

GAUSSIAN WEIGHTS ON FRACTALS AND EDELMAN'S GHOSTS AND SHADOWS

CHATGPT PROMPTED BY DJALIL CHAFAÏ

ABSTRACT. Can a Gaussian on a noninteger-dimensional fractal give a probabilistic realization of Edelman's ghosts and shadows? We distinguish a Gaussian-weighted Hausdorff measure, a radial Gaussian law, and a Gaussian random vector or field. For every real $\beta > 1$, we construct a closed cone of Hausdorff dimension β whose Gaussian-weighted β -dimensional Hausdorff measure has exactly the χ_β radial distribution. The construction relies on exact radial volume growth, not merely on a dimension calculation. We prove a metric obstruction below dimension one and explain why the cone does not supply ordinary Gaussian addition or a scalar algebra. A separate Gram-determinant argument rules out preserving all polynomially continued Euclidean Gaussian Gram identities in a positive model. We then give proofs of the angular beta–gamma identities, associativity of Bessel–Kingman convolution, and its Gaussian convolution semigroup for $\beta \geq 1$. Squared Bessel processes, tridiagonal and bidiagonal matrix models, Dunkl Gaussians, and Gaussian fields on fractals provide complementary constructions with explicitly different meanings of dimension. None of these statements asserts the existence of a full noninteger-dimensional scalar algebra.

1. THE QUESTION AND THE CONVENTIONS

Edelman's note [3] introduces isotropic ghosts through radial laws and calls observable real or complex quantities their shadows. For a standard Gaussian ghost with dimension parameter β , the norm shadow has law χ_β . Section 6 of that note explicitly considers fractal theory in connection with a volume-scaling rule, but does not construct a fractal model. The note also separates operations on independent ghosts from the more difficult issue of dependence.

The question considered here is therefore twofold. Can a fractal carry a natural measure whose Gaussian weighting gives the prescribed norm? Can the same construction realize additional shadows and their joint algebraic relations? The first question has an exact affirmative answer for $\beta > 1$. The second requires additional structure and cannot be answered by a Hausdorff-dimension calculation alone.

Throughout, $\text{Gamma}(a, \theta)$ denotes the gamma distribution with shape $a > 0$ and scale $\theta > 0$, having density

$$\frac{x^{a-1}e^{-x/\theta}}{\Gamma(a)\theta^a}, \quad x > 0.$$

For $\beta > 0$ and $t > 0$, let $\gamma_{\beta,t}$ be the probability measure on $[0, \infty)$ with density

$$(1.1) \quad \gamma_{\beta,t}(dr) = \frac{r^{\beta-1}e^{-r^2/(2t)}}{2^{\beta/2-1}\Gamma(\beta/2)t^{\beta/2}} dr, \quad r > 0.$$

Thus $R \sim \gamma_{\beta,t}$ means

$$(1.2) \quad R \stackrel{d}{=} \sqrt{t} \chi_\beta, \quad R^2 \sim \text{Gamma}(\beta/2, 2t).$$

We set $\gamma_{\beta,0} = \delta_0$. Direct integration gives

$$(1.3) \quad \mathbb{E}R^p = (2t)^{p/2} \frac{\Gamma((\beta+p)/2)}{\Gamma(\beta/2)}, \quad p > -\beta.$$

In particular, $\mathbb{E}R^2 = \beta t$. At integer $\beta = d$, this is the norm distribution of an $N(0, tI_d)$ vector.

We use $\mathcal{H}^s$ for Hausdorff measure, with any one fixed standard normalization. Its normalization affects some constants below, but not the probability measures obtained after normalization. All restrictions of Hausdorff measure below are to Borel sets. No fractional-dimensional Lebesgue measure or exterior algebra is assumed.

2. THE EXACT RADIAL-VOLUME CRITERION

2.1. Gaussian weighting of a metric measure space. Proposition. Let (F, d) be a metric space with a distinguished point o , and let μ be a Borel measure. Write $\rho(x) = d(o, x)$. Suppose that, for constants $C \in (0, \infty)$ and $\beta > 0$,

$$(2.1) \quad \mu\{x : \rho(x) \leq r\} = Cr^\beta, \quad r > 0.$$

Then, for every $t > 0$,

$$(2.2) \quad \nu_t(dx) = \frac{1}{Z_t} e^{-\rho(x)^2/(2t)} \mu(dx), \quad Z_t = C\beta 2^{\beta/2-1} \Gamma(\beta/2) t^{\beta/2},$$

is a probability measure, and

$$(2.3) \quad \rho_\# \nu_t = \gamma_{\beta,t}.$$

Conversely, if the Gaussian weighting of a Borel measure is defined with $0 < Z_t < \infty$ and satisfies (2.3) for one $t > 0$, then (2.1) holds for some $C \in (0, \infty)$.

Proof. The pushforward $\eta = \rho_\# \mu$ has distribution function $\eta([0, r]) = Cr^\beta$. Continuity from above at zero gives $\eta(\{0\}) = 0$, and subtraction on intervals gives

$$\eta(dr) = C\beta r^{\beta-1} dr.$$

Consequently,

$$Z_t = C\beta \int_0^\infty r^{\beta-1} e^{-r^2/(2t)} dr = C\beta 2^{\beta/2-1} \Gamma(\beta/2) t^{\beta/2}.$$

The same pushforward calculation gives (2.3). Conversely,

$$\rho_\# \nu_t(dr) = Z_t^{-1} e^{-r^2/(2t)} \eta(dr).$$

Since the weight is strictly positive, (1.1) implies

$$\eta(dr) = \frac{Z_t}{2^{\beta/2-1} \Gamma(\beta/2) t^{\beta/2}} r^{\beta-1} dr.$$

Integrating from zero to r proves the converse. $\square$

2.2. What a dimension calculation does not imply. For a general radial pushforward $\eta = \rho_\# \mu$, the radius under Gaussian weighting has measure

$$(2.4) \quad Z_t^{-1} e^{-r^2/(2t)} \eta(dr).$$

If $V(r) = \mu\{\rho \leq r\}$, then η is the Lebesgue–Stieltjes measure associated with V , including any mass at zero. Neither $\dim_{\mathbb{H}} F = \beta$ nor two-sided bounds $c_1 r^\beta \leq V(r) \leq c_2 r^\beta$ force the exact law (1.1). Proposition 2.1 identifies the exact necessary and sufficient radial condition for this weighting procedure.

The weighted half-line

$$(2.5) \quad ([0, \infty), |r - s|, r^{\beta-1} dr)$$

already satisfies that condition for every $\beta > 0$. Its support, with the displayed metric, has Hausdorff dimension one, regardless of β . Thus a volume-growth exponent about a distinguished point need not be the Hausdorff dimension of the support.

3. A CONSTRUCTION USING HAUSDORFF MEASURE ON A FRACTAL CONE

3.1. Regular angular sets. Lemma. For every $s > 0$, there is a compact set K in a Euclidean unit sphere and a finite Borel measure σ_0 with support K such that

$$(3.1) \quad c\varepsilon^s \leq \sigma_0(K \cap B(\theta, \varepsilon)) \leq C\varepsilon^s, \quad \theta \in K, \quad 0 < \varepsilon \leq \text{diam } K,$$

for some $0 < c \leq C < \infty$. In particular, $\dim_{\text{H}} K = s$ and $0 < \mathcal{H}^s(K) < \infty$.

Proof. First suppose $0 < s < 1$. Set $a = 2^{-1/s} < 1/2$, and form the compact Cantor set in $[0, 1]$ generated by $x \mapsto ax$ and $x \mapsto 1 - a + ax$. Give each interval at level j mass $2^{-j} = a^{js}$. Choose j so that $a^j \leq \varepsilon < a^{j-1}$. The level- j interval containing a given point is contained in the ball of radius ε about that point. Also, such a ball meets only a bounded number of level- j intervals, with a bound depending on a : distinct level- j intervals are separated by at least $(1 - 2a)a^{j-1}$. These observations give (3.1).

For noninteger $s = k + u$, where $k \geq 0$ is an integer and $0 < u < 1$, take the product of $[0, 1]^k$ with the preceding u -dimensional Cantor set, equipped with the product measure. For integer $s \geq 1$, use the cube $[0, 1]^s$. Product balls for the maximum metric give the required bounds; the Euclidean metric is comparable to that metric. A bounded subset of $\mathbb{R}^\ell$ embeds bi-Lipschitzly into the unit sphere in $\mathbb{R}^{\ell+1}$ by

$$u \mapsto \frac{(u, 1)}{\sqrt{1 + |u|^2}}.$$

This preserves the two-sided estimates, up to constants.

For completeness, the measure bounds imply the Hausdorff-measure assertion as follows. The upper bound on the measure of balls gives a positive lower bound for the s -dimensional cost of any sufficiently fine cover of K . For the reverse estimate, a maximal family of disjoint balls of radius $\varepsilon/2$, centered in K , has at most a constant times ε^{-s} members by the lower measure bound. The corresponding balls of radius ε cover K , giving a uniform upper bound on its s -dimensional covering cost. Hence $0 < \mathcal{H}^s(K) < \infty$, which also gives its Hausdorff dimension. $\square$

3.2. The cone theorem. Theorem. Fix $\beta > 1$, and choose a compact $(\beta - 1)$ -regular angular set $K \subset \mathbb{S}^{m-1}$ as in Lemma 3.1. Define

$$(3.2) \quad F = \{r\theta : r \geq 0, \theta \in K\} \subset \mathbb{R}^m, \quad \mu = \mathcal{H}^\beta \upharpoonright_F.$$

Then F is closed, $\dim_{\text{H}} F = \beta$, and μ is locally finite with support F . There is a constant $C \in (0, \infty)$ such that

$$(3.3) \quad \mathcal{H}^\beta(F \cap \overline{B}(0, r)) = Cr^\beta, \quad r > 0.$$

Consequently the probability measure

$$(3.4) \quad \nu_t(dx) = \frac{1}{Z_t} e^{-|x|^2/(2t)} \mathcal{H}^\beta \upharpoonright_F(dx)$$

satisfies $|X| \sim \gamma_{\beta,t}$ when $X \sim \nu_t$.

Proof. Closedness follows from compactness of K : a convergent sequence of nonzero cone points has convergent radii, and, if the limiting radius is positive, its normalized directions converge in K .

On every annulus with $0 < a \leq r \leq b < \infty$, the polar map is bi-Lipschitz. Indeed, for unit vectors θ, φ ,

$$(3.5) \quad |r\theta - u\varphi|^2 = (r - u)^2 + ru|\theta - \varphi|^2.$$

Since $a^2 \leq ru \leq b^2$, this is comparable to the squared product metric on $[a, b] \times K$. The product of Lebesgue measure on $[a, b]$ and the regular angular measure has two-sided ball bounds of exponent $1 + (\beta - 1) = \beta$. The covering argument in Lemma 3.1 therefore shows

that each nondegenerate closed annular piece of F has positive finite $\mathcal{H}^\beta$ measure, and has Hausdorff dimension β .

Let $A = F \cap \{1/2 \leq |x| \leq 1\}$. The sets $2^{-j}A$, $j \geq 0$, cover $F \cap \overline{B}(0, 1)$ except for the origin. Hausdorff measure scales under similarities, so

$$0 < \mathcal{H}^\beta(A) \leq \mathcal{H}^\beta(F \cap \overline{B}(0, 1)) \leq \mathcal{H}^\beta(A) \sum_{j=0}^{\infty} 2^{-j\beta} < \infty.$$

The origin has zero $\mathcal{H}^\beta$ measure. Thus the middle quantity is an admissible constant C .

Since $rF = F$ and $\mathcal{H}^\beta(rE) = r^\beta \mathcal{H}^\beta(E)$, (3.3) follows. This proves local finiteness. The annular regularity gives positive measure to every relative neighborhood of every nonzero cone point, and (3.3) gives the same conclusion at zero. Thus μ has support F .

Finally, F is a countable union of annular pieces and the origin, so its Hausdorff dimension is exactly β . Applying Proposition 2.1 proves the assertion about (3.4). $\square$

For $1 < \beta < 2$, the angular set can be a regular Cantor set embedded in an arc of the circle, so the cone lies in the plane. For $\beta = 1$, the ordinary line with its one-dimensional Hausdorff measure gives the usual one-dimensional Gaussian. Theorem 3.2 concerns positive measures on actual Euclidean subsets, not a formal continuation of measures.

3.3. A precise angular disintegration. There is a probability measure σ on K such that

$$(3.6) \quad \int_F f(x) \mu(dx) = C\beta \int_K \int_0^\infty f(r\theta) r^{\beta-1} dr \sigma(d\theta)$$

for every nonnegative Borel function f . In particular, under (3.4), the variables $|X|$ and $X/|X|$ are independent, with respective laws $\gamma_{\beta,t}$ and σ .

To prove this, for a Borel set $E \subset K$ define

$$\sigma(E) = C^{-1} \mu\{x : 0 < |x| \leq 1, x/|x| \in E\}.$$

Homogeneity gives

$$\mu\{x : 0 < |x| \leq r, x/|x| \in E\} = Cr^\beta \sigma(E).$$

Subtracting this identity at two radii proves (3.6) on polar rectangles. Equality of the resulting sigma-finite measures and then integration prove the general formula.

This argument does not identify σ with normalized $\mathcal{H}^{\beta-1} \upharpoonright_K$, nor is such an identification needed. More importantly, it does not identify scalar products of independent σ -distributed directions with any universal angular law.

4. GEOMETRIC AND ALGEBRAIC LIMITATIONS

4.1. A metric obstruction below dimension one. Proposition. Let (F, d) have Hausdorff dimension strictly smaller than one. For any $o \in F$, the distance from o of a random point of F cannot have an absolutely continuous probability distribution on $[0, \infty)$.

Proof. The map $\rho(x) = d(o, x)$ is 1-Lipschitz, so

$$\dim_{\mathbb{H}} \rho(F) \leq \dim_{\mathbb{H}} F < 1.$$

Hence $\rho(F)$ is contained in a Borel subset of the real line having Lebesgue measure zero. Every distance distribution is concentrated on that null set and is therefore singular with respect to Lebesgue measure. $\square$

In particular, for $0 < \beta < 1$ one cannot simultaneously require $\dim_{\mathbb{H}} F = \beta$, interpret the ghost norm as distance in that same metric, and obtain χ_β . This does not exclude weighted spaces such as (2.5), arbitrary measurable observables, or other notions of effective dimension.

4.2. A Gaussian weight is not Gaussian vector addition. For independent cone-valued random variables X and Y , ordinary Euclidean addition gives

$$(4.1) \quad |X + Y|^2 = |X|^2 + |Y|^2 + 2|X||Y| \left\langle \frac{X}{|X|}, \frac{Y}{|Y|} \right\rangle.$$

The last factor depends on the angular measure. The sum also need not belong to the cone. For example, choose K in a cap where $\theta_1 \geq c > 0$. For X, Y independent with law ν_1 ,

$$\mathbb{E}|X + Y|^2 = 2\beta + 2|\mathbb{E}X|^2 > 2\beta.$$

Thus even its second radial moment differs from that of $\sqrt{2}\chi_\beta$. Symmetrizing a cone removes this particular nonzero-mean obstruction, but does not determine its angular laws.

There is a stronger obstruction to ordinary vector-valued Gaussian stability. If an $\mathbb{R}^m$ -valued random variable X has finite second moment and, for independent copies,

$$(4.2) \quad X_1 + X_2 \stackrel{d}{=} \sqrt{2}X,$$

then X is a centered, possibly degenerate, Gaussian vector. Indeed, expectations in (4.2) force $\mathbb{E}X = 0$. Iteration shows that the normalized sum of 2^j independent copies has the same law as X . The multivariate central limit theorem identifies this law with $N(0, \text{Cov } X)$.

A Gaussian vector has an affine subspace as its support: it is an affine image of a standard Gaussian in an integer-dimensional space. Therefore a law with full support equal to a noninteger-dimensional cone cannot satisfy (4.2). This is an obstruction to ordinary vector addition, not to a convolution defined directly on radial laws.

4.3. A positivity obstruction for continued Gram identities. Proposition. Fix a noninteger $\beta > 0$. There cannot exist, for every integer $k \geq 1$, a random positive semidefinite real $k \times k$ matrix W_k with integrable determinant such that

$$(4.3) \quad \mathbb{E} \det W_k = \prod_{j=0}^{k-1} (\beta - j).$$

Consequently, no model of real Hilbert-space-valued random vectors can preserve all the Euclidean Gaussian Gram-determinant identities after polynomial continuation of the ambient dimension to β .

Proof. For an integer $d \geq k$, let G be a $d \times k$ matrix of independent $N(0, 1)$ entries. Cauchy–Binet gives

$$\det(G^\top G) = \sum_{\substack{I \subset \{1, \dots, d\} \\ |I|=k}} (\det G_I)^2.$$

For a square $k \times k$ Gaussian matrix, expansion of both determinants shows $\mathbb{E}(\det G_I)^2 = k!$: distinct permutations have zero mixed expectation, and each coincident pair contributes one. Hence

$$(4.4) \quad \mathbb{E} \det(G^\top G) = k! \binom{d}{k} = \prod_{j=0}^{k-1} (d - j).$$

The right side is a polynomial in d , so its polynomial continuation has the value appearing in (4.3).

Now set $k = \lfloor \beta \rfloor + 2$. Exactly one factor in (4.3) is negative, and none is zero. Its right side is therefore negative, whereas positive semidefiniteness implies $\det W_k \geq 0$ almost surely. This is a contradiction. $\square$

For example, $\beta = 3/2$ and $k = 3$ would give $\mathbb{E} \det W_3 = -3/8$. The hypothesis here is explicitly the preservation of *all* the identities (4.4). It is not a consequence of specifying only χ_β radii or selected pairwise shadows. Nor is β being confused with the number of

vectors k . This proposition does not rule out restricted collections of shadows or the matrix models in Section 8.

5. ANGULAR SHADOWS AND BETA-GAMMA ALGEBRA

5.1. The angular law and its parameter range. For $\beta > 1$, let C_β have density

$$(5.1) \quad q_\beta(c) = \frac{\Gamma(\beta/2)}{\sqrt{\pi} \Gamma((\beta-1)/2)} (1-c^2)^{(\beta-3)/2}, \quad -1 < c < 1.$$

This is normalized by the beta integral, and

$$(5.2) \quad C_\beta^2 \sim \text{Beta}\left(\frac{1}{2}, \frac{\beta-1}{2}\right).$$

Its sign is a symmetric random sign independent of its square. If $(a)_0 = 1$ and $(a)_j = a(a+1) \cdots (a+j-1)$ for $j \geq 1$, then

$$(5.3) \quad \mathbb{E}C_\beta^{2j} = \frac{(1/2)_j}{(\beta/2)_j}, \quad \mathbb{E}C_\beta^{2j+1} = 0.$$

For $\beta = 1$, define C_1 to be a symmetric sign; the same moment formula holds.

At an integer $d \geq 2$, C_d is the first coordinate of a uniform point on $\mathbb{S}^{d-1}$, and also the scalar product of two independent uniform points on that sphere. One way to check the coordinate law is to normalize d independent standard Gaussians: the square of the first coordinate is the ratio of independent gamma variables with shapes $1/2$ and $(d-1)/2$. Orthogonal invariance gives the scalar-product interpretation. This is the angular variable used in the independent addition and real-part prescriptions of [3, Section 3].

The density (5.1) is integrable only for $\beta > 1$. Moreover, its proposed moment continuation would give $\mathbb{E}C_\beta^2 = 1/\beta > 1$ for $0 < \beta < 1$, which is impossible for a random variable in $[-1, 1]$. Thus the full parameter range of the radial law must not be transferred without qualification to this angular law.

5.2. A Gaussian coordinate and an independent residual radius. Proposition.

For $\beta > 1$ and $t > 0$, let $R \sim \gamma_{\beta,t}$ and let C_β be independent of R . Then

$$(5.4) \quad Z = RC_\beta \sim N(0, t), \quad W = R\sqrt{1 - C_\beta^2} \sim \gamma_{\beta-1,t},$$

and Z and W are independent. At $\beta = 1$ the same statement holds with $W = 0$ almost surely.

Proof. The beta-gamma splitting identity says that, if $S \sim \text{Gamma}(a+b, \theta)$ and $U \sim \text{Beta}(a, b)$ are independent, then SU and $S(1-U)$ are independent gamma variables of respective shapes a, b and common scale θ . Indeed, changing variables from (S, U) to $(v, w) = (SU, S(1-U))$ has inverse Jacobian $1/(v+w)$ and gives the joint density

$$\frac{v^{a-1} e^{-v/\theta}}{\Gamma(a)\theta^a} \frac{w^{b-1} e^{-w/\theta}}{\Gamma(b)\theta^b}.$$

Apply this with $S = R^2$, $U = C_\beta^2$, $a = 1/2$, $b = (\beta-1)/2$, and $\theta = 2t$. The independent symmetric sign of C_β then turns $\sqrt{SU}$ into an $N(0, t)$ variable. The case $\beta = 1$ follows by independently signing a half-normal variable. $\square$

This gives an actual coupling of a norm shadow and a Gaussian real-part shadow, without a fractal. The residual radius is not an assertion that there are $\beta-1$ sampled real coordinates. Below dimension one, even this pair of shadows is impossible under the natural domination condition: if $R \sim \chi_\beta$, $Z \sim N(0, 1)$, and $|Z| \leq R$ almost surely, then

$$(5.5) \quad 1 = \mathbb{E}Z^2 \leq \mathbb{E}R^2 = \beta.$$

No independence hypothesis is needed for this obstruction.

6. BESSEL–KINGMAN CONVOLUTION: AN EXACT RADIAL ADDITION

The classical Bessel–Kingman convolution extends addition of independent isotropic Euclidean vectors at the level of their norm laws [4, 8]. We give a proof of the properties needed here. In particular, the extension in the dimension parameter will be justified by rational identities, not by an unsupported appeal to analytic continuation from the integers.

6.1. Definition and associativity. Theorem. For $\beta \geq 1$ and $r, s \geq 0$, set

$$(6.1) \quad \delta_r *_{\beta} \delta_s = \text{Law} \left(\sqrt{r^2 + s^2 + 2rsC_{\beta}} \right).$$

Extend this operation to Borel probability measures by

$$(6.2) \quad (\mu *_{\beta} \nu)(A) = \int_0^{\infty} \int_0^{\infty} (\delta_r *_{\beta} \delta_s)(A) \mu(dr) \nu(ds).$$

It is a commutative, associative, probability-preserving convolution, with identity δ_0 . Its point-mass kernel is weakly continuous in (r, s) . At $\beta = 1$,

$$(6.3) \quad \delta_r *_1 \delta_s = \frac{1}{2} \delta_{r+s} + \frac{1}{2} \delta_{|r-s|}.$$

Proof. Nonnegativity under the square root follows from $|C_{\beta}| \leq 1$. Commutativity, preservation of total mass, and the identity property are immediate. Weak continuity of the point-mass kernel follows by bounded convergence against bounded continuous test functions. For $\beta = 1$, convolution is transported from ordinary convolution of symmetric measures on $\mathbb{R}$ by absolute value, so it is associative.

Suppose $\beta > 1$. For a nonnegative integer m , write

$$(6.4) \quad P_m^{(\beta)}(u, v) = \sum_{j=0}^{\lfloor m/2 \rfloor} \binom{m}{2j} (u+v)^{m-2j} (4uv)^j \frac{(1/2)_j}{(\beta/2)_j}.$$

Expanding the m th power and using (5.3) gives

$$\int z^{2m} (\delta_r *_{\beta} \delta_s)(dz) = P_m^{(\beta)}(r^2, s^2).$$

For fixed u, v , this expression is rational in β , with no poles on $(0, \infty)$; for fixed β , it is a polynomial in u, v .

Fix $r, s, t \geq 0$. To compute the $2m$ th moment of $(\delta_r *_{\beta} \delta_s) *_{\beta} \delta_t$, integrate $P_m^{(\beta)}(v, t^2)$ against the law of the squared intermediate radius v . Each moment of v is again given by (6.4). Thus this iterated moment, and likewise the moment from the other association, is rational in β . For every integer $\beta = d \geq 2$, both convolutions are norm laws of sums of independent isotropic vectors in $\mathbb{R}^d$. Their associativity follows from ordinary vector addition. Hence the difference of the two rational moment expressions vanishes at infinitely many integers. Its numerator is a polynomial and must vanish identically. The two iterated convolutions therefore have equal even moments for every $\beta > 1$.

Both measures are supported on $[0, r + s + t]$. Polynomials in the squared radius are dense in the continuous functions on that interval: apply polynomial approximation to $u \mapsto f(\sqrt{u})$ on $[0, (r + s + t)^2]$. Equality of the even moments therefore gives equality of the measures. This proves associativity for three point masses. Integration against arbitrary input probability measures, using Tonelli's theorem, proves the general assertion without moment assumptions on the inputs. $\square$

For $\beta > 1$ and $r, s > 0$, the support of (6.1) is $[|r - s|, r + s]$. In the standard hypergroup terminology the involution is the identity and a Haar measure is, up to a positive constant, $r^{\beta-1} dr$ [8, Section 1]. These statements concern measures on an ordinary half-line; a noninteger-dimensional group is not assumed.

6.2. The Gaussian convolution semigroup. Theorem. For every $\beta \geq 1$ and $s, t \geq 0$,

$$(6.5) \quad \gamma_{\beta,s} *_{\beta} \gamma_{\beta,t} = \gamma_{\beta,s+t}.$$

Proof. The case of a zero parameter follows from the identity element. For $s, t > 0$, choose independent $R \sim \gamma_{\beta,s}$, $S \sim \gamma_{\beta,t}$, and C_{β} . By Proposition 5.2,

$$Z = SC_{\beta} \sim N(0, t), \quad W = S\sqrt{1 - C_{\beta}^2} \sim \gamma_{\beta-1,t}$$

are independent of each other and of R . At $\beta = 1$ take $W = 0$. The squared radius in (6.1) is

$$Q = (R + Z)^2 + W^2.$$

For $\lambda \geq 0$, a one-dimensional Gaussian integral gives

$$\mathbb{E}[e^{-\lambda(R+Z)^2} \mid R] = (1 + 2t\lambda)^{-1/2} \exp\left(-\frac{\lambda R^2}{1 + 2t\lambda}\right).$$

Combining this with the gamma Laplace transforms yields

$$\begin{aligned} \mathbb{E}e^{-\lambda Q} &= (1 + 2t\lambda)^{-\beta/2} \left(1 + \frac{2s\lambda}{1 + 2t\lambda}\right)^{-\beta/2} \\ &= (1 + 2(s+t)\lambda)^{-\beta/2}. \end{aligned}$$

This identifies Q as $\text{Gamma}(\beta/2, 2(s+t))$, proving the theorem. $\square$

This is Gaussian addition of variances at the level of radial laws. Associativity of (6.2) does not supply a single deterministic addition operation on sampled ghosts. A stochastic kernel can be sampled, and iterated kernels define consistent radial Markov chains; this is different from providing all Euclidean inner products and their joint relations. There is also an elementary multiplicative convolution, obtained by taking the law of RS for independent radii. Possessing both such operations on laws does not by itself establish distributivity, conjugation, or matrix factorization in a scalar algebra.

7. SQUARED BESSEL PROCESSES FOR EVERY POSITIVE DIMENSION

The radial process exists on a larger parameter range than the preceding angular convolution. For every $\beta > 0$ and $q \geq 0$, the stochastic differential equation

$$(7.1) \quad dQ_t = 2\sqrt{Q_t} dB_t + \beta dt, \quad Q_0 = q,$$

has a unique nonnegative strong solution, the squared Bessel process of dimension β ; see [6, Section 1]. Its Laplace transform is

$$(7.2) \quad \mathbb{E}_q e^{-\lambda Q_t} = (1 + 2t\lambda)^{-\beta/2} \exp\left(-\frac{q\lambda}{1 + 2t\lambda}\right), \quad \lambda \geq 0.$$

One can verify this formula directly: for a fixed terminal time T , substitute

$$u(s, x) = (1 + 2\lambda(T - s))^{-\beta/2} \exp\left(-\frac{\lambda x}{1 + 2\lambda(T - s)}\right)$$

into $\partial_s u + 2x\partial_{xx}u + \beta\partial_x u = 0$. Itô's formula makes $u(s, Q_s)$ a bounded local martingale and hence a martingale. Evaluation at $s = 0, T$ gives (7.2). In particular,

$$(7.3) \quad Q_0 = 0 \implies Q_t \sim \text{Gamma}(\beta/2, 2t), \quad \sqrt{Q_t} \sim \gamma_{\beta,t}.$$

Away from zero, the generator of $R_t = \sqrt{Q_t}$ is

$$(7.4) \quad L_{\beta}f(r) = \frac{1}{2}f''(r) + \frac{\beta-1}{2r}f'(r) = \frac{1}{2r^{\beta-1}} \frac{d}{dr} \left(r^{\beta-1} f'(r)\right).$$

The expression on $(0, \infty)$ alone does not specify boundary behavior at zero; construction (7.1) fixes the process intended here. In particular, we do not assert an unqualified global singular-drift SDE for R when $0 < \beta < 1$.

For $\beta \geq 1$, the transition kernels of the Bessel process can also be obtained from $\delta_r *_{\beta} \gamma_{\beta,t}$; see [5, Section 5]. Indeed, the Gaussian integration used above, with $R = r$ deterministic, gives exactly the Laplace transform (7.2) with $q = r^2$. For $0 < \beta < 1$, (7.1) still defines the process, even though the angular probability law (5.1) is unavailable.

8. GAMMA IDENTITIES AND GENUINE MATRIX SHADOWS

8.1. Dimension bookkeeping without fractional coordinates. For independent variables $U \sim \chi_a$ and $V \sim \chi_b$, where $a, b > 0$, gamma convolution gives

$$(8.1) \quad U^2 + V^2 \sim \text{Gamma}((a+b)/2, 2), \quad \sqrt{U^2 + V^2} \sim \chi_{a+b}.$$

This identity concerns addition of squared independent radii, not the radial convolution of Section 6.

More generally, if $R_1, \dots, R_n$ are independent χ_{β} variables and $S = \sum_{j=1}^n R_j^2$, then

$$(8.2) \quad \sqrt{S} \sim \chi_{n\beta}, \quad \left(\frac{R_1^2}{S}, \dots, \frac{R_n^2}{S} \right) \sim \text{Dirichlet}(\beta/2, \dots, \beta/2),$$

and the displayed vector is independent of S . To check this, set $y_j = sp_j$ in the product gamma density, with $p_n = 1 - \sum_{j < n} p_j$. The Jacobian is s^{n-1} , and the transformed density factors into a gamma density of shape $n\beta/2$ and the Dirichlet density

$$\frac{\Gamma(n\beta/2)}{\Gamma(\beta/2)^n} \prod_{j=1}^n p_j^{\beta/2-1}$$

on the open simplex, with respect to $dp_1 \cdots dp_{n-1}$. The case $n = 1$ is interpreted as the point mass at (1). These are ordinary probability laws for every $\beta > 0$.

8.2. Hermite and Laguerre matrix models. The following spectral identifications are theorems of Dumitriu and Edelman [2, Theorems 2.12 and 3.4]; their proofs are not reproduced here. The scalings are specified to avoid confusing different Gaussian conventions. In both models, diagonal indices range over $1 \leq j \leq n$, and subdiagonal indices over $1 \leq j < n$.

For $n \geq 1$ and $\beta > 0$, let T be real symmetric tridiagonal, with independent diagonal and subdiagonal variables

$$(8.3) \quad T_{jj} \sim N(0, 1), \quad T_{j,j+1} = T_{j+1,j} \sim 2^{-1/2} \chi_{\beta(n-j)}.$$

Its eigenvalues, labeled by an independent uniform permutation, have density on $\mathbb{R}^n$ proportional to

$$(8.4) \quad \exp \left(-\frac{1}{2} \sum_{j=1}^n \lambda_j^2 \right) \prod_{i < j} |\lambda_i - \lambda_j|^{\beta}.$$

For $a > -1$, let B be lower bidiagonal with mutually independent nonzero entries

$$(8.5) \quad B_{jj} \sim \chi_{2(a+1)+\beta(n-j)}, \quad B_{j+1,j} \sim \chi_{\beta(n-j)}.$$

The eigenvalues of $BB^{\top}$, labeled in the same way, have density on $(0, \infty)^n$ proportional to

$$(8.6) \quad \exp \left(-\frac{1}{2} \sum_{j=1}^n \lambda_j \right) \prod_{j=1}^n \lambda_j^a \prod_{i < j} |\lambda_i - \lambda_j|^{\beta}.$$

All degrees of freedom in (8.5) are positive. The parameter a here is the exponent at the origin; it equals the Laguerre parameter of [2] minus $1 + \beta(n-1)/2$.

These are sampled matrices with ordinary real entries. Their existence supplies spectral shadows; it neither constructs a matrix over a fractional-dimensional scalar system nor follows from the fractal-cone construction. A similarity-scaling identity for Hausdorff measure

is also not a general matrix change-of-variables theorem: additional maps, measures, and Jacobian identities would have to be established.

9. DUNKL GAUSSIANS AND EFFECTIVE DIMENSION

A different rigorous extension of Gaussian analysis is Dunkl theory. For the type- A_{n-1} permutation action on $\mathbb{R}^n$ with nonnegative multiplicity $k = \beta/2$, its Gaussian measure, in the variance convention used here, is

$$(9.1) \quad \nu_{n,\beta}(dx) = \frac{1}{Z} e^{-|x|^2/2} w_\beta(x) dx, \quad w_\beta(x) = \prod_{i < j} |x_i - x_j|^\beta.$$

This theory provides differential–difference operators, generalized Hermite polynomials, and a positive heat semigroup [7, Sections 2–4].

The effective radial dimension in (9.1) is obtained without any special-function argument. The weight is homogeneous of degree

$$h = \frac{\beta n(n-1)}{2},$$

so ordinary polar coordinates give

$$(9.2) \quad |X| \sim \chi_D, \quad D = n + \frac{\beta n(n-1)}{2}.$$

Indeed, writing $x = r\theta$ factors the unnormalized measure into

$$e^{-r^2/2} r^{n-1+h} dr w_\beta(\theta) d\omega(\theta),$$

where ω is surface measure. Its angular integral is finite and strictly positive. Thus the normalized radial factor is $\gamma_{D,1}$. For $n = 1$ the same conclusion is simply the one-dimensional Gaussian calculation.

One can also see D in the radial differential operator. Away from the hyperplanes $x_i = x_j$, write

$$T_i f(x) = \partial_i f(x) + \frac{\beta}{2} \sum_{j \neq i} \frac{f(x) - f(s_{ij}x)}{x_i - x_j},$$

where s_{ij} exchanges coordinates i and j . For a smooth radial function $f(x) = g(|x|)$, direct calculation gives, for $r = |x| > 0$,

$$(9.3) \quad \sum_{i=1}^n T_i^2 f(x) = g''(r) + \frac{D-1}{r} g'(r).$$

To check the additional term, first $T_i f = g'(r)x_i/r$. For each $j \neq i$, exchanging x_i, x_j changes this last function by $g'(r)(x_i - x_j)/r$. The reflection contributions therefore sum to $\beta n(n-1)g'(r)/(2r)$, in addition to the ordinary radial Laplacian.

Here the support of (9.1) is all of $\mathbb{R}^n$, which has integer Hausdorff dimension n . The number D is a homogeneity parameter, not the Hausdorff dimension of the support and not a number of independent coordinates. We work on the full space $\mathbb{R}^n$, including its center-of-mass direction, rather than only the sum-zero hyperplane. The density is also the spectral density (8.4); these two interpretations should not be mistaken for a fractal scalar algebra.

10. GAUSSIAN FIELDS ON FRACTALS AND THE FINAL DISTINCTION

Gaussian fields indexed by a fractal are yet another notion. For a sigma-finite measure space (F, μ) , an isonormal Gaussian family on $L^2(F, \mu)$ is a centered jointly Gaussian family satisfying

$$\mathbb{E}[W(f)W(g)] = \int_F f(x)g(x) \mu(dx).$$

It is not, in general, a random function $x \mapsto W(x)$. Given a nonnegative self-adjoint operator A and $s > 0$, restrict to the orthogonal complement of its kernel and define A^{-s} by spectral calculus. Then $X(f) = W(A^{-s}f)$ is a Gaussian family on the domain of A^{-s} . Regularity and pointwise evaluation require additional estimates.

Baudoin and Lacaux [1] develop such constructions using the Laplacian on the Sierpiński gasket and related fractals. In these constructions the fractal is the index space; a finite collection of well-defined field values is an ordinary Gaussian vector. This does not turn an individual value into a Gaussian scalar with β real components. Likewise, weighting Hausdorff measure by $e^{-|x|^2/2}$ does not identify it with a heat-kernel measure for an intrinsic diffusion on the fractal. An intrinsic diffusion requires a specified Dirichlet form or generator, not merely a Hausdorff dimension.

The conclusions can now be stated without conflating the constructions. For every $\beta > 1$, Theorem 3.2 gives a genuine β -dimensional fractal with a Gaussian-weighted Hausdorff measure and the exact norm shadow χ_β . For every $\beta \geq 1$, Sections 5–6 give a positive angular law, a Gaussian coordinate shadow, and associative radial addition. For every $\beta > 0$, the radial gamma laws, squared Bessel processes, and the displayed matrix models exist.

The limitations concern specified extra requirements. A metric space of Hausdorff dimension below one cannot give an absolutely continuous distance law. Ordinary Gaussian vector stability cannot have a noninteger-dimensional full support in Euclidean space. Preserving all polynomially continued Euclidean Gaussian Gram identities is incompatible with positivity at noninteger dimension. None of these obstructions contradicts the positive radial or spectral constructions.

Thus a fractal cone realizes the exact radial-volume component of the idea, but not, by itself, the full ghosts-and-shadows calculus. A stronger construction must state which observables are jointly realized, which operations are defined, and which positivity and consistency identities they satisfy. No claim of a complete fractional-dimensional scalar algebra, or of novelty for the constructions collected here, is made.

REFERENCES

- [1] F. Baudoin and C. Lacaux, *Fractional Gaussian fields on the Sierpiński gasket and related fractals*, J. Anal. Math. **146** (2022), 719–739. doi:10.1007/s11854-022-0211-x. arXiv:2003.04408.
- [2] I. Dumitriu and A. Edelman, *Matrix models for beta ensembles*, J. Math. Phys. **43** (2002), no. 11, 5830–5847. doi:10.1063/1.1507823. arXiv:math-ph/0206043.
- [3] A. Edelman, *The random matrix technique of ghosts and shadows*, conference note, version dated November 22, 2009, 6 pp. https://math.mit.edu/~edelman/publications/random_matrix_technique.pdf.
- [4] J. F. C. Kingman, *Random walks with spherical symmetry*, Acta Math. **109** (1963), 11–53. doi:10.1007/BF02391808.
- [5] W. Matysiak, *Hypergroups and quantum Bessel processes of non-integer dimensions*, arXiv:1611.04819, 2016 (version with corrections to the published article). doi:10.1007/s10959-016-0688-2.
- [6] J. Pitman and M. Winkel, *Squared Bessel processes of positive and negative dimension embedded in Brownian local times*, Electron. Commun. Probab. **23** (2018), no. 74, 1–13. doi:10.1214/18-ECP174. arXiv:1804.07316.
- [7] M. Rösler, *Dunkl operators: theory and applications*, in *Orthogonal Polynomials and Special Functions* (E. Koelink and W. Van Assche, eds.), Lecture Notes in Math. **1817**, Springer, 2003, 93–135. doi:10.1007/3-540-44945-0_3. arXiv:math/0210366.
- [8] M. Rösler, *Bessel convolutions on matrix cones*, Compos. Math. **143** (2007), no. 3, 749–779. doi:10.1112/S0010437X06002594.