

# A lower bound for two-interval quantum backflow

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Research note for review

## Abstract

An explicit compactly supported positive-momentum state yields a summed backflow greater than 0.06190 over two unit-duration intervals separated by a gap of  $3/2$ , in units where  $\hbar = 1$  and the mass is  $1/2$ . An executable numerical certificate encloses its value approximately between 0.0619551065 and 0.0619551266, conditional on stated arithmetic and library assumptions. The state is defined by stored binary-rational coefficients and is re-integrated with enclosed quadrature and rounding errors. This is a trial-state lower bound, not an optimality result. It exceeds a published conjectural numerical range for equal spacing, but does not establish superiority to the true equal-spacing supremum or publication priority.

**AI work disclosure.** Lyra, an AI assistant powered by OpenAI, performed the literature search, calculations, code development, computational checks, and writing. Daniel A. Pyle initiated the project, requested verification, and reviewed the write-up. No specialist credentials are claimed for the human contributor. The audits and checks were also AI-conducted; no outside human specialist review has taken place. This is not official OpenAI research or an endorsement by OpenAI. Lyra is disclosed as the AI system used, not as a human author or a repository contributor.

## 1 Model and scope

Quantum backflow is an interference effect in which a state with only positive momentum can exhibit an increase in the probability of being to the left of a fixed point. Consider a freely evolving particle with momentum wavefunction  $\psi \in L^2([0, \infty), dp)$  and  $\|\psi\|_2 = 1$ . Write  $P_-(t)$  for its probability of position less than zero at time  $t$ . For  $g > 0$ , set

$$I_1 = [-g/2 - 1, -g/2], \quad I_2 = [g/2, g/2 + 1].$$

The objective is the sum of probability increases over these intervals,

$$R_g(\psi) = P_-(-g/2) - P_-(-g/2 - 1) + P_-(g/2 + 1) - P_-(g/2).$$

This quantity concerns position probabilities calculated from its freely evolving state. It does not model the disturbance caused by repeated measurements on a single particle, and no experimentally practical preparation is established.

Using the momentum-space formulation of repeated backflow in [1], the real symmetric kernel is

$$\begin{aligned} K_g(p, q) &= -\frac{p+q}{2\pi} \int_{I_1 \cup I_2} e^{it(p^2 - q^2)} dt \\ &= -\frac{2(p+q)}{\pi} \frac{\sin((p^2 - q^2)/2)}{p^2 - q^2} \cos\left(\frac{g+1}{2}(p^2 - q^2)\right), \end{aligned} \quad (1)$$

where the singularity is removable and  $K_g(p, p) = -2p/\pi$ . For the real compactly supported state below,

$$R_g(\psi) = \int_0^{320} \int_0^{320} \psi(p) K_g(p, q) \psi(q) dq dp.$$

## 2 The explicit state and certified lower bound

Let  $P_\ell$  denote the Legendre polynomial of degree  $\ell$ . The coefficient archive `explicit_state_L320_g1.5.npz` contains 32 head coefficients  $a_\ell$  and tail coefficients  $b_{\ell j}$  for  $0 \leq \ell \leq 11$  and  $1 \leq j \leq 51199$ . Each stored binary64 coefficient is interpreted as its exact binary-rational value.

Define the unnormalized head on  $0 \leq p \leq \sqrt{2}$  by

$$f_0(p) = \sum_{\ell=0}^{31} a_\ell \sqrt{\frac{2\ell+1}{\sqrt{2}}} P_\ell(\sqrt{2}p - 1).$$

For the tail, use energy coordinate  $x = p^2$ . On  $2j \leq x \leq 2j+2$ , set

$$\phi_j(x) = \sum_{\ell=0}^{11} b_{\ell j} \sqrt{\frac{2\ell+1}{2}} P_\ell(x - 2j - 1).$$

The physical state is

$$\psi(p) = \frac{1}{\sqrt{S}} \begin{cases} f_0(p), & 0 \leq p \leq \sqrt{2}, \\ \sqrt{2p} \phi_j(p^2), & 2j \leq p^2 \leq 2j+2, \quad 1 \leq j \leq 51199, \\ 0, & p < 0 \text{ or } p > 320, \end{cases}$$

with endpoint values immaterial in  $L^2$ . Here

$$S = \sum_{\ell=0}^{31} a_\ell^2 + \sum_{j=1}^{51199} \sum_{\ell=0}^{11} b_{\ell j}^2.$$

Orthonormality and the energy-coordinate Jacobian show that this state is normalized. The verifier recomputes  $S$  exactly using rational arithmetic; it does not trust an approximate norm stored in the archive. Finite momentum support implies finite energy, but the panel joins are not asserted to be experimentally convenient.

The SHA-256 identifier of the coefficient archive is

```
23c0a9394e820932e48be943ad90d3a87
bab6faba6b651e004b99aa8c4460aa9.
```

The two lines concatenate to one hash. Both certificate entrypoints reject a mismatch before loading the native FFT helper or starting verification, and parse the same in-memory bytes that passed the hash check.

**Proposition 1** (Computational certificate, conditional on its trust assumptions). *For the state specified above and in the companion coefficient archive, at gap  $g = 3/2$ ,*

$$R_{3/2}(\psi) > 0.06190.$$

*The intervals are  $[-7/4, -3/4]$  and  $[3/4, 7/4]$ . The reviewed executable reports the enclosure endpoints*

$$[0.06195510650587079, 0.06195512654165395].$$

*These decimal strings report the recorded numerical endpoints; the conservative strict lower bound is the stated result.*

### 3 Certificate construction and trust assumptions

The optimization that selected the coefficients is separate from certification. The verifier holds the coefficient-defined state fixed and re-integrates it, using 64-point Gaussian quadrature on the momentum head and 48-point Gaussian quadrature on each energy tail panel. There is no uncomputed infinite-momentum tail because the state is exactly zero beyond 320.

#### 3.1 Enclosed inputs and arithmetic

SciPy provides approximate Gaussian roots only. Exact dyadic brackets are certified by sign-changing interval evaluations of the Legendre recurrence, with disjoint brackets refined to width at most  $2^{-179}$ . The degree count certifies all roots. The Gaussian weight formula is interval evaluated, as are basis values and coefficient samples.

The arithmetic trust base includes IEEE-754 binary64 round-to-nearest elementary operations and correctly rounded square root, without fast-math reassociation or overflow. Interval elementary operations expand endpoints using `nextafter`. The implementation checks radix two, a 53-bit double significand, round-to-nearest, disabled flush-to-zero and denormals-are-zero, and a long-double significand of at least 64 bits. Compilation disables fast math and floating-point contraction. MPFR is used at 128 bits for trigonometric quantities, and small-node calculations use `mpmath.iv`. These are assumptions about arithmetic and library semantics, not proof-assistant verification of the machine or libraries. Detailed subnormal and summation allowances are stated in the companion proof.

#### 3.2 Kernel evaluation and FFT error control

At  $g = 3/2$ , write

$$c(E) = -\frac{\sin(E/2) \cos(5E/4)}{\pi E}, \quad c(0) = -\frac{1}{2\pi}.$$

Then  $K_{3/2}(p, q) = 2(p + q)c(p^2 - q^2)$ . In the energy tail, quadrature vectors have the form

$$A_{rj} = w_r \phi_j(x_{rj}) x_{rj}^{1/4}, \quad B_{rj} = w_r \phi_j(x_{rj}) x_{rj}^{-1/4}.$$

The transformed kernel gives Toeplitz convolution blocks, with total unnormalized tail contribution  $2 \sum_{r,s} A_r^T T_{rs} B_s$ . MPFR trigonometric tables and certified small-offset addition formulas enclose the kernel values; no uncontrolled large-argument library sine or cosine evaluation is used.

The custom radix-two FFT uses length  $N = 131072$  and 17 stages. With  $u = 2^{-53}$ , the companion proof bounds its forward errors by

$$\begin{aligned} \|F_{\mathfrak{H}}(v) - F(v)\|_2 &\leq \eta\sqrt{N}\|v\|_2, \\ \max_j |F_{\mathfrak{H}}(v)_j - F(v)_j| &\leq \eta\|v\|_1, \end{aligned}$$

where  $\eta = (1 + 32u)^{17} - 1 < 10^{-12}$ . The implemented combined FFT and product-accumulation allowance for a block is  $4 \times 10^{-12}\|A\|_2\|B\|_2\|k\|_1$ , supplemented by explicit input and kernel perturbation bounds. Head/head and mixed terms are separately enclosed. A final  $10^{-8}$  allowance covers scalar tail and error-budget accumulation and endpoint preparation. The proof justifies this reserve using a conservative sequential summation bound, not empirical agreement between runs.

### 3.3 Analytic quadrature remainder

The companion mixed-panel proof applies a Bernstein-ellipse estimate with parameter  $\rho = 2$  to the exact normalized state. Set

$$\begin{aligned} C &= \frac{2\sqrt{320^2 + 1/4}}{\pi} \exp\left(\frac{7}{4} \frac{27}{16}\right), \\ B^2 &= 1024\sqrt{2} + 51199 \cdot 288, \\ A^2 &= 1024\sqrt{2} 2^{-194} + 51199 \cdot 288 2^{-170}. \end{aligned}$$

The normalized double-integral quadrature error is at most

$$16C\sqrt{A^2B^2} < 2.382 \times 10^{-14}.$$

After enclosing the numerator, dividing by the enclosure of the exact  $S$ , and expanding by this analytic remainder, the verifier obtains the reported interval. Its lower endpoint exceeds 0.06190. Full implementation details and the proof of the remainder are supplied with the reproducibility archive.

## 4 Comparison with earlier work and limits

Fewster and Kirk-Karakaya [1] reported an estimate for two equal-duration intervals separated by a gap of that same duration, between 0.0618453024 and 0.0618597497 (published Table 2; see also Table 3.5 of [2]). These are conjectural extrapolation bounds. The present certified trial-state lower bound exceeds that reported numerical range. It does not prove that it exceeds the true equal-spacing maximum, because no rigorous upper bound on that maximum is supplied here.

The framework and accompanying programs of [1, 3] already support arbitrary interval times. Varying the gap is therefore not itself a new method. The possible contribution is the explicit state together with its error-enclosed lower bound. A literature search found no stronger matching two-interval result, but that does not establish priority.

Exploratory direct momentum-space quadrature and composite energy-space FFT quadrature were compared at overlapping cutoffs. The apparent gap advantage persisted as the cutoff grew, and in a separate finite-cutoff check with total elapsed time held fixed. Those explorations are supporting numerical evidence, not proofs of global optimization. A local search at momentum cutoff 320 suggested a slightly better gap near 1.4797; the certified result retains the exact gap  $3/2$ .

The optimal gap and full two-interval maximum remain unknown. No global optimality, equal-spacing upper bound, or experimental validation is claimed. Specialist review of the proof, implementation, and prior literature is needed before describing this as an established new contribution. Experimental relevance would require separate study of state preparation, energy scales, noise, and measurement effects.

## 5 Reproducibility and availability

The companion reproducibility archive is available at the commit-specific GitHub archive link. Its ZIP-file SHA-256 is `9b222cdabeb50bfd1784f8296d053f1bfff78766d7624f383fe97a688b2cf24c`. The two lines concatenate to one hash; this identifies the full archive, separately from the state-file hash above.

The companion archive contains the exact state, verifier sources, proof, recorded results, AI-conducted audit, dependency information, checksums, and earlier numerical history. It excludes compiled binaries. From its extracted top-level directory, run

```
sh certificate/run.sh
```

The tested environment was Linux x86-64 with `gcc`, `libmpfr.so.6`, and Python with NumPy, SciPy, and mpmath. Verification took roughly two minutes in the recorded runs; memory use was not measured. Once dependencies are installed, verification needs no network access. Run Python normally, without optimization flags, because mathematical guards use assertions.

A separate rebuild and rerun is available through

```
sh certificate/run_independent.sh
```

It uses the same underlying certificate algorithm; it is not a second independently designed proof method. The archive’s hash tests check guard behavior and mathematical parity between entrypoints. The reviewed result is a checked computational certificate conditional on its arithmetic assumptions, not human peer review or formal proof-assistant verification.

## References

- [1] C. J. Fewster and H. Kirk-Karakaya, *Repeated quantum backflow and overflow*, Proceedings of the Royal Society A **481** (2025), 20250577. <https://doi.org/10.1098/rspa.2025.0577>.
- [2] H. Kirk-Karakaya, *The Quantum Backflow Phenomenon*, thesis (2026), Table 3.5. [https://etheses.whiterose.ac.uk/id/eprint/38105/8/Kirk-Karakaya\\_203026398\\_CorrectedThesisClean2.pdf](https://etheses.whiterose.ac.uk/id/eprint/38105/8/Kirk-Karakaya_203026398_CorrectedThesisClean2.pdf).
- [3] H. Kirk-Karakaya, *Repeated quantum backflow and quantum overflow*, original spectral data and programs. <https://github.com/hkk506/Repeated-quantum-backflow-and-quantum-overflow/>.