

AN EULER-PRODUCT MOLLIFIER FOR THE JENSEN MOMENT OF ζ AND AN ENERGY CRITERION FOR THE RIEMANN HYPOTHESIS

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ABSTRACT. The Jensen moment of Balazard, Saias and Yor expresses the off-line zeros of ζ through the mean of $\log |\zeta|$ on the critical line, and Jensen's inequality turns any mollifier into an upper bound for them. With Dirichlet polynomials as mollifiers this bound cannot fall below $\frac{1}{2} \log(1+c_0) = 0.0386$, a barrier caused by the pole at $s = 1$. We use instead the reciprocal of a smoothed Euler factor E_X^* , built from the prime powers up to a constant multiple of X and an explicit pole term, for which ζ/E_X^* is a product over the nontrivial zeros alone. We prove that the resulting bound $B_X(s_0)$, at any centre s_0 of the half-plane $\operatorname{Re} s > \frac{1}{2}$, tends to 0 under RH, so that RH is equivalent to $B_X(s_0) \rightarrow 0$; and that, if the supremum Θ of the real parts of the zeros is attained and exceeds $\frac{1}{2}$, it grows at least like $X^r/\log X$ for some $r > 0$. Hence, when Θ is attained, RH holds if and only if the single mean square $\mathcal{E}(X) = \int |\zeta/E_X^*(\frac{1}{2} + it)|^2 dt/(1+t^2)$ stays bounded as $X \rightarrow \infty$. Numerically the bound falls to 0.0015 at $X = 10^4$ for the centre $s_0 = 1$ of Balazard, Saias and Yor, and at $s_0 = 2$ it decays like $0.037/\log X$ up to $X = 10^6$, close to the value predicted by a heuristic with the constant $\kappa = \int_0^\infty (e^{2\operatorname{Ci}(v)} - 1) dv = 1.828\dots$; for the Davenport–Heilbronn function it grows. We describe the geometry of the factorization and discuss, with two control functions, what a proof of the boundedness of $\mathcal{E}(X)$ would have to use.

1. INTRODUCTION

Let $F(s) = (s-1)\zeta(s)/s$ and let μ be the probability measure $dt/(2\pi(\frac{1}{4} + t^2))$ on the critical line. Balazard, Saias and Yor [BSY] proved

$$(1) \quad \int_{-\infty}^{\infty} \log |\zeta(\tfrac{1}{2} + it)| d\mu(t) = \sum_{\operatorname{Re} \rho > \frac{1}{2}} \log \left| \frac{\rho}{1-\rho} \right|,$$

so that RH holds if and only if the left side vanishes. The identity is the Poisson–Jensen formula for F in the half-plane $\operatorname{Re} s > \frac{1}{2}$ at the point $s = 1$; μ is the Poisson measure of that point. For any function M analytic in the half-plane, Jensen's inequality $\int \log |FM| d\mu \leq \frac{1}{2} \log \int |FM|^2 d\mu$ then bounds the right side of (1) by

$$\frac{1}{2} \log \int |\zeta M(\tfrac{1}{2} + it)|^2 d\mu(t) - \log |M(1)|.$$

For Dirichlet polynomials $M(s) = \sum_{n \leq N} a_n n^{-s}$ this quantity is at least $\frac{1}{2} \log(1+c_0) = 0.0386$ with $c_0 = \sum_{k \geq 1} (1 - k(k+1) \log^2(1 + \frac{1}{k})) = 0.0803\dots$ [C, Prop. 3.5'']: the normalization at $s = 1$, where F carries the pole of ζ , forces M to approximate a step function, and the sawtooth costs c_0 whatever the length of M . The barrier is a property of Dirichlet polynomials, not of ζ .

In this paper the mollifier is $1/P$, where $P(s) = (s-1)E_g^*(s)/s$ and E_g^* is the smoothed Euler factor of the explicit formula in [C, Prop. 6.1'] (Lemma 5 below): $\log E_g^*$ is a finite sum over prime powers up to $e^a X$, plus a term that carries the pole and terms for the trivial zeros. It is not a

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Dirichlet polynomial, P is outer (Lemma 6), and $F/P = Z_g$ is a product over the nontrivial zeros alone.

Throughout, $g_0 \in C_c^\infty(\mathbb{R})$ is fixed with $g_0 \geq 0$, $\int g_0 = 1$, $\text{supp } g_0 \subset [-a, a]$, and for $X > e^a$ we put $g_X(u) = g_0(u - \log X)$, $L = \log X$, and write E_X^* , Z_X for $E_{g_X}^*$, Z_{g_X} . Let $\Theta = \sup \text{Re } \rho$.

Theorem 1 (Jensen moment at a general centre). *Let $s_0 = \sigma_0 + it_0$ with $\sigma_0 > \frac{1}{2}$, let $P_{s_0}(t) = (\sigma_0 - \frac{1}{2}) / (\pi((\sigma_0 - \frac{1}{2})^2 + (t - t_0)^2))$ be the Poisson kernel of $\text{Re } s > \frac{1}{2}$ at s_0 , and $b_\rho(s) = (s - \rho)/(s - 1 + \bar{\rho})$. Then*

$$\int \log |\zeta(\frac{1}{2} + it)| P_{s_0}(t) dt = \log \left| \frac{(s_0 - 1)\zeta(s_0)}{s_0} \right| + \sum_{\text{Re } \rho > \frac{1}{2}} \log \frac{1}{|b_\rho(s_0)|},$$

and, if $\zeta(s_0) \neq 0$, for every $X > e^a$

$$\sum_{\text{Re } \rho > \frac{1}{2}} \log \frac{1}{|b_\rho(s_0)|} \leq B_X(s_0) := \frac{1}{2} \log \int \left| \frac{\zeta}{E_X^*}(\frac{1}{2} + it) \right|^2 P_{s_0}(t) dt - \log \left| \frac{\zeta}{E_X^*}(s_0) \right|.$$

Theorem 2 (an off-line zero makes the bound explode). *Suppose that $\Theta > \frac{1}{2}$ and that some zero has real part Θ . Then for every s_0 with $\text{Re } s_0 > \Theta$ (in particular every s_0 with $\text{Re } s_0 > 1$) there are $r \in (0, \Theta - \frac{1}{2})$, C and X_0 with*

$$B_X(s_0) \geq \frac{1}{2} \text{Ei}(r(\log X - a)) - C \log \log X \geq \frac{X^r}{3re^{ra} \log X} \quad (X \geq X_0).$$

Theorem 3 (the converse under RH). *Let*

$$\mathcal{E}(X) = \int_{-\infty}^{\infty} \left| \frac{\zeta}{E_X^*}(\frac{1}{2} + it) \right|^2 \frac{dt}{1 + t^2},$$

which is finite for every $X > e^a$. Assume RH. Then $\mathcal{E}(X) \rightarrow \pi$ as $X \rightarrow \infty$, and $B_X(s_0) \rightarrow 0$ for every s_0 with $\text{Re } s_0 > \frac{1}{2}$.

Corollary 4 (criteria). (a) *For every s_0 with $\text{Re } s_0 > \frac{1}{2}$ and $\zeta(s_0) \neq 0$,*

$$\text{RH} \iff \lim_{X \rightarrow \infty} B_X(s_0) = 0.$$

(b) *If Θ is attained,*

$$\text{RH} \iff \limsup_{X \rightarrow \infty} \mathcal{E}(X) < \infty \iff \liminf_{X \rightarrow \infty} \mathcal{E}(X) < \infty.$$

Part (a) is unconditional, but it is a Jensen-type reformulation: by Theorem 1, $B_X(s_0)$ is bounded below by the off-line contribution, which does not depend on X . The new content is part (b), which rests on Theorem 2: a single weighted mean square, with no normalization at a point, detects an off-line zero by blowing up.

Numerically the Dirichlet barrier disappears: at the centre $s_0 = 1$ of [BSY] the bound $B_X(1)$ is 0.0049, 0.0026, 0.0018, 0.0015 for $X = 10, \dots, 10^4$, against 0.0384 for the optimal Dirichlet polynomial of length 150; at $s_0 = 2$ it decays to 0.0026 at $X = 10^6$ (Section 8). The decay is close to $\kappa W(s_0)/\log X$ with $W(s_0) = \sum_\rho P_{s_0}(\gamma)$ and

$$\kappa = \int_0^\infty (e^{2\text{Ci}(v)} - 1) dv = 1.828\dots,$$

a constant that comes from the profile $e^{\text{Ci}(|v|)}$, $v = (t - \gamma) \log X$, of $|\zeta/E_X^*|$ at a zero on the line. We derive this law only heuristically (Heuristic 7); Theorem 3 proves $B_X \rightarrow 0$ without a rate.

Scope. We do not prove that $\mathcal{E}(X)$ is bounded. $\mathcal{E}(X)$ compares $|\zeta|$ on the critical line with a finite, zero-free factor built from the primes, and Corollary 4(b) says that, when Θ is attained, its boundedness is equivalent to RH; without that hypothesis, boundedness gives only “RH or Θ not attained”. Numerically $\mathcal{E}(X)$ grows for the Davenport–Heilbronn function, which has a functional

equation and no Euler product; we expect it to grow, by the mechanism of Theorem 2, for a Beurling system with a positive prime measure and an off-line zero, which has an Euler product and no functional equation. This suggests that a proof must use the positivity of $\Lambda(n)$ and the exact spacing of the integers $1, 2, 3, \dots$ at the same time (Section 9). Section 10 lists the open points.

2. THE FACTORIZATION

Let $g \geq 0$ be piecewise continuous with compact support in $(0, \infty)$ and $\int g = 1$; put

$$\psi_g(v) = \int_v^\infty g(u) du, \quad K_g(w) = \int_0^\infty g(u) E_1(wu) du,$$

with $E_1(z) = \int_z^\infty e^{-\tau} \tau^{-1} d\tau$ on the principal branch, $|\arg z| < \pi$. We use $E_1(z) = -\gamma_0 - \log z + \text{Ein}(z)$ with Ein entire, and $E_1(z) = e^{-z} \varphi(z)$ with $\varphi(z) = \int_0^\infty e^{-\tau} (z + \tau)^{-1} d\tau$. For $\tau \geq 0$ the distance from $z + \tau$ to 0 is at least $d(z) := \text{dist}(z, (-\infty, 0])$, so

$$(2) \quad |\varphi^{(k)}(z)| \leq k! d(z)^{-k-1} \quad (k \geq 0, z \notin (-\infty, 0]).$$

Across the cut E_1 jumps by $2\pi i$, so K_g jumps by $2\pi i \int g = 2\pi i$ and $e^{\pm K_g}$ is single-valued; $\text{Re } K_g$ is continuous across the cut.

Lemma 5 ([C, Prop. 6.1']). *For $t \neq 0$ and s neither a zero of ζ nor of the form $\rho - r$ with $r > 0$,*

$$\log \zeta(s) = \sum_{n \geq 2} \frac{\Lambda(n) \psi_g(\log n)}{n^s \log n} + K_g(s-1) - \sum_{\rho} K_g(s-\rho) - \sum_{k \geq 1} K_g(s+2k),$$

the zeros counted with multiplicity and in order of increasing $|\text{Im } \rho|$. Consequently $\zeta = E_g^* Z_g$ as meromorphic functions on $\mathbb{C}$, with $Z_g(s) = \prod_{\rho} e^{-K_g(s-\rho)}$ and

$$\log E_g^*(s) = \sum_{n \geq 2} \frac{\Lambda(n) \psi_g(\log n)}{n^s \log n} + K_g(s-1) - \sum_{k \geq 1} K_g(s+2k),$$

and $\log |Z_g(s)| = -\sum_{\rho} \text{Re } K_g(s-\rho)$ for every s that is not a zero.

Sketch. For $x > 1$ not a prime power the truncated explicit formula with the term $\frac{1}{2} \log(1-x^{-2})$ of $\psi(x)$ kept exactly gives $\sum_{n \leq x} \Lambda(n) n^{-s} / \log n + E_1((s-1) \log x) = \log \zeta(s) + \sum_{\rho} E_1((s-\rho) \log x) + \sum_k E_1((s+2k) \log x)$; multiply by $g(\log x)$ and integrate [C]. Each factor $e^{-K_g(s-\rho)}$ is entire with a single simple zero at ρ , since $e^{-E_1(z)} = z e^{\gamma_0} e^{-\text{Ein}(z)}$. For $g = g_X$ the product converges absolutely and locally uniformly by Lemma 8 below (for general g , in the stated order), so both sides of $\zeta = E_g^* Z_g$ are meromorphic and the identity extends from the open set of the statement to $\mathbb{C}$. The formula for $\log |Z_g|$ follows, $\text{Re } K_g$ being continuous across the cut. $\square$

The prime sum is finite when g has compact support, and $e^{K_g(s-1)}$ has a simple pole at $s = 1$. Hence $P(s) = (s-1)E_g^*(s)/s$ has neither zeros nor poles in $\text{Re } s \geq \frac{1}{2}$, and E_g^* itself has no zeros: the zeros of ζ enter only through Z_g .

Lemma 6 (P is outer). *$\log |P|$ is continuous and bounded on $\text{Re } s \geq \frac{1}{2}$. Hence P is an outer function in $\text{Re } s > \frac{1}{2}$, $Z_g = F/P$ lies in the Smirnov class N^+ of the half-plane, and $\log |Z_g(s)| \leq \int \log |Z_g(\frac{1}{2} + it)| |P_s(t)| dt$ for $\text{Re } s > \frac{1}{2}$.*

Proof. Write $\log |P(s)| = -\log |s| + (\log |s-1| + \text{Re } K_g(s-1)) + \text{Re } D(s) - \sum_k \text{Re } K_g(s+2k)$, D the finite prime sum. $|\text{Re } D| \leq \sum \Lambda(n) n^{-1/2} / \log n$ on $\text{Re } s \geq \frac{1}{2}$. Since $\log |z| + \text{Re } E_1(z) = -\gamma_0 + \text{Re } \text{Ein}(z)$, the bracket equals $-\gamma_0 - \int g \log u + \text{Re } \int g(u) \text{Ein}((s-1)u) du$ and is continuous; it is bounded on $|s-1| \leq R$, and for $|s-1| \geq R$ with R large, $z = (s-1)u$ satisfies $\text{Re } z \geq -u/2$ and either $\text{Re } z \geq 0$, $d(z) = |z| \geq Ru$, or $|\text{Im } z| \geq \sqrt{R^2 - 1} u$, so $|E_1(z)| \leq e^{u/2}/d(z)$ by (2)

and $K_g(s-1)$ is bounded, as is $\log |(s-1)/s|$. Finally $|E_1(z)| \leq e^{-\operatorname{Re} z}/|z|$ for $\operatorname{Re} z > 0$ makes $\sum_k K_g(s+2k)$ bounded. A zero-free analytic function with $\log |P|$ bounded, harmonic in the half-plane and continuous up to the line, is the Poisson integral of its boundary values, i.e. outer. By [BSY], F is a Blaschke product times an outer function (no singular inner factor, including at ∞ , since $F(\sigma) \rightarrow 1$ as $\sigma \rightarrow +\infty$); a quotient of outer functions is outer, so $Z_g \in N^+$, and the last inequality is the subharmonic inequality for functions in N^+ . $\square$

3. THE JENSEN MOMENT AT A GENERAL CENTRE

Proof of Theorem 1. The Möbius map $s \mapsto (s-s_0)/(s-1+\bar{s}_0)$, which sends $\operatorname{Re} s > \frac{1}{2}$ to the unit disc and s_0 to 0, carries the Poisson measure of s_0 to normalized arc length. By [BSY] the Poisson–Jensen formula for F holds without singular part: $\int \log |F(\frac{1}{2}+it)| P_{s_0}(t) dt = \log |F(s_0)| + \sum_{\operatorname{Re} \rho > \frac{1}{2}} \log(1/|b_\rho(s_0)|)$, and $|F| = |\zeta|$ on the line. Since P is outer (Lemma 6), $\log |P(s_0)| = \int \log |P| P_{s_0}$; subtracting, $Z_X = F/P$ satisfies the same formula with $\log |Z_X(s_0)|$ on the right. Jensen’s inequality $\int \log |Z_X| P_{s_0} \leq \frac{1}{2} \log \int |Z_X|^2 P_{s_0}$ and $|Z_X| = |\zeta/E_X^*|$ on the line give the bound. $\square$

For $s_0 = 1$ the identity is (1). Moving the centre to $s_0 = \sigma_0 + it_0$ concentrates the sensitivity near height t_0 : an off-line zero at height close to t_0 contributes of order $(\beta - \frac{1}{2})(\sigma_0 - \frac{1}{2})/|s_0 - \rho|^2$ instead of $(\beta - \frac{1}{2})/|\rho|^2$. Table 1 checks the identity numerically, together with its analogue for the Davenport–Heilbronn function f , whose eight off-line zeros below height 400 are listed in [C].

function	s_0	line mean	interior (and pole)	excess
ζ	$2 + 20i$	−0.03463	−0.03408	−0.0006
ζ	$1.5 + 100i$	0.27150	0.27144	+0.0001
ζ	$1 + 85.7i$	0.13177	0.13158	+0.0002
f	$1 + 85.7i$	−0.05385	−1.49465	+1.4408 (off-line zeros: +1.4407)

TABLE 1. Theorem 1, first identity, with the window $|t - t_0| < 4000$; the remaining kernel mass is below $3 \cdot 10^{-4}$.

4. CRATERS: THE CONSTANT κ

Heuristic 7. Assume RH and simple zeros separated at scale $1/\log X$. Then for fixed s_0

$$B_X(s_0) \approx \frac{\kappa W(s_0)}{\log X}, \quad \kappa = \int_0^\infty (e^{2\operatorname{Ci}(v)} - 1) dv = 1.8280\dots, \quad W(s_0) = \sum_\rho P_{s_0}(\gamma).$$

Derivation. On the line, under RH, $\operatorname{Re}(s-\rho) = 0$ and $\operatorname{Re} E_1(ix) = -\operatorname{Ci}(|x|)$, so $\log |Z_X(\frac{1}{2}+it)| = \sum_\gamma \int g_X(u) \operatorname{Ci}(|t-\gamma|u) du$. As $X \rightarrow \infty$ the weight concentrates at $u = \log X$ in relative terms, and near a zero $|Z_X|$ takes the profile $e^{\operatorname{Ci}(|v|)}$ with $v = (t-\gamma) \log X$ (Figure 1b): a well of width of order $1/\log X$, with an overshoot of height $e^{\max \operatorname{Ci}} = e^{\operatorname{Ci}(\pi/2)} = e^{0.4720} = 1.60$. Since $\int_0^\infty \operatorname{Ci}(v) dv = 0$, the mean of $\log |Z_X|$ is unchanged to first order, while each zero adds $\frac{2}{\log X} \int_0^\infty (e^{2\operatorname{Ci}(v)} - 1) dv$ to the energy $\int |Z_X|^2 P_{s_0}$; and $\log |Z_X(s_0)| \rightarrow 0$. Interactions between neighbouring wells enter through the tail $\operatorname{Ci}(v) \sim \sin v/v$; they are neglected, which is reasonable because P_{s_0} weights low heights, where the zeros are sparse. We do not make this rigorous; the numbers below support it.

For $s_0 = 1$, $W = \frac{1}{2\pi} \sum_\rho |\rho|^{-2} = 0.00735$ and $\kappa W/\log 10^4 = 0.00146$, against the measured 0.0015; for $s_0 = 1 + 100i$, $W = 0.2305$ and the ratio of the measured value to $\kappa W/\log X$ is 1.07 at $X = 10^3$ and 1.05 at $X = 10^4$; for $s_0 = 2$, $W = 0.0220$ and $B \log X$ tends to 0.037 against $\kappa W = 0.040$ (Table 3).

5. VOLCANOES: PROOF OF THEOREM 2

Lemma 8 (the kernel). *Let $w = -x + iy$, $|x| \leq x_{\max}$, $X \geq e^{a+1}$, $L = \log X$. For every integer $A \geq 0$ there is C depending on A , $x_{\max}$ and g_0 only, such that:*

- (L1) *if $\operatorname{Re} w \geq \eta > 0$, then $|K_X(w)| \leq C_\eta X^{-\operatorname{Re} w} (1 + |y|)^{-A-1}/L$, with C_η depending also on η ;*
- (L2) *if $\operatorname{Re} w \geq 0$, $w \neq 0$, then $|K_X(w)| \leq C(1 + |\log(|w|L)|)$, and if moreover $|y| \geq 1$, $|K_X(w)| \leq C|y|^{-A-1}/L$;*
- (L3) *if $x > 0$ and $y \neq 0$, then $|K_X(w)| \leq e^{x(L+a)}/(|y|(L-a))$, and if moreover $|y| \geq 1$, $|K_X(w)| \leq C e^{xa} X^x |y|^{-A-1}/L$;*
- (L4) *for $x > 0$, $-\operatorname{Re} K_X(-x) = \int g_X(u) \operatorname{Ei}(xu) du \geq \operatorname{Ei}(x(L-a))$.*

Proof. Write $K_X(w) = e^{-wL} \int g_0(v) e^{-wv} \varphi(w(v+L)) dv$ and $u = v + L \in [L-a, L+a]$; note $d(wu) = u d(w)$, with $d(w) = |w|$ if $x \leq 0$ and $d(w) = |y|$ if $x > 0$. With $A = 0$ and (2), $|K_X(w)| \leq e^{x(L+a)}/(d(w)(L-a))$; for $x > 0$ this is the first bound in (L3). Integrating by parts A times against e^{-wv} (no boundary terms, $g_0 \in C_c^\infty$),

$$K_X(w) = \frac{e^{-wL}}{w^A} \int e^{-wv} \frac{d^A}{dv^A} [g_0(v) \varphi(w(v+L))] dv,$$

and by Leibniz and (2) the v -derivative of order m of $\varphi(w(v+L))$ is at most $m! |w|^m / (u d(w))^{m+1} \leq m! (|w|/d(w))^m / (u d(w))$. If $x \leq 0$, $|w|/d(w) = 1$; if $x > 0$ and $|y| \geq 1$, $|w|/d(w) \leq 1 + x_{\max}$. Hence $|K_X(w)| \leq C e^{x(L+a)} |w|^{-A} / (d(w)(L-a))$, with $e^{x(L+a)} = X^x e^{xa}$, which gives the second bounds in (L2), (L3), and (L1) (there $|w| \geq \max(\eta, |y|)$). For the first bound in (L2): if $|wu| \geq 1$, $|E_1(wu)| \leq |\varphi(wu)| \leq 1$ since $\operatorname{Re}(wu) \geq 0$; if $|wu| < 1$, $|E_1(wu)| \leq \gamma_0 + |\log |wu|| + \pi + e$; and $\log |wu| = \log(|w|L) + O(1)$. For (L4), $E_1(-\xi \pm i0) = -\operatorname{Ei}(\xi) \mp i\pi$ for $\xi > 0$, and Ei is increasing. $\square$

Proof of Theorem 2. Write $Z = Z_X$ and fix s_0 with $\operatorname{Re} s_0 > \Theta$.

(a) *Reduction to a point.* By Lemma 6 and Jensen's inequality, for every s_1 with $\operatorname{Re} s_1 > \frac{1}{2}$, $\log |Z(s_1)| \leq \int \log |Z| P_{s_1} \leq \frac{1}{2} \log \int |Z|^2 P_{s_1}$ (also when the last integral is infinite). Since $P_{s_1} \leq H P_{s_0}$ with $H = \sup_t P_{s_1}(t)/P_{s_0}(t) < \infty$, independent of X ,

$$B_X(s_0) \geq \log |Z(s_1)| - \frac{1}{2} \log H - \log |Z(s_0)|.$$

(b) *Choice of s_1 .* Take a zero $\rho_0 = \Theta + i\gamma_0$, and let $d \in (0, 1]$ be at most the distance from γ_0 to the ordinates of the other zeros of real part Θ (finitely many lie in any bounded window); d depends only on ρ_0 . Let $S = \sum' \min(1/|y|, C|y|^{-3})$ over the zeros $\rho \neq \rho_0$ with $\operatorname{Re} \rho = \Theta$, $y = \gamma_0 - \gamma$, with C the constant of (L3) for $A = 2$, $x_{\max} = \frac{1}{2}$; $S < \infty$ and S does not depend on r . Choose $r \in (0, \Theta - \frac{1}{2})$ with $2re^{2ra}S \leq \frac{1}{4}$ and such that $s_1 = \rho_0 - r$ is not a zero (this excludes a countable set of r). Let $\delta > 0$ be the distance from s_1 to the zeros.

(c) *Lower bound at s_1 .* By Lemma 5, $\log |Z(s_1)| = -\sum_\rho \operatorname{Re} K_X(s_1 - \rho)$, and $w = s_1 - \rho$ has $x = r - (\Theta - \beta) \leq r$.

- $\rho = \rho_0$: $w = -r$, and by (L4) the term is at least $\operatorname{Ei}(r(L-a))$.
- $\operatorname{Re} \rho = \Theta$, $\rho \neq \rho_0$: $x = r$, $|y| \geq d$. By (L3) the sum of $|K_X|$ is at most $X^r e^{ra} \frac{L}{L-a} S/L \leq \frac{1}{2} \cdot \frac{X^r e^{-ra}}{r(L-a)}$ for $X \geq X_0$, by the choice of r .
- $\Theta - r < \beta < \Theta$: $0 < x < r$. If $y = 0$ (finitely many), the term is $\int g_X \operatorname{Ei}(xu) du \geq 0$ for $X \geq X_0$. If $0 < |y| < 1$ (finitely many), each term is $O(e^{x(L+a)}/L) = o(X^r/L)$. If $|y| \geq 1$, by (L3) with $A = 2$ each term divided by X^r/L is at most $C e^{ra} X^{x-r} |y|^{-3} \frac{L}{L-a}$, which tends to 0 and is dominated by the summable $2C e^{ra} |y|^{-3}$. So these zeros contribute $o(X^r/L)$.
- $\beta \leq \Theta - r$: $\operatorname{Re} w \geq 0$ and $|w| \geq \delta$. By (L2), the finitely many with $|y| < 1$ contribute $O(\log L)$, the others $O(1/L)$.

Since $\operatorname{Ei}(\xi) \geq e^\xi/\xi$ for $\xi \geq 1.35$, the main term is at least $X^r e^{-ra}/(r(L-a))$ for $X \geq X_0$, and therefore $\log |Z(s_1)| \geq \frac{1}{2} \operatorname{Ei}(r(L-a)) - C \log L$.

(d) *The centre.* $\operatorname{Re}(s_0 - \rho) \geq \operatorname{Re} s_0 - \Theta =: \eta > 0$ for every zero, so by (L1) with $A = 2$, $|\log |Z(s_0)|| \leq C_\eta X^{-\eta} L^{-1} \sum_\rho (1 + |t_0 - \gamma|)^{-3} \rightarrow 0$. Combining (a), (c) and (d) gives the theorem. $\square$

Remark 9 (Davenport–Heilbronn). The Davenport–Heilbronn function f has an analogue E_f^* , built from the coefficients $b(n)$ of $-f'/f$ and its trivial zeros at $-1, -3, \dots$. Theorem 2 does *not* apply to it as stated: f has infinitely many zeros in $\operatorname{Re} s > 1$ [DH], so $\Theta_f > 1$, its attainment is unknown, and the zero $\rho_f = 0.808517 + 85.699348i$ studied below is not of maximal real part; we also do not prove the factorization and the Smirnov property for f . Numerically, at $s_1 = \rho_f - r$ the dominant term (L4) is nevertheless visible for moderate X : $\log |Z_f(s_1)|$ is 1.0332 and 1.6702 for $X = 10^3, 10^4$ with $r = 0.1$, against $\int g \operatorname{Ei}(ru) du = 1.0373$ and 1.6800, and 2.9489, 4.3721 with $r = 0.2$, against 2.9564, 4.3976.

6. THE CONVERSE UNDER RH

Proof of Theorem 3. Finiteness. On the line $s - 1 = -\frac{1}{2} + it$, so by the proof of Lemma 6 $\log |E_X^*|$ is bounded below there; with $|\zeta(\frac{1}{2} + it)| \ll (|t| + 2)^{1/4}$ this makes $\mathcal{E}(X)$ finite.

Convergence. Under RH, $\operatorname{Re}(s - \rho) = 0$ for $s = \frac{1}{2} + it$, so by Lemma 5, for t not an ordinate,

$$\log |Z_X(\tfrac{1}{2} + it)| = \sum_\gamma c_X(t - \gamma), \quad c_X(y) = \int g_X(u) \operatorname{Ci}(|y|u) du,$$

with multiplicity. Three bounds: (c1) $c_X(y) \leq \max \operatorname{Ci} = \operatorname{Ci}(\frac{\pi}{2}) = 0.4720$, since $g_X \geq 0$ and $\int g_X = 1$; (c2) $|c_X(y)| \leq 2/(|y|(L - a))$, since $|\operatorname{Ci}(x)| \leq 2/x$ (indeed $\leq 1/x$) from $\operatorname{Ci}(x) = \frac{\sin x}{x} - \int_x^\infty \frac{\sin \tau}{\tau^2} d\tau$; (c3) $|c_X(y)| \leq C|y|^{-3}/L$ for $|y| \geq 1$, by (L2). Under RH, $S(T) \ll \log T / \log \log T$ [T, Thm. 14.13], so an interval of length $h \leq 1$ centred at t contains at most $\frac{h}{2\pi} \log(|t| + 3) + C \log(|t| + 3) / \log \log(|t| + 3)$ ordinates. By (c1) the ordinates with $|t - \gamma| < 1/L$ contribute at most $0.472(\frac{1}{\pi L} + \frac{C}{\log \log(|t| + 3)}) \log(|t| + 3)$; by (c2) on dyadic shells $2^k/L \leq |t - \gamma| < 2^{k+1}/L$ the ordinates with $|t - \gamma| < 1$ contribute $\ll (\frac{\log L}{L} + \frac{1}{\log \log(|t| + 3)}) \log(|t| + 3)$; by (c3) the others contribute $\ll \log(|t| + 3)/L$. Treating $|t| \leq T_\varepsilon$ separately, for every $\varepsilon > 0$ there are $X_\varepsilon, C_\varepsilon$ with

$$(3) \quad \log |Z_X(\tfrac{1}{2} + it)| \leq \varepsilon \log(|t| + 3) + C_\varepsilon \quad (t \in \mathbb{R}, X \geq X_\varepsilon).$$

For each t that is not an ordinate, (c2) and (c3) give $\sum_\gamma |c_X(t - \gamma)| \leq (L - a)^{-1} (\sum_{|t - \gamma| < 1} 2/|t - \gamma| + C \sum_{|t - \gamma| \geq 1} |t - \gamma|^{-3}) \rightarrow 0$, so $\log |Z_X(\frac{1}{2} + it)| \rightarrow 0$ almost everywhere. By (3) with $\varepsilon = \frac{1}{4}$, $|Z_X|^2 \leq C(|t| + 3)^{1/2}$, integrable against $dt/(1 + t^2)$; dominated convergence gives $\mathcal{E}(X) \rightarrow \pi$, and in the same way $\int |Z_X|^2 P_{s_0} \rightarrow 1$. Finally $\operatorname{Re}(s_0 - \rho) = \operatorname{Re} s_0 - \frac{1}{2} > 0$ for every zero, so $\log |Z_X(s_0)| \rightarrow 0$ by (L1), and $B_X(s_0) \rightarrow 0$. $\square$

Proof of Corollary 4. (a) If RH fails, the right side of Theorem 1 is a positive number independent of X , so $B_X(s_0)$ does not tend to 0; if RH holds, $B_X(s_0) \rightarrow 0$ by Theorem 3. (b) RH implies $\mathcal{E}(X) \rightarrow \pi$ by Theorem 3. Conversely, suppose $\Theta > \frac{1}{2}$ is attained, and fix s_0 with $\operatorname{Re} s_0 > \max(\Theta, 1)$. Since $P_{s_0}(t) \leq C/(1 + t^2)$, $B_X(s_0) \leq \frac{1}{2} \log(C \mathcal{E}(X)) - \log |Z_X(s_0)|$, and $\log |Z_X(s_0)| \rightarrow 0$ by part (d) of the proof of Theorem 2. As $B_X(s_0) \rightarrow \infty$ by Theorem 2, $\mathcal{E}(X) \rightarrow \infty$; so $\liminf \mathcal{E}(X) < \infty$ already excludes $\Theta > \frac{1}{2}$. $\square$

Numerically, with the weight of Section 8, the ratio $R_X = \max_{t < 600} \log |Z_X(\frac{1}{2} + it)| / \log(t + 2)$ is 0.185, 0.169, 0.170 for $X = 10^2, 10^3, 10^4$, and on $200 \leq t < 600$ it decreases, 0.179, 0.161, 0.146, consistent with (3).

7. THE GEOMETRY OF Z_X

Figure 1 shows the factorization $\zeta = E_X^* Z_X$. On the critical line the truncated Euler product $\prod_{p \leq X} (1 - p^{-s})^{-1}$ does not converge and oscillates around $|\zeta|$, while E_X^* follows $|\zeta(\frac{1}{2} + it)|$ closely and never vanishes: at γ_1 it equals 0.097 for $X = 10^2$ and 0.048 for $X = 10^4$. E_X^* is built from the prime powers up to $e^a X$ and the pole alone; it is a smooth envelope that passes over each zero without touching it, and the zero is carried entirely by Z_X (panels a, b). In the strip (panel c) $\log |Z_X|$ is flat to the right of the line. The factor $e^{-K_X(s-\rho)}$ grows like $\exp(X^{\beta-\sigma}/((\beta-\sigma)\log X))$ where $\text{Re}(s-\rho) < 0$, so every zero has a *volcano* to its left and is seen from the right only as a *crater*; for a zero on the line the volcano lies in $\sigma < \frac{1}{2}$, outside the half-plane of Theorem 1. For the Davenport–Heilbronn function (panel d) the volcano of the off-line zero starts at $\sigma = 0.81$ and crosses the critical line. In these terms, Theorems 2 and 3 say that, when Θ is attained, RH holds if and only if only craters reach the critical line.

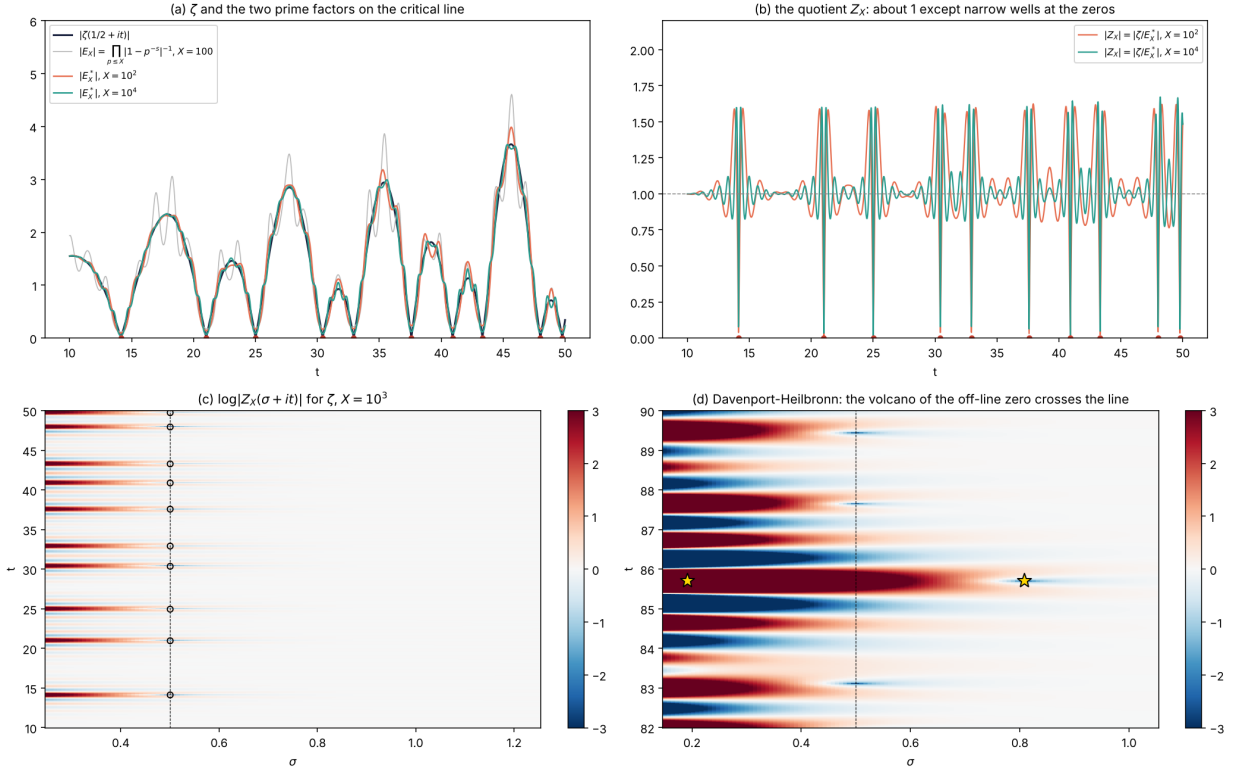


FIGURE 1. The geometry of Z_X . (a) $|\zeta(\frac{1}{2} + it)|$, the truncated Euler product $|E_X|$ with $X = 100$, and $|E_X^*|$ for $X = 10^2, 10^4$; dots: zeros. (b) $|Z_X| = |\zeta/E_X^*|$ on the line: wells at the zeros with the overshoot $e^{0.472}$. (c) $\log |Z_X(\sigma + it)|$ for ζ , $X = 10^3$: craters on the line, volcanoes to the left. (d) The same for the Davenport–Heilbronn function near its off-line zero (stars: the zero and its reflection); its volcano crosses the line.

8. NUMERICAL RESULTS

All computations use ζ and Hurwitz zeta functions by Euler–Maclaurin summation in double precision. For ζ the weight is either the Gaussian of width 0.4 around $\log X$ (Tables 2, 4) or, for large X , the C_c^∞ bump $g_0(v) \propto \exp(-1/(1 - (v/1.2)^2))$ on $[-1.2, 1.2]$, for which the prime sum

is exactly finite (Table 3). With the Gaussian truncated at 4.5 standard deviations the neglected tail of the prime sum grows like $X^{1/2}$ and inflates B from $X = 10^5$ on; this is why the compactly supported weight is used there. The theorems are stated for compactly supported weights.

s_0	$X = 10$	10^2	10^3	10^4
1 (Euler–Cáceres)	0.0049	0.0026	0.0018	0.0015
1 (best Dirichlet polynomial, $N \leq 150$)	0.0384 (floor 0.0386)			
$1 + 100i$ (Euler–Cáceres)	0.131	0.101	0.066	0.048
$1 + 100i$ (best Dirichlet polynomial, $N = 500$)	0.080			

TABLE 2. The bound of Theorem 1 with the mollifier $1/P$ against the optimal Dirichlet polynomial (computed exactly as a quadratic form).

X	10	10^2	10^3	10^4	10^5	10^6
$B_X(2)$	0.0138	0.0074	0.0052	0.0039	0.0032	0.0026
$B_X(2) \log X$	0.032	0.034	0.036	0.036	0.036	0.037

TABLE 3. The limit $X \rightarrow \infty$ at the single centre $s_0 = 2$ with the compactly supported weight; $\kappa W(2) = 0.040$. The Jensen identity holds to $1.3 \cdot 10^{-4}$.

	$X = 10$	10^2	10^3	10^4
$\zeta, s_0 = 1 + 100i$	0.130	0.101	0.065	0.048
$\zeta, s_0 = 2 + 85.7i$	0.241	0.121	0.090	0.071
$f, s_0 = 1 + 85.7i$	1.65	2.55	4.49	7.58
$f, s_0 = 1 + 100i$	0.34	0.31	1.26	5.91
$f, s_0 = 2 + 85.7i$	0.775	1.779	3.822	7.081

TABLE 4. $B_X(s_0)$ for ζ and for the Davenport–Heilbronn function f (Gaussian weight).

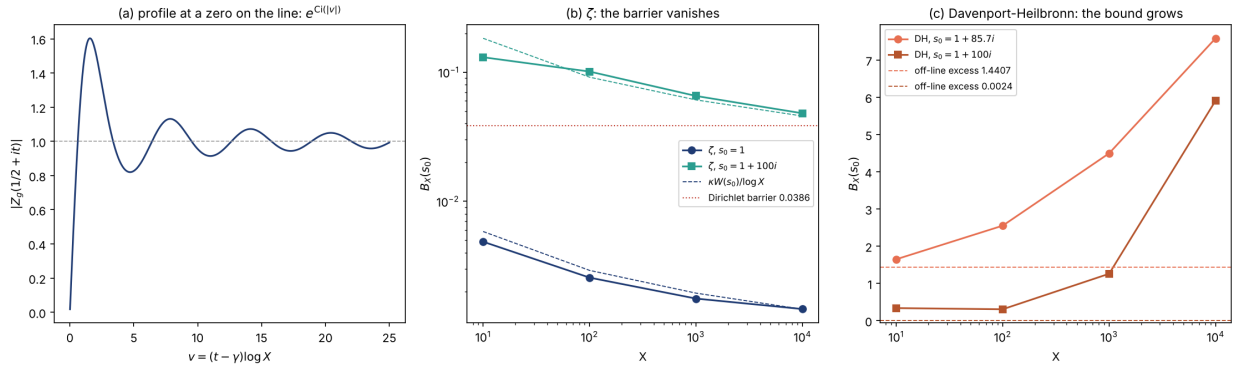


FIGURE 2. (a) Profile $e^{\text{Ci}(|v|)}$ of $|Z_X|$ at a zero on the line, $v = (t - \gamma) \log X$. (b) $B(s_0)$ for ζ at $s_0 = 1$ and $1 + 100i$, the heuristic law $\kappa W(s_0)/\log X$ (dashed), and the floor 0.0386 of Dirichlet mollifiers (dotted). (c) $B(s_0)$ for the Davenport–Heilbronn function, which grows with X ; dashed: its off-line excess.

9. WHAT A PROOF OF THE BOUND WOULD NEED

Positivity alone. For each n , $3 + 4 \cos \theta + \cos 2\theta \geq 0$, so $\Lambda(n) \geq 0$ gives $3 \operatorname{Re} \Sigma(\frac{1}{2}) + 4 \operatorname{Re} \Sigma(\frac{1}{2} + it) + \operatorname{Re} \Sigma(\frac{1}{2} + 2it) \geq 0$ for the prime sum Σ of $\log E_X^*$, and hence an upper bound for $\log |Z_X(\frac{1}{2} + it)|$. The bound is useless: the a_0 term sits at $t = 0$, on the pole, where the prime sum is of size $2\sqrt{X}/\log X$ (6.6, 14.1, 31.0 for $X = 10^2, 10^3, 10^4$). At $t = 20, 30, 40$ the resulting bound for $\log |Z_X|$ is about 5, 10 and 23 for these X , while the true values lie between -0.2 and 0.5 . Every nonnegative trigonometric polynomial has $a_0 > 0$, so every argument of this type pays a correction of size $X^{1/2}$ on the critical line [C, §6, *Positivity and the pole*].

The two controls. Two functions isolate the two halves of the structure of ζ (Table 5).

	$\Lambda \geq 0$, one Euler product	integers exactly $1, 2, 3, \dots$	off-line zeros
ζ	yes	yes (functional equation)	none known
Beurling system [DMV], [C]	yes	no	yes
Davenport–Heilbronn f [DH]	no	yes (functional equation)	yes

TABLE 5. The two controls.

For f the bound fails numerically (Table 4), so an argument that used only the functional equation and mean-value estimates valid for every Dirichlet series with a functional equation would, if it applied to f , prove something false; this is a heuristic constraint, since we have not proved that $\mathcal{E}_f(X)$ is unbounded. For a Beurling system with an off-line zero of maximal real part, the mechanism of Theorem 2 (one dominant term (L4) at $s_1 = \rho_0 - r$) should again force growth; Diamond, Montgomery and Vorhauer [DMV] constructed such systems with zeros as close to $\sigma = 1$ as the region of de la Vallée Poussin allows, and [C] gives a simplified example with a zero at $0.75 + 20i$. We do not prove the analogue of Lemmas 5–6 for these functions. Taken together, the two controls suggest that a proof of the bound must use the positivity of $\Lambda(n)$ and the exact spacing of the integers at the same time.

A quantity in which ζ cancels. For $s^* = 1 - \bar{s}$ the functional equation $|\zeta(s^*)| = |\zeta(s)|/|\chi(s)|$ gives

$$D_X(s) := \log |Z_X(s)| - \log |Z_X(s^*)| = \log |\chi(s)| - \log |E_X^*(s)| + \log |E_X^*(s^*)|,$$

a quantity built from the primes and the gamma factor alone (checked to $2 \cdot 10^{-13}$, also for f). It might seem to combine the two ingredients, but reindexing the zeros by $\rho \mapsto 1 - \bar{\rho}$ (legitimate by absolute convergence) gives $D_X(s) = \sum_{\rho} h(s - \rho)$ with $h(w) = -\operatorname{Re}[K_X(w) - K_X(-\bar{w})]$, the same kernel for every zero; D_X looks alike for ζ , f and a Beurling-type toy (Figure 3), and its prime side contains E_X^* at $\sigma < \frac{1}{2}$, where the pole barrier reappears.

10. OPEN POINTS

The bound. Prove $\limsup_X \mathcal{E}(X) < \infty$. By Corollary 4(b) this is equivalent to RH when Θ is attained, so it is at least as hard as RH in that case. It is a mean-value statement related to the hybrid Euler–Hadamard products of Gonek, Hughes and Keating [GHK, BGM], uniform in the length of the Euler factor at fixed weight. Unconditional mean-value methods with Dirichlet mollifiers are limited by the length of the mollifier relative to the height (the exponent $\frac{4}{7}$ of Conrey [Co]); here the weight $1/(1+t^2)$ is fixed and the length tends to infinity, the opposite regime.

The case Θ not attained. Here is a heuristic outline only. For each X one can choose ρ^* maximizing $\beta \log X - A \log(2 + |\gamma|)$; the zeros at distance at least 1 from ρ^* then contribute at most a small multiple of the main term, and a vertical shift of s_1 aligns the common phase $e^{-i\tau \log X}$. What remains is the cluster amplitude $\sum_{|\gamma - \gamma^*| < 1} X^{\beta - \beta^*} e^{i(\gamma - \gamma^*) \log X} \hat{g}_0(w)/w$, a sum of Cauchy kernels, which can be small only if off-line zeros of nearly maximal real part close to ρ^* cancel one another.

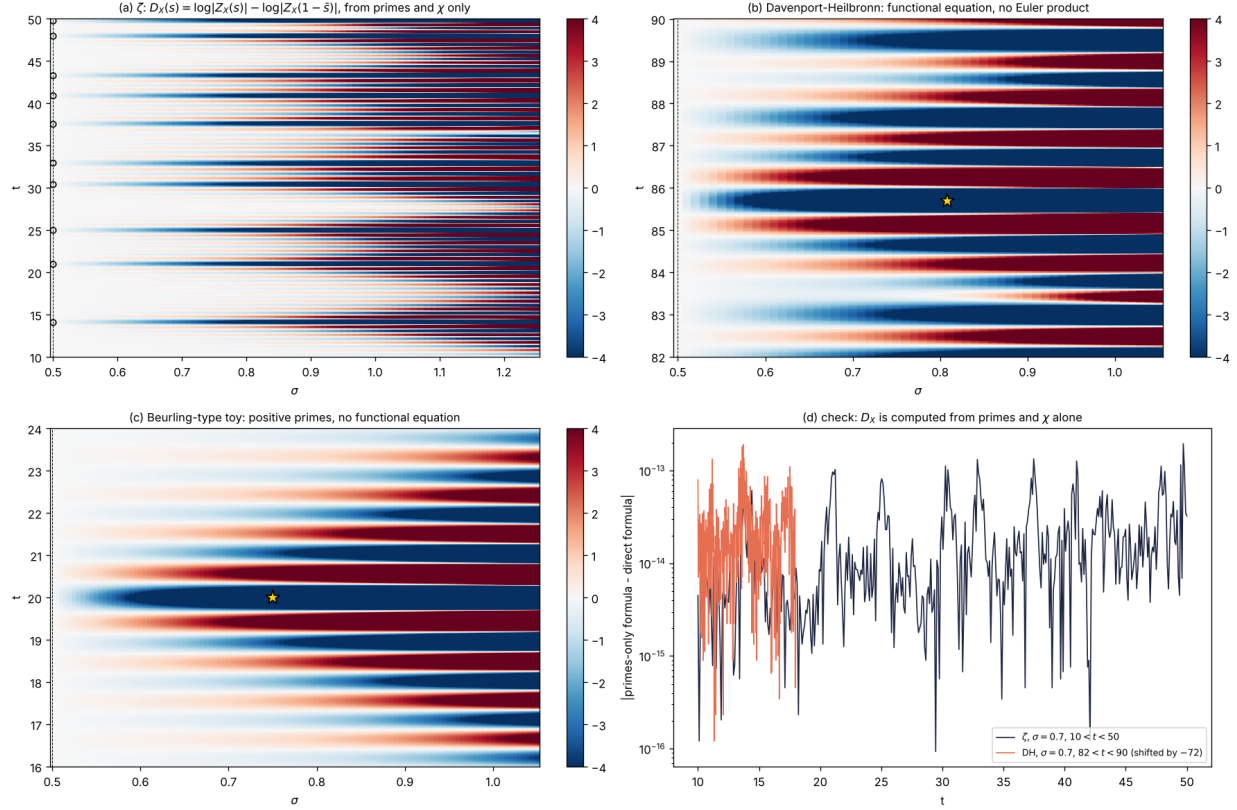


FIGURE 3. The reflection asymmetry $D_X(s)$, $X = 10^3$, computed from the primes and χ only. (a) ζ . (b) The Davenport–Heilbronn function. (c) A Beurling-type toy with the nonnegative prime measure $(1 - 2 \operatorname{Re} x^{\rho_B - 1}) dx / \log x$ on $[16, \infty)$, $\rho_B = 0.75 + 20i$, and no functional equation. (d) Difference between the primes-only formula and the direct one on $\sigma = 0.7$.

For two such zeros at distance Δ we expect a remainder of order $X^r \Delta / (r^2 \log X)$, so a lower bound $\Delta \geq X^{-o(1)}$ for their separation would plausibly close this case; no such bound is known.

DATA AND CODE

All tables and figures are reproduced by the ancillary Python scripts distributed with this paper (NumPy, SciPy, Matplotlib), which are also in the folder `anc/euler_mollifier/` of the source of [C]. The computations are in double precision, without interval arithmetic; they illustrate the theorems and are not used in their proofs.

Use of AI tools. The proofs, computations and drafting of this paper were carried out with the assistance of an AI system (Claude, Anthropic), under the author’s direction; an independent AI review agent refereed the proofs. The author checked the results and takes responsibility for them.

REFERENCES

- [BGM] H. M. Bui, S. M. Gonek, M. B. Milinovich, A hybrid Euler–Hadamard product and moments of $\zeta'(\rho)$, preprint, arXiv:1302.5032, 2013.
- [BSY] M. Balazard, E. Saias, M. Yor, Notes sur la fonction ζ de Riemann, 2, *Adv. Math.* 143 (1999), 284–287.
- [C] P. Caceres, The cross-energy of Dirichlet partial sums and the zeros of the Riemann zeta function, preprint, Zenodo, 2026, doi:10.5281/zenodo.23243641, Hexagon: hexagon:2610.00161; statement numbers refer to version 3 (2026).

- [Co] J. B. Conrey, More than two fifths of the zeros of the Riemann zeta function are on the critical line, *J. Reine Angew. Math.* 399 (1989), 1–26.
- [DH] H. Davenport, H. Heilbronn, On the zeros of certain Dirichlet series I, II, *J. London Math. Soc.* 11 (1936), 181–185 and 307–312.
- [DMV] H. G. Diamond, H. L. Montgomery, U. M. A. Vorhauer, Beurling primes with large oscillation, *Math. Ann.* 334 (2006), 1–36.
- [GHK] S. M. Gonek, C. P. Hughes, J. P. Keating, A hybrid Euler–Hadamard product for the Riemann zeta function, *Duke Math. J.* 136 (2007), 507–549.
- [T] E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, 2nd ed., revised by D. R. Heath-Brown, Oxford University Press, 1986.

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