

Reflected sign patterns of the Liouville function at every large total

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Abstract

For signs s, t , let $N_{s,t}(N)$ count the ordered representations $N = n + (N - n)$ with $\lambda(n) = s$ and $\lambda(N - n) = t$. We prove that, for every fixed $\varepsilon > 0$, all four counts are at least $c_\varepsilon N^{1-\varepsilon}$ at every sufficiently large integer total. The constants are ineffective through Siegel's theorem. The same conclusion holds uniformly for completely multiplicative sign functions whose positive primes satisfy $\sum_{f(p)=1} p^{-\theta} \leq M$, with fixed $0 < \theta < 1$ and M . Separately, at each sufficiently large prime p , both patterns with product $(-1/p)$ have at least $p/(4(\log p)^{96})$ representations. All four satisfy this stronger bound in six reduced classes modulo 24 and outside one very sparse family of primes. The proof combines finite reflection-to-dilation propagation, a short-product spectral test, reconstruction from the units, and signed counting of prime disagreements. For the uniform extension, candidates are united before a common exceptional population is removed. None of the conclusions gives a positive-density bound or cancellation of the reflected correlation at every total.

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1 Introduction and main results

The complete multiplicativity of the Liouville function does not make its additive observations independent. In a reflected pair $(\lambda(n), \lambda(N - n))$, the total N is fixed while both summands vary. The question studied here is quantitative and pointwise in that total: how many representations of each prescribed sign type must occur?

We write $\lambda(n) = (-1)^{\Omega(n)}$, so that $\lambda(p) = -1$ at every prime. For a completely multiplicative function $f : \mathbb{N} \rightarrow \{-1, 1\}$, define

$$N_{s,t}^f(N) = \#\{1 \leq n < N : f(n) = s, f(N - n) = t\}, \quad m_f(N) = \min_{s,t=\pm 1} N_{s,t}^f(N). \quad (1.1)$$

Representations are ordered. We omit the superscript or subscript f when $f = \lambda$. Throughout, all logarithms are natural.

1.1 Every large integer total

Theorem 1.1 (Four patterns at every large total). *For each fixed $\varepsilon > 0$ there are constants $c_\varepsilon > 0$ and N_ε such that*

$$N_{s,t}(N) \geq c_\varepsilon N^{1-\varepsilon} \quad (N \geq N_\varepsilon, s, t \in \{-1, 1\}). \quad (1.2)$$

There is no restriction on the parity, prime support, or prime-power valuations of N . The constants and threshold are ineffective through the use of Siegel's theorem.

Thus $\log m(N)/\log N \rightarrow 1$. The quantifier “for each fixed ε ” is essential: the theorem supplies neither a rate obtained by choosing ε as a function of N , nor a bound $N/(\log N)^A$ with fixed A at all totals. A population of size $N^{1-o(1)}$ can still have density zero.

The function in Theorem 1.1 is the original Liouville function. No prime values are changed, and there is no averaging over totals or over a random multiplicative model.

Qualitative existence of all four patterns at every $N \geq 11$ is established in Meng's recent manuscript [11], following the even-total result [10]. Our purpose is to obtain lower bounds for their number, not to claim the first existence result; the distinction is detailed in Section 1.5.

The four counts are related by one elementary identity. With $A_f(x) = \sum_{n \leq x} f(n)$ and $C_f(N) = \sum_{1 \leq n < N} f(n)f(N - n)$, the indicator expansion recorded in (2.2) is

$$4N_{s,t}^f(N) = N - 1 + (s + t)A_f(N - 1) + stC_f(N).$$

For Liouville, $A_\lambda(N - 1) = o(N)$; cancellation of $C_\lambda(N)$ would therefore give $N_{s,t}(N) \sim N/4$. The present aim is weaker: prevent any individual count from becoming too small, including when the reflected product is almost constant.

1.2 Uniformity under a prime budget

For fixed $0 < \theta < 1$ and $M \geq 0$, put

$$E(f) = \{p \text{ prime} : f(p) = 1\}, \quad \mathcal{F}_{\theta,M} = \left\{ f : \sum_{p \in E(f)} p^{-\theta} \leq M \right\}, \quad (1.3)$$

where all functions in the class are completely multiplicative and sign-valued.

Theorem 1.2 (Uniform prime perturbations). *Fix $0 < \theta < 1$, $M \geq 0$, and $\varepsilon > 0$. There are constants $c_{\theta,M,\varepsilon} > 0$ and $N_{\theta,M,\varepsilon}$ such that*

$$N_{s,t}^f(N) \geq c_{\theta,M,\varepsilon} N^{1-\varepsilon} \quad (f \in \mathcal{F}_{\theta,M}, N \geq N_{\theta,M,\varepsilon}, s, t = \pm 1). \quad (1.4)$$

The constants are uniform in f , which may vary with N within the fixed budget. They remain ineffective through Siegel's theorem. The parameter θ is fixed before N tends to infinity.

Theorem 1.1 is the case $M = 0$. We state it first because it concerns the imposed arithmetic sequence, whereas Theorem 1.2 concerns an entire class of sequences. This extension is not obtained by subtracting a small density of errors from (1.2). For instance, changing only the prime value at 2 gives $f(n) = \lambda(n)(-1)^{v_2(n)}$; the disagreement density with λ is

$$\sum_{\substack{j \geq 1 \\ j \text{ odd}}} 2^{-j-1} = \frac{1}{3}.$$

Such a subtraction would destroy an $N^{1-\varepsilon}$ lower bound.

Corollary 1.3 (A counting formulation of the budget). *Fix $0 \leq \beta < 1$ and $C \geq 0$. The conclusion of Theorem 1.2, with constants depending on β, C, ε , holds uniformly for functions satisfying*

$$\#\{p \leq x : f(p) = 1\} \leq Cx^\beta \quad (x \geq 2).$$

Proof. Choose $\beta < \theta < 1$. Partial summation gives

$$\sum_{p \in E(f)} p^{-\theta} \leq C\theta \int_2^\infty t^{\beta-\theta-1} dt = \frac{C\theta 2^{\beta-\theta}}{\theta - \beta}.$$

Apply Theorem 1.2 with this fixed budget. □

1.3 Logarithmic losses at prime totals

For an odd prime p , let $\chi_p = (\cdot/p)$ and $\rho_p = \chi_p(-1)$. It is useful to separate the counts by their sign product:

$$P_p(f; N) = \#\{1 \leq n < N : f(n)f(N-n) = \rho\}, \quad E_*(f; N) = \min\{P_+(f; N), P_-(f; N)\}. \quad (1.5)$$

Theorem 1.4 (Prime totals with a logarithmic loss). *There is an absolute threshold p_0 such that, for every prime $p \geq p_0$,*

$$st = \rho_p \implies N_{s,t}(p) \geq \frac{p}{4(\log p)^{96}}. \quad (1.6)$$

All four patterns satisfy this bound when

$$p \equiv 1, 7, 11, 13, 17, 23 \pmod{24}. \quad (1.7)$$

After increasing p_0 , define the single family

$$\mathcal{X} = \{p \geq p_0 : p \text{ prime}, E_*(\lambda; p) < p/(\log p)^{96}\}. \quad (1.8)$$

All four patterns satisfy (1.6) outside $\mathcal{X}$. Every member of $\mathcal{X}$ is 5 or 19 modulo 24, and for every sufficiently large T ,

$$\#(\mathcal{X} \cap [\exp((\log T)^{1/8}), T]) \leq 1, \quad (1.9)$$

$$\#(\mathcal{X} \cap [1, T]) \leq \frac{\log \log \log T}{\log 8} + O(1). \quad (1.10)$$

Each $p \in X$ forces a real zero β_p of $L(s, \chi_p)$ with

$$0 < 1 - \beta_p \ll (\log p)^{-10}. \quad (1.11)$$

These thresholds can in principle be made effective from the classical inputs used in the proof; no numerical value is supplied.

For every fixed $\varepsilon > 0$, the quotient of $p/(\log p)^{96}$ by $p^{1-\varepsilon}$ tends to infinity. Nevertheless, Theorem 1.4 does not replace Theorem 1.1: its total is prime, and its all-pattern conclusion retains a possible exceptional family. The family in (1.8) is a sufficient exceptional set for our argument, not a characterization of failure of the four individual lower bounds. We prove neither that it is empty nor that it is finite, and we do not construct a member.

Conclusion	Domain	Effectivity and restriction
$N_{s,t}(N) \gg_{\varepsilon} N^{1-\varepsilon}$	All four signs; every sufficiently large integer N	Ineffective; no fixed logarithmic loss asserted
$N_{s,t}^f(N) \gg_{\theta,M,\varepsilon} N^{1-\varepsilon}$	Same totals and signs; every $f \in \mathcal{F}_{\theta,M}$	Uniform in f ; fixed $\theta < 1$; ineffective
$N_{s,t}(p) \geq p/(4 \log^{96} p)$	Two patterns of product ρ_p at every prime; all four in six classes or outside X	Effective in principle; X is not eliminated

Table 1: The three domains are not interchangeable. Every lower bound is pointwise in its stated total.

1.4 Proof strategy: use the character and its disagreements

The argument first treats totals avoiding a fixed block of small primes. There are two branches. If both sign products have at least $N/(\log N)^{512}$ occurrences, the first moment separates the equal-sign patterns, since

$$N_{+,+}^f(N) - N_{-,-}^f(N) = A_f(N-1).$$

The uniform bound $A_f(N-1) \ll_{\theta,M} N \exp(-c\sqrt{\log N})$ is negligible on this logarithmic scale. It need not be negligible relative to $N^{1-\varepsilon}$ for a fixed $\varepsilon > 0$. Thus, in the remaining branch, counting products and then subtracting a first-moment error would not suffice: the construction must retain each cofactor's sign.

From near-saturation to a positively oriented character. When one sign product is smaller than the threshold, reflection agrees with a constant sign κ except at very few origins. The finite descent of Section 4 transports this error to a fixed set of prime dilations. Averaging over products of these generators reaches polylogarithmic dilations; small-denominator frequencies are paid for, including at composite totals. A high moment over small primes then confines the multiplicative Fourier mass to a small family of characters.

Concentration alone is not the end of the inverse argument. The identity on short *integer* products, $1 = \mathbb{E}_{a,b \in \mathcal{U}_H} f(ab)f(a)f(b) = \sum_{\psi} a_{\psi} b_{\psi}^2$, selects a real character with coefficient close to +1, using Burgess, a restricted Gram estimate and a fourth moment. Here $\mathcal{U}_H$ consists of the units up to $H = \lfloor \sqrt{N-1} \rfloor$; the coefficients and the argument are given in Section 5. The reconstruction of Section 3 then yields agreement on every prefix and small reciprocal mass of prime disagreements. In the budgeted class, the principal character is excluded.

From disagreement primes to signed populations. The remaining quadratic character is *used*, not declared impossible. A prime q with $f(q) = -1$ and $\chi(q) = 1$ reverses one side of the pattern predicted by the character. Sections 6–7 count the resulting pairs with their signs intact, control overlaps, and obtain a supply of such primes from a filtered positive convolution and Siegel’s theorem. Candidates from all marking primes are united *before* one common exceptional population is removed. This is what leaves a power saving for every fixed $\theta < 1$.

From a convenient divisor to every total. Among finitely many disjoint prime blocks, one carries little of the factorization of N . Removing its part leaves a divisor D close to N in exponent and coprime to that block. Simultaneous scaling of the two summands injects representations of D into those of N and permutes the four sign pairs. It transfers the lower bound for their minimum count, not a density; Section 8 accounts for the exponent loss.

The prime logarithmic result takes a different branch after the inverse. The values of Liouville at 2 and 3 restrict the possible primes; any remaining exception gives a small positive $L(1, \chi_p)$ together with a positive derivative, hence a nearby real zero. Landau–Page spaces those exceptions rather than removing them. The exponents 512 and 96 specify the two accuracy budgets; their role is to leave room for these different downstream steps, not to give optimized constants.

1.5 Relation to previous work

Existence and counting. Mangerel’s published nonextremality theorem gives $|C_\lambda(N)| < N - 1$ for $N \geq 11$ [4, Theorem 1.2]. It guarantees both sign products, but does not on its own distinguish the two equal-sign patterns. The release of [10] dated 16 September 2026 proves the negative-negative pattern at every even $N > 2$, and the positive-positive pattern at $2p$ for every prime $p > 3$ (Theorems 1.1–1.2 there). Meng [11, Theorem A] proves all four patterns for every $N \geq 2$ outside the sharp exceptional set $\{2, 3, 4, 5, 6, 9, 10\}$. His Theorem B gives both equal-sign patterns at each prime $p \geq 5$. These are unconditional existence results with explicit small exceptions. Meng explicitly separates them from quantitative counting bounds in his introduction and Section 9.

The point of Theorem 1.1 is instead the lower bound $N^{1-\varepsilon}$ for each fixed $\varepsilon > 0$ at *every* sufficiently large total. It is ineffective and does not replace the sharp finite-range existence statements. The additional quantifier in Theorem 1.2 is uniformity over an entire class of functions, which may vary with N under the fixed budget. Neither theorem imports the recent existence results or their formalizations as a proof input.

Conditional quantitative results. Mangerel’s work on Shusterman’s problem [5] proves under GRH the negative-negative pattern at all sufficiently large even totals (Theorem 1.2), and all four patterns outside a fixed friable domain (Theorem 1.3). Its Proposition 1.4 gives, under GRH, a lower bound of the form $p \exp(-C(\log \log p)^6)$ for every pattern at sufficiently large primes. For fixed $\varepsilon, C > 0$,

$$p^{1-\varepsilon} = o\left(p \exp(-C(\log \log p)^6)\right), \quad p \exp(-C(\log \log p)^6) = o\left(\frac{p}{(\log p)^{96}}\right).$$

Thus our all-total power bound is not quantitatively larger than that conditional prime bound. Its distinguishing features are the absence of GRH and of restrictions on the total. The stronger logarithmic bound in Theorem 1.4 has its own, explicitly smaller all-pattern domain.

The reflection-to-dilation cocycle and Pierce-descent framework have a direct antecedent in [5]. Section 4 proves the propagation used here in full; neither the GRH-dependent result nor the analytic smooth-number propagation of that work is invoked as a black box.

Exact and approximate rigidity. Mangerel’s Gauss-sum rigidity theorem [6, Theorem 1.1 and Corollary 1.2], first available as a preprint in February 2025, already relates approximate dilation symmetry to a real character and, for Liouville, to a real zero near 1. Its prime-dilation hypotheses reach p^c , with $c > 1/4$. Here the hypothesis is a much more accurate reflected saturation, and complete multiplicativity on every short integer product is essential. Comparing the cutoffs alone would therefore misstate the relationship. His Lemma 3.6 is also a specific antecedent for passing from integer disagreements to reciprocal prime disagreement; Section 3 states the differences in our unit-to-prefix reconstruction.

Meng’s exact prime-modulus classification [11, Theorem 7.2] assumes $f(2) = f(3) = -1$, and also $f(5) = -1$ for a missing equal-sign pattern. Under the stated exclusions it forces agreement with the Legendre symbol below $p/2$. Our Theorem 5.3 instead treats quantitative near-saturation at composite as well as prime totals, without prescribing those three prime signs, and allows a principal character or a proper conductor. These are different hypotheses and conclusions; neither comparison is a blanket strengthening of the other.

Version v3 of [9] gives exact extremizer classifications at odd prime powers in Section 5.3; the proper-conductor example in Remark 5.5 belongs to that class. Section 7.2 gives an approximate-dilation rigidity and pattern-scarcity statement at prime moduli, without GRH in that generic statement. Its Liouville separation and quantitative dilation-defect results in Sections 7.3–7.4 explicitly assume GRH. In particular, the scarcity estimate is an upper bound for the patterns forbidden by an approximately imitated character, not an unconditional lower bound for every pattern. These compared statements do not give the all-total conclusion of Theorem 1.1 or the uniform class of Theorem 1.2.

Other quantifiers and analytic context. Krishnamoorthy [3, Theorem 2] bounds the number of totals with a large normalized binary convolution:

$$\#\{N \leq X : |C_\lambda(N)| > \delta N\} \ll_A \delta^{-2} X / (\log X)^A.$$

An exceptional-set estimate of this kind does not by itself give a bound at each total; conversely our lower bounds do not give cancellation of $C_\lambda(N)$. Cantarini, Gambini and Zaccagnini [14] study weighted averages and analytic properties of related convolutions; their Theorem 1 is an explicit-formula result under RH and simplicity of the nontrivial zeta zeros, not a pointwise sign-count theorem. The stronger cancellation problem originates with Corrádi and Kátai [12]. The relationship of Liouville convolution with hypothetical Siegel zeros also predates the present argument; see De Koninck, Germán and Kátai [13] for a result under existence assumptions on such zeros and with its own ranges and coprimality restrictions.

The local attributions below distinguish standard or inherited tools from the quantitative way they are combined: Mangerel’s disagreement lemma, Pollack’s convolution and reciprocal-prime method, the Montgomery–Vaughan inequality, and divisor scaling. The contribution asserted here is the counting mechanism, its losses and its uniform domains, not those general principles in isolation. The comparisons concern the stated hypotheses and conclusions, not a ranking of the methods by a single cutoff or exponent.

1.6 Reading order

Section 2 states the analytic inputs and uniform costs. To reconstruct the inverse, read Sections 4–5, returning to the unit-to-prefix lemma in Section 3 when it is called at the end of

the spectral argument. Sections 6–8 complete the signed count for Liouville and its uniform class together; Sections 9–10 give the separate prime route. Section 11 and Appendix A record the boundaries and parameter budgets. No result about a random multiplicative model or normality is an input.

2 Notation, classical inputs, and uniform costs

For an integer $N \geq 3$, set

$$X = N - 1, \quad L = \log N, \quad \ell = \log(3L), \quad \nu = \frac{\phi(N)}{N}, \quad H_x = \sum_{1 \leq n \leq x} \frac{1}{n}. \quad (2.1)$$

A sum or product indexed by p, q, r is over primes whenever this is indicated or stated locally. Dirichlet characters are extended by zero off the units. The principal primitive character has conductor 1; it is allowed in the generic inverse theorem, but not in the structured branch used for the lower bounds.

For the means A_f and C_f defined in Section 1, expanding the two sign indicators records the identity announced there:

$$4N_{s,t}^f(N) = X + (s + t)A_f(X) + stC_f(N). \quad (2.2)$$

Equivalently,

$$N_{+,-}^f = N_{-,+}^f = \frac{P_-(f; N)}{2}, \quad N_{\pm,\pm}^f = \frac{P_+(f; N) \pm A_f(X)}{2}. \quad (2.3)$$

In the branch where both products have a logarithmic-scale lower bound, these identities and the first moment separate all four counts. The structured branch will instead count each rare signed pair directly. No estimate for $C_f(N)$ is assumed.

2.1 Analytic inputs

We use the following precise forms of classical results. They are external theorems, rather than consequences of the finite diagnostics accompanying this manuscript.

Classical input 2.1 (Prime counts and the Liouville mean). Chebyshev's bounds and Mertens' estimates are used in their usual effective forms. For an absolute $c > 0$,

$$A_\lambda(x) \ll x \exp(-c\sqrt{\log x}) \quad (x \geq 3). \quad (2.4)$$

In particular $\sup_{x \geq 1} |\sum_{n \leq x} \lambda(n)/n| < \infty$. The first-moment input is recalled, in a stronger form, in [5, Section 6.1].

Classical input 2.2 (Burgess at parameter two). If ξ is primitive and nonprincipal with conductor Q , then for every fixed $\eta > 0$,

$$\left| \sum_{n \leq t} \xi(n) \right| \ll_\eta t^{1/2} Q^{3/16+\eta}. \quad (2.5)$$

No cube-free condition is required at this parameter; see [2, equation (1)]. Periodicity extends the bound, if needed, to lengths exceeding the conductor.

Classical input 2.3 (Brun–Titchmarsh). For $x > q$ and $(a, q) = 1$,

$$\pi(x; q, a) \leq \frac{2x}{\phi(q) \log(x/q)}. \quad (2.6)$$

We need the classical Montgomery–Vaughan constant 2 [15], in the form recalled in [7, Introduction].

Classical input 2.4 (A quadratic convolution and Siegel). For each fixed $\eta_1 > 0$, there is $\eta_2 > 0$ such that, uniformly for a primitive nonprincipal quadratic character χ of conductor Q ,

$$B_\chi(t) := \sum_{n \leq t} (1 * \chi)(n) = tL(1, \chi) + O_{\eta_1}(t^{1-\eta_2}) \quad (t \geq Q^{1/4+\eta_1}). \quad (2.7)$$

This is [8, Proposition 3.1]; even conductors are permitted. We also use Siegel’s theorem

$$L(1, \chi) \gg_\zeta Q^{-\zeta} \quad (\zeta > 0 \text{ fixed}), \quad (2.8)$$

with a positive ineffective constant, as used in [8, proof of Theorem 1.3].

Classical input 2.5 (Nonvanishing and Landau–Page). For real primitive nonprincipal χ , $L(1, \chi) > 0$. There is an absolute effective $c_0 > 0$ such that, among primitive real characters of conductors at most T , at most one has a real zero $\beta > 1 - c_0/\log T$, for sufficiently large T . The latter form is recalled in [1, Section 4.2].

Only (2.8) introduces ineffectivity into the all-total argument. The prime logarithmic argument uses Classical inputs 2.1, 2.2 and 2.5, but neither (2.7) nor (2.8) nor Brun–Titchmarsh. Its elementary Gauss-sum truncation is supplied in section 10.

2.2 Costs uniform in the factorization of the total

Define

$$P_N = \prod_{p|N} (1 + p^{-1}), \quad T_N = \sum_{p|N} \frac{\log p}{p+1}.$$

Lemma 2.6. *Uniformly in N ,*

$$\omega(N) \ll \frac{\log(3N)}{\log \log(3N)}, \quad 4^{\omega(N)} = N^{o(1)}, \quad v^{-1}, P_N, T_N \ll \ell. \quad (2.9)$$

Moreover, if

$$u_N(t) = \#\{1 \leq n \leq t : (n, N) = 1\}, \quad J_N = \max_{1 \leq t \leq X} \frac{t}{u_N(t)}, \quad (2.10)$$

where the maximum is over integers, then $J_N \ll \log(3N)$.

Proof. If $k = \omega(N)$, the product of the k distinct prime divisors is at least $(k+1)!$. This gives the first two assertions. Split the prime factors at $L = \log N$. For the smaller ones, Mertens’ estimates give the asserted bounds for the products and for $\sum \log p/p$. For the larger ones, use $\sum_{p|N} \log p \leq L$. Their contribution to $\sum 1/p$ is $O(1)$, and to $\sum \log p/p$ is at most 1. This proves the last three bounds.

For $t \geq C \log N$ with a sufficiently large absolute C , Chebyshev’s lower bound gives $\pi(t) \geq 2\omega(N)$, and therefore $u_N(t) \geq \pi(t)/2 \gg t/\log N$. For smaller t , the unit 1 gives $u_N(t) \geq 1$. $\square$

The factor $4^{\omega(N)}$ will only be absorbed by a fixed power saving in N . It is not treated as bounded, nor absorbed by a fixed power of $\log N$. This distinction matters when the prime support of N is unrestricted.

3 Reconstruction from the units

Agreement on $(\mathbb{Z}/N\mathbb{Z})^\times$ initially says nothing about integers divisible by the primes of N . We first show how complete multiplicativity reconstructs all prefixes, once those prime values have been matched. The fixed-scale passage from few integer disagreements to small reciprocal prime disagreement has a direct antecedent in [6, Lemma 3.6]. The point here is the initial restriction to units, the prescribed quotient values at primes dividing N , and simultaneous control of every prefix with a cost uniform in the factorization of N . The argument is given in full rather than invoking that lemma outside its hypotheses.

In the application, $h = fg$ is the quotient of the source and the character extension g from (5.13); its values at primes dividing N are fixed to be 1. The lemma has two outputs: agreement on *every* prefix, and control of the reciprocal mass of prime disagreements. The first completes the inverse; the second is consumed by the signed inventory and by the value-and-derivative calculation near $s = 1$.

Lemma 3.1 (Rectangular product energy). *Let $A, H \geq 1$ be integers and let $d_{A,H}(n)$ count the pairs $a \leq A, b \leq H$ with $ab = n$. Then*

$$\sum_n d_{A,H}(n)^2 = \sum_{\substack{u,v \leq \min(A,H) \\ (u,v)=1}} \left\lfloor \frac{A}{\max(u,v)} \right\rfloor \left\lfloor \frac{H}{\max(u,v)} \right\rfloor \quad (3.1)$$

$$\leq AH(2H_{\min(A,H)} - 1). \quad (3.2)$$

Proof. In $ab = cd$, write $a = gu, c = gv$, with $(u, v) = 1$. Then $b = tv$ and $d = tu$. The possible g and t give (3.1). The contribution of $\max(u, v) = 1$ is AH ; for each larger maximum j , there are at most $2j$ pairs. Summing $2AH/j$ proves (3.2). $\square$

Lemma 3.2 (Uniform reconstruction). *Let $h : \mathbb{N} \rightarrow \{-1, 1\}$ be completely multiplicative, with $h(p) = 1$ for every prime $p \mid N$. Put*

$$D_U = \#\{n \leq X : (n, N) = 1, h(n) = -1\}, \quad \delta = 2J_N^2 \sqrt{\frac{D_U H_X}{X}}.$$

If $\delta < 1/2$, every unit prefix and every complete prefix of $[1, X]$ has negative proportion at most δ . If $\delta < 3/32$, then in addition

$$\sum_{\substack{q \leq X \text{ prime} \\ h(q) = -1}} \frac{1}{q} \leq 3\delta. \quad (3.3)$$

Proof. Fix an integer $H \leq X$ and put $A = \lfloor X/H \rfloor$, so $AH \geq X/2$. Choose a and b independently and uniformly among the units in $[1, A]$ and $[1, H]$. If $h(ab) = -1$, the product lies among the D_U exceptional unit positions. By Cauchy–Schwarz and Lemma 3.1,

$$\Pr(h(ab) = -1) \leq \frac{\sqrt{2D_U A H H_X}}{u_N(A)u_N(H)} \leq 2J_N^2 \sqrt{D_U H_X / X} = \delta.$$

If t_H is the negative proportion in the unit prefix of length H , independence and multiplicativity give

$$\Pr(h(ab) = -1) = t_A(1 - t_H) + (1 - t_A)t_H \geq \min\{t_H, 1 - t_H\}.$$

Every t_H is therefore at most δ or at least $1 - \delta$. As units enter a prefix one at a time, a change of majority would have to pass through equal positive and negative counts. This is impossible when $\delta < 1/2$. Since $h(1) = 1$, the majority is positive at every prefix.

Every integer $n \leq t$ has a unique factorization $n = am$, where a is supported on the prime divisors of N and $(m, N) = 1$. Our hypothesis gives $h(n) = h(m)$, and hence

$$\#\{n \leq t : h(n) = -1\} \leq \delta \sum_{\substack{a \leq t \\ p|a \Rightarrow p|N}} u_N(\lfloor t/a \rfloor) = \delta t.$$

For the prime assertion, let $S_t = \sum_{q \leq t, h(q)=-1} 1/q$. The integers $qa \leq t$ for which q is negative and a has no negative prime factor are disjoint as q varies. They are all negative. If $S_t \leq 1$, their number is at least

$$(1 - S_t) \sum_{\substack{q \leq t \text{ prime} \\ h(q)=-1}} \lfloor t/q \rfloor \geq \frac{t}{2} S_t (1 - S_t). \quad (3.4)$$

Indeed, for each q , a union bound removes at most an S_t proportion of $a \leq \lfloor t/q \rfloor$.

At a first crossing of $1/4$, the value of S_t lies in $[1/4, 3/4]$, since the increment is at most $1/2$. Formula (3.4) would give a negative proportion at least $3/32$, contradicting $\delta < 3/32$. Thus $S_X < 1/4$. Let e be the negative proportion in the complete prefix $[1, X]$. The number of negative primes up to X is at most eX . Retaining the floors in (3.4) gives

$$e \geq (1 - S_X)(S_X - e).$$

Since $S_X < 1/4$, this implies $S_X \leq 3e \leq 3\delta$. $\square$

Remark 3.3. The hypothesis at primes dividing N cannot be dropped. For $N = 2^k$, the function $h(n) = (-1)^{v_2(n)}$ equals 1 on all units but not on all integers. In our application the quotient of the source and its character extension is deliberately made equal to 1 at those primes.

4 From reflection to polylogarithmic dilations

For a completely multiplicative sign function f and $\kappa \in \{-1, 1\}$, define the reflection error

$$\mathcal{E}_\kappa(f; N) = \#\{1 \leq x < N : f(x)f(N-x) \neq \kappa\}. \quad (4.1)$$

Unlike P_ρ in (1.5), this counts *disagreement* with the displayed sign: exactly $\mathcal{E}_\kappa(f; N) = P_{-\kappa}(f; N)$. When κ is the majority product, it is the minority population $E_*(f; N)$.

Let $r_b(x)$ be the residue of bx in $[0, N)$. Set

$$\Lambda_b(x) = \begin{cases} f(b)f(x)f(r_b(x)), & r_b(x) > 0, \\ 0, & r_b(x) = 0, \end{cases} \quad \widehat{D}_b = \#\{1 \leq x < N : \Lambda_b(x) \neq 1\}.$$

For $(b, N) = 1$, write $\mathcal{D}_b$ for this set and D_b for its cardinality. Defects at noninvertible multipliers, including the zero residues, are retained during the descent.

The example of doubling. Let $\mathcal{E} = \{x : f(N-x) \neq \kappa f(x)\}$ inside $[1, N-1]$. If $x > N/2$, put $y = N-x$. When y and $2y$ are outside $\mathcal{E}$,

$$f(x) = \kappa f(y), \quad f(2x-N) = \kappa f(2y) = \kappa f(2)f(y) = f(2)f(x).$$

For $2x < N$ the dilation identity is just integer multiplicativity. The two maps $x \mapsto y$ and $x \mapsto 2y$ are injective, and $2x = N$ contributes at most one zero state. Thus $\widehat{D}_2 \leq 2|\mathcal{E}| + 1$. This illustrates the transport of error; a general multiplier requires the descent below, whose exponential cost will be used only at fixed generators.

4.1 A finite descent

Lemma 4.1 (Reflection-to-dilation descent). *If $E = \mathcal{E}_\kappa(f; N)$ and $2 \leq b < N$, then*

$$\widehat{D}_b \leq 2^b(b+1)^2(E+1) + b \sum_{r < b} 2^{r-1} \widehat{D}_r \leq 2^{8b^2}(E+1). \quad (4.2)$$

Proof. Put $w(x) = \kappa f(x)f(N-x)$. At a state u whose quotient $r = \lfloor N/u \rfloor$ is less than b , let $\vartheta = N - ru$. If $\vartheta > 0$, its quotient is strictly larger than r . Away from the states where ϑ , $r_b(u)$, or $v = r_b(\vartheta)$ is zero, direct multiplication of signs gives

$$\Lambda_b(u) = \Lambda_r(r_b(u))\Lambda_b(\vartheta)w(ru)w(v), \quad r_r(r_b(u)) = N - v. \quad (4.3)$$

No inverse of r modulo N is used.

A state with quotient r has at most 2^{r-1} earlier origins: the previous quotients form a strictly increasing subsequence of $1, \dots, r-1$, and each such subsequence determines at most one origin through $u_{j-1} = (N - u_j)/r_j$.

At a fixed quotient r , at most $2b+1$ states are singular. The equality $\vartheta = 0$ gives at most one; $r_b(u) = 0$ gives at most b ; and the congruence $bru \equiv 0 \pmod{N}$ within $N/(r+1) < u \leq N/r$ gives at most

$$\lfloor \frac{br}{r(r+1)} \rfloor + 1 \leq b.$$

The failures of the two w factors cost at most $(1+b^2)E$: the first map is injective, and the second has at most $(br, N) \leq b^2$ preimages. Failures of $\Lambda_r(r_b(u))$ cost at most $b\widehat{D}_r$.

At the terminal quotient $r \geq b$, we have $bu \leq N$. The only nontrivial case is $bu = N$, with at most 2^{b-1} origins. Consequently

$$\widehat{D}_b \leq 2^{b-1} + \sum_{r < b} 2^{r-1}(2b+1 + (1+b^2)E + b\widehat{D}_r).$$

This implies the first bound. The second follows by induction from $\widehat{D}_1 = 0$; for example, the recurrence term is at most $b(b-1)2^{b-2+8(b-1)^2}(E+1)$, which is absorbed together with the first term in $2^{8b^2}(E+1)$. $\square$

The cost in (4.2) is too large to apply directly at growing multipliers. We use it only at a fixed finite collection of primes, and then average products of those primes.

4.2 The cocycle and the missing initial interval

For unit multipliers a, b , complete multiplicativity gives the cocycle identity. Its immediate consequences are

$$D_{ab} \leq D_a + D_b, \quad |\mathcal{D}_b \triangle r_a^{-1} \mathcal{D}_b| \leq 2D_a, \quad \mathcal{D}_b \cap [1, \lfloor X/b \rfloor] = \emptyset. \quad (4.4)$$

The first two formulas remain valid when $ab \geq N$: the multiplier is defined using the actual value $f(ab)$ and the residue of its product, not by first reducing ab as an integer.

Lemma 4.2 (Quadratic averaging with small denominators). *Let $\mathcal{A} \subset [1, R] \cap (\mathbb{Z}/N\mathbb{Z})^\times$ be nonempty, put $M_A = |\mathcal{A}|$ and $\overline{D} = M_A^{-1} \sum_{a \in \mathcal{A}} D_a$, and suppose $(b, N) = 1$. If*

$$R^2 H_R \leq \frac{N}{4b},$$

then

$$\frac{D_b}{N} \leq \frac{16b^2 H_N^2}{M_A} + 16b H_N \frac{\overline{D}}{N}. \quad (4.5)$$

Proof. For $k \not\equiv 0 \pmod{N}$, put

$$d_k = \frac{N}{(k, N)}, \quad S_b(k) = \sum_{x \in \mathcal{D}_b} e^{2\pi i k x / N}, \quad W_k(x) = \frac{1}{M_A} \sum_{a \in \mathcal{A}} e^{2\pi i k a x / N}.$$

If $d_k > R$, distinct members of $\mathcal{A}$ are distinct modulo d_k . Additive orthogonality therefore gives

$$\sum_{x \bmod N} |W_k(x)|^2 = \frac{N}{M_A}.$$

Averaging the approximate invariance in (4.4) and applying Cauchy–Schwarz yields

$$|S_b(k)| \leq \sqrt{D_b N / M_A + 2\bar{D}} \quad (d_k > R). \quad (4.6)$$

For $d_k \leq R$, retain the trivial bound D_b .

Let $I = [1, h]$ with $h = \lfloor X/b \rfloor$. At frequencies of reduced denominator d , the sum of the absolute Fourier coefficients of $\mathbf{1}_I$ is at most dH_d , by the geometric-sum bound. Thus all denominators at most R cost at most $R^2 H_R$. The normalized sum over all nonzero frequencies is at most H_N .

Fourier inversion in the empty interval from (4.4), followed by (4.6), now gives

$$\frac{D_b h}{N} \leq \frac{D_b R^2 H_R}{N} + H_N (\sqrt{D_b N / M_A + 2\bar{D}}).$$

Since $h \geq N/(2b)$ and $R^2 H_R \leq N/(4b)$, we have $h - R^2 H_R \geq N/(4b)$. Moving the small-denominator term to the left and writing $z = D_b/N$ gives

$$z \leq 4b H_N \sqrt{z / M_A} + 8b H_N \bar{D} / N.$$

Apply $u\sqrt{z} \leq (z + u^2)/2$ with $u = 4b H_N / \sqrt{M_A}$, and move $z/2$ to the left. The resulting coefficients are $u^2 = 16b^2 H_N^2 / M_A$ and $16b H_N \bar{D} / N$, exactly as in (4.5). $\square$

The low-denominator frequencies are paid for explicitly. Omitting them would restrict the argument to totals with a roughness condition and would not suffice for Theorem 1.1.

4.3 A fixed box of generators

Fix a block $\mathcal{Q}_0 = \{q_1, \dots, q_d\}$ of distinct primes and let $B_0 = \prod_{q \in \mathcal{Q}_0} q$. Suppose $(N, B_0) = 1$. For

$$R = \lfloor N^{1/4} \rfloor, \quad q_{\max} = \max \mathcal{Q}_0, \quad J = \lfloor \frac{\log R}{d \log q_{\max}} \rfloor,$$

the box

$$\mathcal{A} = \{q_1^{j_1} \cdots q_d^{j_d} : 0 \leq j_i \leq J\}$$

is contained in $[1, R]$, consists of units, and has $M_A = (J+1)^d \gg_{B_0} L^d$ elements. Applying Lemma 4.1 only to the fixed q_i and using (4.4), we have

$$\bar{D} \ll_{B_0} L(E+1).$$

Consequently, for unit b in any fixed polylogarithmic range and sufficiently large N ,

$$\frac{D_b}{N} \ll_{B_0} \frac{b^2}{L^{d-2}} + \frac{bL^2(E+1)}{N}. \quad (4.7)$$

The hypothesis of Lemma 4.2 holds in these ranges because $R^2 H_R \ll N^{1/2} L$.

We retain two budgets, rather than obscuring their different uses in one asymptotic symbol:

Use	d	dilation cutoff Y	reflection error	$\max_{b \leq Y, (b, N)=1} D_b/N$
All-total counting	1024	L^{256}	$E \leq N/L^{512}$	$\ll_{B_0} L^{-254}$
Prime logarithmic scale	128	L^{32}	$E \leq N/L^{96}$	$\ll_{B_0} L^{-62}$

For the second row, for example, the two exponents in (4.7) are $64 - 126 = -62$ and $32 + 2 - 96 = -62$. The number of generators is fixed, and their descent costs can be enormous. These choices are not optimized and do not give practical numerical thresholds.

5 The short-product spectral test

The unit group permits characters of proper conductor, which must be kept when N is composite. There are two tasks: confine the Fourier mass to a small family, then use short integer products to select one real character with a coefficient close to $+1$. The second task is not a consequence of small dilation defects alone.

5.1 Prime labels and high moments

Let $G = (\mathbb{Z}/N\mathbb{Z})^\times$, and view $F(x) = f(x)$ on the representatives $1 \leq x < N$. Use the Fourier convention

$$a_\psi = \frac{1}{\phi(N)} \sum_{x \in G} F(x) \overline{\psi(x)}, \quad F(x) = \sum_{\psi \in \widehat{G}} a_\psi \psi(x), \quad \sum_{\psi} |a_\psi|^2 = 1. \quad (5.1)$$

For $Y = L^{256}$ or L^{32} , put

$$M_Y = \#\{q \leq Y : q \text{ prime}, q \nmid N\}, \quad z_\psi = \frac{1}{M_Y} \sum_{\substack{q \leq Y \text{ prime} \\ q \nmid N}} f(q) \psi(q), \quad \mathcal{B} = \{\psi : \Re z_\psi > 1/2\}.$$

The labels are the actual prime signs $f(q)$, not invariably -1 .

Lemma 5.1 (Prime-product moment). *If $Y^j < N$, then*

$$\sum_{\psi \in \widehat{G}} |z_\psi|^{2j} \leq \phi(N) j! M_Y^{-j}. \quad (5.2)$$

For the two cutoffs, respectively,

$$|\mathcal{B}| \leq N^{1/256+o(1)} \leq N^{1/192}, \quad |\mathcal{B}| \leq N^{1/32+o(1)} \leq N^{1/30}. \quad (5.3)$$

Proof. Expand the moment and use character orthogonality. A surviving pair of j -tuples satisfies a congruence between two positive products less than N , and therefore an equality of integers. Unique factorization identifies their prime multisets, so the weights $f(q)$ cancel. Each ordered left tuple permits at most $j!$ right tuples. This proves (5.2).

Take $j = \lfloor \log X / \log Y \rfloor$, so $Y^j \leq X < N$. Chebyshev and Lemma 2.6 give $M_Y \sim \pi(Y) \gg Y / \log Y$. Markov at $1/2$, followed by $j! \leq j^j$, yields

$$|\mathcal{B}| \leq \phi(N) 4^j j! M_Y^{-j}.$$

If $Y = L^a$ with $a = 256$ or 32 , its logarithm is at most

$$L + j(-(a-1) \log L + O(1)) = L/a + o(L),$$

which gives (5.3). $\square$

Multiplicative Parseval gives the exact nonnegative identity

$$\begin{aligned}\Delta &:= \frac{1}{M_Y \phi(N)} \sum_{\substack{q \leq Y \\ q \nmid N}} \sum_{\substack{\text{prime } x \in G}} |F(qx) - f(q)F(x)|^2 \\ &= \sum_{\psi} |a_{\psi}|^2 (2 - 2\Re z_{\psi}), \quad \sum_{\psi \notin \mathcal{B}} |a_{\psi}|^2 \leq \Delta.\end{aligned}\tag{5.4}$$

By the two propagation budgets and $\nu^{-1} \ll \ell$, for sufficiently large N this is bounded by

$$\Delta \leq L^{-253} \quad \text{or} \quad \Delta \leq L^{-61},\tag{5.5}$$

respectively.

5.2 Integer products, induced characters, and orientation

Let $H = \lfloor \sqrt{X} \rfloor$, $\mathcal{U}_H = \{n \leq H : (n, N) = 1\}$ and $h_H = |\mathcal{U}_H|$. Inclusion–exclusion gives

$$h_H = \nu H + O(2^{\omega(N)}) \sim \nu H.$$

Define coefficients with the *opposite* conjugation convention to (5.1):

$$b_{\psi} = \frac{1}{h_H} \sum_{n \in \mathcal{U}_H} f(n) \psi(n).$$

Complete multiplicativity, $ab \leq H^2 \leq X$, and Fourier inversion give the exact identity

$$1 = \mathbb{E}_{a,b \in \mathcal{U}_H} f(ab) f(a) f(b) = \sum_{\psi} a_{\psi} b_{\psi}^2.\tag{5.6}$$

Both factors are drawn from the short unit interval, and their products are integers below N , not merely products of residues. The sign on the left matters: for a real character χ , a table exactly equal to $-\chi$ on these units and products would give $(-\chi(ab))(-\chi(a))(-\chi(b)) = -1$, not $+1$. The estimates below make this orientation argument quantitative on the restricted spectrum. For real characters $b_{\psi}^2 \geq 0$ preserves the sign of a_{ψ} ; nonreal characters are controlled in conjugate pairs.

If a nonprincipal character modulo N is induced by a primitive ξ of conductor $Q \mid N$, then

$$\sum_{n \leq H} \psi(n) = \sum_{\substack{d \mid \text{rad } N \\ (d, Q) = 1}} \mu(d) \xi(d) \sum_{m \leq H/d} \xi(m).\tag{5.7}$$

Using Burgess with $\eta = 1/128$, its mean on $\mathcal{U}_H$ has modulus at most

$$O(2^{\omega(N)} \nu^{-1} H^{-1/2} N^{3/16+1/128}) = N^{-7/128+o(1)} \leq N^{-3/64}.\tag{5.8}$$

The same bound holds for the quotient of any two distinct characters. In particular, the Gram matrix of the vectors $\psi|_{\mathcal{U}_H}$ for $\psi \in \mathcal{B}$, with normalized inner product, has norm at most $1 + \gamma$, where

$$\sum_{\psi \in \mathcal{B}} |b_{\psi}|^2 \leq 1 + \gamma, \quad \gamma \leq \begin{cases} N^{-1/64}, & Y = L^{256}, \\ N^{-1/80}, & Y = L^{32}. \end{cases}\tag{5.9}$$

Indeed, a family of K unit vectors with off-diagonal inner products of modulus at most μ has Gram norm at most $1 + (K - 1)\mu$, by expanding its quadratic form. Combining (5.3) and (5.8), the two exponent margins are

$$\frac{1}{192} - \frac{3}{64} = -\frac{1}{24} < -\frac{1}{64}, \quad \frac{1}{30} - \frac{3}{64} = -\frac{13}{960} < -\frac{1}{80}.$$

They justify both rows of (5.9) with room to absorb fixed constants.

The complete fourth moment costs only a logarithm, apart from the explicit unit density:

$$\sum_{\psi} |b_{\psi}|^4 \leq \frac{\phi(N)}{h_H^4} \#\{a, b, c, d \leq H : ab = cd\} \ll \nu^{-3} L \leq L^2. \quad (5.10)$$

Here orthogonality first gives $ab \equiv cd \pmod{N}$. Both products lie in $[1, X]$, so the congruence is an integer equality, and Lemma 3.1 applies. Dropping the unit restriction only increases the count.

For completeness, the finite inequality behind the argument is worth isolating.

Proposition 5.2 (Oriented non-concentration). *For a real sign table F on G , define a_{ψ}, b_{ψ} as above, and let τ be the proportion of pairs $a, b \in \mathcal{U}_H$ for which $F(ab) \neq F(a)F(b)$. Suppose a family $\mathcal{B}$ satisfies $\sum_{\mathcal{B}} |b_{\psi}|^2 \leq 1 + \gamma$, and put $w = \sum_{\psi \notin \mathcal{B}} |a_{\psi}|^2$ and $\mathcal{E}_H = \sum_{\psi} |b_{\psi}|^4$. Then*

$$1 - 2\tau \leq \max \left\{ \frac{1}{\sqrt{2}}, \max_{\substack{\psi \in \mathcal{B} \\ \psi^2=1}} a_{\psi} \right\} (1 + \gamma) + \sqrt{\mathcal{E}_H w}. \quad (5.11)$$

If the inner maximum is over an empty set, only $1/\sqrt{2}$ is retained.

Proof. The left side is $\sum a_{\psi} b_{\psi}^2$, by the same expansion as (5.6). Since F is real, $a_{\bar{\psi}} = \overline{a_{\psi}}$. Parseval implies $|a_{\psi}| \leq 1/\sqrt{2}$ for nonreal ψ . For real ψ , b_{ψ} is real, so $a_{\psi} b_{\psi}^2$ retains the sign of a_{ψ} . These facts bound the real part of the contribution from $\mathcal{B}$ by the first term of (5.11). Cauchy–Schwarz bounds the complement by $\sqrt{\mathcal{E}_H w}$. $\square$

In our application $\tau = 0$, $w \leq \Delta$, and (5.10) gives

$$1 \leq \max \left\{ 1/\sqrt{2}, \max_{\substack{\psi \in \mathcal{B} \\ \psi^2=1}} a_{\psi} \right\} (1 + \gamma) + L\sqrt{\Delta}.$$

For sufficiently large N , $(1 + \gamma)/\sqrt{2} + L\sqrt{\Delta} < 1$, so the maximum cannot be supplied by the nonreal branch. Choose a real ψ_0 attaining the real maximum. Solving for its positively oriented coefficient gives

$$a_{\psi_0} \geq \frac{1 - L\sqrt{\Delta}}{1 + \gamma}, \quad \frac{1 - a_{\psi_0}}{2} \leq \frac{\gamma + L\sqrt{\Delta}}{2(1 + \gamma)} \leq \frac{\gamma + L\sqrt{\Delta}}{2}.$$

By (5.5), the two bounds for $L\sqrt{\Delta}$ are $L^{-251/2}$ and $L^{-59/2}$, respectively. The power-saving Gram error is smaller still. Thus

$$\frac{D_U}{\phi(N)} := \frac{\#\{x \in G : f(x) \neq \psi_0(x)\}}{\phi(N)} = \frac{1 - a_{\psi_0}}{2} \leq \begin{cases} L^{-124}, & Y = L^{256}, \\ L^{-29}, & Y = L^{32}. \end{cases} \quad (5.12)$$

The orientation is positive in both budgets. An estimate only for $|a_{\psi_0}|$ would not give the required conclusion.

5.3 Two inverse statements from the same proof

Let χ be the primitive character inducing ψ_0 , with conductor $Q \mid N$, and extend it to a completely multiplicative sign function by

$$g(p) = f(p) \ (p \mid N), \quad g(p) = \chi(p) \ (p \nmid N). \quad (5.13)$$

The quotient $h = fg$ equals 1 at primes dividing N . Apply Lemma 3.2 to (5.12). Since $J_N \ll L$ and $H_X \ll L$, the reconstruction loss is at most $O(L^{5/2} \sqrt{D_U/X})$.

Theorem 5.3 (Generic logarithmic inverse). *Fix a block of 128 distinct primes, with product B_0 . For all sufficiently large N depending on this block, if $(N, B_0) = 1$ and*

$$\mathcal{E}_\kappa(f; N) \leq N/(\log N)^{96},$$

then there is a real primitive character χ of conductor $Q \mid N$, allowing $Q = 1$, such that $\chi(-1) = \kappa$ and its extension (5.13) satisfies

$$\max_{1 \leq t < N} \frac{\#\{n \leq t : f(n) \neq g(n)\}}{t} \leq L^{-11}, \quad (5.14)$$

$$\sum_{\substack{q < N \text{ prime} \\ q \nmid N, f(q) \neq \chi(q)}} \frac{1}{q} \leq 3L^{-11}. \quad (5.15)$$

The statement is uniform in the completely multiplicative sign function f .

Proof. The second propagation and spectral budgets give (5.12) with exponent 29. Reconstruction yields $\delta \ll L^{5/2-29/2} = L^{-12} \leq L^{-11}$ for sufficiently large N , proving the two bounds. If $\chi(-1) \neq \kappa$, every unit pair outside at most $2D_U$ disagreements has the wrong reflection product. Thus $\mathcal{E}_\kappa(f; N) \geq \phi(N) - 2D_U \gg N/\ell$, a contradiction. $\square$

Proposition 5.4 (High-precision structured branch). *Fix $0 < \theta < 1$, $M \geq 0$, and a block of 1024 distinct primes with product B_0 . For sufficiently large N , uniformly for $f \in \mathcal{F}_{\theta, M}$, if $(N, B_0) = 1$ and $E_*(f; N) \leq N/L^{512}$, there is a primitive nonprincipal quadratic character χ of conductor $Q \mid N$ with the majority reflection parity. With g as in (5.13),*

$$\frac{\#\{n < N : (n, N) = 1, f(n) \neq \chi(n)\}}{\phi(N)} \leq L^{-124}, \quad (5.16)$$

$$\max_{1 \leq t < N} \frac{\#\{n \leq t : f(n) \neq g(n)\}}{t} \leq L^{-55}, \quad (5.17)$$

$$S := \sum_{\substack{r < N \text{ prime} \\ r \nmid N, f(r) \neq \chi(r)}} \frac{1}{r} \leq 3L^{-55}. \quad (5.18)$$

Proof. Choose the majority sign κ , so $\mathcal{E}_\kappa = E_*$. Use the first budgets, and then reconstruction. The loss is $O(L^{5/2-124/2}) = O(L^{-119/2})$, which is smaller than L^{-55} beyond a fixed threshold. The parity argument is the same as above.

It remains to exclude the principal character. The budget gives $\#\{q \leq Y : q \in E(f)\} \leq MY^\theta$. Also $\omega(N) = o(M_Y)$. Therefore, for the principal character,

$$z_1 = -1 + O_{\theta, M}(Y^{\theta-1} \log Y) = -1 + o(1).$$

It does not belong to $\mathcal{B}$, whereas ψ_0 does. The resulting real primitive character is nonprincipal, hence quadratic. $\square$

Remark 5.5. The principal character in Theorem 5.3 is necessary: $f \equiv 1$ has exact positive reflection. Proper conductors are necessary as well. If χ_3 is the nonprincipal character modulo 3, then

$$f(n) = (-1)^{v_3(n)} \chi_3(n/3^{v_3(n)})$$

has $f(n)f(3^k - n) = -1$ for $1 \leq n < 3^k$. This is a source of conductor 3, not Liouville. It is a special case of the exact odd-prime-power extremizers classified in [9, Section 5.3]; no novelty is claimed for this example. The inverse classifies sufficiently accurate saturation; it does not prohibit saturation for arbitrary multiplicative sources.

6 Uniform means and a signed prime inventory

We now prove the lower bounds for $f \in \mathcal{F}_{\theta, M}$. This also proves the Liouville case without a second copy of the common machinery.

6.1 A positive convolution for the source

Lemma 6.1. *Let $E = E(f)$, and define the nonnegative multiplicative function a_E by $a_E(1) = 1$ and*

$$a_E(p^j) = \begin{cases} 2, & p \in E, j \geq 1, \\ 0, & p \notin E, j \geq 1. \end{cases}$$

Then $f = \lambda * a_E$ and

$$\sum_{d \geq 1} \frac{a_E(d)}{d^\theta} = \prod_{p \in E} \frac{1 + p^{-\theta}}{1 - p^{-\theta}} \leq \exp(C_\theta M), \quad C_\theta = \frac{2}{1 - 2^{-2\theta}}. \quad (6.1)$$

Uniformly in $f \in \mathcal{F}_{\theta, M}$,

$$A_f(x) \ll_{\theta, M} x \exp(-c_1 \sqrt{\log x}), \quad \sup_{x \geq 1} \left| \sum_{n \leq x} \frac{f(n)}{n} \right| \ll_{\theta, M} 1, \quad (6.2)$$

for $x \geq 3$ in the first estimate, with an absolute $c_1 > 0$ after reducing the constant in (2.4).

Proof. The convolution identity is checked at prime powers. The quotient of the two local Dirichlet factors is $(1+z)/(1-z)$ at a prime of E and 1 elsewhere. Positivity justifies the infinite product. For $0 \leq z \leq 2^{-\theta}$,

$$\log \frac{1+z}{1-z} \leq \frac{2z}{1-z^2} \leq C_\theta z,$$

proving (6.1).

In $A_f(x) = \sum_{d \leq x} a_E(d) A_\lambda(x/d)$, split at $d = \sqrt{x}$. The first part is $O_{\theta, M}(x \exp(-c \sqrt{\log x}/\sqrt{2}))$. The second has modulus at most

$$x \sum_{d > \sqrt{x}} \frac{a_E(d)}{d} \leq \exp(C_\theta M) x^{(1+\theta)/2}.$$

For fixed $\theta < 1$, this is absorbed by the first estimate in (6.2) after reducing c_1 . For the harmonic estimate, convolve the truncated harmonic sums and use the bounded Liouville harmonic sums and $\sum a_E(d)/d \leq \exp(C_\theta M)$. $\square$

6.2 Harmonic masks on the units

Fix from now on

$$\alpha = \frac{13}{50}, \quad Z = \lfloor N^\alpha \rfloor, \quad R = \lfloor X/Z \rfloor. \quad (6.3)$$

Compatibility of the two cutoffs. The cutoff must be large enough for Pollack's mean theorem after truncating the filter, but small enough that the Brun–Titchmarsh loss leaves a positive signed population. The inventory and the convolution therefore use the same α . With the convolution cutoff $d \leq N^{1/400}$ and Pollack parameter $\eta_1 = 1/200$, sufficient simultaneous constraints are

$$\frac{103}{400} < \alpha < \frac{\exp(7/16) - 1}{2 \exp(7/16) - 1} \approx 0.2616392277.$$

The left inequality ensures $\alpha - 1/400 > 1/4 + 1/200$. The right is exactly

$$c_\alpha := \frac{1}{2} - \log \frac{1 - \alpha}{1 - 2\alpha} > \frac{1}{16}.$$

Our choice $\alpha = 0.26$ lies strictly between the endpoints. It gives $c_\alpha = \frac{1}{2} - \log(37/24) = 0.0671359177\dots$, with margin $0.0046359177\dots$ above $1/16$. These decimal values are explanatory; only the strict inequalities are used. The remaining tail saving is $(1 - \alpha)(1 - \theta) > 0$ for every fixed $\theta < 1$, not a condition $\alpha + \theta < 1$.

Suppose that Proposition 5.4 applies, and let χ and S be as there. Write $\rho = \chi(-1)$.

Lemma 6.2 (Unit means and sign masks). *Uniformly in $f \in \mathcal{F}_{\theta, M}$ and in the factorization of N ,*

$$\sum_{\substack{n \leq t \\ (n, N)=1}} \frac{1}{n} = v \log t + O(P_N(1 + T_N)), \quad (6.4)$$

$$\left| \sum_{\substack{n \leq t \\ (n, N)=1}} \frac{f(n)}{n} \right| \ll_{\theta, M} P_N. \quad (6.5)$$

For $\frac{1}{2}N^{1-\alpha} \leq t \leq N$, each sign has $(v/2 + o(v))t$ representatives among the units up to t . Furthermore, for a prime $q \leq N^\alpha$ with $q \nmid N$ and $\kappa = \pm 1$,

$$\sum_{\substack{a < q \\ (a, N)=1, \chi(a)=\kappa}} \frac{1}{a \log(N/(aq))} = \frac{v}{2} \log \left(\frac{\log(N/q)}{\log(N/q^2)} \right) + O_{\theta, M} \left(S + \frac{\ell^2}{L} \right). \quad (6.6)$$

Proof. Inclusion–exclusion gives

$$\sum_{\substack{n \leq t \\ (n, N)=1}} \frac{1}{n} = \sum_{d \mid \text{rad } N} \frac{\mu(d)}{d} H_{t/d}.$$

Replacing $H_{t/d}$ by $\log t$ incurs an error bounded by a constant times

$$\sum_{d \mid \text{rad } N} \frac{1 + \log d}{d} = P_N(1 + T_N).$$

This is also valid for $d > t$, since then $\log t \leq \log d$. The coefficient of $\log t$ is ν . For the signed sum, inclusion–exclusion has weights $\mu(d)f(d)/d$, and Lemma 6.1 proves (6.5).

For the unweighted signed sum on units, the same expansion, split at $d = \sqrt{t}$, gives

$$\sum_{\substack{n \leq t \\ (n, N)=1}} f(n) \ll_{\theta, M} t P_N \exp(-c_2 \sqrt{\log t}) + 2^{\omega(N)} \sqrt{t} = o(\nu t) \quad (6.7)$$

in the specified range. Combine this with $u_N(t) = \nu t + O(2^{\omega(N)})$ to separate the signs.

A unit a at which $f(a) \neq \chi(a)$ has a prime divisor appearing in (5.18). Thus the harmonic mass of mismatches below q is at most $H_{q-1}S$. With $\log(N/(aq)) \gg_\alpha L$, replacing the χ mask in (6.6) by the f mask costs $O(S)$.

The f mask is $(1 + \kappa f(a))/2$ on units. Apply partial summation to (6.4) and (6.5) with weight $1/\log(N/(qt))$. Its endpoint size and total variation on $[1, q]$ are $O_\alpha(1/L)$. The main integral is

$$\frac{\nu}{2} \int_1^q \frac{dt}{t \log(N/(qt))} = \frac{\nu}{2} \log \left(\frac{\log(N/q)}{\log(N/q^2)} \right).$$

Changing the endpoint from q to $q - 1$ costs $O(1/L)$. Since $P_N(1 + T_N) \ll \ell^2$, the remaining errors are as stated. $\square$

6.3 Unite candidates before removing a common exceptional set

A useful prime has $q \nmid N$, $f(q) = -1$ and $\chi(q) = 1$. To see what it creates, suppose that $qm < N$ and that m and $N - qm$ are units where f agrees with χ . If $f(m) = \sigma$, then

$$f(qm) = -\sigma, \quad f(N - qm) = \chi(-qm) = \rho\sigma, \quad \rho = \chi(-1).$$

Thus q reverses one side of the character's pattern and gives product $-\rho$. We must retain many distinct instances of this pair after paying for the failures of agreement.

The local families below are *candidates*, not yet successful representations. Their remaining failures, for every marking prime, lie in one common set V_R of origins whose second summand has a large positive prime factor of f . We will unite the families, bound their pair intersections, and only then delete V_R once.

Let

$$\mathcal{B}_f = \{r < N : r \text{ prime}, r \nmid N, f(r) \neq \chi(r)\}, \quad \mathcal{B}_f^- = \{r \in \mathcal{B}_f : f(r) = -1\}.$$

The mass of $\mathcal{B}_f$ is S . Define the useful marking primes

$$\mathcal{P} = \{q \leq Z : q \text{ prime}, q \nmid N, f(q) = -1, \chi(q) = 1\}, \quad U = \sum_{q \in \mathcal{P}} \frac{1}{q} \leq S. \quad (6.8)$$

The following inventory is finite and needs no asymptotic assumptions.

For $q \in \mathcal{P}$ and $\sigma = \pm 1$, put

$$\begin{aligned} T_q &= \lfloor X/q \rfloor, & I_q^\sigma &= \#\{m \leq T_q : (m, N) = 1, f(m) = \sigma\}, \\ K_q^\sigma &= \sum_{\substack{a < q \\ (a, N)=1, \chi(a)=\rho\sigma}} \pi(N/a; q, N\bar{a}), \\ B_q^\sigma &= I_q^\sigma - 3T_q S - K_q^\sigma, & W_E(N, R) &= \sum_{\substack{r \in E(f) \\ R < r < N}} \lfloor X/r \rfloor. \end{aligned} \quad (6.9)$$

Here $\bar{a}$ denotes the inverse modulo q .

Proposition 6.3 (Common-deletion inventory). *With the definitions above,*

$$N_{-\sigma, \rho\sigma}^f(N) \geq \sum_{q \in \mathcal{P}} [B_q^\sigma]_+ - \sum_{\substack{q, r \in \mathcal{P} \\ q < r}} \lfloor \frac{X}{qr} \rfloor - W_E(N, R). \quad (6.10)$$

The same inequality holds for any subfamily of the useful marking primes and for any $2 \leq Z \leq X$, with $R = \lfloor X/Z \rfloor$.

Proof. For each q , start with the m counted by I_q^σ , and regard $n = qm$ as a candidate first summand. Remove first those m with $f(m) \neq \chi(m)$, at a cost at most $T_q S$. Next remove the candidates for which some $r \in \mathcal{B}_f$, $r \leq T_q$, divides $N - qm$. The prime q itself cannot divide $N - qm$. Each other r costs at most $T_q/r + 1$, and the number of such r is at most $T_q S$. This removal costs at most $2T_q S$.

Third, remove candidates with a divisor $r \in \mathcal{B}_f^-$ of $N - qm$ satisfying $r > T_q$. As r is an integer, $r > X/q$. Write $N - qm = ar$. Then $a < q$, $(a, N) = 1$, and $\chi(r) = \chi(q) = 1$. Since $Q \mid N$ and $\chi(m) = \sigma$, we have

$$\chi(a) = \rho\sigma, \quad r \equiv N\bar{a} \pmod{q}.$$

Every such pair (a, r) determines at most one origin. This removal is therefore bounded by K_q^σ .

Let $\mathcal{A}_q^\sigma$ be the surviving candidates $n = qm$. We have $|\mathcal{A}_q^\sigma| \geq [B_q^\sigma]_+$. A survivor still has $f(n) = -\sigma$. If it fails the desired second sign, $N - n$ has a prime of disagreement $r > T_q$ with $f(r) = 1$ and $\chi(r) = -1$. Since $T_q \geq R$, every such failed survivor, whatever its marking prime, belongs to the same set

$$V_R = \{1 \leq n < N : \text{some } r \in E(f), r > R, \text{ divides } N - n\}.$$

The involution $n \mapsto N - n$ gives $|V_R| \leq W_E(N, R)$. Also $|\mathcal{A}_q^\sigma \cap \mathcal{A}_r^\sigma| \leq \lfloor X/(qr) \rfloor$, since an intersection consists of multiples of qr .

For finite sets A_i, V , Bonferroni followed by one deletion gives

$$\left| \left(\bigcup_i A_i \right) \setminus V \right| \geq \sum_i |A_i| - \sum_{i < j} |A_i \cap A_j| - |V|.$$

Apply this to the candidate sets and V_R . No independence is involved. $\square$

Formula (6.10) charges a deleted origin only once, even if it was produced by several marking primes. The next estimate measures the saving from this order of operations.

Lemma 6.4 (Quantitative signed supply). *In the structured branch of Proposition 5.4, for sufficiently large N ,*

$$N_{-\sigma, \rho\sigma}^f(N) \geq \frac{\nu N U}{64} - C_\theta M N^{1-\beta_\theta}, \quad \beta_\theta = \frac{37}{50}(1 - \theta) > 0. \quad (6.11)$$

Both patterns with product ρ have at least $\phi(N)/3$ representatives.

Proof. For $q \in \mathcal{P}$, we have $T_q \geq \frac{1}{2}N^{1-\alpha}$ for large N , and $1/q \leq S$. Thus $q \rightarrow \infty$ uniformly over nonempty families in this branch. The sign count in Lemma 6.2, Brun–Titchmarsh, and (6.6) give

$$B_q^\sigma \geq \frac{N}{q} \left\{ \nu c_\alpha - O_{\theta, M} \left(S + \frac{\ell^2}{L} \right) - o(\nu) \right\},$$

where

$$c_\alpha = \frac{1}{2} - \log \frac{1-\alpha}{1-2\alpha} = \frac{1}{2} - \log \frac{37}{24} > \frac{1}{16}. \quad (6.12)$$

Indeed, the progression is reduced, $N/a > q$, and Brun–Titchmarsh contributes the factor $2N/(q-1)$ times the masked harmonic sum. Replacing $q/(q-1)$ by $1+o(1)$ is legitimate since $q \rightarrow \infty$. The expression inside the logarithm is largest at $q = N^\alpha$.

All errors are $o(v)$ by (5.18) and (2.9). Hence $B_q^\sigma \geq vN/(32q)$. The pair intersections in (6.10) have total at most $NU^2/2$. As $U \leq S = o(v)$, they leave at least $vNU/64$.

The common deleted population satisfies the direct bound

$$W_E(N, R) \leq X \sum_{\substack{r \in E(f) \\ r > R}} \frac{1}{r} \leq MXR^{\theta-1} \ll_\theta MN^{1-\beta_\theta}. \quad (6.13)$$

This proves (6.11).

On the unit group the nonprincipal real character has exactly $\phi(N)/2$ occurrences of each sign, and its reflected product is ρ . Removing at most two unit disagreement sets, whose density is bounded in (5.16), leaves at least $\phi(N)/3$ occurrences of each of those two patterns. $\square$

For $f = \lambda$, $E(f)$ is empty and the tail in (6.13) is exactly zero. For a general source it has a power saving for every fixed $\theta < 1$. No restriction $\alpha + \theta < 1$ is needed after the common deletion.

7 A filtered convolution supplies disagreement mass

It remains to lower-bound U in (6.8). Pollack’s use of the positive convolution $1 * \chi$, including its connection with reciprocal sums of prime residues, is an antecedent of this step [8, Proposition 3.1, proof of Theorem 1.3 and the subsequent remark]. Here we additionally remove both the prime factors of N and the possibly infinite set of positive primes of f . The point of Proposition 7.2 is to retain a uniform error and a positive Euler factor after this filtering, so that only useful disagreements are detected.

Define

$$b_{\chi, N, E}(n) = (1 * \chi)(n) \mathbf{1}_{(n, N)=1} \mathbf{1}_{p|n \Rightarrow p \notin E}, \quad B_{\chi, N, E}(x) = \sum_{n \leq x} b_{\chi, N, E}(n). \quad (7.1)$$

The local coefficients explain what this detects. For $\chi(q) = 1$, $(1 * \chi)(q^j) = j + 1$; for $\chi(q) = -1$, the coefficient is 1 when j is even and 0 when j is odd. All ramified primes are removed because $Q \mid N$. After the filter, a contribution without a useful prime must therefore come from a square. A mean exceeding this square contribution forces a supply of useful primes; the following bound quantifies that observation before we estimate the infinite filter.

Lemma 7.1 (A positive upper bound). *For $x = Z$ and $U \leq 1/2$,*

$$B_{\chi, N, E}(Z) \leq \lfloor \sqrt{Z} \rfloor + 4ZU. \quad (7.2)$$

Proof. If a contributing integer has no useful prime divisor, all its prime divisors have character value -1 . It contributes only when it is a square, with weight 1. For a useful prime q , the local factors give $b_{\chi, N, E}(qm) \leq 2b_{\chi, N, E}(m)$.

For $y \leq Z$, the weight $b_{\chi,N,E}(n)$ is at most the number of divisors of n supported on useful primes. Consequently,

$$B_{\chi,N,E}(y) \leq y \prod_{q \in \mathcal{P}} (1 - q^{-1})^{-1} \leq \frac{y}{1 - U} \leq 2y.$$

A positive union bound over the useful prime divisors now costs at most $\sum_{q \in \mathcal{P}} 2B_{\chi,N,E}(Z/q) \leq 4ZU$, proving the assertion. $\square$

Proposition 7.2 (Uniform infinite-prime filtering). *Fix $0 < \theta < 1$ and $M \geq 0$. There exists $c_\theta > 0$ such that, uniformly for $f \in \mathcal{F}_{\theta,M}$ and for a primitive nonprincipal quadratic character χ of conductor $Q \mid N$,*

$$B_{\chi,N,E}(Z) = ZC_{N,E}(\chi)L(1, \chi) + O_{\theta,M}(ZN^{-c_\theta}), \quad (7.3)$$

where the convergent product

$$C_{N,E}(\chi) = \prod_{p \mid N \text{ or } p \in E} (1 - p^{-1})(1 - \chi(p)p^{-1}) \geq e^{-4M} v^2 \quad (7.4)$$

is not discarded from the main term.

Proof. At each removed prime, the filter polynomial is

$$(1 - z)(1 - \chi(p)z) = 1 - (1 + \chi(p))z + \chi(p)z^2.$$

Let c_N be the filter on the primes of N , and $c_{E \setminus N}$ the filter on the primes of E not dividing N . Their prime supports are disjoint. Coefficientwise,

$$b_{\chi,N,E} = c * (1 * \chi), \quad c = c_N * c_{E \setminus N},$$

and every individual coefficient is a finite sum. The weighted absolute norm is

$$\sum_{d \geq 1} \frac{|c(d)|}{d^\theta} \leq 4^{\omega(N)} \prod_{p \in E} (1 + p^{-\theta})^2 \leq 4^{\omega(N)} e^{2M}. \quad (7.5)$$

This replaces an unusable dependence on the possibly infinite number of removed primes. It also implies absolute convergence of $\sum c(d)/d$, whose value is $C_{N,E}(\chi)$.

At primes of N , the factor in (7.4) is at least $(1 - 1/p)^2$. At the remaining removed primes, use $\log(1 - 1/p) \geq -2/p$ and $\sum_{p \in E} 1/p \leq M$. This proves its lower bound.

Apply (2.7) with $\eta_1 = 1/200$, and decrease its $\eta_2 > 0$ if necessary so that $1 - \eta_2 \geq \theta$. Split the convolution at $d = N^{1/400}$. For the smaller d ,

$$Z/d \geq \frac{1}{2}N^{103/400} \geq Q^{51/200}$$

for all sufficiently large N , since $Q \leq N$ and $103/400 - 51/200 = 1/400 > 0$. This is the lower-end constraint displayed after (6.3); the harmless factor $1/2$ is absorbed by that strict exponent margin. Thus the domain of (2.7) is respected. The sum of its errors is bounded by

$$Z^{1-\eta_2} \sum_d \frac{|c(d)|}{d^{1-\eta_2}} \ll_{\theta,M} Z 4^{\omega(N)} N^{-\alpha\eta_2}.$$

For the tail, use $B_\chi(t) \leq t(1 + \log^+ t)$, $L(1, \chi) \ll 1 + \log Q$, and

$$\sum_{d > N^{1/400}} \frac{|c(d)|}{d} \leq N^{-(1-\theta)/400} \sum_d \frac{|c(d)|}{d^\theta}.$$

The actual tail and the omitted tail of the main term together cost

$$O_{\theta,M}(Z(1+L)4^{\omega(N)}N^{-(1-\theta)/400}).$$

Since $4^{\omega(N)} = N^{o(1)}$ uniformly, both errors have a fixed power saving. For example one may take a sufficiently small positive

$$c_\theta \leq \frac{1}{4} \min\{\alpha\eta_2, (1-\theta)/400\}.$$

This proves (7.3). □

Combining Lemma 7.1 and Proposition 7.2 and decreasing c_θ to absorb $Z^{-1/2}$ yields

$$U \geq \frac{1}{4} [C_{N,E}(\chi)L(1, \chi) - C'_{\theta,M}N^{-c_\theta}]_+. \quad (7.6)$$

In particular, Lemma 6.4 gives the useful hybrid estimate

$$\begin{aligned} N_{-\sigma, \rho\sigma}^f(N) &\geq \frac{\nu N}{256} [C_{N,E}(\chi)L(1, \chi) - C'_{\theta,M}N^{-c_\theta}]_+ \\ &\quad - C_\theta MN^{1-\beta_\theta}. \end{aligned} \quad (7.7)$$

The main term contains both the local factors of the total and the possibly very small value $L(1, \chi)$. The proof does not replace either by a positive absolute constant.

8 Completion at every total

8.1 The fixed-block conclusion

Fix θ, M, ε and a block of 1024 primes. Let N avoid that block and be sufficiently large. If

$$E_*(f; N) \geq N/L^{512},$$

then (2.3) and Lemma 6.1 give

$$m_f(N) \geq \frac{N}{4L^{512}}. \quad (8.1)$$

This exceeds a constant times $N^{1-\varepsilon}$.

Otherwise Proposition 5.4 applies. Both patterns of product ρ have at least $\phi(N)/3$ representatives. For the other two, use (7.7). Choose

$$0 < \zeta < \frac{1}{4} \min\{c_\theta, \beta_\theta, \varepsilon\}.$$

By (2.8) and (7.4),

$$C_{N,E}(\chi)L(1, \chi) \gg_{\theta,M,\zeta} \nu^2 Q^{-\zeta} \geq c_{\theta,M,\zeta} \nu^2 N^{-\zeta}.$$

The positive part of (7.7) therefore has size at least a constant times $\nu^3 N^{1-\zeta}$, and dominates both power-saving error terms. Since $\nu^{-1} \ll \ell$, it is $\gg_{\theta,M,\varepsilon} N^{1-\varepsilon}$. We have proved

$$m_f(N) \gg_{B_0, \theta, M, \varepsilon} N^{1-\varepsilon} \quad ((N, B_0) = 1), \quad (8.2)$$

uniformly in f in the class. The ineffectivity enters at the application of Siegel, and nowhere in the block selection that follows.

8.2 Reserve a block inside a large divisor

Divisor lifting by simultaneous scaling is an elementary established observation; see Bloom's comment in the original discussion [16] and [11, Lemma 2.1]. We use it together with a selection from several disjoint fixed prime blocks. It is this selection and its exponent bookkeeping, rather than scaling alone, that passes from a fixed-block estimate to every large total. The lift preserves a collection of distinct representations; it does not multiply its size by N/D .

Lemma 8.1 (Disjoint-block selection). *Let $Q_1, \dots, Q_J$ be disjoint finite sets of primes, with products B_i . Every $N \geq 2$ has a divisor D and an index i such that*

$$(D, B_i) = 1, \quad D \geq N^{1-1/J}. \quad (8.3)$$

For every completely multiplicative sign function f and every divisor $D \mid N$ with $D \geq 2$,

$$m_f(N) \geq m_f(D). \quad (8.4)$$

Proof. Put $A_i = \prod_{q \in Q_i} q^{v_q(N)}$. These integers are pairwise coprime and $\prod_i A_i \mid N$, so some $A_i \leq N^{1/J}$. Set $D = N/A_i$.

For (8.4), multiply both summands in a representation of D by N/D . This is injective and multiplies both signs by the same number $f(N/D) \in \{-1, 1\}$. It permutes the four sign pairs, proving the minimum inequality. $\square$

Proof of Theorems 1.1 and 1.2. It is enough to consider $0 < \varepsilon < 1$. Fix an integer $J \geq 2/\varepsilon$, then fix J disjoint blocks of 1024 primes. Apply (8.2) with exponent $\varepsilon/2$ to the divisor supplied by Lemma 8.1. There are only finitely many blocks, so take the minimum of their positive constants and the maximum of their thresholds. The divisor tends uniformly to infinity with N . Thus

$$m_f(N) \geq m_f(D) \gg_{\theta, M, \varepsilon} D^{1-\varepsilon/2} \geq N^{(1-1/J)(1-\varepsilon/2)} \geq N^{1-\varepsilon}.$$

The function and its prime budget have not changed on passage to the divisor. This proves Theorem 1.2; taking $M = 0$ proves Theorem 1.1. $\square$

The order of choices is fixed: first θ, M, ε , then the finitely many blocks and the analytic exponents, and only then the threshold for N . No boundedness assumption on $\omega(N)$ or $\Omega(N)$ is used. The argument neither selects a subsequence of totals nor changes the values of Liouville.

9 The logarithmic scale at prime totals

We now return to $f = \lambda$, but use the less stringent reflection threshold and the generic inverse of Theorem 5.3. This is a separate route: the signed inventory, filtered convolution, and Siegel lower bound are not needed.

Fix a block of 128 primes. Every sufficiently large prime p avoids it. Write $L = \log p$ and $T_p = p/L^{96}$. Suppose first that $E_*(\lambda; p) < T_p$, and let κ be the majority product. At a prime modulus the only real characters are the principal character and χ_p . The principal character cannot be supplied by Theorem 5.3: its prefix of length 2 disagrees with λ at 2, giving a proportion $1/2 > L^{-11}$. We obtain

$$\chi_p(-1) = \kappa, \quad \max_{t < p} \frac{\#\{n \leq t : \lambda(n) \neq \chi_p(n)\}}{t} \leq L^{-11}, \quad S_p := \sum_{\substack{q < p \\ \text{prime} \\ \chi_p(q)=1}} \frac{1}{q} \leq 3L^{-11}. \quad (9.1)$$

There is even exact agreement for every integer $n \leq \lfloor L^{11/2} \rfloor$: one error at n would give prefix proportion at least $1/n > L^{-11}$. In particular,

$$\chi_p(2) = \chi_p(3) = -1.$$

The supplementary laws of quadratic reciprocity then force $p \equiv 5$ or $19 \pmod{24}$.

Next, suppose $P_{\rho_p}(\lambda; p) < T_p$. For sufficiently large p , this is the minority product, so Theorem 5.3 applied with majority $\kappa = -\rho_p$ would give $\chi_p(-1) = -\rho_p$, a contradiction. Thus

$$P_{\rho_p}(\lambda; p) \geq T_p \tag{9.2}$$

at every sufficiently large prime. In the six classes (1.7), the branch $E_* < T_p$ was impossible, so both products have at least T_p representations.

Finally, by (2.4), $|A_\lambda(p-1)| \leq T_p/2$ for large p . The exact splitting identities (2.3) therefore give at least $T_p/4$ occurrences of each covered sign pair. They prove (1.6), the six-class conclusion, and the all-pattern conclusion off the fixed family (1.8).

$p \pmod{24}$	$\chi_p(-1)$	$\chi_p(2)$	$\chi_p(3)$	Logarithmic bound guaranteed without $p \notin \mathcal{X}$
1	+	+	+	All four
5	+	-	-	(+, +) and (-, -)
7	-	+	-	All four
11	-	-	+	All four
13	+	-	+	All four
17	+	+	-	All four
19	-	-	-	(+, -) and (-, +)
23	-	+	+	All four

Table 2: Scope of the prime bound. All four patterns still satisfy the all-total power bound in every row.

At a possible $p \in \mathcal{X}$, each of the two patterns of product ρ_p actually has $p/2 - o(p)$ occurrences: the character itself has $(p-1)/2$ of each, and at most twice the total disagreement in (9.1) is lost. No analogous logarithmic conclusion is claimed for the two opposite-product patterns at those primes.

10 An exceptional prime forces a real zero

Prime agreement will first give a *small positive value and a positive slope* at $s = 1$. A second-derivative bound then makes $L(s, \chi_p)$ change sign just to the left of 1, producing a real zero. Only afterwards does Landau–Page space the exceptional primes. A small value alone would not supply this Taylor argument.

The connection between approximate Liouville dilation symmetry and a nearby real zero already occurs in [6, Corollary 1.2]. The calculation below establishes the retained logarithmic precision directly, rather than invoking that antecedent as a black box.

Proposition 10.1 (Value and derivative at a possible exception). *If $p \in \mathcal{X}$ and $L = \log p$, then*

$$0 < L(1, \chi_p) \ll L^{-10}, \quad L'(1, \chi_p) = \zeta(2) + O(L^{-9}). \tag{10.1}$$

Proof. A disagreement between $\lambda(n)$ and $\chi_p(n)$ below p has a prime divisor counted in S_p . The weighted union bound gives

$$\sum_{\substack{n < p \\ \lambda(n) \neq \chi_p(n)}} \frac{1}{n} \leq H_{p-1} S_p.$$

The differences of the truncated harmonic sums, and of the sums weighted by $\log n$, are respectively $O(LS_p)$ and $O(L^2S_p)$.

For $\Re s > 1$, the Liouville Dirichlet series is $\zeta(2s)/\zeta(s)$. Estimate (2.4) and partial summation justify its boundary value and derivative at $s = 1$, giving

$$\sum_{n \geq 1} \frac{\lambda(n)}{n} = 0, \quad -\sum_{n \geq 1} \frac{\lambda(n) \log n}{n} = \zeta(2). \quad (10.2)$$

Their tails after p are $O(\exp(-c'\sqrt{L}))$, after reducing $c' > 0$ to absorb powers of L . No Liouville series is evaluated to the left of 1.

For the character, the Gauss-sum identity

$$\chi_p(n) = \tau(\overline{\chi_p})^{-1} \sum_{a=1}^{p-1} \overline{\chi_p(a)} e^{2\pi i a n / p}, \quad |\tau(\overline{\chi_p})| = \sqrt{p},$$

and the geometric-sum bound yield $\sup_t |\sum_{n \leq t} \chi_p(n)| \ll \sqrt{p} \log p$, after reducing by complete periods. Partial summation therefore bounds the corresponding two character tails by $O(p^{-1/2}L)$ and $O(p^{-1/2}L^2)$.

Combining these estimates with $S_p \leq 3L^{-11}$ gives

$$\begin{aligned} L(1, \chi_p) &= O(LS_p + e^{-c'\sqrt{L}} + p^{-1/2}L), \\ L'(1, \chi_p) &= \zeta(2) + O(L^2S_p + e^{-c'\sqrt{L}} + p^{-1/2}L^2). \end{aligned}$$

Positivity of $L(1, \chi_p)$ follows from the positive Euler product for real $s > 1$ and classical nonvanishing at 1. This is not an ineffective quantitative lower bound. $\square$

Lemma 10.2. *Uniformly for real $1 - 1/L \leq s \leq 1$,*

$$|L''(s, \chi_p)| \ll L^3. \quad (10.3)$$

Proof. Up to p , the absolute sum is at most $e \sum_{n \leq p} (\log n)^2 / n \ll L^3$. In the tail, periodicity bounds partial character sums by p . For large p the weight $(\log t)^2 t^{-s}$ is decreasing on $t \geq p$, and partial summation gives $O(p^{1-s}L^2) = O(L^2)$. The differentiated character series and these tail bounds are valid uniformly on the indicated compact interval in $s > 0$. $\square$

Let $a = L(1, \chi_p) > 0$ and $t = 2a/\zeta(2)$. By Proposition 10.1, $t \ll L^{-10} < 1/L$. Taylor's theorem and Lemma 10.2 give

$$L(1-t, \chi_p) = a - t(\zeta(2) + O(L^{-9})) + O(t^2L^3) = -a + o(a) < 0.$$

The derivative stays positive on $[1-t, 1]$. There is therefore a unique real zero in that interval, and

$$1 - \beta_p = \frac{L(1, \chi_p)}{\zeta(2)} (1 + O(L^{-7})) \ll L^{-10}. \quad (10.4)$$

The uniqueness assertion is confined to this short interval.

Completion of Theorem 1.4. If $p \in X$ lies in $[\exp((\log T)^{1/8}), T]$, then

$$1 - \beta_p \ll (\log p)^{-10} \leq (\log T)^{-5/4} < c_0/\log T$$

for all sufficiently large T . Landau–Page permits at most one such primitive real character of conductor at most T , proving (1.9).

For two sufficiently large successive members $p < r$ of $\mathcal{X}$, the same statement at $T = r$ implies

$$\log r > (\log p)^8, \quad \log \log r > 8 \log \log p.$$

The double logarithms grow geometrically, giving (1.10). This argument allows an infinite but very sparse family; it does not eliminate it. $\square$

11 Boundaries of the conclusions

11.1 Why the reciprocal endpoint is not included

The single common deletion in Proposition 6.3 has error $O_\theta(MN^{1-\beta_\theta})$, with $\beta_\theta = (37/50)(1 - \theta) > 0$ for each fixed $\theta < 1$. The saving disappears at $\theta = 1$, and two separate observations prevent extending the theorem by an informal limiting argument.

First, the required first moment is not uniform under a reciprocal budget, even a budget tending to zero. For a large integer x , flip the prime values of Liouville only in $(x/2, x]$, obtaining f_x . Exactly,

$$\sum_{n \leq x} f_x(n) = A_\lambda(x) + 2(\pi(x) - \pi(x/2)). \quad (11.1)$$

Indeed, the only multiple of such a prime at most x is the prime itself. The prime number theorem gives

$$\sum_{p: f_x(p)=1} \frac{1}{p} \sim \frac{\log 2}{\log x} \rightarrow 0, \quad \sum_{n \leq x} f_x(n) \sim \frac{x}{\log x}.$$

This contradicts a uniform version of (6.2) at $\theta = 1$. It is *not* a counterexample to an all-pattern theorem at that endpoint.

Second, an endpoint theorem would have an additional arithmetic consequence.

Proposition 11.1 (A necessary condition for uniform reciprocal robustness). *Suppose that for every $M > 0$ there exists N_M such that every completely multiplicative sign function with $\sum_{f(q)=1} 1/q \leq M$ possesses all four reflected patterns at every total $N \geq N_M$. Then*

$$\sum_{\substack{q < p \text{ prime} \\ (q/p)=1}} \frac{1}{q} \rightarrow \infty \quad \text{as } p \rightarrow \infty \text{ through primes.} \quad (11.2)$$

Proof. For an odd prime p , prescribe $f_p(q) = (q/p)$ at primes $q < p$ and $f_p(q) = -1$ at primes $q \geq p$. Its budget is the sum in (11.2). For all $n < p$, $f_p(n) = \chi_p(n)$, so $f_p(n)f_p(p-n) = \chi_p(-1)$ and two patterns are absent. Thus any prime whose displayed sum is at most M must lie below N_M . This proves the divergence. $\square$

This is only a necessary condition. We establish neither the divergence in (11.2) nor an infinite sequence on which its left side is bounded. The construction varies with p and does not refute a statement whose threshold is allowed to depend on the entire fixed function.

11.2 Positive density is a different target

Formula (2.2), together with $A_\lambda(X) = o(N)$, shows why a positive-density lower bound needs a uniform separation of the normalized reflected correlation from both extremal values. The inverse theorem begins only when the reflection error is as small as $N/(\log N)^{96}$ or $N/(\log N)^{512}$. It is not a theorem about an arbitrary nonzero reflected bias.

In the structured counting branch the hybrid bound (7.7) still depends on $L(1, \chi)$; Siegel supplies an arbitrarily small fixed power, not a positive absolute constant. In the prime logarithmic branch the potentially resonant conductors are retained and spaced rather than removed. Neither operation proves

$$m(N) \gg N, \quad N_{s,t}(N) \sim N/4, \quad C_\lambda(N) = o(N).$$

The present manuscript does not claim the Corrádi–Kátai conjecture [12] or a general Chowla cancellation result.

11.3 The role of effectivity and computation

The explicit exponents in this paper specify orders of magnitude, not numerical thresholds. The fixed prime blocks have large but fixed descent constants. In the all-total argument, even those constants are followed by the ineffective threshold from Siegel. In the logarithmic prime argument, all inputs can be effective, but the paper does not compute their combined threshold.

Finite examples can test the identities, the allocation of deleted candidates, and the lifting maps. They cannot verify Burgess, Pollack’s mean theorem, Siegel, Landau–Page, or the passage to an asymptotic limit. No numerical exceptional conductor or real zero is produced here.

Authorship, assistance, and status

This work was developed with substantial assistance from GPT-6 Astra in exploratory derivations, proof organization, drafting, bibliographic work and targeted checks. Brice Pouly is the sole author and retains responsibility for the manuscript.

The conclusions rest on the written proofs and the explicitly stated analytic inputs. This manuscript has not been independently verified by a human referee or formalized in Lean; formal verification of other work does not verify these quantitative theorems. The bibliographic comparisons concern identified results and versions and do not certify exhaustive priority.

Finite diagnostics accompanying the sources test identities and bookkeeping. They are not premises of the asymptotic theorems and do not establish the analytic inputs, large- N thresholds or the existence of an exceptional zero.

A Dependency and parameter ledger

The principal dependencies are summarized below. All finite steps used centrally are proved in the manuscript; the analytic inputs are stated in Section 2.

Step	Inputs actually used
Uniform reconstruction	Rectangular product energy; complete multiplicativity; Chebyshev for J_N
Fixed-block propagation	Finite descent with zero states; cocycle; additive Fourier averaging with small denominators
Oriented concentration	Prime-product moment; short integer products; Burgess; Gram and fourth-moment estimates
Source means	Positive convolution $f = \lambda * a_E$; first moment of λ ; fixed $\theta < 1$
Signed supply	Unit harmonic masks; Brun–Titchmarsh with constant 2; common deletion; pair intersections
Supply of useful primes	Positivity of $1 * \chi$; filtered Dirichlet norm; Pollack’s Proposition 3.1; Siegel
Every integer total	Fixed-block conclusion; disjoint blocks; injective divisor lifting
Prime logarithmic bound	Generic inverse; $\lambda(2) = \lambda(3) = -1$; quadratic parity; Liouville first moment
Sparse exceptional family	Prime agreement; Gauss-sum truncation; boundary value and derivative; Taylor; Landau–Page

The two spectral budgets, including their downstream precision, are:

Quantity	All-total branch	Prime logarithmic branch
Number of fixed generators	1024	128
Reflection threshold	N/L^{512}	N/L^{96}
Prime-dilation cutoff	L^{256}	L^{32}
Normalized dilation defect	$O_{B_0}(L^{-254})$	$O_{B_0}(L^{-62})$
Spectral leakage Δ	$\leq L^{-253}$	$\leq L^{-61}$
Exceptional spectral family	$\leq N^{1/192}$	$\leq N^{1/30}$
Gram error γ	$\leq N^{-1/64}$	$\leq N^{-1/80}$
Oriented error $L\sqrt{\Delta}$	$\leq L^{-251/2}$	$\leq L^{-59/2}$
Unit disagreement	$\leq L^{-124}$	$\leq L^{-29}$
Uniform prefix disagreement	$\leq L^{-55}$	$\leq L^{-11}$
Reciprocal prime disagreement	$\leq 3L^{-55}$	$\leq 3L^{-11}$

All inequalities in this table hold beyond their stated fixed-parameter thresholds. The common induced-character estimate $N^{-3/64}$ is sufficient for both branches. The all-total counting stage uses $\alpha = 13/50$, within the simultaneous interval displayed after (6.3); its two filtered-convolution savings are $\alpha\eta_2$ and $(1-\theta)/400$, and the common-deletion saving is $(37/50)(1-\theta)$. The parameters are deliberately not optimized.

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