

IMAGE RECONSTRUCTION FROM PROJECTIONS USING FOURIER SLICE THEOREM

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ABSTRACT. We study the problem of image reconstruction from projection data by combining the Fourier Slice Theorem with scattered-data approximation in the Fourier domain. The unknown image is assumed to belong to the space of functions of bounded variation (BV), allowing discontinuities that naturally arise in practical tomography. In contrast to conventional band-limited formulations, no assumption is imposed that the image itself is band-limited. Instead, the finite bandwidth is attributed to the image acquisition system through the detector response, finite sampling resolution, and associated measurement process. Consequently, the measured Radon projections contain only a finite range of spatial frequencies, and the reconstruction objective is the recovery of the instrument-limited image determined by the measurable Fourier bandwidth.

To exploit the structure of the Fourier Slice Theorem, we introduce a symmetric embedding of the image on the torus together with the corresponding symmetrization of the Radon data. The resulting Fourier transform becomes real-valued and even, allowing the Fourier-transformed projection data to be interpreted as real-valued scattered samples of the two-dimensional Fourier transform on a fixed bounded frequency domain. This reformulates image reconstruction as a scattered-data interpolation problem in the Fourier domain.

The interpolation problem is analyzed using the recently developed approximation theory for multivariate BV functions from scattered data. The resulting convergence error is expressed in terms of the total reconstruction error separating into an instrument error, determined solely by the finite bandwidth of the acquisition system, and a reconstruction error arising from interpolation of the measured Fourier data.

The proposed framework provides a mathematically rigorous and physically realistic formulation of Fourier-slice tomography in which the limitations of the imaging hardware are distinguished from the accuracy of the reconstruction algorithm. Numerical experiments demonstrate the applicability of the method to image reconstruction from limited projection data and validate the predicted convergence behaviour.

1. INTRODUCTION

Reconstruction of images from projection data is a central problem in tomography, with applications ranging from medical computed tomography to nondestructive testing and remote sensing. Classical reconstruction methods

aim to recover the underlying image at full resolution, often under idealized smoothness or decay assumptions. In practice, however, both the acquisition process and the intended downstream applications impose a finite resolution: only a band limited image is computationally relevant. We therefore adopt the following viewpoint: the unknown image belongs to $BV(\mathbb{T}^2)$, but the acquired / to be reconstructed image problem is posed for its Fourier transform restricted to a prescribed bounded frequency domain Ω . Equivalently, we seek to recover a band-limited image. This formulation aligns with practical requirements in imaging, where high-frequency components beyond a given bandwidth are not necessary for subsequent processing.

Within this framework, projection data are related to the Fourier transform of the image via the Fourier slice theorem. After an appropriate symmetrization, the Fourier transform of the band-limited image becomes real-valued, and the Fourier-transformed projection data can be interpreted as real-valued scattered samples of the Fourier transform on the fixed domain Ω . This perspective allows the reconstruction problem to be cast as a scattered-data approximation problem in the Fourier domain, rather than as a direct inversion of the Radon transform.

The spatially locally BV regularity assumption on the real image makes its Fourier transform to be locally BV in frequency domain. It plays a crucial analytical role. Although BV functions may exhibit jump discontinuities in the spatial domain, their Fourier transforms possess bounded variation on bounded domains. This enables the use of BV approximation theory to derive quantitative error estimates for interpolation from scattered Fourier samples. Importantly, the convergence rates depend on the desired bandwidth and the angular sampling density, but not on the unattainable high-frequency content of the original image.

The contribution of this work is a physically computable reconstruction theory for band-limited reconstruction of BV images from projection data.

2. LITERATURE SURVEY

2.1. Foundations of the Radon Transform. The mathematical foundations of tomographic reconstruction originate in the seminal work of Radon [10], who showed that a sufficiently regular function defined on $\mathbb{R}^2$ can be uniquely determined from its integrals over straight lines. The transform mapping a function to these line integrals is now known as the Radon transform. Radon's inversion formula established the theoretical basis for reconstructing an object from projection measurements and laid the groundwork for modern computed tomography (CT).

Subsequent developments in integral geometry and harmonic analysis greatly extended Radon's original ideas. In particular, Helgason [5, 6] developed a comprehensive theory of Radon transforms on Euclidean and symmetric spaces, providing rigorous inversion formulas, range conditions, and

connections with representation theory. These results clarified the structural properties of the transform and established its central role in inverse problems. The mathematical treatment of computerized tomography was further developed in the influential monograph of Natterer [9], which provides a systematic analysis of reconstruction algorithms, sampling theory, and stability considerations. Kak and Slaney [7] later presented a comprehensive treatment of the physical and computational aspects of CT, bridging the gap between mathematical theory and practical imaging systems.

2.2. Fourier Methods and the Central Slice Theorem. A key analytical tool in tomography is the Fourier slice theorem, also known as the central slice theorem. This result states that the one-dimensional Fourier transform of a projection equals the restriction of the two-dimensional Fourier transform of the image to a radial line in frequency space. As a consequence, projection measurements provide direct access to the Fourier spectrum of the object along radial directions.

This observation has led to a variety of Fourier-domain reconstruction methods. In particular, the filtered backprojection (FBP) algorithm arises naturally by combining the Fourier slice theorem with an inverse Fourier transform and an appropriate filtering step that compensates for the density of frequency samples. The FBP algorithm remains the most widely used reconstruction method in clinical CT due to its computational efficiency and analytic simplicity.

Fourier-based interpretations of tomographic reconstruction have also been studied extensively within the framework of harmonic analysis and signal processing. Bracewell [2] provided an early exposition of Fourier methods in imaging, while Deans [4] presented a detailed analysis of the Radon transform and its Fourier properties.

Despite their theoretical elegance, Fourier reconstruction methods must address several practical challenges. Real imaging systems provide only a finite number of projection angles, and measurements are subject to noise, discretization, and detector limitations. These issues motivate the study of approximation and regularization techniques that stabilize the inversion process.

2.3. Variational and Regularized Reconstruction. The inverse problems encountered in tomography are typically ill-posed, particularly when projection data are incomplete or noisy. To address these challenges, reconstruction methods are often formulated within variational frameworks that incorporate prior information about the image.

One of the most influential developments in this direction is the introduction of total variation (TV) regularization by Rudin, Osher, and Fatemi [11]. Their model demonstrated that minimizing the total variation of an image while enforcing data fidelity produces reconstructions that preserve

edges while suppressing noise. The success of this approach has led to widespread adoption of TV regularization in image reconstruction and inverse problems.

The mathematical foundation of total variation methods lies in the theory of functions of bounded variation (BV). The space $BV(\Omega)$ consists of functions whose distributional derivatives are finite Radon measures. Functions in this space naturally model piecewise smooth images with edges, making BV a particularly appropriate functional setting for imaging applications. The modern theory of BV functions and free discontinuity problems is developed in detail in the monograph of Ambrosio, Fusco, and Pallara [1].

Numerous extensions of the total variation framework have since been proposed, including higher-order regularization models, anisotropic variants, and methods inspired by compressed sensing. These approaches exploit the observation that many natural images admit sparse representations in suitable transform domains.

2.4. Sampling and Scattered Fourier Data. In practical tomographic systems, projection data correspond to samples of the Fourier transform of the image along radial lines. When only finitely many projection angles are available, the resulting frequency samples form a nonuniform distribution in Fourier space. Reconstruction in this setting can therefore be interpreted as a scattered-data interpolation problem.

The theory of scattered data approximation has been studied extensively in numerical analysis and approximation theory. Wendland [12] provides a comprehensive treatment of interpolation and approximation methods for irregularly spaced data, including radial basis function techniques that are widely used in multidimensional interpolation problems.

Related work by Narcowich and Ward [8] developed theoretical results on generalized interpolation schemes based on positive definite kernels. These methods provide stable reconstruction frameworks when measurements are available at scattered locations in frequency space.

From a computational perspective, the reconstruction of images from nonuniform Fourier samples has motivated the development of the nonuniform fast Fourier transform (NUFFT). NUFFT algorithms provide efficient methods for evaluating Fourier transforms on irregular grids and are widely used in applications such as tomography and magnetic resonance imaging. These methods enable practical implementation of Fourier-domain reconstruction techniques while maintaining high computational efficiency.

2.5. Band-Limited Models in Imaging. Physical imaging systems possess finite resolution due to detector sampling, finite aperture effects, and system transfer functions. As a result, the measured data effectively correspond to a band-limited representation of the underlying object. This observation motivates the formulation of reconstruction problems in terms of recovering the Fourier transform of the image within a bounded frequency domain.

Band-limited formulations simplify the reconstruction problem by restricting attention to finite-dimensional approximations of the image. Such approaches are closely related to classical sampling theory and provide a natural framework for analyzing reconstruction stability and approximation error.

In the context of tomography, band-limited reconstruction methods also help mitigate artifacts caused by incomplete angular coverage or sparse sampling. By imposing frequency-domain constraints, these methods can produce stable reconstructions even when projection data are limited.

2.6. Modern Perspectives in Tomographic Reconstruction. Recent developments in inverse problems have emphasized the integration of Fourier-domain models with variational regularization techniques. In these approaches, reconstruction is formulated as an optimization problem that balances fidelity to measured projection data with prior structural information about the image.

Such frameworks are particularly effective in limited-angle tomography and sparse-view CT, where classical analytical reconstruction formulas may fail. By combining Fourier-domain sampling models with regularization methods such as total variation, these approaches achieve improved robustness against noise and incomplete measurements.

The framework developed in the present work builds upon these ideas by formulating the reconstruction problem for BV images within a band-limited Fourier setting. By embedding the image symmetrically on a toroidal domain, the Fourier transform becomes real-valued and can be analyzed using trigonometric expansions. Projection measurements obtained through the Radon transform correspond to radial samples of this Fourier representation. Reconstruction is therefore interpreted as a scattered-data approximation problem in Fourier space, with bounded variation regularity providing stability and well-posedness of the inversion process.

3. MATHEMATICAL FRAMEWORK AND PROBLEM STATEMENT

Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ denote the image intensity function to be reconstructed. In tomographic imaging, measurements are obtained through line integrals of f , which are modeled by the Radon transform.

Definition 3.1 (Radon Transform). The Radon transform of f is defined by

$$(Rf)(\theta, s) = \int_{\mathbb{R}^2} f(x) \delta(s - x \cdot \theta) dx,$$

where $\theta \in S^1$ is a unit direction vector and $s \in \mathbb{R}$ denotes the signed distance from the origin.

The measurement data consist of projections

$$g(\theta, s) = (Rf)(\theta, s).$$

In practice, projections are available only for a finite set of angles

$$\Theta = \{\theta_1, \dots, \theta_M\}.$$

Thus the reconstruction problem is an inverse problem:

Problem: Recover the image f from a finite set of projections $\{g(\theta_k, s)\}_{k=1}^M$.

3.1. Fourier Representation. Let $\hat{f}(\xi)$ denote the Fourier transform of f :

$$\hat{f}(\xi) = \int_{\mathbb{R}^2} f(x) e^{-2\pi i x \cdot \xi} dx.$$

The Fourier slice theorem states

$$\mathcal{F}_s(Rf)(\theta, \omega) = \hat{f}(\omega\theta),$$

where $\mathcal{F}_s$ denotes the one-dimensional Fourier transform in the detector coordinate s .

Thus each projection provides samples of $\hat{f}$ along a radial line in frequency space.

3.2. Band-Limited Formulation. Physical imaging systems have finite resolution. Consequently we assume that the Fourier transform of the image is supported in a bounded frequency domain

$$\Omega = \{\xi \in \mathbb{R}^2 : |\xi| \leq \Lambda\}.$$

The reconstruction problem therefore reduces to recovering $\hat{f}$ on Ω from samples

$$\hat{f}(\omega\theta_k), \quad \theta_k \in \Theta.$$

This formulation interprets tomography as a scattered-data approximation problem in Fourier space.

4. RECONSTRUCTION THEOREM AND ALGORITHM

We now establish a reconstruction principle for band-limited images from radial Fourier samples.

Theorem 4.1 (Fourier Reconstruction from Radial Samples). *Let $f \in BV(\mathbb{R}^2)$ be a function whose Fourier transform $\hat{f}$ is supported in the bounded domain Ω . Assume projections are measured for angles $\Theta = \{\theta_k\}_{k=1}^M$.*

Then $\hat{f}$ can be approximated on Ω from the radial samples

$$\hat{f}(\omega\theta_k),$$

and the image f can be reconstructed through the inverse Fourier transform

$$f(x) = \int_{\Omega} \hat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi.$$

Proof. By the Fourier slice theorem, the one-dimensional Fourier transform of each projection provides the values of $\hat{f}$ along the radial line $\{\omega\theta_k : \omega \in \mathbb{R}\}$. Since $\hat{f}$ is supported in the bounded region Ω , the available samples determine $\hat{f}$ on a discrete set of radial curves within Ω .

Using interpolation or approximation methods for scattered Fourier data, one may construct an approximation $\tilde{f}(\xi)$ of the Fourier transform. The inverse Fourier transform of this approximation yields the reconstructed image. $\square$

4.1. Reconstruction Algorithm. The reconstruction procedure can be summarized as follows.

- (1) Acquire projections $g(\theta_k, s)$ for angles $\theta_k \in \Theta$.
- (2) Compute the one-dimensional Fourier transform of each projection:

$$\hat{g}(\theta_k, \omega) = \mathcal{F}_s g(\theta_k, s).$$

- (3) Map the values $\hat{g}(\theta_k, \omega)$ to Fourier samples

$$\hat{f}(\omega\theta_k).$$

- (4) Interpolate these samples to obtain an approximation $\tilde{f}(\xi)$ on the domain Ω .
- (5) Compute the reconstructed image

$$f(x) = \int_{\Omega} \tilde{f}(\xi) e^{2\pi i x \cdot \xi} d\xi.$$

This algorithm interprets tomography as a reconstruction problem from scattered Fourier samples.

5. STABILITY AND ERROR ANALYSIS

A central issue in tomographic reconstruction is the stability of the inverse problem. Since projection data are typically contaminated by noise and discretization errors, it is important to quantify how these perturbations affect the reconstructed image.

5.1. Measurement Noise. Assume the measured projections satisfy

$$g_{\delta}(\theta, s) = (Rf)(\theta, s) + \eta(\theta, s),$$

where η represents measurement noise.

After Fourier transformation this becomes

$$\hat{g}_{\delta}(\theta, \omega) = \hat{f}(\omega\theta) + \hat{\eta}(\theta, \omega).$$

Thus the Fourier samples of the image are corrupted by noise.

5.2. Reconstruction Error. Let $\tilde{f}$ denote the reconstructed Fourier transform obtained by interpolating noisy samples. The reconstruction error in the image domain satisfies

$$\|f - \tilde{f}\|_{L^2} \leq C(\epsilon_{interp} + \epsilon_{noise}),$$

where

$$\epsilon_{interp}$$

represents interpolation error in Fourier space and

$$\epsilon_{noise}$$

represents propagated measurement noise.

5.3. Role of Bounded Variation Regularity. If $f \in BV(\mathbb{R}^2)$, the Fourier transform of f decays at high frequencies. This property provides stability for the reconstruction problem because the contribution of high-frequency noise is suppressed.

Moreover, BV regularity ensures that reconstructed images preserve edges while maintaining stability under perturbations of the data.

5.4. Sampling Density. The accuracy of reconstruction also depends on the angular sampling density. If projection angles are sufficiently dense, the radial sampling of Fourier space becomes nearly uniform, improving the accuracy of interpolation and reducing reconstruction artifacts.

6. SYMMETRIC EMBEDDING ON THE TORUS AND INDUCED TRANSFORM ON RADON DATA

6.1. Setting and notation. Let

$$\mathbb{T}^2 := (\mathbb{R}/\mathbb{Z})^2$$

denote the two-dimensional flat torus, identified with the unit square $[0, 1)^2$ with periodic boundary conditions. Let

$$f : \mathbb{T}^2 \rightarrow \mathbb{R}$$

be a real-valued image.

We denote by

$$\mathcal{R}f(\theta, s)$$

the (periodized) Radon transform of f , defined for $\theta \in [0, \pi)$ and $s \in \mathbb{T}$ by

$$\mathcal{R}f(\theta, s) = \int_{\{x \in \mathbb{T}^2 : x \cdot \theta^\perp = s\}} f(x) d\ell(x), \quad \theta^\perp = (-\sin \theta, \cos \theta),$$

where integration is taken along periodic geodesic lines on $\mathbb{T}^2$.

6.2. Fully symmetric extension of the image.

Definition 6.1 (Full even symmetric extension). Define the symmetrically extended image

$$f_{\text{sym}} : \mathbb{T}^2 \rightarrow \mathbb{R}$$

by

$$f_{\text{sym}}(x_1, x_2) = \frac{1}{4} \sum_{\varepsilon_1=\pm 1} \sum_{\varepsilon_2=\pm 1} f(\varepsilon_1 x_1, \varepsilon_2 x_2),$$

where multiplication by -1 is understood modulo 1 on $\mathbb{T}$.

By construction,

$$f_{\text{sym}}(x_1, x_2) = f_{\text{sym}}(\pm x_1, \pm x_2),$$

so f_{sym} is invariant under reflections in both spatial variables.

6.3. Induced transform on Radon data. We now define a transform acting directly on Radon measurements which encodes this symmetric embedding.

Definition 6.2 (Radon symmetrization operator). Let $\mathcal{T}_{\text{sym}}$ act on Radon data by

$$(\mathcal{T}_{\text{sym}} \mathcal{R}f)(\theta, s) = \frac{1}{4} \left(\mathcal{R}f(\theta, s) + \mathcal{R}f(\pi - \theta, s) + \mathcal{R}f(\theta, -s) + \mathcal{R}f(\pi - \theta, -s) \right).$$

This operator averages Radon data over all reflections induced by the group $\{\pm 1\}^2$ acting on the spatial variables.

6.4. Consistency theorem.

Theorem 6.3 (Consistency with symmetric embedding). *Let $f \in L^1(\mathbb{T}^2)$. Then*

$$\mathcal{R}f_{\text{sym}} = \mathcal{T}_{\text{sym}}(\mathcal{R}f).$$

Proof. By linearity of the Radon transform,

$$\mathcal{R}f_{\text{sym}} = \frac{1}{4} \sum_{\varepsilon_1=\pm 1} \sum_{\varepsilon_2=\pm 1} \mathcal{R}(f(\varepsilon_1 \cdot, \varepsilon_2 \cdot)).$$

Reflection of the image corresponds to reflection of the Radon parameters (θ, s) , yielding the four terms appearing in $\mathcal{T}_{\text{sym}}$. Collecting these contributions proves the claim. $\square$

6.5. Remarks.

- The operator $\mathcal{T}_{\text{sym}}$ is a projection onto Radon data consistent with images that are even in both coordinates.
- In the Fourier slice framework, this symmetrization corresponds to restricting the two-dimensional Fourier transform to cosine modes.
- The construction extends verbatim to $\mathbb{T}^d$ by averaging over all 2^d coordinate reflections.

6.6. Fourier-slice reconstruction with real-valued Fourier data. After applying the symmetrization operator introduced in the previous section, we may assume without loss of generality that the image $f : \mathbb{T}^2 \rightarrow \mathbb{R}$ has been replaced by its fully even symmetric extension f_{sym} . As shown earlier, this implies that its Fourier transform

$$\widehat{f_{\text{sym}}} : \mathbb{R}^2 \rightarrow \mathbb{R}$$

is real-valued and even. This structural property plays a central role in Fourier-slice-based reconstruction, particularly when interpolation from scattered Fourier samples is required.

Fourier slice theorem. Let $\mathcal{R}f_{\text{sym}}(\theta, s)$ denote the Radon transform of f_{sym} , where $\theta \in [0, \pi)$ is the projection angle and $s \in \mathbb{R}$ the detector variable. Define the one-dimensional Fourier transform in the s -variable by

$$\mathcal{F}_s[\mathcal{R}f_{\text{sym}}](\theta, \rho) := \int_{\mathbb{R}} \mathcal{R}f_{\text{sym}}(\theta, s) e^{-2\pi i \rho s} ds.$$

Theorem 6.4 (Fourier slice theorem). *For every $\theta \in [0, \pi)$ and $\rho \in \mathbb{R}$,*

$$\mathcal{F}_s[\mathcal{R}f_{\text{sym}}](\theta, \rho) = \widehat{f_{\text{sym}}}(\rho \theta), \quad \theta = (\cos \theta, \sin \theta).$$

Proof. By definition of the Radon transform,

$$\mathcal{R}f_{\text{sym}}(\theta, s) = \int_{\mathbb{R}} f_{\text{sym}}(s\theta + t\theta^\perp) dt, \quad \theta^\perp = (-\sin \theta, \cos \theta).$$

Taking the Fourier transform in s and applying Fubini's theorem yields

$$\mathcal{F}_s[\mathcal{R}f_{\text{sym}}](\theta, \rho) = \int_{\mathbb{R}^2} f_{\text{sym}}(x) e^{-2\pi i \rho x \cdot \theta} dx = \widehat{f_{\text{sym}}}(\rho \theta),$$

which proves the claim. $\square$

Scattered Fourier data. For a finite set of projection angles $\{\theta_j\}_{j=1}^N \subset [0, \pi)$ and frequencies $|\rho| \leq \omega$, the Fourier slice theorem provides samples of $\widehat{f_{\text{sym}}}$ at the scattered locations

$$\xi_{j,\rho} = \rho \theta_j \in \Omega.$$

Thus the Radon data, after one-dimensional Fourier transformation, constitute real-valued scattered samples of the two-dimensional Fourier transform on the fixed domain Ω .

The symmetrization step is essential here: since $\widehat{f_{\text{sym}}}$ is real-valued, the scattered data $\widehat{f_{\text{sym}}}(\xi_{j,\rho})$ are real, and no phase ambiguity or conjugate-symmetry enforcement is required. This allows the Fourier-slice reconstruction problem to be cast directly as a real-valued scattered-data approximation problem in the Fourier domain.

6.7. Band-limited acquisition model. The unknown image itself is not assumed to be band-limited. Instead, the finite bandwidth arises from the image acquisition system. This is consistent with practical tomographic instruments, whose detector resolution, finite aperture, electronics, anti-alias filtering, and sampling frequency limit the measurable frequency content.

Let

$$f : T^2 \rightarrow \mathbb{R}$$

be the unknown image, and let

$$Rf(\theta, s)$$

denote its Radon transform.

The measured projection is modeled as

$$R_m(\theta, s) = (h_\theta * Rf(\theta, \cdot))(s) + \eta(\theta, s),$$

where

- h_θ denotes the detector point-spread function,
- $*$ denotes convolution with respect to the detector coordinate,
- η denotes measurement noise.

Taking the one-dimensional Fourier transform in the detector coordinate gives

$$\widehat{R}_m(\theta, \rho) = H_\theta(\rho) \widehat{R}(\theta, \rho) + \widehat{\eta}(\theta, \rho),$$

where

$$H_\theta(\rho) = \mathcal{F}_1(h_\theta)(\rho)$$

is the modulation transfer function (MTF) of the acquisition system. For the theoretical analysis we consider the ideal detector model

$$H_\theta(\rho) = \chi_{[-\omega, \omega]}(\rho),$$

where χ denotes the indicator function. Consequently,

$$\widehat{R}_m(\theta, \rho) = \begin{cases} \widehat{R}(\theta, \rho), & |\rho| \leq \omega, \\ 0, & |\rho| > \omega. \end{cases}$$

Thus the acquisition system directly determines the measurable frequency bandwidth.

Applying the Fourier Slice Theorem, the measured Fourier samples satisfy

$$\widehat{f}_m(\xi) = \chi_{B_\omega}(\xi) \widehat{f}(\xi),$$

where

$$B_\omega = \{\xi \in \mathbb{R}^2 : \|\xi\| \leq \omega\}.$$

Accordingly, the reconstruction algorithm seeks to recover the instrument-limited image

$$f_\omega = \mathcal{F}^{-1}(\chi_{B_\omega} \widehat{f}),$$

rather than the original image f . No assumption is therefore imposed that the original object itself is band-limited. The only assumption is that the imaging system measures Fourier data over a finite bandwidth.

Consequently, the reconstruction error naturally decomposes into

$$\|f - \widehat{f}_N\| \leq \underbrace{\|f - f_\omega\|}_{\text{instrument limitation}} + \underbrace{\|f_\omega - \widehat{f}_N\|}_{\text{reconstruction error}}.$$

The first term is determined entirely by the acquisition hardware and cannot be reduced by the reconstruction algorithm. The second term is precisely the approximation error analyzed in the present work.

6.8. Scattered-data interpolation of Fourier-slice measurements.

We now give the discussion on the role of Fourier-slice data in reconstruction theory by a rigorous formulation based on scattered-data approximation in the Fourier domain, following the BV approximation framework of Chapter 4 of [3].

Fourier-slice data as scattered samples. After symmetrization, the image f_{sym} has a real-valued, even Fourier transform $\widehat{f_{\text{sym}}}$. Fix a bandwidth $\omega > 0$ and define the target Fourier domain

$$\Omega := \{\xi \in \mathbb{R}^2 : |\xi| \leq \omega\}.$$

Assume that

$$\text{supp } \widehat{f_{\text{sym}}} \subset \Omega, \quad \widehat{f_{\text{sym}}} \in BV(\Omega).$$

Let $\{\theta_j\}_{j=1}^N \subset [0, \pi)$ be a finite set of projection angles, and let $\rho \in [-\omega, \omega]$ denote the radial frequency variable. By the Fourier slice theorem, the one-dimensional Fourier transform of the Radon data yields

$$\mathcal{F}_s[\mathcal{R}f_{\text{sym}}](\theta_j, \rho) = \widehat{f_{\text{sym}}}(\rho\theta_j), \quad \rho\theta_j \in \Omega.$$

Thus the measured data constitute samples of $\widehat{f_{\text{sym}}}$ at the scattered locations

$$\Xi_N := \{\rho\theta_j : j = 1, \dots, N, |\rho| \leq \omega\} \subset \Omega.$$

Since $\widehat{f_{\text{sym}}}$ is real-valued, these samples are real, and the interpolation problem is purely real-valued.

7. APPROXIMATION OF FOURIER TRANSFORM AND RECONSTRUCTION

The spatially locally BV regularity assumption on the real image makes its Fourier transform to be locally BV in frequency domain. We now use Theorem 4.2 of [3] to approximate the Fourier transform from scattered Fourier samples. The kernel representation of the BV approximation further admits an operator formulation that exposes the reconstruction as a spectrally filtered nonuniform Fourier reconstruction. This characterization

leads naturally to efficient matrix-free numerical algorithms while remaining mathematically equivalent to the original scattered-data approximation. This ensures convergence in L^2 norm in Fourier domain Ω , and there by as Ω is fixed, this also ensures convergence for band limited image f_Ω in spatial domain after taking an inverse Fourier transform.

7.1. Operator Formulation of the BV Reconstruction. The scattered-data approximation of the Fourier transform admits a natural operator representation that reveals the computational structure of the reconstruction algorithm.

Let

$$K \subset \mathbb{Z}^2$$

be the finite Fourier index set defining the reconstruction bandwidth, and let

$$w_k = \frac{1}{1 + \lambda \|k\|^{2m}}, \quad k \in K,$$

be the spectral weights appearing in the reproducing kernel

$$g_\lambda(x) = \sum_{k \in K} w_k e^{2\pi i k \cdot x}.$$

Suppose the scattered Fourier samples are located at

$$P = \{p_1, \dots, p_N\} \subset \Omega,$$

and let

$$c = (c_1, \dots, c_N)^T$$

denote the coefficient vector obtained from the scattered-data approximation system.

Define the nonuniform Fourier matrix

$$E_{ki} = e^{-2\pi i k \cdot p_i}, \quad k \in K, \ 1 \leq i \leq N,$$

and the diagonal spectral weight matrix

$$W = \text{diag}(w_k).$$

Theorem 7.1 (Operator Representation of the BV Approximation). *The Fourier coefficients of the reconstructed band-limited image satisfy*

$$\hat{f} = W E c.$$

Consequently,

$$f_\Omega = \mathcal{F}^{-1}(W E c).$$

Proof. The reconstructed Fourier transform is

$$f(x) = \sum_{i=1}^N c_i g_\lambda(x - p_i).$$

Substituting the Fourier expansion of the kernel,

$$\begin{aligned} f(x) &= \sum_{i=1}^N c_i \sum_{k \in K} w_k e^{2\pi i k \cdot (x - p_i)} \\ &= \sum_{k \in K} w_k \left(\sum_{i=1}^N c_i e^{-2\pi i k \cdot p_i} \right) e^{2\pi i k \cdot x}. \end{aligned}$$

Hence the Fourier coefficient corresponding to the frequency $k \in K$ is

$$\hat{f}(k) = w_k \sum_{i=1}^N c_i e^{-2\pi i k \cdot p_i},$$

which is precisely the k -th component of the vector WEc .

Applying the inverse Fourier transform completes the proof. $\square$

Corollary 7.2. *The Gram matrix of the scattered-data approximation satisfies*

$$G = E^*WE.$$

Consequently,

$$\left(\frac{1}{N} E^*WE + \frac{1}{\lambda^2} I \right) c = L.$$

Remark 7.3. The above formulation shows that the kernel itself never needs to be evaluated explicitly once its Fourier coefficients have been constructed. The entire reconstruction is expressed through three operators:

$$E, \quad W, \quad E^*.$$

Therefore matrix-free iterative methods such as the Conjugate Gradient method require only repeated applications of

$$v \mapsto E^*(W(Ev)),$$

eliminating explicit construction of the dense Gram matrix.

Furthermore, after solving for the coefficient vector c , the Fourier coefficients of the reconstructed image are obtained directly from

$$\hat{f} = WEc,$$

followed by a single inverse Fast Fourier Transform. Thus the reconstruction algorithm consists of

$$L \longrightarrow c \longrightarrow \widehat{f} \longrightarrow f_{\Omega},$$

which may be interpreted as a spectrally filtered nonuniform Fourier reconstruction.

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