

Ghosts in Poisson Brane Gases

Deep Bhattacharjee^{1,2,*} and Ushashi Bhattacharya^{3,†}

¹Formerly, Electro-Gravitational Space Propulsion Laboratory (EGSPL), Bhubaneswar, Odisha 751030, India

²Formerly, University of Calcutta, Senate House, 87/1 College Street, Kolkata 700073, West Bengal, India

³Formerly, National Taiwan University, Taipei 106319, Taiwan

(Dated: 18 October 2026)

A Poisson gas of branes with mass intensity ν dresses the graviton propagator by the random entire function $\prod_k (1 + p^2/t_k)$. We prove that the gas average of the propagator equals $p^{-2} \exp[-\int p^2 \nu(dt)/(p^2 + t)]$, so that the massive states carry total weight $e^{-\nu(0,\infty)} - 1 < 0$. For $\nu = \lambda(t) dt/t$ the averaged spectrum consists of ghosts alone exactly when $\lambda \leq 1$, and every average with a finite four-dimensional tadpole has states of both signs. Frozen gases keep alternating residues.

2020 Mathematics Subject Classification. Primary 81T15, 60G55; Secondary 30D15, 44A15, 30E25, 83C45.

Keywords: Poisson point process; Laguerre–Pólya class; complete Bernstein function; ghost

CONTENTS

I. Introduction	1
II. The gas and its frozen propagator	3
III. The averaged propagator	5
IV. The ghost threshold	6
V. Ultraviolet decay against positivity	7
VI. Discussion	7
Data and code availability	8
Ethical declarations	9
A. Poisson processes	9
B. Proof of Theorem 4(ii) and (iv)	9
C. Calculations	10
References	10

I. INTRODUCTION

Einstein gravity quantized around flat space is not renormalizable, and the two ways of improving its ultraviolet behaviour pull against each other. A kinetic operator $p^2 a(p^2)$ with a a polynomial of degree n makes the propagator fall like p^{-2-2n} , but it also gives the propagator n massive poles, and their residues add up to -1 , so at least one of them describes a state

of negative norm, a ghost [2, 27].¹ Replacing the polynomial by an entire function without zeros, $a = e^H$, removes the extra poles while keeping a fast decay in Euclidean momenta; this is the route of Efimov, Krasnikov, Kuz'min, Tomboulis, Biswas, Mazumdar, Siegel and Modesto [7, 8, 11, 14, 19, 20, 28]. The price is that the entire function has to be chosen by hand.

A brane gas offers a mechanism that produces form factors rather than postulating them [3, 5, 26]. A graviton that crosses a brane of mass scale t picks up a factor $1 + p^2/t$ in its kinetic operator, and a configuration $\Pi = \{t_1 < t_2 < \dots\}$ of branes turns the free propagator $1/p^2$ into

$$G_\Pi(z) = \frac{1}{z F_\Pi(z)}, \quad F_\Pi(z) = \prod_k \left(1 + \frac{z}{t_k}\right), \quad z = p^2. \quad (1)$$

For a Poisson gas the product is a random entire function of genus zero with negative zeros, an element of the Laguerre–Pólya class [18]. Each zero is a pole of the propagator, so a frozen configuration has the same alternating residues as a polynomial form factor. The natural hope is that averaging over the gas removes them: the expectation $\mathbb{E}F_\Pi(z) = \exp(z \int t^{-1} \nu(dt))$ is an exponential and has no zeros at all. But $1/(z \mathbb{E}F_\Pi)$ is not the expectation of any propagator. The two-point function of a field in a random background averages to $\mathbb{E}G_\Pi$, and the average of the reciprocal is not the reciprocal of the average.

This paper computes $\mathbb{E}G_\Pi$ exactly and reads off its spectrum. The setting is a Poisson process Π on $(0, \infty)$ with intensity measure ν , subject to

$$\int_{(0,\infty)} \min(1, t^{-1}) \nu(dt) < \infty, \quad (H)$$

¹ The sum rule is the partial-fraction identity $1/(z \prod_{j=1}^n (1 + z/t_j)) = 1/z + \sum_k r_k/(z + t_k)$ evaluated against the behaviour at infinity: since the left side decays like z^{-1-n} with $n \geq 1$, the coefficient of $1/z$ in the expansion at infinity vanishes, and that coefficient is $1 + \sum_k r_k$. When the masses are distinct and ordered, $t_1 < \dots < t_n$, the residue r_k has the sign $(-1)^k$, so the lightest massive state is always a ghost and the ghosts and healthy states alternate. This is the content of Proposition 2 below, there for infinitely many masses.

* itsdeep@live.com; * Corresponding author. Second address: d.bhattacharjee@erl-forschung.de.

† rai.bhattacharya.in@gmail.com

which is exactly the condition under which the product (1) converges. We write $|v| = v(0, \infty) \in (0, \infty]$ and

$$\varphi(z) = \int_{(0, \infty)} \frac{z}{z+t} v(dt), \quad z \in \mathbb{C} \setminus (-\infty, 0]. \quad (2)$$

Theorem A (Theorem 6). *For every $z \in \mathbb{C} \setminus (-\infty, 0]$ the propagator is integrable and*

$$\overline{G}(z) := \mathbb{E} G_{\Pi}(z) = \frac{e^{-\varphi(z)}}{z}. \quad (3)$$

The function φ is a complete Bernstein function [24], so $x\overline{G}(x) = e^{-\varphi(x)}$ decreases from 1 at $x = 0$ to $e^{-|v|}$ at $x = \infty$. A positive Källén–Lehmann representation [12, 17] would force $x\overline{G}(x)$ to increase. Hence:

Theorem B (Theorem 9). *If $v \neq 0$, then $\overline{G}$ has no representation $c/z + \int \mu(ds)/(z+s)$ with $c \geq 0$ and $\mu \geq 0$. The graviton pole keeps residue 1, and the massive states carry total weight $e^{-|v|} - 1 < 0$.*

The sign of the spectrum is decided by one number. Write $v(dt) = \lambda(t) dt/t$, so that $\lambda(t)$ is the number of branes per unit of $\log t$ near t .

Theorem C (Theorem 11). *If $\lambda \leq 1$ almost everywhere, then $\overline{G}(z) = 1/z - \int \sigma(ds)/(z+s)$ with $\sigma \geq 0$ of total mass $1 - e^{-|v|}$: every massive state in the averaged propagator is a ghost. If λ is continuous and locally Hölder, the spectral density is*

$$\rho(s) = -\frac{e^{-\operatorname{Re} \varphi(-s+i0)} \sin \pi \lambda(s)}{\pi s}, \quad (4)$$

and it is nowhere positive if and only if $\sup \lambda \leq 1$.

The same number controls the ultraviolet behaviour of the average, and the two requirements cannot be met together.

Theorem D (Theorem 13, Corollary 14). *If $\Lambda = \operatorname{ess sup} \lambda < \infty$, then $x\overline{G}(x) \geq e^{-v(0, a]}(1+x/a)^{-\Lambda}$ for all $a, x > 0$. Consequently the D -dimensional tadpole $\int \overline{G}(p^2) d^D p$ diverges whenever $D \geq 2 + 2\Lambda$; in particular it diverges in four dimensions when $\lambda \leq 1$, and for continuous locally Hölder λ a finite four-dimensional tadpole forces the spectral density to take both signs.*

Frozen configurations behave in the opposite way. Every infinite configuration gives a propagator that decays faster than any power, so every Euclidean graph of a massive scalar with polynomial vertices converges (Proposition 3). Its poles, however, stay.

Theorem E (Theorems 4, 5). *For the scale-free gas $v = \theta \mathbf{1}_{[0, \infty)}(t) dt/t$, almost surely the residues alternate in sign starting with a ghost, sum to -1 , and satisfy $\log |r_k| = -(1 + o(1))k^2/(2\theta)$. For every absolutely continuous v and every set A with $v(A) > 0$, the expected total residue $\mathbb{E} \sum_k |r_k| \mathbf{1}_A(t_k)$ is infinite.*

The last statement explains why Theorem A cannot be obtained by averaging the pole expansion term by term: the residues have no mean, and the smooth spectrum (4) appears only after the propagator itself is averaged.

A. What is proved and what is not

Every theorem, proposition and lemma of this paper is proved here by hand, without conditions beyond those stated. The results concern the propagator, that is, the tree-level two-point function. They say that for the multiplicative brane gas neither the frozen nor the averaged propagator is free of ghosts, and that in the averaged theory the cancellation of ultraviolet divergences and the positivity of the spectrum exclude each other in the precise sense of Theorem D. Nothing here proves ultraviolet finiteness or ultraviolet completeness of quantum gravity, and nothing addresses unitarity of the interacting theory; Sec. VI states what remains open. The archive [3] took the propagator of the averaged gas to be $1/(z\mathbb{E}F_{\Pi})$; Theorem A replaces it by the average of the propagator, which is the quantity that enters correlation functions.

B. History

Field theories with higher derivatives were studied by Pais and Uhlenbeck [22], who saw that the extra poles of the propagator come with alternating signs. Lee and Wick [16] tried to keep such theories unitary by moving the ghost poles off the real axis, and the modern form of this idea is Anselmi and Piva's fakeon prescription [1]. Stelle [27] showed that gravity with curvature-squared terms is renormalizable and has a massive spin-two ghost. The spectral representation of a two-point function in terms of a positive measure is due to Källén [12] and Lehmann [17]; positivity of that measure is the statement that the states created by the field have positive norm.²

Efimov [11] introduced nonlocal interactions with entire form factors, and Krasnikov [14], Kuz'min, Tomboulis [28] and Biswas, Mazumdar and Siegel [8] applied the idea to gauge theories and gravity. Biswas, Gerwick, Koivisto and Mazumdar [7] showed that an exponential form factor removes the Newtonian singularity, and Modesto [19] and Modesto and Rachwał [20] constructed super-renormalizable and finite theories along these lines. Shapiro [25] observed that the dressed propagator of such theories can develop complex poles, and Briscese, Calcagni, Modesto and Nardelli [9] analysed when a nonlocal propagator has a Källén–Lehmann representation. Power counting for Euclidean graphs rests on Weinberg's theorem [29].

The mathematics used here is classical. The Poisson process and the formulas of Campbell and Mecke are in Kingman [13] and Last and Penrose [15]. Entire functions of genus zero with

² For a scalar field ϕ with $\langle 0|\phi|0\rangle = 0$, the Källén–Lehmann representation reads $\langle 0|T\phi(x)\phi(0)|0\rangle = \int_0^\infty \Delta_F(x; s) \mu(ds)$ with $\Delta_F(x; s)$ the Feynman propagator of mass $\sqrt{s}$ and μ a positive measure, because $\mu(ds)$ is the norm of the projection of $\phi(0)|0\rangle$ onto states of invariant mass squared s . In Euclidean momentum space this is $G(p^2) = \int \mu(ds)/(p^2 + s)$. A measure that is negative on a set of masses signals states of negative norm there, which is what the word ghost means in this paper. A discrete negative term $-1/(p^2 + m^2)$ is a ghost particle of mass m ; a negative density is a continuum of them.

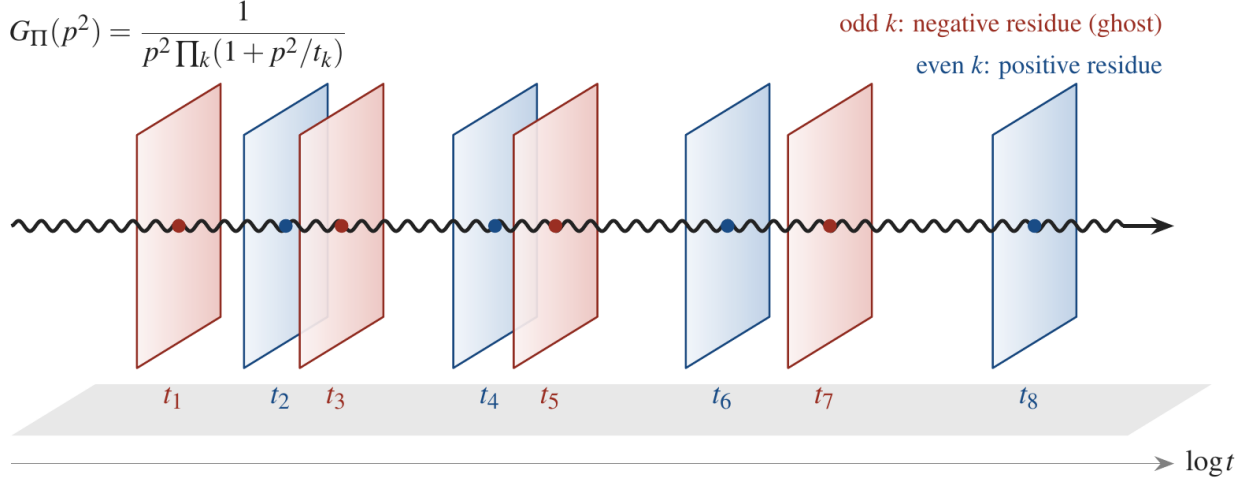


FIG. 1. A frozen brane gas. Each brane of mass scale t_k multiplies the graviton kinetic operator by $1 + p^2/t_k$, so the propagator has a pole at $p^2 = -t_k$. Ordering the masses increasingly, the residue at $-t_k$ is negative for odd k and positive for even k (Proposition 2); the lightest massive state is a ghost. The horizontal axis is $\log t$, the coordinate in which the scale-free gas of Theorem 4 is a homogeneous Poisson process.

real zeros, the Laguerre–Pólya class, are treated by Levin [18]. Complete Bernstein functions and their characterization by the sign of the imaginary part are in Schilling, Song and Vondraček [24].³ The boundary values of Cauchy integrals are given by the formulas of Plemelj [23] and Privalov, in the form of Muskhelishvili [21]. Our earlier work on brane gases and on quantum gravity [3, 5, 26] is the physical motivation, and random coverage questions of the same kind appear in [6].

II. THE GAS AND ITS FROZEN PROPAGATOR

A. Setting

Throughout, ν is a Radon measure on $(0, \infty)$ satisfying (H), and Π is a Poisson process with intensity ν . Condition (H) says that $\nu(0, 1] < \infty$ and $\int_{(1, \infty)} t^{-1} \nu(dt) < \infty$, so ν is finite on bounded sets but may have infinite total mass.

Lemma 1. *Almost surely $\Pi \cap (0, b]$ is finite for every b , and $\sum_{t \in \Pi} \min(1, t^{-1}) < \infty$. On this event F_Π is an entire function, $F_\Pi(x) \geq 1$ for $x \geq 0$, and the zeros of F_Π are the points $-t$, $t \in \Pi$. If ν has no atoms, the points of Π are almost surely distinct and the zeros are simple.*

³ A function $f: (0, \infty) \rightarrow [0, \infty)$ is a complete Bernstein function if $f(x) = a + bx + \int_{(0, \infty)} \frac{x}{x+s} \sigma(ds)$ with $a, b \geq 0$ and σ a positive measure with $\int (1+s)^{-1} \sigma(ds) < \infty$. By [24, Thms. 6.2 and 6.9] this holds if and only if f extends holomorphically to $\mathbb{C} \setminus (-\infty, 0]$, satisfies $\text{Im} f(z) \geq 0$ whenever $\text{Im} z > 0$, and has a finite limit $f(0+)$. Then $a = f(0+)$, $b = \lim_{x \rightarrow \infty} f(x)/x$, and σ is recovered from the boundary values of $\text{Im} f$ on the negative axis. The class is closed under many operations; the one used in Theorem 11 is that a nonnegative bounded holomorphic function whose imaginary part has the sign of $\text{Im} z$ is a Stieltjes transform of a finite positive measure up to the constant and the linear term.

Proof. Campbell’s formula [15, Prop. 2.7] gives

$$\mathbb{E} \sum_{t \in \Pi} \min(1, t^{-1}) = \int \min(1, t^{-1}) \nu(dt) < \infty,$$

so the sum is finite almost surely, and $\Pi \cap (0, b]$ has a Poisson number of points with finite mean $\nu(0, b]$. On compact sets $|z| \leq R$ we have $\sum_{t \in \Pi} |z|/t \leq R \sum_{t \in \Pi} t^{-1} < \infty$ after removing the finitely many points in $(0, 1]$, so the product converges absolutely and uniformly there; a locally uniform limit of polynomials is entire, and an absolutely convergent product vanishes only where a factor does. For $x \geq 0$ every factor is at least 1. If ν has no atoms, the probability that two points coincide is zero, by the same formula applied to the second factorial measure $\nu \otimes \nu$, which gives the diagonal measure zero. $\square$

B. Residues

Assume ν has no atoms and list the points as $t_1 < t_2 < \dots$. The propagator $G_\Pi = 1/(zF_\Pi)$ is meromorphic with simple poles at 0 and at each $-t_k$. Its residue at 0 is 1, and its residue at $-t_k$ is

$$r_k = - \prod_{j \neq k} \left(1 - \frac{t_k}{t_j}\right)^{-1}, \quad (5)$$

the product converging absolutely by Lemma 1. Denote by $r_k^{(N)}$ the same expression with j restricted to $j \leq N$, for $k \leq N$.

Proposition 2. *Let $t_1 < t_2 < \dots$ be the points of a configuration with $\sum_k t_k^{-1} < \infty$.*

- (i) $(-1)^k r_k > 0$ for every k . In particular $r_1 < 0$: the lightest massive state is a ghost.

- (ii) For every N , $\sum_{k=1}^N r_k^{(N)} = -1$.
- (iii) For each k , $|r_k^{(N)}|$ increases with $N \geq k$ and converges to $|r_k|$, and $r_k^{(N)}$ has the sign of r_k .
- (iv) If $\sum_k |r_k| < \infty$, then $\sum_k r_k = -1$.

Proof. (i) In (5) the factors with $j > k$ lie in $(0, 1)$ and the $k-1$ factors with $j < k$ are negative, so the product has the sign $(-1)^{k-1}$ and r_k the sign $(-1)^k$.

(ii) The rational function $R_N(z) = 1/(z \prod_{j \leq N} (1 + z/t_j))$ has simple poles at 0 and $-t_1, \dots, -t_N$ with residues 1 and $r_1^{(N)}, \dots, r_N^{(N)}$, and $R_N(z) = O(|z|^{-N-1})$ at infinity. The sum of its residues is the residue at infinity with the sign reversed, which is the coefficient of z^{-1} in the expansion of R_N at infinity; it vanishes since $N \geq 1$. Hence $1 + \sum_k r_k^{(N)} = 0$.

(iii) Passing from N to $N+1$ multiplies $r_k^{(N)}$ by $(1 - t_k/t_{N+1})^{-1}$, a number larger than 1 because $0 < t_k < t_{N+1}$. The infinite product of these factors converges since $\sum_j t_j^{-1} < \infty$.

(iv) By (iii), $|r_k^{(N)}| \mathbf{1}_{k \leq N} \leq |r_k|$ and $r_k^{(N)} \rightarrow r_k$, so dominated convergence applied to (ii) gives $\sum_k r_k = \lim_N \sum_{k \leq N} r_k^{(N)} = -1$. $\square$

A frozen configuration with infinitely many branes therefore has infinitely many ghosts, one at every odd position. Theorem 4 shows that for the scale-free gas the residues decay fast enough for (iv) to apply.

C. Euclidean finiteness of a frozen configuration

The ultraviolet behaviour of a frozen configuration is as good as it can be.

Proposition 3. Let $\Pi = \{t_1 < t_2 < \dots\}$ be infinite with $\sum_k t_k^{-1} < \infty$.

- (i) For every n and $x \geq 0$, $F_\Pi(x) \geq x^n / (t_1 \cdots t_n)$.
- (ii) Let $m > 0$ and $G_m(x) = 1/((x + m^2)F_\Pi(x))$. For every connected graph Γ in D Euclidean dimensions, with propagators $G_m(p_e^2)$ on the internal lines, vertex factors polynomial in the momenta, and fixed external momenta, the Feynman integral converges absolutely.

Proof. (i) Drop the factors with $k > n$, which are at least 1, and use $1 + x/t_k \geq x/t_k$.

(ii) By (i) with n replaced by any M , $G_m(x) \leq C_M(1+x)^{-M}$, and $G_m \leq m^{-2}$. Choose a spanning tree of Γ ; the L lines outside it, the chords, carry momenta $k_i + a_i$, where $k_1, \dots, k_L \in \mathbb{R}^D$ are independent loop momenta and the a_i are linear in the external momenta. The integrand is bounded by

$$C \prod_{i=1}^L (1 + |k_i|)^d (1 + |k_i + a_i|^2)^{-M} m^{-2(E-L)},$$

where d is the total degree of the vertex polynomials and E the number of internal lines; the tree lines were bounded by m^{-2} . Each factor is integrable over $\mathbb{R}^D$ once $2M > d + D$. $\square$

D. The scale-free gas

Take $v = \theta \mathbf{1}_{[t_0, \infty)}(t) dt/t$ with $\theta, t_0 > 0$. By the mapping theorem [15, Thm. 5.1] the points are $t_k = t_0 e^{u_k}$, where $u_1 < u_2 < \dots$ are the arrival times of a Poisson process of rate θ on $[0, \infty)$. The gaps $g_k = u_{k+1} - u_k$ and u_1 are independent with exponential law of mean $1/\theta$. Put

$$S_k = \sum_{j < k} (u_k - u_j) = \sum_{i=1}^{k-1} i g_i, \quad R_k = - \sum_{j \neq k} \log(1 - e^{-|u_j - u_k|}). \quad (6)$$

The second expression for S_k holds because the gap g_i separates u_j from u_k exactly when $j \leq i < k$.

Theorem 4. For the scale-free gas, almost surely:

- (i) $\log |r_k| = -S_k + R_k$ for every k , with $R_k \geq 0$;
- (ii) $S_k = \frac{k^2}{2\theta}(1 + o(1))$ and $R_k = O((\log k)^2)$ as $k \rightarrow \infty$; in particular $\log |r_k| = -(1 + o(1))k^2/(2\theta)$;
- (iii) $\sum_k |r_k| < \infty$, so the residues alternate in sign starting with a ghost and sum to -1 ;
- (iv) $\log F_\Pi(x) = \frac{\theta}{2}(\log \frac{x}{t_0})^2(1 + o(1))$ as $x \rightarrow \infty$.

Proof. (i) For $j < k$, $|1 - t_k/t_j| = e^{u_k - u_j}(1 - e^{-(u_k - u_j)})$, and for $j > k$, $|1 - t_k/t_j| = 1 - e^{-(u_j - u_k)}$. Taking $-\log$ of the absolute value of (5) gives $-S_k + R_k$, and every term of R_k is positive.

(ii) and (iv) are proved in Appendix B. (iii) follows from (ii) and Proposition 2. $\square$

The residues therefore fall off like a Gaussian in their index (Fig. 2), and the ghosts never thin out: the configuration has a ghost of weight $e^{-k^2/(2\theta)(1+o(1))}$ at every odd k .

E. Residues have no mean

Theorem 5. Let $v(dt) = \lambda(t) dt/t$ with $\lambda \geq 0$ locally integrable and satisfying (H), and let $A \subset (0, \infty)$ be a Borel set with $v(A) > 0$. Then

$$\mathbb{E} \sum_k |r_k| \mathbf{1}_A(t_k) = \infty.$$

Proof. For a configuration π and a point $t \in \pi$ let $r_t(\pi) = -\prod_{u \in \pi \setminus \{t\}} (1 - t/u)^{-1}$, the residue of G_π at $-t$. The Mecke formula [15, Thm. 4.1] gives

$$\mathbb{E} \sum_{t \in \Pi} |r_t(\Pi)| \mathbf{1}_A(t) = \int_A \mathbb{E} |r_t(\Pi \cup \{t\})| v(dt),$$

and $|r_t(\Pi \cup \{t\})| = \prod_{u \in \Pi} h_t(u)$ with $h_t(u) = u/|u - t|$. By independence of a Poisson process on disjoint sets, the expectation is the product of the expectations over $\Pi \cap (0, t/2)$ and $\Pi \cap [t/2, \infty)$. The first factor is positive: it equals $\exp(-\int_{(0, t/2)} (1 - h_t) dv)$ by Lemma 15, and $0 \leq 1 - h_t \leq 1$ there. On $[t/2, \infty)$ we have $h_t \geq 1$, and the same lemma with monotone convergence

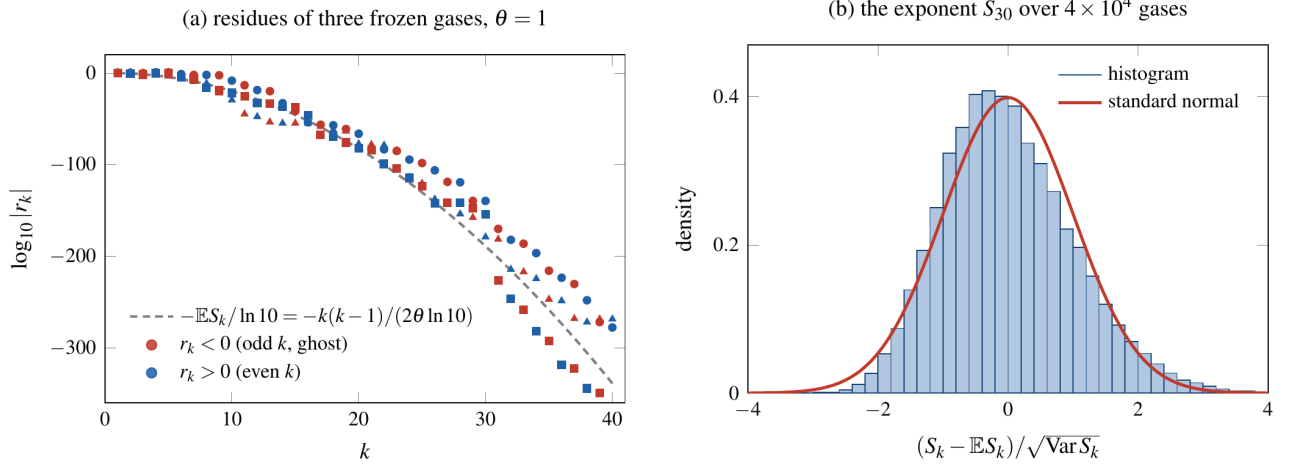


FIG. 2. (a) The residues r_k of three frozen configurations of the scale-free gas with $\theta = 1$, on a logarithmic scale. Circles, squares and triangles mark the three configurations; red marks negative residues and blue positive ones. The dashed curve is $-\mathbb{E}S_k/\ln 10$, the leading term of Theorem 4. (b) The exponent $S_{30} = \sum_{i < 30} i g_i$ over 4×10^4 configurations, centred and scaled; it is a weighted sum of independent exponential gaps, and its skewness is visible.

gives the second factor as $\exp(\int_{[t/2, \infty)} (h_t - 1) dv)$, with value in $(0, \infty]$. Since $h_t(u) \geq t/(2|u - t|)$ for $u \geq t/2$,

$$\int_{[t/2, 2t]} (h_t - 1) dv \geq \frac{1}{4} \int_{|u-t| < t/2} \frac{\lambda(u)}{|u-t|} du - 3 \int_{[t/2, 2t]} dv.$$

If t is a Lebesgue point of λ with $\lambda(t) > 0$, then $\int_{|u-t| < r} \lambda \geq \lambda(t)r$ for $r \leq r_0$, and writing $|u - t|^{-1} = r_0^{-1} + \int_{|u-t|}^{r_0} r^{-2} dr$ on $|u - t| < r_0$ gives

$$\int_{|u-t| < r_0} \frac{\lambda(u)}{|u-t|} du \geq \int_0^{r_0} \frac{1}{r^2} \int_{|u-t| < r} \lambda dr \geq \lambda(t) \int_0^{r_0} \frac{dr}{r} = \infty.$$

Since $v(A) > 0$, the set of such t in A has positive v -measure, and the integral over A is infinite. $\square$

III. THE AVERAGED PROPAGATOR

Theorem 6. Assume (H). For every $z \in \mathbb{C} \setminus (-\infty, 0]$, $\mathbb{E}|G_\Pi(z)| < \infty$ and $\mathbb{E}G_\Pi(z) = e^{-\varphi(z)}/z$, with φ as in (2). The function $\bar{G} = \mathbb{E}G_\Pi$ is holomorphic on $\mathbb{C} \setminus (-\infty, 0]$.

Proof. Fix $z = |z|e^{i\alpha}$ with $|\alpha| < \pi$ and put $g(t) = (1 + z/t)^{-1} = t/(t + z)$, so that $1/F_\Pi(z) = \prod_{t \in \Pi} g(t)$. For $t > 0$,

$$|t + z| \geq \cos(\alpha/2)(t + |z|), \quad (7)$$

because $|t + z|^2 - \cos^2(\alpha/2)(t + |z|)^2 = \frac{1}{2}(1 - \cos \alpha)(t - |z|)^2$. Hence $|g| \leq K := 1/\cos(\alpha/2)$ and $|1 - g(t)| = |z|/|t + z| \leq K \max(1, |z|) \min(1, t^{-1})$, which is v -integrable by (H). For $t > T := 2|z|$ we have $|t + z| \geq t(1 - |z|/t) \geq te^{-2|z|/t}$, since $1 - y \geq e^{-2y}$ for $0 \leq y \leq \frac{1}{2}$; thus $|g(t)| \leq e^{2|z|/t}$. Lemma 15 applies and gives $\mathbb{E} \prod_{t \in \Pi} g(t) = \exp(-\int (1 - g) dv) = e^{-\varphi(z)}$, because $1 - g(t) = z/(z + t)$. The domination in that lemma is uniform for z in compact subsets of $\mathbb{C} \setminus (-\infty, 0]$, so $\bar{G}$ is holomorphic by Morera's theorem and Fubini's theorem. $\square$

The exponent φ has the form of a complete Bernstein function with $a = b = 0$ and $\sigma = v$, and (H) is equivalent to $\int (1 + t)^{-1} v(dt) < \infty$. Its basic properties are read off from (2): $\varphi(0+) = 0$ by dominated convergence; $\varphi(\infty) = |v|$ by monotone convergence; $\varphi'(x) = \int t(x + t)^{-2} v(dt) > 0$; and

$$\text{Im } \varphi(x + iy) = \int \frac{ty}{(x + t)^2 + y^2} v(dt), \quad (8)$$

which has the sign of y .

Remark 7 (Thinning). For $x > 0$, $x\bar{G}(x) = e^{-\varphi(x)}$ is the probability that a thinning of Π , keeping each point t independently with probability $x/(x + t)$, has no points. Indeed the thinned process is Poisson with intensity $\frac{x}{x+t} v(dt)$ and total mass $\varphi(x)$. As x grows the retention probability tends to 1 at every t (Fig. 3(a)), and $x\bar{G}(x)$ decreases to $\mathbb{P}(\Pi = \emptyset) = e^{-|v|}$. For the scale-free gas, $x\bar{G}(x) = (1 + x/t_0)^{-\theta}$ is also the probability that $[t_0, t_0 + x]$ contains no point of Π .

Remark 8 (Two averages). The propagator of the averaged kinetic operator is $1/(z\mathbb{E}F_\Pi(z)) = z^{-1} \exp(-z \int t^{-1} v(dt))$ when $\int t^{-1} v(dt) < \infty$. It has no massive poles, but it is not the average of the propagator. On the positive axis $x/(x + t) \leq x/t$ gives $e^{-\varphi(x)} \geq \exp(-x \int t^{-1} dv)$, which is Jensen's inequality for the convex function $1/F$. The difference is large: for the scale-free gas with $\theta = \frac{3}{2}$ the first decays like $e^{-3x/2}$ and the second only like $x^{-3/2}$ (Fig. 5(b)).

Theorem 9. Assume (H) and $v \neq 0$. There are no $c \geq 0$ and positive measure μ on $(0, \infty)$ with $\int (1 + s)^{-1} \mu(ds) < \infty$ such that $\bar{G}(x) = c/x + \int \mu(ds)/(x + s)$ for all $x > 0$. Moreover $\lim_{x \rightarrow 0} x\bar{G}(x) = 1$ and $\lim_{x \rightarrow \infty} x\bar{G}(x) = e^{-|v|}$.

Proof. If such a representation held, $x\bar{G}(x) = c + \int \frac{x}{x+s} \mu(ds)$ would be nondecreasing in x . But $x\bar{G}(x) = e^{-\varphi(x)}$ and φ is strictly increasing. The limits are $e^{-\varphi(0+)}$ and $e^{-\varphi(\infty)}$. $\square$

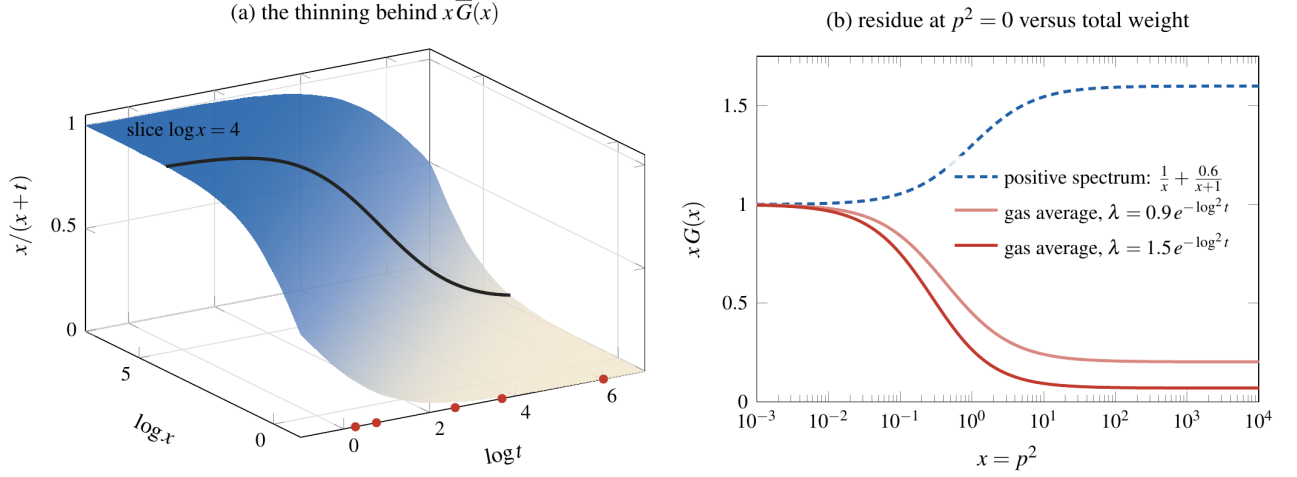


FIG. 3. (a) The retention probability $x/(x+t)$ of the thinning in Remark 7, as a surface over $(\log t, \log x)$; red dots are the points of one configuration, and the black curve is the slice used at $\log x = 4$. The probability that no point survives is $x\bar{G}(x)$. (b) For a positive spectrum $xG(x) = c + \int \frac{x}{x+s} \mu(ds)$ increases with x (dashed). For the gas average it decreases from 1 to $e^{-|v|}$ (solid), for the two intensities $\lambda = 0.9e^{-\log^2 t}$ and $\lambda = 1.5e^{-\log^2 t}$; this is Theorem 9.

For any propagator $c/z + \int \mu(ds)/(z+s)$ with μ a finite signed measure, the residue of the massless pole is $c = \lim_{x \rightarrow 0} xG(x)$ and the total massive weight is $\mu(0, \infty) = \lim_{x \rightarrow \infty} xG(x) - c$. For $\bar{G}$ these limits are 1 and $e^{-|v|} - 1$. The graviton keeps its residue, and the gas contributes a net negative weight, equal to -1 when $|v| = \infty$.

IV. THE GHOST THRESHOLD

From now on $v(dt) = \lambda(t)dt/t$ with $\lambda \geq 0$ measurable; condition (H) reads $\int_0^1 \lambda(t)t^{-1}dt + \int_1^\infty \lambda(t)t^{-2}dt < \infty$. Then (8) becomes

$$\text{Im } \varphi(x+iy) = \int_0^\infty \lambda(t) \frac{y}{(t+x)^2 + y^2} dt, \quad (9)$$

the Poisson integral of λ (extended by 0 to $t \leq 0$) evaluated at $-x+iy$. This is the reason λ , the density of branes per unit $\log t$, is the right variable.

Definition 10. For a function G holomorphic on $\mathbb{C} \setminus (-\infty, 0]$, the spectral density at $s > 0$ is $\rho_G(s) = -\pi^{-1} \lim_{y \downarrow 0} \text{Im } G(-s+iy)$, when the limit exists.

If $G = c/z + \int \mu(ds')/(z+s')$ with μ a signed measure, then $-\pi^{-1} \text{Im } G(-s+iy) = \pi^{-1} \int y((s'-s)^2 + y^2)^{-1} \mu(ds')$, the Poisson integral of μ , and $\rho_G(s)$ is the density of μ at every point where μ has a continuous density. A ghost continuum is a set of s with $\rho_G(s) < 0$.

Theorem 11. Assume (H) and $v(dt) = \lambda(t)dt/t$.

- (i) If $\lambda \leq 1$ almost everywhere, there is a positive measure σ on $(0, \infty)$ of total mass $1 - e^{-|v|}$ such that

$$\bar{G}(z) = \frac{1}{z} - \int_{(0, \infty)} \frac{\sigma(ds)}{z+s}, \quad z \in \mathbb{C} \setminus (-\infty, 0]. \quad (10)$$

- (ii) If λ is Hölder continuous on a neighbourhood of $s > 0$, the limit $\varphi(-s+i0)$ exists, with

$$\begin{aligned} \text{Im } \varphi(-s+i0) &= \pi \lambda(s), \\ \text{Re } \varphi(-s+i0) &= \text{PV} \int_0^\infty \frac{-s}{t-s} \lambda(t) \frac{dt}{t}, \end{aligned}$$

and $\rho_{\bar{G}}(s)$ is given by (4).

- (iii) If λ is continuous and locally Hölder on $(0, \infty)$, then $\rho_{\bar{G}} \leq 0$ on $(0, \infty)$ if and only if $\sup \lambda \leq 1$. If $\sup \lambda > 1$, then $\rho_{\bar{G}}$ is positive on a nonempty open set and negative on another.

Proof. (i) If $v = 0$ there is nothing to prove. Otherwise let $\psi = 1 - e^{-\varphi}$, holomorphic on $\mathbb{C} \setminus (-\infty, 0]$. For $\text{Im } z = y > 0$, (9) and $\lambda \leq 1$ give

$$0 < \text{Im } \varphi(z) \leq \int_0^\infty \frac{y dt}{(t+x)^2 + y^2} = \frac{\pi}{2} + \arctan \frac{x}{y} < \pi,$$

so $\text{Im } \psi(z) = e^{-\text{Re } \varphi(z)} \sin \text{Im } \varphi(z) > 0$. On $(0, \infty)$, ψ takes values in $[0, 1)$ and $\psi(0+) = 0$. By [24, Thms. 6.2 and 6.9], ψ is a complete Bernstein function, $\psi(x) = a + bx + \int \frac{x}{x+s} \sigma(ds)$ with $\sigma \geq 0$. Here $a = \psi(0+) = 0$ and $b = \lim \psi(x)/x = 0$ because ψ is bounded. Monotone convergence as $x \rightarrow \infty$ gives $\sigma(0, \infty) = \psi(\infty) = 1 - e^{-|v|}$. Thus $\bar{G}(x) = (1 - \psi(x))/x = 1/x - \int \sigma(ds)/(x+s)$ for $x > 0$, and both sides are holomorphic on $\mathbb{C} \setminus (-\infty, 0]$.

(ii) With $f(t) = \lambda(t)/t$, (2) reads $\varphi(z) = -w\Phi(w)$ with $w = -z$ and $\Phi(w) = \int_0^\infty f(t)(t-w)^{-1}dt$, a Cauchy integral that converges for $w \notin [0, \infty)$ by (H). As $z \rightarrow -s+i0$, $w \rightarrow s-i0$, and the Plemelj–Privalov formula [21, §§16–17] gives $\Phi(s-i0) = \text{PV} \int f(t)(t-s)^{-1}dt - i\pi f(s)$, valid because f is Hölder near s and integrable against $(1+t)^{-1}$ away from it. Hence $\varphi(-s+i0) = -s\Phi(s-i0)$ has the stated real and imaginary

parts. Finally $\overline{G}(-s+i0) = -s^{-1}e^{-\varphi(-s+i0)}$, so $\text{Im}\overline{G}(-s+i0) = s^{-1}e^{-\text{Re}\varphi}\sin(\pi\lambda(s))$, which is (4).

(iii) If $\sup\lambda \leq 1$, then $\sin\pi\lambda \geq 0$ and $\rho \leq 0$ by (4). Conversely let $\lambda(s_0) > 1$. The integral $\int_0^1 \lambda(t)t^{-1}dt$ is finite, hence $\liminf_{t \rightarrow 0} \lambda(t) = 0$, and by continuity λ takes every value in $(0, \lambda(s_0))$ on $(0, s_0)$. The open set $\lambda^{-1}(1, \min(2, \lambda(s_0)))$ is therefore nonempty, and there $\sin\pi\lambda < 0$ and $\rho > 0$. The open set $\lambda^{-1}(0, 1)$ is nonempty too, and there $\rho < 0$. $\square$

The threshold 1 is sharp in two senses: below it the whole massive spectrum is a ghost continuum, and as soon as λ crosses it somewhere, healthy states appear but ghosts remain (Fig. 4). In the window $1 < \lambda < 2$ the sign of the density is positive; for larger λ the sign of $\sin\pi\lambda$ alternates on the intervals $(n, n+1)$, so the averaged spectrum is a sequence of ghost and healthy bands, the continuum counterpart of the alternating residues of Proposition 2.

Corollary 12. *For the scale-free gas, $\varphi(z) = \theta \log(1+z/t_0)$ and*

$$\overline{G}(z) = \frac{1}{z} \left(1 + \frac{z}{t_0}\right)^{-\theta}.$$

- (i) *For $0 < \theta < 1$, (10) holds with $\sigma(ds) = \frac{\sin\pi\theta}{\pi}(s/t_0 - 1)^{-\theta} \mathbf{1}_{s>t_0} \frac{ds}{s}$, of total mass 1.*
- (ii) *For $\theta = 1$, $\overline{G}(z) = 1/z - 1/(z+t_0)$: a single ghost of residue -1 at t_0 .*
- (iii) *For $1 < \theta < 2$, $\rho_{\overline{G}}(s) = -\frac{\sin\pi\theta}{\pi}(s/t_0 - 1)^{-\theta} s^{-1}$ is positive on (t_0, ∞) and not integrable at t_0 .*

Proof. $\varphi(z) = \theta \int_{t_0}^{\infty} \frac{z dt}{t(z+t)} = \theta \log \frac{z+t_0}{t_0}$. For $s > t_0$, $1 - s/t_0 + i0 = (s/t_0 - 1)e^{i\pi}$, so $(1 + z/t_0)^{-\theta} \rightarrow (s/t_0 - 1)^{-\theta} e^{-i\pi\theta}$ as $z \rightarrow -s + i0$, which gives the densities in (i) and (iii); for $0 < s < t_0$ the boundary value is real and the density vanishes. In (i), $\lambda = \theta \mathbf{1}_{[t_0, \infty)} \leq 1$ and Theorem 11(i) applies; the measure σ there has no mass on $(0, t_0)$ and has the computed density on (t_0, ∞) , and its total mass is $1 = 1 - e^{-\infty}$. The substitution $s = t_0(1+v)$ gives $\int \sigma = \frac{\sin\pi\theta}{\pi} \int_0^{\infty} \frac{v^{-\theta} dv}{1+v} = \frac{\sin\pi\theta}{\pi} \cdot \frac{\pi}{\sin\pi\theta} = 1$ (Appendix C), so σ has no atom at t_0 . (ii) is the partial fraction $t_0/(z(z+t_0)) = 1/z - 1/(z+t_0)$. $\square$

At $\theta = 1$ the continuum of (i) collapses into one ghost. Above it the density is positive, yet by Theorem 9 the propagator is not a positive superposition: the negative weight is concentrated at the threshold t_0 , where the density fails to be integrable (Fig. 5(a)).

V. ULTRAVIOLET DECAY AGAINST POSITIVITY

Theorem 13. *Assume (H), $v(dt) = \lambda(t) dt/t$ and $\Lambda = \text{ess sup } \lambda < \infty$. For all $a, x > 0$,*

$$x\overline{G}(x) \geq e^{-v(0,a)} \left(1 + \frac{x}{a}\right)^{-\Lambda}.$$

Proof. Split (2) at a . On $(0, a]$ the integrand is at most 1. On (a, ∞) ,

$$\int_a^{\infty} \frac{x}{x+t} \lambda(t) \frac{dt}{t} \leq \Lambda \int_a^{\infty} \left(\frac{1}{t} - \frac{1}{x+t}\right) dt = \Lambda \log\left(1 + \frac{x}{a}\right).$$

Thus $\varphi(x) \leq v(0, a] + \Lambda \log(1 + x/a)$. $\square$

For the scale-free gas with $a = t_0$ the bound is an equality, so the exponent Λ cannot be improved.

Corollary 14. *Under the hypotheses of Theorem 13, the tadpole $I_D = \int_{\mathbb{R}^D} \overline{G}(|p|^2) d^D p$ diverges for every $D \geq 2 + 2\Lambda$. Hence:*

- (i) *if $\lambda \leq 1$, then $I_D = \infty$ for all $D \geq 4$: a ghost-only average is not ultraviolet finite at one loop in four dimensions;*
- (ii) *if λ is continuous and locally Hölder on $(0, \infty)$ and $I_D < \infty$ for some $D \geq 4$, then $\rho_{\overline{G}}$ is positive on a nonempty open set and negative on another.*

Proof. By Theorem 13 with $a = 1$, $\overline{G}(x) \geq cx^{-1-\Lambda}$ for $x \geq 1$, so $I_D \geq c' \int_1^{\infty} r^{D-3-2\Lambda} dr = \infty$ when $D - 2 - 2\Lambda \geq 0$. (i) is the case $\Lambda \leq 1$. For (ii), if $\Lambda = \infty$ then $\sup\lambda > 1$; if $\Lambda < \infty$, finiteness of I_D forces $\Lambda > D/2 - 1 \geq 1$, and continuity gives $\sup\lambda > 1$. In both cases Theorem 11(iii) applies. $\square$

The comparison with Proposition 3 is the main point of this section. A frozen configuration decays faster than any power and renders every Euclidean graph finite, at the price of infinitely many ghost poles. Averaging trades the poles for a continuum and, in doing so, gives up the fast decay: the average falls off only like a power, the power is the brane density Λ , and a power large enough for a finite tadpole in four dimensions requires a density above 1, which brings healthy and ghost states together. Table I collects the cases.

VI. DISCUSSION

The results answer, for the multiplicative brane gas, the question whether averaging over the branes removes the ghosts of a polynomial form factor. It does not. The frozen propagator has a ghost at every odd mass (Proposition 2), with weights that decay like a Gaussian in the index for the scale-free gas (Theorem 4); the averaged propagator is the explicit function (3), whose massive spectrum has net weight $e^{-|v|} - 1 < 0$ (Theorem 9) and consists of ghosts alone whenever the gas has at most one brane per unit of $\log t$ (Theorem 11). The two averages of Remark 8 differ: the earlier claim that the averaged propagator keeps only the graviton pole holds for $1/(zEF_{\Pi})$, which is not the average of a propagator.

Three questions are left open. First, the averaged theory has been studied only through its propagator. Whether a Lee–Wick or fakeon prescription [1, 16] can be applied to a ghost continuum such as (10), and what it would give for scattering amplitudes, is not addressed here. Second, quenched and annealed averages of loop amplitudes differ: the average of a product of propagators is not a product of averages, and the correlations between the propagators of one configuration are

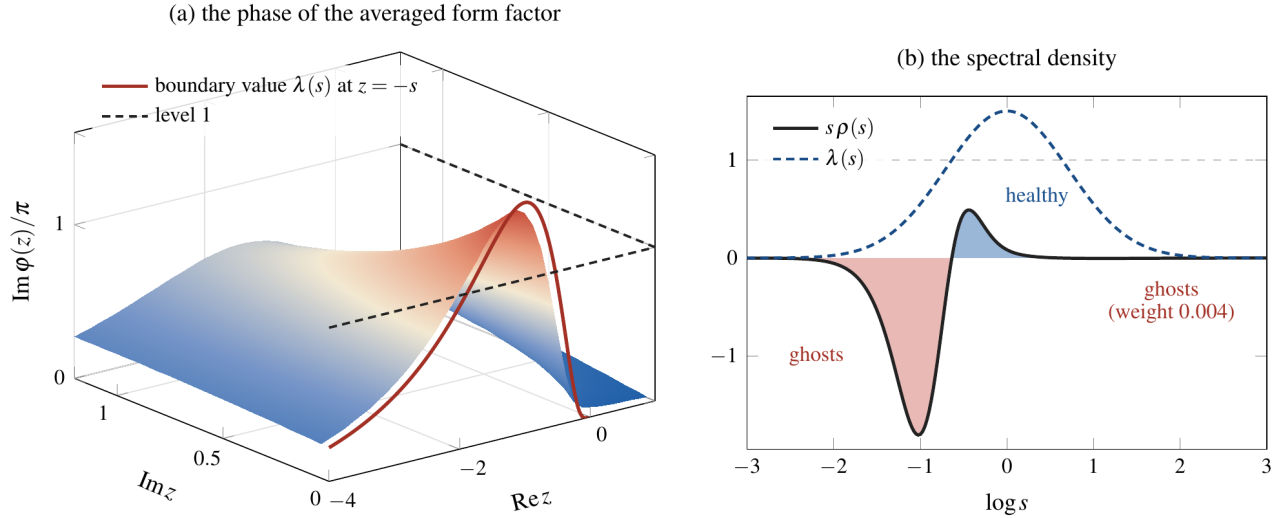


FIG. 4. The averaged spectrum for $\lambda(t) = 1.5e^{-\log^2 t}$. (a) The phase $\text{Im } \phi(z)/\pi$ over the upper half-plane. It is the Poisson integral (9) of λ , and its boundary value on the cut $z = -s$ is $\lambda(s)$ (red curve); the dashed line marks the level 1. (b) The spectral density $s\rho(s)$ of (4) against $\log s$, with λ dashed. The density is positive exactly where $1 < \lambda < 2$ and negative where $0 < \lambda < 1$; its integral is $e^{-|v|} - 1 = e^{-1.5\sqrt{\pi}} - 1 \approx -0.930$, in agreement with Theorem 9.

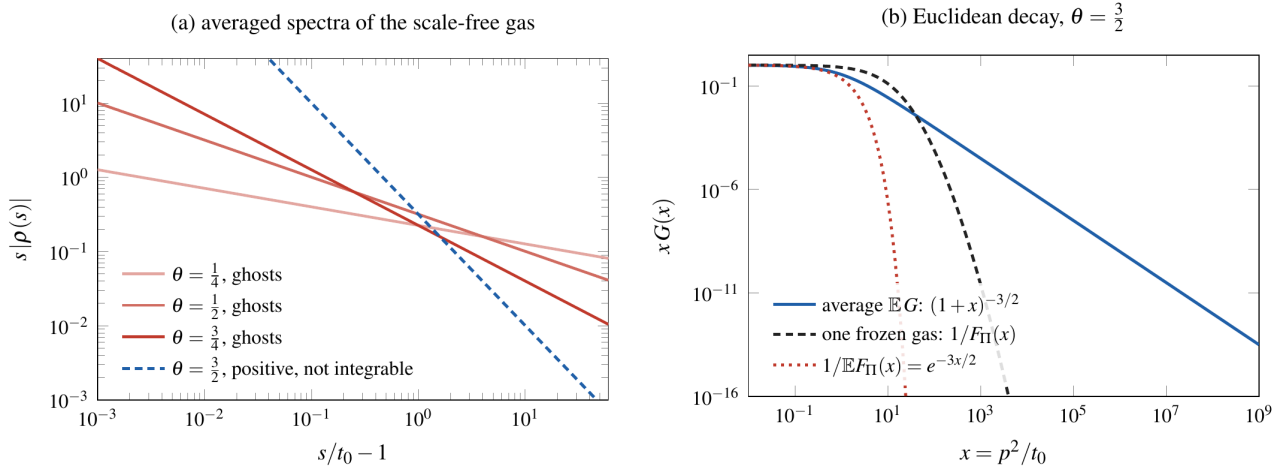


FIG. 5. The scale-free gas. (a) The averaged spectral density of Corollary 12, as $s|\rho(s)|$ against $s/t_0 - 1$: ghost continua for $\theta = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$, each of total weight 1, and for $\theta = \frac{3}{2}$ a positive density that is not integrable at the threshold. (b) Decay along the Euclidean axis for $\theta = \frac{3}{2}$, $t_0 = 1$: the average $x\bar{G}(x) = (1+x)^{-3/2}$ (Theorem 6), one frozen configuration $1/F_\Pi(x)$, which falls like $\exp(-\frac{3}{4}\log^2 x)$ (Theorem 4(iv)), and $1/\mathbb{E}F_\Pi(x) = e^{-3x/2}$ (Remark 8).

not computed in this paper. Third, the gas is multiplicative and Poisson. Interactions between branes, or a gas whose points repel, would change ϕ ; the threshold argument of Theorem 11 uses only that $\text{Im } \phi/\pi$ is a Poisson integral of the brane density, and would apply to any averaged exponent of that form.

Ultraviolet finiteness of perturbative quantum gravity is not proved here, and the results point the other way for this model: Theorem D shows that in the averaged gas the decay needed for a finite four-dimensional tadpole and a positive spectrum cannot both hold.

DATA AND CODE AVAILABILITY

All proofs in this paper are by hand. The finite sum rule and sign pattern of Proposition 2, inequality (7), the elementary bounds of Appendix B, the bound $\text{Im } \phi \leq \pi$ used in Theorem 11, the sign of (4) in the two windows and the partial fraction of Corollary 12(ii) were also verified in Lean 4 with Mathlib. Theorem 6 was checked by simulation, and Theorems 4, 11, 13 and Corollary 12 numerically; these checks, in Python, C and Julia, and the Lean files are archived with the source of this

TABLE I. The averaged propagator $\bar{G}(z) = z^{-1} e^{-\varphi(z)}$ for $v(dt) = \lambda(t)dt/t$. The last column refers to the tadpole $\int \bar{G}(p^2)d^4p$; “finite” requires $\text{ess sup } \lambda > 1$ (Corollary 14).

Intensity	Massive spectrum	Total massive weight	Decay of $x\bar{G}(x)$	$D = 4$ tadpole
$\lambda \leq 1, \nu < \infty$	ghost continuum (Thm. 11)	$e^{- \nu } - 1$	to $e^{- \nu }$	divergent
$\theta \mathbf{1}_{[t_0, \infty)}, 0 < \theta < 1$	ghost continuum on (t_0, ∞)	-1	$(x/t_0)^{-\theta}$	divergent
$\mathbf{1}_{[t_0, \infty)}$	one ghost at t_0	-1	$(x/t_0)^{-1}$	divergent
$\theta \mathbf{1}_{[t_0, \infty)}, 1 < \theta < 2$	positive density, singular threshold	-1	$(x/t_0)^{-\theta}$	finite
continuous, $\text{sup } \lambda > 1$	both signs (Thm. 11(iii))	$e^{- \nu } - 1$	at least $x^{-\Lambda}$	finite only if $\Lambda > 1$

paper in the repository [creelie/brane-cluster-gases](https://github.com/creelie/brane-cluster-gases) and in its Zenodo release 3.0.0 [4]; a PDF of the paper is at <https://github.com/creelie/brane-cluster-gases/blob/main/preprintGhostsBraneGases/GhostsBraneGases.pdf>. Deep Bhattacharjee carried out the computations and the Lean, Python, C and Julia checks with AI assistance from Claude (Anthropic); both authors approved the final text.

ETHICAL DECLARATIONS

Funding. None. *Competing interests.* None.

Appendix A: Poisson processes

We use three standard facts [13, 15]. Campbell’s formula: $\mathbb{E} \sum_{t \in \Pi} f(t) = \int f dv$ for measurable $f \geq 0$. The restrictions of Π to disjoint sets are independent. The Mecke formula: $\mathbb{E} \sum_{t \in \Pi} h(t, \Pi) = \int \mathbb{E} h(t, \Pi \cup \{t\}) v(dt)$ for measurable $h \geq 0$ on pairs of a point and a configuration.

Lemma 15. *Let $g: (0, \infty) \rightarrow \mathbb{C}$ be measurable with $\int |1 - g| dv < \infty$, and suppose there are $T > 0, K \geq 1$ and $c \geq 0$ with $|g(t)| \leq K$ for $t \leq T$ and $|g(t)| \leq e^{c/t}$ for $t > T$. Then $\prod_{t \in \Pi} g(t)$ converges absolutely almost surely, is integrable, and*

$$\mathbb{E} \prod_{t \in \Pi} g(t) = \exp\left(-\int (1 - g) dv\right).$$

If instead $g \geq 1$, the same identity holds with both sides in $(0, \infty]$.

Proof. By Campbell’s formula $\sum_{t \in \Pi} |1 - g(t)| < \infty$ almost surely, so the product converges absolutely. For $b > T$ the restriction $\Pi \cap (0, b]$ has a Poisson number N_b of points with mean $m_b = v(0, b] < \infty$, and given $N_b = n$ the points are independent with law $v|_{(0, b]}/m_b$. Hence

$$\mathbb{E} \prod_{t \in \Pi \cap (0, b]} g(t) = \sum_{n \geq 0} e^{-m_b} \frac{1}{n!} \left(\int_{(0, b]} g dv \right)^n = e^{-\int_{(0, b]} (1 - g) dv}.$$

For every b the partial product is bounded in modulus by $D = K^{N_T} \exp(c \sum_{t \in \Pi, t > T} t^{-1})$. The two factors of D are independent, $\mathbb{E} K^{N_T} = e^{(K-1)v(0, T]}$, and by the first computation

with $g = e^{c/t}$, $\mathbb{E} \exp(c \sum_{t > T} t^{-1}) = \exp \int_{(T, \infty)} (e^{c/t} - 1) dv \leq \exp(c e^{c/T} \int_{(T, \infty)} t^{-1} dv) < \infty$. Let $b \rightarrow \infty$ and use dominated convergence on the left and the integrability of $1 - g$ on the right. If $g \geq 1$, the partial products increase in b and monotone convergence replaces dominated convergence. $\square$

Appendix B: Proof of Theorem 4(ii) and (iv)

The main term. $S_k - \mathbb{E} S_k = \sum_{i < k} i(g_i - \theta^{-1})$ and $\mathbb{E} S_k = k(k-1)/(2\theta)$. The series $\sum_i (g_i - \theta^{-1})/i$ has independent centred terms with $\sum_i \text{Var}(g_i)/i^2 = \theta^{-2} \sum_i i^{-2} < \infty$, so it converges almost surely [10, Thm. 2.5.6]. Kronecker’s lemma [10, Thm. 2.5.9] with weights i^2 gives $k^{-2} \sum_{i < k} i(g_i - \theta^{-1}) \rightarrow 0$, so $S_k = \frac{k^2}{2\theta} (1 + o(1))$. The strong law of large numbers gives $u_k/k \rightarrow \theta^{-1}$.

Two Borel–Cantelli bounds. First, $\mathbb{P}(g_i < i^{-2}) = 1 - e^{-\theta i^{-2}} \leq \theta i^{-2}$, which is summable, so almost surely $g_i \geq i^{-2}$ for all large i . Second, let $N(I)$ be the number of u_k in an interval I and $n_m = \max(\lceil e^2 \theta \rceil, \lceil 2 \log(m+3) \rceil)$. Since $N[m, m+1)$ is Poisson with mean θ and $n! \geq (n/e)^n$,

$$\mathbb{P}(N[m, m+1) \geq n_m) \leq \frac{\theta^{n_m}}{n_m!} \leq \left(\frac{e\theta}{n_m}\right)^{n_m} \leq e^{-n_m} \leq (m+3)^{-2},$$

which is summable. Hence there is a random C_1 with $N[m, m+1) \leq C_1 + 2 \log(m+3)$ for all $m \geq 0$, and every interval I of length at most 1 with $\sup I \leq y$ satisfies $N(I) \leq 2C_1 + 4 \log(y+4)$.

The remainder. Write $\ell(d) = -\log(1 - e^{-d})$, so that $R_k = \sum_{j \neq k} \ell(|u_j - u_k|)$. Since $1 - e^{-d} \geq d/(1+d)$, $\ell(d) \leq \log(1 + 1/d)$, and for $d \geq 1$, $\ell(d) \leq e^{-d}/(1 - e^{-1}) \leq 2e^{-d}$. For the terms with $|u_j - u_k| \geq 1$, group the u_j by the integer part of $|u_j - u_k|$: the group at distance in $[m, m+1)$ lies in two intervals of length 1, so

$$\sum_{|u_j - u_k| \geq 1} \ell \leq 4 \sum_{m \geq 1} e^{-m} (2C_1 + 4 \log(u_k + m + 5)) = O(\log k).$$

The terms with $0 < |u_j - u_k| < 1$ are at most $2(2C_1 + 4 \log(u_k + 5)) = O(\log k)$ in number. Each has $|u_j - u_k| \geq \min(g_{k-1}, g_k) \geq k^{-2}$ for large k , so each is at most $\log(1 + k^2)$. Together, $R_k = O((\log k)^2)$.

The form factor. Put $L = \log(x/t_0)$. Then $\log F_\Pi(x) = \sum_k \log(1 + e^{L - u_k})$, and $a_+ \leq \log(1 + e^a) \leq a_+ + e^{-|a|}$ for real a .

The main part is $\sum_k (L - u_k)_+ = \int_0^L N[0, v] dv = \frac{\theta}{2} L^2 (1 + o(1))$, since $N[0, v]/v \rightarrow \theta$. The error is $\sum_k e^{-|L - u_k|} = O(\log L)$ by the window bound. This proves (iv).

Appendix C: Calculations

The Poisson kernel. For $y > 0$ and real x , $\int_0^\infty y((t+x)^2 + y^2)^{-1} dt = \frac{\pi}{2} + \arctan(x/y)$, which lies in $(0, \pi)$; over the whole line the integral is π . As $y \downarrow 0$ the kernel $\pi^{-1} y((t+x)^2 + y^2)^{-1}$ tends to the point mass at $t = -x$, which gives $\text{Im } \varphi(-s + i0) = \pi \lambda(s)$ at every point of continuity of λ ; Theorem 11(ii) needs Hölder continuity only for the real part.

The Beta integral. For $0 < \theta < 1$, $\int_0^\infty v^{-\theta} (1+v)^{-1} dv = B(1-\theta, \theta) = \Gamma(1-\theta)\Gamma(\theta) = \pi/\sin \pi\theta$ by the reflection formula. It is the total mass in Corollary 12(i).

The Stieltjes form for $0 < \theta < 1$. Directly from the integral,

$$\frac{\sin \pi \theta}{\pi} \int_0^\infty \frac{v^{-\theta}}{(1+v)(1+v+x/t_0)} dv = \frac{t_0}{x} \left(1 - \left(1 + \frac{x}{t_0} \right)^{-\theta} \right),$$

which, divided by t_0 , is $1/x - \overline{G}(x)$. The identity follows from $\frac{1}{(1+v)(1+v+w)} = \frac{1}{w} \left(\frac{1}{1+v} - \frac{1}{1+v+w} \right)$ and $\int_0^\infty v^{-\theta} (c+v)^{-1} dv = c^{-\theta} \pi / \sin \pi \theta$.

Signs on the cut. For $\lambda \in (0, 1)$, $\sin \pi \lambda > 0$ and the density (4) is negative; for $\lambda \in (1, 2)$, $\sin \pi \lambda < 0$ and it is positive; at integer values it vanishes.

An elementary inequality. For $0 \leq y \leq \frac{1}{2}$, $h(y) = 1 - y - e^{-2y}$ satisfies $h(0) = 0$, increases on $[0, \frac{1}{2} \log 2]$ and decreases afterwards to $h(\frac{1}{2}) = \frac{1}{2} - e^{-1} > 0$; hence $1 - y \geq e^{-2y}$ there. This is used in the proof of Theorem 6.

-
- [1] D. Anselmi and M. Piva, A new formulation of Lee–Wick quantum field theory, *J. High Energy Phys.* **06** (2017) 066, doi:10.1007/JHEP06(2017)066.
 - [2] M. Asorey, J. L. López and I. L. Shapiro, Some remarks on high derivative quantum gravity, *Int. J. Mod. Phys. A* **12** (1997) 5711–5734, doi:10.1142/S0217751X97002991.
 - [3] D. Bhattacharjee, Brane cluster gases and ghost-free form factors for quantum gravity: verification suite and figure scripts, version 2.2.0, Zenodo (2026), doi:10.5281/zenodo.23288723; all versions doi:10.5281/zenodo.22866290.
 - [4] D. Bhattacharjee, Brane cluster gases and ghost-free form factors for quantum gravity: verification suite and figure scripts, version 3.0.0, Zenodo (2026), doi:10.5281/zenodo.23449534.
 - [5] D. Bhattacharjee, A coherent approach towards quantum gravity, *Phys. Sci. Int. J.* **26** (2022) 59–78, doi:10.9734/psij/2022/v26i6751.
 - [6] D. Bhattacharjee and U. Bhattacharya, Iterated nerve complexes for k -fold sensor coverage, Preprints (2026), doi:10.20944/preprints202606.1351.v2.
 - [7] T. Biswas, E. Gerwick, T. Koivisto and A. Mazumdar, Towards singularity- and ghost-free theories of gravity, *Phys. Rev. Lett.* **108** (2012) 031101, doi:10.1103/PhysRevLett.108.031101.
 - [8] T. Biswas, A. Mazumdar and W. Siegel, Bouncing universes in string-inspired gravity, *J. Cosmol. Astropart. Phys.* **03** (2006) 009, doi:10.1088/1475-7516/2006/03/009.
 - [9] F. Brischese, G. Calcagni, L. Modesto and G. Nardelli, Form factors, spectral and Källén–Lehmann representation in nonlocal quantum gravity, *J. High Energy Phys.* **08** (2024) 204, doi:10.1007/JHEP08(2024)204.
 - [10] R. Durrett, *Probability: Theory and Examples*, 5th ed., Cambridge University Press, Cambridge, 2019, doi:10.1017/9781108591034.
 - [11] G. V. Efimov, Non-local quantum theory of the scalar field, *Commun. Math. Phys.* **5** (1967) 42–56, doi:10.1007/BF01646357.
 - [12] G. Källén, On the definition of the renormalization constants in quantum electrodynamics, *Helv. Phys. Acta* **25** (1952) 417–434, doi:10.5169/seals-112316.
 - [13] J. F. C. Kingman, *Poisson Processes*, Oxford University Press, Oxford, 1993, doi:10.1093/oso/9780198536932.001.0001.
 - [14] N. V. Krasnikov, Nonlocal gauge theories, *Theor. Math. Phys.* **73** (1987) 1184–1190, doi:10.1007/BF01017588.
 - [15] G. Last and M. Penrose, *Lectures on the Poisson Process*, Cambridge University Press, Cambridge, 2017, doi:10.1017/9781316104477.
 - [16] T. D. Lee and G. C. Wick, Negative metric and the unitarity of the S -matrix, *Nucl. Phys. B* **9** (1969) 209–243, doi:10.1016/0550-3213(69)90098-4.
 - [17] H. Lehmann, Über Eigenschaften von Ausbreitungsfunktionen und Renormierungskonstanten quantisierter Felder, *Nuovo Cim.* **11** (1954) 342–357, doi:10.1007/BF02783624.
 - [18] B. Ja. Levin, *Distribution of Zeros of Entire Functions*, Transl. Math. Monogr. 5, American Mathematical Society, Providence, RI, 1964, doi:10.1090/mmono/005.
 - [19] L. Modesto, Super-renormalizable quantum gravity, *Phys. Rev. D* **86** (2012) 044005, doi:10.1103/PhysRevD.86.044005.
 - [20] L. Modesto and L. Rachwał, Super-renormalizable and finite gravitational theories, *Nucl. Phys. B* **889** (2014) 228–248, doi:10.1016/j.nuclphysb.2014.10.015.
 - [21] N. I. Muskhelishvili, *Singular Integral Equations*, translated by J. R. M. Radok, P. Noordhoff, Groningen, 1953; reprinted by Dover, New York, 2008.
 - [22] A. Pais and G. E. Uhlenbeck, On field theories with non-localized action, *Phys. Rev.* **79** (1950) 145–165, doi:10.1103/PhysRev.79.145.
 - [23] J. Plemelj, Ein Ergänzungssatz zur Cauchyschen Integraldarstellung analytischer Funktionen, Randwerte betreffend, *Monatsh. Math. Phys.* **19** (1908) 205–210, doi:10.1007/BF01736696.
 - [24] R. L. Schilling, R. Song and Z. Vondraček, *Bernstein Functions: Theory and Applications*, 2nd ed., De Gruyter, Berlin, 2012, doi:10.1515/9783110269338.
 - [25] I. L. Shapiro, Counting ghosts in the “ghost-free” non-local gravity, *Phys. Lett. B* **744** (2015) 67–73, doi:10.1016/j.physletb.2015.03.037.
 - [26] S. Singha Roy, A. Sadhu, D. Bhattacharjee, P. Samal, S. Nandi and P. Nath Thakur, Brane clustering as a UV completion to quantum gravity, Preprints (2025), doi:10.20944/preprints202508.0407.v2.
 - [27] K. S. Stelle, Renormalization of higher-derivative quantum gravity, *Phys. Rev. D* **16** (1977) 953–969, doi:10.1103/PhysRevD.16.953.
 - [28] E. T. Tomboulis, Superrenormalizable gauge and gravitational theories, arXiv:hep-th/9702146 (1997), arXiv:hep-th/9702146.
 - [29] S. Weinberg, High-energy behavior in quantum field theory, *Phys. Rev.* **118** (1960) 838–849, doi:10.1103/PhysRev.118.838.