_i(n))).$$

After averaging, the empty term is exactly $2^{-k} \nu_X\{z\}$. Each non-empty term is controlled by (5.1), using the fixed weight $\mathbf{1}_{z(n)=z}$ and the corresponding affine subfamily. Summing over the finitely many states gives (1.7). This also gives the vanishing-delay version with error $O(e^{-u_X \Theta_X} + \Theta_X^{-1})$ for fixed forms and marks. The arithmetic estimates and the concentration argument apply separately to the fixed-prefix and dyadic choices of I_X . $\square$

5.3 Prime density, integer disagreement and harmonic mass

Proposition 5.2 (Prime cost and integer disagreement). *For selective noise, almost surely, for every $T < \infty$,*

$$\sup_{0 \leq t \leq T} \frac{\#\{p \leq x : F_t^f(p) \neq f(p)\}}{\pi(x)} \longrightarrow 0. \quad (5.3)$$

On one probability-one event, for every $0 < u < T < \infty$,

$$\sup_{u \leq t \leq T} \left| \frac{1}{X} \#\{n \leq X : F_t^f(n) \neq f(n)\} - \frac{1}{2} \right| \longrightarrow 0. \quad (5.4)$$

On that event, for all real times $t > 0$ simultaneously,

$$\sum_{p: F_t^f(p) \neq f(p)} \frac{1}{p} = \infty. \quad (5.5)$$

The last assertion does not claim a uniform rate of divergence on time intervals.

Proof. For fixed T , the variables $I_p = \mathbf{1}_{Q_p(T) \geq 1}$ are independent with means $1 - e^{-Ta_p} \rightarrow 0$. Hoeffding's bounded-variable inequality, followed by Borel–Cantelli along the first m primes, gives $m^{-1} \sum_{j \leq m} I_{p_j} \rightarrow 0$. Every prime changed before time T has rung, proving (5.3). Positive integer horizons form a countable family.

The quotient $F_t^f(n)/f(n)$ equals $W_t(n)$, independently of f . Hence

$$\frac{1}{X} \# \{n \leq X : F_t^f(n) \neq f(n)\} = \frac{1}{2} \left(1 - \frac{1}{X} \sum_{n \leq X} W_t(n) \right).$$

Theorem 5.1 applied to the constant source and the single form n proves (5.4), on one event for all positive compact time intervals.

For the simultaneous conclusion (5.5), we use a deterministic fact. If $w \in \mathcal{M}_\pm$ and $E = \{p : w(p) = -1\}$ satisfies $\sum_{p \in E} 1/p < \infty$, then

$$\lim_{X \rightarrow \infty} \frac{1}{X} \sum_{n \leq X} w(n) = \prod_{p \in E} \frac{p-1}{p+1} > 0. \quad (5.6)$$

To prove it, let w_y retain only the negative prime signs in $E \cap [2, y]$. For this finite set, truncate the valuations and apply the Chinese remainder theorem. Passing through the valuation truncations gives

$$\lim_{X \rightarrow \infty} \frac{1}{X} \sum_{n \leq X} w_y(n) = \prod_{\substack{p \in E \\ p \leq y}} \left((1 - p^{-1}) \sum_{j \geq 0} \frac{(-1)^j}{p^j} \right) = \prod_{\substack{p \in E \\ p \leq y}} \frac{p-1}{p+1}.$$

The discarded valuation tails have density tending to zero, which justifies the finite-set limit. If $w(n) \neq w_y(n)$, some prime of E greater than y divides n . Uniformly in X ,

$$\frac{1}{X} \sum_{n \leq X} |w(n) - w_y(n)| \leq 2 \sum_{\substack{p \in E \\ p > y}} \frac{1}{p}. \quad (5.7)$$

This tends to zero as $y \rightarrow \infty$. The infinite product converges to a positive number because its logarithmic terms are $O(1/p)$ in absolute value for large p . This proves (5.6).

On the event already used for (5.4), the mean of W_t is zero for every $t > 0$. Apply the deterministic implication (5.6) to each of these functions W_t . Its negative-prime set cannot have finite harmonic mass. Since $W_t(p) = -1$ precisely when $F_t^f(p) \neq f(p)$, (5.5) follows *without any additional probabilistic intersection over times*.

A separate fixed-time calculation retains quantitative information about the expected truncated mass. For fixed $t > 0$, the modification indicators are independent with probabilities $q_p(t) = (1 - e^{-2ta_p})/2$, and (2.4) gives

$$\mathbb{E} \sum_{\substack{p \leq X \\ F_t^f(p) \neq f(p)}} \frac{1}{p} = \sum_{p \leq X} \frac{q_p(t)}{p} = t\Theta_X + O_t(1). \quad (5.8)$$

The centred series $\sum_p (\mathbf{1}_{Q_p(t) \text{ odd}} - q_p(t))/p$ converges almost surely, by summability of its variances. This is a useful fixed-time description, but the all-time assertion above comes from the deterministic argument. $\square$

The completion is neither a negligible Hamming perturbation of the integers nor a surgery with finite harmonic prime budget. Relative prime density alone does not measure its effect on observations.

5.4 An entire family of protected norms

Root-freeness is sufficient for protection, but stronger than necessary. A prime whose valuation is always even on the protected arguments is equally invisible to a sign flip; this protects the whole family below.

Corollary 5.3 (Sums of two squares). *For every source $f \in \mathcal{M}_\pm$ and every prescribed prefix, the construction can satisfy, at all times,*

$$F_t^f(a^2 + b^2) = f(a^2 + b^2) \quad (a, b \in \mathbb{Z}, a^2 + b^2 > 0),$$

while retaining all the preceding completion and cost conclusions.

Proof. Take $S = \{p > M_0 : p \equiv 3 \pmod{4}\}$, with any further finite removals. It satisfies (2.1) with $\delta = 1/2$. If such a p divides $a^2 + b^2$, then it divides both a and b . Dividing by p^2 and repeating shows that $v_p(a^2 + b^2)$ is even. Any sign change at these primes therefore leaves the norm value unchanged. Affine regularisation uses only the harmonic reservoir estimate and the rates. $\square$

For $f = \lambda$, all the protected values are exactly Liouville values. For $f \equiv 1$, the completion is constantly $+1$ on these norms while belonging to $\mathcal{L}_\pm$. As explained above, the latter existence case follows already from [9, Theorem 2.1, arXiv v2]. It illustrates a previously available separation: affine cancellation alone does not force a vanishing polynomial mean.

6 The irreducible-factor channel

Throughout this section every prime clock has rate one. The model remains (1.8), but it no longer generally preserves the trace singled out by selective completion.

The need for a factor channel is already visible in a small example. Put $A = T^2 + 1$ and $B = T^2 + 2$. The observations (A, B, AB, A^2) always give a sign vector of the form $(x, y, xy, 1)$, not four free signs. A flip in the A factor changes the first and third coordinates; a flip in B changes the second and third; the square does not change. The matrix below records exactly these shared actions. Section 8.1 gives the four states and their limiting frequencies.

6.1 Observation algebra and the rational-constant environment

Fix $R_1, \dots, R_m \in \mathbb{Q}(T)^\times$, positive at all sufficiently large integers. Factor them as

$$R_j(T) = c_j \prod_{\alpha=1}^r P_\alpha(T)^{e_{j\alpha}}, \quad c_j \in \mathbb{Q}_{>0}, \quad e_{j\alpha} \in \mathbb{Z}, \quad (6.1)$$

where the $P_\alpha \in \mathbb{Z}[T]$ are primitive, irreducible, distinct and have positive leading coefficient. Negative exponents are allowed. In the parity matrix, row j describes observation R_j , while column α records which observations change when the sign of factor P_α flips. Put

$$E = (e_{j\alpha} \bmod 2) \in \mathbb{F}_2^{m \times r}, \quad H = \{(-1)^{Ez} : z \in \mathbb{F}_2^r\}, \quad d(u) = \text{wt}(E^T u) \quad (u \in \mathbb{F}_2^m). \quad (6.2)$$

Sign vectors are multiplied coordinatewise. The initial and noisy empirical laws are

$$\mu_X^f = \frac{1}{X} \sum_{n \in I_X} \delta_{(f(R_j(n)))_j}, \quad \hat{\mu}_{X,t}^f = \frac{1}{X} \sum_{n \in I_X} \delta_{(F_t^f(R_j(n)))_j}.$$

The interval convention of Section 3 excludes zeros, poles and negative values. The initial law may vary without converging.

Let $B_\alpha(s) = (-1)^{N_\alpha(s)}$ be independent rate-one telegraph processes starting at $+1$. The channel K_s multiplies an initial sign vector by

$$\left(\prod_{\alpha} B_\alpha(s)^{e_{j\alpha}} \right)_{j \leq m}.$$

Writing v_α for column α of E , its generator and Fourier multipliers are

$$\mathcal{A}\varphi(x) = \sum_{\alpha=1}^r [\varphi(x \cdot (-1)^{v_\alpha}) - \varphi(x)], \quad \widehat{K}_s(u) = e^{-2sd(u)}. \quad (6.3)$$

This is an image of classical Boolean noise [12, Section 2.4, Definition 2.46 and Proposition 2.47], with noise parameter e^{-2s} . Repeated columns retain their multiplicities; zero columns have no visible effect. One irreducible factor, rather than one output observation, supplies each elementary clock. The rank of E determines the size of the compatible group H , while its columns with multiplicity determine the finite-time transition law, as Section 8.2 illustrates. For $a \in \{-1, 1\}^m$, write $\tau_a \mu$ for the law of aV when V has law μ .

Rational constants carry a different kind of information. For the pair $(n, 2n)$, the product of the observed signs is exactly $F_t^f(2)$. This pair is not admissible in (1.1), because its forms are proportional; it belongs to the rational-observation theory here. Its two-state compatible class moves with the clock at 2, which explains the translation in the next theorem.

Theorem 6.1 (Empirical channel with rational constants). *For the fixed family (6.1) and $h = \log \log X$,*

$$\sup_{t \geq 0} \sup_{f \in \mathcal{M}_\pm} \left\| d_{\text{TV}} \left(\hat{\mu}_{X,t}^f, \tau_{(W_t(c_j))_j} (\mu_X^f K_{th}) \right) \right\|_{L^2(\mathbb{P})} \ll_R h^{-1/2}. \quad (6.4)$$

Both suprema are outside the expectation. The source may depend on X as long as it is fixed before the clocks.

The target is random through primes dividing the numerators or denominators of the constants c_j . It is not assumed independent of the field. At time $t = s/h$ with bounded s , the translation equals the identity with probability $1 - O_R(s/h)$. At a fixed positive physical time, it cannot generally be dropped. “Constant environment” refers to constant rational *arguments*; their signs evolve in time.

Proof. For $u \in \mathbb{F}_2^m$, write

$$\prod_j R_j(T)^{u_j} = C_u R_u(T) A_u(T)^2, \quad C_u = \prod_j c_j^{u_j}, \quad R_u = \prod_{\alpha: (E^T u)_\alpha = 1} P_\alpha.$$

Set $C_{u,X}^f(t) = X^{-1} \sum_{n \in I_X} \prod_j F_t^f(R_j(n))^{u_j}$. If $d(u) > 0$, Lemma 3.1 for R_u gives, in Lemma 4.1 with $\mu = d(u)h$,

$$b \ll_R h^{-1}, \quad D_1 \ll_R \sqrt{h}, \quad G \ll_R 1.$$

Consequently,

$$\left\| C_{u,X}^f(t) - W_t(C_u) e^{-2d(u)th} C_{u,X}^f(0) \right\|_{L^2(\mathbb{P})} \ll_R h^{-1/2}. \quad (6.5)$$

Indeed, first apply the lemma with the fixed coefficient $f(R_u(n))$, then multiply the error by $F_t^f(C_u)$. Multiplication by this random sign preserves the L^2 norm exactly, whether or not it depends on the other clocks.

When $d(u) = 0$, there is instead the exact identity

$$C_{u,X}^f(t) = F_t^f(C_u). \quad (6.6)$$

The Fourier coefficients of the target in (6.4) are precisely the target coefficients in (6.5). Finite Fourier inversion and the triangle inequality in L^2 complete the proof, with constants depending on the fixed family. $\square$

6.2 Simultaneous classification at positive times

Theorem 6.2 (Square-compatible equidistribution). *Fix $f \in \mathcal{M}_\pm$ and a positive sequence $u_X \rightarrow 0$, before the clocks, with $u_X \log \log X \rightarrow \infty$. Almost surely, for every fixed family (6.1) and every $T < \infty$,*

$$\sup_{u_X \leq t \leq T} d_{\text{TV}}\left(\hat{\mu}_{X,t}^f, \text{Unif}(F_t^f(c)H)\right) \ll_{R,T} e^{-u_X h} + h^{-1} \quad (6.7)$$

for all sufficiently large X , where $F_t^f(c) = (F_t^f(c_j))_{j \leq m}$. A single realisation satisfies the resulting classification for all fixed rational families and all real $t > 0$.

Proof. For every u with $d(u) > 0$, apply Proposition 4.2 to R_u , with $\Lambda_X = h$ and $c_n = f(R_u(n))$. Lemma 3.1 verifies its assumptions. Multiplication by $F_t^f(C_u)$ does not change absolute values, even at a ring affecting a constant. These characters tend uniformly to zero at the asserted rate.

The characters with $E^T u = 0$ are exactly those annihilating H . By (6.6), they equal $F_t^f(C_u)$ at every size and time. These are the Fourier coefficients of the uniform law on $F_t^f(c)H$; all its other coefficients vanish. Fourier inversion proves (6.7). Rational families and integer horizons are countable, so we can intersect their probability-one events. The event depends on the fixed source, and each $t > 0$ is eventually above u_X . $\square$

Square identities are thus the exact invariant characters of the channel. The initial law is supported *within* $f(c)H$; not every point of that coset need occur at every finite size. The theorem describes the frequencies after regularisation.

Corollary 6.3 (Normality along a polynomial). *Let $P \in \mathbb{Q}[T]$ be eventually positive. If it is not of the form $cQ_0(T)^2$ with $c \in \mathbb{Q}_{>0}$ and $Q_0 \in \mathbb{Q}[T]$, then for every source f fixed before the clocks, almost surely*

$$n \mapsto F_t^f(P(n))$$

is normal for all real times $t > 0$ simultaneously. Word frequencies converge uniformly on compact time intervals contained in $(0, \infty)$, with the vanishing-delay version in (6.7). When $P = cQ_0^2$, the sequence is instead constantly $F_t^f(c)$ outside the omitted initial segment.

Proof. Take a non-empty set of distinct integer shifts A . Among the odd-multiplicity roots of P , choose one, α , with maximal real part, and put $a_0 = \min A$. The root $\alpha - a_0$ occurs to odd multiplicity in $P(T + a_0)$. If it occurred to odd multiplicity in another $P(T + a)$, then $\alpha + (a - a_0)$ would be an odd-multiplicity root of P with strictly greater real part. Any even-multiplicity occurrences do not change the parity. Hence $\prod_{a \in A} P(T + a)$ is not a constant times a square. All non-trivial word characters vanish by Theorem 6.2, giving frequencies $2^{-|A|}$. The square case follows directly from complete multiplicativity. $\square$

This corollary does not follow merely from normality on consecutive integers; the theorem is applied to the polynomial arguments themselves. It is also a first-order statement at scale X , not a central limit theorem at scale $\sqrt{X}$. For comparison, de la Bretèche–Wang–Xu [4, Theorem 1.2] prove a Gaussian limit for the fair completely multiplicative model, called extended Rademacher, when $P \in \mathbb{Z}[T]$ is separable, has positive leading coefficient and total degree at least two, and all its irreducible factors have degree at most three. Our conclusion requires less asymptotic precision and concerns a different, non-stationary prime law. It is not an extension of that central limit theorem to all degrees.

7 Local paths and memory of the source

There are two distinct path questions. Proposition 7.1 compares an entire path at one sampled origin after integrating both the origin and the clocks. Proposition 7.2 instead compares an empirical law at finitely many times after fixing the clocks, with its error averaged over the clocks. The second statement needs an additional variance estimate; a full-path empirical comparison in total variation is obstructed by finite support, as explained below.

7.1 Coupling at a random origin

The noise is still full, with rate one at every prime. Fix $S_0 > 0$ and choose U_X uniformly in I_X , independently of the clocks. For $0 \leq s \leq S_0$, observe

$$\gamma_X(s) = (F_{s/h}^f(R_j(U_X)))_{j \leq m}.$$

The target γ_X° starts from the same vector $(f(R_j(U_X)))_j$ and multiplies it by the factor telegraphs in (6.3).

Proposition 7.1 (Local path comparison, averaged over the clocks). *On the space of càdlàg paths, retaining the jump times,*

$$d_{\text{TV}}(\mathcal{L}(U_X, \gamma_X), \mathcal{L}(U_X, \gamma_X^\circ)) \ll_R S_0 h^{-1/2}. \quad (7.1)$$

Both the origin and the clocks are integrated in this comparison.

Proof. Remove a finite set B containing the primes in the constants and the resultants between distinct irreducible factors. Outside B , the prime supports of the values $P_\alpha(n)$ are disjoint at a fixed origin: a common prime would divide their non-zero resultant. The probability that any prime of B rings before S_0/h is at most $S_0|B|/h$.

At a fixed origin n , on the event of no such ring, the products of signs outside B are independent telegraphs of rates $K_{P_\alpha, \notin B}(n)/h$ in local time. Coupling rate- r and rate-one Poisson processes through their common part costs at most $S_0|r-1|$ on the horizon. Lemma 3.1 and Cauchy–Schwarz give

$$\frac{1}{X} \sum_{n \in I_X} |K_{P_\alpha, \notin B}(n)/h - 1| \ll_R h^{-1/2}.$$

Sum over the factors and average the origins. This proves (7.1). The coupling treats one sampled origin, not all positions simultaneously. $\square$

7.2 Finite-time empirical laws after fixing the clocks

Proposition 7.2 (Empirical finite-time marginals). *Fix distinct integer shifts $a_1, \dots, a_m$, a source $f \in \mathcal{M}_\pm$ before the clocks, and local times $s_1, \dots, s_q \in [0, S_0]$. Use either interval convention from Section 3, with all shifted arguments positive. Define the random empirical law*

$$\hat{\nu}_{X,s}(\omega) = \frac{1}{X} \sum_{n \in I_X} \delta_{(F_{s_\ell/h}^f(n+a_i;\omega))_{i,\ell}} \quad (7.2)$$

and the deterministic target

$$\nu_{X,s}^\circ = \frac{1}{X} \sum_{n \in I_X} \mathcal{L}((f(n+a_i)B_i(s_\ell))_{i,\ell}), \quad (7.3)$$

where $B_1, \dots, B_m$ are independent rate-one telegraphs starting at +1. Then

$$\mathbb{E}_\omega d_{\text{TV}}(\hat{\nu}_{X,s}(\omega), \nu_{X,s}^\circ) \ll_{a,q,S_0} h^{-1/2}. \quad (7.4)$$

For each coordinate i , the same telegraph is observed at all the times s_ℓ ; fresh independent signs are not drawn at successive times.

Proof. This needs a variance estimate beyond the coupling in Proposition 7.1. For a fixed test $0 \leq \Phi \leq 1$ of the local path, replace one whole prime-clock history on $[0, S_0/h]$ by an independent copy. This changes its empirical average over origins by at most

$$b_p \leq m(1/p + 1/X).$$

Only primes $p \leq C_a X$ can occur in the shifted arguments, for either interval convention, and $\sum_p b_p^2 \ll_a 1$ when restricted to these primes. The two histories are both empty with probability $e^{-2S_0/h}$, in which case the replacement changes nothing. The Efron–Stein product variance inequality [5] yields

$$\text{Var}\left(\frac{1}{X} \sum_{n \in I_X} \Phi(\gamma_{X,n})\right) \leq \frac{1}{2}(1 - e^{-2S_0/h}) \sum_p b_p^2 \ll_a S_0/h. \quad (7.5)$$

Apply this to each indicator of a state of the mq sampled signs. Cauchy–Schwarz and summation over the fixed finite state space bound $\mathbb{E} d_{\text{TV}}(\hat{\nu}_{X,s}, \mathbb{E} \hat{\nu}_{X,s})$ by $O_{a,q,S_0}(h^{-1/2})$. Projecting the path coupling (7.1) onto the selected times bounds $d_{\text{TV}}(\mathbb{E} \hat{\nu}_{X,s}, \nu_{X,s}^\circ)$ by the same order. The triangle inequality proves (7.4). $\square$

This assertion retains finitely many times and averages its error over the clocks. It is neither an almost-sure path statement nor total-variation convergence of empirical laws of entire paths. Indeed, once the clocks are fixed, such an empirical path law is supported on at most X paths. For a general factor matrix E , let r_* be the number of its non-zero columns, counted with multiplicity. The target has a nonatomic component of mass $1 - e^{-r_* S_0}$, corresponding to at least one visible jump at a continuous time. Consequently,

$$d_{\text{TV}}(\text{empirical path law}, \mathcal{L}(\gamma_X^\circ)) \geq 1 - e^{-r_* S_0}. \quad (7.6)$$

Repeated non-zero columns do not remove this component: their independent clocks almost surely have distinct jump times. This is a finite-support obstruction, not an additional arithmetic cancellation claim.

7.3 The universal scale and the limitation of returning to time zero

For distinct shifts alone, the irreducible factors are the corresponding affine polynomials and the channel is

$$B_s(v, w) = \prod_{j=1}^m \frac{1 + e^{-2s} v_j w_j}{2}.$$

Theorem 6.1 gives the following regimes, in probability and for a fixed family size.

Scale	Empirical law after noise, up to $o(1)$
$t_X h \rightarrow 0$	Moving initial law μ_X^f .
$t_X h \rightarrow s \in (0, \infty)$	$\mu_X^f B_s$, with non-zero memory.
$t_X h \rightarrow \infty$	Uniform law on the cube.

The last condition is necessary for a guarantee valid for every source: for $f \equiv 1$, at finite local time s , a single-letter mean tends to $e^{-2s} > 0$. If $t_X h$ fails to tend to infinity, one can pass to a subsequence converging to a finite s . This necessity is not asserted for each source individually; a source may already be normal before noise.

For a non-empty set of shifts A and the Liouville source, (6.5) at time s/h reads

$$\frac{1}{X} \sum_{n \in I_X} \prod_{a \in A} F_{s/h}^\lambda(n + a) = e^{-2|A|s} \frac{1}{X} \sum_{n \in I_X} \prod_{a \in A} \lambda(n + a) + O_{L^2}(h^{-1/2}). \quad (7.7)$$

At fixed s , convergence of the left-hand side to zero in probability would therefore be equivalent to cancellation of the deterministic correlation on the right. The theorem calculates the multiplier, not this correlation. When $t_X h \rightarrow \infty$, memory disappears; cancellation of the output cannot then be inverted without additional information. The same distinction holds for selective noise, with Θ_X and (5.2).

Finally, the order of choices is indispensable. Choosing $f(p) = (-1)^{Q_p(t_0)}$ after observing the clocks would give $F_{t_0}^f \equiv 1$. There can be no single event asserting these conclusions for all adaptive sources simultaneously.

8 Channel examples and the limits of protection

8.1 Four observations need not be four free signs

Take $A = T^2 + 1$, $B = T^2 + 2$, and observations (A, B, AB, A^2) . Their parity matrix is

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

The only permitted states are

$$(+, +, +, +), \quad (-, +, -, +), \quad (+, -, -, +), \quad (-, -, +, +).$$

Theorem 6.2 gives each asymptotic frequency $1/4$. At every size and time their law has total-variation distance at least $3/4$ from four independent fair signs, since the other twelve states are forbidden.

For $(n, 2n)$, the constant environment is essential: the product of the two signs equals $F_t^f(2)$ exactly. The compatible limiting law is uniform on

$$\{(1, F_t^f(2)), (-1, -F_t^f(2))\}.$$

8.2 Column multiplicities matter

For $P(T) = T(T + 1)$, the observations $P(n)$ and $P(n + 1)$ have columns

$$(1, 0)^T, \quad (1, 1)^T, \quad (0, 1)^T.$$

Each non-trivial character has $d(u) = 2$. From the constant source, at local time with $e^{-2s} = 1/2$, the probabilities in the order $(++, ++, +-, --)$ are

$$(7, 3, 3, 3)/16.$$

They are not the probabilities of two independent clocks acting directly on the outputs. As time tends to infinity, rank two does give four frequencies equal to $1/4$.

More generally, if m_v is the multiplicity of a non-zero column v , then

$$d(u) = \sum_{v \neq 0} m_v \frac{1 - (-1)^{\langle u, v \rangle}}{2}, \quad m_v = -2^{1-m} \sum_{u \in \mathbb{F}_2^m} d(u) (-1)^{\langle u, v \rangle}.$$

This classical Fourier inversion recovers the visible incidences of the channel. It does not recover zero columns and is not a new polynomial factorisation algorithm.

8.3 Several traces: a sufficient criterion and its specific obstruction

The availability of a common root-free reservoir is a classical Galois question, not a new criterion of this paper. The density description in Berend–Bilu [3, Theorem 2], as restated in Elsholtz–Klahn–Technau [6, Theorem 2.1 and Proposition 2.2], identifies primes where a polynomial has a root by fixed points in the Galois action. The following proposition applies that criterion to our completion construction.

Proposition 8.1 (Completion from a common reservoir). *Let $P_1, \dots, P_r$ be non-constant, separable, eventually positive integer polynomials. If the Galois group of their common splitting field has an element fixing no point in the union of their roots, the conclusions of Theorem 1.1 and Corollary 1.2 hold while preserving all the traces P_j simultaneously.*

Proof. The conjugacy classes with no fixed point in this union provide a positive-density Chebotarev set root-free for every P_j . After finite removals, the harmonic conclusion of Lemma 2.1 and the whole weighted construction apply without further change. $\square$

The examples below give the relative density of the common root-free reservoir, outside exceptional primes.

Protected traces	Reservoir density
$T^2 + 1$	$1/2$
$T^3 - T - 1$	$1/3$
$T^2 - 2$ and $T^2 - 3$	$1/4$
$T^2 - 2, T^2 - 3$ and $T^2 - 6$	0

The exact product in the last row appears in [6, equation (1.1)]. It is *exceptional*, meaning that it has a root modulo all but finitely many primes and no integer root, but is not *intersective*: it has no root modulo 2^6 . Root availability modulo almost all primes is the property relevant here, not root availability modulo every integer.

The cubic is irreducible by the rational-root test, and its discriminant -23 gives Galois group S_3 , with two derangements out of six elements. The two independent quadratics have

group $\mathbb{F}_2^2$. For the last row, all three quadratic symbols cannot be negative simultaneously, since $(6/p) = (2/p)(3/p)$ for $p > 3$.

Absence of a reservoir has a specific, limited consequence. Outside a finite exceptional set, a simple root of P_j modulo p has exactly one root lift among its p lifts modulo p^2 . The other lifts provide arbitrarily large indices with $v_p(P_j(n)) = 1$. An independent clock of positive rate at this prime changes that trace value when it rings: the other prime factors of this fixed value are finite in number and almost surely do not ring simultaneously. If every Galois element fixes some root in the union, only finitely many clocks can remain active while preserving every trace at all times.

Starting from $f \equiv 1$, finitely many negative prime signs do not give a normal function. Its mean is the positive product

$$\prod_{p:g(p)=-1} \frac{p-1}{p+1},$$

a finite case of (5.6). This obstruction concerns independent prime clocks acting separately and pointwise protection at all times. It does not prove that every correlated completion is impossible when the reservoir is empty. For repeated polynomial factors, even exponents should first be discarded, since they impose no sign constraint of this kind.

9 Scope

The central result is a completion theorem, not a method for calculating the protected trace. Even when the source is Liouville, the function with vanishing correlations is F_t^λ , not λ . The marked version retains a moving empirical law without identifying that law.

The observation families, their algebraic coefficients and their sizes are fixed as $X \rightarrow \infty$. Marks and time horizons are fixed as stated in each result. This does not prohibit the size-dependent data explicitly allowed elsewhere: Proposition 4.2 permits a bounded coefficient table $(c_{X,n})$ specified before the clocks, and Theorem 6.1 permits a source depending on X in its L^2 bound, with both suprema outside the expectation. The simultaneous almost-sure conclusions for a multiplicative source fix that source before the noise, together with the weights, marks and delay sequence required by the relevant statement. Their probability-one events may depend on these choices; a uniform constant is not an event valid for arbitrary choices made after observing the clocks.

The results do not provide normality for adaptive sources, Poisson laws for long runs, temporal maxima from a prescribed source, Gaussian fluctuations at scale $\sqrt{X}$, or total-variation convergence of empirical laws of whole paths. No effective algorithm for the infinite function obtained by almost-sure existence is claimed. The two specified interval conventions do not entail arbitrary-origin uniformity. The fixed-family estimates neither import nor supply a relation profile for words of growing length.

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