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CROSS-ENERGY OF DIRICHLET PARTIAL SUMS. PART 18: THE EULER–CACERES BOUND

PEDRO CACERES

Abstract. Gonek, Hughes and Keating factored $\zeta \approx P_X Z_X$, an Euler product times a product over the zeros. We use an exact form $\zeta = E_X^* Z_X$, E_X^* a smoothed Euler factor, and for $\operatorname{Re} s_0 > \frac12$ study the Euler–Caceres bound $B_X(s_0) = \frac12 \log \int |Z_X(\frac12 + it)|^2 P_{s_0}(t)\,dt - \log|Z_X(s_0)|$, P_{s_0} the Poisson kernel. It is the off-line Blaschke term of the Poisson–Jensen formula at s_0, independent of X, plus a positive Jensen gap. Under RH, $(\log X)B_X(s_0) \to \sum_\gamma \kappa_{m(\gamma)} P_{s_0}(\gamma)$, with universal constants κ_m for multiplicity m ($\kappa_1 = 1.8279\ldots$). Hence, for any single s_0, RH holds if and only if $B_X(s_0) = O(1/\log X)$, and RH with simple zeros if and only if $(\log X)B_X(s_0)$ tends to an explicit number ($0.040186\ldots$ at $s_0 = 2$). If $\Theta = \sup \operatorname{Re}\rho > \frac12$, Landau's theorem in the variable $\log X$ gives $B_X(s_0) = \Omega_+(X^{\Theta - \frac12 - \varepsilon})$ for $\operatorname{Re} s_0 > 1$, attained or not, so RH holds iff $B_X(s_0) \ll X^\varepsilon$ at one point. Unconditionally, $B_X(s_0) = O(\sqrt{X}/\log X)$ for $\operatorname{Re} s_0 > 1$, and the off-line term at $s_0 = 2$ is below $2.3 \cdot 10^{-12}$. Under RH we identify the second-order term up to an interaction of the wells (known only to be $o(\log X)$), related heuristically to Montgomery's pair correlation; it is governed by the gaps between zeros, and under a small-gap hypothesis the law holds with error $O(\log\log X/\log X)$. We give a function-field model where the bound holds, and controls (a Beurling system with the zeros of ζ and a zero at 0.6, the Davenport–Heilbronn function) suggesting that a proof must couple the Euler product with the functional equation. Numerically the bound follows the law, stays below the floor 0.0386 of Dirichlet-polynomial mollifiers at the centre $s_0 = 1$, and grows for the Davenport–Heilbronn function and for a planted off-line zero. We do not prove RH.

1. Introduction

1.1. **The Jensen moment and its mollifiers.** Let $F(s) = (s-1)\zeta(s)/s$. Balazard, Saias and Yor [BSY] proved that

$$\int_{-\infty}^{\infty} \log|\zeta(\tfrac12 + it)|\,\frac{dt}{2\pi(\frac14 + t^2)} = \sum_{\operatorname{Re}\rho > \frac12} \log\Big|\frac{\rho}{1-\rho}\Big|, \tag{1}$$

the Poisson–Jensen formula for F in the half-plane $\operatorname{Re} s > \frac12$ at the point $s = 1$. RH holds if and only if the left side vanishes. Jensen's inequality turns any analytic M into an upper bound for the right side, $\frac12 \log\int |\zeta M|^2 d\mu - \log|M(1)|$. For Dirichlet polynomials M this bound cannot fall below $\frac12\log(1+c_0) = 0.0386$, $c_0 = \sum_{k\ge1}(1 - k(k+1)\log^2(1+\frac1k))$ (Proposition 52 in the appendix): the normalization at the pole forces M to approximate a step function. We replace M by the reciprocal of the prime factor of an exact form of the hybrid Euler–Hadamard product of Gonek, Hughes and Keating [GHK], which carries the pole itself; the barrier disappears (Table 7), and the resulting bound decays like $1/\log X$.

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This paper is about that bound. We determine its exact asymptotic behaviour under RH, show that it measures the multiplicities of the zeros, and turn it into criteria for RH and for RH together with the simplicity of the zeros.

1.2. **Setting.** Fix $g_0 \in C_c^\infty(\mathbb{R})$ with $g_0 \ge 0$, $\int g_0 = 1$, $\operatorname{supp} g_0 \subset [-a,a]$. For $X > e^{a+1}$ put $L = \log X$, $g_X(u) = g_0(u-L)$, $\psi_X(v) = \int_v^\infty g_X$, and

(2)
$$\log E_X^*(s) = \sum_{n\ge 2} \frac{\Lambda(n)\,\psi_X(\log n)}{n^s \log n} + K_X(s-1) - \sum_{k\ge 1} K_X(s+2k), \qquad K_X(w) = \int g_X(u) E_1(wu)\,du.$$

Here $E_1(z) = \int_z^\infty e^{-\tau}\tau^{-1}d\tau$ is the exponential integral and, below, $\operatorname{Ci}(v) = -\int_v^\infty \tau^{-1}\cos\tau\,d\tau$ is the cosine integral. The prime sum is finite (it stops at $e^a X$). By the explicit formula (Lemma 10), $\zeta = E_X^* Z_X$ with

$$Z_X(s) = \prod_\rho e^{-K_X(s-\rho)},$$

a product over the nontrivial zeros, each factor entire with a simple zero at ρ. E_X^* has a simple pole at $s=1$ and the trivial zeros $-2k$, and no other zeros or poles; in particular it has no zeros in $\operatorname{Re} s > -2$.

Relation to the hybrid Euler–Hadamard product. This factorization is a variant of the hybrid Euler–Hadamard product of Gonek, Hughes and Keating [GHK, Thm. 1] (see also [BGM]). For $\sigma > 0$, $|t| \ge 2$, they prove $\zeta(s) = P_X(s)Z_X(s)\big(1+O(X^{K+2}(|s|\log X)^{-K})+O(X^{-\sigma}\log X)\big)$ with $P_X(s) = \exp\sum_{n\le X}\Lambda(n)n^{-s}/\log n$ and $Z_X(s) = \exp\big(-\sum_\rho U((s-\rho)\log X)\big)$, $U(z) = \int u(x)E_1(z\log x)\,dx$, u a smooth weight on $[e^{1-1/X}, e]$; and they observe that the second error term disappears if P_X is smoothed. Our Z_X is their zero factor with the weight u replaced by g_X, which has fixed width in $\log x$ around $\log X$; our E_X^* is the corresponding smoothed prime factor, completed by the pole term $K_X(s-1)$ and the trivial zeros so that the identity is exact for every s (Lemma 10). The difference matters here: we fix s and let $X \to \infty$, a limit in which the error term of [GHK] does not decrease, while the moments of Z_X studied in [GHK] are averages over $t \in [T, 2T]$ with $X \ll (\log T)^{2-\varepsilon}$. The profile $\operatorname{Re} E_1(ix) = -\operatorname{Ci}(|x|)$ of Z_X on the critical line under RH (Lemma 13) already appears in their moment computation. The name *Euler–Caceres bound* refers to the quantity below, not to the factorization.

The bound. For $s_0 = \sigma_0 + it_0$, $\sigma_0 > \frac12$, let

$$P_{s_0}(t) = \frac{\sigma_0 - \frac12}{\pi\big((\sigma_0-\frac12)^2 + (t-t_0)^2\big)}$$

be the Poisson kernel of the half-plane $\operatorname{Re} s > \frac12$ at s_0, a probability density on $\mathbb{R}$, and let $\phi_X(t) = \log|Z_X(\frac12 + it)|$.

$$\textit{The Euler–Caceres bound:}\quad B_X(s_0) = \tfrac12 \log \int e^{2\phi_X(t)} P_{s_0}(t)\,dt - \log|Z_X(s_0)|.$$

1.3. **Results.** Let $b_\rho(s) = (s-\rho)/(s-1+\bar\rho)$ and let

$$\mathcal{O}_{\text{off}}(s_0) = \sum_{\operatorname{Re}\rho > \frac12} \log\frac{1}{|b_\rho(s_0)|} \ \ge 0$$

be the off-line contribution; $\mathcal{O}_{\text{off}}(1)$ is the right side of (1).

Theorem 1 (decomposition)**.** *For every s_0 with $\operatorname{Re} s_0 > \frac12$ and $\zeta(s_0) \ne 0$, and every $X > e^{a+1}$,*

$$B_X(s_0) = \mathcal{O}_{\text{off}}(s_0) + J_X(s_0), \qquad J_X(s_0) = \tfrac12 \log \int e^{2\phi_X} P_{s_0} - \int \phi_X\, P_{s_0} \ > 0.$$

The first term does not depend on X; the second is the Jensen gap of ϕ_X under the probability measure $P_{s_0}(t)\,dt$, and it is strictly positive because $\phi_X \to -\infty$ at the ordinates of the zeros on the line. In particular $B_X(s_0) > \mathcal{O}_{\mathrm{off}}(s_0)$ for every X. For the multiplicity m of a zero define

$$\kappa_m = \int_0^\infty \big(e^{2m\,\mathrm{Ci}(v)} - 1 - 2m\,\mathrm{Ci}(v)\big)\,dv, \qquad \mathrm{Ci}(v) = -\int_v^\infty \frac{\cos\tau}{\tau}\,d\tau.$$

The integrand is $O(m^2/v^2)$ at infinity and $O(m\log(1/v))$ at 0, so κ_m is finite; Table 1 gives the first values. Since $\int_0^\infty \mathrm{Ci} = 0$, $\kappa_1 = \int_0^\infty (e^{2\,\mathrm{Ci}(v)} - 1)\,dv$ as an improper integral.

Theorem 2 (the law of the bound)**.** *Assume RH. For every s_0 with $\operatorname{Re} s_0 > \frac12$,*

$$\lim_{X\to\infty} (\log X)\, B_X(s_0) = \Omega(s_0) := \sum_\gamma \kappa_{m(\gamma)} P_{s_0}(\gamma),$$

where γ runs over the distinct ordinates of the zeros and $m(\gamma)$ is the multiplicity of $\frac12 + i\gamma$; the series converges.

m	1	2	3	4
κ_m	1.82792	7.34579	20.036	48.806
κ_m/m	1.82792	3.67289	6.679	12.202

TABLE 1. The constants κ_m.

Corollary 3 (criteria)**.** *Let $\operatorname{Re} s_0 > \frac12$ and $\zeta(s_0) \neq 0$. Unconditionally:*

(a) *RH $\iff$ $B_X(s_0) \to 0$ $\iff$ $B_X(s_0) = O(1/\log X)$;*

(b) *RH and all zeros simple $\iff$ $\lim_{X\to\infty} (\log X)\, B_X(s_0) = \dfrac{\kappa_1}{\pi} \operatorname{Re} \dfrac{\xi'}{\xi}(s_0)$;*

(c) *if RH holds, $\liminf (\log X) B_X(s_0) \geq \frac{\kappa_1}{\pi} \operatorname{Re} \frac{\xi'}{\xi}(s_0)$, and the excess is $\sum_\gamma \big(\kappa_{m(\gamma)} - m(\gamma)\kappa_1\big) P_{s_0}(\gamma)$, which vanishes only if all zeros are simple.*

Here $\xi(s) = \frac12 s(s-1)\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$, and $\operatorname{Re}(\xi'/\xi)(s_0) = \sum_\rho \operatorname{Re}\frac{1}{s_0-\rho}$ is an explicit number; at $s_0 = 2$, $(\kappa_1/\pi)\operatorname{Re}(\xi'/\xi)(2) = 0.040186\ldots$. Both RH and the simple-zeros conjecture are thus encoded in a single limit at a single point, whose value is known in advance.

Remark 4 (a zero detector)*.* Assume RH. As a function of s_0, Ω is the Poisson integral of the positive measure $\sum_\gamma \kappa_{m(\gamma)}\delta_\gamma$ on the critical line; it is harmonic in $\operatorname{Re} s > \frac12$, and its boundary measure recovers every ordinate together with its multiplicity (through κ_m, which is strictly increasing). The Euler–Caceres bound, rescaled by $\log X$, converges to this harmonic function.

The rate of convergence in Theorem 2 is governed by the shape of the weight. Let W, W' be independent random variables with density g_0, so that $E|W - W'|$ (the mean difference of g_0) measures how sharply E_X^* cuts the primes at X, and let $\hat g_0(y) = \int g_0(w)e^{iyw}\,dw$ (we use this convention throughout, except in Theorem 36, where $\widehat{F}_X$ is the boundary value of a Laplace transform, with e^{-itu}; for the even functions of Sections 6 and 7 the sign is immaterial). For a single ordinate γ of multiplicity m, let

$$H_{X,\gamma} = \int h\big(m\,c_X(t-\gamma)\big) P_{s_0}(t)\,dt, \qquad h(x) = e^{2x} - 1 - 2x,$$

be the cost of its well alone, where c_X is the well profile (4), and let

$$\mathcal{I}_X(s_0) = \tfrac12 (\log X)^2 \Big(\int h(\phi_X) P_{s_0} - \sum_\gamma H_{X,\gamma} \Big)$$

be the interaction of the wells.

Theorem 5 (the second-order term)**.** *Assume RH and that g_0 is even. For every s_0 with $\operatorname{Re} s_0 > \frac{1}{2}$,*

$$(\log X)\, B_X(s_0) = \Omega(s_0) + \frac{\Omega_1(s_0) - \Omega(s_0)^2 + \mathcal{I}_X(s_0)}{\log X} + o\Big(\frac{1}{\log X}\Big),$$

$$\Omega_1(s_0) = \frac{1}{2}\sum_{\gamma} m(\gamma)^2 \Big[-\pi E|W - W'|\, P_{s_0}(\gamma) + \mathrm{PV}\int \frac{P_{s_0}(\gamma+y) - P_{s_0}(\gamma)}{y^2}\, |\hat{g}_0(y)|^2\, dy\Big],$$

the series converging absolutely, and $\mathcal{I}_X(s_0) = o(\log X)$.

Since we prove only $\mathcal{I}_X = o(\log X)$, the term $\mathcal{I}_X/\log X$ is only known to be $o(1)$, and it may be larger than the explicit part $(\Omega_1 - \Omega^2)/\log X$. Theorem 5 is therefore an exact decomposition of the error, in which the single wells are computed to second order (Proposition 18) and the interaction is isolated; it becomes a second-order asymptotic exactly when $\mathcal{I}_X = O(1)$. Section 8 shows that $\mathcal{I}_X$ is governed by the gaps between the zeros: it is $O(\log\log X)$ under a hypothesis on small gaps (Theorem 21), and numerically $O(1)$.

The leading part of Ω_1 is $-\frac{\pi}{2}E|W - W'|\sum m^2 P_{s_0}(\gamma)$: a smooth cut-off of the primes costs a deficit proportional to its width, the price of E_X^* being zero-free. The interaction $\mathcal{I}_X$ collects the overlaps of neighbouring wells; its phases depend on the zeros only through their differences $\gamma - \gamma'$, and it is damped, roughly, by $\hat{g}_0(\gamma - \gamma')$ (Section 6); at large height its quadratic part is related, by a heuristic derivation, to the pair correlation of the zeros (Section 7). A wider weight reduces the interaction and makes Ω_1 more negative; numerically, at $s_0 = 2$, $\mathcal{I}_X$ is a fraction of $|\Omega_1|$ that decreases with the width of the weight, from about 35% to about 4% in Table 6.

Theorem 6 (an attained off-line supremum makes the bound explode)**.** *Suppose $\Theta = \sup \operatorname{Re}\rho > \frac{1}{2}$ is attained. Then for every s_0 with $\operatorname{Re} s_0 > \Theta$ there are $r \in (0, \Theta - \frac{1}{2})$, C, X_0 with*

$$B_X(s_0) \;\geq\; \tfrac{1}{2}\operatorname{Ei}\big(r(\log X - a)\big) - C\log\log X \;\geq\; \frac{X^r}{3re^{ra}\log X} \qquad (X \geq X_0).$$

Theorem 6 covers only the case in which the supremum Θ is attained. Without attainment, a Laplace transform in the variable $\log X$ and Landau's theorem on Laplace integrals with an eventually one-signed integrand give the growth along a sequence (Section 9.1).

Theorem 7 (the bound explodes along a sequence)**.** *Suppose $\Theta = \sup \operatorname{Re}\rho > \frac{1}{2}$, attained or not. Then for every s_0 with $\operatorname{Re} s_0 > 1$ and every $\varepsilon > 0$,*

$$\limsup_{X\to\infty} X^{-(\Theta - \frac{1}{2} - \varepsilon)} B_X(s_0) = +\infty; \qquad \textit{in particular} \quad \limsup_{X\to\infty} \frac{\log B_X(s_0)}{\log X} \;\geq\; \Theta - \tfrac{1}{2}.$$

Corollary 8 (the bound at one point)**.** *Let $\operatorname{Re} s_0 > 1$. Then RH $\iff$ $B_X(s_0) \ll_\varepsilon X^\varepsilon$ for every $\varepsilon > 0$ $\iff$ $\limsup_{X\to\infty} \log B_X(s_0)/\log X \leq 0$.*

Corollary 9 (one mean square)**.** *Let $\mathcal{E}(X) = \int |\zeta/E_X^*(\frac{1}{2} + it)|^2\, dt/(1+t^2)$, finite for every X. If RH holds, $\mathcal{E}(X) \to \pi$. Unconditionally, RH $\iff$ $\limsup_X \mathcal{E}(X) < \infty$. If the supremum Θ of the real parts of the zeros is attained (this is automatic under RH, where $\Theta = \frac{1}{2}$), then also RH $\iff$ $\liminf_X \mathcal{E}(X) < \infty$.*

1.4. **What is new, and what is not.** The factorization is the hybrid Euler–Hadamard product of [GHK] in an exact form, and the profile of Z_X on the line under RH is in [GHK]. Theorem 1 and Corollary 3(a) are Jensen-type reformulations: the off-line term does not depend on X. The new content is Theorem 2: under RH the gap J_X is asymptotically $\Omega(s_0)/\log X$ with the universal constants κ_m, which come from the profile $e^{m\,\mathrm{Ci}(|v|)}$, $v = (t-\gamma)\log X$, of $|Z_X|$ at a zero of multiplicity m (Lemma 13). This gives the rate in (a) and the simple-zero criterion (b), and Theorems 6 and 7 give the growth on the other side. Theorem 7 uses no attainment hypothesis: it is Landau's argument for $\psi(x) - x = \Omega_\pm(x^{\Theta-\varepsilon})$ [I, Ch. V], applied to $\log|Z_X(s)|$ as a function of $\log X$, whose

Laplace transform has a logarithmic singularity at $\rho - s$ for every zero ρ (Lemma 23); it makes the bound $B_X(s_0) \ll X^{\varepsilon}$ at one point an equivalent of RH (Corollary 8). Theorem 5 identifies the next term up to the interaction of the wells, which we bound only by $o(\log X)$; Section 8 shows that the interaction is a sum over pairs of zeros of a two-well function, whose excess at coincidence is $\kappa_2 - 2\kappa_1$, and that the rate in Theorem 2 is controlled by the gaps between the zeros (Theorem 21). The closest known analogue of Theorem 2 is the Nyman–Beurling approximation problem: Burnol [Bu] proved $\liminf_N (\log N) d_N^2 \geq \sum m(\rho)^2/|\rho|^2$, the sum over the distinct zeros ρ on the critical line, for the Nyman–Beurling distance d_N, and Bettin, Conrey and Farmer [BCF] showed, under RH and a bound for $\sum_{|\gamma|<T} |\zeta'(\rho)|^{-2}$, that this lower bound is the limit; for simple zeros it is $2 + \gamma_E - \log 4\pi = 2\lambda_1$, the constant of (11) and of Corollary 34. Both problems decay like $1/\log$ with a constant that is a sum over the zeros; there a zero of multiplicity m is weighted by m^2, here by κ_m, and there the approximants are Dirichlet polynomials, here the reciprocal of a smoothed Euler factor. We do not prove that $(\log X) B_X(s_0)$ is bounded; by Corollary 3 this is equivalent to RH. Theorem 26 is elementary: it combines Theorem 1 with the verification of RH up to height $3 \cdot 10^{12}$ [PT], Trudgian's explicit form of $N(T)$ [Tr] and the trivial size of the Euler factor on the critical line, and we include it because it says how far the bound is controlled without RH. Section 7 relates the quadratic part of the interaction of the wells to Montgomery's pair correlation by a derivation that we do not make rigorous, and tests it numerically. Proposition 51 is an elementary rearrangement of Chebyshev's identity in a Beurling system; we include it, with a computation on a system of the type of Broucke, Debruyne and Révész [BDR], because it shows which information about the integers the identity at a zero can use. Section 12.1 looks at the factorization through the cross-energy C_2 of [CE1]: the radius of the helix of C_2 on the critical line is $|E_X^*||Z_X|$, and Z_X is the field of the charges of $\log|\zeta|$, which can be computed from the partial sums alone (Proposition 31); this is a reorganization of Lemma 10, not a new estimate. Section 13 recalls from [CE5, Sect. 2] the explicit formula in the coordinates of the cross-energy, a reparametrization of the ordinate, and uses them because they show that the law of the bound at $s_0 = 1$ and Cramér's mean square of the prime error term are sums over the zeros of one kernel (Corollary 34). Under RH, that kernel gives an isometry, up to a remainder of lower order (Theorem 36), between $\log Z_X$ on the line and the error term of the prime number theorem beyond X, and with it the comparison of the bound with the energy of the primes (Corollary 37); we also record the standard energy criterion for RH in the form we need (Proposition 35). The reflected cross-energy, which pairs s with $1 - s$ and separates the controls of Section 14.5, is studied in [CE3]. Theorem 6, Corollary 9 (in its lim inf form, under attainment), Theorem 1 in a weaker form, and the computations of Tables 2 and 7, first appeared in the preprint [C2], which this paper supersedes; the factorization of Lemma 10 and the material of Section 12.1 first appeared in version 3 of [CP].

1.5. **Place in the book.** This is Part 18 of the series, in block IV (Weil positivity). It builds on Part 1 [CE1] (the C-transform and Theorem A), Part 5 [CE5] (the explicit formula in the coordinates of the cross-energy), Part 8 [CE8] (the Euler product inside the critical strip) and Part 46 [CE46] (the Davenport–Heilbronn and Beurling controls). Its fixed-interval computation of Weil's form is the comparison point of Part 19 [CE19]; its Jensen moment and Euler-product mollifier are cited in Part 17 [CE17], and its floor $\frac{1}{2}\log(1 + c_0) = 0.0386$ for mollifiers is used in Part 33 [CE33] (the de Bruijn–Newman bound). It also uses Part 3 [CE3]. It is also used by Parts 19, 33, 41 and 46 [CE19, CE33, CE41, CE46].

1.6. **Organization.** Section 2 proves the factorization and the bounds for its kernel; Sections 3–5 prove the decomposition, the law and the criteria; Section 6 gives the second-order term, Section 7 its heuristic relation, tested numerically, to pair correlation, and Section 8 the two-well function and the rate under a gap hypothesis. Section 9 proves the explosion theorem and, in Section 9.1, its form without attainment, Section 10 the unconditional bounds, and Sections 11 and 12 report the computations. Section 12.1 shows the factorization through the cross-energy C_2 of consecutive

partial sums of ζ [CE1]: the radius of its helix on the critical line factorizes as $|E_X^*||Z_X|$, only the zeros near the height t matter, and Z_X is recovered from C_2 as the field of the charges of $\log|\zeta|$. Section 13 recalls from [CE5] the explicit formula in the coordinates of C_2, whose limit on the critical line, $1/(\frac14+t^2)$, is 2π times the Poisson kernel P_1 of the Jensen moment. Under RH, the mean square of $\log Z_X$ on the critical line with the weight P_1 equals, up to $O((\log X)^{-3/2})$, the energy $\mathcal{E}_\Delta(X)$ of the normalized prime error $\Delta(u)=e^{-u/2}(\psi(e^u)-e^u)$ beyond $u=\log X$ with the weight u^{-2} (Theorem 36); and, with Theorem 2 and Proposition 35, if all zeros are simple, $B_X(1)/\mathcal{E}_\Delta(X)\to\kappa_1/2\pi$, and $(\log X)B_X(1)$ tends to $\kappa_1/2\pi$ times Cramér's mean square of Δ. Section 14 asks what a proof of the bound would need. Over $\mathbb{F}_q[T]$, where RH is Weil's theorem, the same construction obeys the same law, and the bound is equivalent to the square-root estimate for the prime sums of each degree (Proposition 40); over $\mathbb{Q}$ the corresponding quadratic form in the primes is not absolutely convergent (Lemma 43), and, when Θ is attained, the abscissa of bounded energy of ζ/E_X^* on vertical lines is Θ (Proposition 46). Chebyshev's identity $\log=\Lambda*1$ evaluated at a zero removes the twist from the Euler–Maclaurin and sawtooth terms, and at scales below $|\gamma|/2\pi$ its kernel is approximately the dual sum of the functional equation; Selberg's formula behaves in the same way. In any Beurling system whose integers are better than $\sqrt{x}$ the identity at a zero on the line only restates the prime error (Proposition 51), and a system of the type of Broucke, Debruyne and Révész, which has every zero of ζ and a real zero at 0.6, satisfies it at ρ_1 with a remainder of order $N^{-0.4}$; with the Davenport–Heilbronn function this suggests that a proof must couple the functional equation with the Euler product. Weil's quadratic form, computed from the pole, the Gamma factor and the primes on test functions supported in $[-\frac{\ell}{2},\frac{\ell}{2}]$, $\ell\le5$, shows the pole and Gamma terms alone failing positivity for $\ell\gtrsim0.75$, somewhat beyond the entry of the prime 2, and, with the primes, near-null directions consistent with those found by Connes and Consani. Positivity of Λ alone pays the pole (Remark 45), and the reflection asymmetry of Z_X, built from the primes and the Gamma factor alone, sees every zero through one kernel and does not separate ζ from the controls (Figure 8). Apart from the results stated there (Lemmas 41 and 43, Propositions 48 and 51, Corollary 49, and Propositions 40 and 46), the conclusions of Section 14 about what a proof would need are heuristic or numerical. Section 15 concludes and Section 16 lists open problems.

2. The factorization

We use $E_1(z)=\int_z^\infty e^{-\tau}\tau^{-1}d\tau$ on $|\arg z|<\pi$, $E_1(z)=-\gamma_E-\log z+\mathrm{Ein}(z)$ with Ein entire and $\gamma_E=0.5772\ldots$ Euler's constant, and $E_1(z)=e^{-z}\varphi(z)$, $\varphi(z)=\int_0^\infty e^{-\tau}(z+\tau)^{-1}d\tau$. With $d(z)=\mathrm{dist}(z,(-\infty,0])$,

$$|\varphi^{(k)}(z)|\le k!\,d(z)^{-k-1}\qquad(k\ge0). \tag{3}$$

Across the cut K_X jumps by $2\pi i\int g_X=2\pi i$, so $e^{\pm K_X}$ is single-valued and $\mathrm{Re}\,K_X$ continuous. On the imaginary axis $\mathrm{Re}\,E_1(\pm iy)=-\,\mathrm{Ci}(|y|)$, so

$$-\,\mathrm{Re}\,K_X(iy)=c_X(y):=\int g_X(u)\,\mathrm{Ci}(|y|u)\,du. \tag{4}$$

Lemma 10 (the factorization; cf. [GHK, Thm. 1])**.** *$\zeta=E_X^*Z_X$ as meromorphic functions on $\mathbb{C}$, the product converging absolutely and locally uniformly, and $\log|Z_X(s)|=-\sum_\rho\mathrm{Re}\,K_X(s-\rho)$ (with multiplicity) for every s that is not a zero.*

Proof. For $x>1$ not a prime power, the truncated explicit formula with the term $\frac12\log(1-x^{-2})$ of $\psi(x)$ kept exactly gives $\sum_{n\le x}\Lambda(n)n^{-s}/\log n+E_1((s-1)\log x)=\log\zeta(s)+\sum_\rho E_1((s-\rho)\log x)+\sum_k E_1((s+2k)\log x)$ for $t\ne0$ and s off the horizontal cuts (this is the formula for $\sum_{n\le x}\Lambda(n)n^{-s}$ [T, (14.20)] integrated in s, as in [GHK, Thm. 1]); multiply by $g_X(\log x)$ and integrate in x. Equivalently, and without any conditionally convergent sum, the s-derivative of the identity follows

by Mellin inversion of $\sum_n \Lambda(n)\psi_X(\log n)n^{-s}$ against $-\zeta'/\zeta$ with the smooth weight ψ_X, and one integrates in s from $+\infty$: the transform of ψ_X decays faster than any power on vertical lines, the contour is moved to the left across the pole, the zeros and the trivial zeros, and the sum over the zeros converges absolutely. The symmetric partial sums over $|\gamma| < T$ of the truncated explicit formula are bounded uniformly in T and in $x \in [e^{L-a}, e^{L+a}]$, for s in a compact set off the zeros and the cuts, and they converge for every x that is not a prime power [D, Ch. 17]; the prime powers form a set of measure zero, so dominated convergence justifies integrating the formula term by term against $g_X(\log x)$. Each factor $e^{-K_X(s-\rho)}$ is entire with a simple zero at ρ, since $e^{-E_1(z)} = ze^{\gamma_E}e^{-\operatorname{Ein}(z)}$. Absolute local uniform convergence follows from Lemma 12(L1)–(L3), and the identity extends to $\mathbb{C}$ by analytic continuation. □

Lemma 11 (P is outer)**.** *Let $P(s) = (s-1)E_X^*(s)/s$. Then $\log|P|$ is continuous and bounded on $\operatorname{Re} s \ge \frac12$, P is outer in $\operatorname{Re} s > \frac12$, and $Z_X = F/P$ lies in the Smirnov class N^+.*

Proof. $\log|P(s)| = -\log|s| + \big(\log|s-1| + \operatorname{Re} K_X(s-1)\big) + \operatorname{Re} D(s) - \sum_k \operatorname{Re} K_X(s+2k)$, D the finite prime sum, $|\operatorname{Re} D| \le \sum \Lambda(n)n^{-1/2}/\log n$. Since $\log|z| + \operatorname{Re} E_1(z) = -\gamma_E + \operatorname{Re}\operatorname{Ein}(z)$, the bracket is continuous, and bounded on $|s-1| \le R$. For $|s-1| \ge R$, $z = (s-1)u$ has either $\operatorname{Re} z \ge 0$ and $d(z) = |z|$, or $|\operatorname{Im} z| \ge u\sqrt{R^2-1}$, so $|E_1(z)| \le e^{u/2}/d(z)$ by (3) and $K_X(s-1)$ is bounded, by $e^{(L+a)/2}/((L-a)\sqrt{R^2-1})$, as is $\log|(s-1)/s|$; and $|E_1(z)| \le e^{-\operatorname{Re} z}/|z|$ for $\operatorname{Re} z > 0$ bounds the trivial-zero terms. A zero-free analytic function whose $\log|\cdot|$ is bounded, harmonic, and continuous up to the line is the Poisson integral of its boundary values, i.e. outer. By [BSY] F is a Blaschke product times an outer function (no singular inner factor, also at ∞, since $F(\sigma) \to 1$); hence $Z_X \in N^+$. The bounds depend on X: $|\operatorname{Re} D| \le \mathcal{M}(e^a X) \sim 2\sqrt{e^a X}/\log X$ in the notation of Theorem 26, and $K_X(s-1)$ is of the same order on the line; this is the size used in Theorem 26(b). □

Lemma 12 (the kernel)**.** *Let $w = -x + iy$, $|x| \le x_{\max}$. For every integer $A \ge 0$ there is C depending on A, $x_{\max}$, g_0 only, such that:*

- (L1) *if $\operatorname{Re} w \ge \eta > 0$, then $|K_X(w)| \le C_\eta X^{-\operatorname{Re} w}(1+|y|)^{-A-1}/L$;*
- (L2) *if $\operatorname{Re} w \ge 0$, $w \ne 0$, then $|K_X(w)| \le C(1+|\log(|w|L)|)$, and if moreover $|y| \ge 1$, $|K_X(w)| \le C|y|^{-A-1}/L$;*
- (L3) *if $x > 0$, $y \ne 0$, then $|K_X(w)| \le e^{x(L+a)}/(|y|(L-a))$, and if moreover $|y| \ge 1$, $|K_X(w)| \le Ce^{xa}X^x|y|^{-A-1}/L$;*
- (L4) *for $x > 0$, $-\operatorname{Re} K_X(-x) = \int g_X(u)\operatorname{Ei}(xu)\,du \ge \operatorname{Ei}\big(x(L-a)\big)$.*

Proof. $K_X(w) = e^{-wL}\int g_0(v)e^{-wv}\varphi(w(v+L))\,dv$; put $u = v + L \in [L-a, L+a]$, and note $d(wu) = u\,d(w)$ with $d(w) = |w|$ if $x \le 0$, $d(w) = |y|$ if $x > 0$. Without integration by parts, (3) gives $|K_X(w)| \le e^{x(L+a)}/(d(w)(L-a))$, the first part of (L3). Integrating by parts A times against e^{-wv}, $K_X(w) = w^{-A}e^{-wL}\int e^{-wv}\partial_v^A[g_0(v)\varphi(w(v+L))]\,dv$, and the m-th derivative of $\varphi(w(v+L))$ is at most $m!(|w|/d(w))^m/(u\,d(w))$, with $|w|/d(w) \le 1 + x_{\max}$ in all cases used. Hence $|K_X(w)| \le Ce^{x(L+a)}|w|^{-A}/(d(w)(L-a))$, which gives the second parts of (L2), (L3) and (L1). For the first part of (L2), $|E_1(wu)| \le |\varphi(wu)| \le 1$ if $|wu| \ge 1$, and $|E_1(wu)| \le \gamma_E + |\log|wu|| + \pi + e$ if $|wu| < 1$. For (L4), $E_1(-\xi \pm i0) = -\operatorname{Ei}(\xi) \mp i\pi$ and Ei increases. □

By (4) and (L2), for all real $y \ne 0$,

$$\begin{aligned} c_X(y) &\le \operatorname{Ci}(\tfrac{\pi}{2}) = 0.4720, \qquad |c_X(y)| \le C\big(1 + |\log(|y|L)|\big), \\ |c_X(y)| &\le \frac{2}{|y|(L-a)}, \qquad |c_X(y)| \le \frac{C}{|y|^3 L} \quad (|y| \ge 1), \end{aligned} \tag{5}$$

the first because $g_X \ge 0$, $\int g_X = 1$, the third from $|\operatorname{Ci}(x)| \le 2/x$ (indeed $\le 1/x$).

3. The decomposition

Proof of Theorem 1. The hypotheses of the Poisson–Jensen formula on $\operatorname{Re} s > \frac{1}{2}$ are as follows. $F = (s-1)\zeta(s)/s$ is analytic on $\operatorname{Re} s \geq \frac{1}{2}$ and of bounded type there: $F(s)/(s+1)$ is bounded on $\operatorname{Re} s > \frac{1}{2}$ by the convexity bound. Its zeros satisfy the Blaschke condition, $\log|F(\frac{1}{2}+it)|$ is integrable against P_{s_0}, and F has no singular inner factor, neither on the line nor at infinity [BSY]. The Poisson–Jensen formula at any centre s_0 then follows from this canonical factorization. $P = (s-1)E_X^*/s$ is outer, with $\log|P|$ bounded and continuous up to the line (Lemma 11). Finally $F(s_0) \neq 0$ because $\zeta(s_0) \neq 0$ (and $F(1) = 1$). So $\log|F|$ and $\log|P|$ satisfy the Poisson–Jensen formula at s_0 without singular part, $\log|P|$ without zeros; subtracting,

$$\int \phi_X \, P_{s_0} = \log|Z_X(s_0)| + \mathcal{O}_{\text{off}}(s_0).$$

Substituting $\log|Z_X(s_0)| = \int \phi_X P_{s_0} - \mathcal{O}_{\text{off}}(s_0)$ in the definition of B_X gives $B_X = \mathcal{O}_{\text{off}} + J_X$, and $J_X \geq 0$ is Jensen's inequality, with equality only if ϕ_X is constant almost everywhere. Since $\phi_X(t) \to -\infty$ as $t \to \gamma_1 = 14.1347\ldots$ (the first zero lies on the line), $J_X > 0$. □

Since $|F| = |\zeta|$ on the critical line, the Poisson–Jensen formula for F used in the proof reads, at a general centre,

$$\int \log|\zeta(\tfrac{1}{2}+it)| \, P_{s_0}(t)\, dt = \log\Big|\frac{(s_0-1)\zeta(s_0)}{s_0}\Big| + \mathcal{O}_{\text{off}}(s_0), \tag{6}$$

which is (1) for $s_0 = 1$. Moving the centre to $s_0 = \sigma_0 + it_0$ concentrates the sensitivity near the height t_0: an off-line zero at height close to t_0 contributes of order $(\beta - \frac{1}{2})(\sigma_0 - \frac{1}{2})/|s_0 - \rho|^2$ instead of $(\beta - \frac{1}{2})/|\rho|^2$. Table 2 checks (6) numerically (script `mollifier_tables.py`), together with its analogue for the Davenport–Heilbronn function f (Section 11), which has no pole and whose off-line term we compute from its eight off-line zeros below height 400; we do not prove the Poisson–Jensen formula without singular part for f, and the agreement in the last row is numerical.

function	s_0	line mean	interior (and pole)	excess
ζ	$2 + 20i$	-0.03463	-0.03408	-0.0006
ζ	$1.5 + 100i$	0.27150	0.27144	$+0.0001$
ζ	$1 + 85.7i$	0.13177	0.13158	$+0.0002$
f	$1 + 85.7i$	-0.05385	-1.49465	$+1.4408$ (off-line zeros: $+1.4407$)

Table 2. The identity (6) at three centres, with the window $|t - t_0| < 4000$ and step 0.002; the remaining kernel mass is below $3 \cdot 10^{-4}$, and for ζ the excess, which should be $\mathcal{O}_{\text{off}}(s_0) < 10^{-11}$ (Theorem 26), is of the order of the truncation error. For f the interior term is $\log|f(s_0)|$, and the excess agrees with the off-line Blaschke term of its zeros.

Under RH, $\mathcal{O}_{\text{off}} = 0$, Z_X is outer, and

$$\ell_X := \int \phi_X \, P_{s_0} = \log|Z_X(s_0)|, \qquad |\ell_X| \leq C_{s_0} \frac{X^{-(\sigma_0 - \frac{1}{2})}}{L} \tag{7}$$

by (L1), since $\operatorname{Re}(s_0 - \rho) = \sigma_0 - \frac{1}{2}$ for every zero and $\sum_\rho (1 + |t_0 - \gamma|)^{-3} < \infty$. Writing $h(x) = e^{2x} - 1 - 2x \geq 0$ and $H_X = \int h(\phi_X) P_{s_0}$, we get $\int e^{2\phi_X} P_{s_0} = 1 + H_X + 2\ell_X$ and

$$B_X(s_0) = \tfrac{1}{2}\log(1 + H_X + 2\ell_X) - \ell_X. \tag{8}$$

So under RH everything is in H_X, an integral of a nonnegative function.

4. Proof of the law

Lemma 13 (profile at a zero; cf. [GHK, Sect. 3])**.** *Assume RH and let $\frac12+i\gamma$ be a zero of multiplicity m. Then for every $v\neq 0$, $\phi_X(\gamma+v/\log X)\to m\,\mathrm{Ci}(|v|)$ as $X\to\infty$.*

Proof. By Lemma 10 and (4), under RH $\phi_X(t)=\sum_\rho c_X(t-\gamma_\rho)$ for t not an ordinate. The m terms of γ give $m\,c_X(v/L)=m\int g_0(w)\,\mathrm{Ci}(|v|(1+w/L))\,dw\to m\,\mathrm{Ci}(|v|)$. The other ordinates are at distance $\geq\delta>0$ from γ; by (5) those within distance 1 contribute $O(1/L)$ and the rest $O(\log(|\gamma|+3)/L)$. □

Lemma 14 (uniform bounds under RH)**.** *Assume RH. (i) For every $\varepsilon>0$ there are X_ε, C_ε with $\phi_X(t)\leq\varepsilon\log(|t|+3)+C_\varepsilon$ for all t and $X\geq X_\varepsilon$. (ii) For every interval I of length 1 at distance $\leq n$ from 0, $\int_I\phi_X^2\leq C\log^2(n+3)/L$.*

Proof. (i) Under RH, $S(T)\ll\log T/\log\log T$ [T, Thm. 14.13], so an interval of length $h\leq1$ centred at t contains at most $\frac{h}{2\pi}\log(|t|+3)+C\log(|t|+3)/\log\log(|t|+3)$ ordinates. By (5), the ordinates with $|t-\gamma|<1/L$ contribute at most $0.472\big(\frac{1}{\pi L}+\frac{C}{\log\log(|t|+3)}\big)\log(|t|+3)$, those on the dyadic shells $2^k/L\leq|t-\gamma|<\min(1,2^{k+1}/L)$ contribute $\ll\big(\frac{\log L}{L}+\frac{1}{\log\log(|t|+3)}\big)\log(|t|+3)$, and those with $|t-\gamma|\geq1$ contribute $\ll\log(|t|+3)/L$. Treating $|t|\leq T_\varepsilon$ separately gives (i). (ii) For $t\in I$ split $\phi_X=\phi'+\phi''$, with ϕ' the sum over the $N\ll\log(n+3)$ ordinates (with multiplicity) within distance 1 of I. By (5), $|\phi''|\leq C\log(n+3)/L$, and $\int_{\mathbb R}c_X^2\leq C/L$ (split at $|y|=1/L$). By Cauchy–Schwarz, $\int_I\phi'^2\leq N\sum\int_{\mathbb R}c_X^2\leq CN^2/L$. □

Numerically, with the Gaussian weight of Table 7 and step 0.01, the maximum of $\phi_X(t)/\log(t+2)$ over $0<t<600$ is 0.189, 0.169, 0.170 for $X=10^2,10^3,10^4$, and the same maximum over $200\leq t<600$ decreases, 0.179, 0.163, 0.145 (script `mollifier_tables.py`), consistent with Lemma 14(i).

Proof of Theorem 2. By (7) and (8), $L\,B_X=\frac12LH_X+O\big(L(H_X+|\ell_X|)^2\big)+O(L|\ell_X|)$, so it suffices to prove $\frac12LH_X\to\Omega(s_0)$ (which also gives $H_X\to0$).

We use $h(x)\leq2x^2e^{2\max(x,0)}$ and $h(x)\leq C_Mx^2$ for $x\leq M$, and $P_{s_0}(t)\leq C/(1+t^2)$.

Tail. Fix $\varepsilon=\frac14$. For $X\geq X_\varepsilon$, Lemma 14 gives on each unit interval I at distance n from 0: $L\int_I h(\phi_X)\leq2C(n+3)^{1/2}L\int_I\phi_X^2\leq C'(n+3)^{1/2}\log^2(n+3)$. Hence

$$L\int_{|t|>T}h(\phi_X)P_{s_0}\leq C''\sum_{n\geq T}\frac{\log^2(n+3)}{(n+1)^{3/2}}=:\eta(T)\xrightarrow[T\to\infty]{}0,$$

uniformly in $X\geq X_\varepsilon$.

Bounded part. Fix T such that $\pm T$ are not ordinates. Let $\gamma_1,\dots,\gamma_k$ be the distinct ordinates in $(-T,T)$, with multiplicities m_j, and let $\delta\in(0,1]$ be at most the distance between any two distinct ordinates in $[-T-1,T+1]$ and the distance from $\pm T$ to the ordinates. Let $U_j=(\gamma_j-\delta/2,\gamma_j+\delta/2)$. For $t\in U_j$, write $\phi_X(t)=m_jc_X(t-\gamma_j)+r_X(t)$: every other ordinate is at distance $\geq\delta/2$ from t, so by (5) $|r_X(t)|\leq C_T/L$. For $t\in(-T,T)\setminus\bigcup U_j$ the same bounds give $|\phi_X(t)|\leq C_T/L$, hence $L\int h(\phi_X)P_{s_0}\leq C_T^2/L\to0$ there. On U_j substitute $t=\gamma_j+v/L$:

$$L\int_{U_j}h(\phi_X)P_{s_0}\,dt=\int_{|v|<\delta L/2}h\big(m_jc_X(v/L)+r_X\big)P_{s_0}(\gamma_j+v/L)\,dv.$$

The integrand tends to $h(m_j\,\mathrm{Ci}(|v|))P_{s_0}(\gamma_j)$ for every $v\neq0$ (Lemma 13). For domination we use $h(x)\leq C_M(1+|x|)$ and $h(x)\leq C_Mx^2$ for $x\leq M$ (note $0\leq h(x)<2|x|$ for $x<0$). If $|v|\leq1$, $m_jc_X(v/L)+r_X\leq M$ and $|m_jc_X(v/L)|\leq C(1+|\log|v||)$, so the integrand is at most $C(1+|\log|v||)$. If $|v|\geq1$, $|m_jc_X(v/L)|\leq3m_j/|v|$, and $h(x)\leq C_Mx^2\leq2C_M(9m_j^2/v^2)+2C_Mr_X^2$; the second piece is at most $2C_MC_T^2/L^2$, whose integral over $|v|<\delta L/2$ is $O(1/L)$, and we split it

off. Dominated convergence applies to the rest, with the integrable majorant $C(1+|\log|v||)\mathbf{1}_{|v|\le 1}+18C_M m_j^2 v^{-2}\mathbf{1}_{|v|>1}$ times $\max P_{s_0}$. Hence

$$\lim_{X\to\infty} L\int_{U_j} h(\phi_X)P_{s_0} = P_{s_0}(\gamma_j)\int_{\mathbb{R}} h(m_j\,\mathrm{Ci}(|v|))\,dv = 2\kappa_{m_j}P_{s_0}(\gamma_j).$$

Conclusion. Let $\Omega_T=\sum_{|\gamma|<T}\kappa_{m(\gamma)}P_{s_0}(\gamma)$. We have shown $\limsup|\frac12 LH_X-\Omega_T|\le\frac12\eta(T)$. In particular $\Omega_T\le\frac12\liminf LH_X+\frac12\eta(T)$ is bounded in T, so $\Omega=\lim_T\Omega_T$ converges, and letting $T\to\infty$ gives $\frac12 LH_X\to\Omega(s_0)$. □

Where uniformity enters. The proof sums the local contributions of the zeros in two regimes, and the interactions between wells are controlled in both. (1) For $|t|>T$ no local analysis is used: the bound $\eta(T)$ comes from Lemma 14, which holds uniformly in $X\ge X_\varepsilon$ and uses only the counting function of the zeros under RH ($S(T)\ll\log T/\log\log T$); it bounds the total, wells and interactions together. (2) For $|t|<T$ there are finitely many wells, separated by at least δ. On each U_j the other wells enter only through r_X, with $|r_X|\le C_T/L$, and in the domination this remainder is either absorbed in the majorant ($|v|\le 1$) or contributes $O(1/L)$ ($|v|\ge 1$); between the U_j the integrand is $O(C_T^2/L^2)$, so these regions contribute $O(C_T^2/L)$ after multiplication by L. So the interactions change $\frac12 LH_X$ by $O_T(1/L)+\frac12\eta(T)$ and do not alter the limit. For explicit constants, Littlewood's bound may be replaced by $|S(T)|\le(\frac14+o(1))\log T/\log\log T$ under RH [CChM]. Every constant in the proof depends continuously on s_0, so the convergence in Theorem 2 is locally uniform in $\mathrm{Re}\,s_0>\frac12$; in particular the harmonic function Ω of Remark 4 is the locally uniform limit of $(\log X)B_X$. The constant C_T depends on the smallest gap δ between the ordinates below T; Section 8 shows that this dependence is genuine. The same interactions are what Theorem 5 isolates at the next order, where we can only show that they are $o(\log X)$.

5. The constants κ_m and the criteria

Lemma 15. *κ_m/m is strictly increasing in m; in particular $\kappa_m>m\kappa_1$ for $m\ge 2$.*

Proof. For fixed $x\neq 0$ let $f(\mu)=(e^{2\mu x}-1-2\mu x)/\mu$, $\mu>0$. Then $\mu^2 f'(\mu)=(2\mu x-1)e^{2\mu x}+1>0$, since $(u-1)e^u+1>0$ for $u\neq 0$. Integrate against dv with $x=\mathrm{Ci}(v)$, which vanishes only on a null set. □

Lemma 16. *$\sum_\rho \mathrm{Re}\,\frac{1}{s_0-\rho}=\mathrm{Re}\,\frac{\xi'}{\xi}(s_0)$ for $\mathrm{Re}\,s_0>\frac12$ with $\zeta(s_0)\neq 0$, the sum absolutely convergent and counting multiplicity. Under RH, $\mathrm{Re}\,\frac{1}{s_0-\rho}=\pi P_{s_0}(\gamma)$, so $\sum_\rho P_{s_0}(\gamma)=\varpi(s_0)$, where $\varpi(s_0):=\frac1\pi\mathrm{Re}\,\frac{\xi'}{\xi}(s_0)$.*

Proof. Hadamard's product gives $\xi'/\xi(s)=B+\sum_\rho\left(\frac{1}{s-\rho}+\frac1\rho\right)$ with $\mathrm{Re}\,B=-\sum_\rho\mathrm{Re}\,\frac1\rho$ [D, Ch. 12]. Under RH, $s_0-\rho=(\sigma_0-\frac12)+i(t_0-\gamma)$. □

Proof of Corollary 3. (a) If RH fails, $B_X(s_0)\ge\mathcal{O}_{\mathrm{off}}(s_0)>0$ for all X by Theorem 1. If RH holds, $B_X(s_0)\sim\Omega(s_0)/\log X$ by Theorem 2. (c) Under RH, by Lemma 16, $\frac{\kappa_1}{\pi}\mathrm{Re}\,\frac{\xi'}{\xi}(s_0)=\kappa_1\sum_\gamma m(\gamma)P_{s_0}(\gamma)$, so $\Omega(s_0)-\frac{\kappa_1}{\pi}\mathrm{Re}\,\frac{\xi'}{\xi}(s_0)=\sum_\gamma(\kappa_{m(\gamma)}-m(\gamma)\kappa_1)P_{s_0}(\gamma)$, a sum of nonnegative terms, each positive when $m(\gamma)\ge 2$ (Lemma 15; $P_{s_0}>0$). (b) If RH holds and all zeros are simple, $\Omega=\frac{\kappa_1}{\pi}\mathrm{Re}\,\frac{\xi'}{\xi}(s_0)$ by (c). Conversely, if $(\log X)B_X(s_0)$ has a finite limit, then $B_X\to 0$ and RH holds by (a), and equality of the limit with $\frac{\kappa_1}{\pi}\mathrm{Re}\,\frac{\xi'}{\xi}(s_0)$ forces all zeros simple by (c). □

Remark 17. Corollary 3(b) needs one point s_0, because $P_{s_0}(\gamma)>0$ for every γ: a single multiple zero anywhere raises the limit by $(\kappa_m-m\kappa_1)P_{s_0}(\gamma)$, for instance by $3.69\,P_{s_0}(\gamma)$ for a double zero. The test is quantitative, but the effect of a multiple zero at height γ is of size γ^{-2} at a fixed centre; moving s_0 to height γ makes it of order one.

6. The second-order term

Proposition 18 (one well to second order)**.** *Let g_0 be even, $s_0=\sigma_0+it_0$ with $\sigma_0>\frac12$, $\gamma\in\mathbb{R}$ and $m\ge1$. With $L=\log X$,*

$$L\,H_{X,\gamma}=2\kappa_m P_{s_0}(\gamma)+\frac{m^2}{L}\Big[-\pi E|W-W'|P_{s_0}(\gamma)+\mathrm{PV}\int\frac{P_{s_0}(\gamma+y)-P_{s_0}(\gamma)}{y^2}|\hat g_0(y)|^2dy\Big]+R_{X,\gamma},$$

with $|R_{X,\gamma}|\le C\,m^3e^{2m}(1+|\gamma-t_0|)^{-2}\,L^{-2}\log L$, where C depends on s_0 and g_0 only.

Proof. Write $P_\gamma(y)=P_{s_0}(\gamma+y)$, $c=c_X$, and $h(x)=2x^2+h_3(x)$. Constants C depend on s_0,g_0 only, and we use $\sup_{|y|\le1}(|P_\gamma|+|P_\gamma'|+|P_\gamma''|+|P_\gamma'''|)\le C(1+|\gamma-t_0|)^{-2}=:C\,p_\gamma$ and $\int_{|y|\ge1}P_\gamma(y)y^{-2}dy\le Cp_\gamma$.

(i) The quadratic part at the centre. Let $G_X(\xi)=\int_{u\le\xi}g_X(u)\,du$, which vanishes for $\xi<L-a$. From $\mathrm{Ci}(|y|u)=-\int_u^\infty\xi^{-1}\cos(y\xi)\,d\xi$ (an integral that converges uniformly for $u\in[L-a,L+a]$ when $y\ne0$), Fubini on $\xi\le R$ and $R\to\infty$, using $G_X=1$ for $\xi>L+a$,

$$c(y)=-\int_0^\infty\frac{G_X(\xi)}{\xi}\cos(y\xi)\,d\xi, \tag{9}$$

a cosine transform of the L^2 function $G_X(\xi)/\xi$. By Plancherel, $\int_{\mathbb{R}}c^2=\pi\int_0^\infty G_X(\xi)^2\xi^{-2}d\xi$. Now G_X^2 is the distribution function of $L+M$ with $M=\max(W,W')$, so $\int_0^\infty G_X^2\xi^{-2}d\xi=E[(L+M)^{-1}]=L^{-1}-E[M]L^{-2}+O(L^{-3})$, and $E[M]=\frac12E|W-W'|$ because g_0 is even. Hence

$$2m^2L\,P_\gamma(0)\int c^2=2\pi m^2P_\gamma(0)-\frac{\pi m^2E|W-W'|}{L}P_\gamma(0)+O\Big(\frac{m^2p_\gamma}{L^2}\Big).$$

(ii) The quadratic part off the centre. Let $\vartheta(y)=P_\gamma(y)-P_\gamma(0)-P_\gamma'(0)y\mathbf{1}_{|y|\le1}$; since c is even, $2m^2L\int c^2(P_\gamma-P_\gamma(0))=2m^2L\int c^2\vartheta$, and $|\vartheta(y)|\le Cp_\gamma\min(y^2,1)+P_\gamma(y)$. Integrating (9) by parts twice,

$$yL\,c(y)=\mathrm{Im}\,\big(e^{iyL}\hat g_0(y)\big)+\varrho(y),\qquad|\varrho(y)|\le\frac{C}{L}+\frac{C}{|y|L}\quad(|y|\ge1/L),$$

the first term coming from $L\int g_0(w)\sin(y(L+w))(L+w)^{-1}dw$ and ϱ from the factor $L/(L+w)$ and from $L\int G_X\xi^{-2}\sin(y\xi)\,d\xi$. Thus $(yLc)^2=\frac12|\hat g_0(y)|^2-\frac12\,\mathrm{Re}\,\big(e^{2iyL}\hat g_0(y)^2\big)+O(L^{-1}+(|y|L)^{-1})$ for $|y|\ge1/L$, and

$$2m^2L\int c^2\vartheta=\frac{2m^2}{L}\int\frac{(yLc)^2\vartheta(y)}{y^2}\,dy.$$

On $|y|<1/L$, $|c|\le C(1+|\log(|y|L)|)$ by (5) and $|\vartheta|\le Cp_\gamma y^2$, a contribution $O(m^2p_\gamma L^{-2})$. On $|y|\ge1/L$ the term $\frac12|\hat g_0|^2$ gives $\frac{m^2}{L}\,\mathrm{PV}\int|\hat g_0|^2(P_\gamma-P_\gamma(0))y^{-2}dy+O(m^2p_\gamma L^{-2})$ (the term $P_\gamma'(0)y$ is odd); the oscillating term is $O(m^2p_\gamma L^{-2})$ after one integration by parts in y; and the remainder is at most $\frac{Cm^2}{L}\int_{|y|\ge1/L}(L^{-1}+(|y|L)^{-1})|\vartheta|y^{-2}dy\le Cm^2p_\gamma L^{-2}\log L$.

(iii) The cubic part. Since $\int wg_0(w)\,dw=0$ and $\partial_\varepsilon\,\mathrm{Ci}(x(1+\varepsilon))=\cos(x(1+\varepsilon))/(1+\varepsilon)$, Taylor's formula gives $|c(v/L)-\mathrm{Ci}(|v|)|\le Ca^2(1+|v|)L^{-2}$. For $x\le M_m:=0.472m+1$, $|h_3(x)|\le C_m|x|^3$, $|h_3'(x)|\le C_mx^2$ if $|x|\le M_m$ and $|h_3'(x)|\le4|x|+2$ if $x<0$, with $C_m\le Ce^{2m}$. Hence, substituting $y=v/L$,

$$L\int h_3(mc)P_\gamma=\int_{|v|\le L}h_3(m\,\mathrm{Ci}(|v|))P_\gamma(v/L)\,dv+O\Big(\frac{m^3e^{2m}p_\gamma\log L}{L^2}\Big),$$

the region $|v|>L$ contributing $O(m^3L^{-2}\int_{|y|>1}|y|^{-3}P_\gamma(y)\,dy)$, and the difference of the profiles at most $Cm^3e^{2m}p_\gamma L^{-2}$ times $\int_{|v|\le1}(1+|\log|v||)^2dv+\int_{1\le|v|\le L}(1+|v|)v^{-2}dv=O(\log L)$, since $|\mathrm{Ci}(|v|)|\le\min(C(1+|\log|v||),1/|v|)$. Expanding $P_\gamma(v/L)$ to second order (the first-order term is odd), and using $|h_3(m\,\mathrm{Ci}(|v|))|\le Ce^{2m}m^3|v|^{-3}$ for $|v|\ge1$, the integral is $P_\gamma(0)\int_{\mathbb{R}}h_3(m\,\mathrm{Ci}(|v|))\,dv+O(m^3e^{2m}p_\gamma L^{-2}\log L)$.

(iv) The constant. $\int_{\mathbb{R}} 2m^2\,\mathrm{Ci}(|v|)^2dv = 4m^2\int_0^\infty \mathrm{Ci}^2 = 2\pi m^2$, so $2\pi m^2 + \int h_3(m\,\mathrm{Ci}(|v|))dv = \int h(m\,\mathrm{Ci}(|v|))\,dv = 2\kappa_m$. Adding (i)–(iii) proves the proposition. □

Proof of Theorem 5. Under RH, $m(\gamma) \le N(\gamma+\eta) - N(\gamma-\eta)$ for every $\eta > 0$, and Littlewood's bound for $S(T)$ gives $m(\gamma) \ll \log(|\gamma|+3)/\log\log(|\gamma|+3)$, so $m^3e^{2m} \ll_\varepsilon (|\gamma|+3)^\varepsilon$ and $\sum_\gamma m^3e^{2m}(1+|\gamma-t_0|)^{-2} < \infty$. Summing Proposition 18 over the distinct ordinates,

$$\tfrac12 L\sum_\gamma H_{X,\gamma} = \Omega(s_0) + \frac{\Omega_1(s_0)}{L} + O\Big(\frac{\log L}{L^2}\Big),$$

and by definition $\frac12 L\int h(\phi_X)P_{s_0} = \frac12 L\sum_\gamma H_{X,\gamma} + \mathcal{I}_X/L$. By Theorem 2 both $\frac12 LH_X$ and $\frac12 L\sum H_{X,\gamma}$ tend to Ω, so $\mathcal{I}_X = o(L)$ and $LH_X = 2\Omega + o(1)$. Finally (8) and (7) give $LB_X = \frac12 LH_X - \frac14 LH_X^2 + O(LH_X^3 + L|\ell_X|) = \frac12 LH_X - \Omega^2/L + o(1/L)$. □

Remark 19. The proof uses only $\int w g_0(w)\,dw = 0$, so "even" can be replaced by "of mean zero". For a weight with mean $\mu_1 \neq 0$, $E[M] = \mu_1 + \frac12 E|W - W'|$, and the first moment also enters through the profile in (iii); to first order it amounts to replacing $\log X$ by $\log X + \mu_1$. The interaction $\mathcal{I}_X$ is the only part of the second-order term that depends on more than one zero. By the identity $h(x+y) = h(x)+h(y)+(e^{2x}-1)(e^{2y}-1)$ its leading part is $\frac12 L^2\sum_{\gamma<\gamma'}\int f_\gamma f_{\gamma'}P_{s_0}$ with $f_\gamma = e^{2m(\gamma)c_X(\cdot-\gamma)} - 1$; the tails of f_γ oscillate like $e^{\pm i|t-\gamma|\log X}\hat g_0(t-\gamma)$, so the phases of $\mathcal{I}_X$ are $(\gamma-\gamma')\log X$, with a damping that is roughly $\hat g_0(\gamma-\gamma')$. Proving $\mathcal{I}_X = O(1)$, or $O(\log\log X)$, would need control of small gaps between zeros.

7. The interaction at large height: pair correlation

At a fixed centre the wells are far apart and their interaction is small. With the centre at height T and $X = (T/2\pi)^\alpha$ (which agrees with $X = T^\alpha$ up to a factor $1 + O(1/\log T)$ in α) the wells have width of order $1/(\alpha\log T)$ and the zeros have mean spacing $2\pi/\log(T/2\pi)$, so the interaction becomes a statistic of the spacings. Its quadratic part,

$$\mathcal{Q}_X(s_0) = \int 2\phi_X^2 P_{s_0} - \sum_\gamma m(\gamma)^2\int 2c_X(t-\gamma)^2P_{s_0}(t)\,dt = 2\sum_{\rho\neq\rho'}\int c_X(t-\gamma)c_X(t-\gamma')P_{s_0}(t)\,dt,$$

the sum running over ordered pairs of distinct zeros counted with multiplicity, is a pair sum, and we compare it with Montgomery's form factor

$$F(\beta) = \frac{2\pi}{T\log T}\sum_{0<\gamma,\gamma'\le T} T^{i\beta(\gamma-\gamma')}w(\gamma-\gamma'), \qquad w(u) = \frac{4}{4+u^2}.$$

Under RH, Montgomery proved $F(\beta) = \beta + o(1)$ for $0 < \beta < 1$ [M], and the pair-correlation conjecture is $F(\beta) = 1 + o(1)$ for $\beta \ge 1$. (Gonek, Hughes and Keating model the moments of Z_X by random matrix theory for $X \ll (\log T)^{2-\varepsilon}$ [GHK]; here $X = T^\alpha$ is much larger, and the wells are separated.)

Derivation. By (9) and Plancherel, $\int c_X(y)c_X(y-d)\,dy = \frac{1}{2\pi}\int|\hat c(\xi)|^2e^{i\xi d}d\xi$ with $|\hat c(\xi)|^2 = \pi^2G_X(|\xi|)^2/\xi^2$, a function supported on $|\xi| \ge L - a$. Averaging the centre t_0 over a long interval near T removes the kernel P_{s_0} from each pair, so the average of $\mathcal{Q}_X$ is $\frac1\pi\int|\hat c(\xi)|^2\,\frac1H\sum_{\rho\neq\rho'}e^{i\xi(\gamma-\gamma')}d\xi$, the pair sum being over a window of length H near T. In the normalization of F, the pair sum at frequency $\xi = \beta\ell$, $\ell = \log\frac{T}{2\pi}$, is $\frac{\ell}{2\pi}(F(\beta)-1)$ per unit length, the diagonal $\rho = \rho'$ being excluded. The single wells cost on average $\frac{\ell}{2\pi}\int h(c_X) = \frac{\ell}{2\pi}\cdot\frac{2\kappa_1}{L}(1+O(L^{-1}))$ per unit length. Since $L\int_0^\infty G_X^2\xi^{-2}d\xi = 1 + O(L^{-1})$, this gives

$$\frac{\langle\mathcal{Q}_X\rangle}{\langle\sum_\gamma H_{X,\gamma}\rangle} = \frac{\pi}{\kappa_1}\,\alpha\int_\alpha^\infty\frac{F(\beta)-1}{\beta^2}\,d\beta + O\Big(\frac{1}{\log T}\Big), \qquad X = \Big(\frac{T}{2\pi}\Big)^\alpha, \tag{10}$$

where $\langle\cdot\rangle$ is the average over the centre. By Montgomery's theorem α times the part of the integral over $\alpha<\beta<1$ equals $-(1-\alpha+\alpha\log\alpha)$, so under RH

$$\frac{\langle \mathcal{Q}_X\rangle}{\langle\sum H_{X,\gamma}\rangle}=-\frac{\pi}{\kappa_1}\big(1-\alpha+\alpha\log\alpha\big)+\frac{\pi}{\kappa_1}\,\alpha\int_1^\infty\frac{F(\beta)-1}{\beta^2}d\beta+o(1)\qquad(0<\alpha<1),$$

and the pair-correlation conjecture implies that the last integral vanishes, so that the quadratic interaction is 0 for $\alpha\ge1$. We have not written out the error analysis (the smooth truncation of the pair sums and of the window in t_0), which follows the lines of Goldston and Montgomery [GM]; (10) should be read as a derivation, not a theorem. Conversely, (10) can be inverted: $F(\alpha)-1=-\alpha^2\frac{d}{d\alpha}\big(\alpha^{-1}R(\alpha)\big)\cdot\frac{\kappa_1}{\pi}$, R the left side, so the Euler–Caceres energy determines F.

Numerical test. We computed the zeros in $[T-70,T+470]$, $T=10^6$, with the Riemann–Siegel formula (1030 zeros), built $\phi_X=\sum_\gamma c_X(\cdot-\gamma)$ for $X=(T/2\pi)^\alpha$, $0.3\le\alpha\le3$, the normalization of (10), and averaged over 200 centres $s_0=\sigma_0+it_0$, $t_0\in[T,T+400]$, with $\sigma_0-\frac12=1$ and with Montgomery's weight, $\sigma_0-\frac12=2$. Figure 1a shows the quadratic interaction: it follows Montgomery's curve for $\alpha\le0.7$ (-0.587 ± 0.011 against -0.582 at $\alpha=0.3$, -0.257 ± 0.009 against -0.264 at $\alpha=0.5$, -0.088 ± 0.010 against -0.087 at $\alpha=0.7$) and is close to 0 for $\alpha\ge1.2$, as the pair-correlation conjecture predicts; uncorrelated zeros would give 0 for every α. Around $\alpha=1$ the data lie above both, by up to 0.04 (4σ); lower-order terms in the pair correlation at finite height (Bogomolny and Keating [BK]) and the width of the cut-off, which smooths the kink of F at $\beta=1$, are possible causes that we have not separated. The full interaction (Figure 1b) is positive, with a maximum of 0.15 near $\alpha=0.9$: the nonlinear terms more than compensate the repulsion seen in the quadratic part, and we do not have a formula for them.

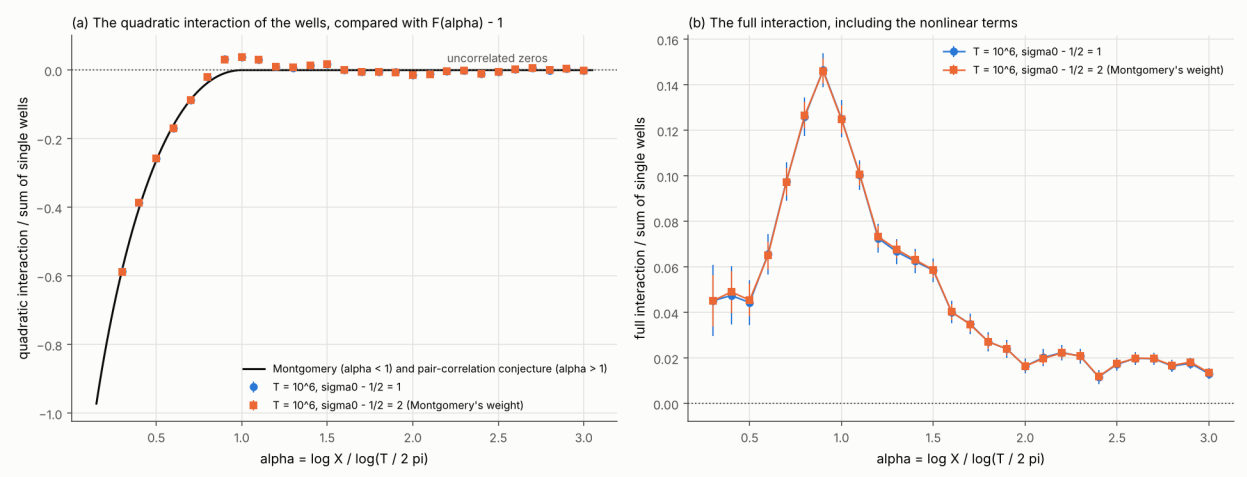


FIGURE 1. The interaction of the wells at height $T=10^6$, $X=(T/2\pi)^\alpha$, averaged over 200 centres, relative to the sum of the single wells. (a) The quadratic part against $-\frac{\pi}{\kappa_1}(1-\alpha+\alpha\log\alpha)$ for $\alpha<1$ (Montgomery's theorem under RH, with the contribution of $\beta\ge1$ set to 0) and 0 for $\alpha>1$ (pair-correlation conjecture). (b) The full interaction.

Remark 20. The total $\int 2\phi_X^2P_{s_0}=\int 2\log^2|\zeta/E_X^*(\frac12+it)|P_{s_0}(t)\,dt$ is computed from ζ and the primes up to e^aX alone, and by Proposition 18 the single-well part is $2\pi E[(L+M)^{-1}]\sum m^2P_{s_0}(\gamma)$ up to $O(L^{-2})$, which under RH and simple zeros is $2E[(L+M)^{-1}]\,\mathrm{Re}\,\frac{\xi'}{\xi}(s_0)$. So (10) expresses the pair correlation through the mean square of the logarithm of the tail of the Euler product beyond X, in the spirit of the equivalences of Goldston, Gonek and Montgomery between pair correlation and mean values of ζ'/ζ [GGM].

8. The interaction of two wells and the gaps between the zeros

The interaction $\mathcal{I}_X$ of Theorem 5 is the only part of the error in Theorem 2 that depends on more than one zero. Its leading part is a sum over pairs of zeros (the remark after the proof of Theorem 5); here we compute the contribution of one pair exactly, show that it is controlled by the distance between the two zeros in the scale $1/\log X$, and deduce a rate in Theorem 2 under a hypothesis on small gaps.

8.1. **The two-well function.** For two simple zeros at distance δ, put $d = \delta \log X$, the distance in the variable $v = (t-\gamma)\log X$ of Lemma 13, and

$$\kappa_{\mathrm{pair}}(d) = \tfrac{1}{2}\int_{\mathbb{R}} h\big(\,\mathrm{Ci}\,|v-\tfrac{d}{2}| + \mathrm{Ci}\,|v+\tfrac{d}{2}|\big)\,dv, \qquad E(d) = \kappa_{\mathrm{pair}}(d) - 2\kappa_1.$$

By dominated convergence, $\kappa_{\mathrm{pair}}(d) \to \kappa_2$ as $d \to 0$ and $\kappa_{\mathrm{pair}}(d) \to 2\kappa_1$ as $d \to \infty$; by the identity $h(x+y) - h(x) - h(y) = (e^{2x}-1)(e^{2y}-1)$, $E(d) = \frac{1}{2}\int (e^{2\,\mathrm{Ci}\,|v-d/2|}-1)(e^{2\,\mathrm{Ci}\,|v+d/2|}-1)\,dv$. The proof of Theorem 2, applied to a configuration in which two simple zeros near height γ are at distance $d/\log X$, shows that the pair contributes $P_{s_0}(\gamma)\kappa_{\mathrm{pair}}(d)$ to the limit of $(\log X)B_X$, in place of $2\kappa_1 P_{s_0}(\gamma)$ for two separated zeros; $P_{s_0}(\gamma)E(d)$ is its interaction. Table 3 gives the function; the script `ec_phase2_controls.py` computes it.

d	0	0.1	0.5	1	2	3	5
$\kappa_{\mathrm{pair}}(d)$	7.3458	7.3092	6.4935	4.6008	2.4720	3.5843	4.3443
$E(d)$	3.690	3.653	2.838	0.945	−1.184	−0.072	0.688
d	8	12	20	40	80	160	∞
$\kappa_{\mathrm{pair}}(d)$	3.2330	3.8722	3.5021	3.5875	3.7013	3.6509	3.6558
$E(d)$	−0.423	0.216	−0.154	−0.068	0.045	−0.005	0

Table 3. The two-well function. $E(0) = \kappa_2 - 2\kappa_1$: two coalescing simple zeros cost what a double zero costs. $E < 0$ for $1.5 \lesssim d \lesssim 3.1$.

Decay. On $10 \le d \le 400$ (781 points) the maximum of $|dE(d)|$ over windows of length 6π is 3.64 to 3.66, constant in d, and the least-squares fit $dE(d) = A\cos d + B\sin d + (C\cos d + D\sin d)\log d$ gives $A = 0.451$, $B = -3.564$, $C = -0.078$, $D = -0.017$, with residual 0.04. So, numerically,

$$E(d) = \frac{0.45\cos d - 3.56\sin d}{d} + O\Big(\frac{\log d}{d^2}\Big).$$

Heuristically, away from the wells $e^{2x}-1 \approx 2x$ and $\mathrm{Ci}(v) \approx \sin v/v$; the region between the wells contributes $-4\cos d\,\log d/d$ and the two outer regions $+4\cos d\,\log d/d$, while the terms of order $\log d/d$ from the neighbourhoods of the wells cancel in pairs. Status: the limits at 0 and ∞ are proved (elementary); the decay is numerical and heuristic.

A planted pair. We replaced the first zero $\pm\gamma_1$ by the pair $\pm(\gamma_1 \pm \delta/2)$, keeping the other zeros below height 1000, and compared $L\Delta B$, the change of $(\log X)B_X(2)$ against the configuration without $\pm\gamma_1$, with $2P_2(\gamma_1)\kappa_{\mathrm{pair}}(\delta L)$ (Table 4). The ratio tends to 1 at the rate $O(1/\log X)$ of the single-well deficit of Proposition 18; for the single zero alone the ratios are 0.899, 0.948, 0.964 at the same X, and the relative deficit m^2/κ_m is the same for $m = 1$ and for a merged pair (0.547 and 0.545).

8.2. **A rate under a gap hypothesis.** For a distinct ordinate γ let $\delta_\gamma \in (0,1]$ be the smaller of 1 and the distance from γ to the nearest other distinct ordinate, and for $\varepsilon > 0$ put

$$\mathcal{G}_\varepsilon(s_0) = \sum_\gamma P_{s_0}(\gamma)\,\frac{(|\gamma|+3)^\varepsilon}{\delta_\gamma} \;\le \infty.$$

X	δ	$d=\delta\log X$	$L\Delta B$	$2P_2(\gamma_1)\kappa_{\rm pair}(d)$	ratio
10^2	0.004	0.018	0.03064	0.03471	0.883
10^2	0.5	2.30	0.01189	0.01250	0.951
10^4	0.02	0.184	0.03213	0.03414	0.941
10^4	0.5	4.61	0.01975	0.02076	0.951
10^6	0.004	0.055	0.03331	0.03467	0.961
10^6	0.1	1.38	0.01485	0.01555	0.955
10^6	2	27.6	0.01695	0.01690	1.003

TABLE 4. A planted close pair at γ_1, $s_0=2$, bump weight $a=1.2$, zeros below height 1000 (script `ec_phase2_controls.py pair`).

$\mathcal{G}_\varepsilon(s_0)<\infty$ follows from any bound $\delta_\gamma\gg|\gamma|^{-1+\eta}$ with $\eta>\varepsilon$, since $P_{s_0}(\gamma)\asymp\gamma^{-2}$ and the zeros have density $\frac{1}{2\pi}\log\gamma$; under the GUE model the smallest gap below T is of order $T^{-1/3}$ and the mean of $1/\delta_\gamma$ is of order $\log\gamma$, so $\mathcal{G}_\varepsilon(s_0)<\infty$ is expected for every $\varepsilon>0$. It is not known, even under RH.

Theorem 21 (a rate for the law)**.** *Assume RH and $\mathcal{G}_\varepsilon(s_0)<\infty$ for some $\varepsilon\in(0,\frac14)$. Then there are C and X_0, depending on s_0, g_0 and ε, with*

$$\big|(\log X)B_X(s_0)-\Omega(s_0)\big|\le C\big(1+\mathcal{G}_\varepsilon(s_0)\big)\,\frac{\log\log X}{\log X}\qquad(X\ge X_0).$$

Proof. By (8) and (7) it suffices to bound $\frac12 LH_X-\Omega$. Under RH, $m(\gamma)\ll\log|\gamma|/\log\log|\gamma|$, so every factor depending on the multiplicity, such as m^3e^{2m}, is $\ll_\varepsilon(|\gamma|+3)^{\varepsilon/2}$; and by Lemma 14(i) with $\varepsilon/4$, $\phi_X\le\frac{\varepsilon}{4}\log(|t|+3)+C$ for $X\ge X_\varepsilon$. Call an ordinate *separated* if $\delta_\gamma L\ge4$ and *crowded* otherwise.

Crowded ordinates. Let $\mathcal{B}$ be the union of the intervals $[\gamma-\frac32,\gamma+\frac32]$ over the crowded γ. On a unit interval I near height n, Lemma 14 gives $L\int_I h(\phi_X)\le C_\varepsilon(n+3)^{\varepsilon/2}\log^2(n+3)$, and the part of Ω carried by the ordinates in I is bounded in the same way. Since a crowded ordinate satisfies $1<4/(\delta_\gamma L)$, the contribution of $\mathcal{B}$ to $|\frac12 LH_X-\Omega|$ is at most $\frac{C}{L}\sum_\gamma P_{s_0}(\gamma)(|\gamma|+3)^\varepsilon/\delta_\gamma=C\mathcal{G}_\varepsilon/L$.

Separated ordinates outside $\mathcal{B}$. On $U_\gamma=(\gamma-\frac{\delta_\gamma}{2},\gamma+\frac{\delta_\gamma}{2})$ write $\phi_X=m\,c_X(\cdot-\gamma)+r_X$. The $\ll\log(|\gamma|+3)$ ordinates within distance 1 are at distance $\ge\delta_\gamma/2$ from U_γ and contribute at most $4/(\delta_\gamma L)$ each by (5), the others $O(\log(|\gamma|+3)/L)$; so $|r_X|\le C\log(|\gamma|+3)/(\delta_\gamma L)$. By the mean value theorem $|h(x+r)-h(x)|\le2|r|\,|e^{2(x+\vartheta r)}-1|$, and, since $|r_X|$ may be as large as $\frac{C}{4}\log(|\gamma|+3)$, Lemma 14(i) caps the exponential by $C(|\gamma|+3)^{\varepsilon/2}$ and gives $\int_{|v|<\delta_\gamma L/2}|e^{2mc_X(v/L)+2\vartheta r_X}-1|\,dv\ll(|\gamma|+3)^{\varepsilon/2}\big(m\log(\delta_\gamma L)+\delta_\gamma L\,|r_X|\big)$: the integrand is at most $C(|\gamma|+3)^{\varepsilon/2}$ for $|v|\le1$ and at most $C(|\gamma|+3)^{\varepsilon/2}(6m/|v|+2|r_X|)$ beyond. Hence

$$L\Big|\int_{U_\gamma}\big(h(\phi_X)-h(mc_X(\cdot-\gamma))\big)P_{s_0}\Big|\ll P_{s_0}(\gamma)\,(|\gamma|+3)^{\varepsilon/2}\,\log(|\gamma|+3)\,\frac{m\log L+\log(|\gamma|+3)}{\delta_\gamma L}\ \ll\ P_{s_0}(\gamma)\,(|\gamma|+3)^{\varepsilon}\,\frac{\log L}{\delta_\gamma L}.$$

By Proposition 18, $L\int_{U_\gamma}h(mc_X(\cdot-\gamma))P_{s_0}=2\kappa_mP_{s_0}(\gamma)+O(m^3e^{2m}P_{s_0}(\gamma)/L)$, up to the truncation of the well at $|v|=\delta_\gamma L/2$, which costs $O(m^2P_{s_0}(\gamma)/(\delta_\gamma L))$ because $h(m\,\mathrm{Ci}(|v|))\ll m^2/v^2$ for $|v|\ge1$.

Between the wells. Outside $\mathcal{B}$ and the U_γ, ϕ_X is a sum of $c_X(t-\gamma')$ with $|t-\gamma'|\ge\delta_{\gamma'}/2$. By Cauchy–Schwarz over the $\ll\log(|t|+3)$ ordinates within distance 1 and $h(x)\le2x^2e^{2\max(x,0)}$, $L\int h(\phi_X)P_{s_0}$ over the gap next to γ is $\ll P_{s_0}(\gamma)(|\gamma|+3)^{\varepsilon/2}\log^2(|\gamma|+3)\,L\int_{|y|\ge\delta_\gamma/2}(yL)^{-2}dy\ll P_{s_0}(\gamma)(|\gamma|+3)^\varepsilon/(\delta_\gamma L)$.

Summing over all ordinates, with $m\log^2(|\gamma|+3)\ll(|\gamma|+3)^{\varepsilon/2}$, the three parts give $|\frac12 LH_X-\Omega|\le C(1+\mathcal{G}_\varepsilon)\log L/L$, with no cut-off in height. □

Status: proved; the constants are not made explicit. The factor $\log\log X$ comes only from the absolute value in the first display of the second step, which discards the oscillation of the pair interaction. If $E(d) = O(1/d)$, as Table 3 indicates, and the interactions of three or more wells are of lower order, the same proof gives

$$(\log X)B_X(s_0) = \Omega(s_0) + \frac{\Omega_1(s_0) - \Omega(s_0)^2 + \mathcal{I}_X(s_0)}{\log X} + o\Big(\frac{1}{\log X}\Big), \qquad \mathcal{I}_X(s_0) = O\big(\mathcal{G}_\varepsilon(s_0)\big),$$

that is, Theorem 5 would be a genuine second-order asymptotic under the gap hypothesis. This is consistent with Table 6, where $\mathcal{I}_X$ is small and does not grow.

Remark 22 (the gaps are the obstruction). Our proof gives no rate in Theorem 2 without information on small gaps. By Table 4, a pair of simple zeros at height γ with gap δ adds about $(\kappa_2 - 2\kappa_1)P_{s_0}(\gamma) = 3.69P_{s_0}(\gamma)$ to $(\log X)B_X(s_0)$ for all X with $\delta \log X \lesssim \frac{1}{2}$: in that range it is indistinguishable from a double zero. The speed of convergence is thus controlled, from both sides, by $\sum_{\delta_\gamma \lesssim 1/\log X} P_{s_0}(\gamma)$, which the hypothesis $\mathcal{G}_\varepsilon < \infty$ makes $O(\mathcal{G}_\varepsilon/\log X)$ by Markov's inequality. Corollary 3(b) is the limit of this picture, and Open problem 3 becomes quantitative: a finite X resolves multiplicities only at the scale $\delta \approx 1/\log X$.

9. Off the line: explosion

Proof of Theorem 6. Write $Z = Z_X$ and fix s_0 with $\operatorname{Re} s_0 > \Theta$.

(a) *Reduction to a point.* By Lemma 11 and Jensen, $\log|Z(s_1)| \le \int \log|Z| P_{s_1} \le \frac{1}{2}\log\int |Z|^2 P_{s_1}$ for $\operatorname{Re} s_1 > \frac{1}{2}$, and $P_{s_1} \le H P_{s_0}$ with H independent of X, so $B_X(s_0) \ge \log|Z(s_1)| - \frac{1}{2}\log H - \log|Z(s_0)|$.

(b) *Choice of s_1.* Take a zero $\rho_0 = \Theta + i\tau_0$; let $d \in (0,1]$ be at most the distance from τ_0 to the other ordinates of zeros of real part Θ, and $S = \sum{}' \min(1/|y|, C|y|^{-3})$ over those zeros, $y = \tau_0 - \gamma$, with C from (L3) ($A = 2$, $x_{\max} = \frac{1}{2}$); S does not depend on r. Choose $r \in (0, \Theta - \frac{1}{2})$ with $2re^{2ra}S \le \frac{1}{4}$ and $s_1 = \rho_0 - r$ not a zero; let $\delta > 0$ be the distance from s_1 to the zeros.

(c) *Lower bound at s_1.* $\log|Z(s_1)| = -\sum_\rho \operatorname{Re} K_X(s_1 - \rho)$, $w = s_1 - \rho$, $x = r - (\Theta - \beta)$. The term of ρ_0 is $\ge \operatorname{Ei}(r(L-a))$ by (L4). The other zeros with $\beta = \Theta$ contribute at most $X^r e^{ra} S/(L-a) \le \frac{1}{8} X^r e^{-ra}/(r(L-a))$ by (L3). The zeros with $\Theta - r < \beta < \Theta$ contribute ≥ 0 if $y = 0$ (finitely many, $X \ge X_0$), $o(X^r/L)$ if $0 < |y| < 1$ (finitely many), and $o(X^r/L)$ in total if $|y| \ge 1$, by (L3) and dominated convergence. The zeros with $\beta \le \Theta - r$ contribute $O(\log L)$ by (L2). Since $\operatorname{Ei}(\xi) \ge e^\xi/\xi$ for $\xi \ge 1.35$ and $X^r e^{-ra}/(r(L-a)) \ge X^r e^{-ra}/(rL)$, the bound $2re^{2ra}S \le \frac{1}{4}$ makes the other zeros of real part Θ cost at most $\frac{1}{8}\operatorname{Ei}(r(L-a))$, and for $X \ge X_0$ the $o(X^r/L)$ terms cost at most another $\frac{3}{8}\operatorname{Ei}(r(L-a))$; hence $\log|Z(s_1)| \ge \frac{1}{2}\operatorname{Ei}(r(L-a)) - C\log L$.

(d) *The centre.* $\operatorname{Re}(s_0 - \rho) \ge \operatorname{Re} s_0 - \Theta > 0$, so $\log|Z(s_0)| \to 0$ by (L1). □

Proof of Corollary 9. $\log|E_X^*|$ is bounded below on the line (proof of Lemma 11) and $|\zeta(\frac{1}{2} + it)| \ll (|t|+2)^{1/4}$, so $\mathcal{E}(X) < \infty$. Under RH, $\phi_X \to 0$ almost everywhere (Lemma 13 and its proof) and $e^{2\phi_X} \le C(|t|+3)^{1/2}$ by Lemma 14(i), so $\mathcal{E}(X) \to \pi$ by dominated convergence. If RH fails, take $\operatorname{Re} s_0 > 1$; then $P_{s_0} \le C/(1+t^2)$ gives $B_X(s_0) \le \frac{1}{2}\log(C\mathcal{E}(X)) - \log|Z_X(s_0)|$, and $\log|Z_X(s_0)| \to 0$ by (L1), since $\Theta \le 1 < \operatorname{Re} s_0$. Theorem 7 makes $B_X(s_0)$ unbounded, so $\limsup \mathcal{E}(X) = \infty$; if Θ is attained, Theorem 6 forces $\mathcal{E}(X) \to \infty$. □

9.1. Θ not attained: Landau's theorem in the variable $\log X$.

The proof of Theorem 6 needs a zero of maximal real part, because the other zeros near the line $\operatorname{Re} s = \Theta$ could cancel its volcano. Landau's argument for $\psi(x) - x = \Omega_\pm(x^{\Theta-\varepsilon})$ [I, Ch. V] avoids this: it treats $\log X$ as a variable, takes a Laplace transform, and uses that every zero is a singularity of it. Write $L = \log X$, $L_0 = a + 1$, $\hat{g}_0(w) = \int g_0(v) e^{wv}\, dv$ (an entire function, real on the real axis, not identically 0), and,

for $s=\sigma+it$ with $\sigma>\frac12$ and $\zeta(s)\neq0$,

$$f_s(L)=\log\left|Z_{e^L}(s)\right|=-\sum_\rho \operatorname{Re}K_{e^L}(s-\rho)\qquad(L\ge L_0).$$

By Lemma 12, f_s is continuous, and $|f_s(L)|\le C_s e^{L/2}$, uniformly for s in compact subsets of $\{\sigma>\frac12\}$ that contain no zero: every zero has $\operatorname{Re}(\rho-s)<\frac12$, the finitely many zeros within distance 1 of s are bounded by (L2) and the first part of (L3), and the others by the second parts of (L2) and (L3).

Lemma 23 (the Laplace transform in $\log X$)**.** *For* $\operatorname{Re}w>2$,

$$\int_{L_0}^\infty f_s(L)\,e^{-wL}\,dL=\tfrac12\big(\Phi_s(w)+\Phi_{\bar s}(w)\big),\qquad \Phi_s(w)=-\frac{\hat g_0(w)}{w}\,\log\big((s+w-1)\zeta(s+w)\big)+R_s(w),$$

where R_s is holomorphic in $\{\operatorname{Re}w>-\sigma\}\backslash\{0\}$. Consequently the left side continues to $\{\operatorname{Re}w>-\sigma\}$ with singular points only at $w=0$, at $w=\rho-s$ and at $w=\rho-\bar s$, ρ running over the zeros; near $w_0=\rho-s\neq0$ the function $\Phi_s(w)+m(\rho)\frac{\hat g_0(w)}{w}\log(w-w_0)$ is holomorphic.

Proof. Let first $\operatorname{Re}s>1$. By (2) and $\log\zeta(s)=\sum\Lambda(n)n^{-s}/\log n$, with $\int_1^\infty G_X(\log x)x^{-s}dx/\log x=K_X(s-1)$ (integrate by parts against $dE_1((s-1)u)=-e^{-(s-1)u}du/u$), we have, with $\theta(u)=\psi(e^u)-e^u$,

$$\log Z_{e^L}(s)=\int_0^\infty G_0(u-L)\,\frac{e^{-su}}{u}\,d\theta(u)+\sum_{k\ge1}K_{e^L}(s+2k),$$

as in the proof of Theorem 36. For $u\ge U:=L_0+a$, the substitution $v=u-L$ and $\int_{-a}^a G_0e^{wv}dv=(e^{wa}-\hat g_0(w))/w$ give

$$\int_{L_0}^\infty G_0(u-L)e^{-wL}dL=-\frac{\hat g_0(w)}{w}\,e^{-wu}+\frac{e^{-wL_0}}{w},$$

and for $u<U$ the inner integral runs over a bounded range of L and is entire in w. Exchanging the integrals (absolutely convergent for $\operatorname{Re}s,\operatorname{Re}w>1$),

$$\begin{aligned}\Phi_s(w):=\int_{L_0}^\infty \log Z_{e^L}(s)e^{-wL}dL&\\ =-\frac{\hat g_0(w)}{w}\int_U^\infty\frac{e^{-(s+w)u}}{u}\,d\theta(u)+\frac{e^{-wL_0}}{w}\int_U^\infty\frac{e^{-su}}{u}\,d\theta(u)+E_s(w)+T_s(w),&\end{aligned}$$

with E_s entire and T_s the transform of the trivial-zero terms. With $z=s+w$, $\int_U^\infty e^{-zu}u^{-1}d\theta=\log\zeta(z)-\sum_{n\le e^U}\Lambda(n)n^{-z}/\log n-E_1((z-1)U)=\log((z-1)\zeta(z))+(\text{entire in }z)$, because $E_1(\xi)=-\gamma_E-\log\xi+\operatorname{Ein}(\xi)$. For the trivial-zero terms, $\int_{L_0}^\infty g_0(u-L)e^{-wL}dL=\hat g_0(w)e^{-wu}$ for $u\ge U$, and $\int_U^\infty E_1(zu)e^{-wu}du$, $z=s+2k$, equals $w^{-1}\log(1+w/z)$ minus an entire function; its singular point $w=-s-2k$ has real part $<-\sigma$, and the sum over k converges locally uniformly in $\operatorname{Re}w>-\sigma$, with a removable singularity at $w=0$. This gives the formula for Φ_s when $\operatorname{Re}s>1$, with R_s holomorphic in $\{\operatorname{Re}w>-\sigma\}\setminus\{0\}$: the trivial zeros of $\zeta(s+w)$ have $\operatorname{Re}w<-\sigma$, and $(z-1)\zeta(z)$ is holomorphic and nonzero at $z=1$.

Taking real parts, $\int f_se^{-wL}dL=\frac12(\Phi_s(w)+\overline{\Phi_s(\bar w)})$, and $\overline{\Phi_s(\bar w)}=\Phi_{\bar s}(w)$ because $Z_X(\bar s)=\overline{Z_X(s)}$. For fixed w with $\operatorname{Re}w>2$, the left side is harmonic in s on the connected set $\{\sigma>\frac12\}\setminus\{\rho\}$, by the locally uniform bound $|f_s(L)|\le C_se^{L/2}$. The right side is the sum of a holomorphic and an antiholomorphic function of s, and it is single-valued: the term of $\Phi_s(w)$ that contains $\log((s+w-1)\zeta(s+w))$ has no monodromy in s, since $\operatorname{Re}(s+w)>\frac52$; the multivalued term is $c(w)\log((s-1)\zeta(s))$ with $c(w)=e^{-wL_0}/w$, so a change of branch adds $2\pi i n\,c(w)$ to $\Phi_s(w)$ and $-2\pi in\,c(w)$ to $\Phi_{\bar s}(w)$, because $\overline{c(\bar w)}=c(w)$. The two sides agree for $\sigma>1$, hence on the whole set.

The statement about singular points follows; a logarithmic branch point multiplied by the entire function $\hat{g}_0(w)/w$, not identically zero, is not removable. □

Theorem 24 ($\log|Z_X|$ at a fixed point)**.** *Let $s=\sigma+it$ with $\sigma>\frac12$ and $\zeta(s)\neq0$, and suppose that $t\notin\mathcal{T}$, the countable set of t for which $t=\pm\gamma$ for some zero, or $\gamma-\gamma'=2t$ for two zeros with $\beta=\beta'$. Then for every zero $\rho=\beta+i\gamma$ with $\beta>\sigma$ and every $\delta\in[0,\beta-\sigma)$,*

$$\limsup_{X\to\infty} X^{-\delta}\log|Z_X(s)|=+\infty,\qquad \liminf_{X\to\infty} X^{-\delta}\log|Z_X(s)|=-\infty.$$

In particular $\log|Z_X(s)|=\Omega_\pm(X^{\Theta-\sigma-\varepsilon})$ for every $\varepsilon>0$, whether or not Θ is attained.

Proof. Suppose $f_s(L)\le Ce^{\delta L}$ for $L\ge L_1$, and let $A(L)=Ce^{\delta L}-f_s(L)\ge0$ on $[L_1,\infty)$. Its Laplace transform $\hat{A}(w)=\int_{L_1}^{\infty}A(L)e^{-wL}dL$ has a finite abscissa of convergence w_c, since $|f_s|\le C_se^{L/2}$, and by Lemma 23

$$\hat{A}(w)=\frac{Ce^{(\delta-w)L_1}}{w-\delta}-\tfrac12\big(\Phi_s(w)+\Phi_{\bar s}(w)\big)+(\text{entire}).$$

Because $t\notin\mathcal{T}$, no point $\rho-s$ or $\rho'-\bar s$ is real, so the right side continues holomorphically to a neighbourhood of every real $w>\max(\delta,0)=\delta$. By Landau's theorem (a Laplace integral with an eventually nonnegative integrand is singular at the real point of its abscissa of convergence; [Wi, Ch. II, Thm. 5b]), $w_c\le\delta$, and $\hat{A}$, hence $\frac12(\Phi_s+\Phi_{\bar s})$, is holomorphic in $\operatorname{Re}w>\delta$. But $w_0=\rho-s$ has real part $\beta-\sigma>\delta$, it is a singular point of Φ_s, and, because $t\notin\mathcal{T}$, it is not a singular point of $\Phi_{\bar s}$, whose singular points are the $\rho'-\bar s$. This is a contradiction. The statement for $\liminf$ follows in the same way from $A(L)=Ce^{\delta L}+f_s(L)$. □

Proof of Theorem 7 and Corollary 8. Let $\varepsilon>0$, choose a zero with $\beta>\Theta-\varepsilon/2$, and let $s_1=\sigma_1+it_1$ with $\sigma_1=\frac12+\varepsilon/4$, $\zeta(s_1)\neq0$ and $t_1\notin\mathcal{T}$. Step (a) of the proof of Theorem 6 gives $B_X(s_0)\ge\log|Z_X(s_1)|-\frac12\log H-\log|Z_X(s_0)|$ with H independent of X, and $\log|Z_X(s_0)|\to0$ by (L1), since $\operatorname{Re}s_0>1\ge\Theta$. Theorem 24 at s_1 with $\delta=\beta-\sigma_1-\varepsilon/4>\Theta-\frac12-\varepsilon$ gives Theorem 7. For Corollary 8: under RH, $B_X(s_0)\to0$ (Theorem 2); if RH fails, Theorem 7 contradicts $B_X(s_0)\ll X^{\varepsilon}$ for $\varepsilon<\frac12(\Theta-\frac12)$. □

Theorem 7 gives growth along a sequence of X; Theorem 6, under attainment, gives it for all large X, which is what the $\liminf$ form of Corollary 9 uses. For an isolated off-line zero the exponent $\Theta-\frac12$ of Theorem 7 is the true one (Table 9); the restriction to a small r in Theorem 6 comes only from the possible cancellation of the volcano of ρ_0 by other zeros of real part Θ.

Remark 25 (Θ not attained: all large X). If $\Theta>\frac12$ is not attained, Theorem 7 gives growth along a sequence, but the proof of Theorem 6, which gives growth for all large X, has no fixed zero ρ_0 to work with. What follows is a heuristic outline only. For each X one can choose a zero $\rho^*=\beta^*+i\gamma^*$ maximizing $\beta\log X-A\log(2+|\gamma|)$, $A>0$ fixed; the zeros at distance at least 1 from ρ^* then contribute at most a small multiple of the main term, and a vertical shift of s_1 aligns the common phase $e^{-i\tau\log X}$. Since $K_X(w)\approx e^{-w\log X}\hat{g}_0(iw)/(w\log X)$ for $|w|\log X$ large, what remains is the cluster amplitude

$$\sum_{|\gamma-\gamma^*|<1} X^{\beta-\beta^*}e^{i(\gamma-\gamma^*)\log X}\,\frac{\hat{g}_0(iw)}{w},\qquad w=s_1-\rho,$$

a sum of Cauchy kernels, which can be small only if off-line zeros of nearly maximal real part close to ρ^* cancel one another. For two such zeros at distance Δ we expect a remainder of order $X^r\Delta/(r^2\log X)$, so a lower bound $\Delta\ge X^{-o(1)}$ for their separation would plausibly close this case; no such bound is known.

10. What holds without RH

The decomposition, the law and the explosion theorem leave open how large the bound can be, unconditionally, for a given X. The answer has three parts: the term $\mathcal{O}_{\mathrm{off}}(s_0)$ that does not depend on X is tiny because RH has been verified to a large height; the bound is at most of the order of the length of the prime sum on the critical line; and between these two the position of the bound is decided by the real parts of the zeros above that height. Let $T_0 = 3 \cdot 10^{12}$; all zeros with $0 < \gamma \le T_0$ lie on the critical line [PT]. (The numbers below are reproduced by the ancillary script `uncond_numbers.py`.)

Theorem 26 (unconditional bounds)**.** (a) *For every $s_0 = \sigma_0 + it_0$ with $\sigma_0 > \frac12$, $\zeta(s_0) \neq 0$ and $|t_0| \le T_0/2$,*

$$0 \le \mathcal{O}_{\mathrm{off}}(s_0) \le (2\sigma_0 - 1)\,\frac{\log(T_0/2\pi) + 2}{\pi T_0} < 3.1 \cdot 10^{-12}\,(2\sigma_0 - 1),$$

and for $t_0 = 0$ the bound improves by the factor $\frac14$; in particular $\mathcal{O}_{\mathrm{off}}(2) < 2.3 \cdot 10^{-12}$.

(b) *For every s_0 with $\operatorname{Re} s_0 > 1$ there is C, depending on s_0 and g_0 only, such that for every $X > e^{a+1}$*

$$0 \le B_X(s_0) \le \mathcal{M}(e^a X) + C\,\frac{e^{(L+a)/2}}{L-a} + C, \qquad \mathcal{M}(Y) = \sum_{n \le Y} \frac{\Lambda(n)}{\sqrt{n}\,\log n} \sim \frac{2\sqrt{Y}}{\log Y};$$

hence $B_X(s_0) \ll \sqrt{X}/\log X$.

Proof. (a) For a zero $\rho = \beta + i\gamma$ with $\beta > \frac12$ and $y = t_0 - \gamma$,

$$\log \frac{1}{|b_\rho(s_0)|} = \frac12 \log\Big(1 + \frac{(2\sigma_0 - 1)(2\beta - 1)}{(\sigma_0 - \beta)^2 + y^2}\Big) \le \frac{(2\sigma_0 - 1)(2\beta - 1)}{2y^2} \le \frac{2\sigma_0 - 1}{2y^2}.$$

Every such zero has $|\gamma| > T_0$, so $|y| \ge |\gamma|/2$ (and $|y| = |\gamma|$ if $t_0 = 0$). The zeros with $\beta > \frac12$ and $\gamma > 0$ are at most half of all the zeros with $\gamma > T_0$, since $\rho \mapsto 1 - \bar\rho$ pairs each of them with a zero of real part $< \frac12$ at the same height, and the same holds for $\gamma < 0$. Hence $\mathcal{O}_{\mathrm{off}}(s_0) \le 2(2\sigma_0 - 1)\Sigma$, and $\mathcal{O}_{\mathrm{off}}(s_0) \le \frac12(2\sigma_0 - 1)\Sigma$ if $t_0 = 0$, with $\Sigma = \sum_{\gamma > T_0} \gamma^{-2}$. By partial summation, $\Sigma = -N(T_0)T_0^{-2} + 2\int_{T_0}^{\infty} N(t)t^{-3}dt$, and the Riemann–von Mangoldt formula with Trudgian's explicit remainder, $|N(t) - \frac{t}{2\pi}\log\frac{t}{2\pi e} - \frac78| \le 0.112\log t + 0.278\log\log t + 2.51$ for $t \ge e$ [Tr], gives $\Sigma = \frac{\log(T_0/2\pi)+1}{2\pi T_0} + O(T_0^{-2}\log T_0) \le \frac{\log(T_0/2\pi)+2}{2\pi T_0}$; numerically $\Sigma < 1.48 \cdot 10^{-12}$.

(b) $B_X(s_0) \ge 0$ by Theorem 1. By (2), on the critical line $-\log|E_X^*(s)| = -\operatorname{Re} D(s) - \operatorname{Re} K_X(s-1) + \sum_k \operatorname{Re} K_X(s+2k)$, with D the finite prime sum. First, $|\operatorname{Re} D(\frac12 + it)| \le \mathcal{M}(e^a X)$, because $0 \le \psi_X \le 1$ and $\psi_X(\log n) = 0$ for $n > e^a X$. Second, $s - 1 = -\frac12 + it$ and $K_X(s-1) = \int g_X(u)E_1(z)\,du$ with $z = (-\frac12 + it)u$, $u \in [L-a, L+a]$; by the asymptotic expansion of E_1, valid uniformly in the closed sector $|\operatorname{ph} z| \le \pi$ with the boundary values $E_1(-x \pm i0) = -\operatorname{Ei}(x) \mp i\pi$ [DLMF, 6.12.1], $|E_1(z)| \le C|e^{-z}|/|z| + \pi \le 2Ce^{u/2}/u + \pi$ for $|z| \ge \frac12$ (on the closed sector E_1 is bounded on $\frac12 \le |z| \le 1$), so $|K_X(s-1)| \le 2Ce^{(L+a)/2}/(L-a) + \pi$ for every t. Third, the trivial-zero terms are bounded (proof of Lemma 11). Hence $e^{2\phi_X(t)} \le |\zeta(\frac12 + it)|^2 \exp\big(2\mathcal{M}(e^a X) + C'e^{(L+a)/2}/(L-a) + C'\big)$, and $\int |\zeta(\frac12 + it)|^2 P_{s_0}(t)\,dt < \infty$ since $\zeta(\frac12 + it) \ll (|t|+2)^{1/4}$ [T]. Finally $\operatorname{Re}(s_0 - \rho) \ge \operatorname{Re} s_0 - 1 > 0$ for every zero, so $|\log|Z_X(s_0)|| \le CX^{-(\operatorname{Re} s_0 - 1)}/L$ by (L1). The asymptotic of $\mathcal{M}$ follows from the prime number theorem by partial summation (Chebyshev's bound already gives $\mathcal{M}(Y) \ll \sqrt{Y}/\log Y$). □

Together with Theorem 6, Theorem 26 brackets the bound. Unconditionally, for $\operatorname{Re} s_0 > 1$ and $|t_0| \le T_0/2$, $B_X(s_0)$ lies between $\mathcal{O}_{\mathrm{off}}(s_0)$, which is below $3.1 \cdot 10^{-12}(2\sigma_0 - 1)$, and $O(\sqrt{X}/\log X)$, the length of the prime sum on the critical line; for $\frac12 < \operatorname{Re} s_0 \le 1$ we prove no upper bound.

Under RH it decays like $\Omega(s_0)/\log X$ (Theorem 2). If $\Theta > \frac{1}{2}$ it exceeds $X^{\Theta-\frac{1}{2}-\varepsilon}$ for arbitrarily large X (Theorem 7), and if Θ is attained it grows at least like $X^r/\log X$ for all large X, for some $r \in (0, \Theta - \frac{1}{2})$; since an attained Θ is < 1, the exponent r of Theorem 6 is always below $\frac{1}{2}$, the exponent of the upper bound. Without information on the zeros above T_0 nothing in between can be excluded, and no finite computation of B_X can confirm RH.

Remark 27 (detection length). In the proof of Theorem 6 the height of the zero $\rho_0 = \Theta + i\tau_0$ enters through the comparison $P_{s_1} \le HP_{s_0}$, with $\frac{1}{2}\log H = \log|\tau_0 - t_0| + O(1)$, and through the zeros near ρ_0, about $\log\tau_0$ per unit length, which affect the constant of the term $C\log\log X$ and X_0; the other constants depend on the configuration of the zeros of real part Θ near ρ_0. The bound at s_0 therefore exceeds a fixed level once $\frac{1}{2}\operatorname{Ei}(r(\log X - a))$ exceeds $\log|\tau_0| + O(\log\log X)$, that is for $\log X$ of order $r^{-1}\log\log|\tau_0|$. For a zero just above T_0, ignoring the constants and with $a = 1.2$, this means $\log X \approx 56$ if the theorem applies with $r = 0.1$ (which needs $\Theta > 0.6$) and $\log X \approx 550$ if it applies with $r = 0.01$. The detection length grows like the logarithm of the logarithm of the height, the same dependence on the height as for Weil's form with two windows in [CE19], which detects an isolated zero with $\beta = \frac{1}{2} + \delta$ with the prime powers up to $(C\log\gamma)^{1/\delta}$; the two mechanisms differ and we have not compared constants. Our computations, with $X \le 10^6$, give $B_X(2) \approx 3\cdot 10^{-3}$, nine orders of magnitude above the largest possible value of $\mathcal{O}_{\text{off}}(2)$ and far below the detection length of any zero above T_0 that is not very far from the line: they agree with the law of Theorem 2, and they would agree with it whether or not RH holds above T_0.

11. Numerical results

Unless stated otherwise (Table 7 and Figure 2), all values are computed in double precision with ζ and Hurwitz zeta functions by Euler–Maclaurin summation, on $|t - t_0| \le 800$ with step 0.01 (the missing kernel mass is added with $\phi_X = 0$). The weight is the bump $g_0(v) \propto \exp(-1/(1-(v/1.2)^2))$ on $[-1.2, 1.2]$, for which the prime sum is exactly finite. The numbers illustrate the theorems; the proofs do not use them.

The law at $s_0 = 2$. Here $\varpi(2) = \frac{1}{\pi}\operatorname{Re}\frac{\xi'}{\xi}(2) = 0.0219844$ (Lemma 16) and $\kappa_1\varpi(2) = 0.040186$. A computation on a finite window only sees the zeros in it: with $|t| \le T$ and $\phi_X = 0$ outside, the limit of $(\log X)B_X(2)$ is $\kappa_1\varpi_T(2)$, $\varpi_T(2) = \sum_{|\gamma|<T} P_2(\gamma)$, computed from the 649 zeros below height 1000. For $T = 800$, $\kappa_1\varpi_{800}(2) = 0.03816$; the zeros above 800 carry the remaining 0.0020. Table 5 shows $(\log X)B_X(2)$ approaching $\kappa_1\varpi_{800}(2)$; with the window $T = 600$ (script `largeX_bump.py`), $B_X(2) = 0.0138, 0.0074, 0.0052, 0.0039, 0.0032, 0.0026$, that is $(\log X)B_X(2) = 0.032, 0.034, 0.036, 0.036, 0.036, 0.037$, for $X = 10, \dots, 10^6$, against $\kappa_1\varpi_{600}(2) = 0.0376$ (Figure 2b), and the identity $\int \phi_X P_2 = \log|Z_X(2)|$ of the proof of Theorem 1 holds to $1.3\cdot 10^{-4}$.

X	$(\log X)B_X(2)$ for ζ	$(\log X)B_X(2)$ for ζ^2	ratio
10^2	0.0347	0.1397	4.025
10^3	0.0361	0.1441	3.995
10^4	0.0363	0.1450	3.992
10^5	0.0368	0.1471	3.998
law, zeros with $\|\gamma\| < 800$	$\kappa_1\varpi_{800}(2) = 0.03816$	$\kappa_2\varpi_{800}(2) = 0.15334$	$\kappa_2/\kappa_1 = 4.0187$
law, all zeros	$\kappa_1\varpi(2) = 0.04019$	$\kappa_2\varpi(2) = 0.16149$	4.0187

Table 5. The law of Theorem 2 at $s_0 = 2$, window $|t| \le 800$, step 0.01, compactly supported weight. For ζ^2 the bound is $\frac{1}{2}\log\int e^{4\phi_X}P_2 - 2\log|Z_X(2)|$. A double zero costs $\kappa_2 = 7.35$, not $2\kappa_1 = 3.66$.

Multiplicities. The function ζ^2 has the factorization $\zeta^2 = (E_X^*)^2 Z_X^2$, and its zeros are those of ζ with doubled multiplicity. Its Euler–Caceres bound is $\frac{1}{2}\log\int e^{4\phi_X}P_{s_0} - 2\log|Z_X(s_0)|$, and the proof

of Theorem 2 (with Lemma 14(i) for $\varepsilon = \frac{1}{8}$ in the tail) gives, under RH, the limit $\sum_\gamma \kappa_{2m(\gamma)} P_{s_0}(\gamma)$; if the zeros of ζ are simple, this is $\kappa_2 \varpi(s_0)$. The ratio of the two bounds should then tend to $\kappa_2/\kappa_1 = 4.0187$, also on a finite window: a double zero is seen with weight κ_2, not $2\kappa_1$. Table 5 shows the ratio within 0.7% of 4.0187 from $X = 10^2$ to 10^5. This is a test of the multiplicity mechanism of Theorem 2, on a function whose zeros are known to be multiple.

The second-order term. Table 6 compares Theorem 5 with the zero sum $\phi_X(t) = \sum_\gamma c_X(t-\gamma)$ over the 649 zeros below height 1000 (which reproduces the direct computation of ζ/E_X^* to within 10^{-3} in $(\log X)B_X$), at $s_0 = 2$, on $|t| < 800$ with step $0.01/\log X$, for bump weights $g_0(v) \propto \exp(-1/(1-(v/a)^2))$ of three widths a. The single-well part $\frac{1}{2}L\sum H_{X,\gamma}$ converges to $\Omega + \Omega_1/L$ with Ω_1 proportional to the width, and the interaction $\mathcal{I}_X$ is small and decreases as the weight widens. With $\Omega_1 - \Omega^2$ included, the deficit of about 4% in Table 5 is reduced to less than 1% of Ω, and to zero within the numerical error once $\mathcal{I}_X$ is added.

a	$E\lvert W-W'\rvert$	Ω_1	$L(\frac{1}{2}L\sum H_{X,\gamma} - \Omega)$			$\mathcal{I}_X$	
			$X = 10^3$	10^4	10^5	10^4	10^5
0.6	0.274	−0.0086	−0.0084	−0.0084	−0.0085	+0.0023	+0.0030
1.2	0.549	−0.0178	−0.0170	−0.0171	−0.0173	+0.0014	+0.0027
2.4	1.098	−0.0359	−0.0329	−0.0335	−0.0339	−0.0002	+0.0013

TABLE 6. Theorem 5 at $s_0 = 2$ for zeros with $|\gamma| < 800$ ($\Omega = 0.03816$, $\Omega^2 = 0.00146$): the single-well part approaches Ω_1, with a remainder of order $\log L/L$; the interaction of the wells is a fraction of $|\Omega_1|$ that decreases as the weight widens; we know only $\mathcal{I}_X = o(\log X)$, and the data do not show convergence.

Other centres and the Dirichlet floor. Table 7 gives $B_X(s_0)$ with a Gaussian weight g_0 of standard deviation 0.4, the prime sum truncated at 4.5 standard deviations (beyond $X = 10^5$ the neglected tail of the prime sum, of size $X^{1/2}$, would inflate B, which is why the compactly supported weight is used elsewhere; the theorems are stated for compactly supported weights). At the centre $s_0 = 1$ of (1), $\varpi(1) = \frac{1}{2\pi}\sum_\rho |\rho|^{-2} = 0.00735$ under RH (by (11)), and $\kappa_1\varpi(1)/\log 10^4 = 0.00146$, against the computed 0.0015; at $s_0 = 1 + 100i$, $\varpi = 0.2305$, and the computed value exceeds $\kappa_1\varpi/\log X$ by the factors 1.07 and 1.05 at $X = 10^3$ and 10^4. At $s_0 = 1$ the exact quadratic form gives, for the best Dirichlet polynomial of length N, the values 0.0404, 0.0396, 0.0392 and 0.0391 for $N = 10$, 30, 150 and 400, above the floor $\frac{1}{2}\log(1+c_0) = 0.0386$ of the introduction as Proposition 52 requires, and the Euler–Caceres bound is below the floor from $X = 10$ on. The other "best polynomial" values of this paragraph and of Table 7 were optimised on a truncated window in t and are biased low by an amount we have not quantified (about 2% in the case $s_0 = 1$, where the windowed optimisation gave 0.0384); at $s_0 = 1 + 100i$ the best Dirichlet polynomial gives 0.090 for $N = 150$ and 0.080 for $N = 500$, and the Euler–Caceres bound is below these from $X = 10^3$ on. The Dirichlet floor depends on the centre: at height 100, with $N = 150$, the best polynomial gives 0.150, 0.133, 0.090, 0.050 and 0.021 at $\operatorname{Re} s_0 = 2$, 1.5, 1, 0.75 and 0.6, and it decreases slowly with N in every case. For f at $1 + 85.7i$ the best polynomial of length 150 gives 1.549, above the off-line term 1.4407, as it must be, while the Euler–Caceres bound of f grows with X (`mollified_barrier.py`).

Davenport–Heilbronn. For the Davenport–Heilbronn function f, which satisfies a functional equation and has zeros off the line, the analogous bound, with E_f^* built from the coefficients of $-f'/f$ and the trivial zeros of f at $-1, -3, \ldots$, grows with X (Table 7, Figure 2c): at $s_0 = 1 + 85.7i$, next to the off-line zero $\rho_f = 0.808517 + 85.699348i$, from 1.65 at $X = 10$ to 7.58 at $X = 10^4$, far above its off-line term 1.4407. Theorem 6 does not apply to f as stated: f has infinitely many zeros in $\operatorname{Re} s > 1$ [DH], so $\Theta_f > 1$ and its attainment is unknown, the zero ρ_f is not of maximal real part,

	s_0	$X=10$	10^2	10^3	10^4
ζ	1	0.0051	0.0027	0.0019	0.0015
ζ	$1+100i$	0.130	0.101	0.065	0.048
ζ	$2+85.7i$	0.241	0.123	0.092	0.072
f	$1+85.7i$	1.65	2.55	4.49	7.58
f	$1+100i$	0.34	0.31	1.26	5.91
f	$2+85.7i$	0.77	1.78	3.82	7.08
best Dirichlet polynomial	1	0.0392 ($N=150$, exact form; floor 0.0386)			
best Dirichlet polynomial	$1+100i$	0.090 ($N=150$), 0.080 ($N=500$)			

TABLE 7. $B_X(s_0)$ for ζ and for the Davenport–Heilbronn function f, Gaussian weight, window $|t-t_0|\le 300$ ($|t|\le 2000$ at $s_0=1$), step 0.01, the kernel mass outside the window added with $\phi_X=0$; and the Jensen bound of the optimal Dirichlet-polynomial mollifier of length N (at $s_0=1$ from the exact Gram matrix on the whole line, `dirichlet_exact_gram.py`; at the other centres as a quadratic form on the window $|t-t_0|<1500$, biased low), which does not depend on X. The off-line terms $\mathcal{O}^f_{\rm off}(s_0)$ of f are 1.4407, 0.0024, 0.4181 at its three centres. Scripts `mollifier_tables.py`, `euler_caceres_barrier.py`, `dh_euler_caceres.py`, `mollified_barrier.py`.

and we do not prove the factorization and the Smirnov property for f. Numerically the mechanism of its proof is nevertheless visible for moderate X: at $s_1=\rho_f-r$, $\log|Z_f(s_1)|$ is 1.0332 and 1.6702 for $X=10^3,10^4$ with $r=0.1$, against the main term $\int g_X(u)\,\mathrm{Ei}(ru)\,du$ = 1.0373 and 1.6800 of (L4), and 2.9489, 4.3721 with $r=0.2$, against 2.9564, 4.3976 (script `explosion.py`).

The decomposition for f. Lemma 10 and the Smirnov property are not proved for f, but the decomposition $B_f=\mathcal{O}^f_{\rm off}+J_f$ of Theorem 1 can be tested. With the bump weight $a=1.2$, the window $|t-t_0|\le 200$, and the eight off-line zeros of f below height 400, which a grid search with Newton refinement returns as $0.808517{+}85.699348i$, $0.650830{+}114.163343i$, $0.574356{+}166.479306i$, $0.724258+176.702461i$, $0.869531+240.404672i$, $0.819550+320.876490i$, $0.768223+331.050259i$ and $0.628508{+}366.640908i$, the identity $\int\phi_X P_{s_0}-\log|Z_f(s_0)|=\mathcal{O}^f_{\rm off}(s_0)$ holds to 10^{-3} at $1+85.7i$ and $2+85.7i$, and to 10^{-2} at $1+100i$, where the window error dominates the small floor (Table 8). The floors do not depend on the weight and agree with Table 7; the bound itself, with this weight, agrees with the Gaussian values of Table 7 to within 0.04. The growth is entirely in the Jensen gap.

s_0	X	$B_f(s_0)$	$\mathcal{O}^f_{\rm off}(s_0)$	$J_f(s_0)$	$\int\phi_X P_{s_0}-\log\lvert Z_f(s_0)\rvert$
$1+85.7i$	10	1.6335	1.4407	0.1933	1.4403
	10^2	2.5790		1.1375	1.4415
	10^3	4.4904		3.0506	1.4398
	10^4	7.6043		6.1637	1.4406
$2+85.7i$	10	0.7555	0.4181	0.3382	0.4173
	10^4	7.1070		6.6890	0.4180
$1+100i$	10	0.3349	0.0024	0.3231	0.0119
	10^4	5.9504		5.9462	0.0042

TABLE 8. The decomposition of Theorem 1 for the Davenport–Heilbronn function, bump weight $a=1.2$ (script `ec_phase2_controls.py dh`).

A planted off-line zero. To see Theorems 1, 6 and 7 at work on a configuration in which Θ is known, we kept the zeros of ζ below height 1000 and moved the first one to $\beta+i\gamma_1$, adding $1-\beta+i\gamma_1$ and the conjugates, and computed $B_X(2)$ from $\phi_X = -\sum_\rho \operatorname{Re} K_X(\frac12 + it - \rho)$ (Table 9). In every case $\int \phi_X P_2 - \log|Z_X(2)|$ equals the floor $\mathcal{O}_{\text{off}}(2)$ to within $4\cdot10^{-4}$, the window error, and $B_X(2) > \mathcal{O}_{\text{off}}(2)$. The maximum of ϕ_X on the line agrees with the main term $\int g_X(u)\,\mathrm{Ei}((\beta-\frac12)u)\,du$ of (L4) to two or three digits once it dominates, and the logarithmic slope of B_X tends to $(\beta-\frac12)-1/\log X$, as $B_X \approx X^{\beta-\frac12}/((\beta-\frac12)\log X)$ predicts: 0.329 against $0.40-0.072=0.328$ at $\beta=0.9$, $X=10^6$. For $\beta=0.55$ the bound first decreases with the on-line wells and turns up only once the main term is positive, near $(\beta-\frac12)\log X \approx 0.37$ where Ei changes sign, with a minimum 0.0081 near $X=10^5$, three times the value 0.0027 of the unperturbed zeros.

β	X	$B_X(2)$	$\mathcal{O}_{\text{off}}(2)$	$\max\phi_X$	$\int g_X\,\mathrm{Ei}((\beta-\frac12)u)du$	$d\log B_X/d\log X$
0.55	10^2	0.0129	0.00149	0.98	−0.65	−0.24
0.55	10^5	0.0081	0.00149	1.17	0.70	−0.02
0.55	10^6	0.0085	0.00149	1.29	1.04	0.02
0.60	10^6	0.0998	0.00297	2.84	2.96	0.354
0.75	10^6	9.363	0.00742	13.54	13.54	0.255
0.90	10^4	11.904	0.01187	15.93	15.93	0.392
0.90	10^6	55.710	0.01187	60.32	60.33	0.329

TABLE 9. The first zero of ζ moved off the line to $\beta+i\gamma_1$, $s_0=2$, bump weight $a=1.2$; the slope is taken over one decade (script `ec_phase2_controls.py planted`).

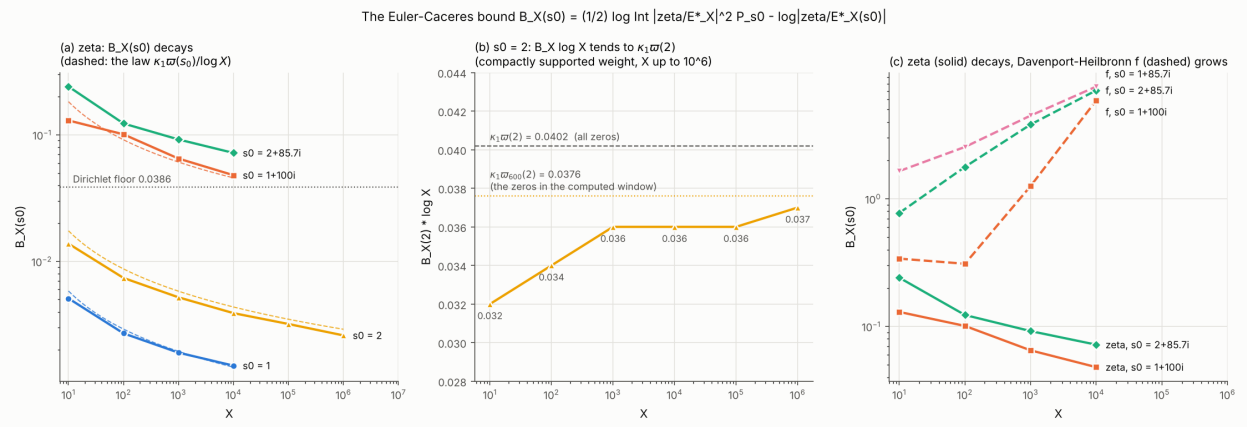


FIGURE 2. The Euler–Caceres bound (data of Table 7 and of the window-600 computation above). (a) $B_X(s_0)$ for ζ at four centres, with $\kappa_1\varpi(s_0)/\log X$ (dashed) and the floor 0.0386 of Dirichlet mollifiers (dotted; proved for $s_0=1$ only). (b) $(\log X)B_X(2)$, window $|t|\le 600$, against $\kappa_1\varpi(2)=0.0402$ and $\kappa_1\varpi_{600}(2)$. (c) ζ (solid) against the Davenport–Heilbronn function (dashed). Panels (a) and (c) use a Gaussian weight g_0 of standard deviation 0.4, with the prime sum truncated at 4.5 standard deviations, on $|t-t_0|\le 300$ ($|t|\le 2000$ at $s_0=1$).

12. THE GEOMETRY BEHIND THE LAW

Figure 3 shows the factorization. On the critical line the truncated Euler product does not converge and oscillates around $|\zeta|$ (the Euler product inside the critical strip is studied in [CE8]),

while E_X^* follows $|\zeta(\frac{1}{2}+it)|$ as a zero-free envelope built from the primes, which passes over each zero without touching it: with the Gaussian weight of Table 7, $|E_X^*(\rho_1)| = 0.097$ for $X = 10^2$ and 0.048 for $X = 10^4$ (script `mollifier_tables.py`), and the zero is carried by Z_X alone; $|Z_X|$ is close to 1 except for a well of width of order $1/\log X$ at each zero, with the profile $e^{m\,\mathrm{Ci}(|v|)}$ of Lemma 13 and an overshoot of height $e^{0.472m}$. Theorem 2 is the statement that each well costs exactly $\kappa_m P_{s_0}(\gamma)/\log X$ in the Jensen gap: the wells narrow as X grows, and their cost in the mean square does not depend on anything else. Off the line the picture changes: the factor of a zero ρ grows like $\exp\left(X^{\beta-\sigma}/((\beta-\sigma)\log X)\right)$ where $\sigma < \beta$, a *volcano* to the left of the zero, which for an off-line zero crosses the critical line; this is the mechanism of Theorem 6. From the right, in $\mathrm{Re}\, s > \frac{1}{2}$, a zero on the line is seen only as a *crater*. In this language, Corollary 3(a) and Theorem 6 say that the bound decays when only craters reach the critical line and, if Θ is attained, explodes when a volcano crosses it.

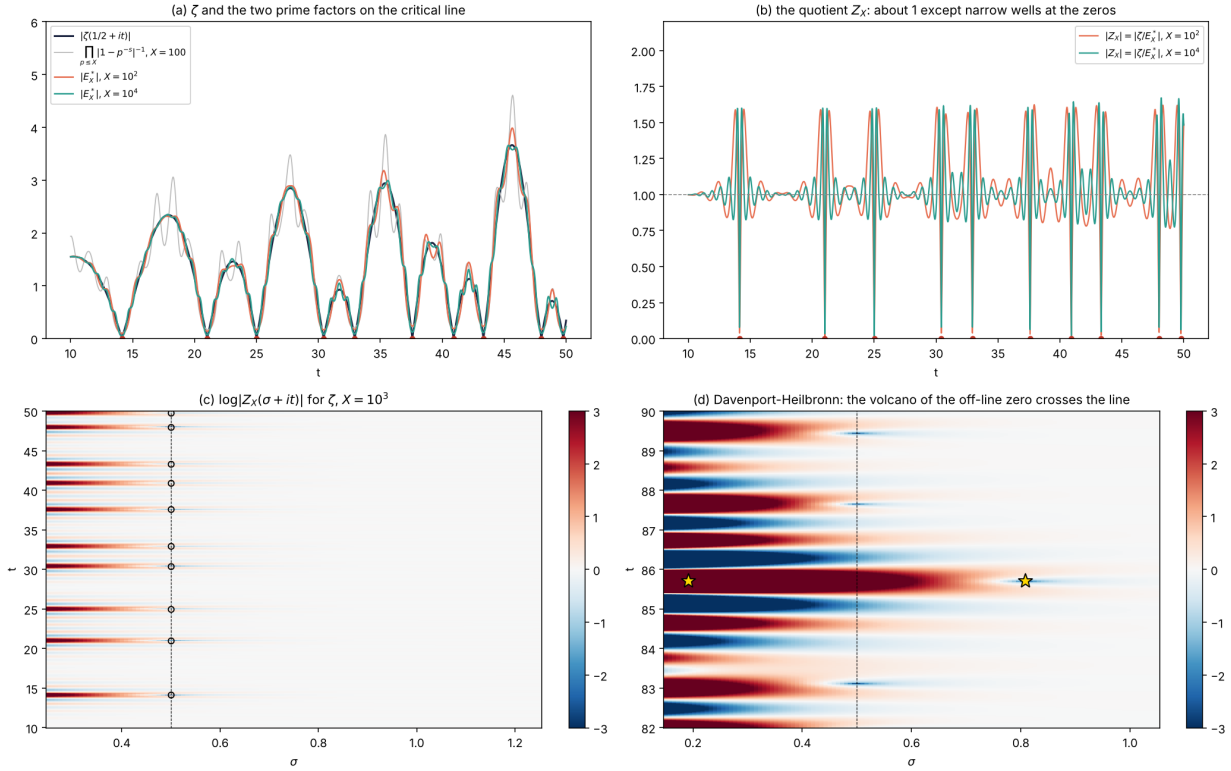


FIGURE 3. (a) $|\zeta(\frac{1}{2}+it)|$, the truncated Euler product $\prod_{p\le X}|1-p^{-s}|^{-1}$ with $X = 100$, and $|E_X^*|$ for $X = 10^2, 10^4$. (b) $|Z_X|$ on the line: wells with the profile $e^{\mathrm{Ci}(|v|)}$. (c) $\log|Z_X(\sigma+it)|$ for ζ, $X = 10^3$: craters on the line, volcanoes to the left. (d) The same for the Davenport–Heilbronn function near its off-line zero $0.8085 + 85.699i$: its volcano crosses the line.

12.1. **The zero factor seen through the cross-energy.** The factorization separates $|\zeta|$ into a zero-free factor built from the primes and a factor carried by the zeros near the point. Here we look at both through the cross-energy C_2 of consecutive partial sums of ζ [CE1]: the radius of the helix that C_2 describes on the critical line is $|E_X^*||Z_X|$, and Z_X can be computed from the partial sums alone, as the field of the charges of $\log|\zeta|$, without dividing by E_X^* and without a list of zeros. The computations of this subsection use for g_0 the Gaussian of standard deviation 0.4 of Table 7, truncated at nine standard deviations (so $a = 3.6$, and the prime sum stops at $e^{3.6}X$; the jumps

of the truncated Gaussian, below 10^{-17}, are invisible in double precision), and $X = 1000.5$; the scripts are in `anc/zero_factor/`.

Remark 28 (the zeros act locally). Let $s = \sigma + it$ with $0 \le \sigma \le 1$. For a zero ρ with $|\gamma - t| \ge 1$, (L2) and (L3) of Lemma 12, with $x_{\max} = 1$, give $|K_X(s-\rho)| \le C_A e^a X^{(\beta-\sigma)_+} |t-\gamma|^{-A-1}/L$ for every A, and since there are $O(\log(|t|+k+2))$ zeros with $k \le |\gamma - t| < k+1$,

$$\Big| \log Z_X(s) + \sum_{|\gamma - t| < W} K_X(s-\rho) \Big| \le C_A \, X^{(\Theta-\sigma)_+} \, \frac{\log(|t|+W+2)}{W^A \, L} \qquad (W \ge 1),$$

the sum taken with multiplicity, modulo $2\pi i$. So in $\zeta = E_X^* Z_X$ only the zeros near the height t matter: a zero at distance $|t-\gamma|$ contributes a term that decays faster than any power of $|t-\gamma|$, and for a Gaussian g_0 of standard deviation σ_g the decay is like $e^{-\sigma_g^2 (t-\gamma)^2/2}$. Table 10 keeps only the zeros with $|\gamma - t| \le W$. With the 6 to 33 zeros within distance 20 of t the factorization holds to the accuracy with which ζ itself is computed; the sharp cut-off at $x = X$, the weight g_X replaced by the point mass at $\log x$ [CE8], leaves errors of order 10^{-4} to 10^{-3} with all the 1517 zeros below height 2000, because each zero then contributes a term that decays only like $1/((t-\gamma)\log x)$. Status: the bound is proved; the Gaussian rate and the table are numerical (`locality_table.py`).

s	$W = 6$	$W = 10$	$W = 15$	$W = 20$	sharp, 1517 zeros
$\frac12 + 20i$	$2.3 \cdot 10^{-6}$	$2.3 \cdot 10^{-6}$	$2.9 \cdot 10^{-13}$	$3.9 \cdot 10^{-13}$	$8.9 \cdot 10^{-4}$
$\frac12 + 33i$	$2.2 \cdot 10^{-4}$	$2.8 \cdot 10^{-6}$	$1.4 \cdot 10^{-10}$	$3.0 \cdot 10^{-12}$	$8.9 \cdot 10^{-4}$
$\frac12 + 100i$	$1.7 \cdot 10^{-4}$	$1.2 \cdot 10^{-6}$	$7.8 \cdot 10^{-11}$	$2.7 \cdot 10^{-13}$	$8.9 \cdot 10^{-4}$
$\frac12 + 1000i$	$2.4 \cdot 10^{-4}$	$2.4 \cdot 10^{-6}$	$1.1 \cdot 10^{-11}$	$3.8 \cdot 10^{-12}$	$1.3 \cdot 10^{-3}$
$0.6 + 20i$	$1.1 \cdot 10^{-6}$	$1.1 \cdot 10^{-6}$	$1.6 \cdot 10^{-13}$	$2.0 \cdot 10^{-13}$	$4.5 \cdot 10^{-4}$
$0.7 + 50i$	$2.8 \cdot 10^{-4}$	$2.8 \cdot 10^{-7}$	$2.7 \cdot 10^{-11}$	$2.9 \cdot 10^{-13}$	$2.2 \cdot 10^{-4}$
$0.3 + 30i$	$7.4 \cdot 10^{-4}$	$3.7 \cdot 10^{-6}$	$6.1 \cdot 10^{-11}$	$4.1 \cdot 10^{-12}$	$3.5 \cdot 10^{-3}$

TABLE 10. The relative error $|E_X^* Z_X/\zeta - 1|$ when Z_X is built from the zeros with $|\gamma - t| \le W$ only; Gaussian weight of standard deviation 0.4, $X = 1000.5$, prime powers up to $3.7 \cdot 10^4$. Last column: the sharp cut-off at $x = 1000.5$ with the 1517 zeros below height 2000. Script `locality_table.py`.

For $s = \sigma + it$ and $n \ge 2$ let $D_n(s) = \sum_{k \le n} k^{-s}$ be the partial sums of the Dirichlet series of ζ and

$$\mathcal{W}(n,s) = C_2(n,s) + i S_2(n,s) = 2 D_{n-1}(s)\, n^{-\bar s},$$

so that $C_2(n,s) = |D_n(s)|^2 - |D_{n-1}(s)|^2 - n^{-2\sigma}$ is the cross-energy of consecutive partial sums [CE1]. ($\mathcal{W}$ is unrelated to the random variables W, W' of Theorem 5.) Write $(s)_j = s(s+1)\cdots(s+j-1)$.

Lemma 29 ([CE1, Sect. 3]). *Let $K \ge 1$, $s \ne 1$ with $\sigma > 1 - 2K$, and $n \ge 2$. Then*

$$\zeta(s) = \tfrac12 n^{\bar s}\, \mathcal{W}(n,s) - \frac{n^{1-s}}{1-s} + \tfrac12 n^{-s} + \sum_{k=1}^{K} \frac{B_{2k}}{(2k)!} (s)_{2k-1} n^{-s-2k+1} + O_K\big(|(s)_{2K+1}|\, n^{-\sigma-2K} \big).$$

In particular, for $0 < \sigma < 1$,

$$\mathcal{W}(n,s) = 2\zeta(s)\, n^{-\bar s} + \frac{2n^{1-2\sigma}}{1-s} - n^{-2\sigma} - \frac{s}{6}\, n^{-2\sigma-1} + O\big(|(s)_3|\, n^{-2\sigma-2} \big),$$

and on the critical line, for $|t| \ge 1$,

$$C_2(n, \tfrac12 + it) = \frac{1}{\frac14 + t^2} + 2n^{-1/2} \operatorname{Re}\big(n^{it} \zeta(\tfrac12 + it) \big) - \frac{1}{n} + O\big(|s|^3 n^{-2} \big).$$

Proof. The Euler–Maclaurin formula for $\sum_{k<n} k^{-s}$ with K Bernoulli terms gives $D_{n-1}(s) = \zeta(s) + \frac{n^{1-s}}{1-s} - \frac{1}{2}n^{-s} - \sum_{k=1}^{K} \frac{B_{2k}}{(2k)!}(s)_{2k-1} n^{-s-2k+1} + R$, with $R = -\frac{(s)_{2K+1}}{(2K+1)!}\int_n^\infty \widetilde{B}_{2K+1}(x) x^{-s-2K-1}dx = O_K(|(s)_{2K+1}|n^{-\sigma-2K})$ for $\sigma > 1-2K$, $\widetilde{B}_j$ the periodic Bernoulli functions; and $D_{n-1}(s) = \frac{1}{2}n^{\bar{s}}\mathcal{W}(n,s)$. Multiply by $2n^{-\bar{s}}$ for the second statement, and take real parts with $\sigma = \frac{1}{2}$ for the third. □

Remark 30 (the radius of the helix). By Lemma 29, on the critical line $\frac{1}{2}\mathcal{W}(n,s)$ winds around the centre $\frac{1}{1-s}$ with radius $|\zeta(\frac{1}{2}+it)|\, n^{-1/2}$ and angle $t\log n$, so that $\sqrt{n}\left(\frac{1}{2}\mathcal{W}(n,s) - \frac{1}{1-s}\right)$ is, up to $O(|s|^3 n^{-1/2})$, a helix of radius $|\zeta(\frac{1}{2}+it)|$ [CE1, Thm. 3.2], and the oscillating part of C_2, rescaled by $\sqrt{n}$, is $2\,\mathrm{Re}(n^{it}\zeta(\frac{1}{2}+it))$, with envelope $\pm 2|\zeta(\frac{1}{2}+it)|$. By Lemma 10 the radius factorizes as $|E_X^*(s)|\,|Z_X(s)|$: the first factor comes from the primes, the pole and the trivial zeros, and by Remark 28 the second is in effect a finite product over the zeros near t. Figure 4(a) shows the oscillating part of C_2 at $n = 2000$ inside the envelope $\pm 2|E_X^*||Z_X|$ computed with the zeros within distance 20; the prime factor $2|E_X^*|$ alone follows the envelope away from the zeros and passes over each zero without touching it. At a simple zero the collapse of the helix is carried by the single factor

$$e^{-K_X(s-\rho)} = e^{\gamma_E + \int g_X(u)\log u\, du}\,(s-\rho)\big(1 + O_X(|s-\rho|)\big),$$

since $E_1(z) = -\gamma_E - \log z + \mathrm{Ein}(z)$, and the remaining factors give

$$\zeta'(\rho) = E_X^*(\rho)\, e^{\gamma_E + \int g_X(u)\log u\, du} \prod_{\rho' \neq \rho} e^{-K_X(\rho-\rho')}$$

(see [CE8, Rem. 4], where this form is checked numerically), the constant that governs the behaviour of C_2 near a zero [CE2, Sect. 2]. Status: the factorization of the radius and the expansion at a zero are proved (Lemmas 10 and 29); the figure is numerical (`fig_local_zeros.py`).

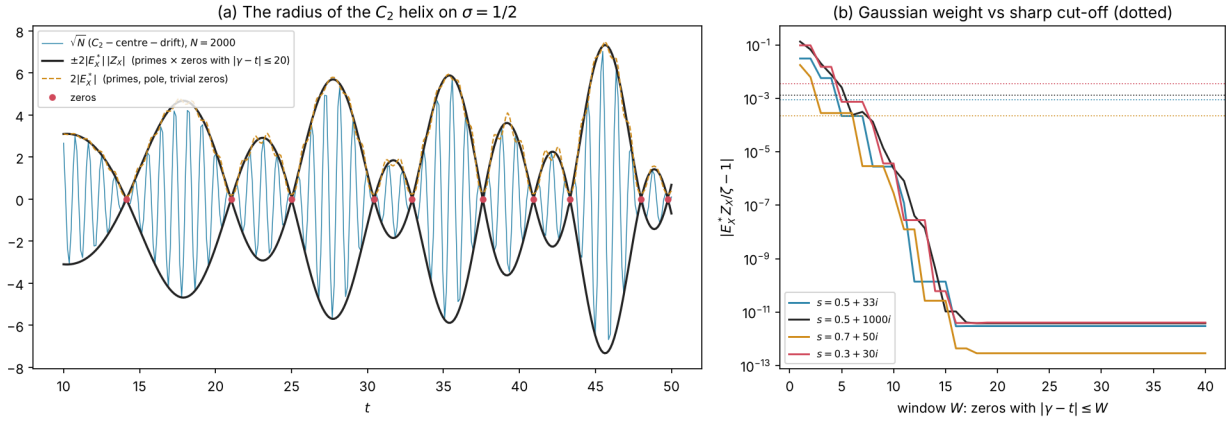


FIGURE 4. The factorization $\zeta = E_X^* Z_X$ seen by C_2, Gaussian weight of standard deviation 0.4, $X = 1000.5$. (a) $C_2(n, \frac{1}{2}+it)$ minus its centre $1/(\frac{1}{4}+t^2)$ and its Bernoulli terms, multiplied by $\sqrt{n}$, for $n = 2000$, against $\pm 2|E_X^*||Z_X|$ with the zeros $|\gamma - t| \le 20$ and against the prime factor $2|E_X^*|$ alone; dots: zeros. (b) The relative error of $E_X^* Z_X$ against the window W of zeros used, at four points (Table 10); dotted: the sharp cut-off at $x = 1000.5$ with the 1517 zeros below height 2000.

The prime factor predicts the gaps. The prime factor also sees how far apart the zeros are. For each of the 1396 gaps between consecutive ordinates $\gamma_k < \gamma_{k+1}$ with $100 < \gamma_k < 1900$, let M_k be the maximum of $\log|E_X^*(\frac12+it)|$ over the gap, with the weight and the value of X of this subsection, and let $d_k = (\gamma_{k+1}-\gamma_k)\frac{1}{2\pi}\log\frac{\gamma_k}{2\pi}$ be the normalized gap. M_k is computed from the prime powers up to $3.7\cdot10^4$ alone; the zeros enter only to delimit the gaps. The Spearman rank correlation of M_k with d_k is 0.951, as high as that of the maximum of $\log|\zeta|$ itself over the gap (0.947). The extreme quartiles do not overlap: every gap in the top quartile of M_k has $d_k \geq 1.00$, and every gap in the bottom quartile has $d_k \leq 0.89$ (the median of d_k is 0.96, its minimum 0.14). This is the explicit formula seen through the factorization, not a constraint on the zeros: $\log|E_X^*| = \log|\zeta| - \log|Z_X|$, and $\log|Z_X|$ is small between the wells, so E_X^* inherits the classical correlation between the size of ζ in a gap and the length of the gap. It does not predict the backward collision times of the zeros under the heat flow, which are fixed by the closest pair in a neighbourhood [CE8, Sect. 6.2]. Status: numerical (`gap_predictor.py`).

Since E_X^* has no zeros in $\operatorname{Re} s > -2$, $Z_X = \zeta/E_X^*$ is determined there by ζ. It can also be computed from the partial sums alone. Let $\Phi = \log|\zeta|$ on $\mathbb{C}\setminus\{1\}$, and let $\mu_\Phi = \frac{1}{2\pi}\Delta\Phi$, in the sense of distributions, be its Riesz measure (its charges); write $D_X(s) = \sum_{n\geq2}\Lambda(n)\psi_X(\log n)n^{-s}/\log n$ for the finite prime sum of (2).

Proposition 31 (the zero factor from the cross-energy). (i) *$\mu_\Phi = \sum_\rho \delta_\rho + \sum_{k\geq1}\delta_{-2k} - \delta_1$, the zeros counted with multiplicity, and for $\sigma > 0$*

$$\Phi(s) = \lim_{n\to\infty} \log\Big|\tfrac12 n^{\bar s}\,\mathcal{W}(n,s) - \frac{n^{1-s}}{1-s}\Big|,$$

locally uniformly on $\{\sigma>0\}$ minus the zeros and the point 1.

(ii) *For every s that is not a zero, a trivial zero or the pole,*

$$\log|\zeta(s)| = \operatorname{Re} D_X(s) - \int \operatorname{Re} K_X(s-w)\,d\mu_\Phi(w),$$

$$\log|Z_X(s)| = -\int_{0<\operatorname{Re} w<1} \operatorname{Re} K_X(s-w)\,d\mu_\Phi(w),$$

$$Z_X(s) = \exp\Big(-\int_{0<\operatorname{Re} w<1} K_X(s-w)\,d\mu_\Phi(w)\Big),$$

the integrals being absolutely convergent sums, the last one defined modulo $2\pi i$.

Proof. (i) Φ is harmonic off the zeros and the pole; near a zero of multiplicity m it is $m\log|s-\rho|$ plus a harmonic function, near 1 it is $-\log|s-1|$ plus a harmonic function, and $\Delta\log|s-w| = 2\pi\delta_w$. By Lemma 29 with $K=1$, $\frac12 n^{\bar s}\mathcal{W}(n,s) - \frac{n^{1-s}}{1-s} = \zeta(s) + O(n^{-\sigma})$, uniformly on compact subsets of $\sigma>0$, and $\log|\cdot|$ is continuous where $\zeta\neq0$. (ii) By Lemma 10 and (2), $\log|\zeta(s)| = \operatorname{Re} D_X(s) + \operatorname{Re} K_X(s-1) - \sum_k \operatorname{Re} K_X(s+2k) - \sum_\rho \operatorname{Re} K_X(s-\rho)$, and $\operatorname{Re} K_X$ is continuous across the cut; the last three terms are $-\int \operatorname{Re} K_X(s-w)\,d\mu_\Phi(w)$ over the charge -1 at the pole, the charges $+1$ at the trivial zeros and the charges in the strip, which are the nontrivial zeros. The sums converge absolutely by Lemma 12. The last statement is the definition of Z_X. □

So the prime factor is the finite prime sum together with the field of the charges outside the critical strip, and Z_X is the field of the charges in the strip. In this form Z_X is computed from C_2 and S_2 alone: Φ from $\mathcal{W}(n,\cdot)$ by (i), its charges by a discrete Laplacian, and Z_X from them by (ii), with the truncation in height controlled by Remark 28. The construction locates the zeros as charges; it does not say where they must be. RH is the statement that the charges of μ_Φ in the strip lie on $\operatorname{Re} w = \frac12$.

From far away an off-line pair is a quadrupole. Let $c = \frac12 + i\gamma_0$ and $0 < \delta < \frac12$. A pair of charges at $c \pm \delta$ and a double charge at c differ by

$$\log|w-c-\delta| + \log|w-c+\delta| - 2\log|w-c| = \log\Big|1 - \frac{\delta^2}{(w-c)^2}\Big| = -\operatorname{Re}\frac{\delta^2}{(w-c)^2} + O\Big(\frac{\delta^4}{|w-c|^4}\Big)$$

for $|w-c| > 2\delta$: they have the same total charge, so the counting function $N(T)$ does not distinguish them, and the same dipole moment (zero, by symmetry), and the first difference is a quadrupole of strength δ^2, which decays like $|w-c|^{-2}$. In the zero factor the same replacement changes $\log Z_X(s)$ by $-[K_X(s-c-\delta) + K_X(s-c+\delta) - 2K_X(s-c)] = -\delta^2 K_X''(s-c) + O(\delta^4 \sup|K_X^{(4)}|)$, a second difference of the kernel, which by Lemma 12 is concentrated within $O(1)$ of the height γ_0. So a symmetric off-line pair enters every quantity built linearly from the charges only at order δ^2; in the same way the product of the cross-energies at a reflected pair studied in [CE3] dips below zero only by the fraction $4\delta^2/(1-4\delta^2)$ of its limit, and the local reading of the real parts in [CE8, Rem. 9] sees symmetric pairs only at second order. Status: proved (elementary).

Remark 32 (the charges, computed). We computed Φ from $\mathcal{W}(N,\cdot)$, $N = 200$, by Lemma 29 with $K = 5$, on a grid of step $h = 0.01$ in $0.2 < \sigma < 0.8$, $5 < t < 55$, shifted by $h/2$ so that no node is a zero. The five-point Laplacian $h^2\Delta_h\Phi/2\pi$ is concentrated in eleven charges (connected clusters of cells where it exceeds 10^{-6}), each of total 1 within $1.2\cdot10^{-5}$, with centres on $\sigma = \frac12$ and at the ordinates of the zeros within $6\cdot10^{-7}$; outside the charges it is the discretization error. The factor $Z_X(\frac12+it) = \exp\big(-\sum_j Q_j\big(K_X(s-w_j) + K_X(s-\bar w_j)\big)\big)$ built from these charges Q_j at w_j and their conjugates agrees with the product over the same eleven zeros within $3.4\cdot10^{-4}$ for $10 \le t \le 50$ (Figure 5). Status: numerical (`zero_charges.py`, `zero_charges_fig.py`).

Remark 33 (primes and charges in the Jensen moment). At $s_0 = 1$ the decomposition of Theorem 1 splits the Jensen moment (1) along the factorization. With $F = PZ_X$, $P(s) = (s-1)E_X^*(s)/s$ outer (Lemma 11), the Poisson–Jensen formula of the proof of Theorem 1 gives

$$\int \log|P(\tfrac12+it)|\,P_1(t)\,dt = \log|P(1)|, \qquad \int \phi_X\,P_1 = \log|Z_X(1)| + \mathcal{O}_{\text{off}}(1),$$
$$\log|P(1)| + \log|Z_X(1)| = 0,$$

the last because $F(1) = 1$, with $\log P(1) = \sum_{n\ge2}\Lambda(n)\psi_X(\log n)/(n\log n) - \gamma_E - \int g_X\log u - \sum_{k\ge1}K_X(1+2k)$, since $\log(s-1) + K_X(s-1) \to -\gamma_E - \int g_X\log u$ as $s\to1$. The moment of the prime factor is a number computed from the primes, and all the information on the position of the zeros is in the factor of the charges: RH holds if and only if $\int\phi_X P_1 = -\log|P(1)|$, and each zero with $\beta > \frac12$ adds $\log|\rho/(1-\rho)| > 0$ to the left side. The weight $P_1(t) = 1/(2\pi(\frac14+t^2))$ is $\frac{1}{2\pi}$ times the limit of $C_2(n,\frac12+it)$ (Lemma 29). For the Gaussian weight, $\log|P(1)| = -6.857\cdot10^{-11}$, and $\log|Z_X(1)| = 6.855\cdot10^{-11}$ from the 1517 zeros below height 2000, which sum to $-1.1\cdot10^{-14}$. This is Theorem 1 at $s_0 = 1$, not a new criterion: it separates the part of the Jensen moment fixed by the primes from the part carried by the charges. Status: exact given Lemmas 10, 11 and [BSY]; the numbers are numerical (`jensen_split.py`).

13. The explicit formula in cross-energy coordinates

The wells of Z_X are the zeros seen through a smoothed Euler factor; the explicit formula for $\psi(x) = \sum_{n\le x}\Lambda(n)$ sees the same zeros through the primes. Part 5 writes that formula in the coordinates of the cross-energy [CE5, Sect. 2], and we recall what we use. By Lemma 29, $\mathcal{W}(n,s)\to 2/\bar s$ at every point $s = \frac12+it$ of the critical line, so the limits $C_2(\infty,s)$, $S_2(\infty,s)$ of $C_2(n,s)$, $S_2(n,s)$ exist there, and

$$\frac1s = \frac{C_2(\infty,s) - iS_2(\infty,s)}{2}, \qquad C_2(\infty,\tfrac12+it) = \frac{1}{\frac14+t^2} = 2\pi P_1(t), \qquad S_2(\infty,s) = 2t\,C_2(\infty,s)$$

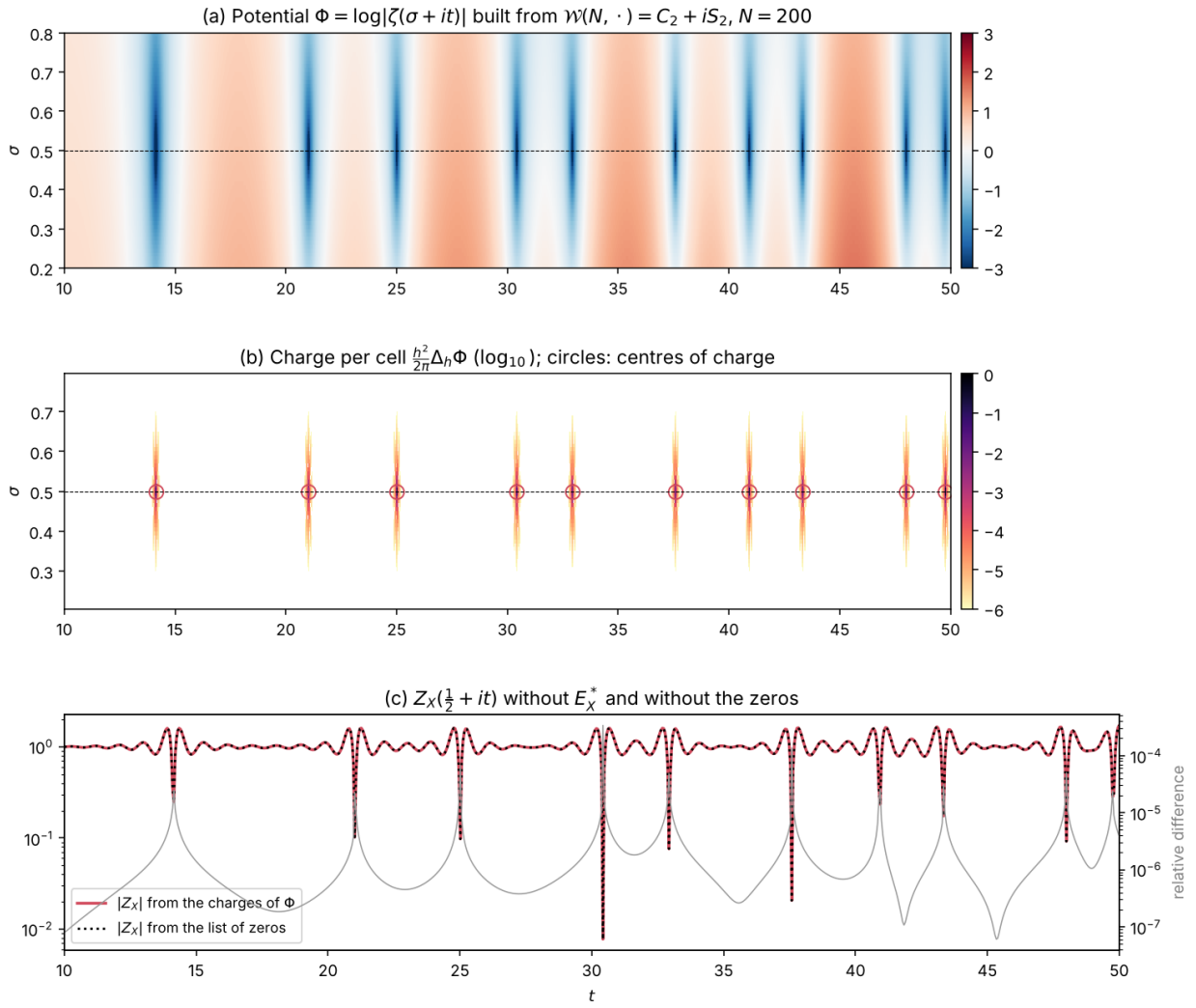


FIGURE 5. The zero factor computed from C_2 (Proposition 31). (a) The potential $\Phi = \log|\zeta|$ built from $\mathcal{W}(N,\cdot) = C_2 + iS_2$, $N = 200$. (b) Its charge density $h^2\Delta_h\Phi/2\pi$, $h = 0.01$, in $\log_{10}$; circles: centres of the eleven charges, each of total 1. (c) $|Z_X(\frac{1}{2}+it)|$ from these charges against the product over the zeros, Gaussian weight of standard deviation 0.4, $X = 1000.5$; grey, right axis: relative difference.

[CE5, Prop. 2.6]. These coordinates hold at every point of the line, zero or not; the cross-energy sees a zero on the line only through the rate of convergence, $1/n$ instead of $n^{-1/2}$, and the Jensen moment (1) integrates $\log|\zeta|$ against $C_2(\infty,\cdot)/2\pi$. In them, von Mangoldt's explicit formula becomes, under RH, a sum of cosines in $\log x$, one for each distinct ordinate $\gamma > 0$, with amplitude $2m(\gamma)\sqrt{C_2(\infty,\rho)} = 2m(\gamma)/|\rho|$, frequency γ and phase $\arctan 2\gamma = \arg\rho$ [CE5, Thm. 2.7]; the triangle inequality over these amplitudes gives Schoenfeld's constant $\frac{1}{8\pi}$ [CE5, Rem. 2.9], and the phases, almost constant, leave the size of $\psi(x) - x$ to the alignment of the numbers $\gamma\log x$ modulo 2π [CE5, Rem. 2.10]. Squaring the cosines in mean is Cramér's theorem [Cr] (see [MV2, Ch. 13]) in these coordinates [CE5, Rem. 2.8]: under RH, with $\Delta(u) = e^{-u/2}\big(\psi(e^u) - e^u\big)$,

$$\lim_{U\to\infty}\frac{1}{U}\int_0^U \Delta(u)^2\,du = \sum_\gamma m(\gamma)^2\,C_2(\infty,\tfrac{1}{2}+i\gamma) \;\geq\; \sum_\rho C_2(\infty,\rho) = 2 + \gamma_E - \log 4\pi, \tag{11}$$

where $2+\gamma_E-\log 4\pi = 0.046191\ldots$, the first sum is over the distinct ordinates of both signs, and the second over the zeros with multiplicity, with equality if and only if all zeros are simple. Since $\sum_\rho 1/\rho = (\xi'/\xi)(1)$ is Li's first coefficient λ_1 [Li], the constant is $2\lambda_1$ [CE5, Prop. 2.4]. Replacing the partial sums of ζ by $\mathcal{P}_N(s)=\sum_{n\le N}\Lambda(n)n^{-s}$ gives the arithmetic cross-energy, which, unlike $C_2(\infty,\cdot)$, sees a zero ρ_0 of multiplicity m: under RH, $\mathcal{P}_x(\rho_0)=\frac{x^{1-\rho_0}}{1-\rho_0}-m\log x+x^{-\rho_0}\big(\psi(x)-x\big)+O_{\rho_0}(1)$, the resonance of Conrad [Con, (6.12)] [CE5, Rem. 2.22]. What follows is specific to the bound: the law of Theorem 2 at $s_0=1$ and (11) are sums over the zeros of the same kernel.

Corollary 34 (the bound and the energy of the primes)**.** *Assume RH. Then*

$$\lim_{X\to\infty}(\log X)\,B_X(1)=\frac{1}{2\pi}\sum_\gamma \kappa_{m(\gamma)}\,C_2(\infty,\tfrac12+i\gamma).$$

If all zeros are simple, this equals $\dfrac{\kappa_1}{2\pi}\lim\limits_{U\to\infty}\dfrac{1}{U}\displaystyle\int_0^U\Delta(u)^2\,du=\frac{\kappa_1}{\pi}\lambda_1=0.013438\ldots$.

Proof. Theorem 2 at $s_0=1$, where $P_1(t)=\frac{1/2}{\pi(\frac14+t^2)}=\frac{1}{2\pi}C_2(\infty,\frac12+it)$ (the Poisson kernel of $\operatorname{Re}s>\frac12$ at $s_0=1$, with $\sigma_0-\frac12=\frac12$), and (11). □

So, under RH and for simple zeros, the Euler–Caceres bound at $s_0=1$ is $\kappa_1/(2\pi)=0.29092$ times the mean square of the normalized error term of the prime number theorem, divided by $\log X$: both are sums over the zeros of the same kernel $C_2(\infty,\rho)=1/|\rho|^2$. They weigh a multiple zero differently, by κ_m and by m^2 per ordinate, and both exceed their simple-zero values when a zero is multiple (Corollary 3(c) and (11)).

An isometry at $s_0=1$. The coincidence of kernels in Corollary 34 is not an accident of the limit. At $s_0=1$ the Poisson kernel is $P_1(t)=1/(2\pi|\frac12+it|^2)$, so it cancels the factor s that an integration by parts produces in the prime representation of $\log Z_X$, and the Mellin transform in the prime variable becomes an isometry. Recall $G_X(u)=\int_{-\infty}^u g_X$ (proof of Proposition 18), which vanishes for $u<L-a$ and equals 1 for $u>L+a$, and define the *energy of the primes beyond X*

$$\mathcal{E}_\Delta(X)=\int_0^\infty G_X(u)^2\,\frac{\Delta(u)^2}{u^2}\,du,$$

so that

$$\int_{Xe^a}^\infty\frac{(\psi(x)-x)^2}{(x\log x)^2}\,dx\le\mathcal{E}_\Delta(X)\le\int_{Xe^{-a}}^\infty\frac{(\psi(x)-x)^2}{(x\log x)^2}\,dx.$$

Proposition 35 (the energy criterion; standard)**.** *The following are equivalent:*

(a) *RH;*
(b) $\displaystyle\int_2^\infty\frac{(\psi(x)-x)^2}{(x\log x)^2}\,dx<\infty$*;*
(c) $\mathcal{E}_\Delta(X)<\infty$ *for some X;*
(d) $\mathcal{E}_\Delta(X)\to0$ *as $X\to\infty$.*

Under RH, $\mathcal{E}_\Delta(X)=c/\log X+o(1/\log X)$ with $c=\sum_\gamma m(\gamma)^2C_2(\infty,\frac12+i\gamma)$ as in (11).

Proof. With $x=e^u$ the integral in (b) is $\int\Delta(u)^2u^{-2}du$. Since $0\le G_X\le1$, $G_X=1$ for $u>L+a$, and Δ is bounded on compact sets, (b)⇔(c), and (d)⇒(c). (c)⇒(a): for $\sigma>\frac12$, by Cauchy–Schwarz, $\int_{L+a}^\infty|\Delta(u)|e^{-(\sigma-\frac12)u}du\le\big(\int\Delta^2u^{-2}\big)^{1/2}\big(\int u^2e^{-(2\sigma-1)u}du\big)^{1/2}<\infty$, so $\int_1^\infty(\psi(x)-x)x^{-s-1}dx=\int_0^\infty\Delta(u)e^{-(s-\frac12)u}du$ converges absolutely and is analytic in $\operatorname{Re}s>\frac12$. For $\operatorname{Re}s>1$ it equals $-\frac{\zeta'(s)}{s\zeta(s)}-\frac{1}{s-1}$ [D, Ch. 17]; hence ζ'/ζ has no pole in $\operatorname{Re}s>\frac12$ other than $s=1$, which is RH. (a)⇒(d) and the asymptotic: under RH, $A(U)=\int_0^U\Delta^2=cU+o(U)$ by (11), and partial integration

gives $\int_{L+a}^{\infty}\Delta^2u^{-2}du=-A(L+a)(L+a)^{-2}+2\int_{L+a}^{\infty}A(u)u^{-3}du=c/L+o(1/L)$. For the range $[L-a,L+a]$, which contributes $O(L^{-2})$, we use the bound $\int_V^{V+1}\Delta^2\ll 1$ uniformly in V, that is, $\int_Y^{2Y}(\psi(x)-x)^2dx\ll Y^2$ under RH [MV2, Ch. 13]. □

The criterion is a form of the classical equivalence between RH and a polynomially bounded mean square of Δ: (c)⇒(a) is the Mellin argument of [I], valid for any weight u^{-A}, and (a)⇒(b) is Cramér's theorem, valid for any weight $u^{-1-\varepsilon}$ but not for u^{-1}. We state it with the weight u^{-2} because that is the weight the isometry below produces.

Theorem 36 (an isometry at $s_0=1$)**.** *Assume RH, let $X>e^{a+1}$, let $\log Z_X(s)=-\sum_\rho K_X(s-\rho)$ (analytic in $\operatorname{Re}s>\frac12$, and defined on the line off the ordinates), and put $F_X(u)=G_X(u)\Delta(u)/u$, $\widehat{F}_X(t)=\int F_X(u)e^{-itu}du$ (the Fourier transform in L^2). Then, for almost every t, with $s=\frac12+it$,*

$$\log Z_X(s)=s\,\widehat{F}_X(t)+R_X(t),\qquad R_X(t)=\int\Delta(u)\Big(\frac{G_X(u)}{u^2}-\frac{g_X(u)}{u}\Big)e^{-itu}\,du+\sum_{k\ge1}K_X(s+2k),$$

and

$$\int\big|s\,\widehat{F}_X(t)\big|^2P_1(t)\,dt=\mathcal{E}_\Delta(X),\qquad\int|R_X(t)|^2P_1(t)\,dt=O\Big(\frac{1}{\log^2X}\Big).$$

Consequently $\displaystyle\int|\log Z_X(\tfrac12+it)|^2P_1(t)\,dt=\mathcal{E}_\Delta(X)+O\big((\log X)^{-3/2}\big)=\frac{c}{\log X}+o\Big(\frac{1}{\log X}\Big)$.

Proof. For $\operatorname{Re}s>1$, (2) and $\log\zeta(s)=\sum\Lambda(n)n^{-s}/\log n$ give, since $\int_1^\infty G_X(\log x)x^{-s}dx/\log x=K_X(s-1)$,

$$\log Z_X(s)=\int_1^\infty G_X(\log x)\frac{x^{-s}}{\log x}\,d\big(\psi(x)-x\big)+\sum_{k\ge1}K_X(s+2k).$$

(Here $\log\zeta-\log E_X^*$ and $-\sum_\rho K_X(s-\rho)$ are both analytic logarithms of Z_X in $\operatorname{Re}s>1$ that tend to 0 as $\operatorname{Re}s\to+\infty$, by (L1) of Lemma 12, so they coincide.) Put $x=e^u$, $\psi(e^u)-e^u=e^{u/2}\Delta(u)$, and integrate by parts; there are no boundary terms, because $G_X(u)=0$ for $u<L-a$ and $\Delta(u)=O(e^{u/2})$ while $\operatorname{Re}s>1$. Since $\frac{d}{du}\big(G_Xe^{-su}/u\big)=\big(g_X/u-sG_X/u-G_X/u^2\big)e^{-su}$, this gives $\log Z_X(s)=s\int F_X(u)e^{-(s-\frac12)u}du+\int F_2(u)e^{-(s-\frac12)u}du+\sum_kK_X(s+2k)$, with $F_2=\Delta(G_X/u^2-g_X/u)$. Under RH, F_X and F_2 lie in $L^2(0,\infty)$ (Proposition 35 and the bound $\int_V^{V+1}\Delta^2\ll1$ [MV2, Ch. 13]), so their Laplace transforms belong to the Hardy space H^2 of $\operatorname{Re}s>\frac12$ (Paley–Wiener); $\log Z_X$ is analytic there because Z_X has no zeros in $\operatorname{Re}s>\frac12$, so the identity holds in $\operatorname{Re}s>\frac12$ by uniqueness. As $\sigma\downarrow\frac12$ the H^2 functions converge in $L^2(dt)$ to $\widehat{F}_X$ and $\widehat{F}_2$, while $\log Z_X(\sigma+it)\to\log Z_X(\frac12+it)$ for every t that is not an ordinate; along a subsequence the convergence is also a.e. on the right, which gives the identity a.e. Since $|s|^2P_1(t)=1/(2\pi)$, Plancherel gives $\int|s\widehat{F}_X|^2P_1=\frac{1}{2\pi}\int|\widehat{F}_X|^2=\int F_X^2=\mathcal{E}_\Delta(X)$. For the remainder, $P_1\le2/\pi$ and Plancherel give $\int|\widehat{F}_2|^2P_1\le4\int F_2^2\le8\int_{L-a}^\infty\Delta^2u^{-4}+8\|g_0\|_\infty^2(L-a)^{-2}\int_{L-a}^{L+a}\Delta^2=O(L^{-2})$, and $|K_X(s+2k)|\le\int g_X(u)e^{-(2k+\frac12)u}du/((2k+\frac12)u)$ is $O(X^{-5/2})$ after summation. The last statement follows by expanding the square, Cauchy–Schwarz for the cross term, and Proposition 35. (The exponent $\frac32$ is not sharp: numerically both correction terms are below $10^{-5}/\log X$ from $\log X=10$ on.) □

Corollary 37 (the bound against the energy of the primes)**.** *Assume RH. Then*

$$\lim_{X\to\infty}\frac{B_X(1)}{\mathcal{E}_\Delta(X)}=\frac{\sum_\gamma\kappa_{m(\gamma)}C_2(\infty,\frac12+i\gamma)}{2\pi\sum_\gamma m(\gamma)^2C_2(\infty,\frac12+i\gamma)},$$

which equals $\kappa_1/(2\pi)=0.29092\ldots$ if all zeros are simple.

Proof. Theorem 2 at $s_0=1$ and Proposition 35. □

What the shared kernel does not give. Corollaries 34 and 37 and Theorem 36 are statements under RH: two sums over the zeros with the same kernel, and an isometry whose proof uses $Z_X \neq 0$ in $\operatorname{Re} s > \frac{1}{2}$. The bounds $\mathcal{E}_\Delta(X) = O(1/\log X)$ and $B_X(1) = O(1/\log X)$ are equivalent, but only because each is equivalent to RH (Proposition 35 and Corollary 3); the equivalence goes through RH, not through the kernel. Without RH, the shared kernel gives no inequality between $\mathcal{E}_\Delta(X)$ and $B_X(1)$ at a given X, nor between their constants.

Remark 38 (the wells and the primes compute the same number). Under RH, the function $\log Z_X = -\sum_\rho K_X(s-\rho)$ is a sum of wells, and $\int_{\mathbb{R}} |E_1(iv)|^2 dv = 2\int_0^\infty (\mathrm{Ci}^2 + \mathrm{si}^2) = 2\pi$, with $\mathrm{si} = \mathrm{Si} - \frac{\pi}{2}$; hence

$$\int_{\mathbb{R}} |K_X(iy)|^2 \, dy = \frac{2\pi}{\log X}\Big(1 + O\Big(\frac{1}{\log X}\Big)\Big).$$

Summing over the zeros by dominated convergence (with (L2) of Lemma 12 and $\sum_\gamma m(\gamma)^2 P_1(\gamma) < \infty$), the diagonal part of $\int |\log Z_X|^2 P_1$ is $\sum_\gamma m(\gamma)^2 P_1(\gamma) \cdot 2\pi/\log X + o(1/\log X) = c/\log X + o(1/\log X)$, the value given by the primes in Theorem 36. Consequently the interaction of the wells in this quadratic mean is $o(1/\log X)$, the counterpart, on the side of the zeros, of the vanishing of the off-diagonal terms in Cramér's theorem. Of the energy $2\pi/\log X$ of a simple well, the real part ϕ_X carries half, $\pi/\log X$; the Jensen gap does not see it quadratically but through $h(x) = e^{2x} - 1 - 2x$, which turns π into $\kappa_1 = 1.828$ because $\phi_X \to -\infty$ at the zero. This is the factor $\kappa_1/(2\pi)$ of Corollary 37: the bound is the energy of the primes beyond X, compressed by the logarithmic singularity of the wells.

Numerics. Table 11 compares the two sides of Theorem 36, with the bump weight of half-width $a = 1$. The primes side is computed from $\psi(x)$ for $x \le 10^9$, on a grid of step $2 \cdot 10^{-4}$ in u: first $\mathcal{E}_\Delta(X)$ alone, then $\mathcal{E}_\Delta(X)$ with the contribution of R_X (its cross term with $s\widehat{F}_X$ and its square, by a discrete Fourier transform on the same grid). The part $u > \log 10^9$ of $\mathcal{E}_\Delta(X)$ cannot be computed and is replaced by its value $c/\log 10^9$ under RH; it is shown in the last column and is a large part of the primes side for large X, so that there the comparison tests the identity only on $[\log X - a, \log 10^9]$. (Moving the cut from $u = 16$ to $u = 20.72$ changes every entry by at most 0.0006: $\int_0^U \Delta^2 - cU$ is nearly constant on $[8, 20.7]$, and a constant offset cancels in the partial integration.) The zeros side is $\int |\log Z_X(\frac{1}{2} + it)|^2 P_1(t)\, dt$ computed from the first 1000 zeros, with the diagonal value for the zeros above them. With R_X included the two sides agree to 1% or better at every X, including $X = 20$; $\mathcal{E}_\Delta(X)$ alone agrees within 2% from $X \approx 400$ on, the difference for small X being mainly the cross term.

Remark 39 (what the coordinates say about the bound). By Corollary 3(a) the bound is equivalent to the location of the zeros, and under RH its leading term $\Omega(s_0)/\log X$ involves only the ordinates and their multiplicities. The simultaneous alignment of the numbers $\gamma \log x$ modulo 2π, which decides the size of $\psi(x) - x$ at a given x [CE5, Rem. 2.10], does not enter $\Omega(s_0)$, and the differences of the ordinates enter B_X only at second order, through the interaction $\mathcal{I}_X$ of Theorem 5. Off the line, if $\beta > \frac{1}{2}$, the coefficient $1/\rho$ sits in the normalized cross-energy at the reflected zero $1 - \bar\rho$, and $\mathcal{W}(n, 1-\bar\rho) \sim 2n^{2\beta-1}/\bar\rho$ grows like the square of the factor $x^{\beta-\frac{1}{2}}$ by which the term of ρ outgrows $\sqrt{x}$ [CE5, Prop. 2.6 and Rem. 2.10]; this is the cross-energy counterpart of Theorem 6, and the criterion of [CE1, Cor. 3.5] (RH holds if and only if $C_2(\cdot, \rho)$ is bounded at every zero) is a restatement of $\beta = \frac{1}{2}$ through this reflection.

14. What a proof of the bound would need

By Corollary 3 the bound $B_X(s_0) = O(1/\log X)$ is RH. In this section we ask what a proof of it would have to contain. We look at a function-field model, where RH is Weil's theorem and the bound holds (Section 14.1); at the prime expansion over $\mathbb{Q}$, which is not absolutely convergent

		primes		zeros	
$\log X$	X	$L\,\mathcal{E}_\Delta(X)$	with R_X	$L\int \lvert\log Z_X\rvert^2 P_1$	tail $u>\log 10^9$
3	20	0.0727	0.0434	0.0437	0.0067
4	55	0.0532	0.0444	0.0446	0.0089
5	148	0.0470	0.0443	0.0444	0.0111
6	403	0.0455	0.0446	0.0447	0.0134
8	2981	0.0448	0.0447	0.0449	0.0178
10	$2.2\cdot10^4$	0.0451	0.0451	0.0453	0.0223
12	$1.6\cdot10^5$	0.0457	0.0457	0.0459	0.0268
14	$1.2\cdot10^6$	0.0455	0.0455	0.0459	0.0312

TABLE 11. Theorem 36, $L=\log X$: the energy of the primes beyond X, alone and with the remainder R_X, against the mean square of $\log Z_X$ on the critical line with the weight P_1; the limit is $c=2+\gamma_E-\log 4\pi=0.046191$. The last column is the part of the primes side that comes from $u>\log 10^9$, where Δ was replaced by its mean square under RH.

(Section 14.2); at the energy of ζ/E_X^* on vertical lines (Section 14.3); at Chebyshev's identity evaluated at a zero (Section 14.4); at Beurling systems that share the zeros of ζ and have one more, real, zero, and at the reflection asymmetry of Z_X (Section 14.5); and at Weil's quadratic form (Section 14.6). The propositions are proved; the conclusions about what a proof would need are heuristic.

14.1. **A function-field model.**

Setting. Let q be prime, $M\in\mathbb{F}_q[T]$ monic irreducible of degree d, and χ a character of $(\mathbb{F}_q[T]/M)^*$ that is nontrivial on $\mathbb{F}_q^*$. Then $L(u,\chi)=\sum_{f\text{ monic}}\chi(f)u^{\deg f}$ is a polynomial of degree $D=d-1$,

$$L(u,\chi)=\prod_{j=1}^{D}(1-\alpha_j u),\qquad \log L(u,\chi)=\sum_{n\ge1}\frac{c_n}{n}u^n,\qquad c_n=\sum_{\deg f=n}\Lambda(f)\chi(f)=-\sum_j\alpha_j^n,$$

and RH, $|\alpha_j|=\sqrt{q}$, is Weil's theorem [W] (applied to the curve of the associated cover, whose zeta function has $L(u,\chi)$ as a factor; see [R]). With $u=q^{-s}$ the critical line is the circle $|u|=R:=q^{-1/2}$, the half-plane $\operatorname{Re}s>\frac12$ is the disc $|u|<R$, and $X=q^N$ corresponds to the degree N.

The exact factorization. Let $w_N(n)=1-G_0(n-N)$, G_0 the distribution function of a smooth bump on $[-a,a]$, and

$$E_N^*(u)=\exp\sum_{n\le N+a}\frac{w_N(n)c_n}{n}u^n,\qquad Z_N(u)=\frac{L(u,\chi)}{E_N^*(u)}=\prod_j e^{-K_N(\alpha_j u)},$$

$$K_N(z)=\sum_{n\ge1}\frac{1-w_N(n)}{n}z^n=-\log(1-z)-\sum_{n\le N+a}\frac{w_N(n)}{n}z^n.$$

E_N^* is the exponential of a polynomial in u, built from the prime powers of degree at most $N+a$; Z_N is a finite product over the zeros (for a general polynomial L we define $Z_N:=L/E_N^*$, which is the same product, continued). On the circle, $\alpha_j u=e^{i(\theta-\theta_j)}$ under RH, and $\operatorname{Re}K_N(e^{i\varphi})=\sum_n(1-w_N(n))\frac{\cos n\varphi}{n}\approx-\operatorname{Ci}(N|\varphi|)$ for $|\varphi|$ of order $1/N$: the wells of Lemma 13 reappear, of width $1/N$ in θ. With $P_{u_0}(\theta)=\frac{1}{2\pi}\frac{R^2-|u_0|^2}{|Re^{i\theta}-u_0|^2}$, a probability density on $[-\pi,\pi]$,

$$B_N(u_0)=\tfrac12\log\int_{-\pi}^{\pi}|Z_N(Re^{i\theta})|^2P_{u_0}(\theta)\,d\theta-\log|Z_N(u_0)|.$$

Proposition 40. *Let $L(u)=\prod_{j\le D}(1-\alpha_j u)$ be any polynomial with $L(0)=1$, $c_n=-\sum_j \alpha_j^n$, and $|u_0|<R$. The following are equivalent:*

(a) *$|\alpha_j|\le\sqrt{q}$ for all j, i.e. no zero of L in the disc $|u|<R$;*
(b) *$\sup_n |c_n|q^{-n/2}<\infty$ (and then $\le D$);*
(c) *$B_N(u_0)\to 0$ as $N\to\infty$;*
(d) *$\sup_N B_N(u_0)<\infty$.*

If they hold, $N\,B_N(u_0)\to\sum_\theta \kappa_{m(\theta)}P_{u_0}(\theta)$, the sum over the distinct arguments θ of the zeros on the circle; a zero outside the closed disc contributes $O(N^{-1}(|\alpha_j|/\sqrt{q})^{N-a})$ to $\log|Z_N|$ on the circle.

Proof. Write $G_0(k)=1-w_N(N+k)$, so G_0 is non-decreasing, $G_0(k)=0$ for $k\le -a$ and $G_0(k)=1$ for $k\ge a$. Then $K_N(z)=\sum_{n\ge1}G_0(n-N)z^n/n$ for $|z|<1$ and, for all $z\ne1$,

$$K_N(z)=-\log(1-z)-\sum_{n\le N+a}\big(1-G_0(n-N)\big)\frac{z^n}{n}, \tag{12}$$

so that $e^{-K_N(z)}=(1-z)\exp\sum_{n\le N+a}(1-G_0(n-N))z^n/n$ is entire, and $Z_N=L/E_N^*$ is a polynomial times the exponential of a polynomial, with the zeros $u_j=1/\alpha_j$.

(a)⇔(b). $\sum_n c_n u^n=-\sum_j \alpha_j u/(1-\alpha_j u)$ is rational, with a pole at each $1/\alpha_j$ whose residue is $1/\alpha_j$ times the multiplicity, so the poles cannot cancel; its radius of convergence is $1/\max|\alpha_j|$, and Cauchy–Hadamard gives (b)⇒(a). Conversely, (a) gives $|c_n|\le\sum_j|\alpha_j|^n\le Dq^{n/2}$.

(a)⇒(c), and the asymptotic. By Lemma 41, (a) gives $0\le B_N(u_0)=O(1/N)$, hence (c). For the asymptotic, put $\phi_N(\theta)=\log|Z_N(Re^{i\theta})|$ and $c_N(\varphi)=-\sum_n G_0(n-N)\frac{\cos n\varphi}{n}$, so that $\phi_N(\theta)=\sum_j{}'c_N(\theta-\theta_j)+\epsilon_N(\theta)$, the sum over the zeros $u_j=1/\alpha_j=Re^{i\theta_j}$ on the circle, so that $\alpha_jRe^{i\theta}=e^{i(\theta-\theta_j)}$, and $|\epsilon_N|\le\sum_{|\alpha_j|<\sqrt{q}}\sum_{n>N-a}(|\alpha_j|R)^n/n=O\big((\max_{|\alpha_j|<\sqrt{q}}|\alpha_j|R)^{N-a}\big)$ for the zeros outside the closed disc. (These bounds, and those for c_N below, do not use Lemma 41, which uses them.) With $T_M(\varphi)=\sum_{n\ge M}\cos(n\varphi)/n$, summation by parts against the non-decreasing G_0 gives $c_N(\varphi)=-\sum_M\big(G_0(M-N)-G_0(M-1-N)\big)T_M(\varphi)$, a convex combination of the $-T_M$ with $M>N-a$. For $0<|\varphi|\le\pi$, Abel summation with $|\sum_{k\le n}\cos k\varphi|\le\frac{\pi}{2|\varphi|}+\frac12$ gives $|T_M(\varphi)|\le(\pi/|\varphi|+1)/M$; and if $M|\varphi|<1$, the terms with $M\le n<1/|\varphi|$ are positive and add up to at most $\log\frac{1}{M|\varphi|}+1$. Hence, for $N>a+1$ and $0<|\varphi|\le\pi$,

$$c_N(\varphi)\le\pi+1,\qquad |c_N(\varphi)|\le\frac{\pi/|\varphi|+1}{N-a},\qquad |c_N(\varphi)|\le\pi+2+\big|\log\big((N-a)|\varphi|\big)\big|,$$

the analogues of (5), and $c_N(v/N)\to\mathrm{Ci}(|v|)$ for every $v\ne0$ (a Riemann sum for $-\int_1^\infty\cos(vy)\,dy/y$, since $G_0(n-N)\to\mathbf{1}_{n>N}$ in the scale $n=Ny$). The Riemann sum converges because the tail bound $|T_M(\varphi)|\le(\pi/|\varphi|+1)/M$ controls the improper integral uniformly. With these four facts the proof of Theorem 2 applies on the circle, with changed constants (for instance $|c_N(v/N)|\le(\pi N/(N-a)+1)/|v|\le5/|v|$ for $1\le|v|\le\pi N$ in place of $3/|v|$): there are finitely many wells, no tail, $\log|Z_N(u_0)|=O(\tau_*^N)$ for some $\tau_*<1$, and $\int h(\phi_N)P_{u_0}\,d\theta$ times N tends to $2\sum_\theta\kappa_{m(\theta)}P_{u_0}(\theta)$.

(c)⇒(d) is trivial.

Not (a) ⇒ not (c). (This also follows from (c)⇒(d)⇒(a), but it gives an explicit lower bound.) If u_0 is a zero, $B_N(u_0)=+\infty$. Otherwise, if some zero u_j lies in $|u|<R$, the Poisson–Jensen formula on the disc (with boundary zeros allowed) gives $\int\phi_N P_{u_0}=\log|Z_N(u_0)|+\sum_{|u_j|<R}\log\big|\frac{R^2-\bar u_j u_0}{R(u_0-u_j)}\big|$, and Jensen's inequality gives $B_N(u_0)\ge\sum_{|u_j|<R}\log\big|\frac{R^2-\bar u_j u_0}{R(u_0-u_j)}\big|$, a positive number independent of N.

(d)⇒(a). Suppose that $\nu=R\max_j|\alpha_j|>1$, and let J be the set of zeros with $R|\alpha_j|=\nu$, $\alpha_j=\nu\sqrt{q}e^{i\theta_j}$, with multiplicities m_j. Fix $r'\in(|u_0|,R)$ with $\rho':=\nu r'/R>1$ such that no zero of L lies on the circle $|u|=r'$; then $\log|1-\alpha_j u|$ is bounded on that circle for every j. We may assume

that u_0 is not a zero (otherwise $B_N(u_0) = +\infty$). *Reduction to a point:* $|Z_N|^2$ is subharmonic and continuous on the closed disc, so $|Z_N(u_1)|^2 \le \int |Z_N|^2 P_{u_1}$ for $|u_1| = r'$, and $P_{u_1} \le H P_{u_0}$ with $H = \frac{(R+r')(R+|u_0|)}{(R-r')(R-|u_0|)}$ (Harnack); hence $B_N(u_0) \ge \log|Z_N(u_1)| - \log|Z_N(u_0)| - \frac12 \log H$. *The dominant zeros:* by (12), for $|z| = \rho'$,

$$-K_N(z) = \log(1-z) + \frac{z^N}{N}\big(\mathfrak{r}(z) + O(N^{-1})\big), \qquad \mathfrak{r}(z) = \sum_{k\le a} \big(1 - G_0(k)\big) z^k,$$

uniformly in $\arg z$, since $\frac{N}{N+k} = 1 + O(\frac{|k|}{N})$ for $|k| \le \frac{N}{2}$ and the terms with $k < -\frac{N}{2}$ are $O(N\rho'^{-N/2})$ relative to z^N/N; $\mathfrak{r}$ is analytic in $|z| > 1$. The zeros with $R|\alpha_j| < \nu$ contribute $O\big((\rho'')^N/N + 1\big)$ to $\log|Z_N(u_1)|$ for some $\rho'' < \rho'$, and every zero contributes $O\big((\nu|u_0|/R)^N/N + 1\big)$ to $\log|Z_N(u_0)|$. *Choice of* $\arg u_1$*:* with $u_1 = r'e^{i\varphi}$, the main term is $\frac{\rho'^N}{N} \operatorname{Re}\big(e^{iN\varphi} G_{a(N)}(\varphi)\big)$, where $G_a(\varphi) = \sum_{j\in J} m_j a_j\, \mathfrak{r}(\rho' e^{i(\theta_j+\varphi)})$ and $a(N) = (e^{iN\theta_j})_j$ lies on the torus $\mathbb{T}^J$. The functions $\varphi \mapsto \mathfrak{r}(\rho' e^{i(\theta_j+\varphi)})$ are linearly independent: a relation $\sum_j b_j \mathfrak{r}(\rho' e^{i(\theta_j+\varphi)}) \equiv 0$ gives $\sum_j b_j e^{ik\theta_j} = 0$ for every $k \le -a$, where $1 - G_0(k) = 1$, and the Vandermonde matrix of the distinct $e^{i\theta_j}$ forces $b_j = 0$. So $\mu := \min_{a\in\mathbb{T}^J} \max_\varphi |G_a(\varphi)| > 0$, by continuity and compactness, and the G_a are uniformly Lipschitz in φ. Taking φ_0 with $|G_{a(N)}(\varphi_0)| \ge \mu$ and then $\varphi \in [\varphi_0, \varphi_0 + 2\pi/N)$ with $e^{iN\varphi} G_{a(N)}(\varphi_0) > 0$ gives $\operatorname{Re}\big(e^{iN\varphi} G_{a(N)}(\varphi)\big) \ge \mu - O(1/N) \ge \mu/2$ for N large. Since $\rho' > \max(1, \nu|u_0|/R, \rho'')$, this gives $B_N(u_0) \ge \frac{\mu \rho'^N}{3N}$ for N large, so $B_N \to \infty$. □

Lemma 41 (one frequency per degree)**.** *Assume (b) and write* $Z_N = \exp\Phi_N$, $\Phi_N(u) = \sum_n b_n u^n$, $b_n = (1 - w_N(n)) c_n/n$. *With* $u_0 = \tau R e^{i\theta_0}$, $0 \le \tau < 1$,

$$\int |\Phi_N(Re^{i\theta})|^2 P_{u_0}(\theta)\, d\theta = \sum_{m,n} b_m \bar b_n R^{m+n} \tau^{|m-n|} e^{i(m-n)\theta_0},$$

the double series converging absolutely, with sum at most $\frac{1+\tau}{1-\tau} D^2 \sum_{n>N-a} n^{-2} = O(1/N)$. *Consequently* $\int |Z_N|^2 P_{u_0} = 1 + O(1/N)$ *and* $0 \le B_N(u_0) = O(1/N)$.

Proof. The Fourier coefficients of P_{u_0} are $\tau^{|k|} e^{-ik\theta_0}$, $|b_n| R^n \le D/n$, and $|b_m b_n| \le \frac12(|b_m|^2 + |b_n|^2)$; sum the geometric series in $|m-n|$. For the energy, (a) gives the one-sided bound $\operatorname{Re}\Phi_N = \log|Z_N| \le (\pi+1)D + o(1)$ on the circle (proof of Proposition 40: $c_N \le \pi + 1$), so with $h(x) = e^{2x} - 1 - 2x \le Cx^2$ for $x \le (\pi+1)D + 1$, $\int |Z_N|^2 P = 1 + 2\operatorname{Re}\Phi_N(u_0) + O(\int |\Phi_N|^2 P) = 1 + O(1/N)$; Jensen gives $B_N \ge 0$. □

Remark 42 (where Weil enters). In the function-field case the whole bound reduces to (b), and Lemma 41 turns (b) into control of the energy. The mechanism is that all prime powers of degree n have the same frequency $n \log q$, so the cancellation among them is already inside the single number c_n, and Weil's theorem bounds it. In the Stepanov–Bombieri method [R, Appendix], (b) comes from an auxiliary polynomial that vanishes to high order at the rational points of a curve, built with the Frobenius map, and from the passage to all extensions $\mathbb{F}_{q^k}$. No counterpart of these ingredients that yields RH is known over $\mathbb{Q}$.

Numerical illustration. We took $q = 3$, $M = T^8 + T^5 + 2$ (irreducible, with T primitive), and χ the character of order $3^8 - 1$ with $\chi(T) = e^{2\pi i/6560}$, which is odd; L has degree 7 and all its zeros on $|u| = 3^{-1/2}$ (checked to 10^{-12}). With $a = 2.5$ and $u_0 = R/3$ (the analogue of $\operatorname{Re} s_0 = \frac32$), $NB_N(u_0) = 1.591, 1.769, 1.906, 1.985, 2.008, 2.042$ for $N = 10, 20, 40, 80, 160, 320$, against the limit $\kappa_1 \sum_j P_{u_0}(\theta_j) = 2.063$; the analogue of Theorem 5 predicts 1.985 at $N = 80$ and 2.043 at $N = 320$. Moving one zero into the disc ($|\alpha_1| = 1.05\sqrt{q}$), the prime sums leave the Weil range from $n \approx 50$ on and B_N grows geometrically: 0.20, 0.28, 1.97, 14.8 for $N = 10, 20, 40, 80$ (Figure 6).

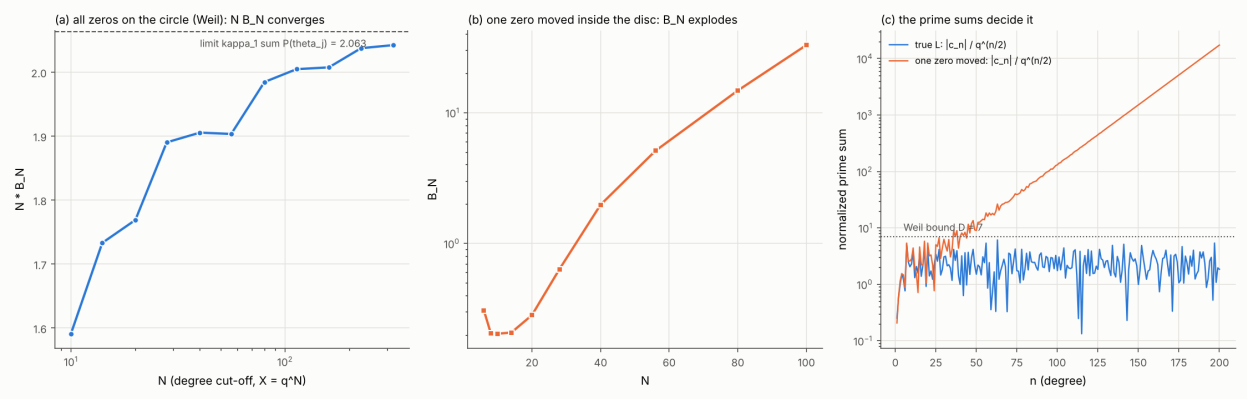


FIGURE 6. The Euler–Caceres bound over $\mathbb{F}_3[T]$, $\deg L = 7$. (a) NB_N for the true L-function (all zeros on the circle, by Weil's theorem). (b) B_N with one zero moved into the disc. (c) The normalized prime sums $|c_n|q^{-n/2}$: bounded by D for the true L, geometrically growing for the perturbed one.

14.2. **Over $\mathbb{Q}$: no frequency is shared.** Over $\mathbb{Q}$ the analogue of Φ_N is the tail of the prime series, which, because E_X^* contains the pole term, is an integral against the signed measure $d\psi(x) - dx$:

$$\log\zeta - \log E_X^* = \int_1^\infty \big(1 - w_X(x)\big)\frac{x^{-s}}{\log x}\,\big(d\psi(x) - dx\big) + (\text{trivial zeros}),$$

with $w_X(x) = \psi_X(\log x) = 1 - G_X(\log x)$ in the notation of (2). Each prime power has its own frequency $\log n$, and no two coincide.

Lemma 43. *Let $b_n = \Lambda(n)n^{-1/2}/\log n$, $\eta > 0$, $X \geq 2$. Then, for $Y > 4X$,*

$$\sum_{X<m,n\leq Y} b_m b_n \Big(\frac{\min(m,n)}{\max(m,n)}\Big)^\eta \;\gg_\eta\; \frac{Y}{\log^2 Y}.$$

Proof. All terms are nonnegative. For a prime $n \in (Y/2, Y]$, $b_n = n^{-1/2}$, and the primes $m \in (n/2, n]$ contribute at least $2^{-\eta}n^{-1/2}\sum_{n/2<m\leq n} m^{-1/2} \gg (\log n)^{-1}$; there are $\gg Y/\log Y$ such n. □

The Fourier transform of P_{s_0} is $e^{i\xi t_0 - \eta|\xi|}$, $\eta = \sigma_0 - \frac{1}{2}$, so the formal mean square of the prime tail on the critical line with the weight P_{s_0} is a double sum over prime powers with the kernel $(m/n)^{it_0}(\min/\max)^\eta$, and Lemma 43 says that its absolute series diverges, for every s_0 and X, whatever the truth of RH. (The absolute series of the signed measure $d\psi - dx$ is larger still.) At $s_0 = 1$, under RH, the signed truncations do converge in $L^2(P_1)$, and $\int |\log Z_X|^2 P_1 = O(1/\log X)$ (Theorem 36); the bound is then a cancellation phenomenon among primes in windows of bounded logarithmic length, which absolute estimates on the prime expansion do not see. In the function-field case the absolute series over individual primes also diverges; the series of Lemma 41 converges because the primes of each degree share one exact frequency and Weil's bound acts inside each c_n. Over $\mathbb{Q}$ there is no such grouping. The Montgomery–Vaughan theorem [MV] $\int_0^T |\sum a_n n^{-it}|^2 dt = \sum |a_n|^2(T + O(n))$, the standard tool for such cancellation, gives nothing when $n \gg T$, which is the regime of the bound (fixed weight, $X \to \infty$).

The coefficients of the tail. The multiplicative counterpart of the prime tail has an exact arithmetic structure, which explains why it cannot carry the cancellation by itself. Let $A_X(s) = \sum_{n\geq 2}\Lambda(n)w_X(n)n^{-s}/\log n$ be the (finite) prime sum of $\log E_X^*$, an entire function, and $\tilde{D}_X(s) = \zeta(s)e^{-A_X(s)} = \sum_{n\geq 1}\tilde{b}_X(n)n^{-s}$ for $\operatorname{Re} s > 1$, so that $Z_X = \tilde{D}_X\, e^{-K_X(s-1)}e^{\sum_k K_X(s+2k)}$. Then, unconditionally:

(i) $\tilde b_X(n)\log n=\sum_{d\mid n}\Lambda(d)\,(1-w_X(d))\,\tilde b_X(n/d)$, $\tilde b_X(1)=1$; hence $\tilde b_X(n)\ge 0$ for all n;
(ii) $\tilde b_X(n)=0$ for $2\le n\le Xe^{-a}$, and $\tilde b_X(p)=1-w_X(p)$ for every prime p, so $\tilde b_X(p)=1$ for $p>Xe^{a}$;
(iii) with $\tilde S_X(x)=\sum_{n\le x}\tilde b_X(n)$ and $\tilde c_X=e^{-A_X(1)}$ (the residue of $\tilde D_X$ at 1), $\tilde S_X(x)-\tilde c_X x$ is bounded for $x\ge 1$; hence the energy $\int_1^\infty|\tilde S_X(x)-\tilde c_X x|^2x^{-2}\,dx$ is finite for every X, and it equals $\frac{1}{2\pi}\int|\Psi(\frac12+it)|^2dt$ with $\Psi(s)=\tilde D_X(s)/s-\tilde c_X/(s-1)$.

(i) is the logarithmic derivative of $\log\tilde D_X=\sum\Lambda(n)(1-w_X(n))n^{-s}/\log n$, a Dirichlet series with nonnegative coefficients; (ii) follows from (i), since $1-w_X(d)=0$ for $d\le Xe^{-a}$ and $=1$ for $d>Xe^{a}$. For (iii), $e^{-A_X(s)}=\sum_m\eta_X(m)m^{-s}$ is a finite product over $p\le Xe^{a}$ of exponentials of polynomials in p^{-s}, so $\sum_m|\eta_X(m)|<\infty$. Since $\tilde b_X=1*a$, $\tilde S_X(x)=\sum_m\eta_X(m)\lfloor x/m\rfloor=\tilde c_X x-\sum_m\eta_X(m)\{x/m\}$, with $\sum_m\eta_X(m)/m=e^{-A_X(1)}=\tilde c_X$, so $|\tilde S_X(x)-\tilde c_X x|\le\sum|\eta_X(m)|$. Then $u\mapsto(\tilde S_X(e^u)-\tilde c_X e^u)e^{-u/2}$ lies in $L^2(0,\infty)$; its Laplace transform, $\int_1^\infty(\tilde S_X(x)-\tilde c_X x)x^{-s-1}dx$, equals $\Psi(s)$ for $\operatorname{Re}s>1$ and belongs to the Hardy space of $\operatorname{Re}s>\frac12$, and Plancherel gives the identity.

So the coefficients of ζe^{-A_X} have no sign changes, and the analogue of Lemma 43 holds for them a fortiori, for $Y>4Xe^{a}$. The energy of their summatory error is finite for every X without any hypothesis, so it carries no information about RH. A finite energy for $\tilde D_X$ says nothing about $Z_X=\tilde D_X e^{-K_X(s-1)}e^{\sum_k K_X(s+2k)}$, because on the critical line the explicit factor $e^{-K_X(s-1)}$ has size up to $\exp(O(\sqrt{X}/\log X))$ (near $t=0$, $-\operatorname{Re}K_X(s-1)\approx\int g_X(u)\operatorname{Ei}(u/2)\,du$), and the bound for Z_X requires a cancellation between the two factors. Status: (i)–(iii) are proved (elementary), and the vanishing, the values at primes and the boundedness of $\tilde S_X(x)-\tilde c_X x$ were checked numerically for $X=100$; the remarks after them are heuristic.

Remark 44 (the signed form). For the logarithmic derivative instead of the logarithm (coefficients $\Lambda(n)$ instead of $\Lambda(n)/\log n$), the analogue of the form in Lemma 43 at $s_0=1$ ($\eta=\frac12$, $t_0=0$), with the range $1\le m,n\le N$ in place of $(X,Y]$, is

$$\sum_{m,n\le N}\frac{\Lambda(m)\Lambda(n)}{\sqrt{mn}}\Big(\frac{\min(m,n)}{\max(m,n)}\Big)^{1/2}=\sum_{m,n\le N}\frac{\Lambda(m)\Lambda(n)}{\max(m,n)}\ \sim\ 2N.$$

Its signed form, with the pole removed, is an exact identity for every real $N\ge1$: expanding $\psi(x)^2=\sum_{m,n}\Lambda(m)\Lambda(n)\mathbf{1}_{\max(m,n)\le x}$,

$$\sum_{m,n\le N}\frac{\Lambda(m)\Lambda(n)}{\max(m,n)}-\frac{\psi(N)^2}{N}-2\sum_{n\le N}\Lambda(n)\log\frac{N}{n}+N-1=\int_1^N\frac{(\psi(x)-x)^2}{x^2}\,dx.$$

Equivalently, with $\mathcal P_N$ as in Section 13 and $\mathfrak D_N(s)=\mathcal P_N(s)-\frac{N^{1-s}}{1-s}+\frac{s}{1-s}-N^{-s}(\psi(N)-N)=s\int_1^N(\psi(x)-x)x^{-s-1}dx$, Plancherel ($\int P_1(t)e^{i\xi t}dt=e^{-|\xi|/2}$ and $|s|^2P_1=1/2\pi$) gives

$$\int|\mathfrak D_N(\tfrac12+it)|^2P_1(t)\,dt=\int_1^N\frac{(\psi(x)-x)^2}{x^2}\,dx,$$

without any hypothesis. This is the analogue, for $-\frac{\zeta'}{\zeta}(s)-\frac{s}{s-1}$ truncated at $x\le N$, of the Plancherel step in the proof of Theorem 36; it is unconditional because the support is compact. The averaged arithmetic cross-energy is elementary, $\int C_2^{\Lambda}(n,\frac12+it)P_1(t)\,dt=2\Lambda(n)\psi(n-1)/n$. Under RH the right side is $c\log N+o(\log N)$ by (11); if RH fails, $\limsup_N\log\big(\int_1^N\big)/\log N=2\Theta-1>0$, by the Mellin argument of Proposition 35 with the weight $x^{-\delta}$ and by $\psi(x)-x\ll x^{\Theta+\varepsilon}$. So the cancellation that the bound requires is this: a sum of positive terms asymptotic to $2N$, minus single prime sums of total size $2N$ whose individual fluctuations are of order $\sqrt N$, must be $O(N^{\varepsilon})$ for every $\varepsilon>0$; this is equivalent to RH, hence to the bound (Corollary 3). Numerically, with the

primes up to 10^6, the two sides of the identity agree to nine digits, and their difference from $c\log N$ is 2.481, 2.504, 2.510, 2.513, 2.511 for $N = 10^2, \dots, 10^6$.

Remark 45 (positivity alone pays the pole). The positivity of $\Lambda(n)$ does give a pointwise bound, but not a useful one. Let $\Sigma_X(s) = \sum_n \Lambda(n)\psi_X(\log n)n^{-s}/\log n$ be the prime sum of $\log E_X^*$ in (2). For each n, $3 + 4\cos\theta + \cos 2\theta \geq 0$, so $3\,\mathrm{Re}\,\Sigma_X(\frac{1}{2}) + 4\,\mathrm{Re}\,\Sigma_X(\frac{1}{2}+it) + \mathrm{Re}\,\Sigma_X(\frac{1}{2}+2it) \geq 0$, which bounds $-\mathrm{Re}\,\Sigma_X(\frac{1}{2}+it)$, and hence $\log|Z_X(\frac{1}{2}+it)| = \log|\zeta(\frac{1}{2}+it)| - \mathrm{Re}\,\Sigma_X(\frac{1}{2}+it) - \mathrm{Re}\,K_X(-\frac{1}{2}+it) + \sum_k \mathrm{Re}\,K_X(\frac{1}{2}+2k+it)$, from above. The bound is useless: the constant term of the trigonometric polynomial sits at $t = 0$, on the pole, where $\Sigma_X(\frac{1}{2}) \asymp \sqrt{X}/\log X$. Numerically, with the Gaussian weight of Table 7, $\Sigma_X(\frac{1}{2}) = 6.6$, 14.0, 30.8 for $X = 10^2, 10^3, 10^4$, the resulting upper bound for $\log|Z_X(\frac{1}{2}+it)|$ at $t = 20, 30, 40$ is about 5, 10 and 23 for these X, and the true values lie between -0.2 and 0.5 (script `mollifier_tables.py`). Every nonnegative trigonometric polynomial $\sum_j a_j \cos j\theta$ has $a_0 > 0$, so every argument of this type pays a term of size $X^{1/2}/\log X$ on the critical line; this is a heuristic statement about a class of arguments, not a theorem.

14.3. **The energy abscissa.** For $\sigma \geq \frac{1}{2}$ let $\mathcal{E}_\sigma(X) = \int |\zeta/E_X^*(\sigma+it)|^2 \frac{dt}{\pi(1+t^2)}$ and

$$\sigma_E = \inf\Big\{\sigma \geq \tfrac{1}{2} : \limsup_{X\to\infty} \mathcal{E}_\sigma(X) < \infty\Big\}.$$

Proposition 46. *(i) Unconditionally, $\mathcal{E}_\sigma(X) \to 1$ for every $\sigma > \Theta$; hence $\sigma_E \leq \Theta$. (ii) If $\Theta > \frac{1}{2}$ is attained, then $\mathcal{E}_\sigma(X) \to \infty$ for every $\sigma \in (\frac{1}{2}, \Theta)$; hence $\sigma_E = \Theta$. In particular, when Θ is attained, RH holds if and only if $\sigma_E = \frac{1}{2}$.*

Proof. (i) For $\mathrm{Re}\,s = \sigma > \Theta$ every zero has $\mathrm{Re}(s-\rho) \geq \sigma - \Theta =: \eta > 0$, so by (L1) of Lemma 12, $|\log|Z_X(\sigma+it)|| \leq C_\eta X^{-\eta} L^{-1} \sum_\rho (1+|t-\gamma|)^{-2} \leq C_\eta X^{-\eta}\log(|t|+3)$; for large X, $|Z_X|^2 \leq (|t|+3)^{1/2}$, which is integrable against $dt/(1+t^2)$, and dominated convergence gives $\mathcal{E}_\sigma \to 1$. Here the pole term of E_X^* is essential: without it the factor $X^{1-\sigma}$ of the pole would make $\mathcal{E}_\sigma$ explode for $\sigma < 1$. (ii) Fix $\sigma \in (\frac{1}{2}, \Theta)$ and X. *Polynomial growth.* $Z_X = \zeta/E_X^*$ is entire (Lemma 10). On $\mathrm{Re}\,s \geq 2$, ζ and $\log E_X^*$ are bounded (for $K_X(s-1)$ and the trivial-zero terms use $|E_1(z)| \leq e^{-\mathrm{Re}\,z}/|z|$, $\mathrm{Re}\,z > 0$). On $\sigma \leq \mathrm{Re}\,s \leq 2$, $|t| \geq 1$: $|\zeta(s)| \ll |t|^{1/2}$ (convexity), the finite prime sum in $\log E_X^*$ is bounded, $K_X(s-1)$ is bounded by (L2) (with $x_{\max} = 1$) if $\mathrm{Re}\,s \geq 1$ and by the first part of (L3) if $\sigma \leq \mathrm{Re}\,s < 1$, and the trivial-zero terms are bounded (proof of Lemma 11); the rest is compact and Z_X is continuous there. So $|Z_X(s)| \leq C_X(|s|+2)^{1/2}$ on $\mathrm{Re}\,s \geq \sigma$. *Sub-mean-value inequality.* Then $f = Z_X/(s+1)$ is bounded and analytic on $\mathrm{Re}\,s > \sigma$ and continuous up to the line, so $\log|f(s_1)| \leq \int \log|f(\sigma+it)| P_{s_1}^{(\sigma)}(t)\,dt$ for $\mathrm{Re}\,s_1 > \sigma$, where $P_{s_1}^{(\sigma)}$ is the Poisson kernel of $\mathrm{Re}\,s > \sigma$; and $\log|s+1|$ is the Poisson integral of its boundary values, because $s+1$ has no zeros in $\mathrm{Re}\,s \geq \sigma$ and grows at most linearly. Hence $\log|Z_X(s_1)| \leq \int \log|Z_X(\sigma+it)| P_{s_1}^{(\sigma)}(t)\,dt \leq \frac{1}{2}\log\int |Z_X(\sigma+it)|^2 P_{s_1}^{(\sigma)}(t)\,dt$, by Jensen's inequality. Since $P_{s_1}^{(\sigma)}(t) \leq C_{s_1}/(1+t^2)$, $\mathcal{E}_\sigma(X) \geq c_{s_1}|Z_X(s_1)|^2$. *Growth at one point.* Take $\rho_0 = \Theta + i\tau_0$, and choose r as in part (b) of the proof of Theorem 6 and moreover $r < \Theta - \sigma$, so that $s_1 = \rho_0 - r$ has $\mathrm{Re}\,s_1 > \sigma$. Part (c) of that proof uses only Lemma 12 and the position of the zeros, not the half-plane, and gives $\log|Z_X(s_1)| \geq \frac{1}{2}\,\mathrm{Ei}(r(\log X - a)) - C\log\log X \to \infty$. Hence $\mathcal{E}_\sigma(X) \to \infty$. The last statement follows: under RH, $\Theta = \frac{1}{2}$ is attained and (i) gives $\sigma_E = \frac{1}{2}$; if $\Theta > \frac{1}{2}$ is attained, $\sigma_E = \Theta > \frac{1}{2}$. □

Remark 47. When Θ is attained, Proposition 46 is a mean-square analogue of the classical fact that Θ is the abscissa of convergence of $\sum(\Lambda(n)-1)n^{-s}$: the energy of the Euler–Caceres quotient on vertical lines sees exactly the real parts of the zeros. It does not make the bound easier: for $\frac{1}{2} < \sigma < 1$ the absolute series of the arithmetic expansion of $\mathcal{E}_\sigma$ diverges as in Lemma 43, so in the whole region where the question lives only cancellation can help.

Numerics. Table 12 gives $\log \mathcal{E}_\sigma(X)$ for ζ and for the Davenport–Heilbronn function f (Gaussian weight g_0 of standard deviation 0.4, which is not compactly supported, with the prime sum truncated at 4.5 standard deviations; $|t| \le 300$). For ζ the energy tends to 1 on every line, as Proposition 46(i) predicts under RH for $\sigma > \frac12$ and Corollary 9 on the critical line, where $\mathcal{E}_{1/2} = \mathcal{E}/\pi$. For f it explodes on $\sigma = 0.5$ and $\sigma = 0.6$, below its off-line zero $\rho_f = 0.8085 + 85.699i$. On $\sigma \ge 0.75$ nothing is visible up to $X = 10^4$, although f has zeros to the right of these lines: the volcano of ρ_f must first overcome the weight $w = 1/(\pi(1+\gamma_f^2)) \approx 4.33 \cdot 10^{-5}$ of its height. Using the lower bound $\mathrm{Ei}(y)$, $y = (\beta_f - \sigma)\log X$, of Lemma 12(L4) for the volcano, a heuristic onset is $2\,\mathrm{Ei}(y) \approx \log \frac{1}{w(\beta_f-\sigma)}$, that is, $\log X \approx 7.0$, 10.6 and 40 for $\sigma = 0.5$, 0.6 and 0.75; this ignores the width of the lobe and the shift $\log X - a$, and is consistent in order of magnitude with the table. The third value is out of numerical reach. The energy abscissa is an asymptotic notion, and a finite computation sees it only for zeros that are low or far from the line.

	$X = 10$	10^2	10^3	10^4
ζ, $\sigma = 0.5$	0.018	0.010	0.007	0.005
ζ, $\sigma = 0.6$	0.009	0.002	0.001	0.000
ζ, $\sigma = 0.75$, 0.9, 1, 1.25	≤ 0.003	0.000	0.000	0.000
f, $\sigma = 0.5$	0.080	0.067	1.876	12.097
f, $\sigma = 0.6$	0.039	0.014	0.034	0.950
f, $\sigma = 0.75$	0.014	0.002	0.001	0.000
f, $\sigma = 0.9$, 1, 1.25	≤ 0.005	0.000	0.000	0.000

TABLE 12. $\log \mathcal{E}_\sigma(X)$ on vertical lines.

14.4. **Chebyshev's identity at a zero.** A proof has to use the integers and the primes together: the Davenport–Heilbronn function has the additive structure and a functional equation, and Beurling systems have an Euler product with $\Lambda \ge 0$, and both can have zeros off the line; Beurling systems whose integers satisfy $N(x) = cx + O(x^\theta)$ with $\frac12 < \theta < 1$ can even have zeros approaching $\mathrm{Re}\, s = 1$ [DMV]. The identity $\log = \Lambda * 1$ joins the two structures. With $L_N(s) = \sum_{n\le N} \log n\, n^{-s}$, $D_y(s) = \sum_{k\le y} k^{-s}$ and $\mathcal{P}_N$ as in Section 13,

$$L_N(s) = \sum_{m\le N} \Lambda(m)\, m^{-s} D_{N/m}(s), \quad \text{equivalently} \quad 2L_N(s)N^{-\bar s} = \sum_{m\le N} \Lambda(m)\, m^{-2\sigma}\, \mathcal{W}^+(N/m, s),$$

where $\mathcal{W}^+(y,s) = 2D_y(s)y^{-\bar s}$ is the cross-energy wave of the integers at the scale y: on the line, the logarithmically weighted wave at the scale N is the sum, with the weights $\Lambda(m)/m$, of the integer waves at the scales N/m. The identity also holds in every discrete Beurling system; what distinguishes ζ is the integer side D_y, whose counting error is $O(1)$, and a proof along these lines must use more of it than $N(x) = cx + O(x^\theta)$ for any fixed $\theta > \frac25$ (Table 15).

At a zero the identity takes a special form. For a nontrivial zero $\rho = \beta + i\gamma$ and $y \ge 1$ write $D_y(\rho) = A_\rho(y) + E_\rho(y)$, with $A_\rho(y) = \frac{y^{1-\rho}}{1-\rho} - \psi_1(y)\, y^{-\rho}$, $\psi_1(y) = \{y\} - \frac12$, the first two terms of the Euler–Maclaurin expansion (there is no constant term because $\zeta(\rho) = 0$). Let $M(N) = \sum_{m\le N} \Lambda(m)/m = \log N - \gamma_E + r(N)$.

Proposition 48 (Chebyshev's identity at a zero)**.** *For every nontrivial zero ρ and every real $N \ge 1$, with $\log\lfloor N\rfloor!$ in place of $\log N!$ when N is not an integer,*

$$\sum_{m\le N} \Lambda(m)\, m^{-\rho} E_\rho(N/m) = L_N(\rho) - \frac{\rho}{1-\rho} N^{1-\rho} M(N) - N^{-\rho}\Big(\frac{\psi(N)}{2} + \log N!\Big),$$

and, as $N \to \infty$ through the integers, with $c_\rho = \frac{1}{2} + \gamma_E \frac{\rho}{1-\rho} - \frac{1}{(1-\rho)^2}$,

$$\sum_{m \le N} \Lambda(m)\, m^{-\rho} E_\rho(N/m) = c_\rho N^{1-\rho} - \zeta'(\rho) - \frac{\rho}{1-\rho} N^{1-\rho} r(N) \\ - \frac{1}{2} N^{-\rho}\big(\psi(N) - N\big) - \frac{\log 2\pi}{2} N^{-\rho} + O_\rho\big(N^{-\beta-1} \log N\big).$$

Proof. In $\sum_m \Lambda(m) m^{-\rho} A_\rho(N/m)$ the power term gives $\frac{N^{1-\rho}}{1-\rho} M(N)$, and in the sawtooth term the twist cancels, $m^{-\rho}(N/m)^{-\rho} = N^{-\rho}$, leaving $-N^{-\rho} \sum_m \Lambda(m) \psi_1(N/m)$. Since $\lfloor y \rfloor = y - \psi_1(y) - \frac{1}{2}$, Chebyshev's identity $\sum_{m \le N} \Lambda(m) \lfloor N/m \rfloor = \log N!$ gives $\sum_m \Lambda(m) \psi_1(N/m) = N M(N) - \frac{1}{2}\psi(N) - \log N!$. Subtracting from $L_N(\rho) = \sum_m \Lambda(m) m^{-\rho} D_{N/m}(\rho)$ gives the identity. For the asymptotic, $L_N(\rho) = -\zeta'(\rho) + \frac{N^{1-\rho} \log N}{1-\rho} - \frac{N^{1-\rho}}{(1-\rho)^2} + \frac{1}{2} N^{-\rho} \log N + O_\rho(N^{-\beta-1} \log N)$ (differentiate the Euler–Maclaurin expansion of D_N; the error is $O(|\rho| N^{-\beta-1} \log N)$ for $N \ge |\rho|^2$), $\log N! = N \log N - N + \frac{1}{2} \log 2\pi N + O(1/N)$, and $\psi(N) = N + (\psi(N) - N)$; the terms $N^{1-\rho} \log N$ cancel because $\frac{1}{1-\rho} - \frac{\rho}{1-\rho} - 1 = 0$, and the terms $\frac{1}{2} N^{-\rho} \log N$ from L_N and from Stirling's formula cancel. □

The zero removes the twist from the sawtooth and from $\zeta(s)\mathcal{P}_N(s)$, and what remains of the primes is untwisted: $r(N) = r_1(N) - r_2(N)$ with $r_1(N) = (\psi(N) - N)/N$ and $r_2(N) = \int_N^\infty (\psi(x) - x) x^{-2} dx$, since $1 + \int_1^\infty (\psi(x) - x) x^{-2} dx = -\gamma_E$. Let $T_\rho(N) = N^{\rho-1} \sum_{m \le N} \Lambda(m) m^{-\rho} E_\rho(N/m)$, a quantity computed from the primes and the integers up to N.

Corollary 49 (a criterion at one zero)**.** *Let $\rho_1 = \frac{1}{2} + i\gamma_1$ be the first zero (which lies on the critical line). Then RH holds if and only if $T_{\rho_1}(N) = c_{\rho_1} + O(N^{-1/2+\varepsilon})$ for every $\varepsilon > 0$.*

Proof. By Proposition 48, $T_{\rho_1}(N) - c_{\rho_1} = -(\omega + \frac{1}{2}) r_1(N) + \omega r_2(N) + O(N^{-1/2})$ with $\omega = \rho_1/(1 - \rho_1) = \rho_1/\bar{\rho}_1$. Since r_1, r_2 are real and $\operatorname{Im} \omega = \gamma_1/|\rho_1|^2 \ne 0$, the imaginary part controls $r_1 - r_2$ and then the real part, $-\frac{1}{2} r_1 - \operatorname{Re} \omega\, (r_1 - r_2)$, controls r_1; so the condition gives $\psi(N) - N = O(N^{1/2+\varepsilon})$, which is RH. Conversely RH gives $r_1, r_2 = O(N^{-1/2} \log^2 N)$. The same proof works at any zero known to lie on the line. □

The criterion is, by this linear algebra, the classical estimate $\psi(x) - x = O(x^{1/2+\varepsilon})$ in another form, and the zero plays no decisive role; we do not claim more. What the identity adds is a description of the twisted sum. At large scales $E_\rho(y) = -\frac{\rho}{2} \tilde{B}_2(y) y^{-\rho-1} + O(|\rho|^3 y^{-\beta-2})$, with $\tilde{B}_2$ the periodic Bernoulli function (a nontrivial statement for $y \gg |\rho|^2$; numerically the error is much smaller), and there the twist cancels again, $m^{-\rho}(N/m)^{-\rho-1} = N^{-\rho-1} m$: the primes $m \le 2\pi N/|\gamma|$ contribute an untwisted sum, of trivial size about $\frac{\pi^2}{6|\gamma|} N^{1/2}$, which is itself a sum of the type of $\psi(x) - x$. At the scales $1 \le y < |\gamma|/2\pi$ the approximate functional equation [T, Thm. 4.13] with $x = y$, $y' = |\gamma|/(2\pi y)$, and $\zeta(\rho) = 0$ give, on the line,

$$E_\rho(y) = -\chi(\rho) D_{|\gamma|/(2\pi y)}(1 - \rho) + O\big(y^{-1/2} + (y/|\gamma|)^{1/2}\big), \qquad \zeta(s) = \chi(s)\zeta(1 - s),$$

so there the kernel is the dual Dirichlet polynomial of the functional equation (the error is $O(1)$ near $y = |\gamma|/2\pi$). Hence, for large $|\gamma|$, the twisted sum is carried, up to an untwisted part of relative size $O(1/|\gamma|)$, by the primes $m \in (2\pi N/|\gamma|, N]$, twisted by $m^{-i\gamma}$ and weighted by the dual sum at the length $|\gamma| m/(2\pi N)$; for ρ_1 the untwisted part is still visible (17.7 against a total of modulus 82.0 in Table 13), for ρ_{100} it is 0.23. RH is the statement that this sum, of size about $|c_\rho| N^{1/2}$, is determined by c_ρ and $\zeta'(\rho)$ up to $O(N^\varepsilon)$. For a single twist $m^{-i\gamma}$ the best known cancellation in sums over primes is that of the Vinogradov–Korobov zero-free region; combinatorial identities, bilinear forms and the large sieve give power savings only for additive twists or on average over characters or heights. Numerically, Vaughan's identity with $U = V = N^{1/3}$ splits the sum at ρ_1 into pieces whose moduli, after the main term $c_\rho N^{1-\rho}$ is removed from the first, are 34.5, 15.6 and

19.3 at $N = 10^7$, while the total differs from $c_\rho N^{1-\rho}$ by a number of modulus 0.74, mostly the term $-\zeta'(\rho_1)$, of modulus 0.79: each piece carries the resonance $\sum_{d\le U}\mu(d)d^{-\rho} \approx \log U/\zeta'(\rho)$ of $1/\zeta$ at the zero (its modulus is 6.27 at $U = 10^2$ and 14.9 at $U = 10^5$, against $\log U/|\zeta'(\rho_1)| = 5.81$ and 14.5), and the information is in the cancellation among the pieces, not in any one of them. Whether the structure of the kernel can be used is the question this formulation raises.

	$\rho_1 = \frac12 + 14.1347i$	$\rho_{100} = \frac12 + 236.524i$
both sides of Prop. 48	$-43.6987 + 69.4008i$	$-70.9887 + 38.9334i$
$\lvert c_\rho\rvert N^{1/2}$	81.5	77.2
right side: part from $r(N)$, $\psi(N) - N$	$-0.157 + 0.118i$	$-0.176 + 0.086i$
|left side, $m > 2\pi N/\gamma$|	92.1	80.7
|left side, $m \le 2\pi N/\gamma$|	17.7	0.23
$\sum_{m\le N}\Lambda(m)m^{-1/2}\lvert E_\rho(N/m)\rvert$	707	2300

TABLE 13. Proposition 48 at $N = 10^6$. The identity holds to all printed digits; the part of the prime sum that carries the error term of the prime number theorem is of order 1, against a main term $c_\rho N^{1-\rho}$ of order 10^2 and a sum of absolute values of order 10^3.

Which part of the integers the kernel uses. At a zero the kernel has the exact form $E_\rho(y) = \rho\int_y^\infty \psi_1(u)u^{-\rho-1}du$ (from $D_y(s) - \zeta(s) = \frac{y^{1-s}}{1-s} - \psi_1(y)y^{-s} + s\int_y^\infty \psi_1(u)u^{-s-1}du$, valid for $\sigma > 0$), and $\psi_1(u) = -\frac1\pi\sum_{k\ge1}\sin(2\pi ku)/k$. Let $E_\rho^{(K)}$ be the kernel built from the first K modes of the sawtooth and $S^{(K)}(N)$ the corresponding twisted sum. When $y < |\gamma|/(2\pi k)$ the mode k has a stationary point at $u = |\gamma|/(2\pi k)$, which produces the k-th term of the dual sum; the modes with $k > |\gamma|/2\pi$ have no stationary point on $[1,\infty)$ and contribute endpoint terms only. Since S grows like $N^{1/2}$, we write $S - S^{(K)} = \delta_K N^{1-\rho} + R_K(N)$. Replacing Λ by its mean shows that the omitted modes shift the constant c_ρ of the main term by

$$\delta_K = c_\rho - c_\rho^{(K)}, \qquad c_\rho^{(K)} = \frac{\rho}{\pi(1-\rho)}\sum_{k\le K}\frac1k\int_1^\infty \sin(2\pi kv)\big(v^{-2} - v^{-\rho-1}\big)dv;$$

summed over all k this gives c_ρ (with 2000 modes, to 10^{-11} at ρ_1 and $3\cdot10^{-9}$ at ρ_{100}), and we compute δ_K in this way, without fitting. R_K is the part that can reach the terms of order 1. These carry the prime error, the part $-\frac{\rho}{1-\rho}N^{1-\rho}r(N) - \frac12 N^{-\rho}(\psi(N) - N)$ of Proposition 48, whose modulus is about 0.085, 0.09, 0.20, 0.22 at $N = 10^4, \dots, 10^7$. Table 14 shows the same pattern at both zeros: R_K is of the size of the prime error for $K \lesssim |\gamma|/2\pi$ ($K \le 2$ for ρ_1, $K \le 38$ for ρ_{100}), below it by a factor of about 3 or more from $K \approx 1.3|\gamma|/2\pi$ ($K = 3$ and $K = 50$), and below it by about an order of magnitude for $K = 10$ and $K = 100$. The count of about $|\gamma|/2\pi$ modes follows from the stationary-phase argument above, whose stationary points $u = |\gamma|/(2\pi k)$ depend only on the height; since the same kernel $s\int_y^\infty \psi_1(u)u^{-s-1}du$ appears at every s, it is a property of the height rather than of ρ being a zero (the computations were done at zeros).

Remark 50 (Selberg's formula at a zero). The same computation works at every order. Let $\Lambda_2 = \Lambda\log + \Lambda * \Lambda = \mu * \log^2 \ge 0$, so that $\log^2 = \Lambda_2 * 1$, and let $Q_N(s) = \sum_{n\le N}\log^2 n\, n^{-s}$, $M_2(N) = \sum_{m\le N}\Lambda_2(m)/m$, $\psi_2(N) = \sum_{m\le N}\Lambda_2(m)$. Since $\sum_{m\le N}\Lambda_2(m)\lfloor N/m\rfloor = \sum_{n\le N}\log^2 n$, the proof of Proposition 48 gives, at a zero,

$$\sum_{m\le N}\Lambda_2(m)\, m^{-\rho}E_\rho(N/m) = Q_N(\rho) - \frac{\rho}{1-\rho}N^{1-\rho}M_2(N) - N^{-\rho}\Big(\frac{\psi_2(N)}{2} + \sum_{n\le N}\log^2 n\Big)$$

	K	$\lvert\delta_K\rvert$	$N=10^4$	10^5	10^6	10^7
ρ_1 ($\lvert\gamma\rvert/2\pi=2.25$)	1	$3.3\cdot10^{-2}$	0.162	0.054	0.425	0.268
	2	$7.3\cdot10^{-3}$	0.062	0.066	0.124	0.170
	3	$2.4\cdot10^{-3}$	0.021	0.011	0.069	0.079
	5	$5.8\cdot10^{-4}$	0.006	0.019	0.011	0.051
	10	$7.9\cdot10^{-5}$	0.009	0.010	0.002	0.017
	20	$1.0\cdot10^{-5}$	0.008	0.001	0.005	0.004
		prime error	0.085	0.094	0.197	0.223
ρ_{100} ($\lvert\gamma\rvert/2\pi=37.6$)	20	$3.0\cdot10^{-3}$	0.167	0.275	0.240	0.276
	38	$1.3\cdot10^{-3}$	0.170	0.054	0.082	0.098
	50	$2.9\cdot10^{-4}$	0.029	0.009	0.005	0.050
	100	$2.6\cdot10^{-5}$	0.006	0.013	0.012	0.011
	200	$3.0\cdot10^{-6}$	0.001	0.003	0.004	0.010
		prime error	0.085	0.093	0.196	0.222

TABLE 14. Truncating the sawtooth at K modes in the kernel of Proposition 48: the shift δ_K of the main-term constant, computed from the omitted modes, and the remainder $|R_K(N)| = |S - S^{(K)} - \delta_K N^{1-\rho}|$; the rows "prime error" give the modulus of the part of Proposition 48 that carries $r(N)$ and $\psi(N)-N$. For comparison $|S(N)|$ is 82.0 and 81.0 at $N=10^6$ and 257 and 241 at $N=10^7$. Quadrature accurate to about 0.003; smaller entries are not significant.

(both sides are $-1115.7505+1838.7384i$ at ρ_1, $N=10^6$): the twist cancels again, and what remains of the primes is Selberg's sums ψ_2, M_2 without twist. The positivity $\Lambda_2 \geq 0$, together with Selberg's formula $\psi_2(x) = 2x\log x + O(x)$, is the engine of the elementary proof of the prime number theorem [Se], [Er]. But the identities $\log^k = \Lambda_k * 1$ and the positivity $\Lambda_k = \mu * \log^k \geq 0$ hold in every discrete Beurling system, where RH can fail.

What the ingredients give. Table 15 collects what is known about the ingredients used in this section, for discrete Beurling systems $1 < p_1 \leq p_2 \leq \cdots \to \infty$. In the notation of [BDR], such a system is an $[\alpha,\beta]$-system if, for some $c>0$, $\psi(x) = x + O(x^{\alpha+\varepsilon})$ and $N(x) = cx + O(x^{\beta+\varepsilon})$ for every $\varepsilon > 0$ and for no $\varepsilon < 0$; the ordinary integers form a $[\Theta, 0]$-system, $\Theta = \sup \operatorname{Re}\rho$, and RH is the statement $\Theta = \frac{1}{2}$. In this and the next subsection we keep the notation of [BDR]: β is the exponent of the integers, not the real part of a zero, which we write $\operatorname{Re}\rho$ (in [CE3] the same exponent is called b). We write $\Lambda_k = \mu * \log^k$, which is ≥ 0 in every such system by $\Lambda_{k+1} = \Lambda_k \log + \Lambda * \Lambda_k$.

So, unconditionally, no estimate $N(x) = cx + O(x^\theta)$ with $\theta > \frac{2}{5}$ gives RH together with the Euler product and the identities above, and if the conjecture of [BDR] holds, no estimate with a fixed $\theta > 0$ does; a proof must then use the integers more precisely: through the sawtooth ψ_1, which by Poisson summation yields the functional equation, together with the Euler product. Chebyshev's identity at a zero uses $\lfloor N/m \rfloor$ exactly, but the identity for all N is just $\Lambda * 1 = \log$, and Chebyshev's argument from it (Stirling's formula and positivity) gives only $\psi(x) \asymp x$ and Mertens' estimate. Table 14 shows which modes of the sawtooth the kernel at height γ needs for the part of order 1: about $|\gamma|/2\pi$ of them, the dual sum of the functional equation, the others changing only the constant of the main term. A mechanism that couples these modes with the positivity of the Euler product is what this section points to, and we do not have one.

14.5. **Beurling systems with the zeros of ζ.** The systems of [BDR, Thm. 1.3] in the third row of Table 15, which are constructed under RH, are not built from a continuous model: they modify the rational primes [BDR, Sect. 5]. For $\frac{1}{2} < \alpha < \frac{2}{3}$ and $\frac{2\alpha}{\alpha+2} \leq \beta < \frac{1}{2}$ the numbers $p^{1/\beta}$, p prime, are added to the primes, and a set $\mathcal{S}$ of rational primes with $\pi_{\mathcal{S}}(x)$ asymptotic to $\operatorname{Li}(x^\alpha)$ is removed

ingredients	do they force RH?
additive structure and functional equation	no: the Davenport–Heilbronn function
Euler product, $\Lambda_k \ge 0$, $\log^k = \Lambda_k * 1$ for every k, integers $N(x) = cx + O(x^{\beta+\varepsilon})$ with $\beta \ge \frac12$	no: $[\alpha,\beta]$-systems exist for every $\alpha \in [0,1)$, $\beta \in [\frac12, 1)$ [BDR, Thm. 1.1]; zeros near $\operatorname{Re} s = 1$ [DMV]
the same with $\frac25 < \beta < \frac12$	no: under RH, [BDR, Thm. 1.3] gives $[\alpha,\beta]$-systems with $\frac12 < \alpha < \frac23$, $\frac{2\alpha}{\alpha+2} \le \beta < \frac12$, whose zeta function has a zero at $s = \alpha$ and every zero of ζ (Section 14.5); if RH fails, the integers themselves
the same with $\beta \le \frac25$	open, even under RH; [BDR] conjecture that every pair with $\max\{\alpha,\beta\} \ge \frac12$ occurs
every discrete system	$\max\{\alpha,\beta\} \ge \frac12$ [Hi]

TABLE 15. Ingredients and what is known about them.

(the precise condition is below), so that

$$\zeta_B(s) = \frac{\zeta(s)\,\zeta(s/\beta)}{\zeta_{\mathcal S}(s)}, \qquad \zeta_{\mathcal S}(s) = \zeta(s+1-\alpha)\,e^{H(s)}.$$

For the set $\mathcal S$ that we use below (a prime p is in $\mathcal S$ when $\lfloor F\rfloor$ jumps at p, $F(x) = \sum_{p\le x} p^{\alpha-1}$, so that $|\pi_{\mathcal S}(x) - F(x)| < 1$), $H(s) = \sum_p \big(-\mathbf 1_{\mathcal S}(p)\log(1-p^{-s}) + \log(1-p^{-(s+1-\alpha)})\big)$ is analytic in $\operatorname{Re} s > \frac{\alpha}{2}$: the terms of first order are $\int u^{-s} d(\pi_{\mathcal S} - F)$, which converges for $\operatorname{Re} s > 0$ by partial summation, and the others converge absolutely, because $\pi_{\mathcal S}(x) \ll x^{\alpha}$ and $2(\operatorname{Re} s + 1 - \alpha) > 1$. (In [BDR] the set is chosen at random, with a bound for H; we do not prove their bound for our deterministic set.) Hence, in $\operatorname{Re} s > \frac{\alpha}{2}$, ζ_B has poles at 1 and β, a simple zero at the real point α, from the pole of $\zeta(s+1-\alpha)$, and the zeros of ζ there, since under RH $\zeta(s/\beta)$ and $\zeta(s+1-\alpha)$ do not vanish in that half-plane. In particular such a system has the zero $\rho_1 = \frac12 + 14.1347i$ of ζ (unconditionally, since $\operatorname{Re}\rho_1/\beta > 1$ and ζ has no zeros off the line below height $3\cdot 10^{12}$), it has $\Lambda \ge 0$ and $\log = \Lambda * 1$, and, under RH and for the random sets of [BDR], its integers satisfy $N(x) = cx + O(x^{\beta+\varepsilon})$ with $\beta < \frac12$ [BDR, Thm. 1.3], the pole at β contributing a term $c'x^{\beta}$. The question of Section 14.4 at ρ_1 is therefore asked of a system that shares ρ_1 with ζ and has an off-line zero. The following proposition shows what the identity at a zero contains in any Beurling system.

Proposition 51 (Chebyshev's identity at a zero of a Beurling zeta function)**.** *Let a discrete Beurling system have integer counting function $N_B(x) = cx + R(x)$ with $c > 0$ and $|R(x)| \le C_R x^{\beta}$ for $x \ge 1$, $0 \le \beta < 1$, so that $\zeta_B(s) = \int_{1^-}^{\infty} x^{-s} dN_B(x)$ is meromorphic in $\operatorname{Re} s > \beta$ with a single pole, at $s = 1$. Let $\rho \ne 1$ be a zero of ζ_B with $\operatorname{Re}\rho > \beta$. With $D_y(s) = \sum_{b\le y} b^{-s}$, $L_N(s) = \sum_{b\le N} \log b\, b^{-s}$ and $T(N) = \sum_{b\le N}\log b$ over the integers b of the system, $M(N) = \sum_{m\le N}\Lambda_B(m)/m$ and $E_\rho(y) = -\rho\int_y^{\infty} R(u)u^{-\rho-1}du$:*

(i) *for every $y \ge 1$, $D_y(\rho) = \frac{c\,y^{1-\rho}}{1-\rho} + R(y)y^{-\rho} + E_\rho(y)$, and*

$$\sum_{m\le N} \Lambda_B(m)\, m^{-\rho} E_\rho(N/m) = L_N(\rho) - \frac{\rho}{1-\rho}\, c\, N^{1-\rho} M(N) - N^{-\rho} T(N);$$

(ii) *if $\int_1^{\infty}(\psi_B(x) - x)x^{-2}dx$ converges, write $M(N) = \log N + \kappa_B + r(N)$ with $r(N) \to 0$; then $T_\rho^B(N)$, $N^{\rho-1}$ times the sum in (i), satisfies*

$$T_\rho^B(N) = C_\rho - \frac{\rho}{1-\rho}\, c\, r(N) - \zeta_B'(\rho)\, N^{\rho-1} + O_\rho\big(N^{\beta-1}\log(N+1)\big), \qquad C_\rho = c\Big(1 - \frac{1}{(1-\rho)^2} - \frac{\rho\,\kappa_B}{1-\rho}\Big);$$

(iii) *if* $\beta < \frac{1}{2}$ *and* $\operatorname{Re}\rho = \frac{1}{2}$, *then* $T_\rho^B(N) - C_\rho = O(N^{-1/2+\varepsilon})$ *for every* $\varepsilon > 0$ *if and only if* $\psi_B(x) - x = O(x^{1/2+\varepsilon})$ *for every* $\varepsilon > 0$.

Proof. (i) Since $N_B(1^-) = 0$, partial summation gives, for $\operatorname{Re} s > 1$ and then by continuation for $\operatorname{Re} s > \beta$, $\zeta_B(s) = c + \frac{c}{s-1} + s\int_1^\infty R(u)u^{-s-1}du$ and

$$D_y(s) = \zeta_B(s) + \frac{c\,y^{1-s}}{1-s} + R(y)y^{-s} - s\int_y^\infty R(u)u^{-s-1}du;$$

put $s = \rho$. Then $\log = \Lambda_B * 1$ gives $L_N(\rho) = \sum_m \Lambda_B(m)m^{-\rho}D_{N/m}(\rho)$; the power terms give $\frac{c\,N^{1-\rho}}{1-\rho}M(N)$, and in the terms $R(N/m)(N/m)^{-\rho}$ the twist cancels, $m^{-\rho}(N/m)^{-\rho} = N^{-\rho}$, leaving $N^{-\rho}\sum_m \Lambda_B(m)R(N/m) = N^{-\rho}\big(T(N) - cNM(N)\big)$ by $\sum_m \Lambda_B(m)N_B(N/m) = T(N)$. (ii) Differentiating the formula for $D_N(s)$ at $s = \rho$,

$$L_N(\rho) = -\zeta_B'(\rho) + \frac{c\,N^{1-\rho}\log N}{1-\rho} - \frac{c\,N^{1-\rho}}{(1-\rho)^2} + R(N)N^{-\rho}\log N + \int_N^\infty \frac{R(u)}{u^{\rho+1}}du - \rho\int_N^\infty \frac{R(u)\log u}{u^{\rho+1}}du,$$

and $T(N) = c(N\log N - N + 1) + R(N)\log N - \int_1^N R(u)u^{-1}du$. Substituting in (i) with $M(N) = \log N + \kappa_B + r(N)$, the terms $N^{1-\rho}\log N$ cancel because $\frac{1}{1-\rho} - \frac{\rho}{1-\rho} - 1 = 0$, the terms $R(N)N^{-\rho}\log N$ cancel, and what remains is $C_\rho N^{1-\rho} - \frac{\rho}{1-\rho}cN^{1-\rho}r(N) - \zeta_B'(\rho) - cN^{-\rho}$ plus $N^{-\rho}\int_1^N R/u$ and the two tail integrals, which are $O(N^{\beta-\operatorname{Re}\rho}\log(N+1))$ by $|R(u)| \le C_R u^\beta$. Multiply by $N^{\rho-1}$. (iii) By (ii), $T_\rho^B(N) - C_\rho = -\frac{\rho}{1-\rho}c\,r(N) + O(N^{-1/2})$, and $\rho \neq 0$. If $r(N) = O(N^{-1/2+\varepsilon})$, then $\psi_B(N) = NM(N) - \int_1^N M(x)\,dx = N + Nr(N) - \int_1^N r(x)\,dx + O(1) = N + O(N^{1/2+\varepsilon})$; conversely, $\psi_B(x) - x = O(x^{1/2+\varepsilon})$ gives $r(N) = \frac{\psi_B(N)-N}{N} - \int_N^\infty \frac{\psi_B(x)-x}{x^2}dx = O(N^{-1/2+\varepsilon})$. □

For the integers ($c = 1$, $R(u) = -\psi_1(u) - \frac{1}{2}$, $\beta = 0$) the sum in (i) is that of Proposition 48 plus $\frac{1}{2}N^{-\rho}\psi(N)$, $C_\rho = c_\rho + \frac{1}{2}$, and (iii) is Corollary 49 at any zero on the line. What (iii) adds is that, as soon as the integers are better than $\sqrt{x}$, they enter the remainder of the identity only at order $N^{\beta-1}\log N$, below $N^{-1/2}$: in that range the identity at a zero on the line is a rewriting of the prime error, whatever the integers are. The size of the integer error can therefore not be what forces RH through this identity. What the identity does contain is the kernel E_ρ, which is built from the exact shape of R, and a proof along these lines has to bound the left side directly.

A computation. We built the system above with $\alpha = 0.6$ and $\beta = 0.47 \ge \frac{2\alpha}{\alpha+2} = 0.4615$ up to $N = 3\cdot 10^7$. The set $\mathcal{S}$ has 6360 primes below 10^8, against $\operatorname{li}(10^{8\alpha}) = 6364$, beginning $3, 7, 13, 23, 41, 61, \ldots$; there are 20 859 734 integers below $3\cdot 10^7$, the products $n\,k^{1/\beta}$ with n free of the primes of $\mathcal{S}$. From the primes up to 10^8, $c = \zeta(1/\beta)\prod_{p\in\mathcal{S}}(1-p^{-1}) = 0.69529746$, which agrees with $\zeta(1/\beta)/(\zeta(2-\alpha)e^{H(1)})$ to $8\cdot 10^{-10}$; $c' = \zeta(\beta)/(\zeta(\beta+1-\alpha)e^{H(\beta)}) = 0.23799$, $\zeta_B'(\rho_1) = \zeta'(\rho_1)\zeta(\rho_1/\beta)/\zeta_{\mathcal{S}}(\rho_1) = 0.89145 - 0.23690i$ and $\zeta_B'(\alpha) = \zeta(\alpha)\zeta(\alpha/\beta)e^{-H(\alpha)} = -11.5733$. The identity (i) holds at $N = 10^6$ to ten digits at ρ_1 and at α. Table 16 and Figure 7 compare the system with the integers at the same zero ρ_1. For the integers $N^{1/2}|T_{\rho_1}(N) - C_{\rho_1}|$ stays near 0.8, and (ii) predicts it to $3\cdot 10^{-4}N^{-1/2}$. For the system it grows like $N^{\alpha-1/2}$: the prime remainder $r(N)$ follows $N^{\alpha-1}/(1-\alpha) - N^{\beta-1}/(1-\beta)$, the contribution of the removed and of the added primes, and (ii) predicts $T_{\rho_1}^B$ to within $0.21N^{-1/2}$, of the size of the error of κ_B, for which ψ_B beyond $3\cdot 10^7$ is replaced by its main terms. The integers of the system follow $cx + c'x^\beta$, with $(N_B(x) - cx)/x^\beta$ fluctuating around c' by about 0.1 at $x = 10^4$ and 0.01 at $3\cdot 10^7$.

At the real zero the identity has no twist at all: every factor $m^{-\alpha}(N/m)^{-\alpha}$ is real, there is no dual sum (the count $|\gamma|/2\pi$ of modes of Section 14.4 is 0), and the zero is seen directly in the untwisted part. By (ii) the remainder is $KN^{\alpha-1}$ up to lower-order terms, with $K = -\frac{\alpha c}{(1-\alpha)^2} - \zeta_B'(\alpha) = -2.607 + 11.573 = 8.966$, the first term from the removed primes and the second from the zero; the convergence is slow, with relative corrections of order $N^{\beta-\alpha}$, and the data follow the full prediction

	$N^{1/2}\lvert T_{\rho_1}(N)-C_{\rho_1}\rvert$		$r(N)$ of the system		
N	integers	system	computed	$\frac{N^{\alpha-1}}{1-\alpha}-\frac{N^{\beta-1}}{1-\beta}$	$\frac{N_B(N)-cN}{N^\beta}$
10^5	0.712	2.251	0.02105	0.02067	0.260
10^6	0.845	3.210	0.00879	0.00878	0.238
10^7	0.784	5.576	0.00356	0.00361	0.245
$3\cdot 10^7$	0.789	7.136	0.00236	0.00235	0.248

TABLE 16. Chebyshev's identity at $\rho_1 = \frac{1}{2} + 14.1347i$ for the integers and for a Beurling system of the type of [BDR, Thm. 1.3] with $\alpha = 0.6$, $\beta = 0.47$, which has the same zero ρ_1 and a real zero at 0.6. Values at the nearest of 160 logarithmically spaced N; $c' = 0.238$.

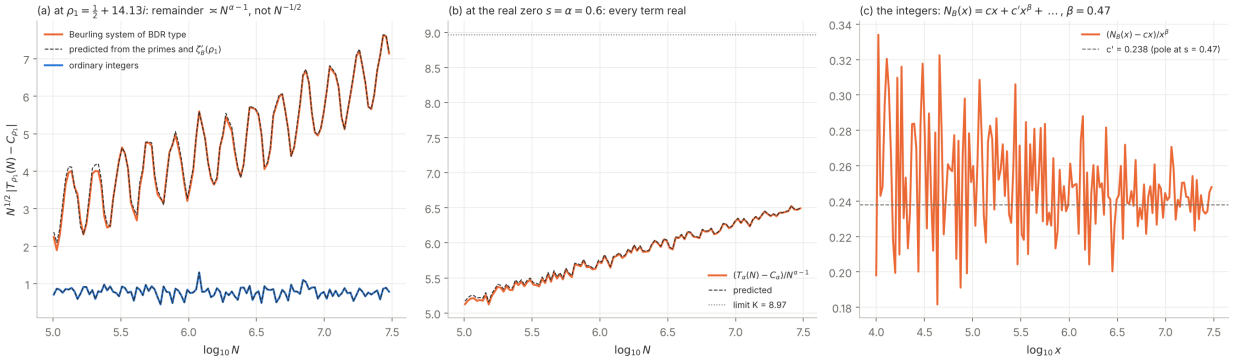


FIGURE 7. A Beurling system with the zeros of ζ and a real zero at $\alpha = 0.6$ ($\beta = 0.47$). (a) $N^{1/2}|T_{\rho_1}(N) - C_{\rho_1}|$ for the system and for the integers, with the prediction of Proposition 51(ii) (dashed). (b) At the real zero: $(T_\alpha^B(N) - C_\alpha)/N^{\alpha-1}$, the prediction and its limit K. (c) $(N_B(x) - cx)/x^\beta$ and c'.

to 1% (Figure 7b). At ρ_1 the same removed primes enter through $r(N)$, which is untwisted, so the system violates the condition of (iii) at a zero that it shares with ζ, although its integers are better than $\sqrt{x}$ and $\Lambda_B \geq 0$.

Three controls. Table 17 sets the system beside the other two controls of the paper. The Davenport–Heilbronn function has the functional equation and no Euler product; the Beurling systems have the Euler product, $\Lambda \geq 0$ and Chebyshev's identity, and those of [BDR] even have integers better than $\sqrt{x}$ and every zero of ζ, but no functional equation of the type of ζ is known for them. Each control has an off-line zero. A proof of the bound, or of RH, must therefore use the functional equation and the Euler product together; by Proposition 51(iii) the size of the integer error, however small, cannot replace the functional equation. The controls are studied in more detail in [CE46].

A quantity in which ζ cancels. The functional equation suggests a combination of the two ingredients that does not contain ζ at all. For $s^* = 1 - \bar{s}$, $|\zeta(s^*)| = |\zeta(s)|/|\chi(s)|$, where $\zeta(s) = \chi(s)\zeta(1-s)$, so the *reflection asymmetry*

$$D_X(s) := \log|Z_X(s)| - \log|Z_X(s^*)| = \log|\chi(s)| - \log|E_X^*(s)| + \log|E_X^*(s^*)|$$

is built from the primes and the Gamma factor alone; numerically the two expressions agree to $2\cdot 10^{-13}$, and also for the Davenport–Heilbronn function (Figure 8d). It might seem to couple the Euler product with the functional equation, but by Lemma 10, reindexing the zeros by $\rho \mapsto 1 - \bar{\rho}$

	functional equation	Euler product, $\Lambda \geq 0$, $\log = \Lambda * 1$	integer error $O(x^\beta)$, $\beta < \frac{1}{2}$	zeros of ζ on the line	off-line zero
Davenport–Heilbronn	yes	no	—	no	yes
Beurling [DMV]	none known	yes	no	no	yes
Beurling [BDR, Thm. 1.3]	none known	yes	yes	yes (under RH)	yes
ζ	yes	yes	yes ($\beta = 0$)	—	RH

TABLE 17. The three controls and ζ (see [CE46]).

(legitimate by absolute convergence),

$$D_X(s) = \sum_\rho h_X(s-\rho), \qquad h_X(w) = -\operatorname{Re}\big[K_X(w) - K_X(-\bar{w})\big],$$

the same kernel for every zero and for every function with such a factorization. So D_X records the zeros in the same way for ζ, for f and for a Beurling-type toy without functional equation (Figure 8): an off-line zero appears in it through the same kernel whatever the function, and the identity alone gives no reason for the zeros to be on the line. Its prime side contains E_X^* at $\operatorname{Re} s^* < \frac{1}{2}$, where the prime sum is large and the pole barrier of Remark 45 reappears. (The identity is proved for ζ; the figure is numerical, script `reflection_asymmetry.py`; for f we do not prove the factorization, and for the toy, whose only zeros are ρ_B and $\bar{\rho}_B$, it is exact by construction.) Weil's form, next, is the classical exact form of the coupling.

14.6. **Weil's positivity: the primes against the Gamma factor.** The coupling that Table 15 asks for has a classical exact form, Weil's criterion [W2]: RH holds if and only if $Q(f) \geq 0$ for every $f \in C_c^\infty(\mathbb{R})$. For real f, with $G = f * \tilde{f}$ ($\tilde{f}(u) = f(-u)$), which is even, and $\Phi(z) = \hat{f}(z)\hat{f}(-z)$, the entire function equal to $|\hat{f}(r)|^2$ for real r,

$$Q(f) = \Phi(\tfrac{i}{2}) + \Phi(-\tfrac{i}{2}) + \mathcal{A}_\infty(G) - 2\sum_{n\geq 2} \Lambda(n) n^{-1/2} G(\log n),$$

and, by the explicit formula, $Q(f) = \sum_\rho \Phi(t_\rho)$, $t_\rho := (\rho - \frac{1}{2})/i$ (so $t_\rho = \gamma_\rho$ for a zero $\rho = \frac{1}{2} + i\gamma_\rho$ on the line, and $t_\rho = \gamma - i\delta$ for a zero $\rho = \beta + i\gamma$ off it, $\gamma > 0$, $\delta = \beta - \frac{1}{2}$). Real f suffice: Q is real and symmetric on real functions and $Q(f_1 + if_2) = Q(f_1) + Q(f_2)$. The archimedean term, which carries the functional equation, is $\mathcal{A}_\infty(G) = \frac{1}{2\pi}\int \Phi(r)\big(\operatorname{Re}\frac{\Gamma'}{\Gamma}(\frac{1}{4} + \frac{ir}{2}) - \log\pi\big)dr$, and for G supported in $[-\ell, \ell]$ it equals

$$\mathcal{A}_\infty(G) = G(0)\big(-\gamma_E - \log\pi - \log(1-e^{-2\ell})\big) + \int_0^\ell \big(G(0)e^{-2x} - e^{-x/2}G(x)\big)\frac{2\,dx}{1-e^{-2x}},$$

from $\operatorname{Re}\frac{\Gamma'}{\Gamma}(\frac{1}{4} + \frac{ir}{2}) = -\gamma_E + \int_0^\infty \frac{e^{-t} - e^{-t/4}\cos(rt/2)}{1-e^{-t}}dt$. With these normalizations and a Gaussian G of width $\frac{1}{2}$, for which the prime term is 0.51, the explicit formula agrees with the first 1000 zeros to $3 \cdot 10^{-16}$. Two limiting cases connect this with the rest of the paper. The kernel of the cross-energy is $\hat{G}$ for $G(u) = e^{-|u|/2}$: $C_2(\infty, \frac{1}{2} + it) = \frac{1}{\frac{1}{4}+t^2} = \hat{G}(t)$, and unconditionally $\hat{G}(t_\rho) = \frac{1}{\rho(1-\rho)}$, so the lower bound $\sum_\rho C_2(\infty, \rho) = 2 + \gamma_E - \log 4\pi$ of (11), which is Cramér's constant if all zeros are simple, is $\sum_\rho \hat{G}(t_\rho)$. This G is not admissible in Weil's criterion ($\hat{G}$ has poles at $\pm\frac{i}{2}$, and $\sum \Lambda(n)n^{-1/2}G(\log n) = \sum \Lambda(n)/n$ diverges); it is the limit of $e^{-(\frac{1}{2}+\varepsilon)|u|}$. Likewise the law of Theorem 2 is $\sum_\gamma \kappa_{m(\gamma)}\hat{G}_{s_0}(\gamma)$ with $\hat{G}_{s_0} = P_{s_0}$, $G_{s_0}(u) = \frac{1}{2\pi}e^{-(\sigma_0 - \frac{1}{2})|u| - it_0 u}$, admissible in the

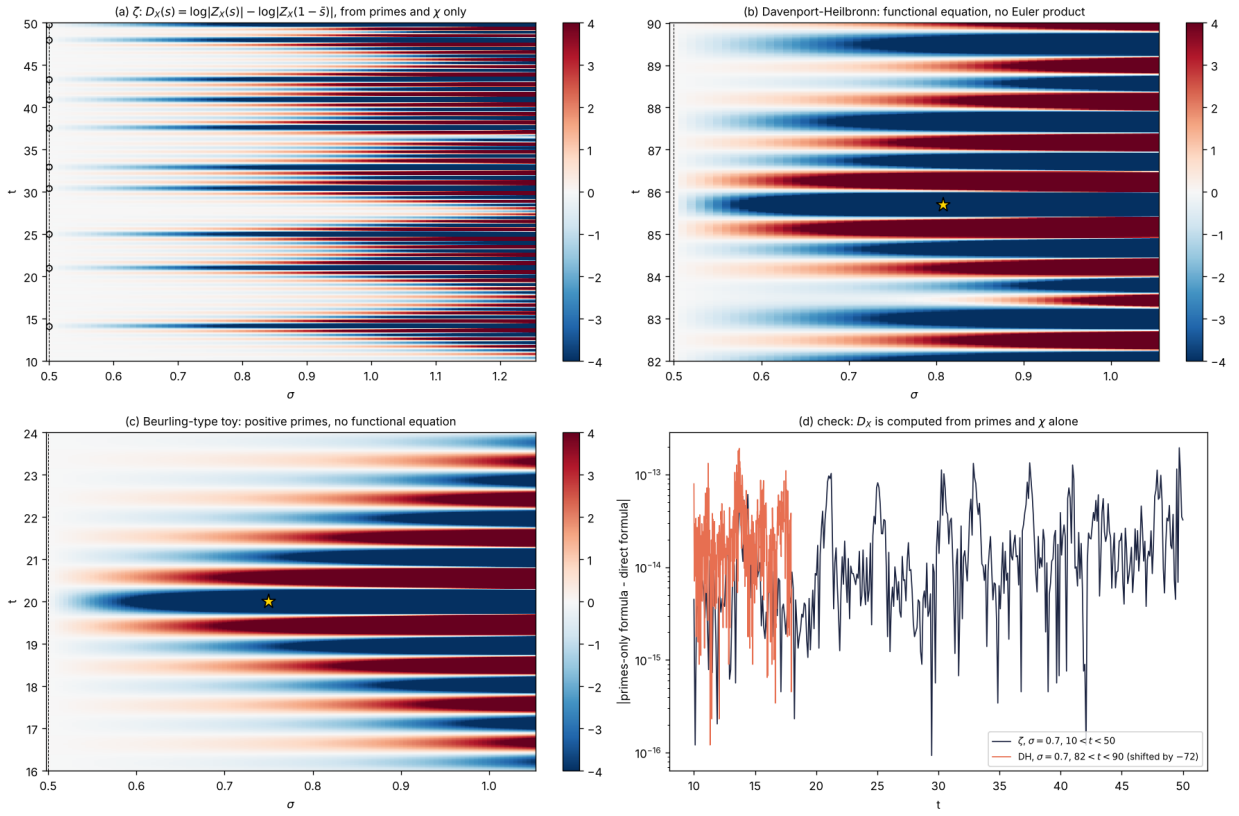


FIGURE 8. The reflection asymmetry $D_X(s)$, $X = 10^3$, Gaussian weight of standard deviation 0.4. (a) ζ, from the primes and χ only; circles: zeros. (b) The Davenport–Heilbronn function near its off-line zero (star), from its coefficients and its χ. (c) A Beurling-type toy with the nonnegative prime measure $(1 - 2\,\mathrm{Re}\,x^{\rho_B - 1})\,dx/\log x$ on $[16, \infty)$, $\rho_B = 0.75 + 20i$ (star), and no functional equation, from its two zeros. (d) Difference between the primes-only formula and the direct one on $\sigma = 0.7$.

explicit formula only for $\sigma_0 > 1$; it weighs an ordinate by κ_m rather than by m, and it is κ_1 times the zero side of the explicit formula only if all zeros are simple.

We computed Q from the pole, the Gamma factor and the primes, without the zeros, on functions supported in $[-\frac{\ell}{2}, \frac{\ell}{2}]$ (so that G is supported in $[-\ell, \ell]$ and the primes up to e^ℓ enter), in the basis $e_j(u) = \sin(j\pi(u+\frac{\ell}{2})/\ell)$, $j \le J$: for $J = 40$ with 80 digits and for $J = 60$ with 130 digits (mpmath). The pole and prime entries are in closed form; the archimedean entries are computed by Gauss–Legendre quadrature (40 nodes on panels of width 0.05). Table 18 gives the smallest eigenvalue of $Q/\|f\|^2$ on the span of $e_1, \dots, e_J$, with and without the prime term. Since these spans increase with J, every entry is an upper bound for the infimum over all f supported in $[-\frac{\ell}{2}, \frac{\ell}{2}]$.

The pole and the Gamma factor alone stay positive somewhat beyond the entry of the prime 2: in double precision with $J = 40$, 80 and 120 their smallest eigenvalue changes sign at $\ell \approx 0.745$ for odd f and at $\ell \approx 0.82$ for even f, and it is negative and decreasing from there on (Table 18). With the prime term the smallest eigenvalue on these spans is positive; it falls from 10^{-6} at $\ell = 1$ to $4.0 \cdot 10^{-64}$ with $J = 40$ and $4.8 \cdot 10^{-87}$ with $J = 60$ at $\ell = 5$, and keeps decreasing as J grows. These near-null directions are consistent with the very small eigenvalues of Weil's form on a fixed interval found by Connes and Consani, whose eigenvectors are built from prolate spheroidal wave functions [CC2] (we have not compared our eigenvectors with theirs). For the archimedean place

		pole + Γ + primes, $J = 40$		pole + Γ + primes, $J = 60$		pole + Γ only, $J = 40$	
ℓ	e^ℓ	f even	f odd	f even	f odd	f even	f odd
1.00	2.7	$1.06 \cdot 10^{-6}$	$2.30 \cdot 10^{-4}$	$1.04 \cdot 10^{-6}$	$2.20 \cdot 10^{-4}$	-0.030	-0.313
1.25	3.5	$3.06 \cdot 10^{-10}$	$1.18 \cdot 10^{-7}$	$3.03 \cdot 10^{-10}$	$1.16 \cdot 10^{-7}$	-0.152	-0.576
1.50	4.5	$5.71 \cdot 10^{-15}$	$4.41 \cdot 10^{-12}$	$4.41 \cdot 10^{-15}$	$3.81 \cdot 10^{-12}$	-0.301	-0.819
2.00	7.4	$4.80 \cdot 10^{-28}$	$5.24 \cdot 10^{-25}$	$1.08 \cdot 10^{-29}$	$3.29 \cdot 10^{-26}$	-0.577	-1.307
2.50	12.2	$3.89 \cdot 10^{-38}$	$2.64 \cdot 10^{-35}$	$5.08 \cdot 10^{-46}$	$3.74 \cdot 10^{-43}$	-0.811	-1.877
3.00	20.1	$2.70 \cdot 10^{-45}$	$4.69 \cdot 10^{-43}$	$9.19 \cdot 10^{-58}$	$5.73 \cdot 10^{-55}$	-1.019	-2.607
3.50	33.1	$1.08 \cdot 10^{-51}$	$6.52 \cdot 10^{-49}$	$6.33 \cdot 10^{-67}$	$2.36 \cdot 10^{-64}$	-1.211	-3.573
4.00	54.6	$5.64 \cdot 10^{-56}$	$5.33 \cdot 10^{-54}$	$1.36 \cdot 10^{-74}$	$4.44 \cdot 10^{-72}$	-1.394	-4.862
5.00	148.4	$3.96 \cdot 10^{-64}$	$3.87 \cdot 10^{-62}$	$4.80 \cdot 10^{-87}$	$1.58 \cdot 10^{-84}$	-1.750	-8.844

TABLE 18. Smallest eigenvalue of Weil's form on the span of $e_1, \dots, e_J$, functions supported in $[-\frac{\ell}{2}, \frac{\ell}{2}]$. The entries are upper bounds for the infimum, and for $\ell \geq 2$ they are far from converged in J. At $\ell = \log 2$, where no prime enters, the smallest eigenvalue is $1.42 \cdot 10^{-3}$ for f even with $J = 40$ ($1.36 \cdot 10^{-3}$ with $J = 120$, double precision).

alone and supports that admit no prime, [CC] give a conceptual reason for positivity, and [CCM] relate the zeros to a semilocal prolate wave operator. These numbers are not evidence for RH beyond what is known: $Q(f)$ for such f is governed by the low zeros, which are known to lie on the line, and positivity on a finite-dimensional subspace at fixed ℓ is not Weil's criterion. They illustrate what the criterion demands: on these directions the prime sum must cancel the polar and archimedean terms, the latter encoding the functional equation, to within these margins, for every ℓ. The positivity of Λ alone does not provide this (Table 15); what matters are the exact values and positions of the prime powers. Weil's form at the windows of the zeros, at large height and with two windows, is studied numerically in [CE19].

15. Conclusion

The function-field model shows what makes the bound true where RH is known: a square-root estimate for the prime sums of each degree (Weil, via Frobenius), turned into control of the energy by an absolutely convergent form because all primes of one degree share one frequency. A proof over $\mathbb{Q}$ that follows this route cannot work: the bound is equivalent to RH, and the corresponding quadratic form in the primes is not absolutely convergent (Lemma 43). Under RH its convergence is a cancellation phenomenon among primes in windows of bounded logarithmic length, and we do not know how to control that cancellation without the zeros. This suggests, but does not prove, that a proof would need a mean-value principle for Dirichlet series whose length is unbounded compared with the height. Chebyshev's identity at a zero (Section 14.4) gives the same requirement a concrete form: square-root cancellation in one twisted sum over the primes in $(2\pi N/|\gamma|, N]$, weighted by the dual polynomial of the functional equation. The Beurling systems of Section 14.5 share the zeros of ζ, have integers better than $\sqrt{x}$ and $\Lambda \geq 0$, and still have an off-line zero: the identity at a zero only restates the prime error, and what separates ζ from them is the functional equation. Weil's form (Section 14.6) makes the demand quantitative on a fixed interval: on its near-null directions the primes up to e^ℓ must cancel the pole and the Gamma factor to within less than 10^{-86} at $\ell = 5$, which the positivity of Λ alone cannot provide. The reflected cross-energy, which separates the two controls and gives a finite-matrix form of the restricted Weil criterion, is studied in [CE3].

Theorem 7 removes the last hypothesis from the equivalence: RH holds if and only if $B_X(s_0) \ll X^\varepsilon$ at one point with $\operatorname{Re} s_0 > 1$, equivalently if and only if $\log |Z_X(s)| \leq C_s X^\varepsilon$ on compact subsets of $\operatorname{Re} s > \frac{1}{2}$, a one-sided bound for the tail of the Euler product beyond X. By Landau's theorem

the one-sided bound is not weaker than the two-sided one, just as $\psi(x) - x \le x^{\frac12+\varepsilon}$ is already equivalent to RH; the problem is unchanged in strength, but it no longer depends on whether Θ is attained. Coupling the bound with the reflected cross-energy of [CE3] does not change this: its oscillating part is ζK_N with an explicit carrier K_N, so the Euler–Caceres bound of $\zeta K_N/E_X^*$ is that of ζ/E_X^* plus the explicit Blaschke term of the zeros of K_N.

16. Open problems

1. The bound. Prove $B_X(s_0) = O(1/\log X)$ for one s_0. By Corollary 3 this is RH; Section 14 explains why the function-field route does not transfer. Unconditionally only $0 \le B_X(s_0) \ll \sqrt{X}/\log X$ is known for $\operatorname{Re} s_0 > 1$ (Theorem 26); by Corollary 8 already a bound $B_X(s_0) \ll_\varepsilon X^\varepsilon$ for every $\varepsilon > 0$, at one such s_0, is equivalent to RH. In the form of Corollary 9 it is a mean-value statement for $|\zeta/E_X^*|^2$ on the critical line with a fixed weight and an Euler factor of growing length, the opposite regime of the mollifier methods of Levinson [Le] and Conrey [Co], and related to the hybrid Euler–Hadamard products of Gonek, Hughes and Keating [GHK]. A proof must distinguish ζ from the Davenport–Heilbronn function, for which the bound grows, and from Beurling systems with an off-line zero, which have an Euler product but not the integers; positivity of $\Lambda(n)$ alone is not enough, since every nonnegative trigonometric polynomial carries a constant term that meets the pole (Remark 45).

2. The interaction of the wells. Bound $\mathcal{I}_X(s_0)$ in Theorem 5 by $O(1)$; under the gap hypothesis of Section 8 it is $O(\log\log X)$ (Theorem 21), and $O(1)$ would follow from $E(d) = O(1/d)$ for the two-well function with a bound for the interactions of three or more wells. Prove the asymptotic of $E(d)$ and identify its constants. Make (10) a theorem for $\alpha < 1$ under RH, explain the excess near $\alpha = 1$ in Figure 1a, and find the nonlinear part of the interaction.

3. Simple zeros at height T. With the centre at height T, Corollary 3(b) becomes a local test for multiple zeros near T. Quantify the dependence of X on T needed to detect a double zero.

4. Θ not attained. Theorem 7 gives growth along a sequence without attainment. Remove the attainment hypothesis in Theorem 6, that is, prove growth for all large X, which would give the $\liminf$ form of Corollary 9 unconditionally; Remark 25 describes the cluster obstruction heuristically.

5. The exponent. Prove $\limsup_X \log B_X(s_0)/\log X = \Theta - \frac12$ for $\operatorname{Re} s_0 > 1$. The lower bound is Theorem 7; the upper bound $B_X(s_0) \ll X^{\Theta-\frac12+\varepsilon}$ would need control of clusters of off-line zeros at large height, which zero-density estimates do not give pointwise.

6. Integers better than $x^{2/5}$. Assume RH. Is there a discrete Beurling system with $N(x) = cx + O(x^\beta)$ for some $\beta \le \frac25$ (in particular with $O(x^\varepsilon)$ for every $\varepsilon > 0$) whose zeta function has a zero with real part $> \frac12$? For $\frac25 < \beta < \frac12$ there is [BDR, Thm. 1.3], with every zero of ζ and a real zero (Section 14.5), and [BDR] conjecture that there is for every β. In the form of Section 14.4: by Proposition 51(iii), for integers better than $\sqrt{x}$ the identity at a zero on the line is a rewriting of the prime error, and a system of the type of [BDR] violates it at the zero ρ_1 that it shares with ζ. The question is therefore whether the left side of Proposition 48 at ρ_1, the twisted sum of Λ with the kernel E_{ρ_1} of the integers, can be shown to be $c_{\rho_1} N^{1-\rho_1} + O(N^{1/2+\varepsilon})$ from $\Lambda \ge 0$ and the structure of that kernel (its first $\asymp |\gamma_1|$ sawtooth modes, the dual sum of the functional equation), without the zeros; the size of the integer error alone is not enough.

Appendix A. The Dirichlet floor

The following barrier, stated in the introduction, is why the mollifier of this paper is not a Dirichlet polynomial. Let μ be the probability measure $dt/(2\pi(\frac14 + t^2))$ on the critical line.

Proposition 52 (the Dirichlet floor)**.** *Let $M(s)=\sum_{n\le N}a_n n^{-s}$ with $M(1)\neq 0$. Then*

$$\sum_{\operatorname{Re}\rho>\frac12}\log\Big|\frac{\rho}{1-\rho}\Big|\le\frac12\log R(M),\qquad R(M)=\frac{\int|\zeta M(\frac12+it)|^2\,d\mu(t)}{|M(1)|^2}\ \ge\ 1+c_0,$$

$$c_0=\sum_{k\ge1}\Big(1-k(k+1)\log^2\big(1+\tfrac1k\big)\Big)=0.0803270\ldots$$

In particular Jensen's inequality with a Dirichlet-polynomial mollifier cannot bound the left side, which vanishes under RH, by less than $\frac12\log(1+c_0)=0.0386$.

Proof. Let $F(s)=(s-1)\zeta(s)/s$. Poisson–Jensen for FM in the half-plane $\operatorname{Re}s>\frac12$ at $s=1$ gives $\int\log|FM|\,d\mu\ge\log|M(1)|+\sum_{\operatorname{Re}\rho>\frac12}\log|\rho/(1-\rho)|$, since the zeros of M and a possible singular factor only increase the left side; and $\int\log|FM|\,d\mu\le\frac12\log\int|FM|^2d\mu$ by Jensen's inequality, with $|F|=|\zeta|$ on the line. For the lower bound, $\zeta(s)/s=-\int_0^\infty\{1/x\}x^{s-1}\,dx$ for $0<\operatorname{Re}s<1$, so by Plancherel $\int|\zeta M|^2d\mu=\int_0^\infty|h(u)|^2u^{-2}\,du$ with $h(u)=\sum_n a_n\{u/n\}=M(1)u-S(u)$, $S(u)=\sum_{m\le u}(a*1)(m)$. On $(0,1)$, $h=M(1)u$ contributes $|M(1)|^2$. On $[k,k+1)$ the step function S is constant, and

$$\int_k^{k+1}\frac{|M(1)u-S|^2}{u^2}\,du\ge|M(1)|^2\min_c\int_k^{k+1}\Big(1-\frac{c}{u}\Big)^2du=|M(1)|^2\Big(1-k(k+1)\log^2\big(1+\tfrac1k\big)\Big).$$

Summing over k gives $R(M)\ge1+c_0$. □

Status: proved (elementary); it first appeared in version 3 of [CP]. The terms of c_0 are $\frac{1}{12k^2}+O(k^{-3})$. With $M(1)=1$ the mollifier must approximate the line u by a step function with jumps at the integers, and the sawtooth costs c_0: the loss comes from the passage from $\log|\cdot|$ to $|\cdot|^2$ and from the discreteness of the integers, not from the arithmetic of ζ. It is the analogue, for the whole critical line, of the constant $c(\theta)>1$ that limits the method of Levinson and Conrey. The coefficients $a=\mu*b$ with $b_m=S_m-S_{m-1}$, $S_m=m(m+1)\log(1+\frac1m)$, attain the minimum on each interval up to N; the truncated choice gives $R-1=0.102$, 0.088, 0.0995, 0.0894, 0.0872 for $N=10$, 30, 100, 300, 1000 (`anc/dirichlet_floor.py`). The Nyman–Beurling mollifiers $\mu(n)(1-\log n/\log N)$ give R growing with N: they approximate $1/\zeta$, while the normalization at $s=1$ requires $1/F$. The mollifier $1/P$ of this paper is not a Dirichlet polynomial, because P carries the pole through $e^{K_X(s-1)}$, and with it the bound falls below the floor (Table 7).

Data and code

The ancillary Python scripts (NumPy, SciPy, Matplotlib; mpmath for the high-precision Weil computations) reproduce all tables and figures (the exact values of the best Dirichlet polynomial at $s_0=1$ come from `dirichlet_exact_gram.py`, added on 10 November 2026); two scripts new in version 2, which need only NumPy and SciPy, reproduce Table 5 and Tables 3–9: `ec_bound_numerics.py` and `ec_phase2_controls.py`. The file `README.txt` lists, for each script, the section, table or figure it reproduces. The computations are in double precision unless stated otherwise, without interval arithmetic; they illustrate the theorems and are not used in their proofs.

Provenance. This paper supersedes the preprint [C2]: its theorems are here in the same or a stronger form (Theorems 1 and 6, Corollaries 3 and 9), its heuristic law is Theorem 2, and its computations and remaining remarks are in Sections 3, 4, 9, 11, 12, 14.2 and 14.5 (Tables 2, 7 and 12, Figures 2, 3 and 8). The material on the Euler–Caceres bound in version 3 of [CP] is also here, including the zero factor seen through the cross-energy (Section 12.1); the explicit formula in cross-energy coordinates of earlier versions of this paper is in [CE5, Sect. 2], and their reflected cross-energy is in [CE3]. Version 2 adds Theorems 7, 24 and 21, Lemma 23, Corollary 8, the unconditional lim sup

form of Corollary 9, Section 8, Tables 3, 4, 8 and 9, the strict positivity of J_X, and the comparison with the Nyman–Beurling problem.

Use of AI tools. The proofs, computations and drafting of this paper, and of the preprint [C2] that it absorbs, were carried out with the assistance of an AI system (Claude, Anthropic), under the author's direction; an independent AI review agent refereed the proofs. The author checked the results and takes responsibility for them.

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Universidad Europea de Valencia, Valencia, Spain
Email address: pedrojesus.caceres@universidadeuropea.es