

Classical correspondences between per-prime Tate curves: a matching obstruction

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Abstract

For the complex Tate curves $E_p = \mathbf{C}^\times/p^{\mathbf{Z}}$ attached to primes, we compute the classical divisorial correspondence groups. For distinct primes, $\text{Hom}(E_p, E_q) \neq 0$ exactly when $\log p \log q \in 4\pi^2 \mathbf{Q}$. If $\log p \log q / (4\pi^2) = u/v$ in lowest positive terms, this group has rank one and its generator has isogeny degree uv . The isogeny relation between distinct primes is a partial matching: unconditionally, each prime has at most one partner. These rigidity statements specialize results of Winkelmann; the explicit Hom generator and degree are included here. Outside the matching, the Néron–Severi group of $E_p \times E_q$ consists of the two fiber classes, which act trivially on H^1 . On every diagonal surface $\text{End}(E_p) = \mathbf{Z}$, by Gelfond–Schneider, and the intersection profile of multiplication graphs is independent of p . Consequently no connected classical correspondence system can involve three distinct prime fibers. This is a restriction on cross-fiber assembly, not a universal prohibition of prime-indexed endomorphism monoids or of explicit-formula realizations with additional coefficient data. Version 2 withdraws that broader no-go assertion of version 1 and states the exact support restriction required by the obstruction.

1 Introduction and scope of the revision

Connes and Consani’s construction supplies the complex elliptic curves $E_p = \mathbf{C}^\times/p^{\mathbf{Z}}$ [1]. Their ordinary products have classical intersection theory. A natural question is how much of a correspondence calculus can be assembled from these products. Here a correspondence from E to E' means a divisor class on $E \times E'$, modulo the two fiber classes. This specific classical category is the subject of the paper.

Theorems 1–5 compute its morphisms and intersection data. The rational-product criterion and matching argument specialize Winkelmann’s rigidity [2], rather than constituting new transcendence theorems. The off-diagonal rational-product question remains open; the matching bound and diagonal non-CM result are unconditional. Theorem 6 gives the resulting connected-component obstruction.

Correction of version 1. Version 1, titled *Products of the per-prime Tate curves of absolute geometry carry no correspondence calculus for the Weil explicit formula*, claimed that its three proposed requirements excluded every assignment of correspondence data. That claim is withdrawn. A multiplicative prime-indexed monoid need not use maps between different fibers, and constancy of an unmarked intersection lattice does not prohibit varying test coefficients. The corrected theorem

*hello.jay.tyagi@gmail.com. *Use of AI.* The original work used Claude (Anthropic). This revision, proof review, and companion checks used OpenAI Codex. The author directed the work and is responsible for the content. Code and publication artifacts: <https://github.com/x67ai/riemann-rh-program>.

requires an explicitly connected cross-fiber system. The consequences for more general explicit-formula proposals are discussed in Section 9; no universal impossibility result is asserted.

2 Lattices and classical correspondences

2.1 The lattice presentation

Throughout, $p, q, \dots$ denote primes; set

$$y_p := \frac{\log p}{2\pi} > 0, \quad \tau_p := i y_p, \quad \Lambda_p := \mathbf{Z} \oplus \mathbf{Z} \cdot \tau_p, \quad E_p := \mathbf{C}/\Lambda_p.$$

Lemma 1 (lattice presentation; [1, §4.1.2]). $\exp_2: \mathbf{C} \rightarrow \mathbf{C}^\times$, $w \mapsto e^{2\pi i w}$, is a surjective homomorphism with kernel $\mathbf{Z}$, and

$$\exp_2(w + \tau_p) = e^{2\pi i w} \cdot e^{2\pi i \cdot (i y_p)} = e^{2\pi i w} \cdot e^{-\log p} = p^{-1} \cdot \exp_2(w).$$

Since $p^{\mathbf{Z}} = (p^{-1})^{\mathbf{Z}}$ as subgroups of $\mathbf{C}^\times$, $\exp_2$ induces an isomorphism $\mathbf{C}/\Lambda_p \cong \mathbf{C}^\times/p^{\mathbf{Z}}$. $\square$

The presentation was confirmed numerically to 55 digits (check A of the numerical suite; see *Provenance*). The sign of τ_p is immaterial: $\mathbf{Z} \oplus \mathbf{Z}(-\tau_p) = \Lambda_p$.

Standard facts used without further comment: every homomorphism of complex tori $\mathbf{C}/\Lambda \rightarrow \mathbf{C}/\Lambda'$ is induced by a unique $\lambda \in \mathbf{C}$ with $\lambda\Lambda \subseteq \Lambda'$ (lift to the universal cover); every holomorphic map between complex tori is a homomorphism followed by a translation (rigidity); $\text{Hom}(E, E')$ denotes the group of holomorphic homomorphisms.

3 The isogeny criterion

Theorems 1–3 specialize to primes a rigidity of Winkelmann [2, Lemma 7 and the proof of Thm. 2, p. 16], who studies the curves $\mathbf{C}^\times/\lambda^{\mathbf{Z}}$ attached to eigenvalues λ of elements of a discrete subgroup of $\text{SL}_2(\mathbf{C})$. He shows that isogeny of two members of that family forces $4\pi^2/(\log \lambda_i \log \lambda_j) \in \mathbf{Q}$, divides two such relations and appeals to multiplicative independence to conclude that each curve has at most one isogenous partner, and runs the argument on three curves; his Lemma 7 is stated as an equivalence, and the converse follows from it in one line, so both directions of the criterion below are his. Proofs are given in full here because the paper is self-contained and the prime-indexed forms differ in presentation from his. What is added is the identification of the group $\text{Hom}(E_p, E_q)$, its rank, and the degree uv of the minimal isogeny.

Theorem 1. *Let $p \neq q$ be distinct primes. Then*

$$\text{Hom}(E_p, E_q) \neq 0 \iff y_p y_q \in \mathbf{Q} \iff \log p \cdot \log q \in 4\pi^2 \mathbf{Q}.$$

Moreover, when the condition holds and $y_p y_q = u/v$ in lowest terms ($u, v \in \mathbf{Z}_{>0}$), the full homomorphism group is

$$\text{Hom}(E_p, E_q) = \mathbf{Z} \cdot \lambda_0, \quad \lambda_0 = i v y_q,$$

of rank 1, and the isogeny induced by λ_0 has degree uv .

Proof. A nonzero homomorphism is multiplication by some $\lambda \in \mathbf{C}^\times$ with $\lambda\Lambda_p \subseteq \Lambda_q$. Two containment conditions:

1. $\lambda \cdot 1 \in \Lambda_q$: so $\lambda = a + b \cdot \tau_q = a + i b y_q$ for some $a, b \in \mathbf{Z}$.

2. $\lambda \cdot \tau_p \in \Lambda_q$: compute $\lambda \cdot \tau_p = (a + iby_q) \cdot (iy_p) = -by_p y_q + ia y_p$. Writing this as $c + id y_q$ with $c, d \in \mathbf{Z}$:

- real part: $-by_p y_q = c \in \mathbf{Z}$;
- imaginary part: $ay_p = d y_q$.

Case $a \neq 0$. Then $d \neq 0$ (as $y_p, y_q > 0$) and $y_p/y_q = d/a$, i.e. $\log p/\log q \in \mathbf{Q}$, i.e. $a \cdot \log p = d \cdot \log q$ with a, d nonzero integers, i.e. $p^a = q^d$ (after clearing signs) — impossible for distinct primes by unique factorization. So $a = 0$, hence $d = 0$.

Case $a = 0$. Then $\lambda = iby_q$ with $b \neq 0$ (else $\lambda = 0$), and the surviving condition is $by_p y_q \in \mathbf{Z}$, which forces $y_p y_q \in \mathbf{Q}$. This proves the forward direction, and identifies $\text{Hom} = \{iby_q : b \in \mathbf{Z}, by_p y_q \in \mathbf{Z}\}$. With $y_p y_q = u/v$ in lowest terms, $bu/v \in \mathbf{Z} \iff v \mid b$, so $\text{Hom} = \mathbf{Z} \cdot (iv y_q)$.

Converse. If $y_p y_q = u/v$, then $\lambda_0 = iv y_q$ satisfies $\lambda_0 \cdot 1 = v \cdot \tau_q \in \Lambda_q$ and $\lambda_0 \cdot \tau_p = (iv y_q)(iy_p) = -v y_p y_q = -u \in \mathbf{Z} \subseteq \Lambda_q$, so $\lambda_0 \Lambda_p \subseteq \Lambda_q$ and $\lambda_0 \neq 0$ — a genuine isogeny. (Numerically confirmed on a synthetic pair with $y y' = 3/7$ exactly — check B.)

Degree. $\text{covol}(\Lambda_p) = y_p$ and $\text{covol}(\lambda_0 \Lambda_p) = |\lambda_0|^2 \cdot y_p = v^2 y_q^2 \cdot y_p$, so $\deg = [\Lambda_q : \lambda_0 \Lambda_p] = v^2 y_q^2 y_p / y_q = v^2 \cdot (y_p y_q) = v^2 \cdot (u/v) = uv$.

Finally $y_p y_q \in \mathbf{Q} \iff \log p \log q = 4\pi^2 \cdot y_p y_q \in 4\pi^2 \mathbf{Q}$. $\square$

Remark 1 (bookkeeping reconciliation). The criterion may be stated as $\log p \cdot \log q \in 4\pi^2 \mathbf{Q}$ or derived in the form $b \cdot \log p \cdot \log q = -4\pi^2 c$ and phrased as $\log p \cdot \log q \in \pi^2 \mathbf{Q}$. These are the *same condition*: $4\pi^2 \mathbf{Q} = \pi^2 \mathbf{Q}$ as subsets of $\mathbf{R}$, because $4 \in \mathbf{Q}^\times$ — a rational multiple of π^2 is a rational multiple of $4\pi^2$ and conversely. The normalization-free statement is $y_p \cdot y_q \in \mathbf{Q}$; we display the constant as $4\pi^2$ because $y_p y_q = \log p \cdot \log q / (4\pi^2)$ makes it the natural one.

Remark 2 (symmetry). The criterion is symmetric in p and q ; hence $\text{Hom}(E_p, E_q) \neq 0 \iff \text{Hom}(E_q, E_p) \neq 0$ (as the general theory also guarantees via the dual isogeny). “Exceptional partner” is therefore a symmetric relation.

4 Two-partner rigidity and the three-prime consequence

Theorem 2 (two-partner rigidity). *For every prime p there is at most one prime $q \neq p$ with $\text{Hom}(E_p, E_q) \neq 0$. Consequently the set of “exceptional pairs” $\{p, q\}$ (pairs with nonzero Hom), if nonempty, forms a partial matching on the primes: no prime appears in two of them.*

Proof. Suppose $q \neq q'$ are two such primes. By Theorem 1, $y_p y_q \in \mathbf{Q}^\times$ and $y_p y_{q'} \in \mathbf{Q}^\times$. Dividing: $y_q/y_{q'} = (y_p y_q)/(y_p y_{q'}) \in \mathbf{Q}$, i.e. $\log q/\log q' \in \mathbf{Q}$, i.e. $q^m = q'^n$ for some positive integers m, n — impossible for distinct primes by unique factorization. $\square$

The proof is unique factorization plus linear algebra over $\mathbf{Q}$ applied to the products $y_p y_q$, and it is agnostic about whether any exceptional pair exists.

Theorem 3 (the three-prime consequence). *Among the three surfaces $E_2 \times E_3$, $E_2 \times E_5$, $E_3 \times E_5$, at most one carries a nonzero correspondence group; in particular no two of them — a fortiori not all three — can simultaneously carry nonzero correspondence groups. (By Theorem 4 below, “correspondence group of $E_p \times E_q$ ” = $\text{Hom}(E_p, E_q)$.)*

Proof. Any two of the three pairs $\{2, 3\}$, $\{2, 5\}$, $\{3, 5\}$ share a prime, so two simultaneously nonzero groups would give some prime two exceptional partners, contradicting Theorem 2. Directly: if $y_2 y_3 = r_1 \in \mathbf{Q}^\times$ and $y_2 y_5 = r_2 \in \mathbf{Q}^\times$ then $y_3/y_5 = r_1/r_2 \in \mathbf{Q}$, i.e. $\log 3/\log 5 \in \mathbf{Q}$, i.e. $3^m = 5^n$ — impossible; the other two cases are identical. $\square$

5 Néron–Severi collapse off the matching

We first fix the classical dictionary. For elliptic curves E, E' over $\mathbf{C}$, the group of *divisorial correspondences* from E to E' is $\mathrm{NS}(E \times E')$ modulo the subgroup generated by the two fiber classes $\xi_1 = [\{0\} \times E']$, $\xi_2 = [E \times \{0\}]$ (Weil’s “trivial correspondences”):

$$\mathrm{Corr}(E, E') := \mathrm{NS}(E \times E') / \langle \xi_1, \xi_2 \rangle.$$

Theorem 4. *Let E, E' be elliptic curves over $\mathbf{C}$. Then there is a canonical isomorphism*

$$\mathrm{NS}(E \times E') \cong \mathbf{Z} \cdot \xi_1 \oplus \mathbf{Z} \cdot \xi_2 \oplus \mathrm{Hom}(E, E'),$$

hence $\mathrm{Corr}(E, E') \cong \mathrm{Hom}(E, E')$ ([3, Ch. 5]; self-contained proof below). Consequently, for distinct primes $p \neq q$ with $\mathrm{Hom}(E_p, E_q) = 0$ (i.e., every pair outside the — conjecturally empty — exceptional matching of Theorems 1–2):

- (a) **Rank collapse.** $\mathrm{NS}(E_p \times E_q) = \mathbf{Z}\xi_1 \oplus \mathbf{Z}\xi_2$ has rank 2, with intersection form $\xi_1^2 = \xi_2^2 = 0$, $\xi_1 \cdot \xi_2 = 1$: every divisor class is numerically $a\xi_1 + b\xi_2$, and $\mathrm{Corr}(E_p, E_q) = 0$.
- (b) **H^1 -inertness.** Every divisor class on $E_p \times E_q$ acts as zero on H^1 (in both directions, $H^1(E_p) \rightarrow H^1(E_q)$ and the transpose). The correspondence calculus is not merely small; as a calculus of operators it is the zero calculus.
- (c) **No graphs, no diagonal.** The only holomorphic maps $E_p \rightarrow E_q$ are constants, so the only graph classes are fiber classes. There is no diagonal: not as a curve (a “diagonal” in a product of nonisomorphic curves means the graph of an isomorphism, and none exists), and not even as a numerical class — no $D \in \mathrm{NS}$ satisfies the diagonal’s numerical triple $(D \cdot \xi_1, D \cdot \xi_2, D^2) = (1, 1, 0)$, since $D \cdot \xi_1 = D \cdot \xi_2 = 1$ forces $D = \xi_1 + \xi_2$ and then $D^2 = 2 \neq 0$.
- (d) **Castelnuovo–Severi vacuous.** The Hodge-index/Castelnuovo–Severi inequality on a product of curves is the statement that the intersection form is negative semidefinite on the primitive part $\langle \xi_1, \xi_2 \rangle^\perp \subset \mathrm{NS}$. Here that primitive part is 0: if $D = a\xi_1 + b\xi_2$ satisfies $D \cdot \xi_1 = b = 0$ and $D \cdot \xi_2 = a = 0$ then $D = 0$. Equivalently: every class of bidegree (d_1, d_2) satisfies $D^2 = 2d_1d_2$ — Castelnuovo–Severi holds with equality everywhere, carrying zero information. The positivity the proposal was to import fiber-by-fiber is free here because there is nothing left for it to constrain.

Proof of the decomposition. All inputs are standard abelian-variety theory ([3, Ch. 2] for the Poincaré bundle, seesaw, and rigidity; Ch. 5 for the endomorphism context; the statement itself is the classical correspondence dictionary). Write $\hat{E}' = \mathrm{Pic}^0(E')$ for the dual, P for the Poincaré bundle on $\hat{E}' \times E'$, normalized so that

$$P|_{\{\alpha\} \times E'} \cong \alpha \quad (\alpha \in \hat{E}'), \quad P|_{\hat{E}' \times \{0\}} \cong \mathcal{O}.$$

Step 1 (the homomorphism φ_L). For $L \in \mathrm{Pic}(E \times E')$ put $L_x := L|_{\{x\} \times E'}$; $\deg L_x$ is constant in x . Normalize:

$$\tilde{L} := L \otimes \mathrm{pr}_1^*(L|_{E \times \{0\}})^{-1} \otimes \mathrm{pr}_2^*(L_0)^{-1},$$

so that $\tilde{L}_x = L_x \otimes L_0^{-1} \in \mathrm{Pic}^0(E')$ for all x , $\tilde{L}_0$ is trivial, and $\tilde{L}|_{E \times \{0\}}$ is trivial. By the universal property of $(\hat{E}', P)$ there is a unique morphism $\varphi_L: E \rightarrow \hat{E}'$ with $(\varphi_L \times \mathrm{id}_{E'})^*P \cong \tilde{L}$ (the a-priori ambiguity by pr_1 -pullbacks is killed by restricting to $E \times \{0\}$, where both sides are trivial). Since

$\tilde{L}_0$ is trivial, $\varphi_L(0) = 0$; a pointed morphism of abelian varieties is a homomorphism (rigidity). So $\varphi_L \in \text{Hom}(E, \hat{E}')$.

Step 2 (kernel; factorization through NS). $L \mapsto \varphi_L$ is additive: normalization and restriction are additive in L , and uniqueness plus the biadditivity of the Poincaré bundle ($((\text{add}) \times \text{id})^* P \cong \text{pr}_{13}^* P \otimes \text{pr}_{23}^* P$ on $\hat{E}' \times \hat{E}' \times E'$, [3, Ch. 2]) give $\varphi_{L \otimes L''} = \varphi_L + \varphi_{L''}$. If $L \in \text{Pic}^0(E \times E')$ then, since the dual of a product is the product of the duals, $L = \text{pr}_1^* M \otimes \text{pr}_2^* M'$ with $M \in \text{Pic}^0(E)$, $M' \in \text{Pic}^0(E')$; the normalization annihilates it, so $\varphi_L = 0$ and the map factors through $\text{NS}(E \times E')$. Conversely if $\varphi_L = 0$ then $\tilde{L}_x$ is trivial for every x and $\tilde{L}|_{E \times \{0\}}$ is trivial, so by the seesaw principle $\tilde{L}$ is trivial; hence

$$L \cong \text{pr}_1^*(L|_{E \times \{0\}}) \otimes \text{pr}_2^*(L_0),$$

whose Néron–Severi class is $\deg(L|_{E \times \{0\}}) \cdot \xi_1 + \deg(L_0) \cdot \xi_2$ (the class of pr_1^* of a degree- d bundle is $d \cdot [\text{pt}] \times E' = d \cdot \xi_1$, and symmetrically). So the kernel of $\text{NS} \rightarrow \text{Hom}(E, \hat{E}')$ is exactly $\mathbf{Z}\xi_1 \oplus \mathbf{Z}\xi_2$ (a free rank-2 group: the intersection numbers against ξ_2, ξ_1 read off the coefficients).

Step 3 (surjectivity and splitting). For $\psi \in \text{Hom}(E, \hat{E}')$ set $L^\psi := (\psi \times \text{id}_{E'})^* P$. Then $L_x^\psi = P|_{\{\psi(x)\} \times E'} = \psi(x) \in \text{Pic}^0(E')$, L_0^ψ is trivial, and $L^\psi|_{E \times \{0\}} = \psi^*(P|_{\hat{E}' \times \{0\}})$ is trivial — so L^ψ is already normalized and $\varphi_{L^\psi} = \psi$ by uniqueness. Additivity of $\psi \mapsto [L^\psi]$ is the Poincaré biadditivity again. This splits the sequence:

$$\text{NS}(E \times E') \cong \mathbf{Z}\xi_1 \oplus \mathbf{Z}\xi_2 \oplus \text{Hom}(E, \hat{E}').$$

Step 4 (eliminating the dual). For an elliptic curve, $x \mapsto [\mathcal{O}_{E'}([x] - [0])]$ is a canonical isomorphism $E' \cong \hat{E}'$ (the Abel–Jacobi isomorphism), so $\text{Hom}(E, \hat{E}') \cong \text{Hom}(E, E')$. $\square$

Proof of (a)–(d). (a) is immediate from the decomposition with $\text{Hom} = 0$, plus $\xi_1^2 = \xi_2^2 = 0$ (disjoint parallel fibers) and $\xi_1 \cdot \xi_2 = 1$ (one transverse point).

(b) The action of a correspondence on cohomology, $\alpha \mapsto \text{pr}_{2*}([D] \cup \text{pr}_1^* \alpha)$, depends only on the class $[D] \in H^2(E \times E', \mathbf{Z})$. Künneth: $H^2 = (H^2 \otimes H^0) \oplus (H^1 \otimes H^1) \oplus (H^0 \otimes H^2)$. A class $\text{pr}_1^* \beta$ ($\beta \in H^2(E)$) acts on $\alpha \in H^1(E)$ by $\text{pr}_{2*}(\text{pr}_1^*(\beta \cup \alpha)) = 0$ since $\beta \cup \alpha \in H^3(E) = 0$; a class $\text{pr}_2^* \beta$ acts by $\beta \cup \text{pr}_{2*}(\text{pr}_1^* \alpha) = 0$ since fiber integration drops the degree of $\text{pr}_1^* \alpha \in H^1$ below zero. Now $\xi_1 = \text{pr}_1^*[\text{pt}]$ and $\xi_2 = \text{pr}_2^*[\text{pt}]$, so with $\text{NS} = \mathbf{Z}\xi_1 \oplus \mathbf{Z}\xi_2$ every algebraic class in H^2 has vanishing $H^1 \otimes H^1$ -component and acts as zero on H^1 . The transpose direction is identical using $\text{Hom}(E_q, E_p) = 0$ (Remark 2).

(c) Rigidity: every holomorphic map $E_p \rightarrow E_q$ is a homomorphism followed by a translation; $\text{Hom} = 0$ leaves the constants, whose graphs are horizontal fibers $E_p \times \{c\} \equiv \xi_2$. The numerical computation is displayed in the statement.

(d) The calculation in the statement gives the orthogonal complement of the fiber classes directly. No use of a Riemann hypothesis is involved. The general Hodge index theorem is not needed to prove that this particular primitive subgroup is zero. $\square$

Remark 3 (what still exists on these surfaces). Rank-2 NS does not mean the surfaces are curve-poor: $\xi_1 + \xi_2$ is ample (Nakai: positive square, positive on every effective class), and general members of $|3(\xi_1 + \xi_2)|$ are irreducible curves dominating both factors. The point of (a)–(b) is that every such curve is *numerically* a fiber combination and acts as zero on H^1 : as a correspondence it is invisible. The collapse is a collapse of the correspondence calculus, not of the surface’s geometry — and it is the calculus that the proposal required.

6 The diagonal surfaces

On the diagonal surfaces $E_p \times E_p$, the endomorphism ring can be computed unconditionally.

Theorem 5 (diagonal residue). *Fix a prime p .*

1. $\text{End}(E_p) = \mathbf{Z}$, unconditionally. (The lattice computation below shows the only alternative would be $(\log p)^2 \in 4\pi^2\mathbf{Q}$ — which would make τ_p imaginary quadratic and E_p a CM curve — and that coincidence is impossible: Gelfond–Schneider, §8, Proposition. No E_p has complex multiplication.)
2. By (1), $\text{NS}(E_p \times E_p) = \mathbf{Z}\xi_1 \oplus \mathbf{Z}\xi_2 \oplus \mathbf{Z} \cdot [\Delta]$ has rank 3 and $\text{Corr}(E_p, E_p) \cong \text{End}(E_p) = \mathbf{Z}$, generated by the diagonal. The available correspondences are (combinations of fiber classes and) the graphs Γ_m of multiplication by $m \in \mathbf{Z}$, with $\Gamma_1 = \Delta$, and their complete intersection profile is:

$$\Gamma_m \cdot \xi_1 = 1, \quad \Gamma_m \cdot \xi_2 = m^2, \quad (\Gamma_m \cdot \Gamma_n) = (m - n)^2 \quad (m \neq n), \quad (\Gamma_m \cdot \Delta) = (m - 1)^2$$

(the last for all m : at $m = 1$ this is $\Delta^2 = 0$, by adjunction with $K = 0$ and $e(E_p) = 0$).

3. Every number in this profile is a universal integer, independent of p . More precisely: under the real-affine identification $E_p \cong \mathbf{R}^2/\mathbf{Z}^2$ sending $(1, \tau_p)$ to the standard basis, the subtori Γ_m, ξ_1, ξ_2 correspond to the same subtori for every p , and intersection numbers of divisor classes are topological (cup products of integral classes). The map

$$p \mapsto (\text{complete intersection profile of } E_p \times E_p)$$

is constant. The prime enters the surface only through its complex structure τ_p , and the complex structure enters the correspondence calculus only through Hom/End — which is the same ring $\mathbf{Z}$ for every p (part 1).

Proof. (1) $\lambda\Lambda_p \subseteq \Lambda_p$ with $\lambda = a + iby_p$ (from $\lambda \cdot 1 \in \Lambda_p$) requires $\lambda \cdot \tau_p = -by_p^2 + iay_p \in \Lambda_p$, i.e. $-by_p^2 \in \mathbf{Z}$ (and the τ_p -coefficient $a \in \mathbf{Z}$, automatic). If $b \neq 0$ then $y_p^2 \in \mathbf{Q}$, i.e. $(\log p)^2 \in 4\pi^2\mathbf{Q}$ — impossible by the §8 Proposition (Gelfond–Schneider; a $b \neq 0$ endomorphism would make $\tau_p^2 = -y_p^2 \in \mathbf{Q}$, i.e. τ_p quadratic imaginary and End a CM order, and no E_p admits this). So $b = 0$ and $\lambda = a \in \mathbf{Z}$: $\text{End}(E_p) = \mathbf{Z}$ with no exceptional case.

(2) Theorem 4 with $E = E' = E_p$ gives $\text{NS} = \mathbf{Z}\xi_1 \oplus \mathbf{Z}\xi_2 \oplus \text{End}(E_p)$, and the graph of id is Δ . Intersection numbers: Γ_m meets a vertical fiber $\{x\} \times E_p$ in the single transverse point (x, mx) , so $\Gamma_m \cdot \xi_1 = 1$. It meets a horizontal fiber $E_p \times \{c\}$ in the fiber $[m]^{-1}(c)$, which has m^2 points, all transverse ($[m]$ is unramified when $m \neq 0$), so $\Gamma_m \cdot \xi_2 = m^2$ for $m \neq 0$. For $m = 0$, the graph is a horizontal fiber and its intersection with a distinct horizontal fiber is zero, giving the same formula. Here $\#\ker([k]) = \#((\frac{1}{k}\Lambda_p)/\Lambda_p) = k^2$ (representatives $(i + j \cdot \tau_p)/k$, $0 \leq i, j < k$; check D of the numerical suite tabulates them for $k \leq 7$). For $m \neq n$, $\Gamma_m \cap \Gamma_n = \{(x, mx) : (m - n)x = 0\} = \ker([m - n])$, with $(m - n)^2$ points, each transverse (tangent directions $(1, m)$ and $(1, n)$ are independent), so $(\Gamma_m \cdot \Gamma_n) = (m - n)^2$; $n = 1$ gives $(\Gamma_m \cdot \Delta) = (m - 1)^2$ — which is also the Lefschetz number $1 - 2m + m^2$ of $[m]$ (trace on $H^0 - H^1 + H^2$), the integer Lefschetz data of the family.

(3) The subtori Γ_m are the images of $x \mapsto (x, mx)$, defined by the *integral* linear data m alone; under the identification by the basis $(1, \tau_p)$ they are the same integral subtori of $\mathbf{R}^4/\mathbf{Z}^4$ for every p . Cup products of integral classes on a fixed topological manifold are p -independent. $\square$

7 The precise connectedness obstruction

Definition 1. The cross-fiber graph has one vertex for each prime, and an undirected edge pq for $p \neq q$ exactly when $\text{Corr}(E_p, E_q) \neq 0$. A connected classical cross-fiber system on a prime set S means that this graph, restricted to S , is connected. In particular, a proposed assembly that requires a nonzero correspondence between every two distinct fibers is a connected system in this sense.

Theorem 6 (matching obstruction). *Every connected component of the cross-fiber graph contains at most two vertices. Hence a connected classical cross-fiber system on three or more distinct prime fibers is impossible. In particular, an assignment requiring nonzero classical correspondences for every pair of distinct primes cannot exist.*

Proof. Theorem 4 identifies the nonzero edge relation with $\text{Hom}(E_p, E_q) \neq 0$. Theorem 2 says that each vertex has degree at most one. A graph of maximum degree one has only singleton and two-vertex connected components. This proves every assertion. $\square$

Remark 4 (what connectedness adds). The hypothesis is about actual cross-fiber support. It is not implied by merely asking for elements indexed by primes satisfying $\Psi_m \circ \Psi_n = \Psi_{mn}$. In any diagonal ring $\text{Corr}(E_p, E_p) \cong \mathbf{Z}$, the classes of multiplication maps satisfy $[m] \circ [n] = [mn]$ for positive integers m, n . These monoids exist regardless of the vanishing of any cross-prime group. They do not by themselves realize the Weil explicit formula, but they show why a multiplicative indexing condition alone is insufficient for a connectedness obstruction.

8 Transcendence status

The diagonal and off-diagonal questions have different answers.

Proposition (no complex multiplication). *For every prime p , $(\log p)^2 \notin 4\pi^2\mathbf{Q}$. In particular $\text{End}(E_p) = \mathbf{Z}$ unconditionally.*

Proof. If $(\log p)^2 = 4\pi^2 r$, then $r \in \mathbf{Q}_{>0}$ and $p = e^{2\pi\sqrt{r}} = (-1)^{-2i\sqrt{r}}$ using the branch $\log(-1) = i\pi$. The base -1 is algebraic and different from $0, 1$, and $-2i\sqrt{r}$ is algebraic irrational. Gelfond–Schneider makes every value of this power transcendental, contradicting that p is an integer. The lattice computation of Theorem 5 then gives the endomorphism statement. $\square$

For distinct primes, whether $\log p \log q \in 4\pi^2\mathbf{Q}$ is possible remains open. Its exclusion is the prime instance of the conjecture labeled (C4E faible), attributed to Bertrand and printed by Diaz [5, p. 231]: for positive real algebraic $a, b \neq 1$, π^2 and $\log a \log b$ are $\mathbf{Q}$ -linearly independent. The factor 4 does not change the rational span.

For completeness the four-exponentials reduction is immediate. If $\log p \log q = 4\pi^2 u/v$, take $x_1 = 2\pi i$, $x_2 = \log p$, $y_1 = 1$, $y_2 = \log q/(2\pi i)$. Both pairs are linearly independent over $\mathbf{Q}$, while the four exponentials are $1, q, p, e^{-2\pi i u/v}$, all algebraic. This contradicts the four-exponentials conjecture [6]. No proven six-exponentials theorem supplies the missing two-by-two conclusion merely by renaming the input. None of the unconditional connectedness or diagonal results assumes this conjecture.

9 Consequences and limits for explicit-formula proposals

If an assembly requires nonzero classical divisorial correspondences joining three distinct prime fibers, Theorem 6 rules it out. On an individual off-matching product the primitive divisor subgroup is zero and no divisor acts nontrivially on H^1 . These are exact restrictions on the specified classical category.

On diagonal products, Theorem 5 gives a constant unmarked intersection profile. For a finite real-coefficient test class,

$$D_{p,f} = \sum_m c_m(p, f) \Gamma_m + a(p, f) \xi_1 + b(p, f) \xi_2, \quad (D_{p,f} \cdot \Delta) = \sum_m c_m(p, f) (m-1)^2 + a(p, f) + b(p, f).$$

This formula identifies the information provided by the intersection lattice. It does not prohibit dependence on p or f in the assignment. Indeed $(\Gamma_p \cdot \Delta) = (p-1)^2$ varies with p inside the same universal profile. To exclude a test-family construction one must specify additional restrictions on its coefficients and naturality, then prove an obstruction under those restrictions.

Similarly, a prime-independent phase factor is not by itself an obstruction to an archimedean contribution, which is a global term. The present results neither construct nor rule out pole or archimedean terms in a more general calculus. They concern classical divisor correspondences on the displayed complex curves. A changed category, an enriched host, a disconnected endomorphism construction, or additional analytic coefficient data requires its own analysis. The characteristic-one constructions of Connes and Consani and dynamical approaches are not excluded by this theorem.

Data, code and revision record

The accompanying file `tate-products-v2-source-and-checks.zip` contains this source, a portable exact-arithmetic check of the synthetic isogeny index and graph intersection determinants, its output, build instructions and the version-1-to-version-2 change record. The main results are proved above; those finite checks are reproducibility diagnostics, not proofs of the open off-diagonal transcendence conjecture. This revision omits the optional decimal/PSLQ evidence because it is not needed by any theorem. Historical version-1 artifacts remain at the publication's original record and repository tag `paper-2026-08-28`. Cite the concept DOI 10.5281/zenodo.22171136; this revised manuscript is version 2.

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