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One Cube Estimate Implies the Yang–Mills Mass Gap

Deep Bhattacharjee* [1,2,*] Ushashi Bhattacharya [3,†] James Montgomery Lockwood,[4,‡] and Tristan Romanov[5,§]

[1] *Formerly, Electro-Gravitational Space Propulsion Laboratory (EGSPL), Bhubaneswar, Odisha 751030, India*
[2] *Formerly, University of Calcutta (Heramba Chandra College), 23/49 Gariahat Road, Golpark, Kolkata 700029, West Bengal, India*
[3] *Formerly, National Taiwan University, Taipei 106319, Taiwan*
[4] *Independent Researcher, San Antonio, USA*
[5] *Independent Researcher, San Francisco, CA, USA*

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We prove that four-dimensional lattice Yang–Mills theory with any compact simple gauge group has a unique infinite-volume state and a mass gap at strong coupling, and that one averaged estimate in a single cube of fixed physical size gives a mass gap to every continuum limit. For SU(2) *the block variables approach the Haar measure in every representation as the coupling is tuned to the continuum, and this proves the estimate at the coarsest scale on every torus of side up to* $e^{2^k/(17k)}$ *after k blocking steps. The estimate along the whole tuning remains open.*

I. INTRODUCTION

Wilson's lattice gauge theory [116] assigns an element Q_e of a compact group $G \subset \mathrm{U}(N)$ to every link e of $\mathbb{Z}^4$ and weighs a configuration by $\exp(N\beta \sum_p \mathrm{Re}\,\mathrm{Tr}\,Q_p)$, where Q_p is the ordered product of the four link variables around the plaquette p.[1] At small $|\beta|$ the theory is under control: it has a unique infinite-volume state, correlations decay exponentially, and the transfer matrix has a gap [97, 105].[2] The question is how far into the coupling range such statements can be proved with explicit constants, since the problem of the continuum limit [73] begins where strong coupling ends and Bałaban's weak-coupling analysis [6, 7, 12] takes over.

Two methods give explicit ranges. The Bakry–Émery method [4, 5] of Shen, Zhu and Zhu [107] bounds the Hessian of the action; in [20] we showed that the sharp bound is $4d = 16$ and that no curvature argument for the ungauged measure goes beyond $|\beta| < 1/32$. The Dobrushin method [42, 43][3] bounds the influence of each link on the conditional law of another; for SU(2) the single-link conditional law is a von Mises–Fisher law on the three-sphere, the sharp Kantorovich modulus[4] of that family is one quarter [20] (Theorem 682 in Appendix K), and every link has eighteen partners, which gives $\beta_W < 2/9$ and nothing beyond, because the coefficients are attained.

Both limits come from the same count. A link of $\mathbb{Z}^4$ lies in six plaquettes and shares each with three other links. Gauge invariance makes part of this redundant: in a box, the variables on a forest of links can be set to the identity without changing any gauge-invariant expectation [38].[5] The forest we use is a

* itsdeep@live.com; * Corresponding author. Secondary address: d.bhattacharjee@erl-forschung.de; GitHub contact: egspl@proton.me, github.com/creelie.
† rai.bhattacharya.in@gmail.com
‡ spectralityworkinggroup@gmail.com; Secondary address: Blaze61029dogQ@proton.me.
§ romanov360@gmail.com

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[1] This is the normalisation of Shen, Zhu and Zhu [107] and of our earlier paper [20]. The conventional inverse coupling is $\beta_W = N^2\beta$, so the weight is $\exp(\frac{\beta_W}{N}\sum_p \mathrm{Re}\,\mathrm{Tr}\,Q_p)$; for SU(2), identified with the unit quaternions, $\frac{1}{2}\mathrm{Tr}\,Q_p = \mathrm{Re}\,Q_p$ and $\beta_W = 4\beta$. Standard accounts of the lattice theory are the books of Montvay and Münster [91] and of Gattringer and Lang [56]; the Hamiltonian version is due to Kogut and Susskind [79].

[2] Strong-coupling results of this kind go back to Osterwalder and Seiler [97], who proved exponential clustering and a gap by a convergent expansion, and to Seiler's monograph [105]; the general framework for lattice gauge fields and their continuum limits is that of Brydges, Fröhlich and Seiler [31]. More recent work treats the strong-coupling regime by probabilistic methods: the survey of Chatterjee [36], his large-N solution of SO(N) lattice gauge theory at strong coupling [37] and its planar extension by Basu and Ganguly [15], the Langevin dynamics of Shen, Zhu and Zhu [107], and the area law in an enlarged region by Cao, Nissim and Sheffield [33]. For finite gauge groups Adhikari and Cao proved correlation decay at weak coupling [1]. In two dimensions the continuum Yang–Mills measure has been constructed by Gross, King and Sengupta [62], Driver [46] and Lévy [81]; in four dimensions Bałaban's renormalisation group, as expounded by Dimock [40], gives ultraviolet stability in finite volume, and Magnen, Rivasseau and Sénéor constructed the theory with an infrared cutoff [86]. None of these results gives a mass gap in physical units.

[3] Dobrushin's theorem: if the conditional law of each variable given all others moves, in Kantorovich distance, by at most C_{xy} times the distance by which the variable at y moves, and $\sup_x \sum_y C_{xy} < 1$, then the Gibbs measure is unique and correlations decay at the rate given by the Neumann series of the matrix C. Dobrushin and Shlosman [44] replaced single variables by finite boxes, which is the form used in Sec. X.

[4] The Kantorovich distance between two probability measures on a metric space is the least expected distance over their couplings, equivalently the largest difference of their integrals over 1-Lipschitz functions [113]. The modulus of a family of laws indexed by a metric space is the best Lipschitz constant of the map from the index to the law in that distance.

[5] Creutz used a maximal tree to remove the gauge freedom completely; the

field of straight lines with one line through every site, chosen so that every plaquette contains exactly one fixed link. Then every free link has exactly twelve free partners, two in each of its plaquettes (Lemma 9), and both methods improve.

Theorem 1 (SU(2))**.** *Let $G=\mathrm{SU}(2)$ and $0\le\beta_W<\frac{1}{3}$. For every gauge-invariant local observable O the expectations $W_\Lambda(O)$ in boxes with free boundary converge, along every sequence of boxes that exhausts $\mathbb{Z}^4$, to a limit $\omega(O)$ that does not depend on the sequence. The state ω is invariant under translations and the hypercubic group and reflection positive;*[6] *truncated correlations of Lipschitz observables decay at every rate below $-\log(3\beta_W)$; and the transfer matrix has spectral gap*

$$m\ge-\log(3\beta_W)>0.$$

For $-\frac{1}{3}<\beta_W<0$ all statements except reflection positivity and the gap hold.

The step behind Theorem 1 is short once the line field is in place. The number $\frac{1}{3}$ is also the end of the road for this kind of argument.

Theorem 2 (Ceiling)**.** *Let F' be any set of links of $\mathbb{Z}^4$ without cycles, and give two distinct links outside F' that share a plaquette the coefficient $|\beta|$, the largest single-link Kantorovich coefficient of* SU(2)*. Then the weighted Dobrushin condition with these coefficients fails for $\beta_W\ge\frac{1}{3}$. For the line field the coefficients are attained, so no single-link Kantorovich argument in that gauge reaches $\beta_W=\frac{1}{3}$.*

The ceiling concerns coefficients between single links. Take instead as a block the three free links of one plaquette. The free links of the line field split into such triples with period two (Lemma 12), the coupling inside the block plaquette is no longer paid for, and each link keeps ten outside partners. The price is the sensitivity of a law on three spheres instead of one, which we bound analytically by integrating out the middle link of the block.

Theorem 3 (SU(2) with plaquette blocks)**.** *Let $G=\mathrm{SU}(2)$ and $0\le\beta_W\le0.348$. The number $\alpha(\beta_W)$ of (4) is increasing, $\alpha(\frac{1}{3})\le0.9456$, $\alpha(0.345)\le0.9837$ and $\alpha(0.348)\le0.9936$. The conclusions of Theorem 1 hold with decay at every rate below $-\frac{1}{2}\log\alpha(\beta_W)$ and a transfer-matrix gap*

$$m\ge-\tfrac{1}{2}\log\alpha(\beta_W)>0;$$

at $\beta_W=\frac{1}{3}$ this gives $m\ge0.0279$. For $-0.348\le\beta_W<0$ all statements except reflection positivity and the gap hold.

At small coupling the rate of Theorem 1 is the better one, so both bounds stand side by side; Theorem 3 matters past $\beta_W=\frac{1}{3}$, where the gap of Theorem 1 has closed.

The curvature method gains as well, and for every N. Let $\mathrm{Hess}\,S_0$ be the Hessian of $\sum_p\mathrm{Re}\,\mathrm{Tr}\,Q_p$ on G^E and F the set of fixed links.

Theorem 4 (Curvature in the gauge)**.** *For every N, every field Q and every tangent vector $v=(X_eQ_e)$ with $X_e=0$ for $e\in F$,*

$$|\,\mathrm{Hess}\,S_0(v,v)|\le\tfrac{29}{2}|v|^2,$$

against the sharp constant 16 without gauge fixing. Consequently, for $G=\mathrm{SU}(N)$ and $|\beta|<\frac{1}{29}$, or $G=\mathrm{SO}(N)$, $N\ge3$, and $|\beta|<\frac{N-2}{58N}$, the conclusions of Theorem 1 hold with the rate and gap of Theorem 5. For SU(3) *this is $\beta_W<\frac{9}{29}=0.310$.*

The constant $\frac{29}{2}$ is a statement about $\mathrm{U}(N)$, so it holds on every closed subgroup, and the curvature of the group does the rest. This gives the theorem for every compact simple gauge group, the exceptional groups included, with an explicit gap. Let $\rho:G\to\mathrm{U}(N)$ be a faithful unitary representation of a compact connected Lie group G, weigh a configuration by $\exp(N\beta\sum_p\mathrm{Re}\,\mathrm{Tr}\,\rho(Q_p))$, and let κ_G be the least Ricci[7] curvature of G for the bi-invariant metric $\langle X,Y\rangle=-\mathrm{Tr}(\rho(X)\rho(Y))$; we identify G with $\rho(G)\subset\mathrm{U}(N)$.

Theorem 5 (Every compact simple group)**.** *Suppose that the Lie algebra of G is semisimple, so that $\kappa_G>0$, and that*

$$N|\beta|<\tfrac{2}{29}\,\kappa_G.$$

Then the conclusions of Theorem 1 hold, for $\beta<0$ all of them except reflection positivity and the gap, with decay at every rate below

$$m_G(\beta)=\operatorname{arccosh}\Big(1+\frac{1}{12}\Big(\frac{\kappa_G}{N|\beta|}-\frac{29}{2}\Big)\Big)>0$$

and a transfer-matrix gap $m\ge m_G(\beta)$. For a simple G, $\kappa_G=h^\vee/(2\ell_\rho)$, where $h^\vee$ is the dual Coxeter number and ℓ_ρ the Dynkin index of ρ; Table I lists the values.

For $\mathrm{SU}(N)$, $\kappa_G=\frac{N}{2}$, and for $\mathrm{SO}(N)$, $\kappa_G=\frac{N-2}{4}$, so Theorem 5 contains the ranges of Theorem 4 and supplies their rate.[8] For small $|\beta|$ the existence of a gap is classical for every compact group, with no explicit range [97, Thm. 3.6]; what

temporal gauge, with all links in one direction fixed, is the special case of a family of straight lines. In a maximal tree almost every link is fixed in some region and none in others, so the number of partners of a free link is not uniform, and the temporal gauge leaves fourteen partners per link. The point here is a pattern that fixes exactly one link in every plaquette.

[6] That is, $\omega(\vartheta(\bar F)F)\ge0$ for every F depending on the links on one side of a lattice hyperplane, ϑ being the reflection in it. It gives the Hilbert space and the transfer matrix of Osterwalder and Schrader [95, 96] and of Osterwalder and Seiler [97], reconstructed in Appendix F.

[7] For a bi-invariant metric on a compact Lie group the Ricci curvature is $-\frac{1}{4}\mathrm{Tr}((\mathrm{ad}X)^2)$ in the direction of X. It is positive for every $X\neq0$ exactly when the Lie algebra is semisimple, and κ_G is its minimum over unit vectors.

[8] The dual Coxeter number is $h^\vee=N$ for $\mathrm{SU}(N)$, $N-2$ for $\mathrm{SO}(N)$, $n+1$ for $\mathrm{Sp}(n)$, $4,9,12,18,30$ for G_2,F_4,E_6,E_7,E_8, and the Dynkin index of a representation is normalised to one on the defining representation of $\mathrm{SU}(N)$ [76, Ch. 6]. The identity $\kappa_G=h^\vee/(2\ell_\rho)$ is the statement that the Ricci tensor of a compact simple group with a bi-invariant metric is one quarter of the negative Killing form, $\mathrm{Ric}(X,X)=-\frac{1}{4}B(X,X)$ [90], together with $B(X,X)=2h^\vee\mathrm{Tr}(\rho(X)^2)/\ell_\rho$. Lower Ricci bounds of the same kind, for special holonomy and for Riemannian submersions, are studied in [24, 57].

TABLE I. Least Ricci curvature $\kappa_G = h^\vee/(2\ell_\rho)$ of the compact simple groups in a faithful representation ρ of least dimension N, and the range of Theorem 5 for the conventional inverse coupling $\beta_W = N^2\beta$.

Group	ρ	$h^\vee$	ℓ_ρ	κ_G	$\beta_W <$
$\mathrm{SU}(N)$, $N \geq 2$	$\mathbb{C}^N$	N	1	$\frac{N}{2}$	$\frac{N^2}{29}$
$\mathrm{SO}(N)$, $N \geq 5$	$\mathbb{R}^N$	$N-2$	2	$\frac{N-2}{4}$	$\frac{N(N-2)}{58}$
$\mathrm{SO}(3)$	$\mathbb{R}^3$	2	4	$\frac{1}{4}$	$\frac{3}{58}$
$\mathrm{Sp}(n)$, $n \geq 2$	$\mathbb{C}^{2n}$	$n+1$	1	$\frac{n+1}{2}$	$\frac{2n(n+1)}{29}$
G_2	7	4	2	1	$\frac{14}{29}$
F_4	26	9	6	$\frac{3}{4}$	$\frac{39}{29}$
E_6	27	12	6	1	$\frac{54}{29}$
E_7	56	18	12	$\frac{3}{4}$	$\frac{84}{29}$
E_8	248	30	60	$\frac{1}{4}$	$\frac{124}{29}$

is new is an explicit range for every group, past the range $N|\beta| < \kappa_G/16$ that the Hessian bound 16 gives without gauge fixing, and an explicit gap.[9] As $\beta \to 0$ the gap grows like $\log \frac{\kappa_G}{6N|\beta|}$.

No Bakry–Émery argument for the ungauged measure can pass $|\beta| = \frac{1}{32}$ [20, Thm. C]; the gauge fixing changes the measure, not the metric, which is why it can. For $\mathrm{SU}(2)$ the curvature range $\beta_W < \frac{4}{29}$ lies inside that of Theorem 1, and no curvature argument for the gauge-fixed $\mathrm{SU}(2)$ measure passes $\beta_W = \frac{1}{3}$: when one active link e is -1 and all others are 1, the six plaquettes through e have variable -1, and along a geodesic through Q_e the Hessian of $-\frac{\beta_W}{2}\sum_p \mathrm{Re}\,\mathrm{Tr}\,Q_p$ is $-6\beta_W$ times the metric of the unit sphere S^3, whose Ricci curvature is 2, so the Bakry–Émery tensor is negative there once $\beta_W > \frac{1}{3}$. For $\mathrm{SU}(3)$ the Dobrushin route goes further.

Theorem 6 ($\mathrm{SU}(3)$)**.** *Let $G = \mathrm{SU}(3)$ and $0 \leq \beta_W \leq 0.515$, put $r = 2\beta_W$ and*

$$\varepsilon(r) = 1 - \frac{5e^{-r/5}}{9Z_*(r)}, \qquad \alpha_1(\beta_W) = \frac{2r\sqrt{\varepsilon(r)}}{\gamma(r)},$$

where $\gamma(r) = \frac{283}{128} - \frac{r}{2}$ and $Z_*(r) = \int_{\mathrm{SU}(3)} e^{r\,\mathrm{Re}\,\mathrm{Tr}\,U}\,dU$. The number $\alpha_1(\beta_W)$ is increasing, $\alpha_1(\frac{1}{2}) \leq 0.9491$ and $\alpha_1(0.515) \leq 0.9924$. The conclusions of Theorem 1 hold with decay at every rate below $-\log\alpha_1(\beta_W)$ and a transfer-matrix gap

$$m \geq -\log\alpha_1(\beta_W) > 0;$$

at $\beta_W = \frac{1}{2}$ this gives $m \geq 0.052$.

The proof of Theorem 6 bounds the Kantorovich modulus of the one-link measure $\nu_b(dU) \propto e^{\mathrm{Re}\,\mathrm{Tr}(Ub)}dU$ on $\mathrm{SU}(3)$ uniformly over staple sums b. It uses a lower bound $\gamma(r)$ for the spectral gap of ν_b through Bochner's formula with a cross term (Lemma 29), the Poincaré inequality twice, and a bound for the mean of $|\nabla\,\mathrm{Re}\,\mathrm{Tr}(Uh)|^2$ under ν_b through Jensen's inequality and Haar integrals of degree three (Lemma 31); Z_* is a series in the moments of $\mathrm{Tr}\,U$, which are counts of standard tableaux (Lemma 30). A sharper bound for the same modulus, evaluated by a computer on boxes of staple sums, would carry the range to $\beta_W = \frac{3}{4}$; Remark 36 describes it, and no result uses it.

The plaquette blocks of Theorem 3 work for $\mathrm{SU}(3)$ as well. They use the same modulus of the one-link measure, but only through the three links of a block, so they need $\Lambda(2\beta_W)$ small and not $2r\Lambda(r) < 1$: near the end of Theorem 6 they give a larger gap, and they go past it.

TABLE II. Explicit ranges of uniqueness, decay and a transfer-matrix gap for four-dimensional lattice Yang–Mills theory. The first three rows hold for the ungauged measure; the last three are the results of this paper.

Method	$\mathrm{SU}(2)$, β_W	$\mathrm{SU}(3)$, β_W
Curvature, Shen–Zhu–Zhu [107]	$< \frac{1}{12}$	$< \frac{3}{16}$
Curvature, sharp [20]	$< \frac{1}{8}$	$< \frac{9}{32}$
Dobrushin, sharp quarter [20]	$< \frac{2}{9}$	
Curvature, forest gauge	$< \frac{4}{29}$	$< \frac{9}{29}$
Dobrushin, forest gauge	$< \frac{1}{3}$	≤ 0.515
Plaquette blocks, forest gauge	≤ 0.348	≤ 0.52

Theorem 7 ($\mathrm{SU}(3)$ with plaquette blocks)**.** *Let $G = \mathrm{SU}(3)$ and $0 \leq \beta_W \leq 0.52$, let r, $\varepsilon(r)$, $\gamma(r)$ be as in Theorem 6, and let $\alpha_3(\beta_W)$ be the number of (15) with $\Lambda(r)$, τ, λ_* and M replaced by $\sqrt{\varepsilon(r)}/\gamma(r)$, $\varepsilon(r)/\gamma(r)$, 0 and $2\sqrt{3}\,\varepsilon(r)/\gamma(r)$. Then $\alpha_3(\beta_W) < 1$, with $\alpha_3(0.515) \leq 0.9742$ and $\alpha_3(0.52) \leq 0.9927$. The conclusions of Theorem 1 hold with decay at every rate below $-\frac{1}{2}\log\alpha_3(\beta_W)$ and a transfer-matrix gap*

$$m \geq -\tfrac{1}{2}\log\alpha_3(\beta_W) > 0;$$

at $\beta_W = 0.515$ this gives $m \geq 0.013$, against 0.0076 from Theorem 6, and at $\beta_W = 0.52$, outside the range of Theorem 6, $m \geq 0.0036$.

The constants of Theorem 7 are the bounds of Lemma 31; Remark 39 records what computed constants would give.

All seven theorems are statements at fixed lattice spacing. They do not touch the continuum limit, the crossover to Bałaban's regime, or a gap in physical units, which together are the Clay problem; Sec. IX says where each method stops and what remains of the four open problems of [20]. The obstruction proved in [21], that a lattice gap uniform in lattice units forces trivial continuum two-point functions, applies to all of them.

The continuum step. Sec. X goes past that obstruction in one respect, through three implications, each proved.

(I1) A Dobrushin–Shlosman condition on one cube of fixed physical size, uniform along the continuum tuning, gives a mass gap in physical units for every continuum limit (Theorems 43 and 45), and so does exponential decay

[9] The computation of Remark 695, from a certificate of T. Romanov, gives $h_F \leq 14.420001$; with it in place of $\frac{29}{2}$, every range of Table I would grow by the factor $29/28.840002 = 1.0055$. No result of the paper uses it.

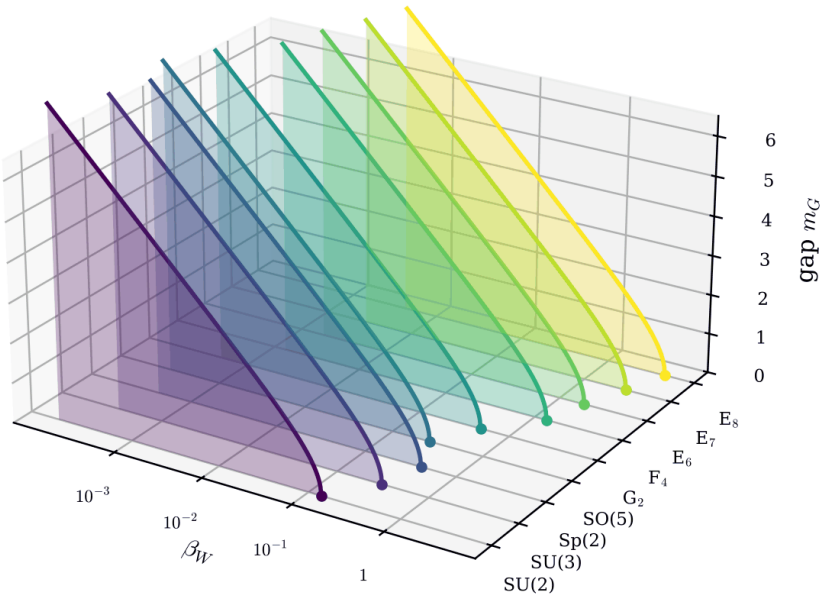


FIG. 1. The gap $m_G(\beta)$ of Theorem 5 as a function of $\beta_W = N^2\beta$ on a logarithmic scale, for nine groups in the representations of Table I. Each curve ends at the edge $\beta_W = 2N\kappa_G/29$ of the range, marked by a dot, where the gap tends to zero; as $\beta_W \to 0$ it grows like $\log(\kappa_G/(6N|\beta|))$. The curves differ only by the factor $N\kappa_G$, since m_G depends on $N|\beta|/\kappa_G$ alone.

with constants of the renormalised correlations of a spanning set of observables (Theorem 46); this holds also for observables smoothed by the spatial gradient flow (Proposition 47). A spectral gap of the dynamics that resamples one such cube at a time, uniform along the tuning and required only on local gauge-invariant functions, gives the same conclusion (Theorems 66 and 84).

(I2) The flowed observables, renormalised by one condition at a single physical separation, have reflection positive continuum limits along subsequences, and such a limit is nontrivial exactly when one ratio of lattice two-point functions stays away from zero (Theorem 48); that ratio is bounded above and below by ratios of correlations of one slice field at fixed physical separations (Proposition 51).

(I3) Tree bounds on the truncated correlations give all the higher Schwinger functions as reflection positive distributions, imply the ratio bound, and carry the conditional mass gap to every polynomial in the smeared fields (Theorem 57); for a smooth packet they give the ratio with an explicit rate and a reference vector of finite energy (Proposition 61).

What is open. The inputs of (I1)–(I3), namely the cube condition along the tuning, the slice ratio in four dimensions and the tree bounds, are not proved, and so the nontrivial existence of the continuum limit and the gap in physical units remain open. Figure 3 shows how the proved statements, the implications and the open inputs fit together.

Sec. XI measures the open inputs. The cube condition holds at strong coupling, fails at every fixed cube size at weak coupling, forces the gauge-fixed variables to be disordered on the scale of the cube, and cannot be reached from estimates on single lines of links; for $\mathrm{SU}(N)$ it needs cubes of side at least of order $(\beta_W/\log\beta_W)^{1/4}$ (Propositions 266 and 267). The spectral gap of the cube dynamics holds on every torus at strong coupling (Corollary 74) and fails on the torus of side $2n+4$ when n is small compared with $(\beta/\log\beta)^{1/4}$ (Theorem 79), but only through functions of nonzero centre charge, which its local form does not involve (Corollary 85). For a Gaussian field it holds at a fixed physical scale when the field has a mass and fails exactly, at every size, when it is massless (Proposition 186); it also fails for the quadratic approximation of the forest gauge itself (Proposition 187), and of every gauge fixed on an invariant forest with infinite trees (Proposition 188). The order-two cases of the tree bounds force the lattice gap below a fixed multiple of the lattice spacing, so no strong coupling expansion gives them (Proposition 60). The slice ratio holds in two dimensions, where it is the exact solution of lattice gauge theory (Remark 54), and fails for a field whose mass is fixed in lattice units; for the free field, conditioning an energy packet on averages over blocks leaves a fixed fraction of its ordinary variance at every lattice spacing, so a reduction to block averages has to be measured in the reflected norm (Proposition 55).

Across scales. The spectral gap of the cube dynamics passes from the fine lattice to the law of the block variables of every local gauge-covariant block-spin map (Theorem 94). In the opposite direction it climbs one scale under a split inequality that bounds the variance of the fine field given the block variables together with the coarse Dirichlet form (Theorem 100), so that along the tuning it follows from the condition at the

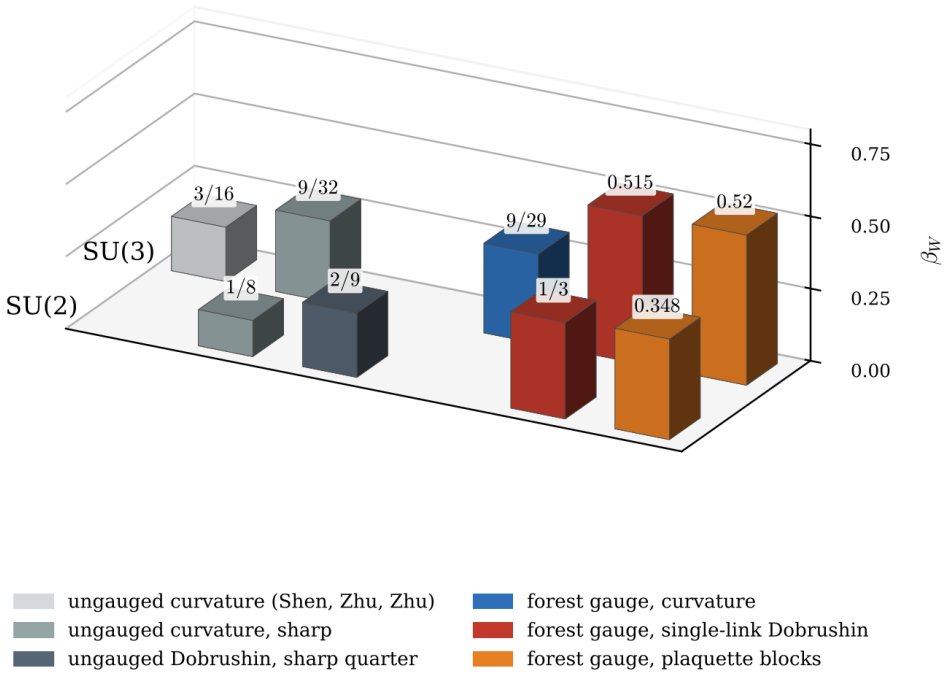


FIG. 2. Ranges of Table II. The height of each bar is the largest inverse coupling β_W covered by the method. Grey bars are methods for the ungauged measure; blue, red and orange bars use the forest gauge. For SU(2) the red bar ends at the ceiling of Theorem 2 and the orange bar, plaquette blocks, passes it (Theorem 3); for SU(3) the red bar is Theorem 6 and the orange bar, plaquette blocks, gives a larger gap there and passes it (Theorem 7).

coarsest scale and the split inequality at every scale with constants bounded in product (Corollary 102). The split inequality holds with the constants of the product measure for every Haar refinement (Theorem 110), for the Wilson measure at strong coupling on every torus with constants within factors $1+O(\beta)$ of those (Theorem 120), and on each torus for every measure with a continuous positive density, with constants that grow with the volume (Proposition 115). The condition at the coarsest scale follows from a bound on the sums of the single-link coefficients of the law of the block variables (Theorem 125), and the closeness of each single-link conditional law to the Haar measure does not suffice (Proposition 133). One step of the straight-path map contracts a weighted Dobrushin sum of these coefficients, so that a bound at one level of the tower suffices (Theorem 137, Corollary 138); but gauge covariance bounds the sum below by two minus four times the normalised spread of the single-link conditional laws, which excludes the Wilson measure at weak coupling and every law of the block variables with concentrated plaquette variables (Theorem 140, Corollaries 141 and 142), while at the coarsest scale the conditional laws are spread out along the tuning and this obstruction is absent (Proposition 143). The coefficients at the coarsest scale are at most $D^2/4$ times the covariances of the forces on distinct marks in the fibre law, uniformly in the coarse data, so that the condition there follows from a decorrelation of these forces; in a Gaussian model they are given by an effective quadratic form of the straight-path map, whose row sums are of order $(\log b)/b$, which gives the decorrelation in that model along the tuning (Proposition 148, Corollaries 149 and 174, Theorem 173). For SU(2) the single-link conditional laws of the block variables have densities within $48(1+6\beta_W)^2(4\beta_W/(4\beta_W+1))^{b-2}$ of the Haar density, with their first two derivatives, because convolution with a one-link density contracts every nontrivial representation; along every tuning with $\beta_k = O(k)$ the single-link curvature of the induced action therefore tends to zero, and the mixed derivatives are covariances of the scores of the interior path products under one posterior law, each tending to zero, which gives the condition at the coarsest scale on every torus of side at most $e^{2^k/(17k)}$ (Theorems 161 and 164, Corollary 166); on tori of every size it needs these covariances to be summable uniformly in the torus (Proposition 168). Along the tuning neither input is proved; the single-link bounds for the laws of the block variables are the effective-action problem of Bałaban.[10] Figure 16 shows the route.

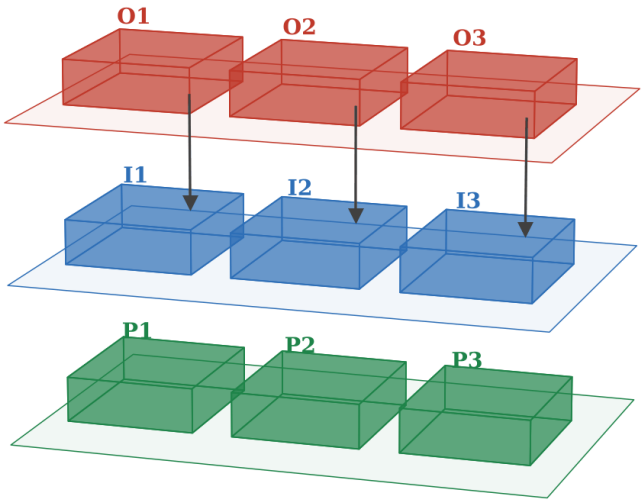
Block averages. Sec. XII poses the cube condition for block averages in a gauge whose fixed links are trees inside blocks of side two. Its quadratic approximation then holds at every coupling once the cube is large (Theorem 334), and integrating out one scale reduces it to a cube condition for the averages, which

[10] Refs. [6, 7, 12] control the effective actions of the small-field region in a box of fixed physical size; the inputs here need them on tori of every size and for all fields, and in the small-field region itself the Dobrushin sums exceed one (Corollary 142).

carry the coarse quadratic form at every scale up to a factor between 1 and 256 that does not depend on the scale, with an exponentially decaying kernel whose constants do not depend on the scale either, and the constrained cube condition holds at every scale with one cube side (Corollary 342, Proposition 335, Theorems 338, 340 and 344); the averages along the lines of the forest gauge, by contrast, leave one class of lines invisible and fail at every size (Proposition 327). For the group-valued links, the straight path between block roots gives a coarse link whose fibres satisfy a coercivity bound for every compact group and every field (Theorem 347), but this map does not compose across scales (Proposition 350), and for every compact connected group no continuous covariant block map treats the positions of a block alike (Proposition 351). A cut-off of Bałaban's averaging operation is continuous, covariant and coercive for every field and agrees with the block averages to first order; it breaks the symmetry only at large fields, as it must, and its constants are those of one step (Theorem 353). On fields with values in a maximal torus whose curvature stays small at every scale it composes, with the energy nonincreasing and the thin paths contracted (Theorem 355), and for every compact group it composes on fields whose curvatures at the successive scales have a small sum (Theorem 359). Under the $\mathrm{SU}(2)$ Wilson measure the plaquettes that are not small lie in clusters whose physical size tends to zero in probability, and the cut-off acts only next to them (Theorem 383, Corollaries 384 and 385); such plaquettes are nevertheless present along the one-loop tuning (Proposition 386). The cut-off average composes through these clusters, which spread by at most three spacings of the current scale per step, up to a coarse spacing of order $(\beta_W/\log\beta_W)^{1/4}a$, the order of the cube side that the cube condition needs (Theorem 388 and Corollary 389). Under every exterior, excess action in a box is rare, restricting one link to a fixed set leaves its Dobrushin response bounded, and one-point rarity of a bad set gives approximate uniqueness of the Gibbs measures but no decay (Lemmas 398 and 399, Proposition 400). The curvature of the block averages costs action independent of the scale (Lemma 401, Proposition 402); the chessboard estimate applies to coarse events and is lost after $O(\log\beta_W)$ scales in the normalisation alone (Proposition 403); a right translation of the Haar measures bounds the tails of the force along any fixed direction by the second variation of the action, at every scale and under every exterior (Theorem 404), which for the abelian theory makes coarse defects rare at the rate $\beta_W x^2$ at every scale (Corollary 405), while for $\mathrm{SU}(2)$ that second variation grows like 4^k at scale k, on the gauge orbit of the vacuum, and the same happens when the direction is transported along a forest, which keeps the Haar measures but leaves the residual gauge freedom of the blocks (Propositions 406, 407); this locates the estimate that the cube condition still needs. In a box whose boundary field has small plaquettes and either lies in a maximal torus or, for $\mathrm{SU}(2)$, has plaquette angles ρ_s with $l^2\rho_s$ below a fixed constant, the fraction of unit cubes with a large plaquette is small with probability exponentially close to one in the volume, uniformly over such boundary fields (Lemmas 390 and 391 and Proposition 392). The second class allows cube sides up to a constant times $(\beta_W/\log\beta_W)^{1/4}$ at the level of Corollary 389, which is the scale a_{k_*} up to a constant factor; for larger boxes with general boundary fields this conditional estimate is open (Remark 393). On the fibres of the cut-off average, at a background whose plaquettes are at most $\kappa \le 10^{-8}$, the gap $\frac{1}{52}$ of the averages[11] persists up to a factor $1+O(\kappa)$, and the Hessian of the action along the fibre is at least $\lambda(\frac{1}{26}-c\kappa)$, however far the links are from I (Theorem 361). On a chart of the fibre with convex domain the constrained law is log-concave for large λ, coarea factor included, and satisfies a Brascamp–Lieb inequality with constant $60/\lambda$ (Theorem 363); on a chart whose domain is a product in the coordinates of the links other than the path links, the covariances of local observables decay exponentially at a rate that depends only on G (Theorem 367). Off the chart the fibre has only points with a large plaquette, and under the $\mathrm{SU}(2)$ Wilson measure their mass is small on most fibres (Theorem 370). Along its private links the cut-off average is a submersion with a constant that depends on nothing (Lemma 371), and its law under Haar measure on them has a bounded density, so a Peierls argument inside the fibre bounds that mass on every fibre through a field whose plaquettes near the blocks are small (Proposition 374). On every fibre, whatever the boundary links, the same argument concentrates the law within any t of the least action of the fields whose averaging sites are near I (Proposition 377); where that action is not small the decay is open (Remark 381). Whether some block map is coercive for all fields and composes across the remaining scales is open.

Plan. Sec. II constructs the line field. Sec. III shows that a Dobrushin bound for the gauge-fixed specification gives the full conclusion of Theorem 1. Secs. IV, V, VI, VII and VIII prove Theorems 1–7, and Sec. IX measures what is left. Sec. X proves (I1)–(I3), Sec. XI states and measures their open inputs, and Sec. XII poses the cube condition for block averages, composes the cut-off average across scales at small fields and locates the large fields under the Wilson measure. Sec. XIII proves what a labelling of the blocks into good and bad gives for the cube condition, where the natural labels fail, the explicit coupling past which the bad islands of one scale are rarer than their entropy (Theorem 465), the ceiling below the scale of the cube that every chessboard bound for these islands meets at every coupling (Theorem 476), the fold across which the path-link law of a bad pair jumps (Proposition 418) and the bounded response of its site (Theorem 421), the Jacobian condition of its relative chart near the identity (Theorem 431), the bounded response of the bad side of a projected block average with no hypothesis (Theorem 443), the factorisation of two bad islands through one kernel (Theorem 483) and the exponential decay of their comparison in the width of the good layer between them (Theorem 485, Corollary 486), and what the single-scale and the multiscale parts of (D) still need. Sec. XIV sets out what is proved and what is open. The appendices contain the residue table of the line field, the comparison lemma and its version for blocks, the blocks of the local Hessian inequality,

[11] By the computation of Remark 331, from a certificate of T. Romanov, this gap is at least $\frac{537}{5000} = 0.1074$, within 0.09 per cent of the best constant on $\mathbb{Z}^4$, a floating-point value. No result of the paper uses it.



open inputs

O1 (D) or local BP along the tuning

O2 slice ratio in four dimensions

O3 tree bounds (T1)-(T3)

proved implications

I1 (D) or local BP: gap for every limit

I2 ratio iff nontrivial flowed limit

I3 tree bounds give the ratio

proved at fixed spacing

P1 gap for every simple group

P2 SU(2), SU(3) with blocks

P3 (D) and BP at strong coupling

arrows: an open input and the implication that consumes it

FIG. 3. The logical structure of the paper. Green: theorems at fixed lattice spacing (P1 the gap for every compact simple group, Theorem 5, P2 the plaquette blocks for SU(2) and SU(3), P3 the cube condition and the averaged condition BP at strong coupling, Corollary 74). Blue: the proved implications (I1)–(I3) of the continuum step, where (I1) takes either (D) or the local form of BP (Theorem 84). Red: their inputs, none of which is proved along the continuum tuning. Each arrow joins an open input to the implication that consumes it; no implication feeds its own input.

the details of the certified computation, the reconstruction of the transfer matrix with the equivalence of a gap and exponential clustering (Appendix F), the facts on spectral measures and on the character expansion that the continuum step uses (Appendices G and I), the Haar measure of SU(2) (Appendix J), and a complete proof of the sharp quarter for the single-link law of SU(2) (Appendices K–M), so that the paper does not depend on our earlier one for any step.

II. THE LINE FIELD

Write $e_0,\dots,e_3$ for the unit vectors of $\mathbb{Z}^4$ and (x,μ) for the link from x to $x+e_\mu$. A *direction field* is a map $\sigma:\mathbb{Z}^4\to\{0,1,2,3\}$; its links are $F=\{(x,\sigma(x)):x\in\mathbb{Z}^4\}$. At each site put $\pi_x(\mu)=\sigma(x+e_\mu)$. The plaquette at x in the plane $\{\mu,\nu\}$ has the links (x,μ), $(x+e_\mu,\nu)$, $(x+e_\nu,\mu)$, (x,ν), so it contains

$$[\sigma(x)\in\{\mu,\nu\}]+[\pi_x(\mu)=\nu]+[\pi_x(\nu)=\mu] \qquad (1)$$

links of F.

Lemma 8. *Every plaquette of $\mathbb{Z}^4$ contains exactly one link of F if and only if, at every site x, π_x fixes $\sigma(x)$ and permutes the other three directions cyclically.*

Proof. Fix x, let $s=\sigma(x)$ and let R be the set of the other three directions. For a plane $\{s,\nu\}$ the count (1) is $1+[\pi_x(s)=\nu]+[\pi_x(\nu)=s]$; it equals one for all $\nu\in R$ exactly when $\pi_x(s)=s$ and $\pi_x(\nu)\neq s$ for $\nu\in R$, that is, when π_x fixes s and maps R into R. For a plane $\{\mu,\nu\}\subset R$ the count is $[\pi_x(\mu)=\nu]+[\pi_x(\nu)=\mu]$. Draw an arrow $\mu\to\pi_x(\mu)$ for every $\mu\in R$ with $\pi_x(\mu)\in R\setminus\{\mu\}$. Each vertex of R has at most one outgoing arrow, and the condition says that each of the three pairs of R carries exactly one arrow. Three arrows on three vertices with out-degree at most one give out-degree exactly one everywhere, and a tournament on three vertices with all out-degrees one is a directed three-cycle. □

From now on σ is the field of period two given in Table VI of Appendix A, read from x modulo two. The table lists π_x at each of the sixteen residue classes; in each row π_x fixes $\sigma(x)$ and is a three-cycle on the rest, so every plaquette contains exactly one link of F by Lemma 8.[12]

[12] There are exactly eight period-two direction fields with this property, all

Lemma 9. (i) *F is a disjoint union of bi-infinite straight lines, one through every site: every site has exactly one outgoing and one incoming link of F, and both point in the direction $\sigma(x)$.*

(ii) *For every box Λ (a product of four finite intervals, with the links and plaquettes it contains), $F \cap E(\Lambda)$ is a disjoint union of segments, hence a forest.*

(iii) *A link $e = (x,\mu) \notin F$ lies in six plaquettes. Four of them contain a link of F that shares an end point with e, and these lie in the planes $\{\mu,\sigma(x)\}$ and $\{\mu,\pi_x(\mu)\}$. In the other two, which lie in the remaining plane $\{\mu,\rho\}$, the link of F is opposite to e.*

(iv) *Every link outside F shares a plaquette with exactly twelve other links outside F, two in each of its plaquettes.*

Proof. (i) Since π_x fixes $\sigma(x)$, $\sigma(x+e_{\sigma(x)}) = \sigma(x)$. Because σ has period two, $x - e_\mu \equiv x + e_\mu$, so also $\sigma(x-e_{\sigma(x)}) = \sigma(x)$, and the link $(x-e_{\sigma(x)},\sigma(x))$ is in F. Conversely, if $(x-e_\nu,\nu) \in F$, then $\sigma(x-e_\nu) = \nu$, and straightness at $x-e_\nu$ gives $\sigma(x) = \nu$. So the incoming link is unique and continues the outgoing one.

(ii) A line meets a box in a segment, and distinct lines are disjoint.

(iii) The plaquettes of e are, for each $\nu \neq \mu$, the one at x and the one at $x-e_\nu$ in the plane $\{\mu,\nu\}$. A link of F sharing an end point with e is the outgoing or the incoming link of F at x or at $x+e_\mu$. At x both point in the direction $s = \sigma(x) \neq \mu$, and they lie in the plaquettes at x and at $x-e_s$ in the plane $\{\mu,s\}$. At $x+e_\mu$ both point in the direction $t = \pi_x(\mu)$, which lies in $R \setminus \{\mu\}$ because π_x is a three-cycle on R, and they lie in the plaquettes at x and at $x-e_t$ in the plane $\{\mu,t\}$. These four plaquettes are distinct because $s \neq t$. Each plaquette contains exactly one link of F, so in the two plaquettes of the third plane $\{\mu,\rho\}$ the link of F is a link of direction μ or ρ not touching e; the links of direction ρ in such a plaquette touch e, so it is the opposite link.

(iv) Each of the six plaquettes contains e, one link of F and two other links, and two distinct links share at most one plaquette. □

III. GAUGE FIXING AND THE FREE-BOUNDARY STATE

Let $G \subset \mathrm{U}(N)$ be compact with Haar measure dU, and let Λ be a box. The Wilson measure with free boundary is

$$W_\Lambda(dQ) = \frac{1}{Z_\Lambda} \exp\Big(N\beta \sum_{p \in P(\Lambda)} \mathrm{Re}\,\mathrm{Tr}\, Q_p\Big) \prod_{e \in E(\Lambda)} dQ_e .$$

The measure is invariant under the gauge transformations $Q_{(x,y)} \mapsto g_x Q_{(x,y)} g_y^{-1}$ with $g \in G^\Lambda$. Call the links outside F *active* and write A for their set. For a function O of the links, O^F is O with every link of F set to the identity.

Lemma 10 (Gauge fixing)**.** *Let μ_Λ^F be the probability measure on $G^{A \cap E(\Lambda)}$ whose density with respect to Haar measure is proportional to $\exp(N\beta \sum_{p \in P(\Lambda)} \mathrm{Re}\,\mathrm{Tr}\, Q_p^F)$. For every gauge-invariant O on $G^{E(\Lambda)}$,*

$$W_\Lambda(O) = \mu_\Lambda^F(O^F).$$

Proof. Each segment of $F \cap E(\Lambda)$ is a path $x_0, x_1, \dots, x_k$ along one direction. Given the variables on F, put $g_{x_0} = 1$ and $g_{x_{i+1}} = g_{x_i} Q_{(x_i,x_{i+1})}$ on every segment. The transformed field has $Q_e^g = 1$ on $F \cap E(\Lambda)$, and for fixed values on F the map $Q_A \mapsto Q_A^g$ is a product of left and right translations, which preserves Haar measure. Gauge invariance of the action and of O and the change of variables give the claim. □

No leftover holonomy appears because $F \cap E(\Lambda)$ is a forest. The measures μ_Λ^F are finite-volume versions of the specification γ^F on G^A: for $e \in A$, γ_e^F resamples Q_e from the law with density proportional to $\exp(N\beta \sum_{p \ni e} \mathrm{Re}\,\mathrm{Tr}\, Q_p^F)$ given all other active links. No infinite-volume gauge transformation is performed; γ^F is a specification in its own right. Fix a metric δ on G of diameter D that is bi-invariant (for $\mathrm{SU}(2)$ the chordal metric on S^3, $D = 2$; for $\mathrm{SU}(N)$ the Hilbert–Schmidt distance). The Kantorovich coefficient of e' on e is

$$C_{ee'} = \sup \frac{W_1\big(\gamma_e^F(\cdot \mid Q), \gamma_e^F(\cdot \mid Q')\big)}{\delta(Q_{e'}, Q'_{e'})},$$

the supremum over pairs of configurations that differ only at e'. It vanishes unless e and e' share a plaquette.

Proposition 11. *Suppose that every row and every column of $(C_{ee'})$ sums to at most $\alpha < 1$. Then:*

(i) *γ^F has exactly one Gibbs measure μ_F;*

(ii) *for every gauge-invariant local O the limit $\omega(O) = \lim W_\Lambda(O)$ exists along every sequence of boxes exhausting $\mathbb{Z}^4$ and equals $\mu_F(O^F)$;*

(iii) *ω is invariant under translations and the hypercubic group;*

(iv) *if $\beta \geq 0$, ω is reflection positive for reflections in hyperplanes through sites and in hyperplanes bisecting links;*

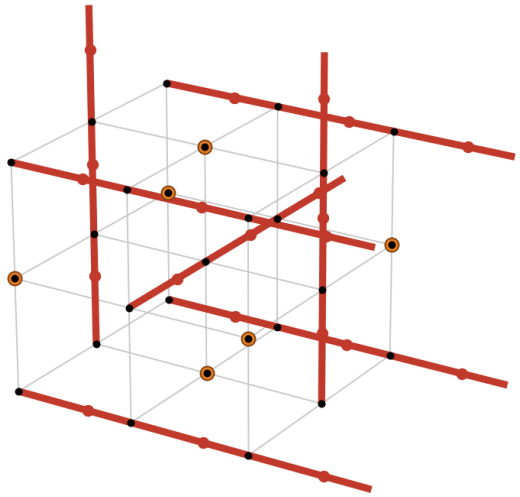
(v) *for gauge-invariant local O, O' with O Lipschitz in each link, $|\mathrm{Cov}_\omega(O,O')| \leq C_{O,O'} e^{-\lambda\, \mathrm{dist}(O,O')}$ for every $\lambda < -\log\alpha$;*

(vi) *if $\beta \geq 0$, the transfer matrix of ω has spectral gap at least $-\log\alpha$.*

---

equivalent to σ under the symmetries of the lattice. A field with one fixed link at every site and one in every plaquette exists only in four dimensions: a link lies in $2(d-1)$ plaquettes and a site carries $d(d-1)/2$ of them, and $2(d-1) = d(d-1)/2$ forces $d = 4$. The statements of this section are also checked in the Lean certificate of the archive.

(a) the line field in the slice $x_3 = 0$



(b) one plaquette in the gauge

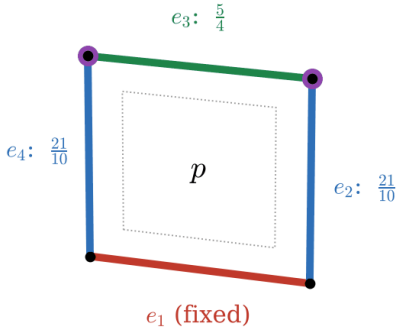


FIG. 4. (a) The line field in the slice $x_3 = 0$ of a box of side three. Thick red segments are the links of F in the directions $0, 1, 2$, with a dot marking the direction of travel; an orange ring marks a site whose line leaves the slice in direction 3. Every line is straight (Lemma 9(i)), and the segments that leave the box are its boundary stubs. (b) One plaquette with its fixed link e_1. The weights are those of the local inequality of Lemma 20 for the three free links: $\frac{21}{10}$ on the two links adjacent to e_1 and $\frac{5}{4}$ on the opposite one. The two purple corners are the ones that enter the corner form (7) once $Y_1 = 0$.

Proof. (i) Gibbs measures exist by compactness, and the comparison lemma of Appendix B with $S = \emptyset$ shows that any two coincide.

(ii) Extend μ_Λ^F by Haar measure on the active links outside Λ, and let S be the set of active links with a plaquette outside $P(\Lambda)$; this includes every active link outside $E(\Lambda)$. For an active link $e \notin S$ all six plaquettes of e lie in $P(\Lambda)$, so the conditional law of Q_e under the extended measure is γ_e^F. Let T be the support of O^F and $\lambda < -\log\alpha$. The comparison lemma with weights $e^{-\lambda\,\mathrm{dist}(\cdot,T)}$ gives

$$|\mu_\Lambda^F(O^F) - \mu_F(O^F)| \le L\,\frac{\theta D}{1-\theta}\sum_{z\in S} e^{-\lambda\,\mathrm{dist}(z,T)},$$

with $\theta = \alpha e^\lambda < 1$ and L a Lipschitz constant of O^F; the sum is at most $C_T R^3 e^{-\lambda R}$ when R is the distance from T to the complement of Λ. For O continuous but not Lipschitz, approximate uniformly. Lemma 10 gives $W_\Lambda(O) = \mu_\Lambda^F(O^F)$.

(iii) $W_\Lambda(\tau_v O) = W_{\Lambda - v}(O)$, and the boxes $\Lambda - v$ also exhaust $\mathbb{Z}^4$; by (ii) the limit does not depend on the sequence. The line field is a device for computing W_Λ and is never shifted. The same argument applies to the hypercubic group.

(iv) For $\beta \ge 0$ the measure W_Λ is reflection positive for every box symmetric under the reflection [97], and reflection positivity passes to the limit along such boxes, which is ω by (ii).

(v) $\mathrm{Cov}_\omega(O, O') = \mathrm{Cov}_{\mu_F}(O^F, O'^F)$, and setting the links of F to the identity keeps the Lipschitz property. The bound is Corollary 666.

(vi) Let ϑ be a time reflection and τ_t a time translation. For gauge-invariant local O that are Lipschitz in each link, (v) gives $|\omega(\vartheta(\bar O)\tau_t O) - |\omega(O)|^2| \le C_O e^{-\lambda t}$. On the support of O these functions are dense in the gauge-invariant part of $L^2(\omega)$: gauge-invariant functions are functions on the compact quotient of G^E by the gauge group, Lipschitz functions for the quotient metric are dense in L^2 of every finite measure on it, and they lift to gauge-invariant Lipschitz functions. By Proposition 669 the spectrum of the transfer matrix on the complement of the vacuum lies in $[0, e^{-\lambda}]$, for every $\lambda < -\log\alpha$. □

The proposition replaces the torus limits and the uniqueness of the ungauged Gibbs measure used in [20, Thm. F] by the free-boundary limit, which has all the properties the reconstruction of the transfer matrix needs.[13]

IV. SU(2) UP TO ONE THIRD

Proof of Theorem 1. Identify $\mathrm{SU}(2)$ with the unit quaternions and use the chordal metric. For an active link e and a plaquette $p \ni e$, the staple s_p is the product of the other three link variables in the order that makes $\mathrm{Re}(Q_e s_p) = \frac{1}{2}\mathrm{Tr}\,Q_p$, with links of F equal to one. Since $N\beta\,\mathrm{Re}\,\mathrm{Tr}\,Q_p = 4\beta\langle Q_e, \bar s_p\rangle$ in $\mathbb{R}^4$, the conditional law of Q_e is the von Mises–Fisher law $\nu_b(dx) \propto e^{\langle x,b\rangle}dx$ on S^3 with $b = 4\beta\sum_{p\ni e}\bar s_p$. Changing a partner e' by δ in the chordal metric changes one staple by an isometric image of that displacement, so b moves by $4|\beta|\delta$,

[13] On a torus the line field closes up into cycles, and the holonomies along them cannot be removed by a gauge transformation; the argument never uses a torus. The state ω is defined only on gauge-invariant observables, which is all that the Osterwalder–Schrader reconstruction [95–97] uses.

and the sharp quarter (Theorem 682), $W_1(\nu_b,\nu_{b'}) \le \frac14|b-b'|$, gives $C_{ee'} \le |\beta|$. By Lemma 9(iv) every row and column has twelve nonzero entries, so $\alpha \le 12|\beta| = 3|\beta_W|$. Proposition 11 applies for $|\beta_W| < \frac13$. □

Proof of Theorem 2. For the line field, every staple of an active link contains two active links, so the six staples can be chosen with sum zero while the staple through a given partner e' moves freely. At $b=0$ the derivative of ν_b in the direction h has Kantorovich norm $\frac14|h|$ by Lemma 681(i), so $C_{ee'} = |\beta|$ exactly, and every row sum is $12|\beta| = 3|\beta_W|$.

Now let F' be any forest and C' the matrix of the statement. Suppose a positive weight w satisfies $C'w \le \alpha w$. For a box Λ_n of side n let C_n be the principal submatrix on the active links of $E(\Lambda_n)$; then $C_n w \le \alpha w$ on these links, and the Collatz–Wielandt inequality for nonnegative matrices gives $\rho(C_n) \le \alpha$. Since C_n is symmetric, $\rho(C_n)$ is at least the mean row sum. The box has $4n^4 - O(n^3)$ links and n^4 sites, and a forest has fewer links than sites, so the number a of active links is at least $3n^4 - O(n^3)$. Let a_p be the number of active links of the plaquette p; the box has $P = 6n^4 - O(n^3)$ plaquettes and $\sum_p a_p = 6a - O(n^3)$. Two links share at most one plaquette, so the row sums add up to $|\beta|\sum_p a_p(a_p-1)$, and by convexity

$$\frac{|\beta|}{a}\sum_p a_p(a_p-1) \ge \frac{|\beta|P}{a}\,\bar a(\bar a-1) = 6|\beta|(\bar a-1) - o(1),$$

with $\bar a = \sum_p a_p/P = 6a/P - o(1) \ge 3 - o(1)$. Hence $\alpha \ge 12|\beta| - o(1)$ for every n, and $\alpha \ge 12|\beta| = 3|\beta_W|$. □

Without gauge fixing the same count gives eighteen partners and the ceiling $\beta_W = \frac29$ of [20, Prop. 13]; the temporal gauge, a family of lines in one direction, leaves fourteen. The line field is optimal among forests: twelve is the mean number of partners forced by the number of free links, and it is attained at every link.

V. SU(2) PAST ONE THIRD: PLAQUETTE BLOCKS

Throughout this section $G = \mathrm{SU}(2)$ is the group of unit quaternions with the chordal metric $|x-y|$, $\bar x$ is the conjugate, $\langle x,y\rangle = \mathrm{Re}(x\bar y)$ is the inner product of $\mathbb{R}^4$, and dx is the normalised Haar measure on S^3. Left and right multiplication by unit quaternions and conjugation are isometries, and $\mathrm{Re}(xyz) = \mathrm{Re}(yzx) = \langle y, \bar z\bar x\rangle$. Put

$$Z_0(c) = \int_{S^3} e^{\langle c,x\rangle}dx = \sum_{j\ge0}\frac{(|c|^2/4)^j}{j!\,(j+1)!}, \qquad c\in\mathbb{R}^4,$$

let $\nu_c(dx) \propto e^{\langle c,x\rangle}dx$ be the von Mises–Fisher law, and write $g = \log Z_0$, so that $\nabla g(c)$ is the mean and $\mathrm{Hess}\, g(c)$ the covariance of ν_c.

A. The blocks

Let $\mathscr{B}$ be the set of plaquettes at x in the plane $\{0,1\}$ with $x_0 \equiv x_1$ and in the plane $\{2,3\}$ with $x_2 \equiv x_3$, congruences modulo two (Table VII of Appendix C lists them modulo $2\mathbb{Z}^4$, and Fig. 5 draws them). A *block plaquette* is a plaquette of $\mathscr{B}$, and its *block* is the set of its three active links.

Lemma 12. *The set $\mathscr{B}$ is invariant under translations by $2\mathbb{Z}^4$ and:*

(i) *the blocks partition the active links;*

(ii) *every plaquette $p \notin \mathscr{B}$ meets three distinct blocks, one in each of its active links;*

(iii) *every active link lies in exactly five plaquettes outside $\mathscr{B}$, so it shares a plaquette outside $\mathscr{B}$ with exactly ten active links, none of them in its own block;*

(iv) *if an active link j and a link of a block B lie in a common plaquette outside $\mathscr{B}$, then every link of B has its midpoint within distance 2 of the midpoint of j.*

Proof. Invariance under $2\mathbb{Z}^4$ is clear from the congruences. We use one fact throughout: two distinct links lie in at most one common plaquette, since they span it.

Every link lies in exactly one plaquette of $\mathscr{B}$. Indeed, a link (y,μ) with $\mu\in\{0,1\}$ lies in no plaquette of the plane $\{2,3\}$, and in exactly two of the plane $\{0,1\}$, those at y and at $y-e_\nu$, where ν is the other element of $\{0,1\}$. The base points y and $y-e_\nu$ differ only in the coordinate ν, so y_0-y_1 has opposite parities at the two, and exactly one of them satisfies $x_0\equiv x_1$. The directions $2,3$ are treated in the same way. Every plaquette of $\mathbb{Z}^4$ contains exactly one link of F (Lemma 8), so a block plaquette contains three active links, and every active link lies in exactly one block. This is (i).

(ii) Let $p\notin\mathscr{B}$ have active links e_1,e_2,e_3, and let q_i be the block plaquette of e_i. Then $q_i \ne p$, so $q_i\cap p = \{e_i\}$ by the fact above. If $q_i = q_k$ with $i\ne k$, then e_i,e_k lie in the two distinct plaquettes p and q_i, which is impossible. So the three blocks are distinct, and each meets p in one link.

(iii) An active link e lies in six plaquettes (Lemma 9), and by (i) exactly one of them is in $\mathscr{B}$. Each of the other five contains two further active links. These ten links are distinct, because a link other than e shares at most one plaquette with e, and none lies in the block of e, because a link f of that block sharing the plaquette $p\notin\mathscr{B}$ with e would put e,f in the two distinct plaquettes p and the block plaquette of e.

(iv) Let $f\in B$ share the plaquette $p\notin\mathscr{B}$ with j. Two links of one unit square have midpoints at distance $2^{-1/2}$ or 1, so the midpoint of j is within distance 1 of that of f, and the midpoint of f is within distance 1 of that of every link of its block plaquette. The triangle inequality gives the bound 2. □

Fix a block plaquette p with active links e_1,e_2,e_3. Reading the plaquette product cyclically and deleting its fixed link, $\frac12\mathrm{Tr}\,Q_p = \mathrm{Re}(y_1y_2y_3)$, where y_i is Q_{e_i} or $Q_{e_i}^{-1}$. Every other plaquette containing e_i meets the block only in e_i by Lemma 12(ii) and contributes $4\beta\langle y_i,\bar s\rangle$ to the exponent, where s is the staple of y_i in it. Hence the conditional law of the block given all other active links is

$$\mu_b(dy) \propto \exp\Big(\sum_{i=1}^3\langle y_i,b_i\rangle + 4\beta\,\mathrm{Re}(y_1y_2y_3)\Big)\,dy_1\,dy_2\,dy_3, \quad (2)$$

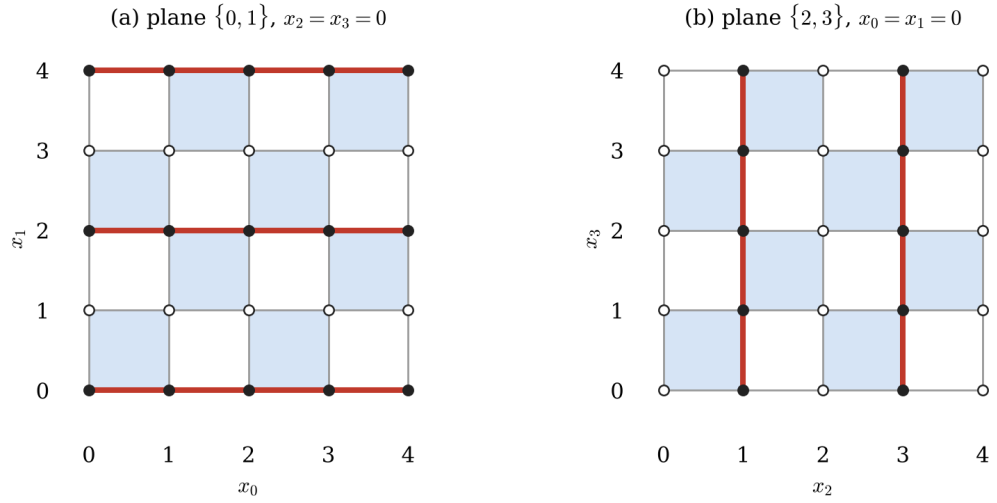


FIG. 5. Block plaquettes, shaded, in (a) a plane $\{0,1\}$ and (b) a plane $\{2,3\}$: the plaquettes at x with $x_0 \equiv x_1$, respectively $x_2 \equiv x_3$, modulo two. Red links belong to the line field F; a filled dot marks a site whose line runs in the plane, an open dot one whose line leaves it. The shaded squares form a checkerboard, so every link of the plane borders exactly one of them, and every square contains exactly one red link (Lemma 8); hence each block plaquette carries three active links and the blocks partition the active links (Lemma 12(i)).

with b_i the sum of $4\beta\bar{s}$ over the five staples, so that $|b_i| \le 20|\beta|$. Inversion is an isometry of S^3 and preserves Haar measure, so we may work with y instead of Q. On $(S^3)^3$ we use the metric $d(y,y') = \sum_i |y_i - y_i'|$.

B. The sensitivity of one block

Let λ_* and τ be a lower and an upper bound for the eigenvalues of $\operatorname{Hess} g(c)$ on the ball $|c| \le 24|\beta|$, and M an upper bound there for the norm of the third derivative $D^3 g(c)$ as a trilinear form. Put

$$w = 8\beta^2(\tau - \lambda_*), \qquad s = \tfrac{1}{4} + (e^w - 1)e^w\sqrt{\tau/3},$$
$$C = 4|\beta|sA + \tfrac{320}{3}|\beta|^3 M e^w, \qquad K = \frac{(1+|\beta|)\,s}{1-C}, \tag{3}$$

assuming $C < 1$, where A is an upper bound for $|\nabla g(c)|$ on the ball $|c| \le 20|\beta|$.

Lemma 13. *Let ν be a probability measure and $\lambda = e^R\nu/\nu(e^R)$ with $\operatorname{osc} R \le \omega$. For bounded f, h,*

$$\operatorname{Cov}_\lambda(f,h) \le \operatorname{Cov}_\nu(f,h) + (e^\omega - 1)e^\omega \operatorname{sd}_\nu(f)\operatorname{sd}_\nu(h).$$

Proof. We may assume $0 \le R \le \omega$. The density $\rho = e^R/\nu(e^R)$ lies in $[e^{-\omega}, e^\omega]$, so $\delta = \|\rho - 1\|_\infty \le e^\omega - 1$. With $\tilde{f} = f - \nu f$ and $\tilde{h} = h - \nu h$,

$$\operatorname{Cov}_\lambda(f,h) - \operatorname{Cov}_\nu(f,h) = \nu\big((\rho-1)\tilde{f}\tilde{h}\big) - \nu\big((\rho-1)\tilde{f}\big)\,\nu\big((\rho-1)\tilde{h}\big),$$

and the right side is at most $(\delta + \delta^2)\operatorname{sd}_\nu(f)\operatorname{sd}_\nu(h)$ by the Cauchy–Schwarz inequality. □

Lemma 14. *Let $\beta \ge 0$, $a, c_0 \in \mathbb{R}^4$ with $|a|, |c_0| \le 20\beta$, and*

$$\kappa(dx) \propto \exp\big(\langle x, a\rangle + g(c_0 + 4\beta\bar{x})\big)dx \quad \text{on } S^3.$$

For every 1-Lipschitz f and every $h \in \mathbb{R}^4$:

(i) $\operatorname{Cov}_\kappa(f, \langle x, h\rangle) \le s|h|$;

(ii) *if κ' is the law with c_0' in place of c_0 and $|c_0'| = |c_0| = \rho$, then $|\kappa(f) - \kappa'(f)| \le (8\beta s\varphi'(\rho^2) + \frac{16}{3}\beta^2 M e^w)|c_0 - c_0'|$.*

Proof. (i) Taylor's formula at c_0 gives $g(c_0 + 4\beta u) = g(c_0) + 4\beta\langle \nabla g(c_0), u\rangle + R(u)$ with $R(u) = 8\beta^2 \operatorname{Hess} g(\xi)(u,u)$ for some ξ on the segment, $|\xi| \le 24\beta$. For $u = \bar{x}$ of norm one, $R(\bar{x}) \in [8\beta^2\lambda_*, 8\beta^2\tau]$, so $\operatorname{osc} R \le w$. Since $\langle \nabla g(c_0), \bar{x}\rangle = \langle x, \overline{\nabla g(c_0)}\rangle$, κ is the tilt $e^R\nu/\nu(e^R)$ of $\nu = \nu_c$ with $c = a + 4\beta\overline{\nabla g(c_0)}$. The sharp quarter, Theorem 682, gives $\operatorname{Cov}_\nu(f, \langle x, h\rangle) = \frac{d}{dt}\nu_{c+th}(f) \le \frac{1}{4}|h|$. A function that is 1-Lipschitz for the chordal metric is 1-Lipschitz for the geodesic one, so $|\nabla f| \le 1$ almost everywhere and the Poincaré inequality of Lemma 679 gives $\operatorname{sd}_\nu(f) \le 1/\sqrt{3}$; and $\operatorname{sd}_\nu\langle x, h\rangle \le \sqrt{\tau}|h|$ because $|c| \le |a| + 4\beta \le 24\beta$. Lemma 13 gives (i). The same argument applies to every law $e^{R'}\nu_{c'}/\nu_{c'}(e^{R'})$ with $|c'| \le 24\beta$ and $\operatorname{osc} R' \le w$.

(ii) We interpolate the exponents rather than the field. For $t \in [0,1]$ let $\kappa_t \propto \exp\big(\langle x, a\rangle + (1-t)g(c_0 + 4\beta\bar{x}) + t g(c_0' + 4\beta\bar{x})\big)dx$, so that $\kappa_0 = \kappa$, $\kappa_1 = \kappa'$ and $\frac{d}{dt}\kappa_t(f) = \operatorname{Cov}_{\kappa_t}(f, D)$ with $D(x) = g(c_0' + 4\beta\bar{x}) - g(c_0 + 4\beta\bar{x})$. Write $g(c + 4\beta u) = g(c) + 4\beta\langle \nabla g(c), u\rangle + R_c(u)$ as in (i), for $c \in \{c_0, c_0'\}$. Then κ_t is the tilt of ν_{c_t}, $c_t = a + 4\beta\,\overline{(1-t)\nabla g(c_0) + t\nabla g(c_0')}$, by $R_t = (1-t)R_{c_0} + tR_{c_0'}$, with $\operatorname{osc} R_t \le w$ and $|c_t| \le 24\beta$ because $|\nabla g| \le 1$; so (i) holds for κ_t, and the density of κ_t with respect to ν_{c_t} is at most e^w. Up to a constant, $D(x) = \langle x, h\rangle + N(\bar{x})$ with $h = 4\beta\,\overline{\nabla g(c_0') - \nabla g(c_0)}$ and $N = R_{c_0'} - R_{c_0}$. Since $\nabla g(c) =$

$2\varphi'(|c|^2)c$ and $|c_0| = |c_0'| = \rho$, $|h| = 8\beta\varphi'(\rho^2)|c_0 - c_0'|$, and by (i) the linear part contributes at most $s|h|$. For the rest, $\nabla N(u) = 16\beta^2 \int_0^1 \big(\operatorname{Hess} g(c_0' + 4\beta\sigma u) - \operatorname{Hess} g(c_0 + 4\beta\sigma u)\big) u \, d\sigma$, both arguments in the convex ball $|c| \le 24\beta$, so N is L-Lipschitz on the unit ball with $L = 16\beta^2 M|c_0 - c_0'|$. With $\nu = \nu_{c_t}$ and the Poincaré inequality of Lemma 679 again, $\operatorname{Var}_{\kappa_t}(N) \le \kappa_t((N - \nu N)^2) \le e^w \operatorname{Var}_\nu(N) \le e^w L^2/3$, and in the same way $\operatorname{Var}_{\kappa_t}(f) \le e^w/3$. So $\operatorname{Cov}_{\kappa_t}(f, N) \le e^w L/3$. Integrating over t gives (ii). □

Lemma 15. *Let $C < 1$. If b and b' satisfy $|b_i|, |b_i'| \le 20|\beta|$ and differ only in one coordinate b_i, then*

$$W_1(\mu_b, \mu_{b'}) \le K\,|b_i - b_i'|$$

in the metric d.

Proof. The map $y_1 \mapsto -y_1$ turns μ_b for β into the law for $-\beta$ with b_1 replaced by $-b_1$ and is an isometry, so we may take $\beta \ge 0$; the cyclic symmetry of $\operatorname{Re}(y_1 y_2 y_3)$ lets us take $i = 1$. Let f be 1-Lipschitz for d and $\varepsilon = |b_1 - b_1'|$.

The middle link. Given y_1, y_3, the law of y_2 is ν_c with $c = b_2 + 4\beta\,\overline{y_3 y_1}$, which does not involve b_1. Changing y_1 by η moves c by $4\beta\eta$, and the sharp quarter moves ν_c by at most $\beta\eta$ in W_1. Hence $F(y_1, y_3) = \mathbb{E}[f \mid y_1, y_3]$ is $(1+\beta)$-Lipschitz in each variable, and $\mu_b(f) = \mu_b^{13}(F)$, where μ_b^{13} is the law of (y_1, y_3):

$$\mu_b^{13} \propto e^{\langle y_1, b_1\rangle + \langle y_3, b_3\rangle} Z_0\big(b_2 + 4\beta\,\overline{y_3 y_1}\big).$$

Since $|b_2 + 4\beta\overline{y_3 y_1}| = |b_2 y_3 + 4\beta \bar{y}_1| = |y_1 b_2 + 4\beta \bar{y}_3|$, its conditional laws are

$$\kappa_1(dy_1 \mid y_3) \propto e^{\langle y_1, b_1\rangle + g(b_2 y_3 + 4\beta\bar{y}_1)} dy_1,$$
$$\kappa_3(dy_3 \mid y_1) \propto e^{\langle y_3, b_3\rangle + g(y_1 b_2 + 4\beta\bar{y}_3)} dy_3,$$

both of the form of Lemma 14, with $|c_0| = |b_2|$.

Two sites. Integrating Lemma 14(i) along the segment from b_1 to b_1' gives $W_1(\kappa_1, \kappa_1') \le s\varepsilon$ for the same y_3. Moving y_3 to y_3' replaces $c_0 = b_2 y_3$ by $c_0' = b_2 y_3'$, of the same norm $\rho = |b_2| \le 20\beta$ and with $|c_0 - c_0'| = \rho|y_3 - y_3'|$. Since $\rho \cdot 2\varphi'(\rho^2) = |\nabla g(c_0)| \le A$, Lemma 14(ii) gives $W_1(\kappa_1(\cdot \mid y_3), \kappa_1(\cdot \mid y_3')) \le (4\beta s A + 20\beta \cdot \frac{16}{3}\beta^2 M e^w)|y_3 - y_3'| = C|y_3 - y_3'|$, and the same for κ_3, which does not depend on b_1. Let π be a coupling of μ_b^{13} and $\mu_{b'}^{13}$ minimising $d_1 + d_3$, $d_k = \mathbb{E}_\pi|y_k - y_k'|$. Replacing (y_1, y_1') by an optimal coupling of $\kappa_1(\cdot \mid y_3)$ and $\kappa_1'(\cdot \mid y_3')$ keeps a coupling, so minimality gives $d_1 \le s\varepsilon + C d_3$; in the same way $d_3 \le C d_1$. Hence $d_1 \le s\varepsilon/(1 - C^2)$ and

$$|\mu_b(f) - \mu_{b'}(f)| \le (1+\beta)(d_1 + d_3) \le \frac{(1+\beta)s}{1 - C}\,\varepsilon = K\varepsilon. \quad \square$$

C. The constants

Lemma 16. *Write $g(c) = \varphi(|c|^2)$ and $r = |c|$. For $c \ne 0$ the eigenvalues of $\operatorname{Hess} g(c)$ are $2\varphi'(r^2)$, three times, and $2\varphi'(r^2) + 4r^2\varphi''(r^2)$, and the norm of $D^3 g(c)$ is*

$$\max\Big(|P + \Gamma|, \tfrac{2}{3}|P|\Big), \qquad P = 12r\varphi''(r^2), \quad \Gamma = 8r^3\varphi'''(r^2).$$

On the ball $r \le 2.088$ every eigenvalue of $\operatorname{Hess} g$ lies between 0.15435 and $\frac{1}{4}$, and the norm of $D^3 g$ is at most 0.1071; on the ball $r \le 1.74$, $|\nabla g| \le 0.38944$.

Proof. Differentiating $\varphi(|c|^2)$ gives $D^2 g(v, v) = 2\varphi'|v|^2 + 4\varphi''\langle c, v\rangle^2$ and $D^3 g(v, v, v) = 12\varphi''\langle c, v\rangle|v|^2 + 8\varphi'''\langle c, v\rangle^3$. For $|v| = 1$ and $t = \langle c/r, v\rangle \in [-1, 1]$ the cubic form is $Pt + \Gamma t^3$, whose largest absolute value is attained at $t = \pm 1$ or at an interior critical point, where it equals $\frac{2}{3}|P||t| \le \frac{2}{3}|P|$; by Banach's theorem [13] the supremum of a symmetric trilinear form on the diagonal of the unit sphere is its norm.

The numbers follow from the Bessel ratio $A = I_2/I_1$ of Lemma 680. Since $Z_0(c) = 2I_1(r)/r$, the radial derivative of g is $A(r)$, so $2r\varphi'(r^2) = A(r)$ and $|\nabla g(c)| = A(r)$. Differentiating $2\varphi'(r^2) = A/r$ in r gives $4r\varphi''(r^2) = (rA' - A)/r^2$, so the eigenvalues are A/r, three times, and A', and $P = 3(rA' - A)/r^2$; differentiating once more gives $4\varphi'' + 8r^2\varphi''' = A''/r - 2(rA' - A)/r^3$, so $\Gamma = A'' - P$ and $P + \Gamma = A''$. Write $\kappa = r$, $u = 1 + \sqrt{\kappa^2 + 9}$ and $v = \frac{3}{2} + \sqrt{\kappa^2 + 25/4}$, so that $\kappa^2 = (u-4)(u+2) = (v-4)(v+1)$. By Lemma 680(ii) and (iv), $\kappa/v \le A \le \kappa/u$, and $A' = 1 - A^2 - 3A/\kappa$.

Eigenvalues. By Lemma 680(iii), $A' \le A/\kappa \le \frac{1}{4}$; at $c = 0$, $\operatorname{Hess} g = \frac{1}{4}I$. The function $1 - a^2 - 3a/\kappa$ decreases in $a \ge 0$, so $A' \ge 1 - \kappa^2/u^2 - 3/u = (8-u)/u^2$, and $(8-u)/u^2$ decreases in $u \in [4, 8]$, hence in κ. At $\kappa = 2.088$, $\kappa^2 + 9 = 13.359744 < 3.6551^2$, so $u < 4.6551$ and $(8-u)/u^2 > 3.3449/21.67 > 0.15435$.

Gradient. $A(r) \le r/(1 + \sqrt{r^2 + 9})$, which increases in r. At $r = 1.74$, $r^2 + 9 = 12.0276 > 3.468^2$, so $A \le 1.74/4.468 < 0.38944$.

Third derivative. First, $\frac{2}{3}|P| = 2|A - \kappa A'|/\kappa^2$ and $A - \kappa A' = 4A - \kappa + \kappa A^2$, which increases in A. At $A = \kappa/v$ it equals $\kappa(v-4)/v^2 \ge 0$, and at $A = \kappa/u$ it equals $2\kappa(u-4)/u^2$; since $u - 4 = \kappa^2/(\sqrt{\kappa^2+9} + 3) \le \kappa^2/6$ and $u \ge 4$, $0 \le \frac{2}{3}|P| \le 4(u-4)/(\kappa u^2) \le \kappa/24 \le 0.087$. Second, differentiating the Riccati equation gives $-A'' = H(A, \kappa)$ with

$$H(a, \kappa) = \Big(1 - a^2 - \frac{3a}{\kappa}\Big)\Big(2a + \frac{3}{\kappa}\Big) - \frac{3a}{\kappa^2},$$

and $\partial_a H = -(2a + 3/\kappa)^2 + 2(1 - a^2 - 3a/\kappa) - 3/\kappa^2 \le 2 - 12/\kappa^2 < 0$ for $a \ge 0$ and $\kappa^2 < 6$. Hence $H(\kappa/u, \kappa) \le -A'' \le H(\kappa/v, \kappa)$. Substituting $\kappa^2 = (u-4)(u+2)$ and $\kappa^2 = (v-4)(v+1)$,

$$H(\kappa/u, \kappa) = \frac{(u-4)(32 + 6u - 2u^2)}{\kappa u^3} \ge 0,$$
$$H(\kappa/v, \kappa) = \frac{(5v+8)(v-4)}{\kappa v^3} = \frac{\kappa(5v+8)}{(v+1)v^3},$$

the first because $4 \le u < 5.7$ for $\kappa \le 2.088$, the last by $v - 4 = \kappa^2/(v+1)$. Call the last expression $G(\kappa)$. As a function of v, $\log G = \frac{1}{2}\log(v-4) + \log(5v+8) - \frac{1}{2}\log(v+1) - 3\log v$, whose derivative is half of

$$\Big(\frac{1}{v-4} - \frac{6}{v}\Big) + \Big(\frac{10}{5v+8} - \frac{1}{v+1}\Big).$$

Both brackets decrease on $(4, 6)$. At $\kappa = 2.088$ one has $\kappa^2 + \frac{25}{4} = 10.609744$, which lies between 3.2572^2 and 3.2573^2, so

$v \in [4.7572, 4.7573]$ and the brackets are at least 0.059 and 0.140. So G increases on $(0, 2.088]$, and

$$0 \le -A'' \le G(2.088) \le \frac{2.088\,(5\cdot 4.7573+8)}{5.7572\cdot 4.7572^3} \le \frac{66.371}{619.81} < 0.1071.$$

The norm of $D^3 g$ is therefore at most $\max(0.1071, 0.087) = 0.1071$ on the ball $r \le 2.088$. □

D. The proof of Theorem 3

Let $\gamma^{\mathscr{B}}$ be the block specification: for a block B, $\gamma_B^{\mathscr{B}}$ resamples the three links of B from (2). It comes from the same interaction as γ^F, so the two specifications have the same Gibbs measures. Define the block coefficients

$$C_{Bj} = \sup \frac{W_1\big(\gamma_B^{\mathscr{B}}(\cdot \mid Q), \gamma_B^{\mathscr{B}}(\cdot \mid Q')\big)}{|Q_j - Q'_j|}$$

for an active link $j \notin B$, the supremum over pairs of configurations that differ only at j, with W_1 for the metric d.

Proposition 17. *Suppose that $\sum_B C_{Bj} \le \alpha < 1$ for every active link j. Then the conclusions (i)–(vi) of Proposition 11 hold with $-\frac{1}{2}\log\alpha$ in place of $-\log\alpha$.*

Proof. The proof of Proposition 11 goes through with Lemma 667 of Appendix B in place of Lemma 665, and with the set S of part (ii), and the support of H in Corollary 666, enlarged to the union of the blocks that meet it. The enlargement multiplies the size of these sets by at most three and moves them by at most one, which changes only the constants. □

Proof of Theorem 3. Let $|\beta| \le 0.348/4 = 0.087$ and take the constants of Lemma 16, whose balls $r \le 2.088$ and $r \le 1.74$ contain $|c| \le 24|\beta|$ and $|c| \le 20|\beta|$: $\lambda_* = 0.15435$, $\tau = \frac{1}{4}$, $M = 0.1071$ and $A = 0.38944$. With these fixed constants every quantity in (3) is increasing in $|\beta|$, and so is

$$\alpha(\beta_W) = 40|\beta|K. \tag{4}$$

Changing an active link j by η changes one staple of each of its ten partners outside its block by an isometric image of the displacement, so the field b_i of each such partner i moves by at most $4|\beta|\eta$, and Lemma 15, applied once for each of these ten fields, gives $\sum_B C_{Bj} \le 40|\beta|K$. At $|\beta| = 0.087$, $8\beta^2 = 0.060552$ and $\tau - \lambda_* = 0.09565$, so $w \le 0.0057918$. Since $e^w \le 1 + w + w^2$ for $0 \le w \le 1$ and $\sqrt{1/12} \le 0.288676$, $e^w \le 1.0058254$ and $s \le \frac{1}{4} + 0.0058594 \cdot 0.288676 \le 0.2516915$. Then

$$\begin{aligned} C &\le 0.348 \cdot 0.2516915 \cdot 0.38944 \\ &\quad + \tfrac{320}{3}\, 0.087^3 \cdot 0.1071 \cdot 1.0058254 \\ &\le 0.0341107 + 0.0075670 \le 0.0416777, \end{aligned}$$

$K \le 1.087 \cdot 0.2516915/0.9583223 \le 0.285490$ and $\alpha \le 3.48 \cdot 0.285490 \le 0.99351$. The same evaluation at $|\beta| = 0.345/4$ gives $w \le 0.0056924$, $s \le 0.2516621$, $C \le 0.041185$, $K \le 0.285110$ and $\alpha \le 0.98363$, and at $|\beta| = \frac{1}{12}$ it gives $w \le 0.0053139$, $s \le 0.2515504$, $C \le 0.0393011$, $K \le 0.2836611$ and $\alpha \le 0.94554$. Proposition 17 applies. At $\beta_W = \frac{1}{3}$, $-\frac{1}{2}\log\alpha \ge \frac{1}{2}(y + \frac{y^2}{2})$ with $y = 1 - 0.9456 = 0.0544$, which is at least 0.0279. □

Remark 18. Most of the range is kept. A second bound for the same sensitivity, which keeps the law of the middle link instead of linearising it, evaluated in floating point on a quadrature grid and maximised over the fields b by a random search followed by the Nelder–Mead method, gives 0.2783 at $\beta_W = 0.34$, and $40|\beta|$ times it crosses one near $\beta_W = 0.357$; this evaluation is an illustration and no result uses it. No bound of this form passes $\beta_W = 0.3667$: at $b = 0$ the function $f = \langle y_1, e\rangle + \mathrm{Re}(y_2 y_3 e)$ is 1-Lipschitz for d and has $\mathrm{Cov}_{\mu_b}(f, \langle y_1, e\rangle) = \frac{1}{4}(1 + I_2(\beta_W)/I_1(\beta_W))$ exactly (Proposition 458), which is at least 0.2718 at $\beta_W = 0.351$, so the sensitivity is at least this number and $40|\beta|$ times it exceeds one for $\beta_W > 0.3667$. Past this point the gain from integrating out one link is spent, and larger blocks are needed, such as the free links of a cube of the line field, whose sensitivity no longer reduces to two sites.

VI. CURVATURE IN THE GAUGE

In this section $G \subset \mathrm{U}(N)$ is a closed connected subgroup with Lie algebra $\mathfrak{g}$ and the inner product $\langle X, Y\rangle = -\mathrm{Tr}(XY)$; the bound of Theorem 4 is proved first for $\mathrm{U}(N)$. A tangent vector at a field Q is $v = (X_e Q_e)$, $|v|^2 = \sum_e |X_e|^2$. For the plaquette at x in the plane $\mu < \nu$ number its links $e_1 = (x, \mu)$, $e_2 = (x + e_\mu, \nu)$, $e_3 = (x + e_\nu, \mu)$, $e_4 = (x, \nu)$, transport their vectors to x,

$$Y_1 = X_{e_1},\quad Y_2 = Q_{e_1} X_{e_2} Q_{e_1}^*,\quad Y_3 = Q_{e_4} X_{e_3} Q_{e_4}^*,\quad Y_4 = X_{e_4}, \tag{5}$$

and write $W = Q_p$. The second derivative of $\mathrm{Re}\,\mathrm{Tr}\,Q_p$ along the geodesic $Q_e(t) = e^{tX_e} Q_e$ is

$$\begin{aligned} f_p = \sum_{i=1}^{4} \mathrm{Re}\,\mathrm{Tr}(Y_i^2 W) + 2\,\mathrm{Re}\,\mathrm{Tr}(Y_1 Y_2 W) + 2\,\mathrm{Re}\,\mathrm{Tr}(Y_3 Y_4 W) \\ - 2 \sum_{i\in\{1,2\},\, j\in\{3,4\}} \mathrm{Re}\,\mathrm{Tr}(Y_i W Y_j), \end{aligned} \tag{6}$$

so that $\mathrm{Hess}\, S_0(v, v) = \sum_p f_p$. To see this, write $Q_p = Q_{e_1} Q_{e_2} Q_{e_3}^* Q_{e_4}^*$. Since $Q_{e_1} e^{tX_{e_2}} = e^{tY_2} Q_{e_1}$ and $e^{-tX_{e_3}} Q_{e_4}^* = Q_{e_4}^* e^{-tY_3}$, the plaquette variable along the geodesic is $Q_p(t) = e^{tY_1} e^{tY_2} W e^{-tY_3} e^{-tY_4}$. Differentiating twice at $t = 0$ gives the four squares, the two products of neighbouring factors and the four products across W, and the squares and the product WY_3Y_4 are moved cyclically under the trace. The corner form of p is

$$c_p = 2\langle Y_1, Y_4\rangle - 2\langle Y_1, Y_2\rangle + 2\langle Y_2, W Y_3 W^*\rangle - 2\langle Y_3, Y_4\rangle. \tag{7}$$

It collects the inner products of the transported vectors of two links that meet at a right angle at a corner of p.

Lemma 19. *On $\mathbb{Z}^4$, or on a box with free boundary, for every field Q and every square-summable X,*

$$\sum_y \Big[|D_y X|^2 + \sum_\mu \big| X_{y,\mu} + Q^*_{y-e_\mu,\mu} X_{y-e_\mu,\mu} Q_{y-e_\mu,\mu} \big|^2 \Big] = 4|X|^2 + \sum_p c_p,$$

where $D_y X = \sum_\mu [X_{y,\mu} - Q^*_{y-e_\mu,\mu} X_{y-e_\mu,\mu} Q_{y-e_\mu,\mu}]$ is the covariant divergence and terms with a link outside the box are omitted.

Proof. At a site y write o_μ for the vector of the outgoing link in direction μ and i_μ for the transported vector of the incoming one, so that $D_y X = \sum_\mu (o_\mu - i_\mu)$. Every link is outgoing at one end and incoming at the other, so the squares $|o_\mu|^2 + |i_\mu|^2$, which appear once in $|D_y X|^2$ and once in $|o_\mu + i_\mu|^2$, add up to $4|X|^2$. The collinear cross term $-2\langle o_\mu, i_\mu\rangle$ of $|D_y X|^2$ cancels the term $+2\langle o_\mu, i_\mu\rangle$ of $|o_\mu + i_\mu|^2$. The remaining cross terms of $|D_y X|^2$ pair two perpendicular ends at y, and each such pair is a corner of exactly one plaquette, with the sign and the transport of (7) (this is the proof of [20, Lemma 1]). In a box two perpendicular ends at y span a plaquette of the box, because a box is a product of intervals. □

Lemma 20. *Let $k \in \{1,2,3,4\}$, let $k \pm 1$ (mod 4) be the positions adjacent to k and $k+2$ the opposite one. For every N, every $W \in \mathrm{U}(N)$ and all $Y_1, \dots, Y_4 \in \mathfrak{u}(N)$ with $Y_k = 0$,*

$$\pm f_p + \tfrac{9}{10} c_p \le \tfrac{21}{10} \big(|Y_{k-1}|^2 + |Y_{k+1}|^2 \big) + \tfrac{5}{4} |Y_{k+2}|^2.$$

Proof. Replacing W by $-W$ changes the sign of f_p and fixes c_p, so the sign $+$ suffices. Conjugation by a unitary matrix preserves both sides, so we may take $W = \mathrm{diag}(w_a)$. The form $\mathscr{F} = \mathrm{RHS} - f_p - \frac{9}{10} c_p$ then splits into a block for each diagonal position a, in the real variables $y_i = -i(Y_i)_{aa}$, and a block for each pair $a \neq b$, in the complex variables $z_i = (Y_i)_{ab}$, since $(Y_i)_{ba} = -\bar{z}_i$. Appendix D shows that every block is positive definite. □

Proof of the bound in Theorem 4. Every plaquette has exactly one link of F, and X vanishes on it. By Lemma 9(iii) each active link is adjacent to the fixed link in four of its plaquettes and opposite to it in two. Summing Lemma 20 over the plaquettes,

$$\sum_p \big(\pm f_p + \tfrac{9}{10} c_p \big) \le \big(4 \cdot \tfrac{21}{10} + 2 \cdot \tfrac{5}{4} \big) |X|^2 = \tfrac{109}{10} |X|^2.$$

By Lemma 19, $\frac{9}{10} \sum_p c_p = \frac{9}{10} \Sigma - \frac{36}{10} |X|^2$, where $\Sigma \ge 0$ is the left side of the lemma. Hence

$$\pm \mathrm{Hess}\, S_0(v,v) + \tfrac{9}{10} \Sigma \le \big(\tfrac{109}{10} + \tfrac{36}{10} \big) |X|^2 = \tfrac{29}{2} |X|^2.$$

In a box, links near the boundary lie in fewer plaquettes and receive less. □

The factor $\frac{9}{10}$ is chosen so that the corners are paid by the divergence identity at the price $\frac{36}{10}$ while the local blocks stay positive; Lemma 19 is the divergence identity of [20, Lemma 1] with the collinear term written as a square, so nothing is lost by Cauchy–Schwarz.

The two lemmas concern vectors in $\mathfrak{u}(N)$ and fields in $\mathrm{U}(N)$, so the bound holds without change on every closed subgroup $G \subset \mathrm{U}(N)$, the vectors X_e ranging over its Lie algebra $\mathfrak{g}$. It is also the Hessian of S_0 on G^E: the metric is bi-invariant, the geodesics of G through Q_e are the curves $e^{tX} Q_e$ with $X \in \mathfrak{g}$, which are geodesics of $\mathrm{U}(N)$, and the Hessian is read off from second derivatives along geodesics.

Lemma 21. *Let $G \subset \mathrm{U}(N)$ be closed and connected with simple Lie algebra $\mathfrak{g}$, let $\mathfrak{t} \subset \mathfrak{g}$ be a maximal abelian subalgebra, and write the eigenvalues of $\mathrm{ad}H$ on $\mathfrak{g} \otimes \mathbb{C}$ and of H on $\mathbb{C}^N$, for $H \in \mathfrak{t}$, as $i\alpha(H)$ and $i\mu(H)$, with α running over the roots and μ over the weights of $\mathbb{C}^N$ counted with multiplicity. Then $\mathrm{Ric}_G = \kappa_G \langle \cdot, \cdot \rangle$ with*

$$\kappa_G = \frac{\sum_\alpha \alpha(H)^2}{4 \sum_\mu \mu(H)^2} \qquad (0 \neq H \in \mathfrak{t}).$$

Proof. For a bi-invariant metric the curvature is $R(X,Y)Z = -\frac{1}{4}[[X,Y],Z]$ on left-invariant fields [90, Sect. 7], so for an orthonormal basis (E_i) of $\mathfrak{g}$,

$$\mathrm{Ric}(X,X) = \sum_i \langle R(E_i,X)X, E_i \rangle = \tfrac{1}{4} \sum_i |[X,E_i]|^2 = -\tfrac{1}{4} \mathrm{Tr}(\mathrm{ad}X)^2,$$

since $\mathrm{ad}X$ is skew. The Killing form $B(X,Y) = \mathrm{Tr}(\mathrm{ad}X\, \mathrm{ad}Y)$ and $\langle \cdot, \cdot \rangle$ are invariant under the adjoint action, and $\mathfrak{g}$ is simple, so $B = -4\kappa \langle \cdot, \cdot \rangle$ for a constant κ and $\mathrm{Ric} = \kappa \langle \cdot, \cdot \rangle$. On $H \in \mathfrak{t}$ the two forms are $B(H,H) = -\sum_\alpha \alpha(H)^2$ and $\langle H, H \rangle = -\mathrm{Tr}\, H^2 = \sum_\mu \mu(H)^2$. □

Table I evaluates the ratio of Lemma 21 for every compact simple Lie algebra and a faithful representation of least dimension, on the root systems of [72, Sect. 12.1]; the weights of the representations of dimensions 27 and 56 of E_6 and E_7 are the roots of E_8 with prescribed inner products with a subsystem A_2, respectively A_1, projected to its orthogonal complement, by the decompositions $248 = (78,1) + (27,3) + (\overline{27},\bar{3}) + (1,8)$ and $248 = (133,1) + (56,2) + (1,3)$ [108]. Taking for H the coroot $\theta^\vee$ of the highest root gives $\sum_\alpha \alpha(\theta^\vee)^2 = 4h^\vee$ and $\sum_\mu \mu(\theta^\vee)^2 = 2\ell_\rho$, so $\kappa_G = h^\vee / (2\ell_\rho)$, with the Dynkin index normalised to one on the defining representation of $\mathrm{SU}(N)$. The ratio depends on $\mathfrak{g}$ and ρ only, so it is the same for every compact group with Lie algebra $\mathfrak{g}$, simply connected or not.

Lemma 22. *Let $e \neq e'$ be two links of a plaquette p, and write*

$$\mathrm{Hess}\, \mathrm{Re}\, \mathrm{Tr}\, Q_p(v,v) = \sum_{e,e' \in p} H^p_{ee'}(X_e, X_{e'})$$

with bilinear forms satisfying $H^p_{ee'}(a,b) = H^p_{e'e}(b,a)$. Then $|H^p_{ee'}(a,b)| \le |a|\,|b|$ for all $a,b \in \mathfrak{u}(N)$ and every field.

Proof. By (6) the terms of f_p that mix two positions $i \neq j$ are $2\,\mathrm{Re}\,\mathrm{Tr}(Y_1 Y_2 W)$, $2\,\mathrm{Re}\,\mathrm{Tr}(Y_3 Y_4 W)$ and $-2\,\mathrm{Re}\,\mathrm{Tr}(Y_i W Y_j)$, so $H^p_{ee'}$ is $\pm\,\mathrm{Re}\,\mathrm{Tr}(abW)$ or $\pm\,\mathrm{Re}\,\mathrm{Tr}(aWb)$ after an isometric transport of a and b, and $|\mathrm{Tr}(abW)| \le \|a\|_{\mathrm{HS}} \|bW\|_{\mathrm{HS}} = |a|\,|b|$. □

Corollary 23. *Let $G \subset \mathrm{U}(N)$ be closed and connected with semisimple Lie algebra and $N|\beta| < \frac{2}{29}\kappa_G$. For every box the measure μ_Λ^F satisfies the Bakry–Émery condition with $K = \kappa_G - \frac{29}{2}N|\beta| > 0$, uniformly in the box, and hence a logarithmic Sobolev inequality with constant $2/K$ and a Poincaré inequality with constant $1/K$.*

Proof. The set $\{Q_F = 1\}$ is a product factor, so its Ricci curvature is at least κ_G on every link, and the Hessian of the restricted action is the restriction of $N\beta \operatorname{Hess} S_0$ to $X_F = 0$. Apply the bound and [5, Cor. 5.7.2]. □

A. Decay with an explicit rate

Let A' be a finite set of active links and P' a finite set of plaquettes whose active links lie in A'. An action $S = \sum_{p\in P'} c_p \operatorname{Re}\operatorname{Tr} Q_p^F$ with $0 \le c_p \le 1$ is *admissible* if $|\operatorname{Hess} S(v,v)| \le \frac{29}{2}|v|^2$ for all v tangent to $G^{A'}$. Two links lie in at most one common plaquette, so by Lemma 22 and Lemma 9(iv) the mixed blocks of its Hessian satisfy $|\operatorname{Hess} S_{ee'}(a,b)| \le |a|\,|b|$, and each link has at most twelve partners. The action $S_\Lambda = \sum_{p\in P(\Lambda)} \operatorname{Re}\operatorname{Tr} Q_p^F$ of a box is admissible by the bound proved above, and so is every convex combination of two admissible actions. Let μ be the probability measure on $G^{A'}$ with density proportional to $e^{N\beta S}$ with respect to Haar measure, let $L = \Delta + N\beta\langle\nabla S, \nabla\cdot\rangle$ be its Langevin generator, which is self-adjoint in $L^2(\mu)$, and $P_t = e^{tL}$. Write ∇_e for the gradient in the factor of the link e, and for weights $w : A' \to (0,\infty)$ put $\Gamma^w(f) = \sum_e w_e |\nabla_e f|^2$; Γ is the case $w \equiv 1$.

Lemma 24. *Let $\lambda \ge 0$ and $w_e = e^{\lambda d_e}$, where $|d_e - d_{e'}| \le 1$ whenever e and e' share a plaquette, and put*

$$K_\lambda = \kappa_G - N|\beta|\Big(\tfrac{29}{2} + 12\big(\cosh\tfrac{\lambda}{2} - 1\big)\Big).$$

Then $\Gamma^w(P_t f) \le e^{-2K_\lambda t} P_t \Gamma^w(f)$ for every smooth f and $t \ge 0$.

Proof. Put $V = -N\beta S$. On the product manifold $M = G^{A'}$ the projection to the tangent space of one factor is parallel, the rough Laplacian commutes with it, and the Ricci tensor is block diagonal. Hence the Bochner–Weitzenböck formula $\nabla\Delta g = \bar\Delta \nabla g + \mathrm{Ric}(\nabla g)$, applied factor by factor, together with the derivative of the drift, gives for smooth g

$$\begin{aligned}\tfrac{1}{2}L\Gamma^w(g) - \Gamma^w(g, Lg) &= \sum_e w_e\big(|\nabla\nabla_e g|^2 + \mathrm{Ric}_G(\nabla_e g, \nabla_e g)\big) \\ &\quad + \sum_{e,e'} w_e \operatorname{Hess} V_{ee'}(\nabla_e g, \nabla_{e'} g),\end{aligned}$$

where $\Gamma^w(g,h)$ is the polarised form.[14] Put $u_e = \nabla_e g$ and $v_e = \sqrt{w_e}\,u_e$. By the symmetry of $\operatorname{Hess} V$,

$$\begin{aligned}\sum_{e,e'} w_e \operatorname{Hess} V_{ee'}(u_e, u_{e'}) &= \operatorname{Hess} V(v,v) \\ &\quad + \sum_{e\ne e'} \Big(\cosh\tfrac{\lambda(d_e - d_{e'})}{2} - 1\Big) \operatorname{Hess} V_{ee'}(v_e, v_{e'}),\end{aligned}$$

because $\frac{w_e + w_{e'}}{2\sqrt{w_e w_{e'}}} = \cosh\frac{\lambda(d_e - d_{e'})}{2}$. The first term is at least $-\frac{29}{2}N|\beta|\,|v|^2$. In the second only partners contribute, each with a coefficient at most $\cosh\frac{\lambda}{2} - 1$ and a block of norm at most $N|\beta|$, and $2|v_e|\,|v_{e'}| \le |v_e|^2 + |v_{e'}|^2$ with twelve partners per link bounds it by $12(\cosh\frac{\lambda}{2} - 1)N|\beta|\,|v|^2$. Since $\mathrm{Ric}_G \ge \kappa_G$ and $|v|^2 = \Gamma^w(g)$,

$$\tfrac{1}{2}L\Gamma^w(g) - \Gamma^w(g, Lg) \ge K_\lambda \Gamma^w(g).$$

For fixed t let $\varphi(s) = P_s \Gamma^w(P_{t-s} f)$, $0 \le s \le t$. Then $\varphi'(s) = P_s\big(L\Gamma^w(g_s) - 2\Gamma^w(g_s, Lg_s)\big) \ge 2K_\lambda \varphi(s)$ with $g_s = P_{t-s} f$, and $\varphi(0) \le e^{-2K_\lambda t}\varphi(t)$ is the claim. □

Lemma 25. *Suppose $K_\lambda > 0$. Let f and g be smooth functions on $G^{A'}$ that depend only on the links of A_f and of B_g, and let D be the least number of steps from A_f to B_g in the graph on A' whose edges join links that share a plaquette. Then*

$$|\operatorname{Cov}_\mu(f,g)| \le \frac{e^{-\lambda D/2}}{K_\lambda} \|\nabla f\|_\infty \sum_{e\in B_g} \|\nabla_e g\|_\infty,$$

where $\|\nabla f\|_\infty^2 = \sup \sum_e |\nabla_e f|^2$.

Proof. Since $K_0 \ge K_\lambda > 0$, Lemma 24 with $\lambda = 0$ gives $\Gamma(P_t f) \le e^{-2K_0 t}\|\nabla f\|_\infty^2$, so $P_t f$ converges uniformly to a constant, which is $\mu(f)$ by invariance, M being connected. Since $\frac{d}{dt}\mu(gP_t f) = \mu(gLP_t f) = -\mu(\Gamma(g, P_t f))$,

$$\operatorname{Cov}_\mu(f,g) = \int_0^\infty \mu\big(\Gamma(P_t f, g)\big)\,dt.$$

Let d_e be the smaller of D and the graph distance from e to A_f, so that $|d_e - d_{e'}| \le 1$ for partners, $w_e = 1$ on A_f, $w_e = e^{\lambda D}$ on B_g, and $\Gamma^w(f) \le \|\nabla f\|_\infty^2$. For $e \in B_g$, Lemma 24 gives $|\nabla_e P_t f|^2 \le w_e^{-1}\Gamma^w(P_t f) \le e^{-\lambda D} e^{-2K_\lambda t}\|\nabla f\|_\infty^2$. Hence $|\Gamma(P_t f, g)| \le e^{-\lambda D/2} e^{-K_\lambda t}\|\nabla f\|_\infty \sum_{e\in B_g}\|\nabla_e g\|_\infty$, and integration in t gives the bound. □

Proof of Theorem 5. Let $\lambda^* > 0$ solve $K_{\lambda^*} = 0$, so that $\frac{\lambda^*}{2} = m_G(\beta)$, and let $0 < \lambda < \lambda^*$; then $K_\lambda > 0$.

Decay in boxes. Lemma 25 with $S = S_\Lambda$ gives

$$|\operatorname{Cov}_{\mu_\Lambda^F}(f,g)| \le C_{f,g} e^{-\lambda D/2},$$

uniformly in the box.

Limit. For boxes $\Lambda \subset \Lambda'$ let $S_s = (1-s)S_\Lambda + sS_{\Lambda'}$ on the active links of Λ', $s \in [0,1]$, and μ_s the corresponding measure; S_s is admissible, μ_0 is μ_Λ^F times Haar measure on the active links outside Λ, and $\mu_1 = \mu_{\Lambda'}^F$. For smooth f of the active links of Λ,

$$\frac{d}{ds}\mu_s(f) = N\beta \sum_{p\in P(\Lambda')\setminus P(\Lambda)} \operatorname{Cov}_{\mu_s}\big(f, \operatorname{Re}\operatorname{Tr} Q_p^F\big),$$

[14] For $w \equiv 1$ this is Bochner's identity $\Gamma_2 = |\nabla^2 g|^2 + (\mathrm{Ric} + \operatorname{Hess} V)(\nabla g, \nabla g)$; the weights enter only through the factor index, which the connection, the Laplacian and the Ricci tensor preserve, so only the off-diagonal blocks of $\operatorname{Hess} V$ see them.

and $|\nabla_e \operatorname{Re}\operatorname{Tr} Q_p| \le \|Q_p\|_{\mathrm{HS}} = \sqrt{N}$. A step in the graph of Lemma 25 moves a link by at most one lattice unit, so a plaquette at graph distance k from the support T of f lies within distance $k+1$ of T, there are at most $C_T(k+1)^4$ of them, and every plaquette outside $P(\Lambda)$ is at graph distance at least $R-1$, R the distance from T to the complement of Λ. Lemma 25 gives

$$|\mu^F_{\Lambda'}(f) - \mu^F_{\Lambda}(f)| \le C_f R^4 e^{-\lambda R/2}.$$

The limit therefore exists along every increasing sequence of boxes, and two sequences are compared through their intersections, which are boxes. Continuous gauge-invariant O are uniform limits of smooth gauge-invariant ones, for instance of $e^{\tau\Delta}O$, since the heat semigroup of G^E commutes with left and right translations; with Lemma 10 the limit $\omega(O)$ of $W_\Lambda(O)$ exists for all of them.

Properties. Invariance and, for $\beta \ge 0$, reflection positivity follow as in Proposition 11(iii)–(iv). For Lipschitz O the functions $e^{\tau\Delta}O$ are smooth with the same Lipschitz constant, since $\mathrm{Ric}_G \ge 0$ [5, Sect. 3.3], so the bound of the first step passes to ω and to Lipschitz observables. Two links of one plaquette have time coordinates that differ by at most one, so observables at time separation t are at graph distance at least $t-c$, and truncated correlations decay at every rate below $m_G(\beta)$. The gap follows from Proposition 669 as in Proposition 11(vi). □

Proof of the rest of Theorem 4. By Lemma 21 and Table I, $\kappa_{\mathrm{SU}(N)} = \frac{N}{2}$ and $\kappa_{\mathrm{SO}(N)} = \frac{N-2}{4}$; for SO(4), whose Lie algebra is $\mathfrak{so}(3)\oplus\mathfrak{so}(3)$ acting on $\mathbb{R}^4\otimes\mathbb{C} = \mathbb{C}^2\otimes\mathbb{C}^2$, each ideal gives $\frac12$ by the same computation. The conditions of Theorem 4 are those of Theorem 5. □

Remark 26. Let h_F be the supremum of $\pm\operatorname{Hess} S_0$ on the gauge-fixed directions, so that uniform curvature reaches $|\beta| < 1/(2h_F)$ and no further. At the identity field $-\operatorname{Hess} S_0$ restricted to $X_F = 0$ has the eigenvalue $9+\sqrt{17} = 13.123$ on the torus of side two, so $9+\sqrt{17} \le h_F \le \frac{29}{2}$. A gradient search over SU(2) fields of periods two, four and six finds nothing above 13.97401, the value at an abelian field; certificates made of plaquette forms and vertex forms on the period-two cell cannot go below 14.384. One of us (T. Romanov) [102] constructed such a certificate, with sixteen positive definite vertex forms of size eight and nonnegative plaquette weights, rational with denominator 10^6. Its positivity is checked by computer, and with it the argument of Appendix N gives $h_F \le 14.420001$ (Remark 695). Like $\frac{29}{2}$ this would be a statement about U(N), so it could replace $\frac{29}{2}$ in Theorems 4 and 5, and the curvature method would then reach $|\beta| < 1/28.840002$, that is $\beta_W < 0.1387$ for SU(2) and $\beta_W < 0.3121$ for SU(3). This is a computation, used by no result of the paper, and concerns the lattice at fixed spacing, not the continuum limit. The theorems of this paper are stated with $\frac{29}{2}$, which is proved by hand. With this gauge the curvature method reaches $|\beta| < \frac{1}{29}$ and stops before $\frac{1}{26.25}$; the computation moves the first value to $1/28.840002$.

VII. SU(3) UP TO 0.515

In this section $G = \mathrm{SU}(3)$, $\beta \ge 0$ and $\beta_W = 9\beta$. The distance on SU(3) is $d(U,V) = \|U-V\|_{\mathrm{HS}}$, which is at most the geodesic distance for $\langle X,Y\rangle = -\operatorname{Tr}(XY)$; $K = \operatorname{conv}\mathrm{SU}(3)$; $A(M) = \frac12(M-M^*)$ and P removes the trace. With this inner product $\mathrm{Ric} = \frac32$, the Casimir of the fundamental representation is $\lambda_F = \frac83$ and that of the adjoint representation is 6.

A. Reduction to one link

The conditional law of an active link is

$$\nu_b(dU) \propto e^{\operatorname{Re}\operatorname{Tr}(Ub)}dU, \qquad b = N\beta\sum_{p\ni e} S_p,$$

where S_p is the staple of e in p, a product of SU(3) matrices and identities, or its adjoint. Six staples give $b \in rK$ with $r = 6N\beta = 2\beta_W$. For a matrix h put $\varphi_h(U) = \operatorname{Re}\operatorname{Tr}(Uh)$ and

$$L_h(b) = \sup\{\mathrm{Cov}_{\nu_b}(g,\varphi_h) : g \text{ 1-Lipschitz for } d\},$$

$$\Lambda(r) = \sup_{b\in rK}\ \sup_{\|h\|_{\mathrm{HS}}=1} L_h(b).$$

Changing a partner e' changes one staple by a matrix of norm $d(Q_{e'}, Q'_{e'})$. Along the segment b_t from b to b', which stays in rK, $\frac{d}{dt}\nu_{b_t}(g) = \mathrm{Cov}_{\nu_{b_t}}(g, \operatorname{Re}\operatorname{Tr}(U(b'-b)))$, so $W_1(\nu_b,\nu_{b'}) \le N\beta\Lambda(r)\,d(Q_{e'},Q'_{e'})$. With twelve partners per link,

$$\alpha \le 12N\beta\Lambda(r) = 2r\Lambda(r) = 4\beta_W\Lambda(2\beta_W). \tag{8}$$

Since K is convex and contains 0, the mean of $I, \omega I, \omega^2 I$ with $\omega^3 = 1$, $r'K \subset rK$ for $r' \le r$ and Λ is nondecreasing.

B. Gradients and one-forms

Lemma 27. *In the left trivialisation, $\nabla\varphi_c(U) = -PA(cU)$; if $V = \varphi_b$ then $\operatorname{Hess} V(X,X) = \operatorname{Re}\operatorname{Tr}(UX^2b)$; and $\Delta\varphi_c = -\lambda_F\varphi_c$.*

Proof. $\frac{d}{dt}\operatorname{Re}\operatorname{Tr}(Ue^{tX}c)|_{t=0} = \operatorname{Re}\operatorname{Tr}(cUX) = -\langle PA(cU), X\rangle$ for $X \in \mathfrak{su}(3)$. On a group with a bi-invariant metric $\nabla_X X = 0$ for left-invariant X, so the Hessian is the second derivative along e^{tX}. The function φ_c is a matrix coefficient of the fundamental representation, on which the Casimir acts by λ_F. □

Lemma 28. *The bottom of the spectrum of the rough Laplacian $\nabla^*\nabla$ on L^2 one-forms of* SU(3) *is $\mu_0 = \frac76$.*

Proof. Trivialise one-forms on the left, so that a one-form is a function $f : \mathrm{SU}(3) \to \mathfrak{su}(3)$. For an orthonormal basis X_a acting as left-invariant vector fields, $\nabla_{X_a}Y = \frac12[X_a, Y]$ and $\nabla_{X_a}X_a = 0$, so $\nabla^*\nabla = -\sum_a(\rho(X_a) + \frac12\operatorname{ad}X_a)^2$. On the isotypic part $V_\pi\otimes V_\pi^*\otimes\mathfrak{su}(3)$ of $L^2\otimes\mathfrak{su}(3)$ put $A_a = \rho_\pi(X_a)$, $B_a = \operatorname{ad}X_a$ and $T_a = A_a + B_a$, the action on the tensor product. Since A_a

and B_a commute, $A_a+\frac12 B_a=\frac12(A_a+T_a)$ and $A_aT_a+T_aA_a=A_a^2+T_a^2-B_a^2$, so on a component $\sigma\subset\pi\otimes\mathrm{ad}$

$$\nabla^*\nabla=\tfrac12(C_\pi+C_\sigma)-\tfrac14 C_{\mathrm{ad}}=\tfrac12(C_\pi+C_\sigma)-\tfrac32,$$

with $C(p,q)=\frac23(p^2+q^2+pq+3p+3q)$ for the representation of highest weight (p,q) [52]. The components of $(p,q)\otimes(1,1)$ are $(p+1,q+1)$, $(p+2,q-1)$, $(p-1,q+2)$, $(p+1,q-2)$, $(p-2,q+1)$, $(p-1,q-1)$ and (p,q), the last with multiplicity two, one or zero as both, one or none of p,q are positive, and only those with nonnegative labels occur. For $\pi=(0,0)$ the only component is $(1,1)$, with value $\frac32$. For $\pi=(1,0)$ the components are $(2,1)$, $(0,2)$, $(1,0)$, with values $\frac{31}{6}$, $\frac{19}{6}$, $\frac{7}{6}$, and $(0,1)$ is conjugate. For $p+q\ge2$ the smallest values occur at $(p-1,q-1)$, (p,q), $(p-2,q+1)$ or $(p+1,q-2)$, and each is at least $\frac13(2(p^2+q^2+pq)+3(p+q)-3)-\frac32\ge\frac32$. □

C. The gap of the one-link measure

Lemma 29. *For $b\in rK$ the spectral gap of $L=\Delta+\langle\nabla V,\nabla\cdot\rangle$, $V=\varphi_b$, on $L^2(\nu_b)$ is at least*

$$\gamma(r)=\tfrac{39}{64}\mu_0+\tfrac32-\tfrac{r}{2}=\tfrac{283}{128}-\tfrac{r}{2}.$$

Proof. Let $-Lf=\lambda f$ with $\lambda>0$, $\omega=df$ and $\eta=e^{V/2}\omega$; norms are in $L^2(dU)$. Bochner's formula integrated against ν_b gives $\lambda\nu_b|\omega|^2=\nu_b|\nabla\omega|^2+\nu_b((\mathrm{Ric}-\mathrm{Hess}\,V)(\omega,\omega))$, and since $e^{V/2}\nabla\omega=\nabla\eta-\frac12 dV\otimes\eta$,

$$\lambda\|\eta\|^2=\|\nabla\eta-\tfrac12 dV\otimes\eta\|^2+\int(\mathrm{Ric}-\mathrm{Hess}\,V)(\eta,\eta).\quad(9)$$

Put $\mathscr{A}=\nabla\eta$ and $\mathscr{B}=\frac12 dV\otimes\eta$. Then

$$\langle\mathscr{A},\mathscr{B}\rangle=\tfrac14\int\langle\nabla V,\nabla|\eta|^2\rangle=\tfrac{\lambda_F}{4}\int V|\eta|^2,$$

because $-\Delta V=\lambda_F V$. Writing $\|\mathscr{A}-\mathscr{B}\|^2=\|\frac58\mathscr{A}-\mathscr{B}\|^2+\frac{39}{64}\|\mathscr{A}\|^2-\frac34\langle\mathscr{A},\mathscr{B}\rangle$ and dropping the first term,

$$\|\mathscr{A}-\mathscr{B}\|^2\ge\tfrac{39}{64}\mu_0\|\eta\|^2-\tfrac12\int V|\eta|^2$$

by Lemma 28. With $\mathrm{Ric}=\frac32$, (9) gives $\lambda\|\eta\|^2\ge(\frac{39}{64}\mu_0+\frac32)\|\eta\|^2-\int(\mathrm{Hess}\,V(\eta,\eta)+\frac12 V|\eta|^2)$, and the claim follows from the pointwise bound

$$\mathrm{Re}\,\mathrm{Tr}(MX^2)+\tfrac12\,\mathrm{Re}\,\mathrm{Tr}\,M\le\tfrac{r}{2}|X|^2,\qquad M=bU\in rK,\quad(10)$$

since $\mathrm{Hess}\,V(X,X)=\mathrm{Re}\,\mathrm{Tr}(MX^2)$ and $V=\mathrm{Re}\,\mathrm{Tr}\,M$. The maximum over $|X|=1$ of the left side is convex in M, so it suffices to take $M=rW$ with $W\in\mathrm{SU}(3)$ and, after a conjugation, $W=\mathrm{diag}(e^{i\theta_k})$ with $\sum_k\theta_k=0$; put $c_k=\cos\theta_k$. Write $X=iH$ with H Hermitian, traceless, $\mathrm{Tr}\,H^2=1$, with diagonal d_k and entries H_{jk}. Then

$$\mathrm{Re}\,\mathrm{Tr}(WX^2)=-\sum_k c_kd_k^2-\sum_{j<k}(c_j+c_k)|H_{jk}|^2,$$

with $\sum_k d_k^2+2\sum_{j<k}|H_{jk}|^2=1$, so the maximum is the larger of $q_{\mathrm{off}}=\max_{j<k}-\frac12(c_j+c_k)$ and $q_{\mathrm{diag}}=\max\{-\sum c_kd_k^2:\sum d_k=0,\ \sum d_k^2=1\}$. If l is the third index, $q_{\mathrm{off}}+\frac12\sum_k c_k=\frac12 c_l\le\frac12$. The value q_{diag} is minus the smaller eigenvalue of $\mathrm{diag}(c)$ compressed to the plane $\sum d_k=0$; the eigenvalues of the compression solve $\sum_k(c_k-\mu)^{-1}=0$, that is $3\mu^2-2e_1\mu+e_2=0$ with e_1,e_2 the elementary symmetric functions of the c_k, so $q_{\mathrm{diag}}=\frac13(\sqrt{e_1^2-3e_2}-e_1)$. The inequality $q_{\mathrm{diag}}+\frac12 e_1\le\frac12$ is $\sqrt{e_1^2-3e_2}\le\frac12(3-e_1)$, that is $e_1^2-4e_2+2e_1-3\le0$. When $\sum\theta_k=0$, $\sum c_k^2=1+2c_1c_2c_3$, and with this identity the left side equals $-2(1-c_1)(1-c_2)(1-c_3)\le0$. □

The weight $\frac58$ is the one that gives the coefficient $\frac12$ of V, and (10) is an equality at $W=I$. At $r=\frac32$, $\gamma=\frac{187}{128}$.[15]

D. The modulus by hand

For $h\in\mathbb{C}^{3\times3}$ let $E_2(h)=\frac12((\mathrm{Tr}\,h)^2-\mathrm{Tr}\,h^2)$, the sum of the principal 2×2 minors of h. All integrals are over $\mathrm{SU}(3)$ with Haar measure; put $t=\mathrm{Tr}\,U$,

$$Z_*(r)=\int e^{r\,\mathrm{Re}\,\mathrm{Tr}\,U}dU,\qquad m_n=\int(\mathrm{Re}\,\mathrm{Tr}\,U)^n dU.$$

Lemma 30. (i) *The integral of a monomial of degree j in the entries of U and k in those of $\bar U$ vanishes unless $j\equiv k$ modulo three; $\int u_{ij}\bar u_{kl}dU=\frac13\delta_{ik}\delta_{jl}$ and $\int u_{ab}u_{cd}u_{ef}dU=\frac16\varepsilon_{ace}\varepsilon_{bdf}$.*

(ii) *For every h the integrals of $\mathrm{Tr}(hUhU)$ and $(\mathrm{Tr}\,hU)^2$ vanish, $\int|\mathrm{Tr}\,hU|^2dU=\frac13\|h\|_{\mathrm{HS}}^2$,*

$$\int\mathrm{Tr}(hUhU)\,\mathrm{Tr}\,U\,dU=-\tfrac13E_2(h),$$
$$\int(\mathrm{Tr}\,hU)^2\,\mathrm{Tr}\,U\,dU=\tfrac13E_2(h),$$

and $|E_2(h)|\le\|h\|_{\mathrm{HS}}^2$.

(iii) *$\int t^a\bar t^b dU$ vanishes unless $a\equiv b$ modulo three, does not change when a and b are exchanged, and equals 1 at $(a,b)=(0,0)$, $(1,1)$, $(3,0)$, 2 at $(2,2)$, 3 at $(4,1)$, 5 at $(6,0)$, 6 at $(3,3)$, 11 at $(5,2)$, 21 at $(7,1)$ and 23 at $(4,4)$. Hence $m_0=1$, $m_1=0$,*

$$(m_2,\dots,m_8)=\left(\tfrac12,\tfrac14,\tfrac34,\tfrac{15}{16},\tfrac{65}{32},\tfrac{231}{64},\tfrac{973}{128}\right),$$

and $m_n\le3^{n-8}m_8$ for $n\ge8$.

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[15] The direct Bochner bound, with Minkowski's inequality in place of the cross term, gives $(\sqrt{\mu_0}-\frac{g}{2}r)^2+\frac32-r$ with $g^2=\max_W\|PA(W)\|^2=2.0655$; it is exact at $b=0$, where the gap is $\frac83$, but falls below γ for $r>0.24$ and vanishes before $r=1$.

Proof. (iii) By the proof of Lemma 34, $t^a = \sum_{\lambda\vdash a} f^\lambda s_\lambda(U)$, and on SU(3) the Schur polynomial s_λ vanishes when λ has more than three rows and is otherwise the irreducible character of highest weight $(\lambda_1-\lambda_3, \lambda_2-\lambda_3)$. By Schur orthogonality [32, Ch. 2], $\int t^a \bar t^b dU$ is the sum of $f^\lambda f^\mu$ over the pairs $\lambda\vdash a$, $\mu\vdash b$ with at most three rows and $\lambda-\mu$ a multiple of $(1,1,1)$; such pairs exist only when $a\equiv b$ modulo three, and the sum is symmetric in a,b. For $a\ge b$ and $a+b\le 8$ the pairs are $((1),(1))$; $((1,1,1),\varnothing)$; $\lambda=\mu\in\{(2),(1,1)\}$; $((2,1,1),(1))$; $((2,2,2),\varnothing)$; $\lambda=\mu\in\{(3),(2,1),(1,1,1)\}$; $((3,1,1),(2))$ and $((2,2,1),(1,1))$; $((3,2,2),(1))$; and $\lambda=\mu\in\{(4),(3,1),(2,2),(2,1,1)\}$. The hook length formula gives $f^{(2,1)}=2$, $f^{(3,1)}=f^{(2,1,1)}=3$, $f^{(2,2)}=2$, $f^{(2,2,2)}=5$, $f^{(3,1,1)}=6$, $f^{(2,2,1)}=5$ and $f^{(3,2,2)}=21$, hence the values, the last one being $1+9+4+9$. Since $(\mathrm{Re}\,\mathrm{Tr}\,U)^n = 2^{-n}\sum_a \binom{n}{a} t^a \bar t^{n-a}$,

$$m_2=\tfrac{2}{4},\quad m_3=\tfrac{2}{8},\quad m_4=\tfrac{6\cdot2}{16},\quad m_5=\tfrac{2\cdot5\cdot3}{32},$$
$$m_6=\tfrac{2\cdot5+20\cdot6}{64},\quad m_7=\tfrac{2\cdot21\cdot11}{128},\quad m_8=\tfrac{2\cdot8\cdot21+70\cdot23}{256}.$$

The eigenvalues of U have modulus one, so $|\mathrm{Re}\,\mathrm{Tr}\,U|\le 3$ and $|\mathrm{Re}\,\mathrm{Tr}\,U|^n \le 3^{n-8}(\mathrm{Re}\,\mathrm{Tr}\,U)^8$ for $n\ge 8$.

(i) The map $U\mapsto \omega U$, $\omega = e^{2\pi i/3}$, preserves Haar measure and multiplies the monomial by ω^{j-k}. For a unitary representation π of SU(3), $\int \pi(U)dU$ is the orthogonal projection onto the invariant vectors, whose dimension is the mean of the character of π. On $\mathbb{C}^3\otimes\overline{\mathbb{C}^3}$ the invariant vectors are the multiples of the identity, which gives the second formula. On $(\mathbb{C}^3)^{\otimes 3}$ the character is t^3, whose mean is 1 by (iii), and ε is invariant because $\det U = 1$; the projection onto the line of ε, whose squared norm is 6, gives the third formula.

(ii) The first two integrals have degree two in U and none in $\bar U$, and $\int |\mathrm{Tr}\,hU|^2 dU = \frac{1}{3}\sum |h_{ji}|^2$ by (i). By (i), $\int h_{pa}u_{aq}h_{qc}u_{cp}u_{ee}dU = \frac{1}{6}h_{pa}h_{qc}\varepsilon_{ace}\varepsilon_{qpe}$, summed over repeated indices, and $\sum_e \varepsilon_{ace}\varepsilon_{qpe} = \delta_{aq}\delta_{cp}-\delta_{ap}\delta_{cq}$, which gives $\frac{1}{6}(\mathrm{Tr}\,h^2-(\mathrm{Tr}\,h)^2)$; the second integral is the same with ε_{pqe} in place of ε_{qpe}, which changes the sign. Finally $E_2(h)$ is the trace of the second exterior power of h, whose singular values are the products $\sigma_i\sigma_j$, $i<j$, of those of h; so $|E_2(h)|\le \sum_{i<j}\sigma_i\sigma_j = \frac{1}{2}((\sum_i\sigma_i)^2-\sum_i\sigma_i^2)\le \sum_i\sigma_i^2$. □

Lemma 31. *Let $r\ge 0$ with $\gamma(r)>0$, put*

$$\varepsilon(r) = 1-\frac{5e^{-r/5}}{9Z_*(r)},$$

and let $b\in rK$ and $\|h\|_{\mathrm{HS}}=1$. Then

$$\nu_b|\nabla\varphi_h|^2\le\varepsilon(r),\quad \mathrm{Var}_{\nu_b}(\varphi_h)\le\frac{\varepsilon(r)}{\gamma(r)},\quad L_h(b)\le\frac{\sqrt{\varepsilon(r)}}{\gamma(r)},$$

so that $\Lambda(r)\le\sqrt{\varepsilon(r)}/\gamma(r)$. The function ε is nondecreasing on $[0,\infty)$.

Proof. By Lemma 27, $\nabla\varphi_h(U) = -PA(M)$ with $M=hU$, and $\|M\|_{\mathrm{HS}}=1$. The anti-Hermitian part has $|A(M)|^2 = \frac{1}{2}(\|M\|^2 - \mathrm{Re}\,\mathrm{Tr}\,M^2)$, and P removes from it the multiple $\frac{1}{3}\mathrm{Tr}\,A(M) = \frac{i}{3}\mathrm{Im}\,\mathrm{Tr}\,M$ of the identity, whose squared norm is $\frac{1}{3}(\mathrm{Im}\,\mathrm{Tr}\,M)^2$. Hence

$$|\nabla\varphi_h|^2 = 1-\tfrac{1}{2}F_h,$$
$$F_h(U) = 1+\mathrm{Re}\,\mathrm{Tr}(hUhU)+\tfrac{2}{3}(\mathrm{Im}\,\mathrm{Tr}\,hU)^2,$$

and $F_h\ge 0$ because $|\mathrm{Re}\,\mathrm{Tr}\,M^2|\le\|M\|^2$. For complex numbers X,Y, $\mathrm{Re}\,X\,\mathrm{Re}\,Y = \frac{1}{2}\mathrm{Re}(XY)+\frac{1}{2}\mathrm{Re}(X\bar Y)$ and $(\mathrm{Im}\,X)^2 = \frac{1}{2}|X|^2-\frac{1}{2}\mathrm{Re}\,X^2$. With these identities, the vanishing of the integrals of degree two in U and one in $\bar U$, or the converse, and Lemma 30(ii),

$$\int F_h\,dU = 1+\tfrac{2}{3}\cdot\tfrac{1}{6} = \tfrac{10}{9},\qquad \int F_h\,\mathrm{Re}\,\mathrm{Tr}\,U\,dU = -\tfrac{2}{9}\mathrm{Re}\,E_2(h),$$

the second as the sum of $-\frac{1}{6}\mathrm{Re}\,E_2(h)$ and $\frac{2}{3}\cdot(-\frac{1}{12})\mathrm{Re}\,E_2(h)$. For $W\in\mathrm{SU}(3)$ the substitution $U\mapsto W^{-1}U$ and Lemma 30(ii) give

$$\int F_h\,\mathrm{Re}\,\mathrm{Tr}(WU)\,dU = \int F_{hW^{-1}}\,\mathrm{Re}\,\mathrm{Tr}\,U\,dU \ge -\tfrac{2}{9}.$$

Write $b = r\sum_i\lambda_i W_i$ as a convex combination of elements of SU(3) and $V = \mathrm{Re}\,\mathrm{Tr}(Ub)$. Under the probability measure $\pi = \frac{9}{10}F_h\,dU$ the mean of V is at least $-\frac{9}{10}\cdot\frac{2}{9}r = -\frac{r}{5}$, so Jensen's inequality gives $\int F_h e^V dU = \frac{10}{9}\pi(e^V)\ge\frac{10}{9}e^{-r/5}$. The function $c\mapsto\log\int e^{\mathrm{Re}\,\mathrm{Tr}(Uc)}dU$ is convex by Hölder's inequality and equals $\log Z_*(r)$ at every $c=rW$, $W\in\mathrm{SU}(3)$, by the invariance of Haar measure; hence $\int e^V dU\le Z_*(r)$ and $\nu_b(F_h)\ge\frac{10}{9}e^{-r/5}/Z_*(r)$, which is the first bound.

The spectral gap of ν_b is at least $\gamma(r)$ by Lemma 29. The Poincaré inequality gives $\mathrm{Var}_{\nu_b}(\varphi_h)\le\nu_b|\nabla\varphi_h|^2/\gamma(r)$, the second bound, and, for g 1-Lipschitz for d, hence for the geodesic distance, $\mathrm{Var}_{\nu_b}(g)\le 1/\gamma(r)$; so $\mathrm{Cov}_{\nu_b}(g,\varphi_h)\le\mathrm{sd}(g)\,\mathrm{sd}(\varphi_h)\le\sqrt{\varepsilon(r)}/\gamma(r)$. Finally $\log Z_*$ is convex in r with derivative $m_1=0$ at $r=0$, so Z_* and ε are nondecreasing on $[0,\infty)$. □

Proof of Theorem 6. Let $r=2\beta_W\le 1.03$, so that $\gamma(r)\ge\gamma(1.03)>1.69$. By (8) and Lemma 31, the Dobrushin constant satisfies $\alpha\le 2r\Lambda(r)\le\alpha_1(\beta_W)$, and α_1 is increasing because ε is nondecreasing and γ is decreasing. By Lemma 30(iii), and since $\sum_{n\ge 9}x^n/n!\le\frac{x^9}{9!}(1-\frac{x}{10})^{-1}$ for $0\le x<10$,

$$Z_*(r)\le\sum_{n=0}^{8}\frac{m_n r^n}{n!}+\frac{m_8}{3^8}\frac{(3r)^9}{9!}\Big(1-\frac{3r}{10}\Big)^{-1}.$$

At $r=1$ the terms of the sum, rounded up, are 1, 0, 0.25, 0.0416667, 0.03125, 0.0078125, 0.0028212, 0.0007162 and 0.0001886, with sum at most 1.3344552, and the last term is $\frac{973}{128\cdot6561}\cdot\frac{19683}{362880}\cdot\frac{10}{7}<9\cdot10^{-5}$; so $Z_*(1)\le 1.334546$. Since $e^{-x}\ge 1-x+\frac{x^2}{2}-\frac{x^3}{6}$ for $x\ge 0$, $e^{-1/5}\ge 0.818666$, so $\frac{5e^{-1/5}}{9Z_*(1)}\ge 0.340800$ and $\varepsilon(1)\le 0.659200<0.81192^2$. With $\gamma(1)=\frac{219}{128}$,

$$\alpha_1(\tfrac{1}{2})\le\frac{2\cdot0.81192\cdot128}{219}\le 0.9491.$$

At $r=1.03$ the terms are at most

$$1,\ 0,\ 0.265225,\ 0.0455303,\ 0.0351722,$$
$$0.0090569,\ 0.0033687,\ 0.0008808,\ 0.0002389,$$

with sum at most 1.3594728, and the last term is less than $1.2\cdot10^{-4}$; so $Z_*(1.03)\le 1.359593$, $e^{-0.206}\ge 0.813760$, $\frac{5e^{-0.206}}{9Z_*(1.03)}\ge 0.332517$ and $\varepsilon(1.03)\le 0.667483<0.817^2$. With $\gamma(1.03)=\frac{5427}{3200}$, $\alpha_1(0.515)\le 2.06\cdot 0.817\cdot\frac{3200}{5427}\le 0.9924$. Proposition 11 applies with the Hilbert–Schmidt metric. With $y=1-\alpha_1$, $-\log\alpha_1\ge y+\frac{y^2}{2}$, which is at least 0.052 at $\beta_W=\frac12$ and 0.0076 at 0.515. □

The losses in this bound are Jensen's inequality, the Poincaré inequality for φ_h, which replaces the true variance of φ_h by $\nu_b|\nabla\varphi_h|^2/\gamma(r)$, and the gap $\gamma(r)$ itself: at $b=0$ the variance is $\frac16$ and the gap $\frac83$, so that $L_h(0)=\frac14$, while the lemma gives $\frac23/\gamma(0)=0.302$. Evaluated in floating point, α_1 reaches one near $\beta_W=0.5176$. The rest of this section describes a sharper bound for $\Lambda(r)$, with a corrector of degree two, the reduction to diagonal staple sums, exact moments and an interpolation over boxes of staple sums; its evaluation by a computer is the subject of Remark 36, and no result uses it.

E. A corrector of degree two

Lemma 32. *For $b\in rK$, $\|h\|=1$ and every smooth real function u on* $\mathrm{SU}(3)$*,*

$$L_h(b)\le\sqrt{\nu_b|\nabla u|^2}+\sqrt{\mathrm{Var}_{\nu_b}(\varphi_h+Lu)/\gamma(r)}.$$

Proof. Let $-L\psi=\varphi_h-\nu_b\varphi_h$. For a 1-Lipschitz g, $|\nabla g|\le1$ almost everywhere, because d is at most the geodesic distance, so $\mathrm{Cov}(g,\varphi_h)=\nu_b\langle\nabla g,\nabla\psi\rangle\le\nu_b|\nabla\psi|$. Write $\psi=u+\chi$. Since $\nu_b(Lu)=0$, χ solves $-L\chi=X-\nu_bX$ with $X=\varphi_h+Lu$, and

$$\nu_b|\nabla\chi|^2=\mathrm{Cov}(\chi,X)\le\sqrt{\mathrm{Var}\,\chi\,\mathrm{Var}X}\le\sqrt{\nu_b|\nabla\chi|^2\,\mathrm{Var}X/\gamma}$$

by Lemma 29. Hence $\nu_b|\nabla\chi|^2\le\mathrm{Var}X/\gamma$, and $\nu_b|\nabla\psi|\le\sqrt{\nu_b|\nabla u|^2}+\sqrt{\nu_b|\nabla\chi|^2}$. □

The best u is ψ itself, and then the second term vanishes. With $u=s\varphi_h$, Lemma 27 gives $\varphi_h+Lu=(1-s\lambda_F)\varphi_h+s\Gamma$ with $\Gamma=\langle\nabla V,\nabla\varphi_h\rangle$; the choice $s=1/\lambda_F$ removes φ_h and leaves the drift. In floating point this blended corrector keeps $2r\sup_h L_h$ below one up to about $\beta_W=0.715$ (Sec. IX). We take u in a space of polynomials of degree two.

Let $\mathscr{P}_2$ be the real vector space of real polynomials of degree one or two in the entries u_{ij} of U and $\bar u_{ij}$ of $\bar U$. It has dimension 189: a basis (B_k) consists of $\frac12(m+\bar m)$ and $\frac{1}{2i}(m-\bar m)$ for each pair $\{m,\bar m\}$ of conjugate monomials with $m\ne\bar m$, and m itself when $m=\bar m$. Identify $h\in\mathbb{C}^{3\times3}$ with $\mathbb{R}^{18}$ through the real and imaginary parts of its entries. For a real 189×18 matrix C put

$$q_h=\sum_k\sum_{m=1}^{18}C_{km}h_mB_k\in\mathscr{P}_2,$$

linear in h. For diagonal $b=\mathrm{diag}(a_1,a_2,x_3+iy_3)$ we have $V=\sum_k b_k\varphi_{F_k}$ with $(b_k)=(a_1,a_2,x_3,y_3)$ and $F=(e_{11},e_{22},e_{33},ie_{33})$, so

$$X_h=\varphi_h+Lq_h=\varphi_h+\Delta q_h+\sum_k b_k\langle\nabla\varphi_{F_k},\nabla q_h\rangle\qquad(11)$$

is a polynomial of degree at most three, affine in b, while ∇q_h does not depend on b. The squares of these functions have degree at most six, so the quadratic forms $h\mapsto\nu_b|\nabla q_h|^2$ and $h\mapsto\nu_b[(X_h-\kappa)^2]$ are combinations of moments of ν_b of degree at most six.

To compute Δ and $\langle\nabla f,\nabla g\rangle$ exactly we use the basis of $\mathfrak{su}(3)$

$$e_{jk}-e_{kj},\quad i(e_{jk}+e_{kj})\quad(j<k),$$
$$i(e_{11}-e_{22}),\quad i(e_{11}+e_{22}-2e_{33}),$$

acting as left-invariant fields $X_af(U)=\frac{d}{dt}f(Ue^{tX_a})|_{t=0}$, with weights $w_a=\frac12$ for the first seven and $\frac16$ for the last. Then $\sum_a w_aX_a\otimes X_a$ is the Casimir tensor of $\langle X,Y\rangle=-\mathrm{Tr}(XY)$, so

$$\langle\nabla f,\nabla g\rangle=\sum_a w_aX_afX_ag,\qquad\Delta=\sum_a w_aX_a^2.$$

The fields map monomials to combinations of monomials of the same degree with coefficients in $\mathbb{Z}[i]$; for instance $\Delta\varphi_h=-\frac83\varphi_h$, as in Lemma 27.

F. Orbits of staple sums

Lemma 33. *Let $V,W\in\mathrm{SU}(3)$.*

(i) *$L_h(VbW)=L_{V^{-1}hW^{-1}}(b)$ and $L_h(\bar b)=L_{\bar h}(b)$.*

(ii) *Every $b\in\mathbb{C}^{3\times3}$ is in the orbit of some $\mathrm{diag}(a_1,a_2,z_3)$ with $a_1\ge a_2\ge|z_3|$ and $\mathrm{Im}\,z_3\ge0$.*

(iii) *A diagonal matrix lies in K if and only if it lies in the convex hull of the diagonal torus T of* $\mathrm{SU}(3)$.

Proof. (i) If U has law ν_{VbW} then WUV has law ν_b; the map preserves d and Haar measure, and $\varphi_h(U)=\varphi_{V^{-1}hW^{-1}}(WUV)$. For $\bar b$ use $U\mapsto\bar U$. (ii) Take a singular value decomposition $b=V_1\Sigma W_1$ and move the determinants of V_1,W_1 into the last entry; a complex conjugation makes $\mathrm{Im}\,z_3\ge0$. (iii) Let c be diagonal with nonzero entries of distinct moduli and let U maximise $\mathrm{Re}\,\mathrm{Tr}(cU)$ on $\mathrm{SU}(3)$. The derivative along $\mathfrak{su}(3)$ vanishes, so the anti-Hermitian part of cU is a multiple $i\lambda I$ of the identity and $cU=H+i\lambda I$ with H Hermitian. Then $H^2+\lambda^2I=cc^*$ is diagonal with distinct entries, H commutes with it and is diagonal, and $U\in T$. Hence $\max_{\mathrm{SU}(3)}\mathrm{Re}\,\mathrm{Tr}(cU)=\max_T\mathrm{Re}\,\mathrm{Tr}(ct)$ for such c, and by continuity for every diagonal c. A diagonal matrix outside $\mathrm{conv}\,T$ is separated from it by such a functional, which then separates it from K. □

Since every $b\in rK$ has operator norm at most r,

$$\Lambda(r)=\sup\Big\{\sup_{\|h\|=1}L_h(b):b\in\mathscr{R}_r\Big\},$$

where $\mathscr{R}_r$ is the set of $\mathrm{diag}(a_1,a_2,x_3+iy_3)\in r\,\mathrm{conv}\,T$ with $a_1\ge a_2\ge|x_3+iy_3|$ and $y_3\ge0$, a subset of the box $[0,r]^2\times[-r,r]\times[0,r]$ in the coordinates (a_1,a_2,x_3,y_3).

G. Exact moments and interpolation

Lemma 34. *For $D, D' \in \mathbb{C}^{3\times 3}$ let $\mathbb{E}_H$ be the Haar mean on* $\mathrm{SU}(3)$ *and*

$$Z(D,D') = \mathbb{E}_H \exp\big(\tfrac{1}{2}\operatorname{Tr}(UD) + \tfrac{1}{2}\operatorname{Tr}(\bar U D')\big).$$

Then

$$Z(D,D') = \sum_{p\ge q\ge 0} \frac{s_{(p,q)}(DD'^{\top})\, G_{p,q}(\det D)\, G_{p,q}(\det D')}{d_{p,q}},$$

with $G_{p,q}(z) = \sum_{k\ge 0} c_{(p+k,q+k,k)} z^k$, $c_\lambda = 2^{-|\lambda|}/\prod \mathrm{hooks}(\lambda)$, $d_{p,q} = \tfrac{1}{2}(p-q+1)(q+1)(p+2)$ and $s_{(p,q)}$ the Schur polynomial. The moments of degree j in U and k in $\bar U$ are $\mathbb{E}_H[u^\alpha \bar u^\beta e^{\mathrm{Re}\operatorname{Tr}(Ub)}] = 2^{|\alpha|+|\beta|}\partial_D^\alpha \partial_{D'}^\beta Z(b,\bar b)$, the entry u_{ij} pairing with D_{ji}.

Proof. We use $(\operatorname{Tr} X)^n = \sum_{\lambda\vdash n} f^\lambda s_\lambda(X)$, where f^λ is the number of standard tableaux of shape λ, given by the hook length formula [109, Cor. 7.21.6]. Expanding the exponential gives

$$e^{\frac{1}{2}\operatorname{Tr} X} = \sum_{n\ge 0} \frac{(\operatorname{Tr} X)^n}{2^n n!} = \sum_\lambda c_\lambda s_\lambda(X),$$

the sum over partitions with at most three parts. For $U \in \mathrm{SU}(3)$ one has $s_{\lambda+(k,k,k)}(UD) = (\det D)^k s_\lambda(UD)$, so

$$e^{\frac{1}{2}\operatorname{Tr}(UD)} = \sum_{p\ge q} G_{p,q}(\det D)\, \chi_{p,q}(UD),$$

where $\chi_{p,q}$ is the irreducible character of highest weight $(p,q,0)$ and $d_{p,q}$ its dimension. Since $\bar U = (U^{-1})^\top$, we have $\chi_{p,q}(\bar U D') = \chi_{p,q}(U^{-1} D'^{\top})$, and Schur orthogonality [32, Ch. 2],

$$\mathbb{E}_H\big[\chi_\rho(UA)\chi_{\rho'}(U^{-1}B)\big] = \delta_{\rho\rho'}\frac{\chi_\rho(AB)}{d_\rho},$$

gives the expansion. The bound below gives absolute convergence. Finally $\mathrm{Re}\operatorname{Tr}(Ub) = \tfrac{1}{2}\operatorname{Tr}(Ub) + \tfrac{1}{2}\operatorname{Tr}(\bar U \bar b)$. □

The computation keeps $p \le 34$ and $k \le 11$. Every omitted partition has more than 34 boxes, and since $|s_\lambda(X)| \le \dim(\lambda)\|X\|^{|\lambda|}$ and $\sum_{\lambda\vdash n} f^\lambda \dim\lambda = 3^n$ by Schur–Weyl duality, the omitted part of each factor is at most $\sum_{n>34}(\frac{3}{2}\|D\|)^n/n!$; Cauchy's estimate on a polydisc around b turns this into a bound for every derivative (Appendix E).

Lemma 35 (Interpolation on a box)**.** *Let $Q = \prod_k [l_k, l_k + w_k]$ in the coordinates (a_1, a_2, x_3, y_3), with corners b_i and multilinear weights $\omega_i(b) \ge 0$, $\sum_i \omega_i(b) = 1$, and let $\varsigma = w_1^2 + w_2^2 + \max(w_3^2, w_4^2)$. For $b \in Q$:*

(i) *$Z(b) \ge e^{-\varsigma/8} \sum_i \omega_i(b) Z(b_i)$;*

(ii) *$\nu_b(f) \le e^{\varsigma/8} \max_i \nu_{b_i}(f)$ for every $f \ge 0$ that does not depend on b;*

(iii) *if $Y_b = (\mathscr{A} + \sum_k (b_k - \beta_k)\mathscr{B}_k)^2$ with functions $\mathscr{A}, \mathscr{B}_k$ of U and a fixed point β, then*

$$\nu_b(Y_b) \le e^{\varsigma/8}\Big(\max_i \nu_{b_i}(Y_{b_i}) + \sum_k \frac{w_k^2}{4}\,\frac{\max_i Z(b_i)\nu_{b_i}(\mathscr{B}_k^2)}{\min_i Z(b_i)}\Big).$$

Proof. Write $E(b) = e^{\mathrm{Re}\operatorname{Tr}(Ub)} = \prod_k e^{c_k b_k}$, where

$$c = (\mathrm{Re}\, u_{11}, \mathrm{Re}\, u_{22}, \mathrm{Re}\, u_{33}, -\mathrm{Im}\, u_{33}),$$

so that $|c_k| \le 1$ and $c_3^2 + c_4^2 \le 1$. In each coordinate the chord of $e^{c_k x}$ over $[l_k, l_k + w_k]$ lies above the function and, by Hoeffding's lemma [70] for a two-point variable, below $e^{c_k x} e^{c_k^2 w_k^2/8}$. Multiplying,

$$E(b) \le \sum_i \omega_i(b) E(b_i) \le E(b)\, e^{\sum_k c_k^2 w_k^2/8} \le E(b)\, e^{\varsigma/8}. \tag{12}$$

Integrating the right inequality gives (i), and (ii) follows from the left one and (i). For (iii) let $\mathscr{F}(b) = Y_b E(b)$ and I_k the linear interpolation in the k-th coordinate, so that $\mathscr{F} - I_1 I_2 I_3 I_4 \mathscr{F} = \sum_k I_1 \cdots I_{k-1}(\mathscr{F} - I_k \mathscr{F})$. Along the k-th coordinate $\mathscr{F}$ is $\ell^2 e^{c_k x}$ times a positive factor, with ℓ affine of slope $\mathscr{B}_k$, and $\partial_k^2 \mathscr{F} = ((c_k \ell + 2\mathscr{B}_k)^2 - 2\mathscr{B}_k^2)E \ge -2\mathscr{B}_k^2 E$. With the Green kernel of the interval, whose integral is at most $w_k^2/8$,

$$\mathscr{F} - I_k \mathscr{F} \le 2\mathscr{B}_k^2 \int G E \le \tfrac{w_k^2}{4}\mathscr{B}_k^2 \sum_i \omega_i^{(k)} E(b_i)$$

by the left inequality of (12), with weights $\omega_i^{(k)} \ge 0$ of sum one that do not depend on U. The operators I_j have nonnegative weights of sum one; integrate over U and divide by $Z(b)$, bounded below by (i). □

H. The cover

Fix a box Q and a matrix C. For any constant κ, $\mathrm{Var}(X_h) \le \nu_b[(X_h - \kappa)^2]$; take $\kappa = \langle \kappa(b), h\rangle$ with $\kappa(b)$ affine in b, fitted to the means at the corners. By (11), $X_h - \kappa = \mathscr{A} + \sum_k (b_k - \beta_k)\mathscr{B}_k$ with β the centre of Q and $\mathscr{B}_k = \langle \nabla\varphi_{F_k}, \nabla q_h\rangle - \langle \partial_k \kappa, h\rangle$, and Lemmas 32 and 35 give, for all $b \in Q \cap \mathscr{R}_r$,

$$\begin{aligned}\sup_h L_h(b) \le{}& \sqrt{e^{\varsigma/8} \max_i \lambda_A(b_i)} \\ &+ \sqrt{\frac{e^{\varsigma/8}}{\gamma(r)}\Big(\max_i \lambda_V(b_i) + \frac{e_N}{\min_i Z(b_i)}\Big)},\end{aligned} \tag{13}$$

where $e_N = \sum_k \frac{w_k^2}{4} \max_i Z(b_i)\lambda_k(b_i)$ and λ_A, λ_V, λ_k are the largest eigenvalues of the quadratic forms $h \mapsto \nu_{b_i}|\nabla q_h|^2$, $h \mapsto \nu_{b_i}[(X_h - \kappa)^2]$ at $b = b_i$, and $h \mapsto \nu_{b_i}[\mathscr{B}_k^2]$ on $\mathbb{R}^{18}$. Their entries are moments of ν_{b_i} of degree at most six, computed with rigorous error bounds (Appendix E); each largest eigenvalue is bounded above by an $LDL^\top$ factorisation of tI minus the computed form with certified positive pivots, plus the norm of its error matrix by Weyl's inequality [67, 74].

The matrix C is chosen in each box in floating point. At the mean of the corner moments and at $b=\beta$, Lemma 32 for $u=q_h$ is a quadratic problem in C: for $\theta>0$ we minimise the trace of $\nu|\nabla q|^2+\theta\,\mathrm{Var}(X)/\gamma$ over the 18 coordinate directions h, a least-squares problem of size 189, keep the θ on a geometric grid of 48 values from 0.05 to 80 that gives the smallest bound, and round C to multiples of 2^{-30}. Any choice gives a valid bound.

A box is discarded when it contains no point with $a_1\ge a_2\ge|z_3|$, or when a linear functional $\ell(b)=\mathrm{Re}\,\mathrm{Tr}(c^*b)$ satisfies $\min_Q\ell>r\max_T\ell$, which separates it from $r\,\mathrm{conv}\,T$ and hence, by Lemma 33(iii), from $\mathscr{R}_r$.

Remark 36 (a computation, used by no theorem). At $r=\frac{3}{2}$, with $\gamma(r)=\frac{187}{128}$, the box $[0,r]^2\times[-r,r]\times[0,r]$ is cut into 8192 boxes of side $r/8$. Of these, 841 survive the two tests, and for 581 the right side B_Q of (13), evaluated with the error bounds of Appendix E, satisfies $2rB_Q<1$. Each of the other 260 is cut into sixteen boxes of side $r/16$; 1760 of these survive the tests, and all satisfy $2rB_Q<1$. The 2341 boxes cover $\mathscr{R}_r$; the largest bound is 0.999887, on the box $[1.3125,1.5]\times[0.9375,1.125]\times[0.5625,0.75]\times[0,0.1875]$, and the median is 0.983 (Fig. 6). This computation gives $2r\Lambda(r)\le0.999887$ at $r=\frac{3}{2}$, and with it (8) and the monotonicity of Λ would give $\alpha\le4\beta_W\Lambda(\frac{3}{2})<1.333183\,\beta_W<1$ for $\beta_W\le\frac{3}{4}$, and so the conclusions of Theorem 1, with $m\ge-\log(1.33319\,\beta_W)$, on that range. The same computation with $\gamma(r)=\frac{4643}{3200}$ and the threshold $2rB_Q<1.03$ at $r=\frac{38}{25}$ (of the 8192 boxes, 784 survive the tests and 690 satisfy the threshold; each of the other 94 is cut into sixteen, of which 373 survive and all satisfy it), and with $\gamma(r)=\frac{4611}{3200}$ and the threshold $2rB_Q<1.0275$ at $r=\frac{77}{50}$ (826 survive and 513 satisfy it; the other 313 are cut into sixteen, of which 2311 survive and 2281 satisfy it; the other 30 are cut into sixteen, of which 31 survive and all satisfy it), gives $2r\Lambda(r)\le1.029967$ at $r=\frac{38}{25}$ and $2r\Lambda(r)\le1.027468$ at $r=\frac{77}{50}$. No result of the paper uses these numbers.

VIII. SU(3) WITH PLAQUETTE BLOCKS

In this section $G=\mathrm{SU}(3)$, $\beta\ge0$, $\beta_W=9\beta$, $N\beta=\beta_W/3$, and the notation is that of Sec. VII: d is the Hilbert–Schmidt distance, $K=\mathrm{conv}\,\mathrm{SU}(3)$, $\nu_c(dU)\propto e^{\mathrm{Re}\,\mathrm{Tr}(Uc)}dU$, $\varphi_h(U)=\mathrm{Re}\,\mathrm{Tr}(Uh)$ and $\Lambda(r)$ is the modulus of (8). Put $Z(c)=\int e^{\mathrm{Re}\,\mathrm{Tr}(Uc)}dU$ and $g=\log Z$ on $\mathbb{C}^{3\times3}$, viewed as $\mathbb{R}^{18}$ with the inner product $\langle x,y\rangle=\mathrm{Re}\,\mathrm{Tr}(xy^*)$. The gradient of g at c is the conjugate mean $m(c)^*$, $m(c)=\nu_c(U)\in K$, and $\mathrm{Hess}\,g(c)(h,h)=\mathrm{Var}_{\nu_c}(\varphi_h)$. Since $Z(VcW)=Z(c)$ for $V,W\in\mathrm{SU}(3)$, the same holds for g.

The blocks are those of Lemma 12, which concerns the line field only and holds for every group. For a block plaquette with active links e_1,e_2,e_3, $\mathrm{Re}\,\mathrm{Tr}\,Q_p=\mathrm{Re}\,\mathrm{Tr}(y_1y_2y_3)$ with $y_i=Q_{e_i}$ or $Q_{e_i}^{-1}$, and the conditional law of the block is

$$\mu_b(dy)\propto\exp\Big(\sum_i\mathrm{Re}\,\mathrm{Tr}(y_ib_i)+N\beta\,\mathrm{Re}\,\mathrm{Tr}(y_1y_2y_3)\Big)dy,\quad(14)$$

where b_i is $N\beta$ times the sum of the five staples of y_i outside the block plaquette, so $b_i\in5N\beta K$ and $\|b_i\|_{\mathrm{op}}\le5N\beta$. On $\mathrm{SU}(3)^3$ we use $d(y,y')=\sum_i\|y_i-y_i'\|_{\mathrm{HS}}$.

A. The sensitivity of one block

Let $r=2\beta_W=6N\beta$. Let λ_* and τ be a lower and an upper bound for the eigenvalues of $\mathrm{Hess}\,g$ on rK, M an upper bound on rK for $|D^3g(c)(u,v,v)|$ over unit vectors u,v, and $\gamma=\gamma(r)$ the gap of Lemma 29. Put

$$\begin{aligned}w&=\tfrac{3}{2}(N\beta)^2(\tau-\lambda_*),\qquad s=\Lambda(r)+(e^w-1)e^w\sqrt{\tau/\gamma},\\ C&=5(N\beta)^2\Big(\tau s+\frac{\sqrt{3}N\beta\,Me^w}{\gamma}\Big),\\ K_3&=\frac{(1+N\beta\Lambda(r))\,s}{1-C},\qquad\alpha_3=10N\beta K_3,\end{aligned}\quad(15)$$

assuming $C<1$.

Lemma 37. *Let $a,c_0\in5N\beta K$ and*

$$\kappa(dx)\propto\exp\big(\mathrm{Re}\,\mathrm{Tr}(xa)+g(c_0+N\beta x)\big)dx\quad\text{on }\mathrm{SU}(3).$$

For every 1-Lipschitz f and every $h\in\mathbb{C}^{3\times3}$:

(i) $\mathrm{Cov}_\kappa(f,\varphi_h)\le s\|h\|_{\mathrm{HS}}$;

(ii) *if c_0 moves in the direction k inside $5N\beta K$, then*

$$\big|\tfrac{d}{dt}\kappa(f)\big|\le N\beta\,(\tau s+\sqrt{3}N\beta Me^w/\gamma)\,\|k\|_{\mathrm{HS}}.$$

Proof. (i) The segment from c_0 to $c_0+N\beta x$ lies in $6N\beta K=rK$, and Taylor's formula gives

$$g(c_0+N\beta x)=g(c_0)+N\beta\,\mathrm{Re}\,\mathrm{Tr}(x\,m(c_0))+R(x)$$

with $R(x)=\frac{1}{2}(N\beta)^2\,\mathrm{Hess}\,g(\xi)(x,x)$. Since $\|x\|_{\mathrm{HS}}^2=3$, $R\in[\frac{3}{2}(N\beta)^2\lambda_*,\frac{3}{2}(N\beta)^2\tau]$ and $\mathrm{osc}\,R\le w$. Hence κ is the tilt $e^R\nu/\nu(e^R)$ of $\nu=\nu_{c'}$ with $c'=a+N\beta m(c_0)\in rK$. By definition of Λ, $\mathrm{Cov}_\nu(f,\varphi_h)\le\Lambda(r)\|h\|$. The function f is 1-Lipschitz for the geodesic distance, which dominates d, so $\mathrm{sd}_\nu f\le\gamma^{-1/2}$ by Lemma 29, and $\mathrm{sd}_\nu\varphi_h\le\sqrt{\tau}\|h\|$. Lemma 13 gives (i).

(ii) The derivative is $\mathrm{Cov}_\kappa(f,\langle\nabla g(c_0+N\beta x),k\rangle)$. Write $\nabla g(c_0+N\beta u)=\nabla g(c_0)+N\beta\,\mathrm{Hess}\,g(c_0)u+N(u)$. The linear term is φ_h with $\|h\|\le N\beta\tau\|k\|$ and contributes at most $N\beta\tau s\|k\|$ by (i). On the segment from c_0 the Hessian moves by at most $N\beta M\|u\|\le\sqrt{3}N\beta M$, so $x\mapsto\langle N(x),k\rangle$ is $\sqrt{3}(N\beta)^2M\|k\|$-Lipschitz for d. The law κ is a tilt of ν by a function of oscillation at most w, so its spectral gap is at least $e^{-w}\gamma$ by the Holley–Stroock lemma [68], and the covariance of two Lipschitz functions under κ is at most e^w/γ times the product of their constants. □

Lemma 38. *Let $C<1$. If b,b' have all fields in $5N\beta K$ and differ only in b_i, then $W_1(\mu_b,\mu_{b'})\le K_3\|b_i-b_i'\|_{\mathrm{HS}}$ for the metric d.*

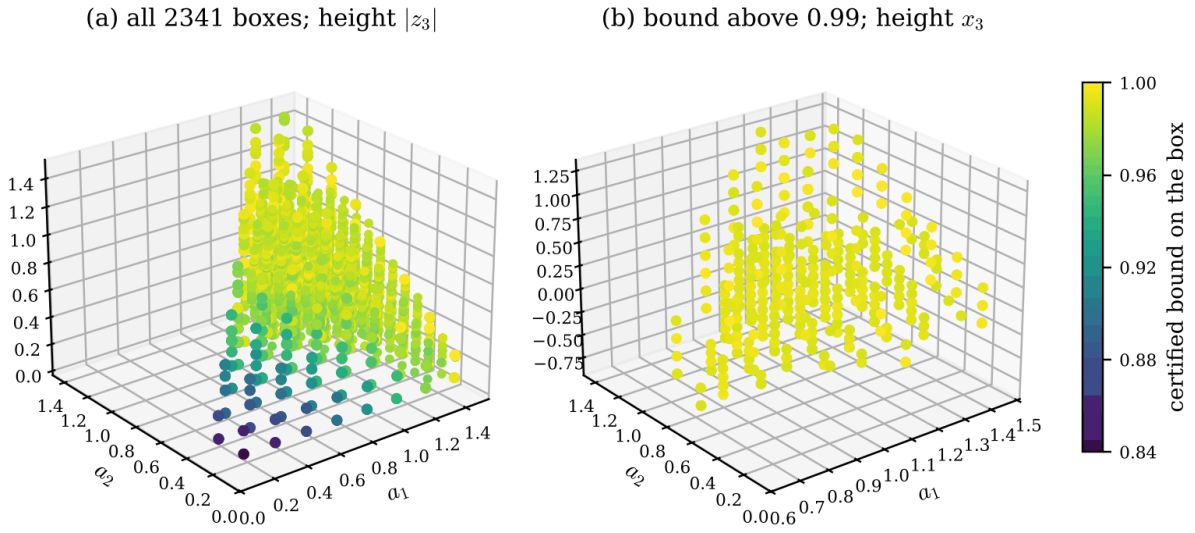


FIG. 6. The 2341 boxes of Remark 36 at $r=\frac{3}{2}$, plotted at their centres in the coordinates (a_1,a_2) of the staple sum $\mathrm{diag}(a_1,a_2,x_3+iy_3)$, coloured by the computed bound on $2r\sup_h L_h$ over the box; the larger points are the boxes of side $r/8$. (a) All boxes, with height $|x_3+iy_3|$. (b) The 419 boxes with bound above 0.99, with height x_3; the largest bounds sit in the interior of $\mathscr{R}_r$, not at its extreme points. No result uses this computation.

Proof. By the cyclic symmetry of $\mathrm{Re}\,\mathrm{Tr}(y_1y_2y_3)$ we may take $i=1$. Given y_1,y_3, the law of y_2 is ν_c with $c=b_2+N\beta y_3y_1\in rK$. Changing y_1 by η in d moves c by $N\beta\eta$ and ν_c by at most $N\beta\Lambda(r)\eta$ in W_1, as in (8); so $F(y_1,y_3)=\mathbb{E}[f\mid y_1,y_3]$ is $(1+N\beta\Lambda(r))$-Lipschitz in each variable. The law of (y_1,y_3) is proportional to $e^{\mathrm{Re}\,\mathrm{Tr}(y_1b_1)+\mathrm{Re}\,\mathrm{Tr}(y_3b_3)}Z(b_2+N\beta y_3y_1)$, and $Z(b_2+N\beta y_3y_1)=Z(y_3^{-1}b_2+N\beta y_1)=Z(b_2y_1^{-1}+N\beta y_3)$, so its two conditional laws have the form of Lemma 37, with $c_0=y_3^{-1}b_2$ and $c_0=b_2y_1^{-1}$. Moving y_3 to y_3' moves $y_3^{-1}b_2$ along a segment in $5N\beta K$ of length at most $\|b_2\|_{\mathrm{op}}\|y_3-y_3'\|\le 5N\beta\|y_3-y_3'\|$. The coupling argument of Lemma 15 gives $d_1\le s\varepsilon+Cd_3$, $d_3\le Cd_1$ and $|\mu_b(f)-\mu_{b'}(f)|\le(1+N\beta\Lambda(r))(d_1+d_3)\le K_3\varepsilon$. □

B. The constants and the proof of Theorem 7

Proof of Theorem 7. Let $r=2\beta_W\le 1.04$ and $\gamma=\gamma(r)$. For $c\in rK$ and unit h, $\mathrm{Hess}\,g(c)(h,h)=\mathrm{Var}_{\nu_c}(\varphi_h)$ lies in $[0,\varepsilon(r)/\gamma]$ by Lemma 31, which also gives $\Lambda(r)\le\sqrt{\varepsilon(r)/\gamma}$; so $\lambda_*=0$ and $\tau=\varepsilon(r)/\gamma$ are admissible. The third derivative $D^3g(c)(u,v,v)$ is the third cumulant $\nu_c[(\varphi_u-\nu_c\varphi_u)(\varphi_v-\nu_c\varphi_v)^2]$. Since U and $m(c)$ lie in K, whose elements have Hilbert–Schmidt norm at most $\sqrt{3}$, $|\varphi_u(U)-\nu_c\varphi_u|=|\mathrm{Re}\,\mathrm{Tr}((U-m(c))u)|\le 2\sqrt{3}$ for unit u, so $|D^3g(c)(u,v,v)|\le 2\sqrt{3}\,\mathrm{Var}_{\nu_c}(\varphi_v)\le 2\sqrt{3}\,\tau=M$. With these constants $\sqrt{\tau/\gamma}=\sqrt{\varepsilon(r)}/\gamma$, the bound used for $\Lambda(r)$, and $\sqrt{3}N\beta Me^w/\gamma=6N\beta\tau e^w/\gamma$. Every field met in Lemmas 37 and 38 lies in rK. Since ε is nondecreasing and γ decreasing in r, every quantity in (15) is nondecreasing in β_W while $C<1$, so it suffices to bound α_3 at $\beta_W=0.52$. Changing an active link j by η changes one staple of each of its ten partners outside its block by a matrix of norm η, so the field b_i of each such partner moves by $N\beta\eta$, and Lemma 38 gives $\sum_B C_{Bj}\le 10N\beta K_3=\alpha_3(\beta_W)$. Proposition 17, whose proof does not use the group, applies with the Hilbert–Schmidt metric.

For the numbers we use $e^w\le(1-w)^{-1}$. At $\beta_W=0.515$, $N\beta=\frac{103}{600}$, and the proof of Theorem 6 gives $\varepsilon(1.03)\le 0.667483<0.817^2$ and $\gamma=\frac{5427}{3200}$; hence $\Lambda(r)\le 0.48174$, $\tau\le 0.39358$, $w\le 0.017398$, $s\le 0.49043$, $C\le 0.064287$, $K_3\le 0.56747$ and $\alpha_3\le 0.9742$. At $\beta_W=0.52$, $N\beta=\frac{13}{75}$ and $\gamma=\frac{5411}{3200}$. The terms of the series for $Z_*(1.04)$ in the proof of Theorem 6 are at most 1, 0, 0.2704, 0.0468694, 0.0365581, 0.0095052, 0.0035697, 0.0009424 and 0.0002581, with sum at most 1.3681029, and the last term is less than $1.31\cdot10^{-4}$; so $Z_*(1.04)\le 1.368234$, $e^{-0.208}\ge 0.812132$, $\frac{5e^{-0.208}}{9Z_*(1.04)}\ge 0.329756$ and $\varepsilon(1.04)\le 0.670244<0.81869^2$. Hence $\Lambda(r)\le 0.48417$, $\tau\le 0.39638$, $w\le 0.017864$, $s\le 0.49314$, $C\le 0.066654$, $K_3\le 0.5727$ and $\alpha_3\le 0.9927$. With $y=1-\alpha_3$, $-\frac{1}{2}\log\alpha_3\ge\frac{1}{2}(y+\frac{y^2}{2})$, which is at least 0.013 at $\beta_W=0.515$ and 0.0036 at 0.52. □

Remark 39 (a computation, used by no theorem). The single-link criterion ends where $2r\Lambda(r)$ reaches one. Before the corrections for the three links the block sum is $\frac{10}{12}$ of the single-link sum $2r\Lambda(r)$, so it can stay below one while $2r\Lambda(r)$ is slightly above one. With the constants of Lemma 31 the corrections use most of that room: evaluated in floating point, α_3 of Theorem 7

reaches one near $\beta_W = 0.522$, and the bound α_1 of Theorem 6 near 0.5176.

Computed constants go further. For $V, W \in \mathrm{SU}(3)$, $g(VcW) = g(c)$ and $x \mapsto VxW$ is an isometry, so the eigenvalues of $\mathrm{Hess}\, g$ and the norm of $D^3 g$ are constant on the orbits, and by Lemma 33 it suffices to bound them on $\mathscr{R}_r$. The derivatives of g are the cumulants of φ under ν_c. Cut the box $[0,r]^2 \times [-r,r] \times [0,r]$ into boxes of side $r/8$ and keep the 841 (784 at $r = \frac{38}{25}$, 826 at $r = \frac{77}{50}$) that survive the two tests of Sec. VII H. On each box Q we compute, with rigorous error bounds, the moments of φ of degree at most three at the sixteen corners b_i and at the centre b_0, and put $k = m(b_0)$, the mean of φ at the centre.

Upper bound. For $b \in Q$ and every h, $\mathrm{Var}_{\nu_b}(\varphi_h) \le \nu_b[(\varphi_h - \langle k,h\rangle)^2] \le e^{\varsigma/8} \max_i \nu_{b_i}[(\varphi_h - \langle k,h\rangle)^2]$ by Lemma 35(ii).

Lower bound. Integrating the right inequality of (12) against $f \ge 0$ and dividing by $Z(b) \le \sum_i \omega_i Z(b_i)$, the left one, gives $\nu_b(f) \ge e^{-\varsigma/8} \min_i \nu_{b_i}(f)$. With $f = (\varphi_h - \nu_b\varphi_h)^2$ this gives $\mathrm{Var}_{\nu_b}(\varphi_h) \ge e^{-\varsigma/8} \min_i \mathrm{Var}_{\nu_{b_i}}(\varphi_h)$.

Third cumulant. Put $\delta = m(b) - k$. The mean value theorem and the upper bound give $\|\delta\| \le \tau\rho + e$, with ρ the largest distance from b_0 in Q and e the error of the computed mean. Expanding the central moment about k,

$$D^3 g(b)(u,v,v) = T_b(u,v,v) - \delta_u \mathrm{Var}(\varphi_v) - 2\delta_v \mathrm{Cov}(\varphi_u,\varphi_v) - \delta_u \delta_v^2,$$

with $T_b = \nu_b[(\varphi - k)^{\otimes 3}]$, $\delta_u = \langle \delta, u\rangle$ and the moments under ν_b. For a function G of either sign, writing $G = G^+ - G^-$ and applying the two bounds above to $G^\pm$ gives $|\nu_b(G)| \le (\max_i |\nu_{b_i}(G)| + \varepsilon \max_i \nu_{b_i}|G|)/(1-\varepsilon)$ with $\varepsilon = 1 - e^{-\varsigma/8}$. For $G = (\varphi_u - k_u)(\varphi_v - k_v)^2$ we use $|\nu_{b_i}(G)| \le \|T_{b_i}\|$, the largest singular value of the 18×324 unfolding of T_{b_i}, and $\nu_{b_i}|G| \le 2\sqrt{3}\, \nu_{b_i}[(\varphi_v - k_v)^2]$, since $\|U - k\|_{\mathrm{HS}} \le 2\sqrt{3}$. The largest eigenvalues and singular values are bounded as in Sec. VII H, and the maxima over these boxes are given below (Appendix C). In this way every eigenvalue of $\mathrm{Hess}\, g$ on rK lies in $[0.056015, 0.247316]$ at $r = \frac{3}{2}$ and $|D^3 g(c)(u,v,v)| \le 0.182126$ for unit u, v; at $r = \frac{38}{25}$ the numbers are $[0.054978, 0.248280]$ and 0.183546, and at $r = \frac{77}{50}$ they are $[0.053955, 0.249240]$ and 0.184989. With these constants and the bounds for $2r\Lambda(r)$ of Remark 36, the quantities of (15) evaluated in ball arithmetic are, at $\beta_W = \frac{3}{4}$, $w \le 0.017935$, $s \le 0.34088$, $C \le 0.043520$, $K_3 \le 0.38609$ and $\alpha_3 \le 0.96521$; at $\beta_W = 0.72$, $\alpha_3 \le 0.91815$; at $\beta_W = 0.76$, $w \le 0.018609$, $s \le 0.34673$, $C \le 0.045770$, $K_3 \le 0.39454$ and $\alpha_3 \le 0.99950$; at $\beta_W = 0.77$, $w \le 0.019298$, $s \le 0.34186$, $C \le 0.047232$, $K_3 \le 0.38953$ and $\alpha_3 \le 0.99978$. They would carry the block criterion to $\beta_W = 0.77$, with a gap of 0.0177 at $\frac{3}{4}$, where the cover of Remark 36 alone gives $1.1 \cdot 10^{-4}$. Evaluated in floating point with the constants of a scan of $\mathscr{R}_r$, the block sum stays below one up to $\beta_W \approx 0.78$. No result of the paper uses these numbers.

IX. WHERE THE METHODS STOP

SU(2). Theorem 2 closes the single-link route in every forest gauge at $\beta_W = \frac{1}{3}$. Blocks of several links, where the influence of a boundary link on a whole block is bounded at once, in the spirit of the Dobrushin–Shlosman criterion [44], pass it: plaquette blocks reach $\beta_W = 0.348$ (Theorem 3), and the floating-point bound of Remark 18, which no result uses, stops near 0.357. Larger blocks are the next step.

Curvature. Remark 26 locates the end of the curvature method in this gauge between $|\beta| = \frac{1}{29}$ and $\frac{1}{26.25}$; the computation of Remark 695, which no result uses, moves the first value to $1/28.840002$.

SU(3). Theorems 6 and 7 use the bound of Lemma 31, whose losses are Jensen's inequality and the Poincaré inequality for φ_h; evaluated in floating point, it keeps α_1 below one up to $\beta_W \approx 0.5176$ and α_3 up to 0.522. The route of Sec. VII with a corrector bounds the $H^{-1}(\nu_b)$ norm of $\varphi_h - \nu_b\varphi_h$, which is at least $L_h(b)$, and needs it uniformly on rK. Its Galerkin values (polynomials of degree two) at matrices of rK give $2r$ times the norm equal to 0.79 at $r = 1.25$, 0.89 at $r = 1.4$, 0.97 at $r = 1.5$ and 1.009 at $r = 1.55$, so the route stops at or below $\beta_W \approx 0.77$ even with exact constants. With $u = s\varphi_h$, Lemma 32 stops earlier: in floating point its largest values on $\mathscr{R}_r$ lie on the face $\mathrm{diag}(r,\rho,\rho)$ with ρ between $0.75r$ and $0.8r$, and $2r$ times the bound is 0.9458 at $r = 1.36$, 0.9767 at $r = 1.40$ and 1.0080 at $r = 1.44$, so this corrector ends near $\beta_W \approx 0.715$. With $u = q_h$ of degree two the largest values on the same face are 0.9305 at $r = 1.44$, 0.9729 at $r = 1.50$ and 1.0018 at $r = 1.54$ (Fig. 7), close to the Galerkin values. Lemma 29 therefore costs little at these couplings, and most of the distance between the floating-point value at $r = 1.5$ and the computed 0.999887 of Remark 36 is the interpolation over boxes. With this corrector the single-link bound would reach $\frac{3}{4}$ and the blocks 0.77 (Remarks 36 and 39), but these evaluations are computations and no result uses them; a bound for the corrector of degree two that holds on all of rK and is derived by hand is the missing step. Past 0.78 one needs larger blocks, whose sensitivity does not go through the one-link modulus.

Other groups. For groups other than SU(2) and SU(3) only the curvature route is carried out here (Theorem 5), and it stops where the bound $\frac{29}{2}$ uses up the curvature of G, at $N|\beta| = \frac{2}{29}\kappa_G$. The Dobrushin route needs a bound for the Kantorovich modulus of the one-link measure of each group, which is not attempted here.

The four open problems. Our earlier paper [20] listed four problems between the strong-coupling results and the Clay problem [73]: (1) uniqueness and a gap at intermediate fixed coupling; (2) the crossover from strong coupling to Bałaban's weak-coupling regime; (3) the continuum limit with the Osterwalder–Schrader axioms; and (4) a gap in physical units. The present paper moves the boundary of (1): for SU(2) from $\beta_W = \frac{2}{9}$ to 0.348, for SU(3) from $\frac{9}{32}$ to 0.52, with a gap at $\frac{1}{2}$ of 0.052 in lattice units, and for every compact simple group, the exceptional groups included, past the curvature ceiling of the ungauged measure, with an explicit gap. It does not close (1); the region between these values and the scaling re-

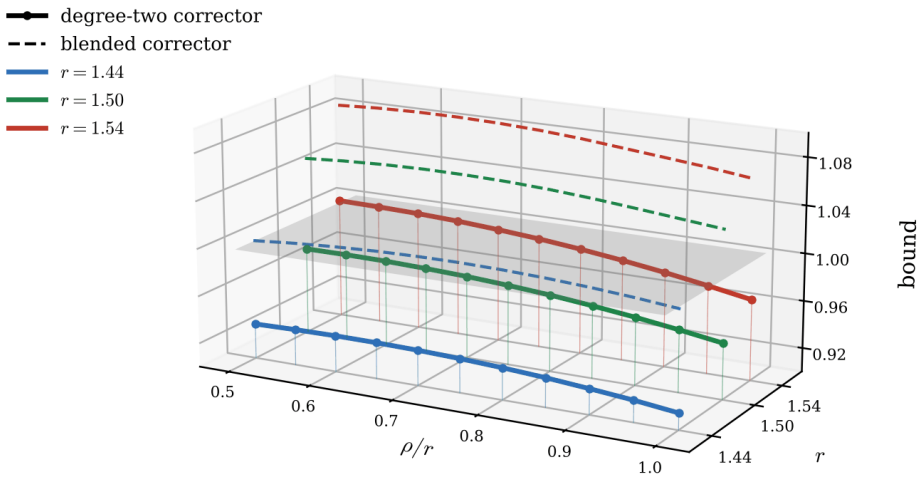


FIG. 7. The floating-point value of $2r\sup_h$ of the bound of Lemma 32 on the face $b = \mathrm{diag}(r,\rho,\rho)$, as a function of ρ/r, for $r = 1.44$, 1.50 and 1.54: dashed with the corrector $u = s\varphi_h$ for the best s, solid with $u = q_h$ of degree two for the best C. The grey plane is the Dobrushin threshold one. The first corrector is above it already at $r = 1.44$; the second stays below it at $r = 1.50$, where the computation of Remark 36 bounds it on all of $\mathscr{R}_r$, and crosses it by $r = 1.54$.

gion, near $\beta_W \approx 2.2$ for SU(2) and $\beta_W \approx 6$ for SU(3), is open, and with it problem (2). Problems (3) and (4) are not solved here. Every bound proved here is uniform in lattice units, so by [21] none of them can by itself produce a continuum mass gap; that requires control of the gap in physical units along the renormalisation group flow [6, 23].[16]

The continuum problem. Two general facts limit what can be imported into problems (3) and (4) above from constructions of continuum measures. Both concern a random distribution F on $\mathbb{R}^d$, a Borel probability measure on the tempered distributions with finite moments, and the growth condition

$$\mathbb{E}\, e^{F(\chi)} \le e^{C(\|\chi\|_1 + \|\chi\|_2^2)} \qquad \text{for all real } \chi \in \mathscr{S}(\mathbb{R}^d), \quad (16)$$

which is the growth condition of the regularity axiom of Glimm and Jaffe at $p = 2$ [60, Sect. 6.1, (6.1.5)]. The next proposition shows that it rules out every field whose two-point function is too singular at short distance.

Proposition 40. *Let F satisfy (16), and suppose that its connected two-point function is, off the diagonal, a locally integrable function $S_2(x,y)$ with $S_2(x,y) \ge c|x-y|^{-2\Delta}\big(\log \frac{1}{|x-y|}\big)^{-k}$ for $0 < |x-y| \le r_0$, where $c > 0$, $k \ge 0$ and $2\Delta > d$. Then F does not exist: no random distribution has both properties.*

Proof. Applying (16) to χ and $-\chi$ and using $\cosh x \ge 1 + \frac{x^2}{2}$ gives $\mathbb{E}F(\chi)^2 \le 2(e^{C(\|\chi\|_1+\|\chi\|_2^2)} - 1)$. Fix nonnegative $\varphi, \psi \in C_c^\infty(\mathbb{R}^d)$, not zero, with supports in a ball of radius $\frac{R}{2}$ and at distance $\delta > 0$ from each other, and put $\varphi_\varepsilon(x) = \varepsilon^{-d/2}\varphi(x/\varepsilon)$, and likewise ψ_ε. Then $\|\varphi_\varepsilon\|_2 = \|\varphi\|_2$ and $\|\varphi_\varepsilon\|_1 = \varepsilon^{d/2}\|\varphi\|_1 \le \|\varphi\|_1$ for $\varepsilon \le 1$, so the covariance of $F(\varphi_\varepsilon)$ and $F(\psi_\varepsilon)$ is bounded uniformly in ε by the Cauchy–Schwarz inequality. On the other hand the supports of φ_ε and ψ_ε are disjoint, and for u, v in the supports $\varepsilon\delta \le |\varepsilon u - \varepsilon v| \le \varepsilon R$. For $\varepsilon R \le r_0$ and $\varepsilon R < 1$ the lower bound for S_2 therefore gives

$$\begin{aligned}\mathrm{Cov}\big(F(\varphi_\varepsilon), F(\psi_\varepsilon)\big) &= \varepsilon^d \iint \varphi(u)\psi(v) S_2(\varepsilon u, \varepsilon v)\, du\, dv \\ &\ge c R^{-2\Delta} \varepsilon^{d-2\Delta} \Big(\log \frac{1}{\varepsilon\delta}\Big)^{-k} \|\varphi\|_1 \|\psi\|_1,\end{aligned}$$

which tends to infinity as $\varepsilon \to 0$. □

The action density $\mathrm{Tr}\, F_{\mu\nu}F_{\mu\nu}$ has dimension $\Delta = 4$ in four dimensions. A two-sided short-distance law of the form $|x-y|^{-8}$ times the fourth power of the running coupling, which is what asymptotic freedom[17] predicts, has $2\Delta = 8 > 4$ and $k = 2$; so that field cannot satisfy (16), and a proof of rotation invariance for it has to use something other than this condition. In three dimensions the free part of the action density already

[16] Every one of these approaches needs an input at physical length scales. Renormalisation group approaches to the ultraviolet behaviour of gauge and gravitational theories, of which [6, 40] are the rigorous ones for Yang–Mills theory, have also been proposed for the effective dimension of space-time [103] and for completions of quantum gravity [16, 22]; they do not bear on the infrared estimates needed here.

[17] For SU(N) the renormalisation group gives [63, 99], with $b_0 = 11N/(48\pi^2)$ and the invariant scale Λ,

$$g(a)^{-2} = 2b_0 \log \tfrac{1}{a\Lambda} + O\big(\log\log \tfrac{1}{a\Lambda}\big),$$

so the inverse coupling $\beta_W = 2N/g(a)^2$ grows like $\frac{11N^2}{12\pi^2} \log \frac{1}{a\Lambda}$ along the continuum tuning.

has $2\Delta = 6 > 3$. Condition (16) suits fields of dimension at most $\frac{d}{2}$, such as the field strength of the free Maxwell field, whose covariance is bounded in momentum space.

Positivity, symmetry and regularity do not produce a gap by themselves.

Proposition 41. *On $\mathbb{R}^4$ let F be the centred Gaussian random distribution with covariance*

$$\widehat{C}(p) = \int_0^1 \frac{m^2}{|p|^2 + m^2}\, dm.$$

Then F is invariant under all Euclidean motions, reflection positive, satisfies (16) with $C = \frac{1}{2}$, and its Osterwalder–Schrader Hamiltonian has no gap above the vacuum.

Proof. Since $0 \le \widehat{C} \le 1$, $\mathbb{E}e^{F(\chi)} = e^{\frac{1}{2}\langle \chi, C\chi\rangle} \le e^{\frac{1}{2}\|\chi\|_2^2}$, with the unitary Fourier transform. Invariance is clear. The covariance is the integral over $m \in (0,1)$, with the weight $m^2 dm$, of the covariances of free fields of mass m, each of which is reflection positive [60, Sect. 6.2]; a positive combination of reflection positive covariances is reflection positive, and so is the Gaussian measure with such a covariance [60, Sect. 6.2]. For $\chi(t,\mathbf{x}) = h(t)g(\mathbf{x})$ with $h \ge 0$ supported in $t > 0$, not zero, and $\hat{g}(0) \ne 0$, the vector $v = F(\chi)$ is orthogonal to the vacuum, and after the Fourier transform in $\mathbf{x}$,

$$\langle v, e^{-sH} v\rangle = \int_0^1 m^2\, dm \int \frac{|\hat{g}(\mathbf{k})|^2}{2\omega} e^{-\omega s} \times \Big(\int_0^\infty h(t) e^{-\omega t} dt\Big)^2 d\mathbf{k},$$

with $\omega = \sqrt{|\mathbf{k}|^2 + m^2}$ and constants absorbed in $\hat{g}$. If H had a gap $m_0 > 0$, the left side would be at most $e^{-m_0 s}\|v\|^2$. For $0 < \eta < m_0$ the part of the integral with $m, |\mathbf{k}| \le \eta/2$ is at least $c_\eta e^{-\eta s}$ with $c_\eta > 0$, a contradiction as $s \to \infty$. □

The Gaussian field of Proposition 41 is not a gauge theory; it shows only that an infrared estimate, such as a bound $\langle v_a, T_a^{\lfloor t/a\rfloor} v_a\rangle \le C_v e^{-m_* t}$ uniform in the spacing a along the continuum tuning, has to be proved in addition to the axioms. The lattice bounds of this paper are of this form only with m_* proportional to $1/a$, and by [21] such bounds give trivial limits. The routes we know to a continuum gap therefore all stop at the same two estimates: the construction of the infinite-volume continuum measure past Bałaban's small-field analysis [6, 12], and a decay estimate in physical units at large scales.

X. THE CONTINUUM STEP AT ONE PHYSICAL SCALE

The bounds of the preceding sections hold at fixed coupling and give a gap that is at least a fixed number in lattice units, and [21] shows that no such bound can produce a continuum gap. This section replaces the single-link condition of Proposition 11 by a condition on one cube of n^4 sites, averaged over its translates in the manner of Dobrushin and Shlosman [44], and proves that the condition gives a gap of order $1/n$ in lattice units. If it holds with n proportional to $1/a$, that is, in a cube of fixed physical size, the gap survives in physical units and passes to every continuum limit (Theorem 45). The two statements that remain unproved, the cube estimate and the existence of the limit, are stated in Sec. XI.

A. Averaged cubes

For a finite set $V \subset A$ let $\gamma_V(\cdot \mid Q)$ be the probability measure on G^V with density proportional to $\exp\big(N\beta \sum_{p \cap V \ne \emptyset} \mathrm{Re}\,\mathrm{Tr}\, Q_p^F\big)$ with respect to Haar measure, the links outside V being those of Q. Its conditional law of Q_e, $e \in V$, given the other links of V is γ_e^F. Write $d_V(Q_V, Q'_V) = \sum_{i \in V} \delta(Q_i, Q'_i)$ and, for $j \in A \setminus V$,

$$C_{V,j} = \sup \frac{W_1^{d_V}\big(\gamma_V(\cdot \mid Q), \gamma_V(\cdot \mid Q')\big)}{\delta(Q_j, Q'_j)},$$

the supremum over pairs that differ only at j; $W_1^{d_V}$ is the Kantorovich distance for d_V. The coefficient vanishes unless j shares a plaquette with a link of V.

Let $n \ge 2$ be even and let V be the set of active links (x,μ) with $x \in [0,n)^4$. The direction field has period two, so the translations by $2\mathbb{Z}^4$ preserve F, A and γ^F, and the translates $V_x = V + x$, $x \in 2\mathbb{Z}^4$, satisfy $C_{V_x, j+x} = C_{V,j}$. Every active link lies in exactly $N_n = (n/2)^4$ of them. A *class* is an orbit of $2\mathbb{Z}^4$ on A, that is, a residue of the base point modulo two together with a direction; there are at most 64 classes. For a class c put

$$K_c = \sum_{j \in c \setminus V} C_{V,j}.$$

Definition 42. For $0 < \alpha < 1$, the condition $D(n,\alpha)$ at the coupling β is $K_c \le \alpha N_n$ for every class c.

Since $\sum_{x \in 2\mathbb{Z}^4,\, j \notin V_x} C_{V_x, j} = \sum_x C_{V, j-x}[j - x \notin V] = K_{c(j)}$ for the class $c(j)$ of j, the condition says that the total influence of a link on the translates that do not contain it is at most the fraction α of the number of translates that do. Figure 8 shows the cubes.

For $n = 2$ it is a bound on blocks of 48 links. Distances between links are those of their midpoints; a link of V and a link sharing a plaquette with V are at distance at most $\rho_n = 2n$, and c_n denotes the number of active links within distance ρ_n of a given one.

Theorem 43. *Let $\beta \ge 0$ and suppose that $D(n,\alpha)$ holds. Then statements* (i)–(iv) *of Proposition 11 hold,* (v) *holds for every $\lambda < \log(1/\alpha)/\rho_n$, and the transfer matrix of ω has spectral gap at least* $\log(1/\alpha)/(2n)$.

Proof. The proof of Proposition 11 uses the coefficients $C_{ee'}$ only through Lemma 665 and Corollary 666. Lemma 668 of Appendix B gives the same conclusions under $D(n,\alpha)$, for every $\lambda > 0$ with $\alpha e^{\lambda\rho_n} < 1$, with the constant $\theta D/(1-\theta)$ replaced by $D(1 + c_n e^{\lambda\rho_n})/(1-\theta)$ and $\theta = \alpha e^{\lambda\rho_n}$. The measures used there are invariant under every γ_{V_x} with V_x disjoint from the exceptional set: a Gibbs measure of γ^F and its conditional measures given the links of a finite set by Lemma 668(a), and the extended finite-volume measure of

(a) averaged cubes, $n = 4$ (b) one physical scale

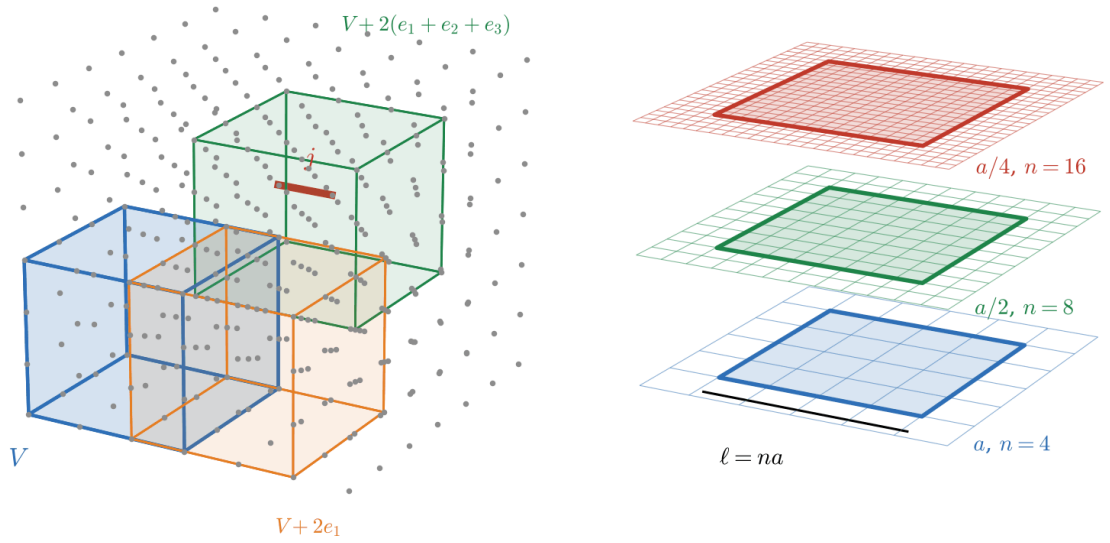


FIG. 8. (a) The cube V of side $n = 4$ and two of its translates by $2\mathbb{Z}^4$, drawn in three of the four dimensions. A link j outside V enters K_c through every translate whose boundary it touches, and $D(n,\alpha)$ compares that total with the $N_n = (n/2)^4$ translates that contain j. (b) The cube of Theorem 45: its physical side $\ell = na$ is held fixed while the spacing is halved, so n doubles at each step and the lattice gap $\log(1/\alpha)/(2n)$ is of order a/ℓ.

part (ii) because every plaquette that meets such a V_x lies in $P(\Lambda)$. Part (vi) then follows from Proposition 669 as before, with every $\lambda < \log(1/\alpha)/(2n)$. □

B. The continuum limit

Let $a_k = 2^{-k}a_0$ and let $\beta_k \geq 0$ be couplings, typically with $\beta_k \to \infty$. Suppose that the lattice states ω_k at β_k exist as in Proposition 11, with transfer matrices $\mathbb{T}_k$ on the reconstructed spaces $\mathscr{H}_k$ and vacua Ω_k (Appendix F). Let $\mathscr{T}$ be a complex vector space of test functions on $\mathbb{R}^4$ supported in $\{t > 0\}$ and, for each k, let $f \mapsto O_k(f)$ be a linear map from $\mathscr{T}$ to gauge-invariant local observables supported in positive lattice times; the example to keep in mind is $O_k(f) = Z_k \sum_x a_k^4 f(a_k x)\Phi(x)$ for a local gauge-invariant field Φ, a renormalisation constant Z_k and $\Phi(x)$ the translate of Φ to x. Write ϑ for the time reflection and τ_t for the time translation by t lattice steps.

Definition 44. A *continuum limit* of (ω_k, O_k) is a Hilbert space $\mathscr{H}$ with a unit vector Ω, a self-adjoint operator $H \geq 0$ with $H\Omega = 0$, and a linear map $f \mapsto \psi(f)$ from $\mathscr{T}$ to $\mathscr{H}$ with dense range, such that for all $f, g \in \mathscr{T}$ and all dyadic $\tau \geq 0$,

$$\langle \psi(f), e^{-\tau H}\psi(g)\rangle = \lim_{k\to\infty} \omega_k\big(\vartheta(\overline{O_k(f)})\,\tau_{\tau/a_k}O_k(g)\big),$$
$$\langle \Omega, \psi(f)\rangle = \lim_{k\to\infty} \omega_k\big(O_k(f)\big),$$

the limits taken over the k for which τ/a_k is an integer. It is *nontrivial* if $\mathscr{H} \neq \mathbb{C}\Omega$.

The Osterwalder–Schrader reconstruction [95, 96] produces such a limit from convergent renormalised Schwinger functions that satisfy the axioms; the definition records only what the next theorem uses.

Theorem 45. *Suppose that there are $\alpha < 1$, $\ell > 0$ and even n_k with $n_k a_k \leq \ell$ such that $D(n_k, \alpha)$ holds at β_k for every k. Then every continuum limit of (ω_k, O_k) satisfies*

$$\sigma(H) \subset \{0\} \cup [m_*, \infty), \qquad m_* = \frac{\log(1/\alpha)}{2\ell},$$

and 0 is a simple eigenvalue. If the limit is nontrivial, $\sigma(H) \cap [m_*, \infty) \neq \emptyset$, so H has a mass gap of at least m_*.

Proof. By Theorem 43 the state ω_k exists and $\mathbb{T}_k$ has gap at least $\log(1/\alpha)/(2n_k)$ on the complement of Ω_k. Let $f \in \mathscr{T}$, put $O = O_k(f)$ and $\varphi_k = \Psi_O - \omega_k(O)\Omega_k$. In the reconstruction of Appendix F, $\langle \Psi_F, \mathbb{T}_k^t \Psi_H\rangle = \omega_k(\vartheta(\bar{F})\tau_t H)$, so $\langle \Omega_k, \Psi_O\rangle = \omega_k(O)$, and $\mathbb{T}_k\Omega_k = \Omega_k$; hence, for integers $t \geq 0$,

$$\langle \varphi_k, \mathbb{T}_k^t \varphi_k\rangle = \omega_k\big(\vartheta(\bar{O})\,\tau_t O\big) - |\omega_k(O)|^2$$
$$\leq e^{-t\log(1/\alpha)/(2n_k)}\,\|\varphi_k\|^2.$$

Take $\tau \geq 0$ dyadic and k large, so that $t = \tau/a_k$ is an integer; then $t/(2n_k) = \tau/(2n_k a_k) \geq \tau/(2\ell)$, and the right side is at most $e^{-m_*\tau}\|\varphi_k\|^2$. Both sides converge by Definition 44. With $\varphi = \psi(f) - \langle \Omega, \psi(f)\rangle\Omega$ and $e^{-\tau H}\Omega = \Omega$ the limit reads

$$\langle \varphi, e^{-\tau H}\varphi\rangle \leq e^{-m_*\tau}\|\varphi\|^2.$$

The maps $f \mapsto O_k(f)$ and $f \mapsto \psi(f)$ are linear, so the vectors φ form a linear subspace, and it is dense in $\Omega^\perp$ because the range of ψ is dense. The inequality extends to $\Omega^\perp$ by continuity, and from dyadic τ to all $\tau \geq 0$ by the strong continuity of $e^{-\tau H}$. For $\varphi \in \Omega^\perp$ with spectral measure μ_φ of H this says

$\int e^{-\tau E}d\mu_\varphi(E) \le e^{-m_*\tau}\mu_\varphi([0,\infty))$ for all $\tau \ge 0$; if $\mu_\varphi([0,m_*-\eta]) > 0$ for some $\eta > 0$, the left side is at least $\mu_\varphi([0,m_*-\eta])e^{-(m_*-\eta)\tau}$, which exceeds the right side for large τ. So μ_φ lives on $[m_*,\infty)$ for every $\varphi \in \Omega^\perp$, which gives both claims about 0, and the spectrum on $\Omega^\perp \neq \{0\}$ lies in $[m_*,\infty)$ and is not empty. □

The proof uses $D(n_k,\alpha)$ only through the decay of the two-point functions, and the same argument accepts any decay with constants that depend on the observable.

Theorem 46. *Let $m_0 > 0$ and let $\mathscr{T}_0$ span $\mathscr{T}$. Suppose that for each $f \in \mathscr{T}_0$ there are $C_f < \infty$, $\tau_f \ge 0$ and k_f such that*

$$C_k^f(\tau) = \omega_k\big(\vartheta(\overline{O_k(f)})\,\tau_{\tau/a_k}O_k(f)\big) - |\omega_k(O_k(f))|^2 \le C_f e^{-m_0\tau}$$

for all $k \ge k_f$ and all dyadic $\tau \ge \tau_f$ with $\tau/a_k \in \mathbb{Z}$. Then every continuum limit of (ω_k, O_k) satisfies $\sigma(H) \subset \{0\} \cup [m_0,\infty)$, and 0 is a simple eigenvalue. If the limit is nontrivial, H has a mass gap of at least m_0.

Proof. For $f \in \mathscr{T}$ put $\varphi(f) = \psi(f) - \langle\Omega,\psi(f)\rangle\Omega$, which is linear in f. Since $e^{-\tau H}$ is self-adjoint and fixes Ω, Definition 44 gives $\langle\varphi(f), e^{-\tau H}\varphi(f)\rangle = \lim_k C_k^f(\tau)$. For $f \in \mathscr{T}_0$ let μ be the spectral measure of H at $\varphi(f)$. Then $\int e^{-\tau E}d\mu(E) \le C_f e^{-m_0\tau}$ for dyadic $\tau \ge \tau_f$. If $\mu([0,m_0-\eta]) > 0$ for some $\eta \in (0,m_0)$, the left side is at least $\mu([0,m_0-\eta])e^{-(m_0-\eta)\tau}$, which is impossible for large τ. Hence the spectral projection $E_{[0,m_0)}$ of H vanishes on $\varphi(\mathscr{T}_0)$. By linearity it vanishes on $\varphi(\mathscr{T})$, and since its kernel is closed, it vanishes on the closure of $\varphi(\mathscr{T})$, which is $\Omega^\perp$ because $\psi(\mathscr{T})$ is dense. So $E_{[0,m_0)} = \langle\Omega,\cdot\rangle\Omega$, which gives both claims. □

Theorem 45 is the case $m_0 = m_*$, $\tau_f = 0$ and $C_f = \sup_{k\ge k_f}\|\varphi_k\|^2$, which is finite because $\|\varphi_k\|^2 = C_k^f(0)$ converges. Theorem 46 needs neither a lattice gap nor the gauge of this paper. It applies to the output of a cluster expansion of any coarse effective theory, provided the expansion bounds the correlations of a spanning set of renormalised observables in physical units.

Theorem 45 bypasses the obstruction of [21]: under the hypothesis the lattice gap is at least $\log(1/\alpha)/(2n_k)$, of order a_k/ℓ in lattice units, which tends to zero, and only its value in physical units is bounded below. In the language of [21] the hypothesis implies the gap part (R2) of Hypothesis R, with $m = m_*$, and the conclusion holds for every continuum limit without a separate assumption on the gap.

C. Smoothed observables

The observables $O_k(f)$ of Definition 44 need not be local. The renormalised fields that one expects to converge are smoothed by the gradient flow of the Wilson action [84][18]; at positive flow time their correlation functions are finite to all orders of perturbation theory once the coupling is renormalised [85]. Flow in all four directions mixes the two sides of the reflection plane, but the spatial flow of [104, Sec. II.B] does not: it flows the links of each time slice under the Wilson action of that slice and leaves the temporal links alone.

Fix a bi-invariant Riemannian metric on G. For a configuration U of all links of $\mathbb{Z}^4$ (for the states of this paper, the links of F equal to 1), let $\Phi_s(U)$, $s \in \mathbb{R}$, solve

$$\frac{d}{ds}U_e = -\nabla_e \sum_{p\ni e,\ p \text{ spatial}} \big(N - \mathrm{Re}\,\mathrm{Tr}\,U_p\big)$$

for every spatial link e, the sum running over the four plaquettes of the time slice of e that contain it, with the temporal links held fixed and ∇_e the gradient in the variable U_e.

Proposition 47. (i) *Φ_s is defined on every configuration of the infinite lattice, $\Phi_s \circ \Phi_{s'} = \Phi_{s+s'}$, and Φ_s commutes with gauge transformations, with spatial and temporal translations and with the reflections in time. The link $\Phi_s(U)_e$ depends only on the spatial links of the time slice of e, and there are constants κ, c depending only on G such that, if Φ_s^R denotes the flow of the links of that slice within distance R of e with all other links held fixed,*

$$\delta\big(\Phi_s(U)_e, \Phi_s^R(U)_e\big) \le D e^{\kappa|s|}\frac{(c|s|)^{R+1}}{(R+1)!}$$

for every configuration U.

(ii) *Let ω be a state as in Appendix F and O a bounded gauge-invariant function of finitely many links at times in $[t_0,t_1]$, continuous in them. Then $O\circ\Phi_s$ is a bounded gauge-invariant function of the links at times in $[t_0,t_1]$ and a uniform limit of gauge-invariant local functions at those times.*

(iii) *Theorem 45 holds with $O_k(f)$ replaced by $O_k(f)\circ\Phi_{s_k}$ for any $s_k \in \mathbb{R}$, in particular for $s_k = \tau/a_k^2$ at a fixed flow time τ in physical units.*

(iv) *The image measure $(\Phi_s)_*\omega$ is reflection positive, and its transfer matrix is unitarily equivalent to that of ω, the unitary fixing the vacuum.*

Proof. (i) Embed G in a matrix algebra. The right side of the equation is a smooth function of the links of the four plaquettes containing e, tangent to G at U_e, so on the Banach space of bounded assignments of matrices to the spatial links it is a vector field that is bounded and Lipschitz on a neighbourhood of the configurations with values in G, with a Lipschitz constant that does not depend on the volume because every link meets finitely many others. The Picard–Lindelöf theorem gives a unique local solution; it stays on G because the field is tangent there, and since the field is bounded on G the solution exists for all s; uniqueness gives the group law. The action of a time slice and the metric are invariant under gauge transformations,

[18] The flow moves the gauge field along the negative gradient of the Wilson action. At positive flow time the flowed fields are smooth on the scale of the square root of the flow time, and in perturbation theory their gauge-invariant composite fields need no further renormalisation [84, 85].

translations and the reflections, so the flow commutes with them by uniqueness. Since the equation of a spatial link involves only links of its own slice, so does the solution. For the comparison, let $d_{e'}(s)$ be the distance between the two flows at a link e' of the slice. Smoothness on the compact group gives constants κ_0, c_1 with $d_{e'}(s) \le \int_0^{|s|} \big(\kappa_0 d_{e'} + c_1 \sum_{e''\sim e'} d_{e''}\big)$ for e' within distance R of e, where $e'' \sim e'$ means that the two links share a spatial plaquette (twelve neighbours), while $d_{e'} \le D$ beyond. Iterating the integral form of this inequality, a term that reaches the links beyond distance R needs at least $R+1$ steps of $\sim$, and $\sum_{k>R} x^k/k! \le e^x x^{R+1}/(R+1)!$; this gives the bound with $c = 12c_1$ and $\kappa = \kappa_0 + c$.

(ii) The truncated flow Φ_s^R commutes with gauge transformations for the same reason as Φ_s, so $O \circ \Phi_s^R$ is a gauge-invariant function of finitely many links at times in $[t_0, t_1]$. By (i) and the uniform continuity of O, it converges to $O \circ \Phi_s$ uniformly as $R \to \infty$.

(iii) The proof of Theorem 45 uses $O_k(f)$ only through the vectors $\Psi_{O_k(f)}$ and the identity $\langle \Psi_F, \mathbb{T}^t \Psi_H \rangle = \omega(\vartheta(\bar F)\tau_t H)$. By (ii) and the footnote in Appendix F, both extend to uniform limits of local functions at positive times, and $f \mapsto O_k(f) \circ \Phi_{s_k}$ is linear.

(iv) By (i), Φ_s and Φ_{-s} preserve the functions of the links at positive times and commute with ϑ and τ_t. Hence $(\Phi_s)_* \omega\big(\vartheta(\bar F)\tau_t H\big) = \omega\big(\vartheta(\overline{F \circ \Phi_s})\,\tau_t(H \circ \Phi_s)\big)$, which for $t = 0$ and $H = F$ is nonnegative, and $F \mapsto F \circ \Phi_s$ induces an isometry between the reconstructed spaces that intertwines the transfer matrices and fixes the vacuum; Φ_{-s} gives its inverse. □

Part (iv) says that smoothing changes the variables and not the theory: it neither creates nor removes a gap. Its use is in the choice of the observables. With $O_k(f)$ the spatially flowed fields at a fixed physical flow time, for instance $Z_k \sum_x a_k^4 f(a_k x)\big(\mathscr{E}_\tau(x) - \omega_k(\mathscr{E}_\tau(x))\big)$ with $\mathscr{E}_\tau(x)$ the flowed action density of the slice, statement (E) asks for the convergence of correlation functions that are known to be finite in perturbation theory [85], and Theorem 45 applies to them. Theorem 48 below constructs their continuum limits along subsequences; their nontriviality is not proved here.

D. Renormalised flowed observables

The spatially flowed fields can be renormalised by one condition at a single physical separation, and their correlation functions then have continuum limits along subsequences without further assumptions. What remains open is whether such a limit is trivial, and that question reduces to one ratio of lattice two-point functions.

Fix a bounded gauge-invariant function Φ of finitely many spatial links of the slice $\{x_0 = 0\}$, continuous in them, a flow time $\tau > 0$ in physical units, and a nonzero test function $f_0 \in C_c(\mathbb{R}^4)$ supported in $\{t > 0\}$. Put $s_k = \tau/a_k^2$, write Φ_x for the translate of Φ to $x \in \mathbb{Z}^4$, and let

$$A_k = \sum_{x \in \mathbb{Z}^4} a_k^4 f_0(a_k x)\big(\Phi_x \circ \Phi_{s_k} - \omega_k(\Phi_x \circ \Phi_{s_k})\big),$$

$$C_k(\sigma) = \omega_k\big(\vartheta(\bar A_k)\,\tau_{\sigma/a_k} A_k\big)$$

for dyadic $\sigma \ge 0$ (the sum is finite, and σ/a_k is an integer for large k). By reflection positivity $C_k(0) \ge 0$; assume $C_k(0) > 0$ and set $Z_k = C_k(0)^{-1/2}$. Let $\mathscr{D}$ be the set of points $y = (t, y') \in \mathbb{R}^4$ with dyadic coordinates (in units of a_0) and $t > 0$, and for $y \in \mathscr{D}$ and k with $y/a_k \in \mathbb{Z}^4$ let A_k^y be the translate of A_k by y/a_k.

Theorem 48. (i) *For $y, y' \in \mathscr{D}$ and every k with $y/a_k, y'/a_k \in \mathbb{Z}^4$,*

$$Z_k^2 \,\big|\omega_k\big(\vartheta(\bar A_k^y) A_k^{y'}\big)\big| \le 1.$$

(ii) *Every sequence of k has a subsequence along which $K(y,y') = \lim Z_k^2\, \omega_k(\vartheta(\bar A_k^y) A_k^{y'})$ exists for all $y, y' \in \mathscr{D}$ and $\rho(\sigma) = \lim C_k(\sigma)/C_k(0)$ exists for all dyadic $\sigma > 0$. Along it there is a probability measure μ on $[0,\infty]$ with*

$$\rho(\sigma) = \int_{[0,\infty]} e^{-\sigma E}\, d\mu(E) \qquad (\sigma > 0),$$

the integrand being 0 at $E = \infty$.

(iii) *Along such a subsequence, let $\mathscr{T}$ be spanned by the translates $f_0(\cdot - y)$, $y \in \mathscr{D}$, and one more function $f_* \in C_c(\mathbb{R}^4)$ supported in $\{t > 0\}$ outside their span, and let O_k be the linear map with $O_k(f_0(\cdot - y)) = Z_k A_k^y$ and $O_k(f_*) = 1$. Then (ω_k, O_k) has a continuum limit in the sense of Definition 44, on which the dyadic spatial translations act by unitary operators commuting with H.*

(iv) *This limit is nontrivial if and only if $\mu([0,\infty)) > 0$, if and only if $\rho(\sigma) > 0$ for one dyadic $\sigma > 0$, and then $\rho(\sigma) > 0$ for all of them. Consequently a nontrivial limit exists along some subsequence if and only if $\limsup_k C_k(\sigma)/C_k(0) > 0$ for some dyadic $\sigma > 0$.*

(v) *If the hypothesis of Theorem 45 holds, then μ is carried by $[m_*, \infty]$ and $\rho(\sigma) \le e^{-m_* \sigma}$.*

Proof. By Proposition 47(ii) each $\Phi_x \circ \Phi_{s_k}$, and hence each A_k^y, is a uniform limit of gauge-invariant local functions at times ≥ 1, so the reconstruction of Appendix F applies to them (its footnote). Write $\Psi_k = Z_k \Psi_{A_k}$. Since $\langle \Omega_k, \Psi_{A_k} \rangle = \omega_k(A_k) = 0$ and $\|\Psi_k\|^2 = Z_k^2 C_k(0) = 1$, Ψ_k is a unit vector orthogonal to Ω_k. The relation $\langle \Psi_F, \mathbb{T}_k^t \Psi_H \rangle = \omega_k(\vartheta(\bar F)\tau_t H)$ for all F gives $\Psi_{\tau_t H} = \mathbb{T}_k^t \Psi_H$. Spatial translations commute with ϑ and with τ_t and leave ω_k invariant, so they act on the reconstructed space by unitaries $U_k(x)$ that commute with $\mathbb{T}_k$. Hence, for $y = (t, y')$,

$$Z_k \Psi_{A_k^y} = U_k(y'/a_k)\, \mathbb{T}_k^{t/a_k} \Psi_k.$$

(i) The left side of the inequality is $|\langle Z_k \Psi_{A_k^y}, Z_k \Psi_{A_k^{y'}} \rangle| \le 1$, because $\|U_k\| = 1$ and $0 \le \mathbb{T}_k \le 1$.

(ii) The set $\mathscr{D} \times \mathscr{D}$ is countable and the numbers are bounded by (i), so a diagonal subsequence gives all the limits K; the limits $\rho(\sigma)$ are among them, since $C_k(\sigma)/C_k(0) = \langle \Psi_k, \mathbb{T}_k^{\sigma/a_k} \Psi_k \rangle$ is the value of K at $y = y' = (\sigma/2, 0)$. Let ν_k be the spectral measure (Lemma 670) of $\mathbb{T}_k$ at Ψ_k, a probability measure on $[0,1]$, and μ_k its image under $\lambda \mapsto -a_k^{-1} \log \lambda \in [0,\infty]$. Then $C_k(\sigma)/C_k(0) = \int e^{-\sigma E} d\mu_k(E)$. The space $[0,\infty]$ is compact, so a further subsequence has $\mu_k \to \mu$ weakly (Lemma 671), and $E \mapsto e^{-\sigma E}$ is continuous on $[0,\infty]$ for $\sigma > 0$.

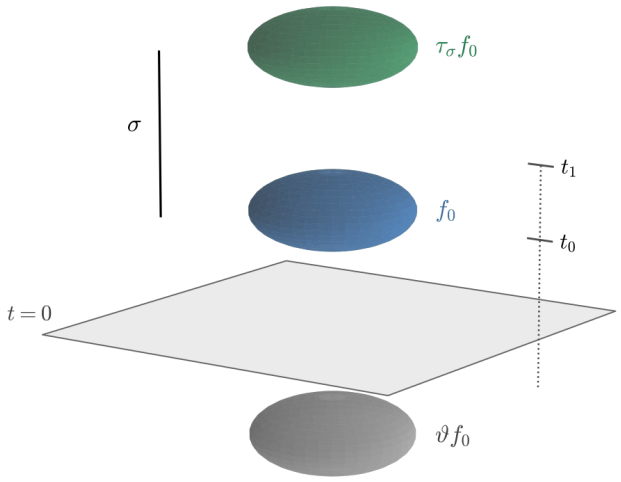


FIG. 9. The packet f_0 of Theorem 48, supported in the time window $[t_0,t_1]$ above the reflection plane $t=0$, its reflection ϑf_0 and its translate $\tau_\sigma f_0$ by σ in time. The correlation $C_k(\sigma)$ pairs the reflected packet with the translated one; $C_k(0)$ is the reflected norm that renormalises the packet.

(iii) A finite combination $\sum_j c_j f_0(\cdot - y_j)$ with distinct y_j that vanishes has Fourier transform $\hat f_0(p)\sum_j c_j e^{-ip\cdot y_j}=0$; since $\hat f_0$ is analytic and not zero, the exponential sum vanishes on an open set and every c_j is zero. So the translates are linearly independent and O_k is well defined. On the space V of finite sums $v=\sum_y c_y\delta_y$ over $\mathscr{D}$ put $B(v,w)=\sum \bar c_y c'_{y'}K(y,y')$. It is the limit of $\langle v_k,w_k\rangle$ with $v_k=\sum_y c_y Z_k\Psi_{A_k^y}$, hence positive semidefinite; let $\mathscr{H}_0$ be the completion of V modulo its null space. For dyadic $\sigma\geq 0$ and dyadic spatial x define $T(\sigma)\delta_y=\delta_{y+(\sigma,0)}$ and $U(x)\delta_y=\delta_{y+(0,x)}$. At level k these are $\mathbb{T}_k^{\sigma/a_k}$ and $U_k(x/a_k)$, so in the limit $B(T(\sigma)v,w)=B(v,T(\sigma)w)$, $B(T(\sigma)v,T(\sigma)v)\leq B(v,v)$, $B(v,T(\sigma)v)\geq 0$, and $U(x)$ preserves B. Thus $T(\sigma)$ extends to commuting positive contractions on $\mathscr{H}_0$ with $T(\sigma)T(\sigma')=T(\sigma+\sigma')$, and $U(x)$ to unitaries commuting with them. For dyadic $\sigma>0$ the kernel of $T(\sigma)=T(\sigma/2)^2$ equals that of $T(\sigma/2)$, so all these kernels coincide; and every δ_y with $y=(t,y')$ equals $T(t/2)\delta_{(t/2,y')}$, a vector in the range of a positive operator and so orthogonal to its kernel. Hence each $T(\sigma)$ is injective on $\mathscr{H}_0$, and $H_0=-\log T(1)$ is a self-adjoint operator with $H_0\geq 0$ and $e^{-\sigma H_0}=T(\sigma)$ for dyadic σ, by the uniqueness of positive roots. Put $\mathscr{H}=\mathbb{C}\Omega\oplus\mathscr{H}_0$, $H=0\oplus H_0$, $\psi(f_0(\cdot-y))=\delta_y$ and $\psi(f_*)=\Omega$. The range of ψ is dense, and the identities of Definition 44 hold: for two translates they are the limits defining K, with $\tau_{\sigma/a_k}A_k^{y'}=A_k^{y'+(\sigma,0)}$; and for f_* they read $\omega_k(1)=1$ and $\omega_k(\vartheta(\bar A_k^y))=\omega_k(\tau_{\sigma/a_k}A_k^y)=0$.

(iv) The limit is nontrivial exactly when $\mathscr{H}_0\neq 0$, that is when $\|\delta_y\|^2=K(y,y)=\rho(2t)$ is positive for some $y=(t,y')$. By (ii) and Lemma 672(b), $\rho(\sigma)>0$ for one $\sigma>0$ if and only if $\mu([0,\infty))>0$, and then $\rho(\sigma)>0$ for every $\sigma>0$. For the last statement choose a subsequence along which $C_k(\sigma)/C_k(0)$ tends to its lim sup, and refine it by (ii).

(v) Under that hypothesis the proof of Theorem 45 gives $\nu_k([0,e^{-\log(1/\alpha)/(2n_k)}])=1$ because $\Psi_k\perp\Omega_k$, so μ_k is carried by $[\log(1/\alpha)/(2n_ka_k),\infty]\subset[m_*,\infty]$, and so is the weak limit μ. □

The measure μ is the spectral measure of the renormalised flowed field in the limit: its part on $[0,\infty)$ is what survives as a state of the continuum theory, and its atom at ∞ is the weight that the field puts on energies that diverge in physical units. The theorem constructs the renormalised continuum observables and their reconstruction for every tuning, and it reduces their nontriviality to the lattice inequality

$$\limsup_{k\to\infty}\frac{C_k(\sigma)}{C_k(0)}>0$$

for one physical separation σ: the flowed two-point function of the smeared packet at a fixed physical distance must not become negligible against its reflected norm $C_k(0)$. If the lattice gap stayed above some $m>0$ in lattice units, the ratio would be at most $e^{-m\sigma/a_k}$ and the limit would be trivial, so the inequality is a condition on the tuning $\beta_k\to\infty$; it is not proved here. Nor are the independence of the limit from the subsequence, Euclidean invariance, or bounds on products of more than two renormalised fields, for which (i) gives nothing because Z_k may diverge (Theorem 57 reduces them to tree bounds); (E) requires all of these.

The denominator $C_k(0)$ is the reflected norm of the packet smeared by f_0, not a coincidence limit of point fields, and the distinction matters. For the free massless scalar field in four dimensions with spatial momenta cut off at Λ, the covariance at spatial separation 0 is $(1-(1+\Lambda t)e^{-\Lambda t})/(4\pi^2 t^2)$ at time $t>0$ and $\Lambda^2/(8\pi^2)$ at coincidence, so its ratio $2(1-(1+\Lambda t)e^{-\Lambda t})/(\Lambda t)^2$ tends to zero as $\Lambda\to\infty$, although the limiting field is the nontrivial Gaussian field; for packets smeared by f_0 the reflected norms of that field have finite positive limits and the ratio stays positive. The criterion of Theorem 48(iv) is about the smeared packets A_k only.

When the ratio is bounded below, renormalising at the separation σ instead of 0 removes the escape to infinite energy altogether, and one slab of the transfer matrix gives a sufficient condition for the bound. Write μ_k for the measures in the proof of Theorem 48.

Proposition 49. (i) *Suppose $C_k(\sigma)\ge cC_k(0)$ for a dyadic $\sigma>0$, a $c>0$ and every k of a sequence. Let μ_k' be the spectral measure, in energies $E=-a_k^{-1}\log\lambda$, of the unit vector $C_k(\sigma)^{-1/2}\Psi_{A_k'}$ with $A_k'=\tau_{\sigma/2a_k}A_k$. Then μ_k' is a probability measure on $[0,\infty)$,*

$$\mu_k'([\Lambda,\infty))\le c^{-1}e^{-\sigma\Lambda},\qquad \int E^m\,d\mu_k'\le c^{-1}\Big(\frac{m}{e\sigma}\Big)^m,$$

and $t\mapsto C_k(\sigma+t)/C_k(\sigma)$ is Lipschitz on $[0,\infty)$ with constant $(e\sigma c)^{-1}$. A subsequence of the μ_k' converges weakly to a probability measure on $[0,\infty)$ with the same bounds.

(ii) *Suppose $\langle\Psi_k,(1-\mathbb{T}_k)\Psi_k\rangle\le Ca_k$ for a constant C and every k of a sequence. Then $C_k(\sigma)/C_k(0)\ge e^{-2C\sigma}$ whenever $Ca_k\le\frac12$ and σ/a_k is an integer, and $\mu_k([\Lambda,\infty])\le C(a_k+\Lambda^{-1})$ for every $\Lambda>0$.*

Proof. (i) We have $\Psi_{A_k'}=\mathbb{T}_k^{\sigma/2a_k}\Psi_{A_k}$, so the measure μ_k' is $e^{-\sigma E}d\mu_k(E)$ divided by its mass $C_k(\sigma)/C_k(0)\ge c$, and $e^{-\sigma E}$ vanishes at $E=\infty$. For $E\ge\Lambda$ we have $e^{-\sigma E}\le e^{-\sigma\Lambda}$, and $E^m e^{-\sigma E}\le(m/e\sigma)^m$ for all $E\ge0$. The ratio in question is the integral of e^{-tE} against μ_k', and the inequality $|e^{-tE}-e^{-sE}|\le|t-s|E$ with the moment bound for $m=1$ gives the Lipschitz bound. The tail bound makes the sequence tight on $[0,\infty)$, so Prokhorov's theorem gives the weak limit, and the bounds pass to it because $\{E>\Lambda\}$ is open and E^m is a monotone limit of bounded continuous functions.

(ii) Let ν_k be the spectral measure of $\mathbb{T}_k$ at Ψ_k on $[0,1]$, a probability measure, and $n=\sigma/a_k$. By Jensen's inequality for the convex function λ^n,

$$\frac{C_k(\sigma)}{C_k(0)}=\int\lambda^n d\nu_k\ge\Big(\int\lambda\,d\nu_k\Big)^n\ge(1-Ca_k)^n\ge e^{-2Ca_kn},$$

since $\log(1-x)\ge-2x$ on $[0,\frac12]$. Moreover $Ca_k\ge\int(1-e^{-a_kE})d\mu_k\ge(1-e^{-a_k\Lambda})\mu_k([\Lambda,\infty])$, and $1-e^{-x}\ge x/(1+x)$. □

Part (ii) asks for an upper bound on a one-slab Dirichlet form, the opposite direction to a mass gap, which bounds such forms from below. In the box of Proposition 50, with g smooth, the time smearing supplies the factor a_k up to one ratio: in the notation of its proof, $\mathbb{T}_k=A$ and $\Psi_{A_k}=\sum_n w_n\mathbb{T}_k^n u$ with $w_n=a_kg(a_kn)$ and $u=a_k^3F^c\psi_0$, and the identity $(1-\mathbb{T}_k)\Psi_{A_k}=\sum_n(w_n-w_{n-1})\mathbb{T}_k^n u$ gives $\langle\Psi_k,(1-\mathbb{T}_k)\Psi_k\rangle\le a_k\|g'\|_{L^1}\|u\|/\|\Psi_{A_k}\|$, so the hypothesis of (ii) holds when the vector u of a single slice stays comparable to the time-smeared packet. That comparison is not proved; Proposition 51 below bounds the ratio without it.

A one-slab bound of this kind also limits the mass that $D(n_k,\alpha)$ can supply. Let e^{-m_k} be the norm of $\mathbb{T}_k$ on $\Omega_k^\perp$. If a unit vector $\varphi_k\perp\Omega_k$ has $\langle\varphi_k,(1-\mathbb{T}_k)\varphi_k\rangle\le\kappa a_k<1$, then the spectral measure of $\mathbb{T}_k$ at φ_k lives on $[0,e^{-m_k}]$, so $1-e^{-m_k}\le\kappa a_k$ and $m_k\le-\log(1-\kappa a_k)$. If in addition $D(n_k,\alpha)$ holds with $n_ka_k\le\ell$, Theorem 43 gives $m_k\ge\log(1/\alpha)/(2n_k)\ge a_k\log(1/\alpha)/(2\ell)$, and therefore

$$\frac{\log(1/\alpha)}{2\ell}\le\frac{-\log(1-\kappa a_k)}{a_k}=\kappa+O(a_k).$$

So the physical mass m_* of Theorem 45 is at most the constant κ of any one-slab bound. The two hypotheses are compatible because both are statements in physical units. The one-slab bound forces $m_k\to0$, so it cannot come from an expansion whose convergence gives a lattice gap bounded below in lattice units.

In a periodic spatial box the assumption $C_k(0)>0$ is a theorem. Let $G=\mathrm{SU}(N)$ with its defining representation, and replace $\mathbb{Z}^4$ by $\mathbb{Z}\times(\mathbb{Z}/L_k\mathbb{Z})^3$ with $L_k\ge3$ and ω_k by the infinite-time Wilson state of the box. The proof of Theorem 48 uses only reflection positivity and the transfer matrix once the spatial translations are dropped, and parts (i)–(iv) hold in the box with $\mathscr{D}$ replaced by its points $(t,0)$, the time translates of f_0 being independent by the same Fourier argument in t.

Proposition 50. *In the box, let $f_0(t,x)=g(t)h(x)$ with $g,h\ge0$, and let $\Phi=\sum_{i<j}(N-\mathrm{Re}\,\mathrm{Tr}\,Q_{ij})$ be the energy of the three spatial plaquettes at the origin. If $g(a_kn)h(a_kx)>0$ for some $n\ge1$ and some x, then $C_k(0)>0$.*

Proof. Lüscher proved that the transfer matrix is strictly positive [83]; we recall the argument because the proof uses it in this form.

The one-link kernel. Write χ for the character of the defining representation and $b=\beta_W>0$. Then

$$e^{\frac{b}{N}\mathrm{Re}\,\chi}=\sum_{m\ge0}\frac{(b/2N)^m}{m!}(\chi+\bar\chi)^m,$$

and $(\chi+\bar\chi)^m$ is the character of $(\rho\oplus\bar\rho)^{\otimes m}$. Every irreducible representation σ occurs in one of these tensor powers: their matrix coefficients span the algebra of polynomials in the entries of q and $\bar q$, which contains the constants and separates points, so it is dense in $C(G)$ by the Stone–Weierstrass theorem, while the matrix coefficients of a σ that never occurred would be orthogonal to it by Schur orthogonality. So the coefficient of every irreducible character in $e^{\frac{b}{N}\mathrm{Re}\,\chi}$ is positive, and convolution with this function acts on each isotypic part of $L^2(G)$ by a positive scalar.

The transfer matrix. In the temporal gauge the transfer matrix of the box acts on the gauge-invariant part of $L^2(G^E)$,

E the spatial links of one slice, as MKM, where K is the tensor product over E of the convolutions above and M is multiplication by the exponential of half the spatial plaquette weight [83, 97]. Both commute with gauge transformations, K is positive and injective by the first step, and M is positive and invertible, so $\langle v, MKMv\rangle = \langle Mv, KMv\rangle > 0$ for $v \neq 0$. Its kernel is a continuous positive function, and K is compact, so its largest eigenvalue λ_0 is simple with an eigenfunction $\psi_0 > 0$, and with $A = MKM/\lambda_0$, a function F of the slice at time $n \geq 0$ has $\Psi_F = A^n F \psi_0$ in the reconstruction, Ω being ψ_0.

Conclusion. Let $F = \sum_x h(a_k x)\, \Phi_x \circ \Phi_{s_k}$ and $F^c = F - \omega_k(F)$. Then $\Psi_{A_k} = a_k^4\, p(A) F^c \psi_0$ with $p(z) = \sum_{n\geq 1} g(a_k n) z^n$, a polynomial with nonnegative coefficients, not all zero, so $p > 0$ on $(0,1]$ and $p(A)$ is injective. The function $\sum_x h(a_k x)\Phi_x$ vanishes at the configuration with every link 1 and is positive at one with a single nontrivial link at a site where $h(a_k x) > 0$; the flow Φ_{s_k} is a bijection, so F is not constant, and as F is continuous and $\psi_0 > 0$, $F^c\psi_0 \neq 0$. Hence $C_k(0) = \|\Psi_{A_k}\|^2 > 0$. □

The proposition removes the one assumption of Theorem 48 in a box; it says nothing about the ratio of part (iv), on which nontriviality depends.

The ratio of Theorem 48(iv) can be compared with correlations of a single slice field. Suppose that $f_0(t,x) = g(t)h(x)$ with $g \geq 0$ supported in $[t_0, t_1]$, $t_0 < t_1$ positive and dyadic, let

$$U_k = \sum_{x\in\mathbb{Z}^3} a_k^3 h(a_k x)\big(\Phi_{(0,x)} \circ \Phi_{s_k} - \omega_k(\Phi_{(0,x)} \circ \Phi_{s_k})\big)$$

be the centred field of the slice at time 0, smeared in space only, and put $u_k = \Psi_{U_k}$ and

$$D_k(r) = \langle u_k, \mathbb{T}_k^{r/a_k} u_k\rangle = \omega_k\big(\vartheta(\bar U_k)\, \tau_{r/a_k} U_k\big)$$

for $r \in a_k\mathbb{Z}$, $r \geq 0$. Let $n_0 = \lceil t_0/a_k\rceil$ and $n_1 = \lfloor t_1/a_k\rfloor$.

Proposition 51. *Assume $C_k(0) > 0$, and let $\sigma \geq 0$ with $\sigma/a_k \in \mathbb{Z}$.*

(i) *$D_k(r) > 0$ for every r, and*

$$\frac{D_k(2a_k n_1 + \sigma)}{D_k(2a_k n_0)} \leq \frac{C_k(\sigma)}{C_k(0)} \leq \frac{D_k(2a_k n_0 + \sigma)}{D_k(2a_k n_1)}.$$

(ii) *$\log D_k$ and $\log C_k$ are convex and nonincreasing on $a_k\mathbb{Z}_{\geq 0}$. If $D_k(r') \geq c D_k(r)$ with $r < r'$ and $c \leq 1$, then $D_k(\beta) \geq c^{\max(1,(\beta-\alpha)/(r'-r))} D_k(\alpha)$ whenever $r \leq \alpha < \beta$; if $C_k(\sigma) \geq c C_k(0)$ with $\sigma > 0$, then $C_k(\sigma') \geq c^{\sigma'/\sigma} C_k(0)$ for $\sigma' \geq \sigma$.*

(iii) *Let k run through a sequence. If $D_k(r') \geq c D_k(r)$ for dyadic $0 < r < r'$ with $r \leq 2t_0$, a $c \in (0,1]$ and every large k, then for every dyadic $\sigma \geq 0$ and every large k*

$$C_k(\sigma) \geq c^{\max(1,(2t_1 - 2t_0 + \sigma)/(r'-r))}\, C_k(0).$$

Conversely, if $C_k(\sigma) \geq c C_k(0)$ for a dyadic $\sigma > 0$ and every k, then $D_k(2t_0 + \sigma') \geq c^{\sigma'/\sigma} D_k(2t_1)$ for every dyadic $\sigma' \geq \sigma$ with $\sigma' > 2(t_1 - t_0)$ and every large k.

Proof. For a function F of the slice at time 0 and an integer $n \geq 0$ we have $\Psi_{\tau_n F} = \mathbb{T}_k^n \Psi_F$, since both have inner product $\omega_k(\vartheta(\bar G)\tau_n F)$ with every Ψ_G. By time translation invariance of ω_k, $A_k = \sum_n w_n \tau_n U_k$ with $w_n = a_k g(a_k n) \geq 0$, and $w_n = 0$ unless $n_0 \leq n \leq n_1$; so $\Psi_{A_k} = \sum_n w_n \mathbb{T}_k^n u_k$. Let ν be the spectral measure of $\mathbb{T}_k$ at u_k, on $[0,1]$, and $W(\lambda) = \sum_n w_n \lambda^n$. Then

$$C_k(\sigma) = \int W(\lambda)^2 \lambda^{\sigma/a_k}\, d\nu(\lambda), \qquad D_k(r) = \int \lambda^{r/a_k}\, d\nu(\lambda).$$

(i) Since $t_0 > 0$, $n_0 \geq 1$ and $W(0) = 0$; so $C_k(0) > 0$ forces $\nu((0,1]) > 0$, and then $D_k(r) > 0$. For $0 \leq \lambda \leq 1$, $\|w\|_1 \lambda^{n_1} \leq W(\lambda) \leq \|w\|_1 \lambda^{n_0}$; inserting these bounds in $C_k(\sigma)$ and $C_k(0)$ gives the two inequalities.

(ii) Convexity of $j \mapsto \log \int \lambda^j\, d\rho$ for a positive measure ρ on $[0,1]$ is Hölder's inequality, and the integrals decrease in j because $\lambda \leq 1$; take $\rho = \nu$ and $\rho = W^2\nu$. If $\beta \leq r'$, monotonicity gives $D_k(\beta) \geq D_k(r') \geq c D_k(r) \geq c D_k(\alpha)$. If $\beta > r'$, the difference quotients of the convex function $\log D_k$ increase in both end points, so $(\log D_k(\beta) - \log D_k(\alpha))/(\beta - \alpha) \geq (\log D_k(r') - \log D_k(r))/(r' - r) \geq \log c/(r' - r)$. The statement on C_k is the same comparison of quotients on $[0,\sigma]$ and $[0,\sigma']$.

(iii) For large k the times $2t_0$, $2t_1$ are multiples of a_k, and $2a_k n_0 \geq 2t_0$, $2a_k n_1 \leq 2t_1$. Since D_k is nonincreasing, (i) gives

$$\frac{D_k(2t_1 + \sigma)}{D_k(2t_0)} \leq \frac{C_k(\sigma)}{C_k(0)} \leq \frac{D_k(2t_0 + \sigma)}{D_k(2t_1)}.$$

The first claim is the left inequality with (ii) for $\alpha = 2t_0 \geq r$ and $\beta = 2t_1 + \sigma$. For the second, (ii) gives $C_k(\sigma') \geq c^{\sigma'/\sigma} C_k(0)$, and the right inequality at σ' gives the claim, the two times being ordered because $\sigma' > 2(t_1 - t_0)$. □

So for the packet $g \otimes h$ the open input of Theorem 48(iv) follows from an inequality between two correlations of one slice field,

$$\liminf_k D_k(r')/D_k(r) > 0 \quad \text{for one pair } 0 < r < r' \text{ with } r \leq 2t_0,$$

and by (iii) a uniform bound for one packet gives one for every packet $g' \otimes h$ with g' supported in $[t_1, \infty)$. The condition $r \leq 2t_0$ cannot be dropped: a bound at a later pair says nothing about an earlier packet, as $D_k(r) = (1 - 4^{-k})2^{-kr} + 4^{-k}2^{-r}$ shows, with $D_k(4)/D_k(3) \to \frac{1}{2}$ and $D_k(2)/D_k(1) \to 0$. A bound at a later pair does apply to the packet shifted forward in time, renormalised by its own reflected norm. The left inequality of (i) bounds the ratio without the comparison of u with the packet that was needed after Proposition 49(ii). Every bound on such ratios proved in this paper, at strong coupling or through Theorem 45, is an upper bound of the form $e^{-m(r'-r)/a_k}$; the inequality above is the opposite one and says that the slice field does not put its weight on energies that diverge in physical units. Reflection positivity and positivity of the transfer matrix do not give it, since a lattice theory with a fixed mass in lattice units has both and its ratio tends to zero. It holds for the free field.

Remark 52. Let D_k be defined as above for the massless free lattice field on $a_k\mathbb{Z}^4$ in place of Φ, with $h \in C_c^\infty(\mathbb{R}^3)$ and the

lattice heat flow for a physical time $t_f > 0$ in place of the gradient flow. With $\hat p^2 = \sum_i (2a_k^{-1}\sin(a_k p_i/2))^2$, $\cosh(a_k\omega) = 1 + a_k^2\hat p^2/2$ and $\hat h_k(p) = \sum_x a_k^3 h(a_k x) e^{-ia_k p\cdot x}$,

$$D_k(r) = \int_{[-\pi/a_k,\pi/a_k]^3} |\hat h_k(p)|^2 e^{-2t_f\hat p^2} \frac{e^{-r\omega}}{2a_k^{-1}\sinh(a_k\omega)} \frac{d^3p}{(2\pi)^3}.$$

On the Brillouin zone $\hat p \geq 2|p|/\pi$ and $a_k^2\hat p^2/2 \leq 6$; as $\operatorname{arccosh}(1+y)/\sqrt{2y}$ decreases on $(0,6]$, $a_k\omega \geq \kappa a_k\hat p$ with $\kappa = \operatorname{arccosh}7/\sqrt{12} > 0.76$, and $a_k^{-1}\sinh(a_k\omega) \geq \omega$. Since $|\hat h_k|$ is bounded uniformly in k, the integrands are dominated by a multiple of $e^{-8t_f|p|^2/\pi^2}/|p|$, which is integrable on $\mathbb{R}^3$, and they converge pointwise to $|\hat h(p)|^2 e^{-2t_f|p|^2} e^{-r|p|}/(2|p|)$. By dominated convergence $D_k(r')/D_k(r)$ tends to the corresponding positive ratio of continuum integrals; for $h = e^{-|x|^2}$, $t_f = 0.05$, $r = \frac12$ and $r' = 1$ it is 0.6337.

Remark 53. The slice field of Proposition 51 is not linear: it is a gauge-invariant function of the spatial links of one slice. Its free analogue is the magnetic energy, and the conclusion of Remark 52 survives for it. Take the abelian free lattice field on $a_k\mathbb{Z}\times a_k\mathbb{Z}^3_{L_k}$ with $a_kL_k = \ell$ fixed, Feynman gauge, the lattice heat flow of the spatial links for a physical time $t_f \geq 0$, and $U_k = \sum_x a_k^3 \sum_{i<j}(F_{ij}(0,x)^2 - \omega_k(F_{ij}^2))$, the zero spatial momentum part of h (for $\mathrm{SU}(N)$ at tree level D_k is multiplied by N^2-1). Since F_{ij} is Gaussian, $\operatorname{Cov}(F^2,F'^2) = 2\operatorname{Cov}(F,F')^2$, and the mixed propagator of the magnetic field at spatial momentum p is $N_{ij,lm}(p)\,K_p(n)$, with a numerator that does not depend on the time momentum and $K_p(n) = e^{-\omega|n|a_k}/(2\sinh(a_k\omega))$. Hence

$$D_k(r) \propto \sum_{p\in\frac{2\pi}{\ell}\mathbb{Z}^3_{L_k}\setminus 0} S_k(p)\, e^{-4t_f\hat p^2} \frac{e^{-2r\omega}}{4a_k^{-2}\sinh^2(a_k\omega)},$$

with $S_k(p) = \sum_{i<j,l<m}|N_{ij,lm}(p)|^2 \leq 36\,\hat p^4$, and the bounds of Remark 52 dominate the summands by $(9/\kappa^2)\,\hat p^2 e^{-4t_f\hat p^2 - 2r\kappa\hat p}$, summable over $\frac{2\pi}{\ell}\mathbb{Z}^3$ for $r > 0$ uniformly in k, also for $t_f = 0$. The summands converge to the continuum ones, so for every $0 < r < r'$ the ratio $D_k(r')/D_k(r)$ converges to a positive limit, with or without the flow. For $\ell = 6$ the computed values of $D_k(\frac12)/D_k(\frac14)$ at $t_f = 0.05$ and $a_k^{-1} = 8, 12, 16, 24$ are 0.27341, 0.27239, 0.27201, 0.27174, against 0.27152 in the continuum; at $t_f = 0$ they are 0.0224, 0.0265, 0.0290, 0.0304, against $(r/r')^5 = \frac{1}{32}$ in infinite volume.

The behaviour at $r = 0$ separates the cases. With the flow, $D_k(0)$ stays bounded and the first-step energy $-a_k^{-1}\log(D_k(a_k)/D_k(0))$ is 6.37, 6.50, 6.56, 6.62 at the four spacings, bounded. Without the flow, $D_k(0) \sim a_k^{-5}$ and $D_k(a_k)/D_k(0) = 0.0198$ at every spacing, so the first-step energy grows like $3.92/a_k$ although every ratio at $r > 0$ converges: the condition of Proposition 51 at $r > 0$ is weaker than a bound on the first step. For the full energy density, which contains F_{0i} and is not a function of one slice, the electric numerator depends on the time momentum, and the second time difference of K_p is $-\delta_{n0}$ plus a multiple of K_p, a contact term: $D_k(0) \sim a_k^{-2}$ even with the flow, $D_k(a_k)/D_k(0) = 0.042$, 0.023, 0.0145 at $a_k^{-1} = 8, 12, 16$, while the ratios at $r > 0$ again converge to the same continuum values (0.27092, 0.27126, 0.27137). The values are numerical evaluations of the free-field formulas.

At next order the spatial flow, being nonlinear, fuses two hard modes into a soft one, and the state Ψ_{U_k} acquires a component of energy about $2q$ for each $q \lesssim a_k^{-1}$. Its norm is of order g^2 and ultraviolet finite, dominated by $q \approx \tau^{-1/2}$, but its mean energy is not: in the free vacuum the first-step energy is $E_{\mathrm{LO}}(1 + g^2(c + \frac{N}{3\pi^2}\log(\sqrt{\tau}/a_k)))$, and the logarithm has coefficient $\frac{8}{11}$ of $2b_0$, $b_0 = 11N/48\pi^2$. A first-step bound at fixed coupling therefore fails logarithmically, while the ratio at a fixed $r > 0$ only loses the norm of that component, damped by e^{-2qr}. This is why Proposition 51 is stated at $r > 0$; the decomposition $\|\mathbb{T}_k^{n/2}u\| \geq c^{n/2}\|v\| - \|w\|$ for $u = v + w$ with a first-step bound on the soft part v alone is the form a four-dimensional argument would use.

Remark 54. The inequality also holds along the right tuning in two-dimensional $\mathrm{SU}(N)$ lattice gauge theory, which is interacting and nonabelian but has no propagating gluons. Take the lattice $a_k\mathbb{Z}\times a_k\mathbb{Z}_{L_k}$ with $a_kL_k = \ell$ fixed, the weight $\prod_p \exp(\frac{\beta_k}{N}\operatorname{Re}\operatorname{Tr}U_p)$ with $\beta_k = 2N/(g^2a_k^2)$, and the temporal gauge. A gauge-invariant function of the links of one slice depends only on the conjugacy class of their holonomy V around the circle; this includes every spatially flowed and spatially smeared slice field, since the flow is gauge covariant. Let w_β be the probability density proportional to $e^{\frac{\beta}{N}\operatorname{Re}\operatorname{Tr}U}$ with respect to Haar measure and, for an irreducible representation λ of dimension d_λ and character χ_λ, $\hat c_\lambda(\beta) = d_\lambda^{-1}\int \chi_\lambda w_\beta\, dU$. Between two slices put $W_x = V_x'V_x^{-1}$; the W_x are independent with density w_{β_k}, which is invariant under conjugation and inversion, and

$$V_1'\cdots V_L' = W_1\,(V_1W_2V_1^{-1})\cdots \\ \cdots(V_1\cdots V_{L-1}W_L(V_1\cdots V_{L-1})^{-1})\,V_1\cdots V_L,$$

so the new holonomy is QV with Q distributed as the L_k-fold convolution of w_{β_k}, independent of V. By Schur's lemma (Lemma 676) the transfer matrix acts on class functions by $\chi_\lambda \mapsto \hat c_\lambda(\beta_k)^{L_k}\chi_\lambda$. The coefficients are real, since w_β is invariant under $U \mapsto \bar U$, nonnegative, since the exponential of $\frac{\beta}{2N}(\chi_{\mathrm F} + \bar\chi_{\mathrm F})$ is a sum of characters with nonnegative coefficients, and below 1 for $\lambda \neq 0$. For a class function f with $\int f\,dU = 0$ and isotypic components of norms $|\hat f_\lambda|$, the slice correlation is therefore

$$D_k(r) = \sum_{\lambda\neq 0} |\hat f_\lambda|^2 e^{-rE_{\lambda,k}}, \qquad E_{\lambda,k} = -\frac{\ell}{a_k^2}\log\hat c_\lambda(\beta_k).$$

Laplace's method[19] at the unique maximum $U = 1$ of $\operatorname{Re}\operatorname{Tr}U$, with $U = e^X$, $\operatorname{Re}\operatorname{Tr}e^X = N + \frac12\operatorname{Tr}X^2 + O(|X|^4)$,

[19] For a smooth function with a unique nondegenerate maximum, $\int e^{\beta\varphi}h$ has an expansion in powers of β^{-1} whose coefficients are Gaussian moments at the maximum; Lemma 677 carries it out here.

$\operatorname{Re}\operatorname{Tr}X^3=0$, $\operatorname{Tr}\rho_\lambda(X)=0$ and $\sum_a\rho_\lambda(T_a)^2=-2C_2(\lambda)$ for an orthonormal basis of $\mathfrak{su}(N)$ in the norm $-\operatorname{Tr}X^2$, gives $\hat c_\lambda(\beta)=1-NC_2(\lambda)/\beta+O(\beta^{-2})$, with the Casimir normalised by $C_2=(N^2-1)/(2N)$ on the fundamental representation (Lemma 677). Hence $E_{\lambda,k}\to g^2\ell C_2(\lambda)/2$, and since $D_k(r)\le\|f\|^2$,

$$\liminf_k\frac{D_k(r')}{D_k(r)}\ge\frac{|\hat f_\mu|^2}{\|f\|^2}e^{-r'g^2\ell C_2(\mu)/2}>0$$

for every $\mu\neq0$ with $\hat f_\mu\neq0$; for $f=\operatorname{Re}\operatorname{Tr}V$ the ratio is exactly $e^{-(r'-r)E_{\mathrm F,k}}\to e^{-(r'-r)g^2\ell(N^2-1)/(4N)}$. This is the exact solution of two-dimensional lattice gauge theory, whose continuum limit is the measure of [46, 62, 81], written as the inequality. In two dimensions g^2 has the dimension of a mass squared and there is no ultraviolet problem, so the argument has no four-dimensional analogue. Nor does the spatial flow give the inequality by itself: it damps the spectral weight at large spatial momentum and leaves the weight at zero spatial momentum unchanged. For the free lattice field of Remark 52 with a mass M_k and the same flow, every energy is at least $m_k=a_k^{-1}\operatorname{arccosh}(1+a_k^2M_k^2/2)$, so $D_k(r')\le e^{-m_k(r'-r)}D_k(r)$, which tends to zero relative to $D_k(r)$ when a_kM_k is fixed.

A route to the slice ratio in four dimensions replaces the packet by its conditional expectations given averages of the field over blocks of increasing size and controls the error of each step. The free field shows in which norm such errors can be small. In the ordinary L^2 norm the first step already fails: conditioning on the averages over blocks of side two leaves a fixed fraction of the variance of an energy packet, at every lattice spacing.

Proposition 55. *Let $d\ge2$, let Λ be a periodic box in $\mathbb{Z}^d$ with even sides, let $\mu>0$, and let ϕ be the Gaussian field on Λ with density proportional to $\exp\bigl(-\frac12\sum_{x,\nu}(\nabla_\nu\phi(x))^2-\frac{\mu^2}{2}\sum_x\phi(x)^2\bigr)$, where $\nabla_\nu\phi(x)=\phi(x+e_\nu)-\phi(x)$. Let $\mathscr Z$ be a set of blocks $B=2z+\{0,1\}^d$, let $\mathfrak B$ be the σ-algebra generated by the averages of ϕ over the blocks in $\mathscr Z$, let $f\ge0$ on Λ, and let*

$$F=\sum_{x\in\Lambda}f(x)\sum_{\nu=1}^d{:}(\nabla_\nu\phi(x))^2{:},$$

where ${:}X^2{:}=X^2-\mathbb EX^2$. For $B=2z+\{0,1\}^d$ put $x_B=2z$, $y_B=2z+e_1+e_2$ and

$$\alpha_B=\frac14\sum_{w\in\{x_B,y_B\}}\Bigl(d\,f(w)+\sum_{\nu=1}^d f(w-e_\nu)\Bigr),$$

and $\sigma^2=2/(2d+\mu^2)$. (i) $\mathbb E\operatorname{Var}(F\mid\mathfrak B)\ge2\sigma^4\sum_{B\in\mathscr Z}\alpha_B^2$.

(ii) $\operatorname{Var}F\le2d^2\sum_x f(x)^2$.

(iii) Let $g\ge0$ be a Lipschitz function on $\mathbb R^d$ with compact support and $\int g^2>0$. For $n\ge1$ let Λ_n contain $n\operatorname{supp}g$ in a fundamental domain, $f_n(x)=g(x/n)$, let $\mathscr Z_n$ contain every block that meets $n\operatorname{supp}g$, and let $\mu=\mu_n\to0$. Then for the packets F_n of f_n,

$$\liminf_{n\to\infty}\frac{\mathbb E\operatorname{Var}(F_n\mid\mathfrak B_n)}{n^d\int g^2}\ge2^{1-d},\qquad\limsup_{n\to\infty}\frac{\operatorname{Var}F_n}{n^d\int g^2}\le2d^2,$$

so $\mathbb E\operatorname{Var}(F_n\mid\mathfrak B_n)\ge(2^{-d}d^{-2}-o(1))\operatorname{Var}F_n$.

Proof. (i) No two of the sites x_B,y_B, $B\in\mathscr Z$, are neighbours: their coordinate sums are even, and a neighbour of a site has a coordinate sum of the other parity. Let $\mathfrak H\supseteq\mathfrak B$ be generated by $\mathfrak B$ and by $\phi(w)$ for every w outside $\{x_B,y_B:B\in\mathscr Z\}$. By the law of total variance, $\operatorname{Var}(F\mid\mathfrak B)=\mathbb E[\operatorname{Var}(F\mid\mathfrak H)\mid\mathfrak B]+\operatorname{Var}(\mathbb E[F\mid\mathfrak H]\mid\mathfrak B)$, so $\mathbb E\operatorname{Var}(F\mid\mathfrak B)\ge\mathbb E\operatorname{Var}(F\mid\mathfrak H)$. In the exponent of the density, a site w enters through $(d+\frac{\mu^2}{2})\phi(w)^2$ and through $-\phi(w)\phi(w')$ for each of its $2d$ neighbours w'. Given $\mathfrak H$, the block average fixes $s_B=\phi(x_B)+\phi(y_B)$, and with $u_B=\phi(x_B)-\phi(y_B)$ we have $\phi(x_B)^2+\phi(y_B)^2=\frac12(s_B^2+u_B^2)$. Since the free sites have only $\mathfrak H$-measurable neighbours, the conditional density of $(u_B)_{B\in\mathscr Z}$ is proportional to $\prod_B\exp(-\frac14(2d+\mu^2)u_B^2+c_Bu_B)$ with $\mathfrak H$-measurable c_B: the u_B are conditionally independent Gaussians of variance σ^2. A term $(\nabla_\nu\phi(w))^2$ contains at most one free site, and if it contains x_B, then $\phi(x_B)=\frac12(s_B+u_B)$ shows that it is $\frac14u_B^2$ plus a term linear in u_B plus an $\mathfrak H$-measurable term; the same holds for y_B with $-u_B$. The terms containing x_B are those with $w=x_B$ and those with $w=x_B-e_\nu$, d of each kind, so

$$F=\sum_{B\in\mathscr Z}\bigl(\alpha_Bu_B^2+\beta_Bu_B\bigr)+F_0$$

with $\mathfrak H$-measurable β_B and F_0. Writing $u_B=m_B+v_B$ with m_B the conditional mean, $\alpha_Bu_B^2+\beta_Bu_B=\alpha_Bv_B^2+(2\alpha_Bm_B+\beta_B)v_B+\text{const}$, and for a centred Gaussian v of variance σ^2, $\operatorname{Var}(\alpha v^2+\gamma v)=2\alpha^2\sigma^4+\gamma^2\sigma^2$. By conditional independence, $\operatorname{Var}(F\mid\mathfrak H)\ge2\sigma^4\sum_B\alpha_B^2$.

(ii) For centred jointly Gaussian X,Y, $\operatorname{Cov}({:}X^2{:},{:}Y^2{:})=2\operatorname{Cov}(X,Y)^2$. With $G_{\nu\rho}(x,y)=\mathbb E[\nabla_\nu\phi(x)\nabla_\rho\phi(y)]$,

$$\begin{aligned}\operatorname{Var}F&=2\sum_{x,y}f(x)f(y)\sum_{\nu,\rho}G_{\nu\rho}(x,y)^2\\&\le\sum_{x,y}\bigl(f(x)^2+f(y)^2\bigr)\sum_{\nu,\rho}G_{\nu\rho}(x,y)^2.\end{aligned}$$

On the dual torus Λ^*, with $\hat p^2=\sum_\nu|e^{ip_\nu}-1|^2$,

$$G_{\nu\rho}(x,y)=\frac{1}{|\Lambda|}\sum_{p\in\Lambda^*}\frac{(e^{ip_\nu}-1)(e^{-ip_\rho}-1)}{\hat p^2+\mu^2}e^{ip\cdot(x-y)},$$

and by Parseval $\sum_yG_{\nu\rho}(x,y)^2=|\Lambda|^{-1}\sum_p|e^{ip_\nu}-1|^2|e^{ip_\rho}-1|^2(\hat p^2+\mu^2)^{-2}\le1$, since each factor $|e^{ip_\nu}-1|^2$ is at most $\hat p^2$. The same holds for the sum over x, and (ii) follows.

(iii) If $\operatorname{Lip}g=K$, every site entering α_B lies within distance 3 of $2z$, so $|\alpha_B-d\,g(2z/n)|\le3dK/n$. Hence $\sum_{B\in\mathscr Z_n}\alpha_B^2\ge\sum_z(d\,g(2z/n)-3dK/n)_+^2$, the sum running over z with $2z\in n\operatorname{supp}g$, and this is $d^2(n/2)^d\int g^2\,(1+o(1))$ as a Riemann sum on the grid $(2/n)\mathbb Z^d$. Since $\sigma^4=4/(2d+\mu_n^2)^2\to d^{-2}$, (i) gives the first bound. By (ii) and $\sum_xg(x/n)^2=n^d\int g^2(1+o(1))$, the second follows. □

For $d=4$ and the energy density, $\operatorname{Var}F_n$ grows like $(\ell/a)^4$ at a fixed physical size ℓ of the packet, and no step of conditioning on block averages removes more than a fraction $1-2^{-d}d^{-2}+o(1)$ of it. The reflected norm $C_k(0)=\omega_k(\vartheta(\bar A_k)A_k)$, which renormalises the packet in Theorem 48, involves the covariance

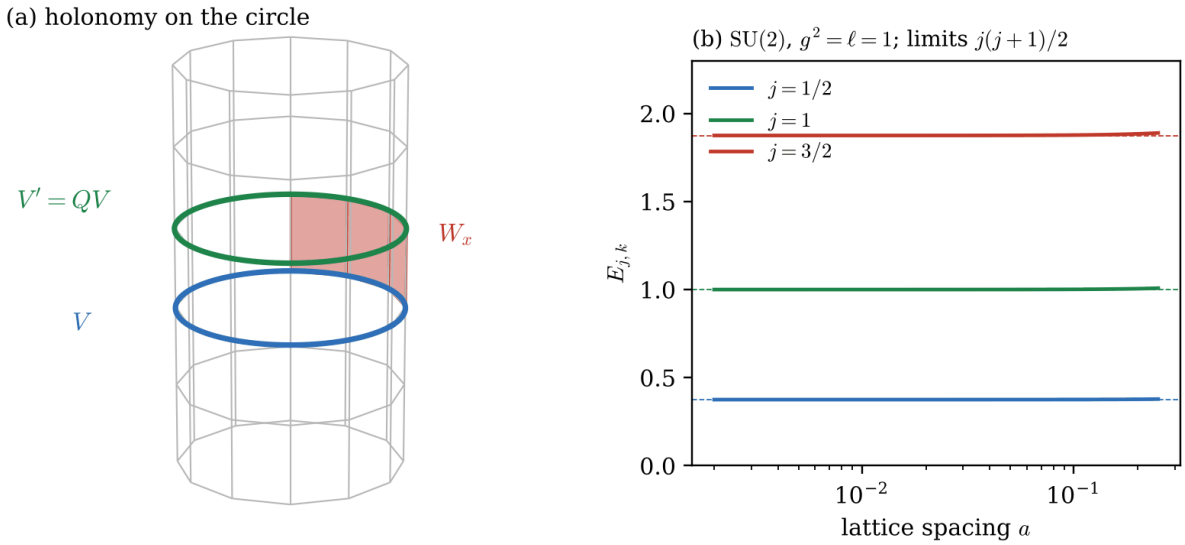


FIG. 10. Two-dimensional SU(N) (Remark 54). (a) The lattice $a_k\mathbb{Z} \times a_k\mathbb{Z}_{L_k}$ drawn as a cylinder: the holonomy V of one slice, the holonomy $V' = QV$ of the next, and the plaquette variables W_x between them. (b) The energies $E_{j,k} = -(\ell/a_k^2)\log \hat{c}_j(\beta_k)$ of SU(2) for $g^2 = \ell = 1$, with $\hat{c}_j(\beta) = I_{2j+1}(\beta)/I_1(\beta)$, against the lattice spacing; the dashed lines are the limits $j(j+1)/2$.

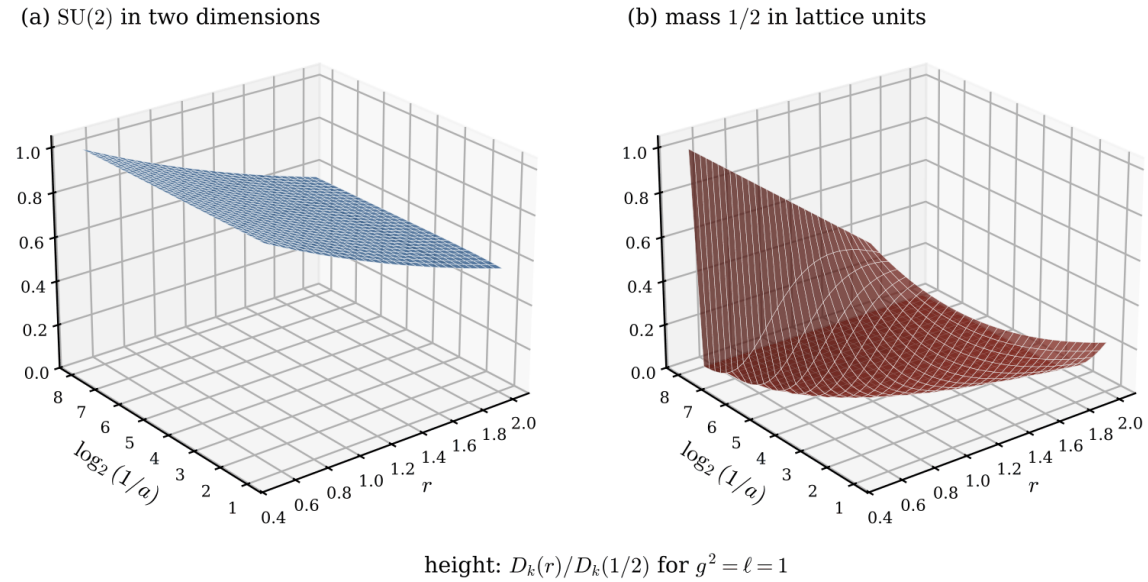
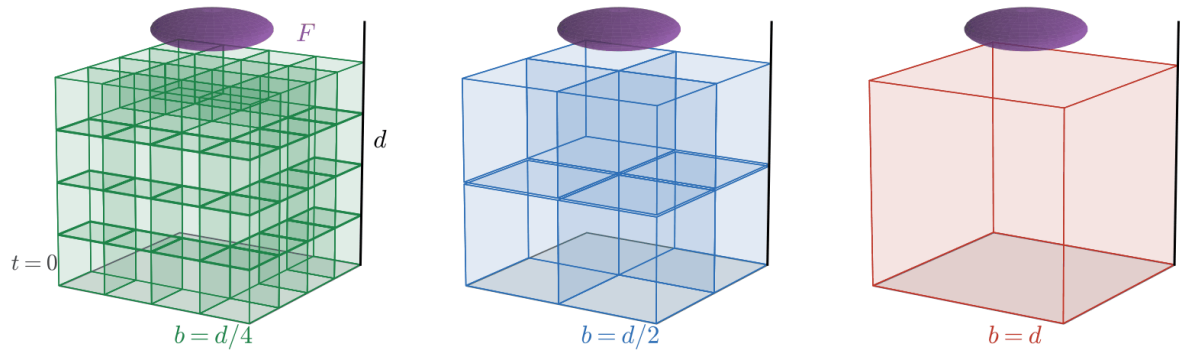


FIG. 11. The ratio $D_k(r)/D_k(\frac{1}{2})$ over the separation r and the lattice spacing a_k. (a) Two-dimensional SU(2) with $f = \mathrm{Re}\,\mathrm{Tr}\,V$ and $g^2 = \ell = 1$, where the ratio is $e^{-(r-1/2)E_{1/2,k}}$ and converges as $a_k \to 0$. (b) The term $e^{-(r-1/2)m_k}$ of the lowest energy m_k of a free field with mass $\frac{1}{2}$ in lattice units, which bounds the ratio from above and tends to zero at every $r > \frac{1}{2}$.

only between the support of ϑf_0 and that of f_0, at a fixed physical distance. The errors of a blocking route have therefore to be measured in that norm; Figure 12 shows the blocks between the reflection plane and the packet.

E. Higher correlations

Theorem 48 controls two renormalised fields at a time, and part (i) of it gives nothing for more: Z_k may diverge while the fields $\Phi_x \circ \Phi_{s_k}$ stay bounded, so a bound on the norm of the renormalised packet as an operator is not available. A reflection positive process with a fixed gap and a two-point function independent of the cutoff can have divergent four-



block averages of side b between the plane $t=0$ and the packet F at height d

FIG. 12. Block averages between the reflection plane $t=0$ and a packet F at physical height d, for blocks of side $d/4$, $d/2$ and d. By Proposition 55 the first conditioning step already leaves a fixed fraction of the ordinary variance of the packet, so the errors of such steps have to be measured in the reflected norm.

point functions at separated times (Remark 59), so no statement about the two-point function alone can replace the following hypothesis on the truncated correlations. In this subsection Φ is real-valued, as the energy of the plaquettes is; for a complex field the conjugate field has to be adjoined and (T1) imposed on all mixed cumulants. Write $\varphi_k(x)=\Phi_x\circ\Phi_{s_k}$, ω_k^T for the truncated correlations (joint cumulants) under ω_k, and for $f\in C_c(\mathbb{R}^4)$

$$A_k(f)=\sum_{x\in\mathbb{Z}^4} a_k^4 f(a_k x)\big(\varphi_k(x)-\omega_k(\varphi_k(x))\big),$$

so that $A_k=A_k(f_0)$ and $\bar A_k(f)=A_k(\bar f)$. Let $w_k:\mathbb{Z}^4\to[0,\infty)$ be even functions with $W_k=\sum_x w_k(x)<\infty$, let $r_k>0$ be amplitude scales, and consider:

(T1) *Tree bounds.* For every $n\ge 2$ there is K_n, independent of k, with

$$\big|\omega_k^T\big(\varphi_k(x_1);\dots;\varphi_k(x_n)\big)\big|\le K_n r_k^n\sum_T\prod_{\{i,j\}\in T} w_k(x_i-x_j)$$

for all $x_1,\dots,x_n\in\mathbb{Z}^4$, the sum running over the trees T on $\{1,\dots,n\}$.

(T2) *Saturation.* $C_k(0)\ge c\,r_k^2 a_k^4 W_k$ for a $c>0$ and every k.

(T3) *Physical volume.* $B_*=\sup_k a_k^4 W_k<\infty$.

The amplitude r_k separates the size of the field from the volume over which it is correlated; Remark 58 shows why it is needed. For a compact $Q\subset\mathbb{R}^4$ let $|Q|_k=a_k^4\#\{x\in\mathbb{Z}^4: a_k x\in Q\}$; for a cube of side L it is at most $(L+a_0)^4$.

Lemma 56. *Under* (T1)*, for $n\ge 2$ and bounded $f_1,\dots,f_n$ with compact supports, f_1 supported in Q,*

$$\big|\omega_k^T\big(A_k(f_1);\dots;A_k(f_n)\big)\big|\le K_n n^{n-2} r_k^n\,|Q|_k\,(a_k^4 W_k)^{n-1}\prod_{j=1}^n\|f_j\|_\infty.$$

Proof. Joint cumulants of order $n\ge 2$ are multilinear and do not change when constants are added to the variables, so the left side is at most $a_k^{4n}\sum_{x_1,\dots,x_n}\prod_j|f_j(a_k x_j)|$ times the absolute value of the truncated correlation, the sums being finite. Insert (T1) and fix a tree T. With x_1 fixed, sum over the other points one at a time, each time over a leaf of what remains of T other than 1; each such sum is at most W_k. The remaining sum over x_1 has at most $|Q|_k a_k^{-4}$ nonzero terms. There are n^{n-2} trees. □

Theorem 57. *Assume* (T1)–(T3).

(i) *For $n\ge 2$, f_j as in the lemma,*

$$\big|\omega_k^T\big(Z_kA_k(f_1);\dots;Z_kA_k(f_n)\big)\big|\le K_n n^{n-2}c^{-n/2}B_*^{n/2-1}\,|Q|_k\prod_{j=1}^n\|f_j\|_\infty,$$

and the moments $\omega_k\big(\prod_j Z_kA_k(f_j)\big)$ with all f_j supported in a cube Q are bounded by $M_n(Q)\prod_j\|f_j\|_\infty$, with $M_n(Q)$ independent of k.

(ii) *Every sequence of k has a subsequence along which*

$$S_n(f_1,\dots,f_n)=\lim_k\omega_k\Big(\prod_{j=1}^n Z_kA_k(f_j)\Big)$$

exists for all n and all $f_j\in C_c(\mathbb{R}^4)$. Each S_n is symmetric, multilinear, bounded by $M_n(Q)\prod_j\|f_j\|_\infty$ on functions supported in Q, hence a distribution on $(\mathbb{R}^4)^n$; it is invariant

under all translations of $\mathbb{R}^4$ if the ω_k are translation invariant, and the kernel K of Theorem 48 along the subsequence is $K(y,y') = S_2\big(\overline{f_0(\cdot - y)} \circ \theta, f_0(\cdot - y')\big)$, with θ as in (iii).

(iii) *Put $S_0 = 1$, $S_1 = 0$, and let θ be the reflection $t \mapsto -t$ of $\mathbb{R}^4$. For every finite family of $c_\alpha \in \mathbb{C}$ and $f_{\alpha,j} \in C_c(\mathbb{R}^4)$ supported in $\{t > 0\}$,*

$$\sum_{\alpha,\beta} \bar{c}_\alpha c_\beta \, S\Big(\prod_j \overline{f_{\alpha,j}} \circ \theta \cdot \prod_j f_{\beta,j}\Big) \geq 0,$$

where S of a product of n test functions means S_n. Hence the Osterwalder–Schrader construction gives a Hilbert space $\mathscr{H}$ with vacuum Ω and a dense set of vectors $\Psi(P)$ indexed by polynomials P in the smeared fields at positive times, with $\langle \Psi(P), \Psi(P') \rangle = S(\theta \bar{P} \cdot P')$; in particular the separated higher correlations of the renormalised fields are finite. The vector $\Psi(P_0)$ of the field smeared by f_0 is a unit vector orthogonal to Ω.

(iv) *Suppose the ω_k are translation invariant. Then the time translations by $t \geq 0$ define on $\mathscr{H}$ a strongly continuous semigroup e^{-tH} with $H \geq 0$ self-adjoint, and the spatial translations a strongly continuous unitary group commuting with H. Moreover there is a dyadic $\sigma > 0$ with $C_k(\sigma) \geq \frac{1}{2} C_k(0)$ for all large k, so the limit of Theorem 48 is nontrivial.*

(v) *If $a_k^4 W_k \to 0$ along the subsequence, every S_n is given by Wick's rule from S_2: the limit is the Gaussian field with covariance S_2.*

(vi) *If the ω_k are translation invariant and the hypothesis of Theorem 45 holds, then 0 is a simple eigenvalue of H and $\sigma(H) \subset \{0\} \cup [m_*, \infty)$ with $m_* = \log(1/\alpha)/(2\ell)$, and $\Omega^\perp \neq 0$.*

Proof. (i) By (T2), $Z_k^n = C_k(0)^{-n/2}$ is at most $(c\, r_k^2 a_k^4 W_k)^{-n/2}$; multiply the lemma by it, which cancels r_k^n, and use $(a_k^4 W_k)^{n/2-1} \leq B_*^{n/2-1}$ by (T3) for $n \geq 2$. A moment is the sum over partitions of $\{1,\dots,n\}$ of the products of the cumulants of the blocks; blocks of size one vanish because $A_k(f)$ is centred, and each block contributes the bound just proved with $|Q|_k \leq (L + a_0)^4$ for a cube of side L. This gives $M_n(Q)$.

(ii) For each cube $Q_L = [-L, L]^4$, $L \in \mathbb{N}$, the continuous functions supported in Q_L form a separable space under the supremum norm. Choose a countable dense set $\mathscr{F}$ of functions; by (i) the countably many sequences $\omega_k(\prod Z_k A_k(f_j))$, $f_j \in \mathscr{F}$, are bounded, and a diagonal subsequence makes all of them converge. By multilinearity,

$$\Big|\omega_k\Big(\prod_j Z_k A_k(f_j)\Big) - \omega_k\Big(\prod_j Z_k A_k(g_j)\Big)\Big| \leq M_n(Q) \sum_i \|f_i - g_i\|_\infty \prod_{j \neq i} \max(\|f_j\|_\infty, \|g_j\|_\infty)$$

for functions supported in Q, so the limits exist for all $f_j \in C_c(\mathbb{R}^4)$ and obey the same bound. Symmetry and multilinearity pass to the limit. A multilinear form bounded in this way is separately continuous on $C_c^\infty(\mathbb{R}^4)$, so the Schwartz kernel theorem makes it a distribution. For dyadic y and large k, $y/a_k \in \mathbb{Z}^4$ and translation invariance of ω_k and of the flow (Proposition 47(i)) give $\omega_k(\prod Z_k A_k(f_j(\cdot - y))) = \omega_k(\prod Z_k A_k(f_j))$. For a real y take dyadic $y_m \to y$; the functions $f_j(\cdot - y_m)$ converge uniformly to $f_j(\cdot - y)$ with supports in one cube, so the bound above carries the invariance to y. The formula for K follows from $A_k^y = A_k(f_0(\cdot - y))$ and the argument of (iii) with $n = 2$.

(iii) Let ϑ be the lattice reflection of Appendix F and θ_k the corresponding reflection of $a_k\mathbb{Z}^4$; it differs from θ by a translation of length at most a_k. Since Φ is real and ϑ commutes with the flow, $\vartheta(\bar{A}_k(f)) = A_k(\bar{f} \circ \theta_k)$. Let $P_k = \sum_\beta c_\beta \prod_j Z_k A_k(f_{\beta,j})$ and $R_k = \sum_\alpha \bar{c}_\alpha \prod_j Z_k A_k(\overline{f_{\alpha,j}} \circ \theta)$, so that $\vartheta(\bar{P}_k)$ is R_k with θ replaced by θ_k. Reflection positivity gives $\omega_k(\vartheta(\bar{P}_k) P_k) \geq 0$. By (i), applied in a cube containing all the supports and their reflections, and by multilinearity, $|\omega_k(\vartheta(\bar{P}_k)P_k) - \omega_k(R_k P_k)|$ is at most a constant independent of k times $\max_{\alpha,j} \|f_{\alpha,j} \circ \theta_k - f_{\alpha,j} \circ \theta\|_\infty$, which tends to zero by uniform continuity, while $\omega_k(R_k P_k)$ tends to the left side of (iii) by (ii). So the limit of the nonnegative numbers is the left side of (iii). The construction of $\mathscr{H}$ is the usual quotient of the polynomials at positive times by the null space of this form [95]. By the same approximation of θ by θ_k, $\|\Psi(P_0)\|^2 = \lim_k Z_k^2 C_k(0) = 1$, and $\langle \Omega, \Psi(P_0) \rangle = S_1(f_0) = 0$.

(iv) For a polynomial P at positive times let $\tau_t P$ be its time translate by $t \geq 0$, again at positive times. By translation invariance and $\theta \circ \tau_t = \tau_{-t} \circ \theta$, $\langle \Psi(\tau_t P), \Psi(P') \rangle = \langle \Psi(P), \Psi(\tau_t P') \rangle$, and $q_P(t) = \|\Psi(\tau_t P)\|^2 = \langle \Psi(P), \Psi(\tau_{2t} P) \rangle$. If P has degree at most d, every partition in (i) of the products in $q_P(t)$ has at most d blocks, each bounded by a multiple of $|Q|_k$ for a cube Q of side $O(1+t)$, so $q_P(t) \leq C_P(1+t)^{4d}$. The Cauchy–Schwarz inequality gives $q_P(t) \leq q_P(0)^{1/2} q_P(2t)^{1/2}$, hence by induction $q_P(t) \leq q_P(0)^{1-2^{-m}} q_P(2^m t)^{2^{-m}}$ for every m, and letting $m \to \infty$, $q_P(t) \leq q_P(0)$. So τ_t maps null vectors to null vectors and extends to a symmetric contraction $\mathsf{T}(t)$ of $\mathscr{H}$; $\mathsf{T}(s)\mathsf{T}(t) = \mathsf{T}(s+t)$, and $\mathsf{T}(t) = \mathsf{T}(t/2)^2 \geq 0$. By the bound of (ii) and uniform continuity of the translated test functions, $\|\Psi(\tau_t P) - \Psi(P)\| \to 0$ as $t \to 0$, so the semigroup is strongly continuous on a dense set, hence everywhere, and $\mathsf{T}(t) = e^{-tH}$ with $H \geq 0$ self-adjoint [95]. Spatial translations preserve the form, have the opposite translations as inverses, commute with τ_t, and are strongly continuous by the same argument.

For the ratio let $f_{0,t} = f_0(\cdot - (t,0))$. For dyadic $t > 0$ and large k, translation invariance gives $\tau_{t/a_k} A_k = A_k(f_{0,t})$, so by (iii)

$$C_k(t) - C_k(0) = \omega_k\big(A_k(\bar{f}_0 \circ \theta_k) A_k(f_{0,t} - f_0)\big),$$

a covariance of two centred fields. The case $n = 2$ of (i) bounds Z_k^2 times it by $L_0 \|f_{0,t} - f_0\|_\infty$ with $L_0 = (K_2/c) \sup_k |Q|_k \|f_0\|_\infty$, Q a cube containing the supports of $f_0 \circ \theta_k$ for all k. By uniform continuity of f_0 there is a dyadic $\sigma > 0$ with $L_0 \|f_{0,\sigma} - f_0\|_\infty \leq \frac{1}{2}$; then $C_k(\sigma)/C_k(0) \geq \frac{1}{2}$, and Theorem 48(iv) applies.

(v) For $n \geq 3$ the exponent $n/2 - 1$ in (i) is positive, so with $(a_k^4 W_k)^{n/2-1}$ in place of $B_*^{n/2-1}$ the cumulants of order $n \geq 3$ of the renormalised fields tend to zero. The moments are the sums over partitions of products of cumulants, so in the limit only the partitions into pairs remain, which is Wick's rule.

(vi) As in the proof of Theorem 45, $\mathbb{T}_k$ has gap at least $\log(1/\alpha)/(2n_k)$ on the complement of Ω_k. For a polynomial P at positive times let $v_k = \Psi_{P_k} - \omega_k(P_k)\Omega_k$ with P_k its lattice version. For dyadic t and large k,

$$\omega_k\big(\vartheta(\bar{P}_k)\tau_{t/a_k} P_k\big) - |\omega_k(P_k)|^2 = \langle v_k, \mathbb{T}_k^{t/a_k} v_k \rangle \leq e^{-m_* t} \|v_k\|^2,$$

and both sides are finite combinations of moments, which converge by (ii) and the approximation of θ in (iii). The limit is $\langle v, e^{-tH}v\rangle \le e^{-m_* t}\|v\|^2$ for $v = \Psi(P) - \langle\Omega,\Psi(P)\rangle\Omega$. These v are dense in $\Omega^\perp$, so the inequality holds on $\Omega^\perp$ for all $t \ge 0$, and the spectral theorem gives the spectral statement as in Theorem 45. By (iii), $\Psi(P_0)$ is a unit vector in $\Omega^\perp$. □

Part (iv) shows that the order-two cases of (T1) and (T2) already contain the lower bound on the ratio that Theorem 48(iv) needs; they are not harmless combinatorial assumptions, and proving them along the continuum tuning is at least as hard as that lower bound. Part (vi) passes the conditional mass gap of Theorem 45 to every polynomial in the smeared fields. Neither part proves Euclidean invariance, independence of the limit from the subsequence, or the reconstruction of the field operators, for which the growth of $M_n(Q)$ in n would also be needed.

Remark 58. The amplitude scale cannot be dropped for a bounded field. Suppose $\operatorname{Var}\varphi_k(0) \to 0$, as for the energy of the flowed plaquettes in lattice units when the flowed links approach the identity. Reflection invariance of ω_k, the Cauchy–Schwarz inequality and stationarity give

$$C_k(0) \le \omega_k(|A_k|^2) \le \Big(a_k^4 \sum_x |f_0(a_k x)|\Big)^2 \operatorname{Var}\varphi_k(0) \to 0.$$

With $r_k = 1$, (T2) then forces $a_k^4 W_k \to 0$, and by part (v) the limit is Gaussian. With r_k^2 of the order of $\operatorname{Var}\varphi_k(0)$ this is no longer forced. Choosing r_k does not prove (T1) or (T2).

Remark 59. Let $0 < \varepsilon < 1$, $m > 0$, let π be the probability measure on $\{-1,0,1\}$ with weights $\varepsilon/2$, $1-\varepsilon$, $\varepsilon/2$, and let $P_t = e^{-mt}I + (1-e^{-mt})\Pi$ on $L^2(\pi)$, where $\Pi f = \pi(f)$ is the projection onto the constants. This is a symmetric Markov semigroup with $P_t \ge 0$ and gap m, so its stationary process is reflection positive. For $X(s) = s/\sqrt{\varepsilon}$ we have $\pi(X) = \pi(X^3) = 0$, $\pi(X^2) = 1$, $\pi(X^4) = 1/\varepsilon$ and $P_t X = e^{-mt}X$, so the two-point function is e^{-mt} for every ε, while for $t_1 \le t_2 \le t_3 \le t_4$ and $q_i = e^{-m(t_{i+1}-t_i)}$ the four-point function is $q_3\,\pi\big(XP_{t_2-t_1}(XP_{t_3-t_2}X^2)\big) = q_1q_3(q_2/\varepsilon + 1 - q_2)$, with truncated part $q_1q_3(q_2/\varepsilon - q_2 - 2q_2^2)$; both diverge as $\varepsilon \to 0$. By part (i), no choice of amplitude scales makes (T1)–(T3) hold for this family with constants independent of ε.

The hypotheses (T1)–(T3) are uniform cluster estimates for the continuum tuning $\beta_k \to \infty$, and the next proposition shows that they are estimates at a critical point: with the order-two cases alone the lattice gap is at most a fixed multiple of a_k, and the tree majorant carries a fixed fraction of its weight at a fixed physical distance. Let $m_k \in [0,\infty]$ be the gap of $\mathbb{T}_k$ on the complement of Ω_k, so that $\|\mathbb{T}_k v\| \le e^{-m_k}\|v\|$ for $v \perp \Omega_k$, and let $t_0 > 0$ be the least time in the support of f_0.

Proposition 60. (i) *For every dyadic $\sigma > 0$ and every large k, $C_k(\sigma) \le e^{-m_k\sigma/a_k}C_k(0)$. If $m_k/a_k \to \infty$, every limit of Theorem 48 is trivial; this is the case for $\beta_k = \beta$ fixed in the range of Theorem 5, where $m_k \ge m_G(\beta)$.*

(ii) *Suppose the ω_k are translation invariant and* (T1) *for $n = 2$ and* (T2) *hold. Then $m_k \le a_k \log 2/\sigma$ for every large k, where σ is the dyadic number of Theorem 57*(iv)*, which depends only on K_2/c and f_0. If also the hypothesis of Theorem 45 holds, then $m_* a_k \le m_k \le a_k \log 2/\sigma$, and in particular $m_* \le \log 2/\sigma$.*

(iii) *Under* (T1) *for $n = 2$ and* (T2)*, let Q be a cube containing the support of f_0 and $\gamma = c/(K_2\|f_0\|_\infty^2 \sup_k |Q|_k)$. Then for every k*

$$\sum_{z\in\mathbb{Z}^4,\ |z_0|\ge R_k} w_k(z) \ge \gamma W_k, \qquad R_k = \frac{2t_0}{a_k} - 1.$$

If $w_k(z) = e^{-\mu_k|z_0|}u_k(\bar z)$ with $\bar z$ the spatial part of z and $u_k \ge 0$ summable on $\mathbb{Z}^3$, then $\mu_k R_k \le \log(2/\gamma)$, so the physical time decay rate of the majorant is at most $\log(2/\gamma)/(2t_0 - a_k)$.

Proof. (i) In the proof of Theorem 48, $C_k(\sigma)/C_k(0) = \langle\Psi_k, \mathbb{T}_k^{\sigma/a_k}\Psi_k\rangle$ with Ψ_k a unit vector orthogonal to Ω_k. The operator $\mathbb{T}_k$ is self-adjoint and fixes Ω_k, so it preserves the complement, and the inequality follows. If $m_k/a_k \to \infty$ then $\rho(\sigma) = 0$ for every dyadic $\sigma > 0$ along every subsequence, and the limit is trivial by Theorem 48(iv). At fixed β in the range of Theorem 5 the gap is at least $m_G(\beta) > 0$ in lattice units, and $a_k \to 0$; the spatial flow does not change the transfer matrix (Proposition 47(iv)).

(ii) Theorem 57(iv) uses (T1) and (T2) only for $n = 2$, since the exponent $n/2 - 1$ in part (i) vanishes there, and gives $C_k(\sigma) \ge \frac12 C_k(0)$ for large k. With (i), $e^{-m_k\sigma/a_k} \ge \frac12$. Under the hypothesis of Theorem 45, the proof of that theorem gives $m_k \ge \log(1/\alpha)/(2n_k) \ge m_* a_k$.

(iii) $C_k(0) = \omega_k(A_k(\bar f_0 \circ \theta_k)A_k(f_0))$, a covariance of two centred fields, by the proof of Theorem 57(iv). A point x in the support of the first sum has $a_k x_0 \le -t_0 + a_k$, and a point y in the support of the second has $a_k y_0 \ge t_0$, so $|x_0 - y_0| \ge R_k$. By (T1) for $n = 2$,

$$\begin{aligned} C_k(0) &\le K_2 r_k^2 \|f_0\|_\infty^2 a_k^8 \sum_{a_k y\in Q} \sum_{|x_0-y_0|\ge R_k} w_k(x-y) \\ &\le K_2 r_k^2 \|f_0\|_\infty^2 |Q|_k\, a_k^4 \sum_{|z_0|\ge R_k} w_k(z), \end{aligned}$$

and (T2) bounds the left side below by $c\,r_k^2 a_k^4 W_k$. For the product form let $R' = \lceil R_k \rceil$; if $R' \le 0$ there is nothing to prove, and otherwise

$$\frac{\sum_{|j|\ge R'} e^{-\mu_k|j|}}{\sum_{j\in\mathbb{Z}} e^{-\mu_k|j|}} = \frac{2e^{-\mu_k R'}}{1+e^{-\mu_k}} \le 2e^{-\mu_k R_k},$$

while the factor $\sum_{\bar z} u_k(\bar z)$ cancels. So $2e^{-\mu_k R_k} \ge \gamma$. □

So reflection positivity and decay do not supply (T2): at fixed β in the range of Theorem 5 the measures are reflection positive with exponential clustering, and $C_k(\sigma)/C_k(0) \to 0$, so no choice of w_k, r_k and constants satisfies the order-two cases of (T1) and (T2) together there. For the same reason no estimate whose constants give a lattice gap bounded below, as a convergent strong coupling expansion does [97], can give the tree bounds along the tuning: (T1) and (T2) hold, if at all, only along couplings where the lattice gap closes at least as fast as a_k. A mass gap in physical units, which bounds the decay of the two-point function, does not give the constants

K_n either. By Theorem 57(v) the limit can be non-Gaussian only if $\liminf_k a_k^4 W_k > 0$; for the product majorants of part (iii) the time factor of W_k is $\coth(\mu_k/2) \geq 2/\mu_k$, which is of order $1/a_k$ at least, so the time direction alone is compatible with this. At a fixed physical flow time the flowed fields are smooth on the scale $\sqrt{\tau}$; majorants of product form for them are not proved here.

The order-two cases of (T1) and (T2) say more about the reference vector than part (iv) of Theorem 57 records: for a smooth f_0 they bound every time difference of the packet, which makes its energy distribution tight with finite moments and gives the ratio with an explicit rate. Recall the spectral measures μ_k on $[0,\infty]$ of the proof of Theorem 48, so that $C_k(\sigma)/C_k(0) = \int e^{-\sigma E} d\mu_k(E)$.

Proposition 61. *Suppose the ω_k are translation invariant,* (T1) *holds for $n = 2$ and* (T2) *holds, and f_0 has $m \geq 1$ continuous time derivatives. Let Q be a cube containing the supports of $f_0(\cdot - (s,0))$ for $0 \leq s \leq m a_0$, and $B_m = (K_2/c)\sup_k |Q|_k \, \|\partial_t^m f_0\|_\infty^2$.*

(i) *$\|(1-\mathbb{T}_k)^m \Psi_k\|^2 \leq B_m a_k^{2m}$ for every k, where $\Psi_k = Z_k \Psi_{A_k}$.*

(ii) *$\int \big((1-e^{-a_k E})/a_k\big)^{2m} d\mu_k(E) \leq B_m$ and $\mu_k([\Lambda,\infty]) \leq B_m(a_k + \Lambda^{-1})^{2m}$ for $\Lambda > 0$. Every weak limit μ of the μ_k satisfies $\mu(\{\infty\}) = 0$ and $\int E^{2m} d\mu(E) \leq B_m$. If* (T1)–(T3) *hold, the unit vector $\Psi(P_0)$ of Theorem 57*(iii)–(iv) *lies in the domain of H^m, with $\|H^m \Psi(P_0)\|^2 \leq B_m$.*

(iii) *If $a_k\sqrt{B_1} \leq \frac{1}{2}$, then $C_k(\sigma) \geq e^{-2\sqrt{B_1}\sigma} C_k(0)$ for every $\sigma \in a_k \mathbb{Z}_{\geq 0}$, and the gap m_k of Proposition 60 satisfies $m_k \leq -\log(1 - a_k\sqrt{B_1}) \leq 2\sqrt{B_1}\, a_k$.*

Proof. (i) Let $D_k f_0 = \sum_{j=0}^m (-1)^j \binom{m}{j} f_0(\cdot - (j a_k, 0))$. By translation invariance $\mathbb{T}_k^j \Psi_{A_k} = \Psi_{\tau_j A_k}$ and $\tau_j A_k = A_k(f_0(\cdot - (j a_k, 0)))$, so $(1-\mathbb{T}_k)^m \Psi_{A_k} = \Psi_{A_k(D_k f_0)}$, and as in the proof of Theorem 57(iv) its squared norm is the covariance $\omega_k(A_k(\overline{D_k f_0} \circ \theta_k) A_k(D_k f_0))$. The function $D_k f_0$ is supported in Q and is an m-th difference with step a_k, so $\|D_k f_0\|_\infty \leq a_k^m \|\partial_t^m f_0\|_\infty$. Lemma 56 with $n = 2$ bounds the covariance by $K_2 r_k^2 |Q|_k \, a_k^4 W_k \|D_k f_0\|_\infty^2$, and (T2) bounds $C_k(0) = Z_k^{-2}$ below by $c\, r_k^2 a_k^4 W_k$.

(ii) By the spectral theorem the left side of (i) is $\int (1 - e^{-a_k E})^{2m} d\mu_k$, the integrand being 1 at $E = \infty$. For $E \geq \Lambda$, $(1-e^{-a_k E})/a_k \geq (1 - e^{-a_k \Lambda})/a_k \geq \Lambda/(1 + a_k \Lambda)$, because $1 - e^{-x} \geq x/(1+x)$; this gives the tail bound. The set $(\Lambda, \infty]$ is open in $[0,\infty]$, so $\mu((\Lambda,\infty]) \leq \liminf_k \mu_k((\Lambda,\infty]) \leq B_m \Lambda^{-2m}$, and $\mu(\{\infty\}) = 0$. For a continuous χ with $0 \leq \chi \leq 1$ and support in $[0, L]$, the same inequality gives $E \leq (1 + a_k L)(1 - e^{-a_k E})/a_k$ on that support, so $\int \chi E^{2m} d\mu = \lim_k \int \chi E^{2m} d\mu_k \leq B_m$; letting χ increase to 1 gives the moment bound. The vector $\Psi(P_0)$ has $\langle \Psi(P_0), e^{-\sigma H} \Psi(P_0)\rangle = \lim_k C_k(\sigma)/C_k(0) = \int e^{-\sigma E} d\mu$ for dyadic $\sigma > 0$ along the subsequence, so its spectral measure for H is μ.

(iii) By (i) with $m = 1$ and the Cauchy–Schwarz inequality, $\langle \Psi_k, (1 - \mathbb{T}_k)\Psi_k\rangle \leq a_k \sqrt{B_1}$, which is the hypothesis of Proposition 49(ii) with $C = \sqrt{B_1}$. Since Ψ_k is a unit vector orthogonal to Ω_k, the same quantity is at least $1 - e^{-m_k}$. □

Part (iii) makes Proposition 60(ii) explicit: the order-two tree and saturation bounds pin the lattice gap below $2\sqrt{B_1}\, a_k$, a physical mass of at most $2\sqrt{B_1}$, and give the ratio of Theorem 48(iv) at every separation. The open input of Theorem 48 is therefore implied by these two bounds for one smooth f_0; nothing here proves them for the Wilson action.

F. An averaged cube condition

The condition $D(n,\alpha)$ enters the proof of Theorem 45 through the comparison lemma, which contracts Lipschitz seminorms and therefore needs the Kantorovich response of a cube for every exterior. It is posed in the forest gauge, and Propositions 187 and 188 and Theorem 262 show what that gauge costs. This subsection replaces it by a condition in the mean over the Wilson measure itself, on gauge-invariant functions and without gauge fixing: a Poincaré inequality for the dynamics that resamples cubes of side n. A finite-speed bound for that dynamics, which uses only the locality of its moves, turns the inequality into decay at a rate of order $1/n$ in lattice units, and the continuum step goes through as in Theorem 45.

Setting. Let $G \subset \mathrm{U}(N)$ be compact and nontrivial, δ a bi-invariant metric of diameter D on G, and φ a positive continuous class function on G, such as $e^{N\beta \operatorname{Re}\operatorname{Tr}(\cdot)}$ or the Villain weight of Proposition 269. The lattice Λ is either the torus $\Lambda_L = (\mathbb{Z}/L\mathbb{Z})^4$ with L even and $L \geq 2n+4$, or a box of $\mathbb{Z}^4$ with all sides at least $2n$; E is its set of links, P its set of plaquettes (on a box, those with four corners in it), and

$$\mu(dU) = \frac{1}{Z} \prod_{p \in P} \varphi(U_p) \prod_{e \in E} dU_e .$$

A link (y, ν) has base point y; on Λ_L differences of base points are taken in $(-L/2, L/2]^4$ and $|\cdot|_\infty$ is the corresponding distance. For $x \in \mathbb{Z}^4$ (or $x \in \Lambda_L$) let V_x be the set of links of E with base point in $x + [0,n)^4$, and let X be the set of x with $V_x \neq \emptyset$. Every link lies in exactly n^4 of the sets V_x, $x \in X$, and $|X| \geq 4n^4$. For $V \subset E$ let ∂V be the set of links of $E \setminus V$ that share a plaquette of P with a link of V; the conditional law $\gamma_V^0(\cdot \mid U)$ of U_V under μ given the other links has density proportional to $\prod_{p \cap V \neq \emptyset} \varphi(U_p)$ and depends only on $U_{\partial V}$. Write $\mathscr{P}_x$ for the conditional expectation given the links outside V_x, $\mathscr{P}_x f(U) = \int f \, d\gamma_{V_x}^0(\cdot \mid U)$, an orthogonal projection on $L^2(\mu)$, and $\mathscr{A}$ for the closed subspace of functions invariant under the gauge group G^Λ. By gauge covariance, $\gamma_{V_x}^0(\cdot \mid gU)$ is the image of $\gamma_{V_x}^0(\cdot \mid U)$ under the action of g on the links of V_x, so $\mathscr{P}_x$ maps $\mathscr{A}$ into itself.

Definition 62. For $n \geq 2$ and $\varpi > 0$, the condition $\mathrm{BP}(n, \varpi)$ for μ is

$$\varpi \operatorname{Var}_\mu(f) \leq \frac{1}{n^4} \sum_{x \in X} \mu\big((f - \mathscr{P}_x f)^2\big)$$

for every real $f \in \mathscr{A}$.

The right side is the Dirichlet form of the dynamics that resamples each cube at rate n^{-4}, so that every link is resampled at rate one, and $\mathrm{BP}(n, \varpi)$ says that this dynamics has spectral

gap at least ϖ on gauge-invariant functions. It contains no supremum over exteriors and no gauge. Put $q_n=(2+1/n)^4$, so that $16<q_n\le\frac{625}{16}$ for $n\ge 2$, and let $\mathscr{K}=|X|^{-1}\sum_{x\in X}\mathscr{P}_x$ be the chain that resamples one cube chosen uniformly. For $Y\subset X$ let $\mathscr{K}_Y=|X|^{-1}\big(\sum_{x\in Y}\mathscr{P}_x+(|X|-|Y|)I\big)$, the same chain with the moves outside Y suppressed.

Write $y\rightsquigarrow z$ if $\partial V_z\cap V_y\neq\emptyset$, that is, if a move at z reads a link of V_y. A link of ∂V_z shares a plaquette with a link based in $z+[0,n)^4$, so its base point lies in $z+[-1,n]^4$; hence $y\rightsquigarrow z$ implies $|z-y|_\infty\le n$, and for each z at most $(2n+1)^4=q_nn^4$ points y have $y\rightsquigarrow z$. For a finite set $S\subset E$ put $Y_1(S)=\{y\in X: V_y\cap S\neq\emptyset\}$ and $Y_{j+1}(S)=Y_j(S)\cup\{y\in X: y\rightsquigarrow z \text{ for some } z\in Y_j(S)\}$.

Lemma 63 (finite speed of the cube chain)**.** *Let f be a bounded function of the links of a finite set $S\subset E$, let $m_0\ge 0$ be an integer and $Y=Y_{m_0}(S)$, with $Y_0(S)=\emptyset$. Then $\mathscr{K}_Y^tf$ depends only on the links of $S\cup\bigcup_{y\in Y}\partial V_y$, and for every integer $t\ge 0$ and every configuration U*

$$\big|\mathscr{K}^tf(U)-\mathscr{K}_Y^tf(U)\big|\le\frac{2|S|}{q_n}\,\|f\|_\infty\sum_{m>m_0}\frac{(q_n\tau)^m}{m!},$$

where $\tau=tn^4/|X|$.

Proof. If h depends only on the links of W, then $\mathscr{P}_xh=h$ when $V_x\cap W=\emptyset$, and $\mathscr{P}_xh$ depends only on the links of $(W\setminus V_x)\cup\partial V_x$ otherwise; induction on t gives the first claim.

Run two chains from U. At each step choose $x_s\in X$ uniformly, the choices independent. If $x_s\notin Y$, the first chain resamples V_{x_s} from $\gamma^0_{V_{x_s}}$ and the second does nothing. If $x_s\in Y$ and the two configurations agree on ∂V_{x_s}, both laws of V_{x_s} are the same, and both chains receive one common sample; otherwise they resample independently. The two marginal processes are the chains $\mathscr{K}$ and $\mathscr{K}_Y$, so $\mathscr{K}^tf(U)-\mathscr{K}_Y^tf(U)$ is the expectation of the difference of f at time t, which vanishes unless the set $\mathscr{D}_t$ of links where the two configurations differ meets S.

We claim that every link of $\mathscr{D}_t$ lies in $V_{x_{s_m}}$ for times $s_1<\cdots<s_m\le t$ with $x_{s_1}\notin Y$ and $x_{s_l}\rightsquigarrow x_{s_{l+1}}$ for $l<m$. For $t=0$, $\mathscr{D}_0=\emptyset$. At step t the links outside V_{x_t} keep their status. If $x_t\notin Y$, the links of V_{x_t} have the chain (t). If $x_t\in Y$, the links of V_{x_t} are in $\mathscr{D}_t$ only if a link $e'\in\partial V_{x_t}$ lies in $\mathscr{D}_{t-1}$; its chain ends at some $s_m<t$ with $e'\in V_{x_{s_m}}$, so $x_{s_m}\rightsquigarrow x_t$, and appending t gives a chain.

If $\mathscr{D}_t$ meets S, there is such a chain with $V_{x_{s_m}}\cap S\neq\emptyset$, so $x_{s_m}\in Y_1(S)$ and $x_{s_{m-l}}\in Y_{l+1}(S)$; since $x_{s_1}\notin Y$, $m>m_0$. For fixed m there are at most $\binom{t}{m}\le t^m/m!$ choices of the times, at most $n^4|S|$ of x_{s_m} and at most q_nn^4 of each earlier cube given the next, and each prescribed sequence occurs with probability $|X|^{-m}$. So the probability that $\mathscr{D}_t$ meets S is at most $\frac{|S|}{q_n}\sum_{m>m_0}(q_nn^4t/|X|)^m/m!$, and the difference of f is at most $2\|f\|_\infty$. □

For finite $S,S'\subset E$ let $d(S,S')$ be the least $|b-b'|_\infty$ over base points b of links of S and b' of links of S'.

Theorem 64 (decay from the averaged condition)**.** *Suppose that* $\mathrm{BP}(n,\varpi)$ *holds for μ. Then $\varpi\le 4$, and for bounded $f,g\in\mathscr{A}$ that depend only on the links of finite sets S_f,S_g with $d=d(S_f,S_g)\ge 3n-2$,*

$$|\mathrm{Cov}_\mu(f,g)|\le\Big(e+\frac{k|S_f||S_g|}{64e^2}\Big)\|f\|_\infty\|g\|_\infty\,e^{-\varpi k/(e^2q_n)},$$
$$k=\Big\lfloor\frac{d+2}{n}\Big\rfloor-2.$$

In particular, for every d, $|\mathrm{Cov}_\mu(f,g)|\le C(1+d)e^{-\varpi d/(e^2q_nn)}$ with C depending on $|S_f|$, $|S_g|$, $\|f\|_\infty$, $\|g\|_\infty$ only, and complex f,g satisfy both bounds with an extra factor four.

Proof. The bound $\varpi\le 4$. The function $f=\mathrm{Re}\,\mathrm{Tr}\,U_p$ of one plaquette is gauge invariant and not constant on G^E, and μ has a positive density, so $\mathrm{Var}_\mu(f)>0$. Each $I-\mathscr{P}_x$ is an orthogonal projection that annihilates constants, so $\mu((f-\mathscr{P}_xf)^2)\le\mathrm{Var}_\mu(f)$, and it vanishes unless V_x contains one of the four links of p, which happens for at most $4n^4$ points x.

The spectral part. Let $\mathscr{A}^0$ be the functions of $\mathscr{A}$ with mean zero. $\mathscr{K}$ is an average of orthogonal projections that preserve $\mathscr{A}^0$, so it is self-adjoint with $0\le\mathscr{K}\le I$ on $\mathscr{A}^0$, and $\mathrm{BP}(n,\varpi)$ reads $\langle h,(I-\mathscr{K})h\rangle\ge\varepsilon\|h\|^2$ on $\mathscr{A}^0$ with $\varepsilon=\varpi n^4/|X|$. Hence the spectrum of $\mathscr{K}$ on $\mathscr{A}^0$ lies in $[0,1-\varepsilon]$ and $\|\mathscr{K}^Th\|_2\le e^{-\varepsilon T}\|h\|_2$ for every integer $T\ge 0$.

The decomposition. Since $(I-\mathscr{K})$ annihilates constants,

$$f-\mu(f)=\mathscr{K}^T\big(f-\mu(f)\big)+\sum_{t=0}^{T-1}\mathscr{K}^t(I-\mathscr{K})f,$$

and pairing with g in $L^2(\mu)$,

$$\mathrm{Cov}_\mu(f,g)=\langle\mathscr{K}^T(f-\mu f),g-\mu g\rangle+\sum_{t=0}^{T-1}\langle\mathscr{K}^tf,(I-\mathscr{K})g\rangle,$$

where the self-adjointness of $\mathscr{K}$ was used. The first term is at most $e^{-\varepsilon T}\|f\|_\infty\|g\|_\infty$.

The local part. Let $Z_g=Y_1(S_g)$, so $|Z_g|\le n^4|S_g|$ and $(I-\mathscr{K})g=|X|^{-1}\sum_{x\in Z_g}(g-\mathscr{P}_xg)$. If h does not depend on the links of V_x, then $\mathscr{P}_xh=h$ and $\langle h,g-\mathscr{P}_xg\rangle=\langle h,g\rangle-\langle\mathscr{P}_xh,g\rangle=0$. Put $m_0=\lfloor(d+2)/n\rfloor-3\ge 0$, so $k=m_0+1$, and $Y=Y_{m_0}(S_f)$. Suppose a link w lies in V_x with $x\in Z_g$, and either $w\in S_f$ or $w\in\partial V_y$ with $y\in Y_j(S_f)$, $1\le j\le m_0$. In the second case there are $y=y_j\rightsquigarrow y_{j-1}\rightsquigarrow\cdots\rightsquigarrow y_1$ with $y_1\in Y_1(S_f)$, so $|y-y_1|_\infty\le n(m_0-1)$; with a base point b of a link of S_f in $y_1+[0,n)^4$, the base point of w in $y+[-1,n]^4\cap(x+[0,n)^4)$, and a base point b' of a link of S_g in $x+[0,n)^4$,

$$|b-b'|_\infty\le(n-1)+n(m_0-1)+n+2(n-1)=n(m_0+3)-3,$$

and the first case gives at most $2(n-1)$. Both are at most $n\lfloor(d+2)/n\rfloor-3<d$, which is impossible. So $\mathscr{K}_Y^tf$, which by Lemma 63 depends only on $S_f\cup\bigcup_{y\in Y}\partial V_y$, does not depend on the links of V_x for $x\in Z_g$, and

$$\big|\langle\mathscr{K}^tf,(I-\mathscr{K})g\rangle\big|=\big|\langle\mathscr{K}^tf-\mathscr{K}_Y^tf,(I-\mathscr{K})g\rangle\big|$$
$$\le\|\mathscr{K}^tf-\mathscr{K}_Y^tf\|_\infty\frac{2|Z_g|}{|X|}\|g\|_\infty.$$

The bound of Lemma 63 increases with t; summing over $t<T$ and putting $\tau=Tn^4/|X|$, the second term is at most

$$\frac{4|S_f||S_g|\,\tau}{q_n}\|f\|_\infty\|g\|_\infty\sum_{m\ge k}\frac{(q_n\tau)^m}{m!}.$$

The choice of T. Take $T=\lfloor k|X|/(e^2q_nn^4)\rfloor$, so that $z=q_n\tau\le k/e^2$ and $\varepsilon T=\varpi\tau\ge \varpi k/(e^2q_n)-\varpi n^4/|X|\ge \varpi k/(e^2q_n)-1$, because $\varpi\le 4$ and $|X|\ge 4n^4$. Since $k!\ge (k/e)^k$,

$$\sum_{m\ge k}\frac{z^m}{m!}\le\frac{z^k}{k!}e^z\le\Big(\frac{ez}{k}\Big)^k e^z\le e^{-k(1-e^{-2})},$$

and $\tau/q_n=z/q_n^2\le k/(256e^2)$. As $\varpi/(e^2q_n)\le 4/(16e^2)<1-e^{-2}$, both terms are at most the stated bound. For the second form, $k>d/n-3$, $k\le d+1$ and $3\varpi/(e^2q_n)<1$; for $d<3n-2$ it holds with $C=e\|f\|_\infty\|g\|_\infty$, since $|\mathrm{Cov}_\mu(f,g)|\le\|f\|_\infty\|g\|_\infty$ and $\varpi d/(e^2q_nn)<1$. Real and imaginary parts give the complex case. □

The proof uses the condition only on $\mathscr{A}$ and only in $L^2(\mu)$; the locality of the moves enters through the counting of chains, at speed e^2q_nn links per unit of the resampling time. The theorem gives the lattice gap and the continuum step. On the torus a *torus limit* at β is a state ω on the continuous gauge-invariant local functions of $\mathbb{Z}^4$ with $\omega(O)=\lim_j\mu_{L_j}(O)$ along a sequence $L_j\to\infty$, each O evaluated on Λ_{L_j} through a box of $\mathbb{Z}^4$ centred at the origin once L_j is large; by compactness of the marginals and a diagonal sequence every sequence $L_j\to\infty$ has a subsequence along which such a limit exists.

Corollary 65 (the lattice gap)**.** *Let $\mathscr{L}$ be an infinite set of even integers $L\ge 2n+4$ and suppose that $\mathrm{BP}(n,\varpi)$ holds for the Wilson measure of Λ_L at $\beta\ge 0$ for every $L\in\mathscr{L}$. Then every torus limit ω along $\mathscr{L}$ is invariant under translations and the hypercubic group and reflection positive, and its transfer matrix has spectral gap at least*

$$m_n=\frac{\varpi}{e^2q_nn}\ge\frac{16\varpi}{625e^2n}.$$

Proof. The measures μ_L are invariant under the translations and the hypercubic group of Λ_L, and for $\beta\ge 0$ and even L they are reflection positive for reflections in planes through sites and in planes bisecting links [97, 105]. Each of these properties is a set of identities or inequalities for finitely many local functions, so it passes to ω, and Appendix F applies. Let F be a continuous gauge-invariant function of finitely many links at positive times. The supports of $\vartheta(\bar F)$ and τ_tF are at distance d with $t-c_F\le d\le t+c_F$, c_F depending only on F, and Theorem 64 on Λ_L, $L\in\mathscr{L}$ large, bounds $|\mathrm{Cov}_{\mu_L}(\vartheta(\bar F),\tau_tF)|$ by $C_F(1+t)e^{-m_nt}$ with C_F independent of L. The limit gives $|\omega(\vartheta(\bar F)\tau_tF)-|\omega(F)|^2|\le C_F(1+t)e^{-m_nt}$, using $\omega(\vartheta(\bar F))=\overline{\omega(F)}$ and translation invariance. These F are dense in the gauge-invariant part of $L^2(\omega)$ on each support, so Proposition 669 with every $m<m_n$ places the spectrum of the transfer matrix on $\Omega^\perp$ in $[0,e^{-m}]$, hence in $[0,e^{-m_n}]$. □

Theorem 66 (the continuum step from the averaged condition)**.** *Suppose that there are $\varpi>0$, $\ell>0$ and integers $n_k\ge 2$ with $n_ka_k\le\ell$ such that, for every k, $\mathrm{BP}(n_k,\varpi)$ holds for the Wilson measure of Λ_L at β_k for all L in an infinite set $\mathscr{L}_k$ of even integers $L\ge 2n_k+4$, and let ω_k be a torus limit along $\mathscr{L}_k$. Then every continuum limit of (ω_k,O_k) satisfies*

$$\sigma(H)\subset\{0\}\cup[m_*,\infty),\qquad m_*=\frac{16\varpi}{625e^2\ell},$$

and 0 is a simple eigenvalue. If the limit is nontrivial, H has a mass gap of at least m_*.

Proof. The proof of Theorem 45 uses $D(n_k,\alpha)$ only through the gap of $\mathbb{T}_k$ on $\Omega_k^\perp$. By Corollary 65 that gap is at least $16\varpi/(625e^2n_k)$, and for $t=\tau/a_k$, $16\varpi t/(625e^2n_k)\ge m_*\tau$; the rest of that proof is unchanged. □

Proposition 47(iii) holds with Theorem 66 in place of Theorem 45, by the same proof. The averaged condition follows from its supremum form, posed for the same measure without gauge fixing. On Λ_L let $C^0_{V,j}$ be defined as $C_{V,j}$ in Sec. X with γ^0_V in place of γ_V and the links of E in place of the active links, put $K^0_\nu=\sum_j C^0_{V_0,j}$, the sum over the links $j\notin V_0$ of direction ν, and let $D^0_L(n,\alpha)$ be the condition $K^0_\nu\le\alpha n^4$ for $\nu=0,\dots,3$. The translations of Λ_L preserve μ, so $C^0_{V_x,j+x}=C^0_{V_0,j}$.

Proposition 67. *On Λ_L:*

(i) *if $D^0_L(n,\alpha)$ holds with $\alpha<1$, then $\mathrm{BP}(n,1-\alpha)$ holds, for every real $f\in L^2(\mu)$;*

(ii) *for $\varphi\equiv 1$, that is $\beta=0$, $\mathrm{BP}(n,1)$ holds for every n and every real $f\in L^2(\mu)$.*

Proof. (i) Let $d_E(U,U')=\sum_{e\in E}\delta(U_e,U'_e)$ and $\kappa_0=(1-\alpha)n^4/L^4$. Let U,U' differ only at a link i. Couple $\mathscr{K}(U,\cdot)$ and $\mathscr{K}(U',\cdot)$ by choosing the same x. If $i\in V_x$, the two exteriors agree, the laws of the resampled cube coincide, and the identity coupling gives distance 0. If $i\notin V_x$, an optimal coupling of the two cube laws for d_{V_x} costs at most $C^0_{V_x,i}\delta(U_i,U'_i)$, and the link i adds $\delta(U_i,U'_i)$. The x with $i\notin V_x$ number L^4-n^4, and $\sum_{x:\,i\notin V_x}C^0_{V_x,i}=K^0_\nu$ for the direction ν of i, so

$$W_1^{d_E}\big(\mathscr{K}(U,\cdot),\mathscr{K}(U',\cdot)\big)\le\frac{L^4-n^4+\alpha n^4}{L^4}\,\delta(U_i,U'_i)=(1-\kappa_0)\,\delta(U_i,U'_i).$$

Changing the links of two arbitrary configurations one at a time and using the triangle inequality for $W_1^{d_E}$ extends this to $(1-\kappa_0)d_E(U,U')$. Hence a function h with Lipschitz constant $\mathrm{Lip}(h)$ for d_E has $\mathrm{Lip}(\mathscr{K}^th)\le(1-\kappa_0)^t\mathrm{Lip}(h)$, and since $d_E\le D|E|$,

$$\mathrm{Var}_\mu(\mathscr{K}^th)=\tfrac12\iint\big(\mathscr{K}^th(U)-\mathscr{K}^th(U')\big)^2d\mu\,d\mu\le\tfrac12(1-\kappa_0)^{2t}\mathrm{Lip}(h)^2D^2|E|^2.$$

Let ν_h be the spectral measure of the self-adjoint operator $0\le\mathscr{K}\le I$ at $h-\mu(h)$. Then $\mathrm{Var}_\mu(\mathscr{K}^th)=\int\lambda^{2t}d\nu_h$, and if

$\nu_h((1-\kappa_0+\eta,1])>0$ for some $\eta>0$ this exceeds the bound for large t; so ν_h lives on $[0,1-\kappa_0]$. Lipschitz functions are uniformly dense in $C(G^E)$ by the Stone–Weierstrass theorem, hence dense in $L^2(\mu)$, so the spectral projection of $\mathscr{K}$ for $(1-\kappa_0,1]$ vanishes on the functions of mean zero, and $\langle f,(I-\mathscr{K})f\rangle \geq \kappa_0 \operatorname{Var}_\mu(f)$ for every f. Multiplying by L^4/n^4 gives $\mathrm{BP}(n,1-\alpha)$.

(ii) For $\varphi\equiv 1$, μ is the product of Haar measures. Write $f=\sum_{S\subset E} f_S$ with f_S depending only on the links of S and $\int f_S\,dU_e=0$ for $e\in S$; the f_S are orthogonal, $\operatorname{Var}_\mu(f)=\sum_{S\neq\emptyset}\|f_S\|^2$, and $\mathscr{P}_x f=\sum_{S\cap V_x=\emptyset} f_S$. Hence

$$\sum_x \mu\big((f-\mathscr{P}_x f)^2\big)=\sum_{S\neq\emptyset}\|f_S\|^2\,\#\{x: S\cap V_x\neq\emptyset\}\geq n^4\operatorname{Var}_\mu(f),$$

because a link of S lies in n^4 cubes. □

Whether $D(n,\alpha)$ in the forest gauge implies BP is not decided here: the forest gauge does not close on a torus, and on $\mathbb{Z}^4$ the two conditions concern different specifications. The averaged condition carries the full content of a gap, since by Theorem 64 it gives decay of every bounded gauge-invariant covariance at the rate $\varpi/(e^2 q_n n)$; like $D(n,\alpha)$, it fails in the Coulomb phase of the abelian theory at every cube size.

Corollary 68. *Let $G=\mathrm{U}(1)$ with the Villain weight, and let β_0 be as in Proposition 269. For $\beta>\beta_0$ there are no $n\geq 2$ and $\varpi>0$ such that $\mathrm{BP}(n,\varpi)$ holds for the measures with free boundary of all boxes of side at least $2n$.*

Proof. The field strength Ψ_p of the proof of Proposition 269 is a bounded gauge-invariant function of the four links of p, and its mean vanishes under every box measure by the symmetry $U\mapsto \bar U$. If $\mathrm{BP}(n,\varpi)$ held for all boxes, Theorem 64 would give $|W_\Lambda(\Psi_p\Psi_{p'})|\leq C(1+d)e^{-\varpi d/(e^2 q_n n)}$ for parallel plaquettes at distance d, with C independent of Λ and p'. The state of [55] is the limit of these measures (proof of Proposition 269), so $\sum_{p'}|\omega(\Psi_p\Psi_{p'})|<\infty$, which contradicts [55, Sect. 2] as in that proof. □

Remark 69 (what the averaged condition settles and what it leaves). Proved: Theorem 66 gives a mass gap of at least $16\varpi/(625e^2\ell)$ in every nontrivial continuum limit of torus limits from the condition $\mathrm{BP}(n_k,\varpi)$ with $n_k a_k\leq\ell$, a spectral gap of the cube dynamics on gauge-invariant functions in L^2 of the Wilson measure, uniform in the torus; the condition follows from its supremum form $D_L^0(n,\alpha)$ with $\varpi=1-\alpha$, holds at $\beta=0$ with $\varpi=1$ (Proposition 67), holds on every torus at strong coupling for every cube size past an explicit $n_0(\beta)$, with $\varpi\to 1$ as $n\to\infty$ (Corollary 74), fails on the torus of side $2n+4$ once $\beta\gg n^4\log\beta$, already on the functions of nonzero centre charge (Theorem 79, Corollary 85), and fails in the Coulomb phase of the Villain theory at every cube size (Corollary 68); the conclusion of Theorem 66 follows from the weaker local form $\mathrm{BP}_{\mathrm{loc}}$, which for $\mathrm{SU}(N)$ is implied by the condition on the neutral sector (Theorem 84). The obstructions of the forest gauge, Propositions 187, 188 and 264, Theorem 262 and Corollary 263, bound K_c from below through Lipschitz functions of the gauge-fixed links, transported by gauge transformations of $\mathbb{Z}^4$, which in the Gaussian models are the flat fields dh; the averaged condition involves only gauge-invariant functions, on which every gauge transformation acts trivially, so these proofs do not apply to it. Nor do Theorem 505 and Proposition 455, which concern bounds uniform in the coarse field and arguments from rarity, since the condition contains no supremum and no labels. Not proved: the condition $\mathrm{BP}(n_k,\varpi)$ along the continuum tuning for any non-abelian group, or outside the strong-coupling ranges of Corollary 74, and its local form $\mathrm{BP}_{\mathrm{loc}}$ along the tuning; the relation between BP and $D(n,\alpha)$; and (E). Remark 81 lists what Sec. X G proves about the dependence on n and β. Corollary 68 shows that a proof of BP has to use the non-abelian structure of G, as for (D). Neither the averaged condition nor the mass gap is proved here.

G. The averaged condition at strong coupling and at fixed cube size

This subsection proves $\mathrm{BP}(n,\varpi)$ on every torus, for every cube size, in explicit ranges of the coupling, with ϖ tending to its value at $\beta=0$ as $n\to\infty$, and proves that at weak coupling the condition fails on the torus of side $2n+4$ when n is small compared with $(\beta/\log\beta)^{1/4}$. The first statement comes from the supremum form $D_L^0(n,\alpha)$ of Proposition 67, which the single-link coefficients of the ungauged measure control through a Dobrushin–Shlosman estimate for one cube [44]. The second uses the center of $\mathrm{SU}(N)$ and the holonomies of the torus, which the cube moves can change only through the curvature of the configuration. Throughout, $\Lambda=\Lambda_L$ with L even, δ is a bi-invariant metric on G, and $C^0_{V,j}$, K^0_V are as in Proposition 67. Two distinct links are *partners* if they lie in a common plaquette; for $L\geq 3$ every link has exactly eighteen partners, three in each of its six plaquettes, and two links share at most one plaquette. Write $C^0_{ee'}=C^0_{\{e\},e'}$ for the single-link coefficients; $C^0_{ee'}=0$ unless e and e' are partners.

Lemma 70. *Suppose that $C^0_{ee'}\leq c$ for all partners e,e', with $18c<1$. Then for every finite $V\subset E$ and every link $j\notin V$,*

$$C^0_{V,j}\leq \frac{1}{1-18c}\sum_{e\in V} C^0_{ej}.$$

Proof. Let U,U' differ only at j, and let $\gamma=\gamma^0_V(\cdot\mid U)$, $\gamma'=\gamma^0_V(\cdot\mid U')$. Under γ the conditional law of σ_e, $e\in V$, given the other links of V is $\gamma^0_e(\cdot\mid\sigma,U_{E\setminus V})$, and likewise for γ'. The couplings of γ and γ' form a weakly compact set on which $\Psi(\pi)=\sum_{e\in V}\mathbb{E}_\pi\delta(\sigma_e,\sigma'_e)$ is continuous; let π^* be a minimiser and $\rho(e)=\mathbb{E}_{\pi^*}\delta(\sigma_e,\sigma'_e)$. For $e\in V$ replace (σ_e,σ'_e) by a sample of an optimal coupling of the two single-link laws, chosen measurably [113, Cor. 5.22]; both marginals are invariant, so the result is a coupling of γ and γ'. Changing the partners of e one at a time bounds the new expected distance at e by $\sum_{k\in V}C^0_{ek}\rho(k)+C^0_{ej}\delta(U_j,U'_j)$, and minimality gives

$$\rho(e)\leq\sum_{k\in V}C^0_{ek}\rho(k)+C^0_{ej}\delta(U_j,U'_j).$$

Sum over $e \in V$. Since k has eighteen partners, $\sum_{e\in V} C^0_{ek} \le 18c$, so $(1-18c)\sum_e \rho(e) \le \delta(U_j,U_j')\sum_{e\in V} C^0_{ej}$, and $W_1^{d_V}(\gamma,\gamma') \le \sum_e \rho(e)$. □

Lemma 71. *Let $n \ge 2$, $L \ge 2n+4$ and let j be a link of direction ν.*

(i) $\sum_{x:\, j\notin V_x} \#\{e \in V_x : e \text{ a partner of } j\} = 18n^3 - 3n^2$.

(ii) *The number B_n of x with $j \notin V_x$ such that V_x contains a partner of j is $7n^3+3n^2$.*

Proof. Let b be the base point of j. For each $\mu \ne \nu$ the plaquette at b in the plane $\{\mu,\nu\}$ gives the partners $(b+e_\nu,\mu)$, $(b+e_\mu,\nu)$, (b,μ), and the plaquette at $b-e_\mu$ gives $(b-e_\mu,\nu)$, $(b-e_\mu,\mu)$, $(b-e_\mu+e_\nu,\mu)$, so the base points of the partners are $b+\Delta$ with $\Delta \in \{e_\nu, e_\mu, 0, -e_\mu, -e_\mu, e_\nu - e_\mu\}$. A link with base point $b+\Delta$ lies in V_x exactly when $z = b-x$ satisfies $z+\Delta \in Q = [0,n)^4$, and $j \in V_x$ exactly when $z \in Q$; since $L \ge 2n+4$ these conditions are read in $\mathbb{Z}^4$.

(i) Exchanging the sums, the left side is $\sum_\Delta \#\{z : z+\Delta \in Q,\ z \notin Q\} = \sum_\Delta (n^4 - |Q \cap (Q-\Delta)|)$. This is 0 for $\Delta = 0$, $n^4 - n^3(n-1) = n^3$ for $\Delta = \pm e_\mu, e_\nu$, and $n^4 - n^2(n-1)^2 = 2n^3 - n^2$ for $\Delta = e_\nu - e_\mu$. Each μ contributes $6n^3 - n^2$.

(ii) The set of $z \notin Q$ with $z+\Delta \in Q$ for some Δ consists of the layer $\{z_\nu = -1\}$ of $Q - e_\nu$, the layers $\{z_\mu = -1\}$ and $\{z_\mu = n\}$ of $Q \mp e_\mu$, each with the other coordinates in $[0,n)$, and the new points $\{z_\nu = -1,\ z_\mu = n\}$ of $Q + e_\mu - e_\nu$, with the two remaining coordinates in $[0,n)$. These sets are disjoint, with $n^3 + 3(2n^3+n^2)$ points. □

Theorem 72 (the averaged condition from single links)**.** *Let $n \ge 2$ and $L \ge 2n+4$.*

(i) *If $C^0_{ee'} \le c$ for all partners, with $18c < 1$, then $K^0_\nu \le \alpha_n n^4$ for $\nu = 0,\dots,3$, with*

$$\alpha_n = \frac{c\,(18n-3)}{(1-18c)\,n^2},$$

and if $\alpha_n < 1$, $\mathrm{BP}(n, 1-\alpha_n)$ holds for every real $f \in L^2(\mu)$.

(ii) *For every β,*

$$K^0_\nu \le (7n^3+3n^2)\,\Xi_n, \qquad \Xi_n = \max_j C^0_{V_0,j}.$$

The number Ξ_n and the condition $D^0_L(n,\alpha)$ do not depend on $L \ge 2n+4$, and $\mathrm{BP}(n, 1-(7n^{-1}+3n^{-2})\Xi_n)$ holds whenever $(7n^3+3n^2)\Xi_n < n^4$.

The number α_n decreases in n and tends to 0.

Proof. Fix a link j of direction ν. By translation invariance, $K^0_\nu = \sum_{x:\, j\notin V_x} C^0_{V_x,j}$ (proof of Proposition 67).

(i) By Lemma 70 and Lemma 71(i),

$$K^0_\nu \le \frac{c}{1-18c} \sum_{x:\, j\notin V_x} \#\{e \in V_x \text{ partner of } j\} = \frac{c\,(18n^3-3n^2)}{1-18c}.$$

This is $D^0_L(n,\alpha_n)$, and Proposition 67(i) gives the condition $\mathrm{BP}(n, 1-\alpha_n)$. Finally $(18n-3)/n^2 = 18/n - 3/n^2$ decreases for $n \ge 1$.

(ii) $C^0_{V_x,j}$ vanishes unless V_x contains a partner of j, which by Lemma 71(ii) happens for $B_n = 7n^3+3n^2$ points x with $j \notin V_x$. The law $\gamma^0_{V_0}(\cdot \mid U)$ depends on U only through the links of ∂V_0, whose base points lie in $[-1,n]^4$; for $L \ge 2n+4$ the links of $V_0 \cup \partial V_0$ and the plaquettes that meet V_0 are those of $\mathbb{Z}^4$, and the links of ∂V_0 take all values in $G^{\partial V_0}$. So every $C^0_{V_0,j}$, hence Ξ_n and K^0_ν, is the same for all $L \ge 2n+4$. The last claim is Proposition 67(i) with $\alpha = B_n \Xi_n / n^4$. □

Lemma 70 is Lemma 259 for the ungauged measure, where every link has eighteen partners instead of twelve. Part (ii) of the theorem is the exact form of the count of Remark 185 for that measure: if one link moves the law of a cube of side $n \ge 3$ by a total amount at most Ξ in Kantorovich distance, $D^0_L(n,\alpha)$ holds once $n \ge 8\Xi/\alpha$, since $7n^3+3n^2 \le 8n^3$. The single-link coefficients are bounded as in Secs. IV–VII, and for a general group by curvature.

Lemma 73. *Let $G \subset \mathrm{U}(N)$ be closed and connected with semisimple Lie algebra, δ the geodesic distance of $\langle X,Y\rangle = -\mathrm{Tr}(XY)$, and $6N|\beta| < \kappa_G$. Then $C^0_{ee'} \le N|\beta|/(\kappa_G - 6N|\beta|)$ for all partners.*

Proof. Given the other links, the law of U_e is $\nu \propto e^{-V}dU$ with $V(U) = -N\beta \sum_{p\ni e} \mathrm{Re}\,\mathrm{Tr}\,U_p$. Along $e^{tX}U$ the second derivative of $\mathrm{Re}\,\mathrm{Tr}\,U_p$ is $\mathrm{Re}\,\mathrm{Tr}(Y^2W)$ with $|Y| = |X|$, by (6), and $|\mathrm{Tr}(Y^2W)| \le |Y|^2$, so $\mathrm{Ric}_G + \mathrm{Hess}\,V \ge K_1 = \kappa_G - 6N|\beta| > 0$. The proof of Lemma 24 with one factor and $w \equiv 1$ gives $\Gamma(P_t h) \le e^{-2K_1 t} P_t \Gamma(h)$ for the Langevin semigroup of ν, and as in the proof of Lemma 25,

$$|\mathrm{Cov}_\nu(g,h)| \le \int_0^\infty |\nu(\Gamma(P_t h, g))|\,dt \le \frac{\|\nabla g\|_\infty \|\nabla h\|_\infty}{K_1}.$$

Move the partner e' along a unit-speed geodesic $e^{sX}U_{e'}$, $0 \le s \le \delta(U_{e'}, U'_{e'})$, and let $\nu_s \propto e^{-V_s}$. Then $\frac{d}{ds}\nu_s(g) = -\mathrm{Cov}_{\nu_s}(g, \partial_s V_s)$; $\partial_s V_s$ comes from the one plaquette p that contains e and e', and its gradient in U_e is $-N\beta$ times the mixed block $H^p_{ee'}(\cdot, X)$, of norm at most one by Lemma 22. For smooth g that are 1-Lipschitz for δ this gives $|\nu_0(g) - \nu_\delta(g)| \le N|\beta|\delta/K_1$. Convolution with a probability measure on G preserves the Lipschitz constant of a function for a bi-invariant metric, so smooth 1-Lipschitz functions approximate every 1-Lipschitz one uniformly, and the Kantorovich–Rubinstein duality gives the bound. □

Corollary 74 (explicit ranges)**.** *For every even $L \ge 2n+4$, $\mathrm{BP}(n, 1-\alpha_n)$ holds for every real $f \in L^2(\mu)$ in the following cases.*

(i) *$G = \mathrm{SU}(2)$, $|\beta_W| < \frac{2}{9}$, with*

$$\alpha_n = \frac{|\beta_W|\,(18n-3)}{(4-18|\beta_W|)\,n^2},$$

whenever $|\beta_W| < 4n^2/(18n^2+18n-3)$*; for* $n=2$ *this is* $|\beta_W| < \frac{16}{105}$.[20] *At* $|\beta_W| \le \frac{1}{8}$, $\alpha_n \le \alpha_2 \le 0.59$ *for every* $n \ge 2$*; at* $|\beta_W| \le \frac{1}{5}$, $\alpha_n \le 0.982$ *for every* $n \ge 9$.

(ii) $G = \mathrm{SU}(3)$ *and* $\beta_W \ge 0$ *with*

$$\zeta(\beta_W) = \frac{3r\sqrt{\varepsilon(r)}}{\gamma(r)} < 1, \qquad r = 2\beta_W,$$

where ε *and* γ *are as in Theorem 6, with*

$$\alpha_n = \frac{\zeta(\beta_W)\,(18n-3)}{18\,(1-\zeta(\beta_W))\,n^2}.$$

The function ζ *is increasing,* $\zeta(0.35) \le 0.8584$*, and for* $\beta_W \le 0.35$ *and* $n \ge 6$, $\alpha_n \le 0.3368\,(18n-3)/n^2 \le 0.9824$.

(iii) $G \subset \mathrm{U}(N)$ *closed and connected with semisimple Lie algebra,* $24N|\beta| < \kappa_G$*, with*

$$\alpha_n = \frac{N|\beta|(18n-3)}{(\kappa_G - 24N|\beta|)n^2},$$

whenever $N|\beta| < \kappa_G n^2/(24n^2+18n-3)$.

In each case, for fixed β *in the range,* $\mathrm{BP}(n,\varpi)$ *holds with one* $\varpi > 0$ *for all* n *from some* $n_0(\beta)$ *on, uniformly in the torus, and the best constant tends to* 1 *as* $n \to \infty$.

Proof. (i) As in the proof of Theorem 1, without the gauge: the law of U_e is the von Mises–Fisher law ν_b with $b = 4\beta\sum_{p\ni e}\bar{s}_p$, a partner moved by δ in the chordal metric moves b by $4|\beta|\delta$, and Theorem 682 gives $C^0_{ee'} \le |\beta| = |\beta_W|/4$. Theorem 72(i) applies with $c/(1-18c) = |\beta_W|/(4-18|\beta_W|)$. At $|\beta_W| = \frac{1}{8}$, $\alpha_2 = \frac{33}{56}$; at $|\beta_W| = \frac{1}{5}$, $\alpha_n = (18n-3)/(2n^2)$ and $\alpha_9 = \frac{159}{162}$; α_n increases with $|\beta_W|$.

(ii) The six staples of a link are products of matrices of SU(3), so $b \in rK$, and as in (8) and Lemma 31, $C^0_{ee'} \le N\beta\Lambda(r) \le c$ with $18c = 3r\sqrt{\varepsilon(r)}/\gamma(r)$ in the Hilbert–Schmidt metric; ζ increases because ε is nondecreasing and γ decreasing. By Lemma 30(iii) and the tail bound in the proof of Theorem 6, at $r = 0.7$ the terms $m_n r^n/n!$, rounded up, are 1, 0, 0.1225, 0.0142917, 0.0075032, 0.0013131, 0.000332, 0.000059 and 0.0000109, with sum at most 1.1460099, and the tail is less than $4\cdot 10^{-6}$; so $Z_*(0.7) \le 1.146014$, $e^{-0.14} \ge 0.869342$, $\frac{5e^{-0.14}}{9Z_*(0.7)} \ge 0.421432$ and $\varepsilon(0.7) \le 0.578568 < 0.76064^2$. With $\gamma(0.7) = \frac{1191}{640}$, $\zeta(0.35) \le 2.1\cdot 0.76064\cdot\frac{640}{1191} \le 0.8584$, and $\zeta/(18(1-\zeta)) \le 0.3368$; at $n=6$, $(18n-3)/n^2 = \frac{35}{12}$.

(iii) Lemma 73 gives $c = N|\beta|/(\kappa_G - 6N|\beta|)$, so $18c < 1$ is $24N|\beta| < \kappa_G$ and $c/(1-18c) = N|\beta|/(\kappa_G - 24N|\beta|)$.

In each case $\alpha_n \to 0$; Proposition 75(ii) below bounds the best constant by $1+2/n$. □

[20] With $c = |\beta_W|/4$ the condition $18c < 1$ of Theorem 72 reads $|\beta_W| < \frac{2}{9}$, and $\alpha_n < 1$ reads $|\beta_W|(18n-3) < (4-18|\beta_W|)n^2$, that is, $|\beta_W|(18n^2+18n-3) < 4n^2$; at $n=2$ this is $105|\beta_W| < 16$. As $n \to \infty$ the threshold $4n^2/(18n^2+18n-3)$ increases to $\frac{2}{9}$.

For SU(3), the condition $\zeta < 1$ of (ii) holds up to $\beta_W \approx 0.391$ by a numerical evaluation that no statement uses, against $\frac{3}{16}$ from (iii); for SU(2), (i) reaches $\frac{2}{9}$ against $\frac{1}{12}$. These ranges lie inside those of Secs. IV–VIII, where the forest gauge gives a gap directly; what Corollary 74 adds is the averaged condition itself, on every torus, for every n past $n_0(\beta)$ and on all of $L^2(\mu)$. The gap it gives through Corollary 65 is of order $1/n$ and is not used. The dependence on n is constrained in general as follows. Write ϖ^*_n for the best constant in $\mathrm{BP}(n,\cdot)$ on Λ_L, and P_V for the conditional expectation given the links outside V.

Proposition 75. *Let* $L \ge 2m+4$ *and* $2 \le n \le m$.

(i) *If* $\mathrm{BP}(n,\varpi)$ *holds, then so does* $\mathrm{BP}(m,(n/m)^4\varpi)$*; that is,* $n^4\varpi^*_n \le m^4\varpi^*_m$.

(ii) $\varpi^*_n \le 1 + 2/n$.

(iii) *Let* W_x *be the cubes of side* m*. Suppose that* $\mathrm{BP}(m,\varpi)$ *holds and that for every* x*, every real* $f \in \mathscr{A}$ *and* μ*-almost every* U*,*

$$\varpi_{\mathrm{in}} P_{W_x}\big((f-P_{W_x}f)^2\big) \le \frac{1}{n^4}\sum_{y:\,V_y\subset W_x} P_{W_x}\big((f-P_{V_y}f)^2\big).$$

Then $\mathrm{BP}(n,\varpi\varpi_{\mathrm{in}}(m/(m-n+1))^4)$ *holds.*

Proof. (i) If $V \subset W$, the σ-algebra of the links outside W is contained in that of the links outside V, so $P_W = P_W P_V$ and $\|f-P_W f\|^2 = \|f-P_V f\|^2 + \|P_V f - P_W f\|^2$. The cube V_y of side n lies in W_x exactly when $y - x \in [0,m-n]^4$, so for each x, $\mu((f-P_{W_x}f)^2)$ is at least the mean of $\mu((f-P_{V_y}f)^2)$ over these $(m-n+1)^4$ points y, and each y occurs for $(m-n+1)^4$ points x. Summing, $\sum_x \mu((f-P_{W_x}f)^2) \ge \sum_y \mu((f-P_{V_y}f)^2)$.

(ii) Let $f = \mathrm{Re}\,\mathrm{Tr}\,U_p$ for a plaquette p at y in the plane $\{\mu,\nu\}$; $\mathrm{Var}_\mu(f) > 0$. As in the proof of Theorem 64, $\mu((f-\mathscr{P}_x f)^2) \le \mathrm{Var}_\mu(f)$, with equality to zero unless V_x contains a link of p, that is, unless $x \in b - [0,n)^4$ for a base point $b \in \{y, y+e_\mu, y+e_\nu\}$. By inclusion and exclusion these x number $3n^4 - 2n^3(n-1) - n^2(n-1)^2 + n^2(n-1)^2 = n^4 + 2n^3$.

(iii) Integrating the hypothesis, $\varpi_{\mathrm{in}}\mu((f-P_{W_x}f)^2) \le n^{-4}\sum_{y:V_y\subset W_x}\mu((f-P_{V_y}f)^2)$. Sum over x, use $\mathrm{BP}(m,\varpi)$ and count each y $(m-n+1)^4$ times: $\varpi\varpi_{\mathrm{in}}\,\mathrm{Var}_\mu(f) \le (m-n+1)^4 m^{-4} n^{-4}\sum_y \mu((f-P_{V_y}f)^2)$. □

Part (iii) is the two-scale argument for block dynamics [88]; it transfers the condition from larger cubes to smaller ones, given a gap of the small-cube dynamics inside each large cube uniform in its exterior. Part (i) transfers it upwards only with the loss $(n/m)^4$, which, through Theorem 64, gives the rate $\varpi n^4/(e^2 q_m m^5)$ and nothing in physical units. So BP at one cube size implies no useful bound at larger sizes by itself.

The remaining results concern $G = \mathrm{SU}(N)$, $N \ge 2$, with center element $z = e^{2\pi i/N} I$, and use the holonomies of the torus. For a site y let P_y be the ordered product of the L links $(y+te_0,0)$, $0 \le t < L$, the Polyakov loop of direction 0 through y, and $A = P_0$, B the analogous product of the links $(te_1,1)$; A and B are based at the site 0, so the traces $F_{ab} = \mathrm{Tr}(A^a B^b)$ are

gauge invariant. For $c,d\in\mathbb{Z}$ the map $Z_{c,d}$ that multiplies the links $(y,0)$ with $y_0=0$ by z^c and the links $(y,1)$ with $y_1=0$ by z^d preserves μ: a plaquette that contains one of these links contains a second one of the same direction with the same base coordinate, entering with the opposite orientation, and z is central. It maps P_y to z^cP_y, A to z^cA, B to z^dB and F_{ab} to $z^{ac+bd}F_{ab}$.

Lemma 76. *Let $R=[0,a]\times[0,b]$ be a rectangle of $\mathbb{Z}^2$ and ϕ a map of its sites into Λ_L that sends unit steps in the two directions to links in two directions $\mu\neq\nu$ and is injective on plaquettes. The holonomy of the boundary of R, traversed counterclockwise from $(0,0)$, is an ordered product $\prod_k g_kU_{p_k}^{\pm1}g_k^{-1}$ over the ab plaquettes of $\phi(R)$, with $g_k\in G$. For every unitarily invariant norm it is within $\sum_{p\in\phi(R)}\|U_p-I\|$ of I.*

Proof. Holonomy is multiplicative under concatenation of paths and does not change when a step followed by its reverse is removed. Let ℓ_j be the boundary of $[0,a]\times[0,j]$ from $(0,0)$. After cancellation, $\ell_{j-1}^{-1}\ell_j$ is the left edge from $(0,0)$ to $(0,j-1)$, the boundary of the row $[0,a]\times[j-1,j]$ from $(0,j-1)$, and the left edge back; so $\mathrm{Hol}(\ell_b)$ is an ordered product of conjugates of the row holonomies. The same argument along a row writes the holonomy of its boundary as an ordered product of conjugates of the holonomies of the boundaries of its plaquettes, each a conjugate of U_p or U_p^*. For unitary X,Y, $\|XY-I\|\le\|X-I\|+\|Y-I\|$ and $\|gXg^{-1}-I\|=\|X-I\|$. □

Lemma 77. *For $A,B\in\mathrm{SU}(N)$ put*

$$M(A,B)=\sum_{(a,b)\in I_N}|\mathrm{Tr}(A^aB^b)|^2,$$

where $I_N=\{0,\dots,N-1\}^2\setminus\{(0,0)\}$. There are constants $c_N>0$ and $\eta_N>0$ such that $M(A,B)\ge c_N$ whenever $\|ABA^{-1}B^{-1}-I\|_{\mathrm{HS}}\le\eta_N$. For $N=2$ one may take $\eta_2=1$ and $c_2=0.64$.

Proof. Let A,B commute and suppose $M(A,B)=0$. Diagonalise them together, with eigenvalues α_k,β_k. The power sums of the α_k of orders $1,\dots,N-1$ vanish, so by Newton's identities the characteristic polynomial of A is $x^N+(-1)^N\det A$, and $\alpha_k=\lambda\omega^k$ after reordering, with $\omega=e^{2\pi i/N}$; likewise $\beta_k=\lambda'\omega^{\tau(k)}$ for a permutation τ of $\mathbb{Z}/N\mathbb{Z}$. Then $\mathrm{Tr}(A^aB^b)=\lambda^a\lambda'^b\sum_k\omega^{ak+b\tau(k)}$ is a multiple of the Fourier coefficient at (a,b) of the indicator of the graph of τ in $(\mathbb{Z}/N\mathbb{Z})^2$. All coefficients except the one at 0 vanish, so the indicator is constant, which is impossible for a set of N points among N^2. Hence $M>0$ on the compact set of commuting pairs. If no η_N,c_N existed, there would be pairs with $\|ABA^{-1}B^{-1}-I\|\to0$ and $M\to0$, and a limit point would be a commuting pair with $M=0$.

For $N=2$ put $x=\mathrm{Tr}A$, $y=\mathrm{Tr}B$, $w=\mathrm{Tr}AB$, real numbers in $[-2,2]$. By the Cayley–Hamilton theorem, $A^{-1}=x-A$, $B^{-1}=y-B$, $\mathrm{Tr}(AB^2)=yw-x$, $\mathrm{Tr}(A^2B)=xw-y$ and $\mathrm{Tr}(ABAB)=w^2-2$, so

$$\mathrm{Tr}(ABA^{-1}B^{-1})=x^2+y^2+w^2-xyw-2.$$

Since $|\mathrm{Tr}(C-I)|\le\sqrt{2}\|C-I\|_{\mathrm{HS}}$, the commutator bound gives $x^2+y^2+w^2-xyw\ge4-\sqrt{2}$, and with $m=\max(|x|,|y|,|w|)$, $3m^2+m^3\ge4-\sqrt{2}>2.432=3(0.8)^2+0.8^3$. So $m>0.8$ and $M=x^2+y^2+w^2>0.64$. □

Lemma 78. *Let $G=\mathrm{SU}(N)$ and $\beta\ge1$. There is A_N such that $\mu(N-\mathrm{Re}\,\mathrm{Tr}\,U_p)\le A_N(1+\log\beta)/\beta$ for every plaquette and every L; for $N=2$, $\mu(2-\mathrm{Tr}\,U_p)\le(\log\beta+6.11)/\beta$.*

Proof. Put

$$H=\sum_p(N-\mathrm{Re}\,\mathrm{Tr}\,U_p)\ge0,\qquad\phi(t)=\log\int e^{-NtH}dU.$$

Then μ has density proportional to $e^{-N\beta H}$, $\phi'(\beta)=-N\mu(H)$, ϕ is convex and $\phi\le0$. Convexity gives $\phi(\beta/2)\ge\phi(\beta)-\frac{\beta}{2}\phi'(\beta)$, hence $\mu(H)\le\frac{2}{N\beta}\log(1/e^{\phi(\beta)})$. Let δ be the geodesic distance of $-\mathrm{Tr}(XY)$. If every link lies within r of I, then $\delta(U_p,I)\le4r$ by bi-invariance, and $N-\mathrm{Re}\,\mathrm{Tr}\,U\le\frac12\delta(U,I)^2$, so $H\le8r^2|P|$. With $v(r)$ the Haar measure of the ball of radius r about I, $|E|=4L^4$ and $|P|=6L^4$,

$$\mu(N-\mathrm{Re}\,\mathrm{Tr}\,U_p)=\frac{\mu(H)}{6L^4}\le\frac{2}{N\beta}\Big(\tfrac23\log\tfrac{1}{v(r)}+8N\beta r^2\Big),$$

the plaquettes being equivalent under the symmetries of Λ_L. Haar measure has a smooth positive density in exponential coordinates, so $v(r)\ge v_Nr^{N^2-1}$ for $r\le1$; $r^2=1/(N\beta)$ gives the first claim. For $\mathrm{SU}(2)$, $U=\cos\theta+u\sin\theta$ in quaternions has $\delta(U,I)=\sqrt{2}\theta$ and the law of θ has density $\frac{2}{\pi}\sin^2\theta$, so $v(\sqrt{2}s)\ge\frac{2}{\pi}\cdot\frac{4}{\pi^2}\cdot\frac{s^3}{3}=\frac{8s^3}{3\pi^3}$ for $s\le\frac{\pi}{2}$. With $r=\sqrt{2}s$ and $s^2=1/(32\beta)$ the bracket times $\frac{1}{\beta}$ is at most $\frac{1}{6\beta}(6\log(32\beta)+6+4\log\frac{3\pi^3}{8})\le(\log\beta+6.11)/\beta$. □

Theorem 79 (the averaged condition at fixed cube size)**.** *Let $G=\mathrm{SU}(N)$, $N\ge2$, $\beta\ge0$ and $n\ge2$.*

(i) *If $\mathrm{BP}(n,\varpi)$ holds on Λ_L, then*

$$\frac{1}{L^3}\sum_{y:\,y_0=0}\mu\big(\mathrm{Tr}\,P_0\,\overline{\mathrm{Tr}\,P_y}\big)\le\frac{2N^2n^2}{\varpi L^2}.$$

(ii) *Let $N=2$ and $\beta\ge1$. If $\mathrm{BP}(n,\varpi)$ holds on Λ_{2n+4}, then*

$$\varpi\le\frac{38400\,n^4(\log\beta+6.11)}{\beta}.$$

(iii) *There is C_N such that for $\beta\ge1$, $\mathrm{BP}(n,\varpi)$ on Λ_{2n+4} implies $\varpi\le C_Nn^4(1+\log\beta)/\beta$.*

In particular, along any sequence (β_k,n_k) with $n_k^4(1+\log\beta_k)/\beta_k\to0$, no $\varpi>0$ satisfies $\mathrm{BP}(n_k,\varpi)$ uniformly in the torus.

Proof. (i) Let $g_y=\mathrm{Re}\,\mathrm{Tr}\,P_y$ and $f=L^{-3}\sum_{y:\,y_0=0}g_y$. By the symmetry $Z_{1,0}$, $\mu(f)=0$. The loop P_y meets V_x only if (y_1,y_2,y_3) lies in the projection of $x+[0,n)^4$, which holds for n^3 loops, and $\|g_y-\mathscr{P}_xg_y\|_2\le\|g_y\|_2\le N$. So $\|f-\mathscr{P}_xf\|_2\le$

Nn^3/L^3 for each of the L^4 points x, the right side of $\mathrm{BP}(n,\varpi)$ is at most N^2n^2/L^2, and $\mu(f^2)\le N^2n^2/(\varpi L^2)$. The same holds for the imaginary parts, and $\mu(|L^{-3}\sum_y \mathrm{Tr}\,P_y|^2)$ is the left side, by translation invariance.

(ii), (iii) Let $L=2n+4\le 4n$, $\varepsilon=\mu(\|U_p-I\|_{\mathrm{HS}}^2)=2\mu(N-\mathrm{Re}\,\mathrm{Tr}\,U_p)$, the same for every plaquette, and let f be the real or the imaginary part of some F_{ab}, $(a,b)\in I_N$. Pick (c,d) with $ac+bd\not\equiv 0$ modulo N; the symmetry $Z_{c,d}$ gives $\mu(F_{ab})=0$.

The right side of BP. If V_x meets neither A nor B, $\mathscr{P}_x f=f$. Otherwise the coordinate x_2 satisfies $0\in x_2+[0,n)$ modulo L, and one of $s=x_2-1$, $s=x_2+n$, read in $(-L/2,L/2]$, has $|s|\le n$ and $s\notin x_2+[0,n)$. The loops A', B' of directions 0, 1 through $y'=se_2$ avoid V_x, so $F'_{ab}=\mathrm{Tr}(A'^aB'^b)$ depends only on the links outside V_x, and $\|f-\mathscr{P}_x f\|_2\le\|F_{ab}-F'_{ab}\|_2$. Let Γ be the holonomy of the straight path from 0 to y'. Lemma 76, for the rectangle $[0,L]\times[0,s]$ mapped onto the plane $\{0,2\}$ through 0, whose boundary is A, the path to y', A' reversed and the path back, gives $A=H_0\Gamma A'\Gamma^{-1}$ with $\|H_0-I\|_{\mathrm{op}}\le S_0$, the sum of $\|U_p-I\|_{\mathrm{op}}$ over the $L|s|$ plaquettes of the strip; likewise $B=H_1\Gamma B'\Gamma^{-1}$ with the strip in the plane $\{1,2\}$. Telescoping, $|F_{ab}-F'_{ab}|\le N(a\|H_0-I\|_{\mathrm{op}}+b\|H_1-I\|_{\mathrm{op}})\le N(N-1)(S_0+S_1)$, and by the Cauchy–Schwarz inequality over the at most $2Ln$ plaquettes of the two strips, $\mu((S_0+S_1)^2)\le(2Ln)^2\varepsilon$. At most $2Ln^3$ points x have V_x meeting $A\cup B$, so

$$\frac{1}{n^4}\sum_x\mu\big((f-\mathscr{P}_x f)^2\big)\le 8N^2(N-1)^2L^3n\varepsilon.$$

The variance. Lemma 76 for the square $[0,L]^2$ mapped onto the plane $\{0,1\}$ through 0, whose boundary reads A, B, A^{-1}, B^{-1}, gives $\|ABA^{-1}B^{-1}-I\|_{\mathrm{HS}}\le\sum_p\|U_p-I\|_{\mathrm{HS}}$ over L^2 plaquettes, so this commutator exceeds η_N with probability at most $L^2\sqrt{\varepsilon}/\eta_N$. By Lemma 77, $\sum_{(a,b)\in I_N}\mu(|F_{ab}|^2)\ge c_N(1-L^2\sqrt{\varepsilon}/\eta_N)$. As $\mu(F_{ab})=0$, some f of the form above has $\mathrm{Var}_\mu(f)\ge c_N(1-L^2\sqrt{\varepsilon}/\eta_N)/(2(N^2-1))$. For $N=2$ the traces are real, and some F_{ab} has variance at least $0.64(1-L^2\sqrt{\varepsilon})/3$.

Conclusion. If $L^2\sqrt{\varepsilon}\le\eta_N/2$, $\mathrm{BP}(n,\varpi)$ gives $\varpi\le 2048(N^2-1)N^2(N-1)^2n^4\varepsilon/c_N$, and for $N=2$, $\varpi\le 300L^3n\varepsilon\le 19200n^4\varepsilon$. In the other case $\eta_N^2<4L^4\varepsilon\le 1024n^4\varepsilon$, and since $\varpi\le 2$ by Proposition 75(ii), $\varpi\le(2048/\eta_N^2)n^4\varepsilon$, which for $N=2$ is again at most $19200n^4\varepsilon$. Lemma 78 bounds ε by $2(\log\beta+6.11)/\beta$ for $N=2$ and by $2A_N(1+\log\beta)/\beta$ in general. □

Corollary 80 (the response of a cube at weak coupling)**.** *Let $G=\mathrm{SU}(2)$, $\beta\ge 1$ and $n\ge 2$, and put $X_n(\beta)=38400n^4(\log\beta+6.11)/\beta$. Then*

$$\Xi_n\ge\frac{n^4}{7n^3+3n^2}\big(1-X_n(\beta)\big),$$

and $D_L^0(n,\alpha)$ for one $L\ge 2n+4$, hence for all, implies $1-\alpha\le X_n(\beta)$.

Proof. By Theorem 72(ii), $D_L^0(n,\alpha)$ does not depend on L and holds with $\alpha=(7n^3+3n^2)\Xi_n/n^4$. If $\alpha<1$, Proposition 67(i) gives $\mathrm{BP}(n,1-\alpha)$ on Λ_{2n+4}, and Theorem 79(ii) gives $1-\alpha\le X_n(\beta)$; if $\alpha\ge 1$ the bound on Ξ_n holds trivially. □

So for $2\le n\le(\beta/(76800(\log\beta+6.11)))^{1/4}$ one link outside the cube moves its law by at least $n/17$ in total, in the supremum over exteriors: the response grows linearly with the cube, the behaviour that Remark 185 assigns to cubes smaller than the correlation length.

Fig. 13 places the two results of this subsection in the plane of the coupling and the cube side.

Remark 81 (what is proved about the averaged condition). Proved: $\mathrm{BP}(n,1-\alpha_n)$ on every torus and on all of $L^2(\mu)$ for SU(2) with $|\beta_W|<\frac{2}{9}$, for SU(3) with $\zeta(\beta_W)<1$ (in particular $\beta_W\le 0.35$, $n\ge 6$, $\varpi\ge 0.0176$), and for every compact connected group with semisimple Lie algebra with $24N|\beta|<\kappa_G$, for all n with $\alpha_n<1$, where $\alpha_n=O(1/n)$ is explicit (Corollary 74); the supremum form D^0 is independent of the torus and follows from a bound $\Xi_n<n/(7+3/n)$ on the total response of a cube to one link (Theorem 72); the best constant satisfies $n^4\varpi_n^*\le m^4\varpi_m^*$ and $\varpi_n^*\le 1+2/n$, and it passes from large cubes to small ones given a uniform inner gap (Proposition 75). For $\mathrm{SU}(N)$ the condition forces the spatially averaged Polyakov correlator to be $O(n^2/L^2)$, and on the torus Λ_{2n+4} it fails once $\beta\gg n^4\log\beta$; for SU(2), $\varpi\le 38400n^4(\log\beta+6.11)/\beta$, and the cube response Ξ_n is at least of order n in that regime (Theorem 79, Corollary 80). Hence $\mathrm{BP}(n_k,\varpi)$ uniform in the torus requires $n_k^4\ge\varpi\beta_k/(C_N(1+\log\beta_k))$ for $\beta_k\ge 1$; the continuum tuning, with n_k of order ℓ/a_k and β_k of order $\log(1/a_k)$, meets this requirement, and the obstruction comes from tori of side $2n+4$, which Theorem 66 does not need, since it uses an infinite set of L. Not proved: $\mathrm{BP}(n_k,\varpi)$ along the continuum tuning for any non-abelian group, or at any coupling outside the ranges above; a transfer from small cubes to large ones without the loss $(n/m)^4$; an obstruction on large tori at weak coupling. Sec. X H shows that the obstruction on Λ_{2n+4} lies in the centre-charged part of $\mathscr{A}$, that the continuum step needs only the local form $\mathrm{BP}_{\mathrm{loc}}$, and that the comparison of cube sizes loses exactly $(n/m)^4$ (Remark 90); Sec. X I reduces the failure of the neutral part on Λ_{2n+4} to a lower bound on the law of the Polyakov loop (Theorem 92).

H. The averaged condition at weak coupling: centre sectors and local slow modes

The condition $\mathrm{BP}(n_k,\varpi)$ along the continuum tuning is the spectral gap that Theorem 66 needs, and it is not proved here. This subsection separates the part of the condition that the continuum step uses from the part that Theorem 79 obstructs, and it records what each available route to the condition gives. Three facts organise it. For $\mathrm{SU}(N)$ the cube moves commute with the centre twists of the torus, so the condition splits into centre sectors; the test functions of Theorem 79 are charged, and every local gauge-invariant function is neutral. The proof of Theorem 64 uses the condition only through the spectral measures of local functions, so a local form of it suffices for the continuum step. Functions of a bounded set of links satisfy a Poincaré inequality for the cube chain uniformly in the torus, so the slow functions that the condition excludes are spread over growing sets of links.

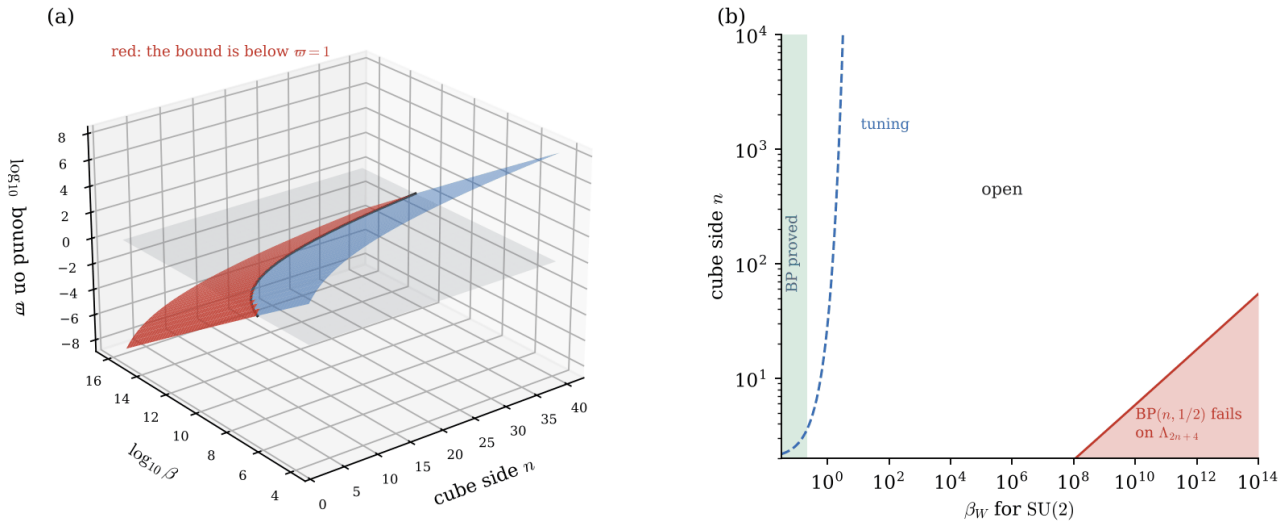


FIG. 13. The averaged condition at strong coupling and at fixed cube size. (a) The bound $\log_{10}\big(38400\,n^4(\log\beta+6.11)/\beta\big)$ of Theorem 79(ii) for SU(2) over the cube side n and $\log_{10}\beta$; where it lies below the plane $\varpi = 1$ (red), BP$(n,1)$ fails on Λ_{2n+4}, and the black curve is the crossing. (b) For SU(2) in the variables $\beta_W = 4\beta$ and n: the range $|\beta_W| < \frac{2}{9}$ of Corollary 74, where BP(n,ϖ) holds on every torus for all $n \geq n_0(\beta_W)$ (green); the region where Theorem 79(ii) excludes BP$(n,\frac{1}{2})$ on Λ_{2n+4} (red); and the continuum tuning $n \propto e^{3\pi^2\beta_W/11}$ at fixed physical cube size (dashed, schematic, the power prefactor omitted), which runs through the region where neither result applies.

The setting is that of Sec. X F on the torus Λ_L, $L \geq 2n+4$ even, with $\mathscr{K} = L^{-4}\sum_x \mathscr{P}_x$. Put

$$\mathscr{E}_n(f) = \frac{1}{n^4}\sum_{x\in X}\mu\big(|f-\mathscr{P}_x f|^2\big), \qquad f \in L^2(\mu),$$

so that $\langle f,(I-\mathscr{K})f\rangle = n^4L^{-4}\mathscr{E}_n(f)$, and BP$(n,\varpi)$ reads $\varpi\,\mathrm{Var}_\mu(f) \leq \mathscr{E}_n(f)$ for real $f \in \mathscr{A}$; here $\mathrm{Var}_\mu(f) = \mu(|f-\mu(f)|^2)$. Let $\mathscr{A}_\mathbb{C}$ be the space of complex gauge-invariant functions in $L^2(\mu)$. For $G = \mathrm{SU}(N)$, $N \geq 2$, put $\omega = e^{2\pi i/N}$ and $z = \omega I$. For a direction ν and $t \in \mathbb{Z}/L\mathbb{Z}$ let Z_t^ν be the map that multiplies every link (y,ν) with $y_\nu = t$ by z and fixes the other links; the maps $Z_{c,d}$ of Sec. X G are $(Z_0^0)^c(Z_0^1)^d$.

Proposition 82 (centre sectors)**.** *Let $G = \mathrm{SU}(N)$, $N \geq 2$.*

(i) *Each Z_t^ν preserves μ, maps $\mathscr{A}_\mathbb{C}$ onto itself, and satisfies $(\mathscr{P}_x f)\circ Z_t^\nu = \mathscr{P}_x(f\circ Z_t^\nu)$ for every x and every $f \in L^2(\mu)$.*

(ii) *For $f \in \mathscr{A}_\mathbb{C}$ the function $f\circ Z_t^\nu$ does not depend on t; write $f\circ Z^\nu$ for it.*

(iii) *For $r \in (\mathbb{Z}/N\mathbb{Z})^4$ let $\mathscr{A}_r = \{f \in \mathscr{A}_\mathbb{C} : f\circ Z^\nu = \omega^{r_\nu} f,\ \nu = 0,\dots,3\}$. Then $\mathscr{A}_\mathbb{C}$ is the orthogonal sum of the $\mathscr{A}_r$, each $\mathscr{A}_r$ is invariant under every $\mathscr{P}_x$, the constants lie in $\mathscr{A}_0$, and complex conjugation maps $\mathscr{A}_r$ onto $\mathscr{A}_{-r}$.*

(iv) *If $f \in \mathscr{A}_\mathbb{C}$ depends only on the links with base points in a box $b+[0,s)^4$ with $s \leq L-1$, then $f \in \mathscr{A}_0$.*

(v) BP(n,ϖ) *holds if and only if $\varpi\,\mathrm{Var}_\mu(f) \leq \mathscr{E}_n(f)$ for every r and every $f \in \mathscr{A}_r$; for $r \neq 0$ and $f \in \mathscr{A}_r$, $\mu(f) = 0$.*

Proof. (i) A plaquette that contains a link (y,ν) with $y_\nu = t$ lies in a plane $\{\nu,\nu'\}$ and contains exactly one other link of direction ν, with the same ν-coordinate and the opposite orientation; a plaquette in a plane without ν contains no link of direction ν. Since z is central, Z_t^ν leaves every U_p unchanged, and it preserves Haar measure, so it preserves μ. Since z is central, it also commutes with every gauge transformation, so it maps $\mathscr{A}_\mathbb{C}$ into itself, and onto it, as $(Z_t^\nu)^N$ is the identity. Write Z_t^ν as $Z_V \times Z'$ on $G^{V_x}\times G^{E\setminus V_x}$. The density of $\gamma^0_{V_x}(\cdot\,|\,U)$ at σ is proportional to the product of φ over the plaquettes meeting V_x of the configuration equal to σ on V_x and to U elsewhere, and Z_t^ν leaves these plaquettes unchanged; with the invariance of Haar measure under Z_V, the law $\gamma^0_{V_x}(\cdot\,|\,Z_t^\nu U)$ is the image of $\gamma^0_{V_x}(\cdot\,|\,U)$ under Z_V. Hence

$$\begin{aligned}\mathscr{P}_x(f\circ Z_t^\nu)(U) &= \int f(Z_V\sigma, Z'U_{E\setminus V_x})\,\gamma^0_{V_x}(d\sigma\,|\,U)\\ &= (\mathscr{P}_x f)(Z_t^\nu U).\end{aligned}$$

(ii) Let g be the gauge transformation equal to z^{-1} at the sites y with $y_\nu = t+1$ and to I elsewhere. It multiplies the links (y,ν) with $y_\nu = t$ by z and those with $y_\nu = t+1$ by z^{-1}, and it fixes every other link, because z is central and the two ends of a link of direction $\nu' \neq \nu$ have the same ν-coordinate. Hence Z_t^ν is Z_{t+1}^ν followed by g, and $f\circ Z_t^\nu = f\circ Z_{t+1}^\nu$ for gauge-invariant f.

(iii) By (i) and (ii) the maps $f \mapsto f\circ Z^\nu$ are unitary on $\mathscr{A}_\mathbb{C}$, commute with one another and have order N. So the operators $\Pi_r f = N^{-4}\sum_j \omega^{-r\cdot j} f\circ(Z^0)^{j_0}\cdots(Z^3)^{j_3}$, $j \in (\mathbb{Z}/N\mathbb{Z})^4$, are orthogonal projections onto the $\mathscr{A}_r$ with $\sum_r \Pi_r = I$ and $\Pi_r\Pi_{r'} = 0$ for $r \neq r'$, and by (i) they commute with every $\mathscr{P}_x$. Constants are fixed by every Z^ν, and $\bar f\circ Z^\nu = \overline{f\circ Z^\nu}$.

(iv) The residue $t = b_\nu - 1$ is not among $b_\nu, \dots, b_\nu + s - 1$ modulo L, since $s \le L-1$. So Z_t^ν fixes every link on which f depends, and $f \circ Z^\nu = f \circ Z_t^\nu = f$ for each ν.

(v) Let $f \in \mathscr{A}_{\mathbb{C}}$ and $f_r = \Pi_r f$. Since Π_r commutes with $\mathscr{P}_x$ and with the projection onto the constants, and the $\mathscr{A}_r$ are orthogonal, $\mathrm{Var}_\mu(f)$ and $\mathscr{E}_n(f)$ are the sums over r of $\mathrm{Var}_\mu(f_r)$ and $\mathscr{E}_n(f_r)$. Since $\mathscr{P}_x$ maps real functions to real functions, both are also the sums of their values at $\mathrm{Re}\, f$ and $\mathrm{Im}\, f$. So the inequality for real $f \in \mathscr{A}$, the inequality for all $f \in \mathscr{A}_{\mathbb{C}}$ and the inequality on every $\mathscr{A}_r$ are equivalent. For $r \neq 0$, $\mu(f) = \langle 1, f\rangle = 0$, since $1 \in \mathscr{A}_0$. □

A local gauge-invariant function, such as a Wilson loop of a contractible loop or a function of the plaquettes of a box, is therefore neutral, while the trace of a Polyakov loop of direction ν lies in $\mathscr{A}_r$ with $r = e_\nu$, since the loop crosses each slice $\{y_\nu = t\}$ once.[21] The continuum step needs the condition only on neutral functions, and only through local ones. Call $f \in L^2(\mu)$ *local* if it depends only on the links with base points in some box of side at most $L/2$.

Definition 83. Let G, φ and Λ_L be as in Sec. X F, $n \ge 2$ and $\varpi > 0$, and put $\varepsilon = \varpi n^4 / L^4$.

(a) $\mathrm{BP}_{\mathrm{loc}}(n, \varpi)$: for every local real $f \in \mathscr{A}$ and every integer $t \ge 0$,

$$\big\| \mathscr{K}^t \big(f - \mu(f) \big) \big\|_2 \le (1-\varepsilon)^t \big\| f - \mu(f) \big\|_2 .$$

(b) For $G = \mathrm{SU}(N)$, $\mathrm{BP}_0(n, \varpi)$: $\varpi \,\mathrm{Var}_\mu(f) \le \mathscr{E}_n(f)$ for every real $f \in \mathscr{A}_0$.

For a self-adjoint $0 \le \mathscr{K} \le I$ and $g \in L^2(\mu)$ with spectral measure ν_g, $\|\mathscr{K}^t g\|_2^2 = \int \lambda^{2t} d\nu_g$; so the inequality of (a) for all t says that ν_g is carried by $[0, 1-\varepsilon]$ for $g = f - \mu(f)$, the spectral gap of the cube chain on the cyclic subspace of each local function. Condition (a) involves no function of links spread over the torus.[22]

Theorem 84 (the local form of the averaged condition)**.** *Let G, φ and Λ_L be as in Sec. X F.*

(i) $\mathrm{BP}(n, \varpi)$ *implies* $\mathrm{BP}_{\mathrm{loc}}(n, \varpi)$*; for* $G = \mathrm{SU}(N)$, $\mathrm{BP}(n, \varpi)$ *implies* $\mathrm{BP}_0(n, \varpi)$*, which implies* $\mathrm{BP}_{\mathrm{loc}}(n, \varpi)$.

(ii) *If* $\mathrm{BP}_{\mathrm{loc}}(n, \varpi)$ *holds on* Λ_L*, then* $\varpi \le 4$*, and both bounds of Theorem 64 hold for bounded* $f, g \in \mathscr{A}$ *with* f *local.*

(iii) *Corollary 65 and Theorem 66 hold with* $\mathrm{BP}_{\mathrm{loc}}(n, \varpi)$*, respectively* $\mathrm{BP}_{\mathrm{loc}}(n_k, \varpi)$*, in place of* BP*; so does the form of Proposition 47*(iii) *stated after Theorem 66.*

(iv) *If* $\mathrm{BP}_{\mathrm{loc}}(n, \varpi)$ *holds, then every local real* $f \in \mathscr{A}$ *and every real* $h \in \mathscr{A}$ *satisfy*

$$\mathrm{Cov}_\mu(f,h)^2 \le \mathrm{Var}_\mu(f)\, \frac{\mathscr{E}_n(h)}{\varpi} .$$

If it fails, there are a local real $f \in \mathscr{A}$ *and a real* $h \in \mathscr{A}$ *with* $\mathscr{E}_n(h) < \varpi \,\mathrm{Var}_\mu(h)$ *and* $\mathrm{Cov}_\mu(f,h) = \mathrm{Var}_\mu(h) > 0$*; for* $G = \mathrm{SU}(N)$, $h \in \mathscr{A}_0$.

Proof. (i) Let $\mathscr{B}$ be $\mathscr{A}$, or $\mathscr{A}_0$ for $G = \mathrm{SU}(N)$, and $\mathscr{B}^0$ its functions of mean zero. Every $\mathscr{P}_x$ maps $\mathscr{A}$ into itself (Sec. X F) and $\mathscr{A}_0$ into itself (Proposition 82(iii)), and $\mu(\mathscr{P}_x g) = \mu(g)$; so $\mathscr{K}$ maps $\mathscr{B}^0$ into itself, and there it is self-adjoint with $0 \le \mathscr{K} \le I$. If the inequality of $\mathrm{BP}(n, \varpi)$ holds on the real functions of $\mathscr{B}$, then $\langle g, (I - \mathscr{K}) g\rangle \ge \varepsilon \|g\|_2^2$ on $\mathscr{B}^0$, the spectrum of $\mathscr{K}$ on $\mathscr{B}^0$ lies in $[0, 1-\varepsilon]$, and $\|\mathscr{K}^t g\|_2 \le (1-\varepsilon)^t \|g\|_2$ for $g \in \mathscr{B}^0$. A local real $f \in \mathscr{A}$ lies in $\mathscr{A}_0$ by Proposition 82(iv), since $L/2 \le L-1$, so $g = f - \mu(f) \in \mathscr{B}^0$ in both cases.

(ii) Let $f = \mathrm{Re}\,\mathrm{Tr}\, U_p$ for a plaquette p, a local function with $\mathrm{Var}_\mu(f) > 0$, and $g = f - \mu(f)$. By $\mathrm{BP}_{\mathrm{loc}}(n, \varpi)$ with $t = 1$, $\langle g, \mathscr{K} g\rangle \le \|g\|_2 \|\mathscr{K} g\|_2 \le (1-\varepsilon)\|g\|_2^2$, so $\langle g, (I-\mathscr{K}) g\rangle \ge \varepsilon \|g\|_2^2$. As in the proof of Theorem 64, $\langle g, (I-\mathscr{K}) g\rangle = L^{-4} \sum_x \mu((g - \mathscr{P}_x g)^2) \le 4 n^4 L^{-4} \|g\|_2^2$. Hence $\varpi \le 4$. The proof of Theorem 64 uses $\mathrm{BP}(n, \varpi)$ in two places only: for the bound $\varpi \le 4$, just obtained, and in its spectral part, for $\|\mathscr{K}^T (f - \mu(f))\|_2 \le e^{-\varepsilon T} \|f - \mu(f)\|_2$, which $\mathrm{BP}_{\mathrm{loc}}(n, \varpi)$ gives for local real f. The real and imaginary parts of a local function are local.

(iii) In the proof of Corollary 65, Theorem 64 is applied on Λ_L, $L \in \mathscr{L}$ large, to $\vartheta(\bar{F})$ and $\tau_t F$, which depend on links with base points in a box whose side depends on F and t only, hence are local once L is large; (ii) applies. Theorem 66 and the form of Proposition 47(iii) use the condition only through Corollary 65.

(iv) Let E be the spectral projection of $\mathscr{K}$ on $L^2(\mu)$ for the interval $(1-\varepsilon, 1]$, f local, real and in $\mathscr{A}$, and $g = f - \mu(f)$. Under $\mathrm{BP}_{\mathrm{loc}}(n, \varpi)$, $\int \lambda^{2t} d\nu_g \le (1-\varepsilon)^{2t} \nu_g([0,1])$ for all t, so $\nu_g((1-\varepsilon+\eta, 1]) = 0$ for every $\eta > 0$, and $Eg = 0$. On the range of $I - E$, $I - \mathscr{K} \ge \varepsilon$, so for real $h \in \mathscr{A}$,

$$\begin{aligned} \mathrm{Cov}_\mu(f,h)^2 &= \langle g, (I-E)h\rangle^2 \\ &\le \|g\|_2^2 \, \frac{\langle (I-E)h, (I-\mathscr{K})(I-E)h\rangle}{\varepsilon} \\ &\le \|g\|_2^2 \, \frac{\langle h, (I-\mathscr{K})h\rangle}{\varepsilon} \\ &= \mathrm{Var}_\mu(f)\, \frac{\mathscr{E}_n(h)}{\varpi} . \end{aligned}$$

If $\mathrm{BP}_{\mathrm{loc}}(n, \varpi)$ fails, some local real $f \in \mathscr{A}$ has $\nu_g((1-\varepsilon, 1]) > 0$ for $g = f - \mu(f)$; put $h = Eg$. Spectral projections of $\mathscr{K}$ are strong limits of polynomials in $\mathscr{K}$, so h is real and lies in $\mathscr{A}$, and in $\mathscr{A}_0$ for $G = \mathrm{SU}(N)$, since g does. As $\mathscr{K} 1 = 1$, $E1 = 1$ and $\mu(h) = \langle E1, g\rangle = 0$; then $\mathrm{Var}_\mu(h) = \|Eg\|_2^2 = \langle g, Eg\rangle = \mathrm{Cov}_\mu(f,h) > 0$, and since ν_h is carried by $(1-\varepsilon, 1]$, $\langle h, (I - \mathscr{K})h\rangle = \int (1-\lambda)\, d\nu_h < \varepsilon \|h\|_2^2$, that is, $\mathscr{E}_n(h) < \varpi \,\mathrm{Var}_\mu(h)$. □

Fig. 14 shows the sectors and the spectral picture behind Theorem 84.

[21] The twists Z_t^ν are the lattice form of the centre symmetry of the pure gauge theory on a torus. They change no plaquette, so they are invisible to the action, and the trace of a Polyakov loop, which they multiply by ω, is the order parameter of the deconfinement transition at finite temperature; by Proposition 82(v) its mean is zero on every torus.

[22] The distinction is the one between a spectral gap of the whole generator and a gap on the cyclic subspaces of the observables that are measured. Only the latter enters a bound on their correlations, and a lower bound on the Rayleigh quotient of a function does not give it; Appendix H proves both statements in the generality of a self-adjoint contraction.

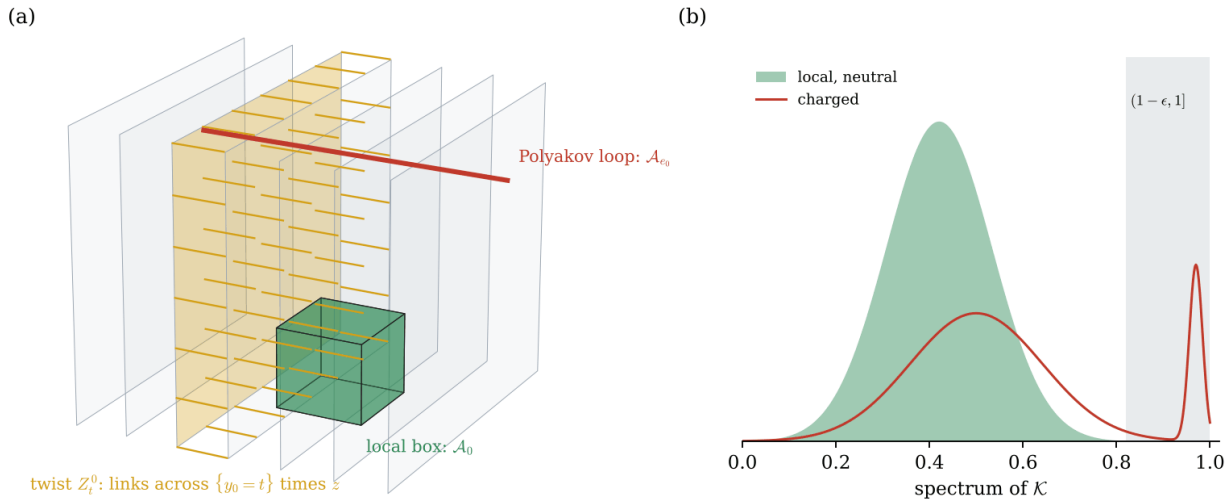


FIG. 14. Centre sectors and the local form of the averaged condition. (a) A three-dimensional section of the torus with the slices $\{y_0 = t\}$. The twist Z_t^0 multiplies the links that cross one slice by the central element z (gold) and leaves every plaquette unchanged; a Polyakov loop of direction 0 crosses every slice once and is charged, while a gauge-invariant function of a box that misses one slice is neutral (Proposition 82). (b) Spectral measures of the cube chain $\mathcal{K}$, schematic: under $\mathrm{BP}_{\mathrm{loc}}$ the measure of a local function puts no mass on $(1-\varepsilon, 1]$ (green), while a charged function may carry mass there (red), as the test functions of Theorem 79 do at weak coupling (Theorem 84, Corollary 85).

By (iv), under $\mathrm{BP}_{\mathrm{loc}}$ the functions on which the cube chain is slow are nearly orthogonal to every local gauge-invariant function, with the defect of orthogonality bounded by their Dirichlet form, and a failure of $\mathrm{BP}_{\mathrm{loc}}$ produces a slow function correlated with a local one. Slow functions orthogonal to all local ones, and for $\mathrm{SU}(N)$ slow functions in the charged sectors, do not affect the continuum step. The Villain obstruction persists for the local form: the proof of Corollary 68 uses the condition only through Theorem 64 for the functions Ψ_p, and through the bound $\varpi \le 4$, which holds with Ψ_p in place of $\mathrm{Re}\,\mathrm{Tr}\,U_p$, a nonconstant function of the four links of p. So for the Villain theory at $\beta > \beta_0$ there are no n and $\varpi > 0$ such that $\|\mathcal{K}^t(\Psi_p - \mu(\Psi_p))\|_2 \le (1 - \varpi n^4/|X|)^t \|\Psi_p - \mu(\Psi_p)\|_2$ for all t, all plaquettes and the measures with free boundary of all boxes of side at least $2n$. The local form, like the full one, has to use the non-abelian structure of G.

For $\mathrm{SU}(N)$ the obstruction of Sec. X G lies in the charged sectors. Let $\mathscr{C}$ be the set of real $f \in \mathscr{A}$ with $\Pi_0 f = 0$, in the notation of the proof of Proposition 82(iii); it is orthogonal to $\mathscr{A}_0$ and preserved by every $\mathscr{P}_x$.

Corollary 85 (the fixed-size obstruction is charged)**.** *Let $G = \mathrm{SU}(N)$, $N \ge 2$. Theorem 79 holds, with the same constants in* (i) *and* (ii) *and with another constant C_N in* (iii)*, when the inequality of $\mathrm{BP}(n, \varpi)$ is assumed only for $f \in \mathscr{C}$. Its hypotheses and conclusions concern only the restriction of the condition to $\mathscr{C}$; neither $\mathrm{BP}_0(n, \varpi)$ nor $\mathrm{BP}_{\mathrm{loc}}(n, \varpi)$ enters it.*

Proof. In the proof of Theorem 79(i) the condition is applied to $L^{-3}\sum_y \mathrm{Re}\,\mathrm{Tr}\,P_y$ and to the corresponding imaginary part. The trace of P_y lies in $\mathscr{A}_{e_0}$ and its conjugate in $\mathscr{A}_{-e_0}$, and $\pm e_0 \ne 0$ in $(\mathbb{Z}/N\mathbb{Z})^4$, so both functions lie in $\mathscr{C}$. In the proof of (ii) and (iii), $F_{ab} = \mathrm{Tr}(A^a B^b)$ lies in $\mathscr{A}_r$ with $r = (a, b, 0, 0) \ne 0$ for $(a, b) \in I_N$, so the real and imaginary parts of F_{ab} lie in $\mathscr{C}$. The only other use of the condition is the bound $\varpi \le 2$ from Proposition 75(ii), whose test function is neutral. In its place take $f_* = \mathrm{Re}\,\mathrm{Tr}\,P_0 \in \mathscr{C}$, a nonconstant continuous function, so $\mathrm{Var}_\mu(f_*) > 0$. $\mathscr{P}_x f_* = f_*$ unless V_x meets the loop P_0, which happens for at most Ln^3 points x, and $\mu((f_* - \mathscr{P}_x f_*)^2) \le \mathrm{Var}_\mu(f_*)$; so $\varpi \le L/n \le 4$ on Λ_{2n+4}. In the case $\eta_N^2 < 4L^4\varepsilon \le 1024 n^4 \varepsilon$ of that proof this gives $\varpi \le (4096/\eta_N^2) n^4 \varepsilon$, which for $N = 2$ is at most $19200\, n^4 \varepsilon$, so the constant 38400 of (ii) is unchanged. $\mathscr{C}$ is orthogonal to $\mathscr{A}_0$, which contains every local function of $\mathscr{A}$. □

Corollary 85 constrains every argument that proves the condition through a contraction of the cube chain on gauge orbits. On G^E let $d(U, U') = \min_g d_E(U, gU')$, the minimum over gauge transformations g, with d_E as in the proof of Proposition 67; since gauge transformations act by isometries of d_E, d is a pseudometric, constant on pairs of gauge orbits. Write W_1^d for the Kantorovich distance of d. By gauge covariance, $\mathcal{K}(gU', \cdot)$ is the image of $\mathcal{K}(U', \cdot)$ under g, so a contraction of $W_1^{d_E}$ by a factor $1 - \kappa$, as in the proof of Proposition 67(i), gives one of W_1^d by the same factor.

Corollary 86 (couplings on gauge orbits)**.** *Let $G = \mathrm{SU}(2)$, $\beta \ge 1$, $n \ge 2$ and $L = 2n + 4$. If there is $\kappa > 0$ with*

$$W_1^d\big(\mathcal{K}(U, \cdot), \mathcal{K}(U', \cdot)\big) \le (1 - \kappa)\, d(U, U')$$

for all configurations U, U' of Λ_L, then

$$\kappa \le \frac{38400\, n^8 (\log \beta + 6.11)}{L^4 \beta}.$$

Proof. Let h be gauge invariant and Lipschitz for d with constant $\mathrm{Lip}(h)$. For a coupling of $\mathscr{K}(U,\cdot)$ and $\mathscr{K}(U',\cdot)$ with expected distance W_1^d, $|\mathscr{K}h(U)-\mathscr{K}h(U')| \le \mathrm{Lip}(h)W_1^d$, so $\mathscr{K}h$ is Lipschitz with constant $(1-\kappa)\mathrm{Lip}(h)$, and it is gauge invariant. As in the proof of Proposition 67(i), with $d \le D|E|$, the spectral measure of $\mathscr{K}$ at $h-\mu(h)$ is carried by $[0,1-\kappa]$. Such h are dense in $\mathscr{A}$: averaging over the compact gauge group is the orthogonal projection of $L^2(\mu)$ onto $\mathscr{A}$ and maps continuous functions to continuous ones, so the continuous functions of $\mathscr{A}$ are dense in it; a continuous gauge-invariant f is uniformly continuous for d, since $d(U,U')$ small gives g with $d_E(U,gU')$ small and $f(gU')=f(U')$; and $f_k(U)=\inf_V(f(V)+k\,d(U,V))$ is gauge invariant, Lipschitz for d with constant k, and converges to f uniformly. So the spectral projection of $\mathscr{K}$ for $(1-\kappa,1]$ vanishes on the functions of $\mathscr{A}$ of mean zero, $\langle f,(I-\mathscr{K})f\rangle \ge \kappa\,\mathrm{Var}_\mu(f)$ on $\mathscr{A}$, and $\mathrm{BP}(n,\kappa L^4/n^4)$ holds on Λ_{2n+4}. Theorem 79(ii) gives the bound. □

The proof uses only that d-Lipschitz functions are dense in the charged part $\mathscr{C}$. If d is replaced by its minimum over the centre twists as well, $d'(U,U')=\min d_E(U,gZU')$ over gauge transformations g and products Z of the maps Z_t^ν, the d'-Lipschitz functions are neutral, and the same argument, with the average over the gauge group and the group generated by the Z_t^ν, gives $\mathrm{BP}_0(n,\kappa L^4/n^4)$, which Theorem 79 does not constrain. A coupling argument for the cube chain at weak coupling and small n therefore has to contract a pseudometric that does not see the centre twists. Since $n^8/(2n+4)^4 \ge n^4/256$, the bound of Corollary 86 is at least $150\,n^4(\log\beta+6.11)/\beta$, which exceeds one once $n^4 \ge \beta/150$; along the tuning, where n_k^4 grows faster than every power of β_k, the corollary does not exclude such couplings.

The remaining results hold for every G and φ of Sec. X F, on tori and on boxes. Put $\omega_\varphi=\log(\max\varphi/\min\varphi)$, so that $\omega_\varphi \le 2N^2|\beta|$ for the Wilson weight, and for a finite set S of links let P_S be the set of plaquettes of P that contain a link of S; $|P_S| \le 6|S|$.

Proposition 87 (no local slow modes)**.** *Let $S \subset E$ be finite and let $f \in L^2(\mu)$ be real and depend only on the links of S. Then*

$$\mathrm{Var}_\mu(f) \le e^{5\omega_\varphi|P_S|}\,\mathscr{E}_n(f).$$

In particular, if $\mathscr{E}_n(f) \le \varpi\,\mathrm{Var}_\mu(f)$ with $0<\varpi<1$ and $\mathrm{Var}_\mu(f)>0$, then $|P_S| \ge \log(1/\varpi)/(5\omega_\varphi)$; for the Wilson weight, $|S| \ge \log(1/\varpi)/(60N^2|\beta|)$.

Proof. For a finite $W \subset E$ write H_W for the normalised Haar measure on G^W and $\rho_W(\cdot\,|\,U)$ for the density of $\gamma_W^0(\cdot\,|\,U)$ with respect to H_W. It is proportional to a product of $|P_W|$ values of φ, so the ratio of two of its values is at most $e^{\omega_\varphi|P_W|}$, and as its integral is one, $e^{-\omega_\varphi|P_W|} \le \rho_W \le e^{\omega_\varphi|P_W|}$. Put $a=\omega_\varphi|P_S|$, and write P_W also for the conditional expectation given the links outside W; then $P_{V_x}=\mathscr{P}_x$.

Step 1. Let ν_S be the law of U_S under μ. Its density with respect to H_S is the mean of $\rho_S(\cdot\,|\,U)$ over U, so it lies in $[e^{-a},e^a]$, and $r_U=\rho_S(\cdot\,|\,U)/(d\nu_S/dH_S) \ge e^{-2a}$. Let $g=f-\mu(f)$, a function on G^S with $\nu_S(g)=0$. Then $P_Sg(U)=\int g\,r_U\,d\nu_S=\int g\,(r_U-e^{-2a})\,d\nu_S$, and the Cauchy–Schwarz inequality for the positive measure $(r_U-e^{-2a})\,d\nu_S$ of mass $1-e^{-2a}$ gives

$$(P_Sg)^2(U) \le (1-e^{-2a})\int g^2 r_U\,d\nu_S = (1-e^{-2a})P_S(g^2)(U).$$

Integrating, $\mu((P_Sg)^2) \le (1-e^{-2a})\mu(g^2)$, and since P_S is an orthogonal projection, $\mu((f-P_Sf)^2)=\mu(g^2)-\mu((P_Sg)^2) \ge e^{-2a}\,\mathrm{Var}_\mu(f)$.

Step 2. $\mu((f-P_Sf)^2)$ is the mean over U of the variance of f under $\gamma_S^0(\cdot\,|\,U)$, which is at most $\int(f-H_S(f))^2\rho_S\,dH_S \le e^a\,\mathrm{Var}_{H_S}(f)$. So $\mathrm{Var}_\mu(f) \le e^{3a}\,\mathrm{Var}_{H_S}(f)$.

Step 3. For $W \subset S$ let v_W be the variance of f in the links of W under H_W, the links of $S\setminus W$ fixed, a function of these links. With the decomposition $f=\sum_{T\subset S}f_T$ of the proof of Proposition 67(ii) for H_S, $H_S(v_W)=\sum_{T\cap W\neq\emptyset}\|f_T\|^2$, and since every link lies in exactly n^4 of the sets V_x,

$$\sum_{x\in X} H_S(v_{S\cap V_x}) = \sum_T \|f_T\|^2\,\#\{x : T\cap V_x\neq\emptyset\} \ge n^4\,\mathrm{Var}_{H_S}(f).$$

Step 4. Let $W=S\cap V_x\neq\emptyset$. Since $d\nu_S/dH_S \ge e^{-a}$ and $v_W \ge 0$, $H_S(v_W) \le e^a\nu_S(v_W)=e^a\mu(v_W)$. For every configuration U, $v_W \le \int(f-c)^2dH_W$ with c the mean of f under $\gamma_W^0(\cdot\,|\,U)$, and $dH_W/d\gamma_W^0 \le e^{\omega_\varphi|P_W|} \le e^a$, so v_W is at most e^a times the variance of f under $\gamma_W^0(\cdot\,|\,U)$. Hence $\mu(v_W) \le e^a\mu((f-P_Wf)^2) \le e^a\mu((f-\mathscr{P}_xf)^2)$, the last step because the links outside V_x are among those outside W.

Steps 3 and 4 give $\mathrm{Var}_{H_S}(f) \le n^{-4}e^{2a}\sum_x\mu((f-\mathscr{P}_xf)^2) = e^{2a}\mathscr{E}_n(f)$, and with Step 2, $\mathrm{Var}_\mu(f) \le e^{5a}\mathscr{E}_n(f)$. For the Wilson weight, $5\omega_\varphi|P_S| \le 60N^2|\beta||S|$. □

So a sequence of functions with $\mathscr{E}_n(f_j)/\mathrm{Var}_\mu(f_j)\to 0$, on tori of growing side at fixed n and β, depends on growing sets of links,[23] and no subsequence converges to a function of finitely many links. This does not give $\mathrm{BP}_{\mathrm{loc}}$: Proposition 87 bounds the Rayleigh quotient of a local function, which is the mean of $1-\lambda$ under its spectral measure, while $\mathrm{BP}_{\mathrm{loc}}$ asks that this measure put no mass near $\lambda=1$; by Theorem 84(iv) a failure of it produces a slow function correlated with a local one.

The transfer of the condition between cube sizes is limited in the same way. Write W_x for the cubes of side m and $\mathscr{E}_m$ for the corresponding form; the proof of Proposition 75(i) shows $\mathscr{E}_m \ge (n/m)^4\mathscr{E}_n$ on $L^2(\mu)$.

Proposition 88. *Let $\varphi\equiv 1$, $2\le n\le m$, $R\ge 2$, and let L be even with $L\ge\max(R+m,2m+4)$. There is a real $f\in\mathscr{A}$ with $\mathrm{Var}_\mu(f)=1$ and*

$$\mathscr{E}_m(f) \le \Big(\frac{n}{m}\Big)^4\Big(\frac{R+m}{R-1}\Big)^4\mathscr{E}_n(f).$$

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[23] The exponent $5\omega_\varphi|P_S|$ is the cost of comparing the conditional laws of the links of S with Haar measure, in the manner of the perturbation lemma of Holley and Stroock [68]; it depends on S through the plaquettes that touch it, and not on the torus. At fixed β it forces $|S|\to\infty$ as $\varpi\to 0$, linearly in $\log(1/\varpi)$.

So no inequality $\mathscr{E}_m \geq c\,\mathscr{E}_n$ on $\mathscr{A}$ with $c > (n/m)^4$ holds on all tori, although $\mathrm{BP}(n,1)$ *and* $\mathrm{BP}(m,1)$ *both hold for this measure.*

Proof. Since G is compact and nontrivial, it has a nontrivial irreducible unitary representation, whose character has Haar mean zero; its real or imaginary part, normalised, is a real class function h with $\int h\,dU = 0$ and $\int h^2\,dU = 1$. Let Q be the set of plaquettes of the plane $\{0,1\}$ with base point $y \in [0,R)^4$ and y_0, y_1 even. The plaquette at y has the links $(y,0)$, $(y+e_0,1)$, $(y+e_1,0)$, $(y,1)$. A link $(w,0)$ of such a plaquette determines y, which is w or $w-e_1$ according to the parity of w_1, and a link $(w,1)$ determines it through the parity of w_0; so two plaquettes of Q share no link. Put $f = \prod_{p\in Q} h(U_p)$, a gauge-invariant function. For one plaquette, integrating $h(U_p)$ over any one of its four links gives $\int h\,dU = 0$, by the invariance of Haar measure under translation and inversion; so in the decomposition of the proof of Proposition 67(ii), $h(U_p)$ has the single term $T = p$, and since the plaquettes of Q involve disjoint links, $f = f_T$ with T the union of their links, and $\|f\|_2 = 1 = \mathrm{Var}_\mu(f)$. That proof gives $k^4\mathscr{E}_k(f) = N_k$, the number of x with $V_x \cap T \neq \emptyset$ for cubes of side k, $k = n,m$.

The base points of the links of T lie in $[0,R]^4$, so $N_k \leq (R+k)^4$, as $R+k \leq L$. If $x \in [0,R-2]^4$, the cube of side $k \geq 2$ at x contains the four points $x + ae_0 + be_1$, $a,b \in \{0,1\}$, of $[0,R)^4$, one of which has even coordinates 0 and 1 and is the base point of a plaquette of Q; so $N_k \geq (R-1)^4$. Hence $\mathscr{E}_m(f)/\mathscr{E}_n(f) = (n/m)^4 N_m/N_n \leq (n/m)^4((R+m)/(R-1))^4$. The last claim is Proposition 67(ii) and $R \to \infty$. □

At $\beta = 0$ the gaps at the two sizes are both at least one, while the Dirichlet forms differ by the factor $(n/m)^4$ on functions spread over large boxes. An upward transfer without that loss has to use more than the comparison of the forms, for instance an orthogonal decomposition such as that of Proposition 67(ii), which is specific to product measures.

Remark 89 (the routes to the averaged condition at weak coupling). The results above settle how far the following routes go.

Constructions. A test function bounds the best constant from above. A lower bound needs a Poincaré inequality, and Proposition 87 gives one for every function of a fixed finite set S of links, uniformly in the torus, with constant $e^{-5\omega_\varphi|P_S|}$. A Lyapunov argument derives a Poincaré inequality from a local one on a set of configurations and a drift towards that set; for the cube chain on Λ_L the set consists of configurations of all $4L^4$ links, a small-field set for instance, and the local inequality it needs is one for the restriction of μ to that set, for functions of all links, which Proposition 87 does not give. A coupling that contracts d_E or the pseudometric d on gauge orbits gives the condition on $\mathscr{A}$ on every torus to which it applies, and Corollaries 80 and 86 bound its rate on Λ_{2n+4} for $\mathrm{SU}(2)$; a coupling that contracts a pseudometric blind to the centre twists gives BP_0, which suffices by Theorem 84 and is open. The comparison with the cube chain of larger cubes reduces to the inner gap of Proposition 75(iii); that proof works sector by sector, so it gives $\mathrm{BP}_0(n,\cdot)$ from $\mathrm{BP}_0(m,\cdot)$ and the inner inequality for $f \in \mathscr{A}_0$. The comparison with the cube chain of the same size for another measure μ' through the density $d\mu/d\mu'$ costs the exponential of the oscillation of its logarithm, a sum of $6L^4$ plaquette terms; a bound uniform in the torus has to localise the comparison, which leads back to the inner gap. The comparison of Dirichlet forms between cube sizes loses exactly $(n/m)^4$ (Proposition 88).

Forced blow-up. If $\mathrm{BP}_{\mathrm{loc}}(n_k,\varpi)$ fails for every $\varpi > 0$ along the tuning, Theorem 84(iv) gives local functions f_k and slow functions h_k, in $\mathscr{A}_0$ for $\mathrm{SU}(N)$, with $\mathrm{Cov}_\mu(f_k,h_k) = \mathrm{Var}_\mu(h_k)$ and $\mathscr{E}_{n_k}(h_k)/\mathrm{Var}_\mu(h_k) \to 0$, and Proposition 87 gives $60N^2\beta_k|S| \geq \log(\mathrm{Var}_\mu(h_k)/\mathscr{E}_{n_k}(h_k))$ for every set S of links on which h_k depends. A contradiction has to come from a property of these slow functions. The properties proved here for the tuned measures, tightness of the marginals, reflection positivity, the transfer matrix, gauge invariance and centre symmetry, are properties of the measures and of their torus limits, not of the cube chain, and the paper proves no implication from any of them to $\mathrm{BP}_{\mathrm{loc}}$. Such an implication is a converse of Theorem 64, from decay of correlations to relaxation of the cube chain. For local dynamics of spin systems the known converses assume a mixing condition uniform in the boundary condition [88], of the type of the supremum form D^0, which Corollary 80 excludes for n below a multiple of $(\beta/\log\beta)^{1/4}$. A limit object along the tuning would be a slow mode of a dynamics of continuum gauge fields, and no such dynamics is constructed here. Continuity in β does not connect the regimes: Corollary 74 holds at fixed β for $n \geq n_0(\beta)$, and on Λ_{2n+4} the best constant of the charged part passes from at least one at $\beta = 0$ to $O(n^4\log\beta/\beta)$ (Theorem 79, Corollary 85); no monotonicity in β is known.

Residual sets. Theorem 84 and Corollary 85 identify the residual: the condition may fail on charged functions and on slow functions orthogonal to all local ones without affecting the continuum step, and what has to be excluded is a slow function correlated with a local neutral one. The Villain theory beyond β_0 has such slow functions, by the local form of Corollary 68. The failure of $\mathrm{BP}_{\mathrm{loc}}$ does not by itself give a trivial or Gaussian limit, since $\mathrm{BP}_{\mathrm{loc}}$ is sufficient for a gap and is not shown to be necessary; a trivialisation argument needs the converse of Theorem 64 again.

Perturbative and static estimates. By Theorem 84(ii), $\mathrm{BP}_{\mathrm{loc}}(n,\varpi)$ on infinitely many tori gives exponential decay of every local gauge-invariant covariance in the torus limit. The quadratic approximation of the Wilson measure at weak coupling consists of $\dim G$ copies of the free lattice Maxwell field, whose field strengths have correlations that decay as a power of the distance. An argument whose estimates apply equally to that Gaussian approximation, or order by order in an expansion around it, therefore cannot give $\mathrm{BP}_{\mathrm{loc}}$; the Villain theory, to which the setting applies, is a nonperturbative instance of the same failure. The renormalisation group bounds of [6, 7, 12] concern effective actions in a box of fixed physical size and do not address the cube chain. The Bakry–Émery bound of Lemma 73 requires $6N|\beta| < \kappa_G$. Reflection positivity gives the transfer matrix and the chessboard estimate bounds probabilities of local events uniformly in the torus; both are statements about the measure, not about the cube chain.

Remark 90 (what is proved about the averaged condition at weak coupling). Proved: for $\mathrm{SU}(N)$ the cube moves preserve

the centre sectors of the torus, local gauge-invariant functions are neutral, and the condition splits into sectors (Proposition 82); the conclusions of Theorems 64 and 66 and of Corollary 65 follow from the local form $\mathrm{BP}_{\mathrm{loc}}$, which is implied by the neutral form BP_0; under $\mathrm{BP}_{\mathrm{loc}}$ slow functions are nearly orthogonal to local ones, and a failure of it produces a slow function correlated with a local one (Theorem 84); the obstruction of Theorem 79 holds for the charged part alone, and neither BP_0 nor $\mathrm{BP}_{\mathrm{loc}}$ enters it (Corollary 85); for $\mathrm{SU}(2)$ a contraction of the cube chain on the gauge orbits of Λ_{2n+4} has rate at most $38400\,n^8(\log\beta+6.11)/(L^4\beta)$ (Corollary 86); every function of a finite set S of links satisfies $\mathrm{Var}_\mu(f)\le e^{5\omega_\varphi|P_S|}\mathscr{E}_n(f)$ on every torus and box (Proposition 87); and the upward comparison of the Dirichlet forms loses exactly $(n/m)^4$, already at $\beta=0$ (Proposition 88). Not proved: BP, BP_0 or $\mathrm{BP}_{\mathrm{loc}}$ along the continuum tuning for any non-abelian group; whether BP_0 or $\mathrm{BP}_{\mathrm{loc}}$ fails on Λ_{2n+4} at weak coupling, of which Theorem 92 reduces the first, for $\mathrm{SU}(2)$, to a lower bound on the law of the Polyakov loop; a converse of Theorem 64, from decay of correlations to the relaxation of the cube chain; a coupling that contracts a pseudometric blind to the centre twists; and the mass gap.

I. The neutral sector on small tori

Theorem 79 proves that the averaged condition fails on the torus Λ_{2n+4} at weak coupling, and Corollary 85 places that failure in the charged part of $\mathscr{A}$. This subsection treats the neutral part for $\mathrm{SU}(2)$. The neutral functions of a Polyakov loop change under the cube moves only through the curvature, as the charged ones do, so the right side of BP_0 is small for them; what is missing is a lower bound on their variance. A flow that turns every Polyakov loop of one direction inside its own one-parameter subgroup changes the Wilson action only through the curvature, and integration by parts along it bounds the concentration of the law of the Polyakov loop. The outcome is that BP_0 at weak coupling on Λ_{2n+4} forces the Polyakov loop towards the centre and towards the traceless elements of $\mathrm{SU}(2)$, and that the failure of BP_0 there reduces to one lower bound on the law of the Polyakov loop.

Throughout, $G=\mathrm{SU}(2)$ with the Wilson weight $\varphi=e^{2\beta\,\mathrm{Tr}(\cdot)}$, $\Lambda=\Lambda_L$ with L even, and $H=\sum_p(2-\mathrm{Tr}\,U_p)$, so that μ has density proportional to $e^{-2\beta H}$. Every $W\in\mathrm{SU}(2)$ can be written $W=\cos\theta\,I+\sin\theta\,u$ with $\theta\in[0,\pi]$ and u traceless anti-Hermitian with $u^2=-I$, and $\mathrm{Tr}\,W=2\cos\theta$; in these polar coordinates of the three-sphere the Haar measure is $\frac{2}{\pi}\sin^2\theta\,d\theta$ times the uniform measure of u. For $W\in\mathrm{SU}(2)$ put

$$J(W)=\mathrm{Tr}(W)\,\tfrac12(W-W^*)=2\cos\theta\sin\theta\,u,$$

a traceless anti-Hermitian matrix that commutes with W and satisfies $J(gWg^{-1})=gJ(W)g^{-1}$ for $g\in\mathrm{SU}(2)$ and $J(-W)=J(W)$. As in Sec. X G, P_b is the Polyakov loop of direction 0 through the site b, the ordered product of the L links $(b+te_0,0)$, $0\le t<L$, and we put $T_b=\mathrm{Tr}\,P_b$ and $T=T_0$. By the symmetries of the torus every T_b, and the trace of every Polyakov loop of another direction, has the law of T under μ.

On the configuration space G^E let $\mathscr{V}$ be the vector field whose component at a link $(b,0)$ is $L^{-1}J(P_b)U_{(b,0)}$ and whose components at the links of the directions $1,2,3$ vanish; it is the velocity at $s=0$ of $U_{(b,0)}\mapsto e^{sJ(P_b)/L}U_{(b,0)}$. Its entries are polynomials in the entries of the links, so $\mathscr{V}$ is smooth, and for a function h of class C^1 we write $\mathscr{V}h$ for the derivative of h along $\mathscr{V}$. Put

$$\Theta=\mathscr{V}S-\mathrm{div}\,\mathscr{V},\qquad S=-2\beta\sum_p\mathrm{Tr}\,U_p,$$

where div is the divergence for the product of the Haar measures.

Lemma 91 (the holonomy flow)**.** *Let $L\ge8$ and $\beta\ge2$.*

(i) *For every site y, $\mathscr{V}P_y=J(P_y)P_y$; in particular $\mathscr{V}T=-\frac12T(4-T^2)$.*

(ii) $\mathrm{div}\,\mathscr{V}=\sum_{y:y_0=0}(2T_y^2-2)$, *so* $|\mathrm{div}\,\mathscr{V}|\le6L^3$.

(iii) $|\mathscr{V}S|\le16\beta\sum_q(2-\mathrm{Tr}\,U_q)$, *the sum over the plaquettes q of the planes $\{0,\nu\}$, and* $\|\Theta\|_2\le192L^4(\log\beta+5.44)$.

(iv) *For every real h of class C^1, $\mu(\mathscr{V}h)=\mu(h\Theta)$ and $\mu(\mathscr{V}h)^2\le\mathrm{Var}_\mu(h)\|\Theta\|_2^2$.*

Proof. (i) For $0\le t<L$ write U_t for the link $(y+te_0,0)$ and put $G_t=U_0\cdots U_{t-1}$, so that the loop based at $y+te_0$ is $P^{(t)}=U_t\cdots U_{L-1}U_0\cdots U_{t-1}=G_t^{-1}P_yG_t$. Then $J(P^{(t)})=G_t^{-1}J(P_y)G_t$, and differentiating the product,

$$\mathscr{V}P_y=\sum_{t=0}^{L-1}G_t\,\frac{J(P^{(t)})}{L}\,U_t\cdots U_{L-1}=\sum_{t=0}^{L-1}\frac{J(P_y)}{L}\,P_y=J(P_y)P_y.$$

With $P_y=\cos\theta+\sin\theta\,u$,

$$\begin{aligned}\mathrm{Tr}(J(P_y)P_y)&=2\cos\theta\sin\theta\,\mathrm{Tr}(u\cos\theta+u^2\sin\theta)\\&=-4\cos\theta\sin^2\theta=-\tfrac12T_y(4-T_y^2).\end{aligned}$$

(ii) The divergence of a vector field on G^E is the sum over the links e of the divergence in U_e of its component at e, the other links fixed. At $e=(y+te_0,0)$ the component is $L^{-1}J(U_eR)U_e$ with $R=U_{t+1}\cdots U_{L-1}U_0\cdots U_{t-1}$ independent of U_e. The right translation $U_e\mapsto W=U_eR$ preserves Haar measure and carries this field to $W\mapsto L^{-1}J(W)W$. Since $J(W)$ commutes with W, $e^{sJ(W)}W=\cos(\theta+s\sin2\theta)+\sin(\theta+s\sin2\theta)\,u$, so in polar coordinates the field $J(W)W$ is $\sin2\theta\,\partial_\theta$, with divergence

$$\frac{1}{\sin^2\theta}\,\partial_\theta\big(\sin^2\theta\sin2\theta\big)=8\cos^2\theta-2=2(\mathrm{Tr}\,W)^2-2$$

for $0<\theta<\pi$, and by continuity on all of $\mathrm{SU}(2)$. As $\mathrm{Tr}\,W=T_y$, the L links of the loop through y contribute $2T_y^2-2$, and there are L^3 such loops.

(iii) Only the plaquettes of the planes $\{0,\nu\}$ contain links of direction 0. The plaquette p at b in the plane $\{0,\nu\}$ is $U_p=U_{(b,0)}U_{(b+e_0,\nu)}U_{(b+e_\nu,0)}^{-1}U_{(b,\nu)}^{-1}$, and with $J_b=J(P_b)$ and $V=U_{(b,\nu)}$,

$$\mathscr{V}\,\mathrm{Tr}\,U_p=\frac1L\,\mathrm{Tr}\big(U_p\Delta_p\big),$$

where

$$\Delta_p = J_b - V J_{b+e_\nu} V^{-1} = J(P_b) - J(V P_{b+e_\nu} V^{-1}).$$

The strip σ of the L plaquettes at $b+te_0$ in the plane $\{0,\nu\}$ has boundary holonomy $P_b V P_{b+e_\nu}^{-1} V^{-1}$, so by Lemma 76, $\|P_b - V P_{b+e_\nu} V^{-1}\|_{\mathrm{HS}} \le Q_\sigma = \sum_{q\in\sigma}\|U_q - I\|_{\mathrm{HS}}$. For $P, P' \in \mathrm{SU}(2)$, $|\mathrm{Tr}\,P - \mathrm{Tr}\,P'| \le \sqrt{2}\|P-P'\|_{\mathrm{HS}}$, $\|\frac12(P-P^*)\|_{\mathrm{HS}} \le \sqrt{2}$ and $\|\frac12(P-P^*) - \frac12(P'-P'^*)\|_{\mathrm{HS}} \le \|P-P'\|_{\mathrm{HS}}$, so J is 4-Lipschitz for the Hilbert–Schmidt norm and $\|\Delta_p\|_{\mathrm{HS}} \le 4Q_\sigma$. As Δ_p is anti-Hermitian, $\mathrm{Tr}\,\Delta_p$ is imaginary, and $\mathrm{Tr}(U_p\Delta_p)$ is real because $\mathscr{V}\,\mathrm{Tr}\,U_p$ is; hence $|\mathrm{Tr}(U_p\Delta_p)| = |\mathrm{Re}\,\mathrm{Tr}((U_p - I)\Delta_p)| \le 4\|U_p - I\|_{\mathrm{HS}} Q_\sigma$. Every plaquette p of the planes $\{0,\nu\}$ lies in exactly one strip σ, so

$$|\mathscr{V} S| \le \frac{8\beta}{L}\sum_\sigma Q_\sigma^2 \le 8\beta \sum_\sigma \sum_{q\in\sigma} \|U_q - I\|_{\mathrm{HS}}^2,$$

by the Cauchy–Schwarz inequality over the L plaquettes of σ; with $\|U-I\|_{\mathrm{HS}}^2 = 2(2-\mathrm{Tr}\,U)$ this is the first claim, and $|\mathscr{V} S| \le 16\beta H$.

Put $\phi(t) = \log\int e^{-2tH}dU$, convex and decreasing, and $M = \log\mu(e^{\beta H}) = \phi(\beta/2) - \phi(\beta) \ge 0$. Convexity gives $M \le -\frac{\beta}{2}\phi'(\beta/2) = \beta\mu_{\beta/2}(H)$, where $\mu_{\beta/2}$ is the Wilson measure at $\beta/2 \ge 1$, and Lemma 78 at $\beta/2$ bounds each of the $6L^4$ terms of $\mu_{\beta/2}(H)$ by $2(\log\beta + 6.11 - \log 2)/\beta$; so $M \le 12L^4(\log\beta + 5.42)$. For $y \ge 0$ and $a \ge 0$, $2ay + y^2 \le (2a+2)(e^y - 1)$, since $y \le e^y - 1$ and $y^2 \le 2(e^y - 1 - y)$; so for every $X \ge 0$, $X^2 \le a^2 + (2a+2)e^{X-a}$, and with $X = \beta H$ and $a = M$, $\beta^2\mu(H^2) \le M^2 + 2M + 2 \le (M+2)^2$. Hence $\|\mathscr{V} S\|_2 \le 16(M+2) \le 192L^4(\log\beta + 5.43)$ for $L \ge 8$, and with (ii), $\|\Theta\|_2 \le 192L^4(\log\beta + 5.43) + 6L^3 \le 192L^4(\log\beta + 5.44)$.

(iv) On the compact manifold G^E,

$$\int \mathscr{V} g\, dU = -\int g\,\mathrm{div}\,\mathscr{V}\, dU$$

for every g of class C^1. Applied to $g = he^{-S}$ this gives $\mu(\mathscr{V} h) = \mu(h(\mathscr{V} S - \mathrm{div}\,\mathscr{V})) = \mu(h\Theta)$. With $h = 1$, $\mu(\Theta) = 0$, so $\mu(\mathscr{V} h) = \mu((h - \mu(h))\Theta)$, and the Cauchy–Schwarz inequality gives the bound. □

Part (iv) with $h = \psi(T)$ and part (i) give, for every real ψ of class C^1 on $[-2,2]$,

$$\mu\big(T(4-T^2)\psi'(T)\big)^2 \le 4\,\mathrm{Var}_\mu\big(\psi(T)\big)\|\Theta\|_2^2. \qquad (17)$$

The twist Z^0 preserves μ and reverses the sign of T, so the left side vanishes for odd ψ. With $\psi(t) = t^2/4$ it says that $\mathrm{Var}_\mu(T^2/4) \ge \mu(T^2(4-T^2))^2/(16\|\Theta\|_2^2)$: a law of the Polyakov loop with $\mu(T^2(4-T^2))$ bounded away from zero gives T^2 a variance at least of order $1/(L^8\log^2\beta)$.

Theorem 92 (the neutral sector on small tori)**.** *Let $G = \mathrm{SU}(2)$, $\beta \ge 2$, $n \ge 2$, $L \ge 2n+4$ even, and $T = \mathrm{Tr}\,P_0$.*

(i) *If $\mathrm{BP}_0(n,\varpi)$ holds on Λ_L, then*

$$\mu\big(T^2(4-T^2)\big)^2 \le \frac{9\cdot 2^{19} L^{11} n\,(\log\beta + 6.11)^3}{\varpi\beta}.$$

(ii) *On Λ_{2n+4}, $\mathrm{BP}_0(n,\varpi)$ implies*

$$\mu\big(T^2(4-T^2)\big) \le \frac{4.45\cdot 10^6\, n^6(\log\beta + 6.11)^{3/2}}{\sqrt{\varpi\beta}}.$$

(iii) *If $\mu(T^2(4-T^2)) \ge c > 0$ on Λ_{2n+4}, then $\mathrm{BP}_0(n,\varpi)$ on Λ_{2n+4} implies*

$$\varpi \le \frac{9\cdot 2^{41} n^{12}(\log\beta + 6.11)^3}{c^2\beta}.$$

Proof. (i) Let $h = T^2/4$. It is real and gauge invariant; Z^0 maps P_0 to $-P_0$ and Z^ν, $\nu \ne 0$, fixes it, so $h \in \mathscr{A}_0$. If V_x meets no link of P_0, $\mathscr{P}_x h = h$, and V_x meets a link of P_0 for Ln^3 points x. For each of these choose s as in the proof of Theorem 79(ii), so that $|s| \le n$ and the loop $A' = P_{se_2}$ avoids V_x; that proof gives $P_0 = H_0 \Gamma A' \Gamma^{-1}$ with $\|H_0 - I\|_{\mathrm{op}} \le S_0$, the sum of $\|U_p - I\|_{\mathrm{op}}$ over at most Ln plaquettes. Then $|T - \mathrm{Tr}\,A'| = |\mathrm{Tr}((H_0 - I)\Gamma A'\Gamma^{-1})| \le 2S_0$, and as $|\frac{d}{dt}\frac{t^2}{4}| \le 1$ on $[-2,2]$, $h' = (\mathrm{Tr}\,A')^2/4$ satisfies $|h - h'| \le 2S_0$. Since h' depends only on the links outside V_x and $\mathscr{P}_x$ is the orthogonal projection onto such functions,

$$\mu\big((h - \mathscr{P}_x h)^2\big) \le \mu\big((h-h')^2\big) \le 4\mu(S_0^2) \le 4L^2n^2\varepsilon,$$

with $\varepsilon = \mu(\|U_p - I\|_{\mathrm{HS}}^2) \le 2(\log\beta + 6.11)/\beta$ by Lemma 78. So $\mathscr{E}_n(h) \le n^{-4} Ln^3 \cdot 4L^2n^2\varepsilon = 4L^3 n\varepsilon$, and $\mathrm{BP}_0(n,\varpi)$ gives $\mathrm{Var}_\mu(h) \le 8L^3 n(\log\beta + 6.11)/(\varpi\beta)$. By (17) with $\psi(t) = t^2/4$, $\mu(T^2(4-T^2))^2 \le 16\,\mathrm{Var}_\mu(h)\|\Theta\|_2^2$, and Lemma 91(iii) gives $\|\Theta\|_2^2 \le 36864\,L^8(\log\beta + 6.11)^2$. Since $16\cdot 8\cdot 36864 = 9\cdot 2^{19}$, (i) follows.

(ii), (iii) For $L = 2n+4 \le 4n$, $L^{11}n \le 2^{22}n^{12}$, and $3\cdot 2^{41/2} \le 4.45\cdot 10^6$. □

Under the Haar measure,

$$\mu\big(T^2(4-T^2)\big) = 16\cdot\frac{2}{\pi}\int_0^\pi \sin^4\theta\cos^2\theta\, d\theta = 2.$$

The quantity vanishes exactly when P_0 lies almost surely in $\{\pm I\} \cup \{W : \mathrm{Tr}\,W = 0\}$, and Theorem 92(ii) says that BP_0 with a fixed ϖ on Λ_{2n+4} at $\beta \gg n^{12}\log^3\beta$ forces the law of the Polyakov loop towards that set. A coupling of the cube chain that contracts a pseudometric blind to the centre twists at rate κ gives $\mathrm{BP}_0(n, \kappa L^4/n^4)$, so on Λ_{2n+4} it forces $\mu(T^2(4-T^2)) \le 4.45\cdot 10^6 n^8(\log\beta + 6.11)^{3/2}/(L^2\sqrt{\kappa\beta})$.

The exceptional set is that of the method. Consider fields whose component at each link of direction 0 is $L^{-1}K(P)U$, with U the link and P the Polyakov loop based at its initial site. The cancellation in Lemma 91(iii) uses that K is equivariant under conjugation, and the part of the field that is odd under the centre contributes nothing, since for such a field $\mathscr{V} h$ is odd under Z^0 and has mean zero for every neutral h. An equivariant field $W \mapsto K(W)W$ on $\mathrm{SU}(2)$ has $K(W)$ fixed by conjugation with the stabiliser of W, so $K(\pm I) = 0$ and, for $W = \cos\theta + \sin\theta\, u$ with $0 < \theta < \pi$, $K(W) = k(W)u$ with k real. If $\mathrm{Tr}\,W = 0$, then $-W = vWv^{-1}$ for every $v \in \mathrm{SU}(2)$ that anticommutes with u, so evenness and equivariance give

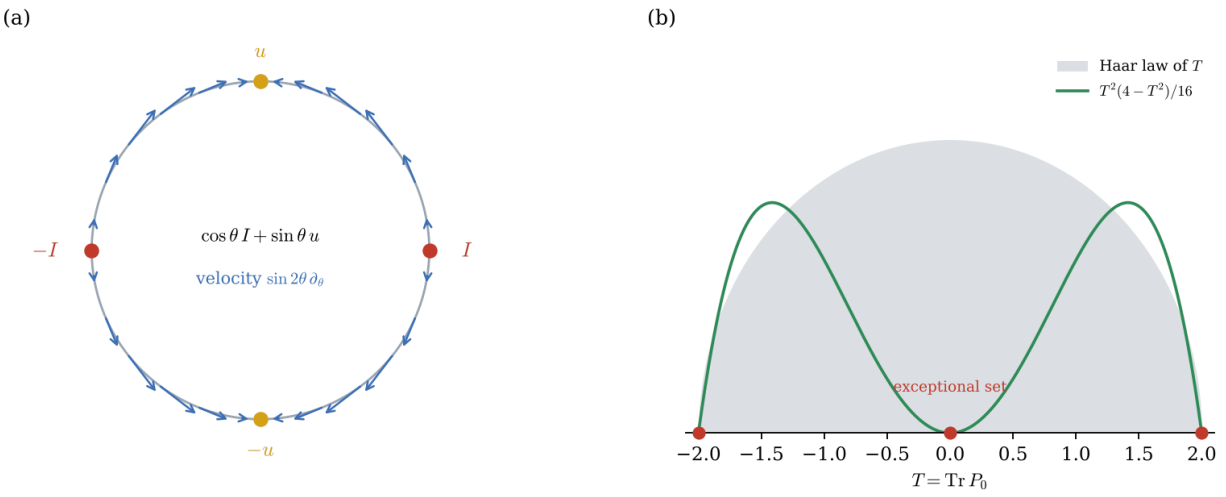


FIG. 15. The neutral sector on small tori. (a) A great circle of SU(2) through I, u, $-I$, $-u$, with the velocity $\sin 2\theta\,\partial_\theta$ by which the flow of Lemma 91 moves a Polyakov loop $P = \cos\theta\, I + \sin\theta\, u$. The flow fixes the centre $\pm I$ (red) and the traceless elements $\pm u$ (gold). (b) The Haar law of $T = \mathrm{Tr}\,P_0$, with density $\sqrt{4-T^2}/(2\pi)$, and the weight $T^2(4-T^2)$ of Theorem 92; the weight has Haar mean 2 and vanishes exactly at $T \in \{-2,0,2\}$, the set on which no flow of this kind gives a bound.

$k(W)u = K(-W) = vK(W)v^{-1} = -k(W)u$ and $K(W) = 0$. So every even equivariant K vanishes on $\{\pm I\} \cup \{\mathrm{Tr}\,W = 0\}$, and $\mu(\mathscr{Y}h)$ tends to zero for every neutral h of class C^1 when the law of the Polyakov loop concentrates there; integration by parts along such fields then gives no lower bound on the variance of a neutral function. The law of the Polyakov loop on Λ_{2n+4} at weak coupling is not determined here.

Remark 93 (what is proved about the neutral sector and what remains). Proved: for SU(2) on every torus with $L \geq 8$ and $\beta \geq 2$, the law of the Polyakov loop satisfies the anti-concentration bound (17) with $\|\Theta\|_2 \leq 192L^4(\log\beta + 5.44)$ (Lemma 91); $\mathrm{BP}_0(n,\varpi)$ on Λ_{2n+4} forces $\mu(T^2(4-T^2)) \leq 4.45\cdot 10^6 n^6(\log\beta + 6.11)^{3/2}(\varpi\beta)^{-1/2}$, so BP_0 fails there once $\beta \gg n^{12}\log^3\beta$ whenever $\mu(T^2(4-T^2))$ is bounded below (Theorem 92); and no vector field of the kind used can treat a Polyakov loop concentrated on the centre and the traceless elements. Not proved: a lower bound on $\mu(T^2(4-T^2))$ on Λ_{2n+4} at weak coupling, and so whether BP_0 fails there; whether $\mathrm{BP}_{\mathrm{loc}}$ fails there; BP_0 or $\mathrm{BP}_{\mathrm{loc}}$ along the continuum tuning for any non-abelian group; and the mass gap.

The routes to $\mathrm{BP}_{\mathrm{loc}}(n_k,\varpi)$ along the tuning stand as follows. A coupling blind to the centre twists is constrained on Λ_{2n+4} as above and not on the large tori that Theorem 66 uses. The comparison with larger cubes gives $\mathrm{BP}_0(n,\cdot)$ from $\mathrm{BP}_0(m,\cdot)$ and the inner inequality of Proposition 75(iii) for $f \in \mathscr{A}_0$, which must hold for almost every configuration outside the cube of side m, that is, a gap of the chain of cubes of side n for the measure of that cube under every boundary configuration; nothing is proved about it at weak coupling, and $\mathrm{BP}_0(m,\cdot)$ is the same problem at a larger cube size. Theorem 92 is an implication from a property of the measure alone, the spread of the Polyakov loop, to the failure of a gap of the cube chain; a converse of Theorem 64 needs an implication in the opposite direction, from properties of the measure such as reflection positivity and the transfer matrix to a gap of the cube chain, and none is proved. A comparison with a chain on block variables in the range of Corollary 74 needs two estimates on tori of every size: single-link bounds as in Theorem 72 for the effective measure of the block variables, and a Poincaré inequality for the fine links given the block variables, uniform in them, which is an inner inequality of the kind of Proposition 75(iii); neither is proved here, and the bounds of [6, 7, 12] concern effective actions in a box of fixed physical size. What remains for the continuum step is $\mathrm{BP}_{\mathrm{loc}}(n_k,\varpi)$ itself on infinitely many tori for each k, with one $\varpi > 0$; by Theorem 84(iv) it bounds the correlation of every slow function with every local one, and its failure produces a slow neutral function correlated with a local one.

J. The averaged condition under block-spin maps

A bootstrap of the averaged condition across scales compares the cube chain of a fine lattice with a cube chain of block variables. This subsection proves the comparison in the direction of the block-spin transformation, for every local gauge-covariant block-spin map and every gauge-invariant measure: BP passes from the fine lattice to the image measure of the block variables, with a loss that tends to one as the cube grows. Along the tuning, $\mathrm{BP}(n_k,\varpi)$ therefore implies the condition at one cube size, with one constant, for the image measures at a fixed coarser spacing. The local form passes only in an integrated form, which bounds the integrated autocorrelation time of a local function and gives no spectral gap. The opposite direction, which a bootstrap needs, is not obtained.

Setting. The definitions of Sec. X F apply to every probability measure ρ on G^E, E the links of Λ_L, that is invariant under the gauge group: P_x^ρ is the conditional expectation un-

der ρ given the links outside V_x, an orthogonal projection on $L^2(\rho)$; $\mathscr{A}^\rho$ is the space of gauge-invariant functions in $L^2(\rho)$; $\mathscr{E}_n^\rho(f) = n^{-4}\sum_x \rho((f - P_x^\rho f)^2)$, $\mathscr{K}^\rho = L^{-4}\sum_x P_x^\rho$, and $\mathrm{BP}_\rho(n,\varpi)$ and $\mathrm{BP}_{\mathrm{loc},\rho}(n,\varpi)$ are Definitions 62 and 83(a) with ρ in place of μ. A gauge transformation g maps the functions of the links outside V_x onto themselves and preserves ρ, so $(P_x^\rho f)\circ g = P_x^\rho(f\circ g)$ and P_x^ρ maps $\mathscr{A}^\rho$ into itself. The proofs of Proposition 75(i) and of the first statement of Theorem 84(iv) use only that the P_x^ρ are orthogonal projections that preserve $\mathscr{A}^\rho$ and the constants, with $P_W = P_W P_V$ for $V \subset W$; so both hold for every such ρ.

Let $b \geq 1$ and $r \geq 1$ be integers. The fine torus is Λ_{bL} with links E', the coarse torus is Λ_L with links E, and the block of $y \in \Lambda_L$ is $B_y = by + [0,b)^4 \subset \Lambda_{bL}$. A measurable map $c : G^{E'} \to G^E$ is a *block-spin map of range r* if

(B1) for every coarse link (y,ν), $c_{(y,\nu)}(U)$ depends only on the links of E' with base point in a block B_w with $|w-y|_\infty \leq r$;

(B2) for every gauge transformation g of Λ_{bL} there is a gauge transformation $\hat{g}$ of Λ_L with $c(gU) = \hat{g}c(U)$ for all U, and every gauge transformation of Λ_L is of this form.

The map that assigns to (y,ν) the product of the b links along the straight path from by to $by + be_\nu$ is a block-spin map of range one, with $\hat{g}(y) = g(by)$. For $b = 2$ so is the cut-off average (35), evaluated on the links put in the tree gauge of Definition 328 in every block by the gauge transformation equal to I at the root $2y$: that gauge transformation is a function of the tree links of the block, so the gauge-fixed links of the paths that start in $B(y)$ are functions of the links with base point in $B(y)\cup B(y+e_\nu)$; a gauge transformation g changes the gauge-fixed field by the transformation equal to $g(2y)$ on each block $B(y)$, which acts on the cut-off average as $\hat{g}(y) = g(2y)$ by Theorem 353.

For a gauge-invariant probability measure μ' on $G^{E'}$ let $\mu^c = c_*\mu'$ be the law of the block variables. It is invariant under the gauge group of Λ_L: given $\hat{g}$, choose g as in (B2); then $c^{-1}(\hat{g}^{-1}A) = g^{-1}c^{-1}(A)$ for every Borel set A, and μ' is invariant under g. Write $\mathscr{E}'_{n'}$, $\mathscr{A}'$, $P'_{x'}$ for the objects of μ' on Λ_{bL} and $\mathscr{E}^c_n$, $\mathscr{A}^c$, P^c_y for those of μ^c on Λ_L.

Theorem 94 (descent along block-spin maps)**.** *Let c be a block-spin map of range r with blocks of side b, let $n \geq 2r+2$, let $L \geq 2n+4$ be even, let $2 \leq n' \leq b(n-2r-1)+1$ with $bL \geq 2n'+4$, let μ' be a gauge-invariant probability measure on $G^{E'}$, and put $\varpi_c = \varpi'(n'/(bn))^4$.*

(i) *For every real $F \in \mathscr{A}^c$, $F\circ c \in \mathscr{A}'$, $\mathrm{Var}_{\mu'}(F\circ c) = \mathrm{Var}_{\mu^c}(F)$ and*

$$\mathscr{E}'_{n'}(F\circ c) \leq \Big(\frac{bn}{n'}\Big)^4 \mathscr{E}^c_n(F).$$

(ii) $\mathrm{BP}_{\mu'}(n',\varpi')$ *implies* $\mathrm{BP}_{\mu^c}(n,\varpi_c)$. *For $n' = b(n-2r-1)+1$, $\varpi_c \geq \varpi'(1-(2r+1)/n)^4$.*

(iii) *If $\mathrm{BP}_{\mathrm{loc},\mu'}(n',\varpi')$ holds, then every real $F \in \mathscr{A}^c$ that depends only on the coarse links with base points in a box of side at most $L/2-2r$, and every real $H \in \mathscr{A}^c$, satisfy*

$$\mathrm{Cov}_{\mu^c}(F,H)^2 \leq \mathrm{Var}_{\mu^c}(F)\,\frac{\mathscr{E}^c_n(H)}{\varpi_c}.$$

Proof. (i) For every gauge transformation g of Λ_{bL}, $F\circ c\circ g = F\circ\hat{g}\circ c$, and $F\circ\hat{g} = F$ holds μ^c-almost everywhere, hence $F\circ\hat{g}\circ c = F\circ c$ holds μ'-almost everywhere; so $F\circ c \in \mathscr{A}'$. The law of $F\circ c$ under μ' is that of F under μ^c.

For $x' \in \Lambda_{bL}$ with coordinates $x'_i \in \{0,\dots,bL-1\}$ let $y(x') \in \Lambda_L$ have coordinates $\lfloor x'_i/b\rfloor - r$ modulo L, and for $y \in \Lambda_L$ let $Q_y = by + br\mathbf{1} + [0, b(n-2r))^4$, the union of the blocks B_w with $w \in y + [r, n-r)^4$. Each y is $y(x')$ for exactly b^4 points x'. With $y = y(x')$, $b(y_i+r) = b\lfloor x'_i/b\rfloor \leq x'_i$ and $x'_i + n' - 1 \leq b\lfloor x'_i/b\rfloor + b - 2 + n' \leq b(y_i+r) + b(n-2r) - 1$, by the bound on n'; so the base points of the links of $V'_{x'}$ lie in Q_y. Let (z,ν) be a coarse link with $z \notin y + [0,n)^4$. By (B1), $c_{(z,\nu)}$ depends only on links with base point in blocks B_w with $|w-z|_\infty \leq r$; if such a block lay in Q_y, then $w-y$ would be congruent to a vector in $[r, n-r)^4$ and $z-y$ to one in $[0,n)^4$, which is excluded. The blocks partition the sites, so $c_{(z,\nu)}$ depends only on links with base point outside Q_y, which lie outside $V'_{x'}$.

Let $\mathscr{G}_y$ be the σ-algebra on $G^{E'}$ generated by the $c_{(z,\nu)}$ with $z \notin y + [0,n)^4$. It is contained in the σ-algebra of the links outside $V'_{x'}$, onto whose L^2 space $P'_{x'}$ projects orthogonally. For bounded ψ of the coarse links outside V_y, $\mu'((F\circ c)(\psi\circ c)) = \mu^c(F\psi) = \mu^c((P^c_y F)\psi) = \mu'(((P^c_y F)\circ c)(\psi\circ c))$, so $\mathbb{E}_{\mu'}[F\circ c \mid \mathscr{G}_y] = (P^c_y F)\circ c$. Hence, with $f = F\circ c$,

$$\mu'\big((f - P'_{x'}f)^2\big) \leq \mu'\big((f - (P^c_y F)\circ c)^2\big) = \mu^c\big((F - P^c_y F)^2\big),$$

and summing over x', each y occurring b^4 times,

$$n'^4 \mathscr{E}'_{n'}(f) \leq b^4 \sum_{y\in\Lambda_L} \mu^c\big((F - P^c_y F)^2\big) = b^4 n^4 \mathscr{E}^c_n(F).$$

(ii) For real $F \in \mathscr{A}^c$, (i) and $\mathrm{BP}_{\mu'}(n',\varpi')$ give

$$\varpi' \mathrm{Var}_{\mu^c}(F) = \varpi' \mathrm{Var}_{\mu'}(F\circ c) \leq \mathscr{E}'_{n'}(F\circ c) \leq \Big(\frac{bn}{n'}\Big)^4 \mathscr{E}^c_n(F).$$

For $n' = b(n-2r-1)+1$, $n'/(bn) \geq (n-2r-1)/n$.

(iii) Each coarse link with base point in the box $y_0 + [0,s)^4$, $s \leq L/2-2r$, depends by (B1) on links with base point in the blocks of $y_0 - r\mathbf{1} + [0, s+2r)^4$, a box of Λ_{bL} of side $b(s+2r) \leq bL/2$; so $F\circ c$ is local. The first statement of Theorem 84(iv) for μ', with $f = F\circ c$ and $h = H\circ c \in \mathscr{A}'$, gives $\mathrm{Cov}_{\mu^c}(F,H)^2 = \mathrm{Cov}_{\mu'}(f,h)^2 \leq \mathrm{Var}_{\mu^c}(F)\,\mathscr{E}'_{n'}(h)/\varpi'$, and (i) bounds $\mathscr{E}'_{n'}(h)$ by $(bn/n')^4\mathscr{E}^c_n(H)$. □

The inequality of (iii) is weaker than $\mathrm{BP}_{\mathrm{loc},\mu^c}(n,\varpi_c)$. For a self-adjoint operator $0 \leq K \leq I$ and a vector g with spectral measure ν_g, the sum $\sum_{t\geq 0}\langle g, K^t g\rangle = \int (1-\lambda)^{-1} d\nu_g$ is the integrated autocorrelation time of g.

Lemma 95. *Let ρ be a gauge-invariant probability measure on G^E, $n \geq 2$, $\varpi > 0$, $\varepsilon = \varpi n^4/L^4 < 1$, $f \in \mathscr{A}^\rho$ real, $g = f - \rho(f)$, and ν_g the spectral measure of $\mathscr{K}^\rho$ at g.*

(i) *The following are equivalent:*

(a) $\mathrm{Cov}_\rho(f,h)^2 \le \mathrm{Var}_\rho(f)\,\mathscr{E}_n^\rho(h)/\varpi$ *for every real* $h \in \mathscr{A}^\rho$;

(b) $\int_{[0,1]} \varepsilon(1-\lambda)^{-1} d\nu_g(\lambda) \le \|g\|_2^2$, *the integrand being* $+\infty$ *at* $\lambda = 1$.

(ii) *The inequality of* $\mathrm{BP}_{\mathrm{loc},\rho}(n,\varpi)$ *for* f *implies* (b).

(iii) (b) *implies* $\nu_g((1-\delta,1]) \le \delta\|g\|_2^2/\varepsilon$ *for* $0<\delta\le 1$ *and* $\|(\mathscr{K}^\rho)^t g\|_2^2 \le \|g\|_2^2/(\varepsilon(2t+1))$ *for every integer* $t \ge 0$.

(iv) *For every* $0<\eta<\varepsilon$ *there are a self-adjoint* $0 \le K \le I$ *on* $\mathbb{R}^2$ *and a unit vector* g *that satisfies* (b) *for* K *and has* $\nu_g((1-\varepsilon,1]) > 0$.

Proof. Let $\mathscr{H}$ be the real Hilbert space of the real functions of $\mathscr{A}^\rho$ with mean zero. Each P_x^ρ maps it into itself, so $K = \mathscr{K}^\rho$ restricts to a self-adjoint operator on $\mathscr{H}$ with $0 \le K \le I$, and so does every bounded Borel function of K; the spectral measure of g for this restriction has the moments $\langle g, K^m g\rangle$, so it is ν_g. Put $A = I - K$. For real $h \in \mathscr{A}^\rho$ and $h_0 = h - \rho(h) \in \mathscr{H}$, $\mathrm{Cov}_\rho(f,h) = \langle g, h_0\rangle$ and $\langle h_0, Ah_0\rangle = n^4 L^{-4}\mathscr{E}_n^\rho(h)$, since P_x^ρ fixes the constants. So (a) says $\langle g,h\rangle^2 \le \varepsilon^{-1}\|g\|_2^2\langle h, Ah\rangle$ for every $h \in \mathscr{H}$.

(i) Assume (b). Then $\nu_g(\{1\}) = 0$. Let $\psi_M(\lambda) = \min\{(1-\lambda)^{-1/2}, M\}$. By dominated convergence, with the integrable bound $(1-\lambda)^{-1}$, the vectors $\psi_M(K)g$ converge as $M \to \infty$ to some $v \in \mathscr{H}$ with $\|v\|_2^2 = \int(1-\lambda)^{-1} d\nu_g \le \|g\|_2^2/\varepsilon$, and $A^{1/2}\psi_M(K)g \to g$, since $(1-\lambda)^{1/2}\psi_M(\lambda) \to 1$ boundedly on $[0,1)$. For $h \in \mathscr{H}$, $\langle g,h\rangle = \lim_M \langle \psi_M(K)g, A^{1/2}h\rangle = \langle v, A^{1/2}h\rangle$, and the Cauchy–Schwarz inequality gives (a). Conversely, assume (a) and let $h = \varphi_M(K)g$ with $\varphi_M(\lambda) = \min\{(1-\lambda)^{-1}, M\}$. Then $\langle g,h\rangle = \int \varphi_M d\nu_g$ and $\langle h, Ah\rangle = \int(1-\lambda)\varphi_M^2 d\nu_g \le \int \varphi_M d\nu_g$, so (a) gives $\int \varphi_M d\nu_g \le \|g\|_2^2/\varepsilon$, and monotone convergence as $M \to \infty$ gives (b).

(ii) By Lemma 673, ν_g is carried by $[0, 1-\varepsilon]$, where $\varepsilon(1-\lambda)^{-1} \le 1$.

(iii) On $(1-\delta,1]$ the integrand of (b) is at least ε/δ. For $\lambda \in [0,1]$, $\lambda^{2t}(1-\lambda) \le (2t/(2t+1))^{2t}/(2t+1) \le 1/(2t+1)$, the maximum being at $\lambda = 2t/(2t+1)$, so $\|K^t g\|_2^2 = \int \lambda^{2t} d\nu_g \le \varepsilon^{-1}(2t+1)^{-1}\int \varepsilon(1-\lambda)^{-1} d\nu_g$.

(iv) Take $K = \mathrm{diag}(0, 1-\eta)$ and $g = (\cos\theta, \sin\theta)$ with $0 < \sin^2\theta \le \eta(1-\varepsilon)/(\varepsilon(1-\eta))$. Then $\int \varepsilon(1-\lambda)^{-1} d\nu_g = \varepsilon + \varepsilon\sin^2\theta(\eta^{-1}-1) \le 1$, and $\nu_g(\{1-\eta\}) = \sin^2\theta$ with $1-\eta \in (1-\varepsilon, 1]$. □

So under $\mathrm{BP}_{\mathrm{loc},\mu'}(n',\varpi')$, if $\varpi_c n^4 < L^4$, every function F as in Theorem 94(iii), with $G = F - \mu^c(F)$ and $\mathscr{K}^c$ the coarse cube chain, satisfies $\sum_{t\ge 0}\langle G, (\mathscr{K}^c)^t G\rangle \le L^4\|G\|_2^2/(\varpi_c n^4)$ and $\|(\mathscr{K}^c)^t G\|_2 \le (2\varpi_c\tau)^{-1/2}\|G\|_2$ at the time $\tau = tn^4/L^4 > 0$; in the proof of Theorem 64 this bound replaces $e^{-\varepsilon T}$ by $(\varepsilon(2T+1))^{-1/2}$, which gives no exponential decay. Along the continuum tuning the full condition gives the following.

Corollary 96. *Let* $a_k = 2^{-k}a_0$, *let* $0 < \ell_0 \le \ell$ *and* $n_k \ge 2$ *with* $\ell_0 \le n_k a_k \le \ell$, *and suppose that* $\mathrm{BP}(n_k,\varpi)$ *holds for the Wilson measure* $\mu_{k,L}$ *of* Λ_L *at* β_k *for every* L *in a set* $\mathscr{L}_k$ *of even integers* $L \ge 2n_k + 4$. *Fix* $j \ge 0$, *put* $n_j^* = 4 + \lfloor \ell/a_j \rfloor$, *and for* $k \ge j$ *let* c_k *be a block-spin map of range one with blocks of side* 2^{k-j}. *Then for every* $k \ge j$ *and every* $L \in \mathscr{L}_k$ *divisible by* 2^{k-j+1} *with* $L \ge 2^{k-j}(2n_j^*+4)$, *the measure* $(c_k)_*\mu_{k,L}$ *on* $\Lambda_{L/2^{k-j}}$ *satisfies*

$$\mathrm{BP}\Big(n_j^*,\ \varpi\Big(\frac{\ell_0}{\ell+4a_j}\Big)^4\Big).$$

Proof. Put $b = 2^{k-j}$, $L_c = L/b$, even, and $n = 3 + \lceil (n_k - 1)/b \rceil \ge 4$. Since $(n_k-1)/b < n_k a_k/a_j \le \ell/a_j$, $n \le n_j^*$, so $L_c \ge 2n+4$, and $n_k \le b(n-3)+1$. Theorem 94(ii) with $r=1$, $n' = n_k$ and the fine torus $\Lambda_L = \Lambda_{bL_c}$ gives $\mathrm{BP}(n, \varpi(n_k/(bn))^4)$ for $(c_k)_*\mu_{k,L}$, and Proposition 75(i), which holds for this measure, gives $\mathrm{BP}(n_j^*, \varpi(n_k/(bn_j^*))^4)$. Finally $n_k/b = n_k a_k/a_j \ge \ell_0/a_j$ and $n_j^* \le 4 + \ell/a_j$. □

Remark 97 (what the descent gives and what a bootstrap needs). Proved: for every local gauge-covariant block-spin map and every gauge-invariant measure, the averaged condition passes from the fine lattice to the law of the block variables with the factor $(n'/(bn))^4$, which is $(1-(2r+1)/n)^4$ for the largest admissible fine cube (Theorem 94); along the tuning, with $n_k a_k$ between ℓ_0 and ℓ, $\mathrm{BP}(n_k,\varpi)$ gives BP at the one cube size $4 + \lfloor \ell/a_j \rfloor$ with the constant $\varpi(\ell_0/(\ell+4a_j))^4$ for the laws of the block variables at the spacing a_j, for every $k \ge j$ (Corollary 96). A proof of $\mathrm{BP}(n_k,\varpi)$ along the tuning therefore contains a proof of the condition at a fixed scale for the sequence of measures that the block-spin transformation produces from the tuned couplings; the results of Secs. X G and X I are proved for Wilson measures and do not apply to these measures as they stand. The local form passes only in the integrated form of Lemma 95, which bounds the integrated autocorrelation time of a local function and gives no spectral gap (Lemma 95(iv)).

Not proved: the opposite direction. For $f \in \mathscr{A}'$,

$$\mathrm{Var}_{\mu'}(f) = \mu'\big(\mathrm{Var}_{\mu'}(f \mid c)\big) + \mathrm{Var}_{\mu^c}\big(\mathbb{E}_{\mu'}[f \mid c]\big),$$

and a passage from BP_{μ^c} to $\mathrm{BP}_{\mu'}$ needs two bounds by $\mathscr{E}'_{n'}(f)$: of the first term, the Poincaré inequality for the fine links given the block variables named in Remark 93, here in the mean over the block variables but for every $f \in \mathscr{A}'$; and of the coarse Dirichlet form of $\mathbb{E}_{\mu'}[f \mid c]$, an inequality of the kind of Theorem 94(i) in the reverse direction. With $\mathscr{F}_{x'}$ the σ-algebra of the links outside $V'_{x'}$ and $y = y(x')$ as in the proof of (i), $\mathscr{G}_y \subset \mathscr{F}_{x'}$ gives, for $f = F \circ c$,

$$\mu'\big(\mathrm{Var}(f \mid \mathscr{G}_y)\big) = \mu'\big(\mathrm{Var}(f \mid \mathscr{F}_{x'})\big) + \mu'\big(\mathrm{Var}(\mathbb{E}[f \mid \mathscr{F}_{x'}] \mid \mathscr{G}_y)\big);$$

the proof of (i) drops the last term, and a reverse inequality has to bound it, that is, the information that the fine field outside the fine cube carries on F beyond the coarse field outside the coarse cube. Sec. X K poses the two bounds as one split inequality, proves it for the product of Haar measures and shows that the two bounds posed separately lose a fixed factor at each scale; Sec. X L proves it for the Haar refinements, for which that information vanishes, and Sec. X M bounds the information by a factor of order β^2 for the Wilson measure at strong coupling (Theorem 119). The bounds of [6, 7, 12] concern effective actions in a box of fixed physical size, while Corollary 96 poses the condition at the scale a_j on tori of every size.

K. The upward step

This subsection treats the passage from the law of the block variables to the fine measure. It proves that the two bounds of Remark 97 give the upward step when they are posed jointly, as one inequality whose two terms are paid by the fine Dirichlet form together, and that the step then runs across any number of scales with a constant controlled by the product of the coarse constants. For the product of Haar measures and the straight-path map the joint inequality holds with the constants that reproduce $\mathrm{BP}(n,1)$ at every scale. The two bounds posed separately, as in Remark 97, lose a fixed amount at each scale: already for the product of Haar measures, where $\mathrm{BP}(n,1)$ holds on every torus, the constant they give after K scales of physical cube size between ℓ_0 and ℓ is at most $2(\ell/\ell_0)^4/K$.

The setting is that of Sec. X J: c is a block-spin map of range r with blocks of side $b \geq 2$ from Λ_{bL} to Λ_L, μ' is a gauge-invariant probability measure on $G^{E'}$, $\mu^c = c_*\mu'$, and $\mathscr{E}'_{n'}$, $\mathscr{A}'$, $P'_{x'}$ and $\mathscr{E}^c_n$, $\mathscr{A}^c$, P^c_y are the objects of μ' and μ^c. For $f \in L^2(\mu')$ write $\mathbb{E}[f \mid c]$ for the conditional expectation under μ' given the σ-algebra $\sigma(c)$ of the block variables, and $\mu'(\mathrm{Var}(f \mid c)) = \mu'(f^2) - \mu'(\mathbb{E}[f \mid c]^2)$. As in the proof of Theorem 94(i), $\mathscr{G}_y \subset \sigma(c)$ is generated by the $c_{(z,v)}$ with $z \notin y + [0,n)^4$.

Lemma 98. *Let $f \in \mathscr{A}'$ be real. There is a real $F_f \in \mathscr{A}^c$ with $\mathbb{E}[f \mid c] = F_f \circ c$ μ'-almost everywhere, and*

$$\mathrm{Var}_{\mu'}(f) = \mu'\big(\mathrm{Var}(f \mid c)\big) + \mathrm{Var}_{\mu^c}(F_f),$$
$$\mu^c\big((F_f - P^c_y F_f)^2\big) = \mu'\big(\mathrm{Var}(\mathbb{E}[f \mid c] \mid \mathscr{G}_y)\big)$$

for every $y \in \Lambda_L$.

Proof. A $\sigma(c)$-measurable function is of the form $F \circ c$ with F Borel on G^E; let F_f be such a function for $\mathbb{E}[f \mid c]$, real, with $\mu^c(F_f^2) = \mu'(\mathbb{E}[f \mid c]^2) < \infty$. Let $\hat g$ be a gauge transformation of Λ_L and g as in (B2), and let ψ be bounded on G^E. The invariance of μ' under g and $f \circ g^{-1} = f$ almost everywhere give $\mu'(f\,\psi \circ \hat g \circ c) = \mu'(f\,\psi \circ c \circ g) = \mu'(f\,\psi \circ c)$, so

$$\mu^c(F_f\,\psi \circ \hat g) = \mu'(f\,\psi \circ \hat g \circ c) = \mu'(f\,\psi \circ c) = \mu^c(F_f\,\psi),$$

and the invariance of μ^c under $\hat g$ turns the left side into $\mu^c((F_f \circ \hat g^{-1})\psi)$. As ψ is arbitrary, $F_f \circ \hat g^{-1} = F_f$ μ^c-almost everywhere, so $F_f \in \mathscr{A}^c$. The first identity is the decomposition of the variance along $\sigma(c)$, with $\mathrm{Var}_{\mu'}(\mathbb{E}[f \mid c]) = \mathrm{Var}_{\mu^c}(F_f)$. The proof of Theorem 94(i) gives $\mathbb{E}[F_f \circ c \mid \mathscr{G}_y] = (P^c_y F_f) \circ c$, so $\mu^c((F_f - P^c_y F_f)^2) = \mu'((\mathbb{E}[f \mid c] - \mathbb{E}[\mathbb{E}[f \mid c] \mid \mathscr{G}_y])^2)$, which is the second identity. □

Definition 99. For $n, n' \geq 2$ and $C_A, C_B > 0$, the *split inequality* $\mathrm{S}(n', n; C_A, C_B)$ for (μ', c) is

$$\frac{\mu'(\mathrm{Var}(f \mid c))}{C_A} + \frac{\mathscr{E}^c_n(F_f)}{C_B} \leq \mathscr{E}'_{n'}(f)$$

for every real $f \in \mathscr{A}'$, with F_f as in Lemma 98. The *separate inequalities* are (A) $\mu'(\mathrm{Var}(f \mid c)) \leq C_A \mathscr{E}'_{n'}(f)$ and (B) $\mathscr{E}^c_n(F_f) \leq C_B \mathscr{E}'_{n'}(f)$ for every real $f \in \mathscr{A}'$.

By Lemma 98, $n^4 \mathscr{E}^c_n(F_f) = \sum_y \mu'(\mathrm{Var}(\mathbb{E}[f \mid c] \mid \mathscr{G}_y))$, so (B) bounds the coarse Dirichlet form of $\mathbb{E}[f \mid c]$, and with it the term dropped in the proof of Theorem 94(i), named in Remark 97. The split inequality implies (A) and (B) with the same constants, since each of its terms is at most $\mathscr{E}'_{n'}(f)$; (A) and (B) imply it with the constants $2C_A$ and $2C_B$.

Theorem 100 (the upward step)**.** *Let c be a block-spin map with blocks of side b, $n, n' \geq 2$, $L \geq 2n+4$ and $bL \geq 2n'+4$ even, and μ' a gauge-invariant probability measure on $G^{E'}$.*

(i) *If $\mathrm{S}(n', n; C_A, C_B)$ holds for (μ', c) and $\mathrm{BP}_{\mu^c}(n, \varpi_c)$ holds, then $\mathrm{BP}_{\mu'}(n', \varpi')$ holds with*

$$\varpi' = \min\Big(\frac{1}{C_A}, \frac{\varpi_c}{C_B}\Big).$$

(ii) *If* (A) *and* (B) *hold and $\mathrm{BP}_{\mu^c}(n, \varpi_c)$ holds, then $\mathrm{BP}_{\mu'}(n', (C_A + C_B/\varpi_c)^{-1})$ holds.*

(iii) *If $\mathrm{BP}_{\mu'}(n', \varpi')$ holds, then* (A) *holds with $C_A = 1/\varpi'$.*

Proof. Let $f \in \mathscr{A}'$ be real, $a = \mu'(\mathrm{Var}(f \mid c))$ and $e = \mathscr{E}^c_n(F_f)$. By Lemma 98 and $\mathrm{BP}_{\mu^c}(n, \varpi_c)$ for the real function $F_f \in \mathscr{A}^c$,

$$\mathrm{Var}_{\mu'}(f) = a + \mathrm{Var}_{\mu^c}(F_f) \leq a + \frac{e}{\varpi_c}.$$

(i) The right side is $C_A \frac{a}{C_A} + \frac{C_B}{\varpi_c}\frac{e}{C_B} \leq \max(C_A, C_B/\varpi_c)\,\mathscr{E}'_{n'}(f)$, and $1/\max(C_A, C_B/\varpi_c) = \varpi'$.
(ii) The right side is at most $(C_A + C_B/\varpi_c)\,\mathscr{E}'_{n'}(f)$.
(iii) $\mu'(\mathrm{Var}(f \mid c)) \leq \mathrm{Var}_{\mu'}(f) \leq \mathscr{E}'_{n'}(f)/\varpi'$. □

The step composes along a tower of block-spin maps. Let $b \geq 2$, $r \geq 1$, $K \geq 1$, L_0 even and $L_j = b^j L_0$; for $1 \leq j \leq K$ let c_j be a block-spin map of range r with blocks of side b from Λ_{L_j} to $\Lambda_{L_{j-1}}$. Given a gauge-invariant probability measure μ_K on the links of Λ_{L_K}, put $\mu_{j-1} = (c_j)_*\mu_j$ for $j = K, \dots, 1$; each μ_j is gauge invariant (Sec. X J).

Theorem 101 (the upward step across scales)**.** *Let $n_0, \dots, n_K \geq 2$ with $L_j \geq 2n_j + 4$. Suppose that $\mathrm{BP}_{\mu_0}(n_0, \varpi_0)$ holds and that, for $1 \leq j \leq K$, $\mathrm{S}(n_j, n_{j-1}; C_{A,j}, C_{B,j})$ holds for (μ_j, c_j). Put $\Gamma_j = \prod_{i=j}^K C_{B,i}$ and $\Gamma_{K+1} = 1$. Then $\mathrm{BP}_{\mu_K}(n_K, \varpi_K)$ holds with*

$$\varpi_K = \min\Big(\frac{\varpi_0}{\Gamma_1}, \min_{1 \leq j \leq K} \frac{1}{C_{A,j}\Gamma_{j+1}}\Big).$$

In particular, if $C_{A,j} \leq C_A$ and $\Gamma_j \leq \Gamma$ for all j, with $\Gamma \geq 1$, then $\varpi_K \geq \min(\varpi_0, 1/C_A)/\Gamma$.

Proof. Let

$$m_j = \min\Big(\frac{\varpi_0}{\prod_{i=1}^j C_{B,i}}, \min_{1 \leq i \leq j} \frac{1}{C_{A,i}\prod_{l=i+1}^j C_{B,l}}\Big),$$

so that $m_K = \varpi_K$, and $m_j = \min(1/C_{A,j}, m_{j-1}/C_{B,j})$ for $j \geq 1$, with $m_0 = \varpi_0$. If $\mathrm{BP}_{\mu_{j-1}}(n_{j-1}, m_{j-1})$ holds, Theorem 100(i) for $\mu' = \mu_j$, $c = c_j$ gives $\mathrm{BP}_{\mu_j}(n_j, m_j)$; induction on j gives the claim. For the last statement, $\varpi_0/\Gamma_1 \geq \varpi_0/\Gamma$ and $C_{A,j}\Gamma_{j+1} \leq C_A\Gamma$, also for $j = K$ since $\Gamma \geq 1$. □

Along the tuning this reduces $\mathrm{BP}(n_k,\varpi)$ to the condition at the coarsest scale and the split inequality at every scale, for the measures that the block-spin transformation produces from the tuned couplings.

Corollary 102 (conditional bootstrap)**.** *Let $a_k = 2^{-k}a_0$, let $n_0 \geq 2$, $\ell = n_0 a_0$ and $n_k = 2^k n_0$, so that $n_k a_k = \ell$. For each k let $\mathscr{L}_k$ be an infinite set of integers $L = 2^k L_0$ with L_0 even and $L_0 \geq 2n_0+4$; for $L \in \mathscr{L}_k$ let $\mu_{k,L}$ be the Wilson measure of Λ_L at β_k, let $c_1,\dots,c_k$ be block-spin maps of range r with blocks of side two as above, with $L_j = 2^j L_0$, and let $\mu_{k,L}^{(j)}$ be the measures of the tower with $\mu_{k,L}^{(k)} = \mu_{k,L}$. Suppose that there are $\varpi_0 > 0$, C_A and $\Gamma \geq 1$ such that for all k and $L \in \mathscr{L}_k$:*

(H1) $\mathrm{BP}(n_0,\varpi_0)$ *holds for* $\mu_{k,L}^{(0)}$;

(H2) *for $1 \leq j \leq k$, $\mathrm{S}(n_j,n_{j-1};C_{A,j},C_{B,j})$ holds for $(\mu_{k,L}^{(j)},c_j)$, with $C_{A,j} \leq C_A$ and $\prod_{i=j}^{k} C_{B,i} \leq \Gamma$.*

Then $\mathrm{BP}(n_k,\varpi)$ holds for $\mu_{k,L}$ for all k and $L \in \mathscr{L}_k$, with $\varpi = \min(\varpi_0, 1/C_A)/\Gamma$, and every continuum limit as in Theorem 66 satisfies $\sigma(H) \subset \{0\} \cup [16\varpi/(625e^2\ell),\infty)$.

Proof. $L \geq 2^k(2n_0+4) \geq 2n_k+4$, and likewise $L_j \geq 2n_j+4$ on each torus of the tower. Theorem 101 with $K = k$ gives $\mathrm{BP}(n_k,\varpi)$ for $\mu_{k,L}$, and Theorem 66 applies with $n_k a_k = \ell$. □

The hypotheses are not proved for the tuned Wilson measures. Part of them is contained in the conclusion. For straight-path maps c_j, whose composites are straight-path maps of range one, and $L_0 \geq 2n_0+12$, Corollary 96 with $\ell_0 = \ell$ turns $\mathrm{BP}(n_k,\varpi)$ for all k into $\mathrm{BP}(n_j+4,\varpi(\ell/(\ell+4a_j))^4)$ for $\mu_{k,L}^{(j)}$, $0 \leq j \leq k$, and by Theorem 100(iii) it gives (A) at every step of the tower, with the cube sizes n_j+4; for $j = 0$ Proposition 130 gives (H1) at the cube size n_0+1 when $L_0 \geq 2n_0+6$. The split inequality with a bounded product of the constants $C_{B,j}$ is not contained in the conclusion as far as is proved here. For $\varphi \equiv 1$ and the straight-path maps every hypothesis of Theorem 101 holds with $\varpi_0 = 1$, $C_{A,j} = 1$ and $\Gamma_j = 1$ when $n_j = 2n_{j-1}$; there μ^c is again a product of Haar measures and the computation is exact.

Proposition 103 (the product measure)**.** *Let $\varphi \equiv 1$, so that μ' is the product of the Haar measures on the links of Λ_{bL}, $b \geq 2$, let c be the straight-path map, and let $n,n' \geq 2$, $L \geq 2n+4$ and $bL \geq 2n'+4$ even.*

(i) *μ^c is the product of the Haar measures on the links of Λ_L, and if $n' \geq bn$, then every real $f \in L^2(\mu')$, with $\mathbb{E}[f \mid c] = F_f \circ c$, satisfies*

$$\mu'\big(\mathrm{Var}(f \mid c)\big) + \Big(\frac{bn}{n'}\Big)^4 \mathscr{E}_n^c(F_f) \leq \mathscr{E}'_{n'}(f);$$

so $\mathrm{S}(n',n;1,(n'/(bn))^4)$ holds.

(ii) *There is a real $f \in \mathscr{A}'$ with $\mathbb{E}[f \mid c] = 0$ and $\mu'(\mathrm{Var}(f \mid c)) = (1+2/n')^{-1}\mathscr{E}'_{n'}(f) > 0$. For every $\eta > 0$ there is L_* such that for every $L \geq L_*$ there is a real $f \in \mathscr{A}'$ with $\mathbb{E}[f \mid c] = f$ and $\mathscr{E}_n^c(F_f) \geq ((n'/(bn))^4 - \eta)\,\mathscr{E}'_{n'}(f) > 0$. So constants for which* (A), (B) *or* $\mathrm{S}(n',n;C_A,C_B)$ *hold for all L in an infinite set satisfy $C_A \geq (1+2/n')^{-1}$ and $C_B \geq (n'/(bn))^4$.*

(iii) *For $n' = bn$, Theorem 100*(i) *with $\mathrm{BP}_{\mu^c}(n,1)$ gives $\mathrm{BP}_{\mu'}(bn,1)$, while Theorem 100*(ii) *with $\mathrm{BP}_{\mu^c}(n,1)$ and any constants valid for all L in an infinite set gives a constant at most $(1+2/n')/(2+2/n')$.*

Proof. (i) Every link of Λ_{bL} lies on at most one of the paths $\pi(e)$ of b links that c multiplies, $e \in E$, and for $T \subset E$ write $\pi(T) = \bigcup_{e\in T}\pi(e)$. The coarse links are products over disjoint sets of independent Haar variables, so μ^c is the product of the Haar measures. Decompose $f = \sum_{S'\subset E'} f_{S'}$ as in the proof of Proposition 67(ii) for μ'.

Step 1: $\mathbb{E}[f_{S'} \mid c] = 0$ *unless $S' = \pi(T)$ for some $T \subset E$, and $F_T = \mathbb{E}[f_{\pi(T)} \mid c]$ is a function of c in the component T of the decomposition for μ^c.* If S' contains a link e' that lies on no path, $U_{e'}$ is independent of the σ-algebra $\mathscr{H}$ generated by c and the other links, and $\mathbb{E}[f_{S'} \mid \mathscr{H}] = \int f_{S'}\,dU_{e'} = 0$. If S' contains a link $e' \in \pi(e)$ and $\pi(e) \not\subset S'$, pick $e'' \in \pi(e)\setminus S'$. In the product c_e the link $U_{e''}$ enters once, between products of other links, so by the invariance of Haar measure under translations c_e is a Haar variable independent of all links other than $U_{e''}$. Hence the links of $\pi(e)\setminus\{e''\}$ are independent of the σ-algebra $\mathscr{H}$ generated by c_e and the links outside $\pi(e)$, which contains $\sigma(c)$, and $\mathbb{E}[f_{S'} \mid \mathscr{H}]$ is the integral of $f_{S'}$ over the links of $\pi(e)\setminus\{e''\}$; integrating first over $U_{e'}$ gives zero. In both cases $\mathbb{E}[f_{S'} \mid c] = \mathbb{E}[\mathbb{E}[f_{S'} \mid \mathscr{H}] \mid c] = 0$. For $S' = \pi(T)$, the links of $\pi(T)$ and the coarse links of T are independent of the coarse links of $E\setminus T$, so F_T depends only on c_T. For $e \in T$, the links of $\pi(e)$ are independent of those of $\pi(T\setminus\{e\})$, which determine $c_{T\setminus\{e\}}$, so $\mathbb{E}[F_T\circ c \mid c_{T\setminus\{e\}}] = \mathbb{E}[f_{\pi(T)} \mid c_{T\setminus\{e\}}] = 0$ after integration over the links of $\pi(e)$; as μ^c is a product measure, this says that F_T has Haar mean zero in each variable c_e, $e \in T$. Moreover $\|F_T\|_2 \leq \|f_{\pi(T)}\|_2$, and $F_\emptyset = f_\emptyset$.

Step 2: the two sides. By Step 1, $F_f = \sum_T F_T$ is the decomposition of F_f for μ^c, $\mu'(\mathbb{E}[f \mid c]^2) = \sum_T \|F_T\|_2^2$ and

$$\mu'\big(\mathrm{Var}(f \mid c)\big) = \sum_{S'\notin\pi(\cdot)} \|f_{S'}\|_2^2 + \sum_{T\neq\emptyset}\big(\|f_{\pi(T)}\|_2^2 - \|F_T\|_2^2\big),$$

where the first sum runs over the sets S' not of the form $\pi(T)$. By the proof of Proposition 67(ii), $n'^4\mathscr{E}'_{n'}(f)$ is $\sum_{S'}\|f_{S'}\|_2^2 N'(S')$ and $n^4\mathscr{E}_n^c(F_f) = \sum_T \|F_T\|_2^2 N^c(T)$, where $N'(S')$ is the number of $x' \in \Lambda_{bL}$ with $V'_{x'} \cap S' \neq \emptyset$ and $N^c(T)$ the number of $y \in \Lambda_L$ with $V_y \cap T \neq \emptyset$. For $S' \neq \emptyset$, $N'(S') \geq n'^4$, since a link lies in n'^4 cubes.

Step 3: $N'(\pi(T)) \geq b^4 N^c(T)$. Let $V_y \cap T \neq \emptyset$, with $(z,\nu) \in T$ and $z = y+d$, $d \in [0,n)^4$ modulo L. For $w \in [0,b)^4$ put $x' = by - w$ modulo bL. Then $bz - x' = bd + w \in [0,bn)^4 \subset [0,n')^4$, so the first link (bz,ν) of $\pi(z,\nu)$ lies in $V'_{x'}$. The map $(y,w) \mapsto by - w$ is injective: if $by - w \equiv by_2 - w_2$ modulo bL, then $w - w_2$ is divisible by b with entries in $(-b,b)$, so $w = w_2$ and $y \equiv y_2$ modulo L.

Step 4. The left side of (i) is the sum of $\|f_{S'}\|_2^2$ over $S' \notin \pi(\cdot)$

and of

$$\big(\|f_{\pi(T)}\|_2^2 - \|F_T\|_2^2\big) + \|F_T\|_2^2 \,\frac{b^4 N^c(T)}{n'^4}$$
$$\le \|f_{\pi(T)}\|_2^2 \max\Big(1, \frac{b^4 N^c(T)}{n'^4}\Big)$$

over $T \neq \emptyset$, by $\|F_T\|_2 \le \|f_{\pi(T)}\|_2$. By Steps 2 and 3 each term is at most the corresponding term $\|f_{S'}\|_2^2 N'(S')/n'^4$ of $\mathscr{E}'_{n'}(f)$.

(ii) Let h be a real class function with Haar mean zero and $\int h^2 dU = 1$, as in the proof of Proposition 88. For a plaquette p of Λ_{bL}, $f = h(U_p)$ is gauge invariant and its decomposition has the single term $S' = p$, the four links of p. A nonempty set $\pi(T)$ contains, with each link (x,ν), the link $(x+e_\nu,\nu)$ or $(x-e_\nu,\nu)$, because $b \ge 2$, and p does not; so $\mathbb{E}[f \mid c] = 0$ and $\mu'(\mathrm{Var}(f \mid c)) = 1$. By the count in the proof of Proposition 75(ii), $N'(p) = n'^4 + 2n'^3$, so $\mathscr{E}'_{n'}(f) = 1 + 2/n'$.

For the second function let $R \ge 2$, $L \ge R+1$, and let $F = \prod_{P\in Q} h(U_P)$ be the function of the proof of Proposition 88 on Λ_L, with Q the plaquettes of the plane $\{0,1\}$ with base point in $[0,R)^4$ and even coordinates 0 and 1, and T the union of their links; its decomposition has the single term T, $\|F\|_2 = 1$, and $N^c(T) \ge (R-1)^4$ by that proof. Put $f = F \circ c$, gauge invariant by (B2), with $\mathbb{E}[f \mid c] = f$ and $F_f = F$. For $P \in Q$, c_P, the holonomy of c around P, is the product of the $4b$ links of π of the four links of P, each entering once; integrating $h(c_P)$ over any one of them gives $\int h\,dU = 0$, and distinct $P \in Q$ involve disjoint links, so the decomposition of f has the single term $\pi(T)$ and $\|f\|_2 = 1$. The base points of the links of T lie in $[0,R]^4$, those of $\pi(T)$ in $[0, bR+b-1]^4$, and a cube $V'_{x'}$ meets $\pi(T)$ only if $x' \in [-(n'-1), bR+b-1]^4$ modulo bL; so $N'(\pi(T)) \le (bR+b+n'-1)^4$. Hence

$$\frac{\mathscr{E}^c_n(F)}{\mathscr{E}'_{n'}(f)} = \Big(\frac{n'}{n}\Big)^4 \frac{N^c(T)}{N'(\pi(T))} \ge \Big(\frac{n'}{n}\Big)^4 \Big(\frac{R-1}{bR+b+n'-1}\Big)^4,$$

which tends to $(n'/(bn))^4$ as $R \to \infty$. Choose R with the right side at least $(n'/(bn))^4 - \eta$ and $L_* = R+1$. The first function bounds the constant in (A) and in the first term of S, which it does not see in the second term; the second bounds the constant in (B) and in the second term of S, as $\mu'(\mathrm{Var}(f \mid c)) = 0$ for it.

(iii) By (i) and Proposition 67(ii), Theorem 100(i) applies with $C_A = C_B = 1$ and $\varpi_c = 1$. By (ii), with $n' = bn$, Theorem 100(ii) gives $(C_A + C_B)^{-1} \le ((1+2/n')^{-1} + 1)^{-1}$. □

In the proof of (i) the fine Dirichlet form pays each term of the decomposition once: a component $f_{S'}$ that is not a union of paths contributes only to the fibre term, and a component $f_{\pi(T)}$ splits into its fibre part and its image F_T in proportions that add up to its norm. The separate inequalities charge the whole Dirichlet form to each term. Over many scales this costs a factor that grows with the number of scales.

Corollary 104 (the separate inequalities across scales)**.** *Let $\varphi \equiv 1$, $b \ge 2$, $K \ge 1$, $0 < \ell_0 \le \ell$, $a_j = b^{-j} a_0$ and $n_0, \dots, n_K \ge 2$ with $\ell_0 \le n_j a_j \le \ell$ and $L_j \ge 2n_j + 4$. Let c_j be the straight-path maps of the tower above, and suppose that for $1 \le j \le K$ the inequalities* (A) *and* (B) *hold for (μ_j, c_j) with cube sizes (n_j, n_{j-1}) and constants $C_{A,j}$, $C_{B,j}$, for all L_0 in an infinite set. Let $\psi_0 > 0$ and $\psi_j = (C_{A,j} + C_{B,j}/\psi_{j-1})^{-1}$, the constant that Theorem 100*(ii) *gives at the scale j from $\mathrm{BP}_{\mu_0}(n_0, \psi_0)$. Then*

$$\psi_K \le \frac{2}{K}\Big(\frac{\ell}{\ell_0}\Big)^4,$$

while $\mathrm{BP}_{\mu_K}(n_K, 1)$ holds on every torus. With $n_j = b^j n_0$, the split inequalities $\mathrm{S}(n_j, n_{j-1}; 1, 1)$ hold, and Theorem 101 with $\mathrm{BP}_{\mu_0}(n_0, 1)$ gives $\mathrm{BP}_{\mu_K}(n_K, 1)$.

Proof. Unrolling the recursion and dropping the term with ψ_0,

$$\frac{1}{\psi_K} \ge \sum_{j=1}^{K} C_{A,j} \prod_{i=j+1}^{K} C_{B,i}.$$

Each μ_j is a product of Haar measures, so by Proposition 103(ii), $C_{A,j} \ge (1+2/n_j)^{-1} \ge \frac{1}{2}$ and $C_{B,i} \ge (n_i/(bn_{i-1}))^4$. The product telescopes:

$$\prod_{i=j+1}^{K} \frac{n_i}{bn_{i-1}} = \frac{n_K}{b^{K-j} n_j} = \frac{n_K a_K}{n_j a_j} \ge \frac{\ell_0}{\ell}.$$

So $1/\psi_K \ge \frac{K}{2}(\ell_0/\ell)^4$. $\mathrm{BP}(n_K, 1)$ is Proposition 67(ii), and the last statement is Proposition 103(i) with Theorem 101, with $C_{A,j} = 1$ and $\Gamma_j = 1$. □

Remark 105 (what the upward step gives and what it needs). Proved: for every block-spin map and every gauge-invariant measure, the split inequality $\mathrm{S}(n', n; C_A, C_B)$ and BP for the law of the block variables give $\mathrm{BP}(n', \min(1/C_A, \varpi_c/C_B))$ for the fine measure, and the fibre inequality (A) is implied by BP for the fine measure (Theorem 100); along a tower of block-spin maps the constant after K steps is controlled by the constants $C_{A,j}$ and the products of the $C_{B,j}$ (Theorem 101); along the tuning, the condition at the coarsest scale (H1), which a proof of $\mathrm{BP}(n_k, \varpi)$ has to contain by Corollary 96, and the split inequality at every scale (H2) give $\mathrm{BP}(n_k, \varpi)$ with an explicit ϖ and the conclusion of Theorem 66 (Corollary 102). For the product of Haar measures and the straight-path map, $\mathrm{S}(n', n; 1, (n'/(bn))^4)$ holds for $n' \ge bn$, the constants are the best ones uniform in the torus, and the tower reproduces $\mathrm{BP}(n, 1)$ at every scale (Proposition 103). The separate inequalities (A) and (B) of Remark 97, composed through Theorem 100(ii), give after K scales of physical cube size between ℓ_0 and ℓ at most $2(\ell/\ell_0)^4/K$, already for the product of Haar measures, where the condition holds with constant one (Corollary 104); a bootstrap across the scales of the tuning has to use the split form.

Not proved: (H1) and (H2) for the tuned Wilson measures; the split inequality, with C_B close to $(n'/(bn))^4$, for the Wilson measure outside the strong-coupling range, for U(1) at $\beta \neq 0$, and for SU(2) at weak coupling on calm fibres; Sec. X L proves it, with the constants of the product measure, for the Haar refinements, which are not Wilson measures, and Sec. X M proves it for the Wilson measure at strong coupling, with constants within factors $1 + O(\beta)$ of those (Theorem 120). The proof of Proposition 103 uses the orthogonal decomposition of the product measure, in which the components of f that

the fibre and the coarse field see are orthogonal; for an interacting measure they are not, and the term dropped in the proof of Theorem 94(i), the information that the fine field outside a fine cube carries on the block variables inside the coarse cube beyond the coarse field outside it, vanishes for the product measure and its Haar refinements (Proposition 109(ii), Lemma 108(ii)), is bounded at strong coupling by a factor q^2 of order β^2 times the coarse form of the refinement (Theorem 119), and is not bounded here for any other measure. Theorem 423 and Proposition 434 concern bounds uniform in the conditioning on block variables, while (A) and S are posed in the mean over μ', so they do not decide them. Neither BP along the tuning nor the mass gap is proved here.

L. Haar refinements

This subsection proves the split inequality, with the constants of Proposition 103(i), for a class of interacting measures: those whose conditional law given the block variables of the straight-path map is the Haar law of the fibres, while the law of the block variables is an arbitrary gauge-invariant measure. Along a tower of such refinements the upward step carries the condition at the coarsest scale to every finer scale without loss. The proof rests on an exact splitting of the fine Dirichlet form into a fibre form and a coarse form, valid for every measure; for a general measure the coarse form is reduced by the information that the fine field outside a fine cube carries on the block variables, and the Wilson measure at $\beta \neq 0$ is not a refinement.

Throughout, c is the straight-path map with blocks of side $b \geq 2$ from Λ_{bL} to Λ_L, $\pi(e)$ is the path of the b links that c multiplies for $e \in E$, and $E'_0 = E' \setminus \bigcup_{e\in E} \pi(e)$ is the set of links on no path. A gauge transformation acts by $(gU)_{(x,\mu)} = g(x)U_{(x,\mu)}g(x+e_\mu)^{-1}$, and λ_S denotes the product of the Haar measures on G^S. For $W \in G$ and $m \geq 1$ let $\nu_W^{(m)}$ be the law of $(U_1,\dots,U_m)$ with $U_1,\dots,U_{m-1}$ independent Haar variables and $U_m = (U_1\cdots U_{m-1})^{-1}W$, the uniform law on the tuples with product W. For $V \in G^E$ let ν_V be the product of the laws $\nu_{V_e}^{(b)}$ of the links of $\pi(e)$, taken in their order along the path, over $e \in E$, and of $\lambda_{E'_0}$; then $c = V$ holds ν_V-almost surely.

Lemma 106. *Let $W \in G$ and $m \geq 2$, and let $(U_1,\dots,U_m)$ have law $\nu_W^{(m)}$.*

(i) *Any $m-1$ of the U_k are independent Haar variables, and so are the partial products $A_k = U_1\cdots U_k$, $1 \leq k \leq m-1$.*

(ii) *For $h_0,\dots,h_m \in G$ the law of $(h_{k-1}U_kh_k^{-1})_{k\leq m}$ is $\nu^{(m)}_{h_0Wh_m^{-1}}$.*

Proof. (i) For the coordinates $k \neq m$ this is the definition. Leave out U_i with $i < m$ and condition on the U_k with $k \neq i,m$. Then $U_m = QU_i^{-1}R$ with Q,R fixed, and U_m is a Haar variable independent of the conditioning variables, by the invariance of Haar measure under $U \mapsto U^{-1}$ and under translations. The map $(U_1,\dots,U_{m-1}) \mapsto (A_1,\dots,A_{m-1})$ is the composite of the translations $A_k = A_{k-1}U_k$, $A_0 = I$, so the A_k are independent Haar variables.

(ii) The variables $h_{k-1}U_kh_k^{-1}$, $k \leq m-1$, are independent Haar variables, and the product of all m of them is $h_0Wh_m^{-1}$. □

Definition 107. For a probability measure ρ on G^E, the *Haar refinement* of ρ along c is the probability measure $R(\rho) = \int \rho(dV)\,\nu_V$ on $G^{E'}$.

For $x' \in \Lambda_{bL}$ write $\mathscr{F}_{x'}$ for the σ-algebra of the links outside $V'_{x'}$, $\mathscr{H}_{x'} = \mathscr{F}_{x'} \vee \sigma(c)$, and $T(x')$ for the set of $e \in E$ with $\pi(e) \cap V'_{x'} \neq \emptyset$. A link of $\pi(e)$ with $e \notin T(x')$ lies outside $V'_{x'}$, so $\sigma(c_{E\setminus T(x')}) \subset \mathscr{F}_{x'}$. For a probability measure ρ on G^E and $T \subset E$ let P_T^ρ be the conditional expectation under ρ given the coarse links outside T. Every site of Λ_{bL} that is not a block corner by and lies on a path is an inner point $s = by + ke_\nu$, $1 \leq k \leq b-1$, of exactly one path $\pi(y,\nu)$. For a configuration U let g_U be the gauge transformation equal to the partial product $U_{(by,\nu)}U_{(by+e_\nu,\nu)}\cdots U_{(by+(k-1)e_\nu,\nu)}$ at each such inner point and to I at all other sites, put $M(U) = (g_UU)_{E'_0}$, and let $\Theta(V,m) \in G^{E'}$, for $V \in G^E$ and $m \in G^{E'_0}$, have the links $I,\dots,I,V_e$ along each path $\pi(e)$ and m on E'_0.

Lemma 108. *Let ρ be a gauge-invariant probability measure on G^E and $\mu' = R(\rho)$.*

(i) *μ' is gauge invariant, $c_*\mu' = \rho$, and the conditional law of U given c is $\nu_{c(U)}$. $R(\lambda_E) = \lambda_{E'}$, and if ρ has density r with respect to λ_E, then μ' has density $r\circ c$ with respect to $\lambda_{E'}$.*

(ii) *For every x' and every $f \in L^2(\mu')$, $\mathbb{E}[\mathbb{E}[f \mid \mathscr{F}_{x'}] \mid c] = (P^\rho_{T(x')}F)\circ c$, where $\mathbb{E}[f \mid c] = F \circ c$.*

(iii) *$g_UU = \Theta(c(U),M(U))$; under μ', M is independent of c and of the partial products along the paths, with law $\lambda_{E'_0}$; and every $f \in \mathscr{A}'$ satisfies $f = \tilde f(c,M)$ μ'-almost everywhere for a Borel function $\tilde f$ on $G^E \times G^{E'_0}$.*

Proof. (i) Let g be a gauge transformation of Λ_{bL} and $\hat g(y) = g(by)$. By Lemma 106(ii) with $h_k = g(by+ke_\nu)$ on each path $\pi(y,\nu)$, and by the invariance of $\lambda_{E'_0}$, the image of ν_V under g is $\nu_{\hat g V}$; hence the image of μ' is $\int \rho(dV)\nu_{\hat gV}$, which is μ' because ρ is invariant under $\hat g$. As $c = V$ holds ν_V-almost surely, $\mu'(\psi(c)k) = \int \rho(dV)\psi(V)\nu_V(k)$ for bounded Borel ψ,k, which gives the law of c and the conditional law. Under $\lambda_{E'}$ the links of a path satisfy: U_b is a Haar variable independent of $U_1,\dots,U_{b-1}$, hence so is $U_1\cdots U_b$, and given $U_1,\dots,U_{b-1}$ and the product the last link is determined; so $\lambda_{E'} = \int \lambda_E(dV)\nu_V$. The last claim follows, since $\int r(V)\lambda_E(dV)\nu_V = (r\circ c)\int \lambda_E(dV)\nu_V$.

(ii) Let $T = T(x')$ and $\mathscr{F} = \mathscr{F}_{x'}$. Under ν_V the links outside $V'_{x'}$ are independent families: the links of $\pi(e)$ for $e \notin T$, with law $\nu_{V_e}^{(b)}$; for $e \in T$, the links of $\pi(e)$ outside $V'_{x'}$, a proper subset of the path, which are independent Haar variables by Lemma 106(i); and the links of E'_0 outside $V'_{x'}$, with Haar law. So for bounded $\mathscr{F}$-measurable k, $\nu_V(k) = \kappa(V_{E\setminus T})$ depends only on $V_{E\setminus T}$, and for bounded Borel ψ on G^E,

$$\mu'(k\,\psi\circ c) = \int \rho(dV)\psi(V)\kappa(V_{E\setminus T}) = \mu'\big(k\,(P_T^\rho\psi)\circ c\big).$$

By density this holds for $k \in L^2(\mathscr{F})$. With $k = \mathbb{E}[f \mid \mathscr{F}]$, and since $(P_T^\rho \psi)\circ c$ is $\mathscr{F}$-measurable,

$$\begin{aligned}\mu'(\mathbb{E}[f \mid \mathscr{F}]\,\psi\circ c) &= \mu'\big(f\,(P_T^\rho\psi)\circ c\big)\\ &= \rho(F\,P_T^\rho\psi) = \rho\big((P_T^\rho F)\psi\big),\end{aligned}$$

which is the claim.

(iii) On each path $\pi(y,v)$ with links $U_1,\dots,U_b$ and partial products A_k, $A_0 = I$, the transformed links are $A_{k-1}U_kA_k^{-1} = I$ for $k \le b-1$ and $A_{b-1}U_b = c_{(y,v)}$, since $g_U = I$ at the block corners; this is the first claim. Under ν_V the partial products along the paths and the links of E_0' are independent Haar variables (Lemma 106(i)), and given the partial products each $M_{e'}$, $e' \in E_0'$, is a translate on both sides of $U_{e'}$ by fixed elements; so given V and the partial products, M has law $\lambda_{E_0'}$, for every V. For $f \in \mathscr{A}'$ let $\bar f(U) = \int f(gU)\,dg$, the integral over the Haar measure of the compact gauge group, where it converges absolutely, and $\bar f(U) = 0$ elsewhere. Since μ' is gauge invariant, $\iint |f(gU) - f(U)|\,dg\,d\mu' = \int \|f\circ g - f\|_{L^1(\mu')}dg = 0$, so $\bar f = f$ almost everywhere, and $\bar f(hU) = \bar f(U)$ for every gauge transformation h and every U. Hence $f = \bar f(g_U) = \bar f(\Theta(c,M))$ almost everywhere, and $\tilde f = \bar f\circ\Theta$. □

The following splitting holds for every measure. Write $\Pi_{\mathscr{B}}$ for the conditional expectation given a σ-algebra $\mathscr{B}$ under the measure at hand, and put

$$\Phi(f) = \frac{1}{n'^4}\sum_{x'\in\Lambda_{bL}} \mu'\big(\mathrm{Var}(f \mid \mathscr{H}_{x'})\big),$$

the Dirichlet form of the cube dynamics on the fibres of c, which resamples the links of a fine cube given the links outside it and the block variables.

Proposition 109 (the fine form splits)**.** *Let μ' be a gauge-invariant probability measure on $G^{E'}$, $\mu^c = c_*\mu'$, $n' \ge 2$ and $bL \ge 2n'+4$ even.*

(i) *For every real $f \in L^2(\mu')$, with $\mathbb{E}[f \mid c] = F_f\circ c$,*

$$\begin{aligned}\mathscr{E}'_{n'}(f) &= \Phi(f) + \frac{1}{n'^4}\sum_{x'}\mu'\big((\Pi_{\mathscr{H}_{x'}}f - \Pi_{\mathscr{F}_{x'}}f)^2\big)\\ &\ge \Phi(f) + \frac{1}{n'^4}\sum_{x'}\mu'\big((F_f\circ c - \Pi_{\sigma(c)}\Pi_{\mathscr{F}_{x'}}f)^2\big).\end{aligned}$$

(ii) *For every real $F \in L^2(\mu^c)$,*

$$\begin{aligned}\mathscr{E}'_{n'}(F\circ c) = \frac{1}{n'^4}\sum_{x'}\mu^c\big((F - P^{\mu^c}_{T(x')}F)^2\big)\\ - \frac{1}{n'^4}\sum_{x'}\mu'\big((\Pi_{\mathscr{F}_{x'}}(F\circ c) - (P^{\mu^c}_{T(x')}F)\circ c)^2\big).\end{aligned}$$

In particular, if $\Phi(f) \ge \mu'(\mathrm{Var}(f \mid c))/C_A$ and the last sum in (i) *is at least $n'^4\mathscr{E}_n^c(F_f)/C_B$ for every real $f \in \mathscr{A}'$, then $\mathrm{S}(n',n;C_A,C_B)$ holds for (μ',c).*

Proof. (i) Since $\mathscr{F}_{x'} \subset \mathscr{H}_{x'}$, $f - \Pi_{\mathscr{H}_{x'}}f$ is orthogonal to $\Pi_{\mathscr{H}_{x'}}f - \Pi_{\mathscr{F}_{x'}}f$, and $\mu'((f - P'_{x'}f)^2) = \mu'(\mathrm{Var}(f \mid \mathscr{H}_{x'})) + \mu'((\Pi_{\mathscr{H}_{x'}}f - \Pi_{\mathscr{F}_{x'}}f)^2)$. The projection $\Pi_{\sigma(c)}$ does not increase the norm, and $\Pi_{\sigma(c)}\Pi_{\mathscr{H}_{x'}} = \Pi_{\sigma(c)}$ because $\sigma(c) \subset \mathscr{H}_{x'}$.

(ii) For bounded ψ of the coarse links outside $T = T(x')$, $\mu'((F\circ c)(\psi\circ c)) = \mu^c(F\psi) = \mu^c((P_T^{\mu^c}F)\psi)$, so $\Pi_{\sigma(c_{E\setminus T})}(F\circ c) = (P_T^{\mu^c}F)\circ c$. As $\sigma(c_{E\setminus T}) \subset \mathscr{F}_{x'}$, Pythagoras' theorem gives $\mu'((F\circ c - (P_T^{\mu^c}F)\circ c)^2) = \mu'((F\circ c - \Pi_{\mathscr{F}_{x'}}(F\circ c))^2) + \mu'((\Pi_{\mathscr{F}_{x'}}(F\circ c) - (P_T^{\mu^c}F)\circ c)^2)$. The last statement follows from (i). □

The two conditions of the last statement bound disjoint parts of the fine form, so, unlike (A) and (B) of Definition 99, they give the split inequality with their own constants. The term subtracted in (ii) is the information that the fine field outside $V'_{x'}$ carries on F beyond the block variables of the links whose paths avoid $V'_{x'}$, the exact form of the term dropped in the proof of Theorem 94(i) and named in Remark 97. By Lemma 108(ii) it vanishes for a Haar refinement, which therefore has, among all measures with the same law of the block variables, the largest fine Dirichlet form on every function of the block variables.

Theorem 110 (the split inequality for Haar refinements)**.** *Let ρ be a gauge-invariant probability measure on G^E, $\mu' = R(\rho)$, and $n, n' \ge 2$ with $n' \ge bn$, $L \ge 2n+4$ and $bL \ge 2n'+4$ even. Then $\mu^c = \rho$, every real $f \in \mathscr{A}'$ satisfies*

$$\Phi(f) \ge \mu'\big(\mathrm{Var}(f \mid c)\big), \qquad \mathscr{E}'_{n'}(f) \ge \Phi(f) + \Big(\frac{bn}{n'}\Big)^4\mathscr{E}_n^c(F_f),$$

and $\mathrm{S}(n',n;1,(n'/(bn))^4)$ holds for (μ',c). Constants for which $\mathrm{S}(n',n;C_A,C_B)$ or (A) *holds satisfy $C_A \ge (1+2/n')^{-1}$.*

Proof. $\mu^c = \rho$ by Lemma 108(i). *The coarse term.* By Proposition 109(i) and Lemma 108(ii),

$$\mathscr{E}'_{n'}(f) \ge \Phi(f) + \frac{1}{n'^4}\sum_{x'}\rho\big((F_f - P^\rho_{T(x')}F_f)^2\big).$$

For $y \in \Lambda_L$ and $w \in [0,b)^4$ let $x' = by - w$. By Step 3 of the proof of Proposition 103, the map $(y,w) \mapsto by - w$ is a bijection onto Λ_{bL} and $V'_{x'}$ contains the first link of $\pi(z,v)$ for every $z \in y + [0,n)^4$, because $n' \ge bn$; so $V_y \subset T(x')$, the coarse links outside $T(x')$ lie outside V_y, $P^\rho_{T(x')} = P^\rho_{T(x')}P^c_y$, and by Pythagoras' theorem $\rho((F - P^\rho_{T(x')}F)^2) \ge \rho((F - P^c_yF)^2)$. Summing over (y,w), the last sum is at least $b^4\sum_y\rho((F_f - P^c_yF_f)^2) = b^4n^4\mathscr{E}_n^c(F_f)$.

The fibre term. Let $\tilde f$ and M be as in Lemma 108(iii). Under μ' the pair (c,M) has law $\rho\otimes\lambda_{E_0'}$, so the decomposition of the proof of Proposition 67(ii), in the variables $m \in G^{E_0'}$ with V as a parameter, writes $\tilde f = \sum_{B\subset E_0'}\tilde f_B$, where $\tilde f_B(V,m)$ depends on m only through m_B and has integral zero in each $m_{e'}$, $e' \in B$; the terms are orthogonal in $L^2(\rho\otimes\lambda_{E_0'})$, $\mathbb{E}[f \mid c] = \tilde f_\emptyset(c)$, and $\mu'(\mathrm{Var}(f \mid c)) = \sum_{B\ne\emptyset}\|\tilde f_B\|^2$. Fix x', let $J = E_0' \cap V'_{x'}$, and let $\mathscr{G}$ be the σ-algebra generated by c, the partial products along all

paths and the links of $E'_0 \setminus J$. Each link of a path is a function of c and of the partial products along its path, so $\mathscr{H}_{x'} \subset \mathscr{G}$, and $M_{E'_0\setminus J}$ is $\mathscr{G}$-measurable. Given $\mathscr{G}$, the links of J are independent Haar variables, by Lemma 108(i) and Lemma 106(i), and M_J is their translate on both sides by $\mathscr{G}$-measurable elements; so M_J has law λ_J and is independent of $\mathscr{G}$, and $\mathbb{E}[f \mid \mathscr{G}] = \int \tilde f(c, m_J, M_{E'_0\setminus J})\,\lambda_J(dm_J) = \sum_{B\cap J=\emptyset} \tilde f_B(c, M)$. Hence

$$\mu'\big(\mathrm{Var}(f \mid \mathscr{H}_{x'})\big) \geq \mu'\big(\mathrm{Var}(f \mid \mathscr{G})\big) = \sum_{B\cap J\neq\emptyset} \|\tilde f_B\|^2.$$

A link lies in n'^4 of the sets $V'_{x'}$, so summing over x' gives $n'^4\Phi(f) \geq n'^4 \sum_{B\neq\emptyset} \|\tilde f_B\|^2$.

The constant C_A. A plaquette p of Λ_{bL} contains a link of E'_0: of its two links of direction μ, based at x' and $x'+e_\nu$, at most one has its ν-coordinate divisible by b. Under μ' a link of E'_0 is a Haar variable independent of all other links. Let h be a real class function with Haar mean zero and $\int h^2 dU = 1$ and $f = h(U_p)$; then $f \in \mathscr{A}'$, $\mathbb{E}[f \mid c] = 0$ by integration over that link, and $\mu'(\mathrm{Var}(f \mid c)) = 1$. Each term $\mu'((f - P'_{x'}f)^2)$ is at most 1 and vanishes unless $V'_{x'}$ meets p, which happens for $n'^4 + 2n'^3$ points (proof of Proposition 103(ii)); so $\mathscr{E}'_{n'}(f) \leq 1 + 2/n'$, and (A) and S force $1 \leq C_A(1+2/n')$. □

For $\rho = \lambda_E$ the theorem is Proposition 103(i) on $\mathscr{A}'$. In the proof, the gauge transformation g_U puts the fine field in an axial gauge along the paths, in which a refinement becomes the product of the law of the block variables and of independent Haar variables on E'_0; the fibre form then only resamples Haar variables, and the coarse form sees the block variables through the sets $T(x')$, which contain the coarse cubes. The refinements compose along a tower: with c_j and the tori Λ_{L_j} as before Theorem 101, c_j the straight-path maps and R_j the refinement along c_j, put $\mu_0 = \rho$ and $\mu_j = R_j(\mu_{j-1})$; then $(c_j)_*\mu_j = \mu_{j-1}$ by Lemma 108(i), so (μ_j) is the tower of μ_K.

Corollary 111 (towers of refinements)**.** *Let $b \geq 2$, $K \geq 1$, $n_0 \geq 2$, $L_0 \geq 2n_0+4$ even, $n_j = b^j n_0$, and let ρ be a gauge-invariant probability measure on the links of Λ_{L_0} for which $\mathrm{BP}_\rho(n_0, \varpi_0)$ holds. Then $\mathrm{BP}_{\mu_K}(n_K, \min(\varpi_0, 1))$ holds. For $G = \mathrm{SU}(2)$ and ρ the Wilson measure of Λ_{L_0} at $|\beta_W| < \frac{16}{105}$, μ_K is the probability measure on the links of $\Lambda_{b^K L_0}$ with density proportional to $\prod_P \varphi(c_P^{(K)}(U))$ with respect to $\lambda_{E'}$, where $c^{(K)} = c_1 \circ \cdots \circ c_K$ and $c_P^{(K)}(U)$ is the product of the links of U around the square of side b^K with corner $b^K y$ that $c^{(K)}$ assigns to the plaquette P of Λ_{L_0} at y, and $\mathrm{BP}_{\mu_K}(n_K, 1-\alpha_{n_0})$ holds with α_{n_0} as in Corollary 74(i), for every K and every such L_0.*

Proof. For $1 \leq j \leq K$, $L_j = b^j L_0 \geq 2n_j + 4$, and Theorem 110 with $n' = n_j = bn_{j-1}$ gives $\mathrm{S}(n_j, n_{j-1}; 1, 1)$ for (μ_j, c_j). Theorem 101 with $C_{A,j} = 1$ and $\Gamma_j = 1$ gives the first claim. For the Wilson measure, Corollary 74(i) gives $\mathrm{BP}_\rho(n_0, 1-\alpha_{n_0})$, since the threshold $4n^2/(18n^2+18n-3)$ increases with n and equals $\frac{16}{105}$ at $n=2$; the density follows from Lemma 108(i) by induction on K, and $c^{(K)}$ is the straight-path map with blocks of side b^K. □

The measures μ_K are interacting, with Wilson weights on the loops of side b^K, but on all finer scales they are Haar: every plaquette has expectation zero. The tuned Wilson measures are of the opposite kind.

Proposition 112. *Let $G = \mathrm{SU}(N)$, $N \geq 2$, and μ'_β the Wilson measure of Λ_{bL} with $\varphi = e^{N\beta\,\mathrm{Re\,Tr}(\cdot)}$. Every Haar refinement along c gives every plaquette expectation $\mu'(\mathrm{Re\,Tr}\,U_p) = 0$, while $\mu'_\beta(\mathrm{Re\,Tr}\,U_p)$ has the sign of β. So μ'_β is not a Haar refinement for $\beta \neq 0$; in particular $\mu'_\beta \neq R(c_*\mu'_\beta)$.*

Proof. By the proof of Theorem 110, p contains a link of E'_0, which under a refinement is a Haar variable $U_{e'}$ independent of the other links. Up to a cyclic permutation, $U_p = AU_{e'}^{\pm 1}B$ with A, B products of the other links of p, and integrating over $U_{e'}$ first gives $\mathrm{Re\,Tr}(A(\int U^{\pm 1} dU)B) = 0$, because $\int U\,dU = 0$ on $\mathrm{SU}(N)$. For the Wilson measure, the translations and the hypercubic group of Λ_{bL} act transitively on the plaquettes and preserve μ'_β, so $m(\beta) = \mu'_\beta(\mathrm{Re\,Tr}\,U_p)$ does not depend on p. With $S = \sum_p \mathrm{Re\,Tr}\,U_p$ and $Z(\beta) = \int e^{N\beta S} d\lambda_{E'}$,

$$\frac{d}{d\beta}\log Z = N|P|\,m(\beta), \qquad \frac{d^2}{d\beta^2}\log Z = N^2\,\mathrm{Var}_{\mu'_\beta}(S) > 0,$$

the variance being positive because S is continuous and not constant, with $S = N|P|$ at $U \equiv I$ and Haar mean zero, and μ'_β has a positive continuous density. So m is strictly increasing, and $m(0) = 0$. □

Remark 113 (what the refinements give and what they leave). Proved: for every gauge-invariant law ρ of the block variables, its Haar refinement along the straight-path map satisfies the split inequality with the constants $C_A = 1$ and $C_B = (n'/(bn))^4$ of the product of Haar measures, and $C_A = 1$ is optimal up to the factor $1 + 2/n'$ (Theorem 110); along a tower of refinements, (H2) holds with $C_{A,j} = 1$ and $\Gamma_j = 1$ and (H1) is the condition for ρ, so the condition at the coarsest scale passes to every finer scale without loss, and for ρ a Wilson measure in the range of Corollary 74 it holds at every scale with the constant $1 - \alpha_{n_0}$, uniformly in the number of scales and the torus (Corollary 111); for every measure the fine Dirichlet form splits into the fibre form Φ and a coarse form, and the split inequality follows from one bound on each with the same constants, while on functions of the block variables the fine form falls short of that of the refinement by the information of the fine exterior on the block variables (Proposition 109).

Not proved: the split inequality for the Wilson measure, which is not a refinement (Proposition 112), outside the strong-coupling range treated in Sec. XM, for U(1) at $\beta > 0$, or for SU(2) on calm fibres; for them neither the bound $\Phi(f) \geq \mu'(\mathrm{Var}(f \mid c))/C_A$, a Poincaré inequality for the fine links given the block variables in the mean over μ', nor a bound on the subtracted term of Proposition 109(ii) is proved, and whether that term vanishes for them is not decided. Whether the split inequality with C_B near $(n'/(bn))^4$ holds or fails along the tuning is not decided; no obstruction to it is proved here, and the obstructions on small tori give none (Proposition 122). For a general ρ the optimal C_B in Theorem 110 is not determined; for $\rho = \lambda_E$ it is $(n'/(bn))^4$ by Proposition 103(ii). The condition (H1) does not follow from the results of this paper along

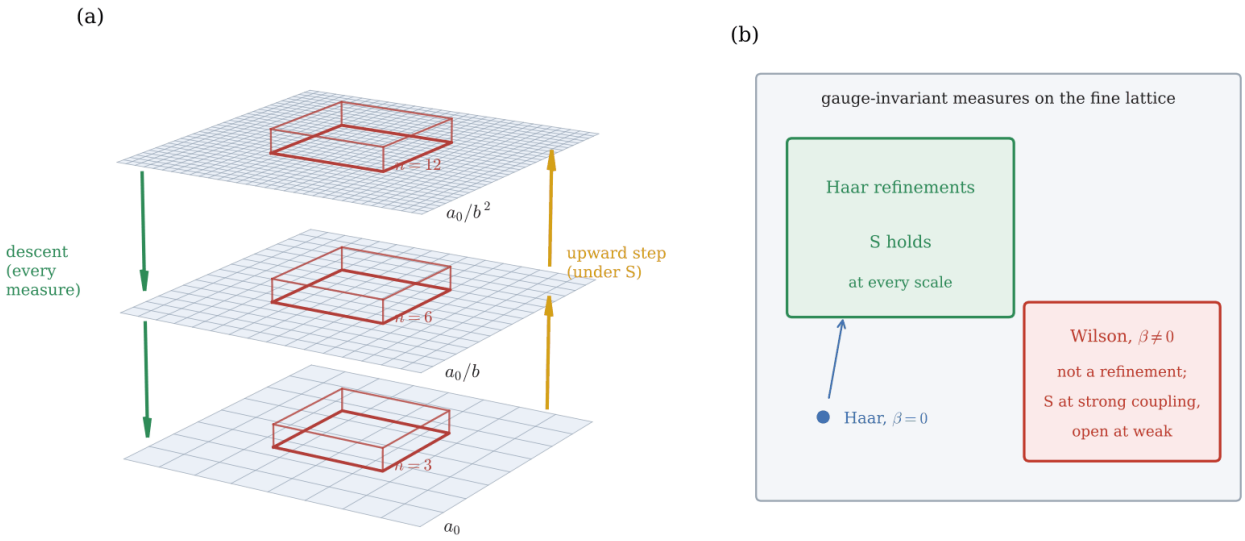


FIG. 16. The averaged condition across scales. (a) Three lattices of spacings a_0, a_0/b, a_0/b^2, $b = 2$, each with a cube of the same physical side, of $n = 3, 6, 12$ sites. The condition passes down to the block variables for every gauge-invariant measure (Theorem 94) and up one scale under the split inequality S (Theorem 100). (b) S holds for every Haar refinement (Theorem 110), the product of Haar measures included; a Wilson measure at $\beta \neq 0$ is not a Haar refinement (Proposition 112); for it S holds at strong coupling (Theorem 120) and is open at weak coupling.

the tuning: Sec. X N reduces it to bounds on the single-link coefficients of the laws $\mu^{(0)}_{k,L}$ of the block variables, which are not proved outside the strong-coupling range, and gives it at the cube size $n_0 + 1$ for SU(2) and those k with $|\beta_{k,W}| < \frac{16}{105}$, which along a tuning with $\beta_k \to \infty$ are finitely many (Proposition 130). Corollary 111 concerns measures that are Haar on all scales below the coarsest, and says nothing about the tuned Wilson measures; it shows that the split mechanism tolerates an arbitrary interaction of the block variables, and Proposition 109 places the remaining difficulty in the interaction at the finer scales. Neither BP along the tuning nor the mass gap is proved here.

M. The split inequality at strong coupling

This subsection proves the split inequality for the Wilson measure in the strong-coupling range of Corollary 74, on every torus, with constants that tend to those of Theorem 110 as $\beta \to 0$. The two parts of the fine Dirichlet form in Proposition 109 are bounded separately. In the axial gauge of Lemma 108(iii) the conditional law of the fine field given the block variables is a Gibbs measure on the links of E'_0 with the Wilson specification, and the single-link coefficients give a Poincaré inequality for the fibre form Φ. The information that the fine exterior of a cube carries on the block variables, which vanishes for the refinements, is bounded by the square of the response of the cube to one coarse link, and the contributions of distant cubes are almost orthogonal because correlations in the fibres decay. The split inequality is first derived from three bounds that can be posed for every measure. The subsection closes with the weak-coupling obstructions of Theorems 79 and 92: carried through Theorem 100, they bound the condition for the law of the block variables and put no constraint on the split inequality.

Throughout, c, $\pi(e)$, E'_0, $\mathscr{F}_{x'}$, $\mathscr{H}_{x'}$, $T(x')$, Φ, λ_S and ν_V are as in Sec. X L, μ' is a gauge-invariant probability measure on $G^{E'}$, $\mu^c = c_*\mu'$, $Q = \Pi_{\sigma(c)}$, so that $Qf = F_f \circ c$, and $\mathscr{G}'_{x'} = \sigma(c_{E\setminus T(x')}) \subset \mathscr{F}_{x'}$. Norms are those of $L^2(\mu')$ or of $L^2(\mu^c)$. The projection $P^{\mu^c}_{T(x')}$ maps $\mathscr{A}^c$ into itself, since a gauge transformation of Λ_L acts on each coarse link separately and preserves μ^c. For a real $g \in \mathscr{A}^c$ with $P^{\mu^c}_{T(x')} g = 0$ put

$$w_{x'}(g) = (I - Q)\,\Pi_{\mathscr{F}_{x'}}(g \circ c),$$

the part of the information of the fine exterior of $V'_{x'}$ on g that is not a function of the block variables.

Theorem 114 (the split inequality from three bounds)**.** *Let $n, n' \geq 2$ with $n' \geq bn$, $L \geq 2n+4$ and $bL \geq 2n'+4$ even. Let $C_F > 0$, $\rho \geq 0$ and $R > 0$ be such that:*

(F) $\mu'(\mathrm{Var}(f \mid c)) \leq C_F\,\Phi(f)$ *for every real $f \in \mathscr{A}'$;*

(X) $\|\Pi_{\mathscr{F}_{x'}}(g \circ c)\| \leq \rho \|g\|$ *for every x' and every real $g \in \mathscr{A}^c$ with $P^{\mu^c}_{T(x')} g = 0$;*

(D) $\|\sum_{x'} w_{x'}(g_{x'})\|^2 \leq R \sum_{x'} \|w_{x'}(g_{x'})\|^2$ *for every family of such $g_{x'}$.*

Put $\eta = \rho\,(RC_F)^{1/2}/n'^2$. If $\rho^2 + \eta < 1$, then every real $f \in \mathscr{A}'$ satisfies

$$\mathscr{E}'_{n'}(f) \geq (1-\eta)\,\Phi(f) + (1-\rho^2-\eta)\left(\frac{bn}{n'}\right)^4 \mathscr{E}^c_n(F_f),$$

and $\mathrm{S}(n',n;C_F/(1-\eta),(n'/(bn))^4/(1-\rho^2-\eta))$ *holds for* (μ',c).

Proof. Let $f\in\mathscr{A}'$ be real, $F=F_f$, $g=F\circ c=Qf$ and $k=f-g$, so that $Qk=0$. Fix x' and write $T=T(x')$, $\mathscr{F}=\mathscr{F}_{x'}$, $\mathscr{H}=\mathscr{H}_{x'}$. Then $g_{x'}=F-P_T^{\mu^c}F$ is a real element of $\mathscr{A}^c$ with $P_T^{\mu^c}g_{x'}=0$, and $g=g_{x'}\circ c+(P_T^{\mu^c}F)\circ c$, where the second term is $\mathscr{G}'_{x'}$-measurable, hence $\mathscr{F}$-measurable. Put $u_{x'}=\Pi_{\mathscr{F}}(g_{x'}\circ c)$ and $w_{x'}=w_{x'}(g_{x'})=u_{x'}-Qu_{x'}$.

The function $g-\Pi_{\mathscr{F}}g=g_{x'}\circ c-u_{x'}$ is orthogonal to $L^2(\mathscr{F})$, which contains $\Pi_{\mathscr{F}}k$, and $\langle g,k\rangle=0$; so

$$\|f-\Pi_{\mathscr{F}}f\|^2=\|g-\Pi_{\mathscr{F}}g\|^2+\|k-\Pi_{\mathscr{F}}k\|^2 -2\langle\Pi_{\mathscr{F}}g,k\rangle.$$

By Pythagoras' theorem and (X), $\|g-\Pi_{\mathscr{F}}g\|^2=\|g_{x'}\|^2-\|u_{x'}\|^2\ge(1-\rho^2)\|g_{x'}\|^2$. Since $\mathscr{F}\subset\mathscr{H}$ and g is $\mathscr{H}$-measurable, $\|k-\Pi_{\mathscr{F}}k\|^2\ge\|k-\Pi_{\mathscr{H}}k\|^2=\|f-\Pi_{\mathscr{H}}f\|^2=\mu'(\mathrm{Var}(f\mid\mathscr{H}))$. The functions $(P_T^{\mu^c}F)\circ c$ and $Qu_{x'}$ are $\sigma(c)$-measurable and $Qk=0$, so $\langle\Pi_{\mathscr{F}}g,k\rangle=\langle u_{x'},k\rangle=\langle w_{x'},k\rangle$. Summing over x', with $W=\sum_{x'}w_{x'}$ and $\Sigma_0=\sum_{x'}\|g_{x'}\|^2$,

$$n'^4\mathscr{E}'_{n'}(f)\ge(1-\rho^2)\Sigma_0+n'^4\Phi(f)-2\langle W,k\rangle.$$

By (D) and (X), $\|W\|^2\le R\sum_{x'}\|w_{x'}\|^2\le R\sum_{x'}\|u_{x'}\|^2\le R\rho^2\Sigma_0$, and by (F), $\|k\|^2=\mu'(\mathrm{Var}(f\mid c))\le C_F\Phi(f)$. Hence

$$2|\langle W,k\rangle|\le 2\rho(RC_F)^{1/2}(\Sigma_0\Phi(f))^{1/2}\le\eta\Sigma_0+\eta n'^4\Phi(f),$$

because $2\rho(RC_F)^{1/2}=2\eta n'^2$. So $\mathscr{E}'_{n'}(f)\ge(1-\eta)\Phi(f)+(1-\rho^2-\eta)n'^{-4}\Sigma_0$. As in the proof of Theorem 110 (the coarse term), $V_y\subset T(by-w)$ for $w\in[0,b)^4$, so $\|F-P^{\mu^c}_{T(by-w)}F\|^2\ge\mu^c((F-P_y^cF)^2)$, and $n'^{-4}\Sigma_0\ge(bn/n')^4\mathscr{E}_n^c(F)$. With (F), $(1-\eta)\Phi(f)\ge(1-\eta)\mu'(\mathrm{Var}(f\mid c))/C_F$. □

Hypothesis (D) holds with $R=(bL)^4$ by the Cauchy–Schwarz inequality, and (X) with $\rho=1$; the content of (D) is a constant R independent of the torus, an almost orthogonality of the exterior information of different cubes, which (F) and (X), each posed for one cube, do not give. For a Haar refinement, (X) holds with $\rho=0$ by Lemma 108(ii), so $w_{x'}=0$ and (D) is empty, and (F) holds with $C_F=1$ by the proof of Theorem 110; then $\eta=0$ and the theorem is Theorem 110.

On a single torus a bounded density with respect to a Haar refinement already gives the split inequality, at every coupling, with constants that depend on the torus.

Proposition 115 (the split inequality on one torus)**.** *Let* $n,n'\ge 2$ *with* $n'\ge bn$, $L\ge 2n+4$ *and* $bL\ge 2n'+4$ *even, let* ρ *be a gauge-invariant probability measure on* G^E, $\nu=R(\rho)$, *and* $\mu'=h\nu$ *with* h *gauge invariant,* $0<m\le h\le M$ *and* $\mathbb{E}_\nu[h\mid c]=1$. *Put* $q=(bn/n')^4$, $B=(L/n)^4$ *and* $v=(M-1)(1-m)$. *Then* $\mu^c=\rho$, *and* $\mathrm{S}(n',n;C_A,C_B)$ *holds for* (μ',c) *with*

$$C_A=\frac{2M}{m},\qquad C_B=\frac{1+2qBv}{mq}.$$

Every gauge-invariant probability measure μ' *on* $G^{E'}$ *with a continuous positive density* D *with respect to* $\lambda_{E'}$ *satisfies these hypotheses with* $\rho=\mu^c$, $h=D/\nu_c(D)$, $m=e^{-\Omega}$ *and* $M=e^{\Omega}$, *where* $\Omega=\log(\max D/\min D)$; *for the density* $D=Z^{-1}\prod_p\varphi(U_p)$ *of Sec. XF,* $\Omega\le 6(bL)^4\log(\max\varphi/\min\varphi)$.

Proof. Since $c_*\nu=\rho$ (Lemma 108(i)) and $\mathbb{E}_\nu[h\mid c]=1$, $\mu^c=\rho$. Let $f\in\mathscr{A}'$ be real, and let F_μ, F_ν be the functions with $F_\mu\circ c=\mathbb{E}_{\mu'}[f\mid c]$ and $F_\nu\circ c=\mathbb{E}_\nu[f\mid c]$. For every σ-algebra $\mathscr{G}$, $\mu'(\mathrm{Var}_{\mu'}(f\mid\mathscr{G}))=\inf_g\mu'((f-g)^2)$ over the $\mathscr{G}$-measurable g, and $m\nu\le\mu'\le M\nu$, so

$$\mu'\big(\mathrm{Var}_{\mu'}(f\mid c)\big)\le M\,\nu\big(\mathrm{Var}_\nu(f\mid c)\big),\qquad \mathscr{E}'_{n'}(f)\ge m\,\mathscr{E}^\nu_{n'}(f),$$

with $\mathscr{E}^\nu_{n'}$ the fine form of ν. As $\mathbb{E}_\nu[h\mid c]=1$,

$$(F_\mu-F_\nu)\circ c=\mathbb{E}_\nu\big[(h-1)(f-F_\nu\circ c)\;\big|\;c\big],$$

and $(h-m)(M-h)\ge 0$ gives $\mathbb{E}_\nu[(h-1)^2\mid c]=\mathbb{E}_\nu[h^2\mid c]-1\le M+m-mM-1=v$; the conditional Cauchy–Schwarz inequality gives $\rho((F_\mu-F_\nu)^2)\le v\,\nu(\mathrm{Var}_\nu(f\mid c))$. Each of the L^4 terms of $n^4\mathscr{E}_n^c(F_\mu-F_\nu)$ is at most $\rho((F_\mu-F_\nu)^2)$, so $\mathscr{E}_n^c(F_\mu-F_\nu)\le Bv\,\nu(\mathrm{Var}_\nu(f\mid c))$. If $v=0$, then $F_\mu=F_\nu$ ρ-almost everywhere; if $v>0$, the inequality $(x+y)^2\le(1+\varepsilon)x^2+(1+\varepsilon^{-1})y^2$ in each term, with $\varepsilon=2qBv$, gives

$$\mathscr{E}_n^c(F_\mu)\le(1+\varepsilon)\mathscr{E}_n^c(F_\nu)+(1+\varepsilon^{-1})Bv\,\nu\big(\mathrm{Var}_\nu(f\mid c)\big).$$

Since $mq(1+\varepsilon^{-1})Bv/(1+2qBv)=m/2$, in both cases

$$\frac{\mu'(\mathrm{Var}_{\mu'}(f\mid c))}{C_A}+\frac{\mathscr{E}_n^c(F_\mu)}{C_B}\le m\Big(\nu\big(\mathrm{Var}_\nu(f\mid c)\big)+q\,\mathscr{E}_n^c(F_\nu)\Big)\le m\,\mathscr{E}^\nu_{n'}(f)\le\mathscr{E}'_{n'}(f),$$

by Theorem 110 for ν.

For the second statement, Lemma 108(i) with $\rho=\lambda_E$ gives $\lambda_{E'}=\int\lambda_E(dV)\,\nu_V$ with $c=V$ ν_V-almost surely, so $\mu'=\int\lambda_E(dV)\,D\,\nu_V$, μ^c has the density $\nu_V(D)$ with respect to λ_E, and $\mu'=\int\mu^c(dV)\,(D/\nu_V(D))\,\nu_V=hR(\mu^c)$. Then $\mathbb{E}_{R(\mu^c)}[h\mid c]=1$, and $\min D\le\nu_V(D)\le\max D$ gives $e^{-\Omega}\le h\le e^{\Omega}$. The density $Z^{-1}\prod_p\varphi(U_p)$ is a product over the $6(bL)^4$ plaquettes of Λ_{bL}. □

The proposition applies to the laws $\mu^{(j)}$ of the block variables in a tower, which have continuous positive densities (Lemma 128), as well as to the Wilson measure at every coupling, so the split inequality holds on each torus at every scale. Its constants grow like $e^{2\Omega}$ and $B=(L/n)^4$, and for the Wilson measure Ω is proportional to the volume; it gives no constants independent of the torus, which (H2) needs, since Corollary 102 uses infinitely many tori at each scale. Being upper bounds, these constants say nothing about the optimal ones.[24]

For the Wilson measure the three bounds are proved from the single-link coefficients. A *marking* l assigns to each $e\in E$

[24] The proof compares the conditional mean F_μ of μ' with the conditional mean F_ν of the refinement; their difference is bounded by the conditional variance, but its coarse form only by $B=(L/n)^4$ times its norm, which is where the dependence on the torus enters C_B.

a link $l(e) \in \pi(e)$. For a configuration U let g^l_U be the gauge transformation equal to I at the block corners and at the sites on no path and, at an inner point s of a path $\pi(e)$, equal to the product of the links of $\pi(e)$ from its first site to s if s lies before $l(e)$ on the path, and to the inverse of the product of the links of $\pi(e)$ from s to its last site if s lies after $l(e)$. Put $M^l(U) = (g^l_U U)_{E'_0}$, and for $V \in G^E$ and $m \in G^{E'_0}$ let $\Theta^l(V,m) \in G^{E'}$ have the link V_e at $l(e)$, I at the other links of $\pi(e)$, $e \in E$, and m on E'_0. For the marking by the last links of the paths these are g_U, M and Θ of Lemma 108(iii).

Lemma 116 (the fibres of the Wilson measure)**.** *Let μ' have the density $D = Z^{-1}\prod_p \varphi(U_p)$ with respect to $\lambda_{E'}$, with φ a positive continuous class function, and let l be a marking.*

(i) *$V \mapsto \nu_V(D)$ is continuous and positive, μ^c has the density $\nu_V(D)$ with respect to λ_E, and $\nu^\varphi_V = D\,\nu_V/\nu_V(D)$ is a conditional law of U given $c = V$, defined for every V.*

(ii) *$g^l_U U = \Theta^l(c(U), M^l(U))$. Under ν^φ_V the values of g^l_U at the inner points of the paths are independent Haar variables, independent of M^l, and M^l has the law $\varsigma^l_V = \gamma^0_{E'_0}(\cdot \mid \Theta^l(V,\cdot))$, the conditional law of the links of E'_0 given those of the paths, at the path links of $\Theta^l(V,\cdot)$; its conditional law of m_J given $m_{E'_0\setminus J}$, $J \subset E'_0$, is $\gamma^0_J(\cdot \mid \Theta^l(V,m))$. Under μ' the pair (c, M^l) has the law $\mu^c(dV)\,\varsigma^l_V(dm)$.*

(iii) *Every real $f \in \mathscr{A}'$ satisfies $f = \tilde f^l(c, M^l)$ μ'-almost everywhere, with $\tilde f^l = \bar f \circ \Theta^l$ and $\bar f$ the gauge average of the proof of Lemma 108(iii), and $\mathbb{E}[f \mid c] = \varsigma^l_c(\tilde f^l(c,\cdot))$.*

Proof. Let $U_1, \dots, U_b$ be the links of a path $\pi(e)$, s_k the site after U_k, $A_k = U_1 \cdots U_k$, $A_0 = I$, and $l(e) = U_{k_e}$. The values of g^l_U at the inner points are A_k for $k < k_e$ and $(U_{k+1}\cdots U_b)^{-1} = c_e^{-1} A_k$ for $k_e \le k \le b-1$, and I at s_0, s_b. So the transformed links are $A_{k-1}U_k A_k^{-1} = I$ for $k < k_e$, $(U_k \cdots U_b)^{-1} U_k (U_{k+1}\cdots U_b) = I$ for $k > k_e$, and $A_{k_e-1}U_{k_e}U_{k_e+1}\cdots U_b = c_e$ for $k = k_e$; this is the first claim of (ii). Under ν_V the A_k, $k \le b-1$, are independent Haar variables (Lemma 106(i)), so are their left translates by the fixed elements V_e^{-1}, and the links of E'_0 are Haar variables independent of the paths; each $M^l_{e'}$ is a translate of $U_{e'}$ on both sides by values of g^l_U. Hence under ν_V the values of g^l_U at the inner points and M^l are independent with Haar laws, and $U = (g^l_U)^{-1}\Theta^l(V, M^l)$, so ν_V is the image of $\lambda \otimes \lambda_{E'_0}$ under $(h,m) \mapsto h^{-1}\Theta^l(V,m)$, which depends continuously on V.

(i) By Lemma 108(i),

$$\lambda_{E'} = \int \lambda_E(dV)\,\nu_V, \qquad \mu' = \int \lambda_E(dV)\,D\,\nu_V.$$

As D is continuous and positive and ν_V depends continuously on V, $\nu_V(D)$ is continuous and positive; as $c = V$ holds ν_V-almost surely, $\mu'(\psi(c)k) = \int \lambda_E(dV)\nu_V(D)\psi(V)\nu^\varphi_V(k)$ for bounded Borel ψ, k.

(ii) D is gauge invariant, so $D(U) = D(\Theta^l(V, M^l))$ ν_V-almost surely, and under ν^φ_V the values of g^l_U keep their Haar law and remain independent of M^l, whose law has density proportional to $D(\Theta^l(V,m))$ with respect to $\lambda_{E'_0}$. Every plaquette contains a link of E'_0 (proof of Theorem 110), so $D(\Theta^l(V,m))$ is proportional to $\prod_{p \cap E'_0 \ne \emptyset} \varphi(\Theta^l(V,m)_p)$, the density of $\gamma^0_{E'_0}(\cdot \mid \Theta^l(V,m))$; its conditional laws are those of the specification. The law of (c, M^l) follows from (i).

(iii) $f = \bar f$ almost everywhere and $\bar f$ is invariant under every gauge transformation, so $\bar f(U) = \bar f(g^l_U U) = \bar f(\Theta^l(c, M^l))$ for every U; the conditional expectation follows from (i) and (ii). □

For sets A, B of links of a torus let $d(A,B)$ be the least k for which there are links $e_0 \in A$, $e_1, \dots, e_k$ with $e_k \in B$ and e_{i-1}, e_i partners, and $d(j,A) = d(\{j\},A)$. The coefficients of Sec. X G are those of the measure at hand.

Lemma 117 (decay of the response of a block)**.** *Let μ be as in Sec. X F on Λ_L, $L \ge 3$, with $C^0_{ee'} \le \mathfrak{c}$ for all partners and $\xi = 18\mathfrak{c} < 1$.*

(i) *Let V be finite, $A \subset V$, $j \notin V$, and U, U' differ only at j. The laws of σ_A under $\gamma^0_V(\cdot \mid U)$ and $\gamma^0_V(\cdot \mid U')$ are at Kantorovich distance at most $\xi^{d(j,A)}\delta(U_j, U'_j)/(1-\xi)$ for d_A.*

(ii) *Let V_0 be finite, $\gamma = \gamma^0_{V_0}(\cdot \mid U)$ for some U, $A, B \subset V_0$ with $d(A,B) \ge 1$, $\phi \in L^2(\gamma)$ a function of σ_A with $\gamma(\phi) = 0$ and $\psi \in L^2(\gamma)$ a function of σ_B. Then*

$$|\gamma(\phi\psi)| \le \frac{\xi^{d(A,B)}}{1-\xi}\|\phi\|_{L^2(\gamma)}\|\psi\|_{L^2(\gamma)}.$$

Proof. (i) As in the proof of Lemma 70, a minimising coupling of the two laws for $\sum_{e\in V}\mathbb{E}\delta(\sigma_e, \sigma'_e)$ exists, and $\rho(e) = \mathbb{E}\delta(\sigma_e, \sigma'_e)$ satisfies $\rho \le \mathbf{C}\rho + \delta(U_j, U'_j)\mathbf{c}$ entrywise, with $\mathbf{C} = (C^0_{ek})_{e,k\in V}$ and $\mathbf{c} = (C^0_{ej})_{e\in V}$. Each column of $\mathbf{C}$ and the vector $\mathbf{c}$ have sums at most $18\mathfrak{c} = \xi$, since a link has eighteen partners; iterating, and since $\sum_e(\mathbf{C}^N\rho)(e) \le \xi^N \sum_e \rho(e) \to 0$, $\rho \le \delta(U_j, U'_j)\sum_{m\ge 0}\mathbf{C}^m\mathbf{c}$. The entry of $\mathbf{C}^m\mathbf{c}$ at e is a sum over chains $e, k_1, \dots, k_m, j$ of consecutive partners, so it vanishes unless $m+1 \ge d(j,e)$, and $\sum_{e\in A}(\mathbf{C}^m\mathbf{c})(e) \le \xi^{m+1}$. Hence $\sum_{e\in A}\rho(e) \le \delta(U_j, U'_j)\sum_{m\ge d(j,A)-1}\xi^{m+1} = \delta(U_j, U'_j)\xi^{d(j,A)}/(1-\xi)$, and the coupling restricted to A bounds the distance.

(ii) Let $\theta = (\xi^d/(1-\xi))^2$, $d = d(A,B)$, and let Π_A, Π_B be the conditional expectations under γ given σ_A, σ_B, with the versions given by the specification: given σ_B, the links of $V_0 \setminus B$ have the law $\gamma^0_{V_0\setminus B}(\cdot \mid \sigma_B, U)$. If h is a function of σ_A with Lipschitz constant Λ for d_A and σ_B, σ'_B differ at one link $j \in B$, then (i) with $V = V_0 \setminus B$ and $d(j,A) \ge d$ gives $|\Pi_B h(\sigma_B) - \Pi_B h(\sigma'_B)| \le \Lambda\xi^d\delta/(1-\xi)$; so Π_B maps such functions to functions of σ_B with Lipschitz constant $\Lambda\xi^d/(1-\xi)$ for d_B, and likewise Π_A in the other direction. The operator $\mathscr{R} = \Pi_A\Pi_B$ on the functions of σ_A in $L^2(\gamma)$ equals $\Pi_A\Pi_B\Pi_A$ there, so it is self-adjoint with $0 \le \mathscr{R} \le I$ and $\mathscr{R}1 = 1$, and it multiplies Lipschitz constants by at most θ. As in the proof of Proposition 67(i), $\mathrm{Var}_\gamma(\mathscr{R}^t h) \le \frac{1}{2}\theta^{2t}\Lambda^2 D^2|A|^2$, the

spectral measure of $h-\gamma(h)$ lives on $[0,\theta]$, and by density of the Lipschitz functions $\langle\phi,\mathscr{R}\phi\rangle\le\theta\|\phi\|^2$ for every ϕ as in the statement. Then $\gamma(\phi\psi)=\langle\Pi_B\phi,\psi\rangle$ and $\|\Pi_B\phi\|^2=\langle\phi,\mathscr{R}\phi\rangle\le\theta\|\phi\|^2$. □

For $A=V$, (i) is Lemma 70 with the bound $\sum_{e\in V}C^0_{ej}\le 18\mathfrak{c}$. In the rest of this subsection μ' is as in Lemma 116, its single-link coefficients on Λ_{bL} satisfy $C^0_{ee'}\le\mathfrak{c}$ for all partners, $\xi=18\mathfrak{c}<1$, $q=\xi/(1-\xi)$, and for $n'\ge 2$

$$\alpha_{n'}=\frac{\mathfrak{c}\,(18n'-3)}{(1-18\mathfrak{c})\,n'^2},$$

the number of Theorem 72(i).

Theorem 118 (the fibre inequality at strong coupling)**.** *Let $n'\ge 2$, $bL\ge 2n'+4$ and $\alpha_{n'}<1$. Then every real $f\in\mathscr{A}'$ satisfies $\Phi(f)\ge(1-\alpha_{n'})\,\mu'(\mathrm{Var}(f\mid c))$.*

Proof. Use the marking by the last links and write ς_V, $\tilde f$ for ς^l_V, $\tilde f^l$ and $\tilde f_V=\tilde f(V,\cdot)$. Fix x', let $J=E'_0\cap V'_{x'}$ and $\mathscr{G}$ the σ-algebra of the links outside J. It contains $\mathscr{F}_{x'}$ and $\sigma(c)$, so $\mu'(\mathrm{Var}(f\mid\mathscr{H}_{x'}))\ge\mu'(\mathrm{Var}(f\mid\mathscr{G}))$. The links outside J determine c, the values of g_U at the inner points, which are partial products of path links, and $M_{E'_0\setminus J}$, and are determined by them; by Lemma 116(ii) the conditional law of M_J given $\mathscr{G}$ is $\gamma^0_J(\cdot\mid\Theta(c,M))$. Let Π^V_J be the conditional expectation under ς_V given $m_{E'_0\setminus J}$, $\Pi^V_J h(m)=\int h(\sigma,m_{E'_0\setminus J})\,\gamma^0_J(d\sigma\mid\Theta(V,m))$. By Lemma 116(ii),(iii), $\mathbb{E}[f\mid\mathscr{G}]=(\Pi^c_J\tilde f_c)(M)$,

$$\mu'\big(\mathrm{Var}(f\mid\mathscr{G})\big)=\int\mu^c(dV)\,\varsigma_V\big((\tilde f_V-\Pi^V_J\tilde f_V)^2\big),$$
$$\mu'\big(\mathrm{Var}(f\mid c)\big)=\int\mu^c(dV)\,\mathrm{Var}_{\varsigma_V}(\tilde f_V).$$

It suffices to prove, for every V and every real $h\in L^2(\varsigma_V)$,

$$\sum_{x'}\varsigma_V\big((h-\Pi^V_{J(x')}h)^2\big)\ge(1-\alpha_{n'})n'^4\,\mathrm{Var}_{\varsigma_V}(h),$$

with $J(x')=E'_0\cap V'_{x'}$. Let $N=(bL)^4$, $\mathscr{K}=N^{-1}\sum_{x'}\Pi^V_{J(x')}$ and $d(m,m')=\sum_{e\in E'_0}\delta(m_e,m'_e)$, and let m, m' differ only at $i\in E'_0$. Couple $\mathscr{K}(m,\cdot)$ and $\mathscr{K}(m',\cdot)$ by choosing the same x'. If $i\in V'_{x'}$, then $i\in J(x')$, $\Theta(V,m)$ and $\Theta(V,m')$ agree outside $J(x')$, and the two laws coincide; this happens for n'^4 points x'. Otherwise $\Theta(V,m)$ and $\Theta(V,m')$ differ only at i, an optimal coupling costs at most $C^0_{J(x'),i}\delta(m_i,m'_i)$, and the link i adds $\delta(m_i,m'_i)$. By Lemma 70, $C^0_{J(x'),i}$ is at most $\mathfrak{c}/(1-18\mathfrak{c})$ times the number of partners of i in $V'_{x'}$, and by Lemma 71(i) on Λ_{bL} the sum of these bounds over the x' with $i\notin V'_{x'}$ is at most $\mathfrak{c}(18n'^3-3n'^2)/(1-18\mathfrak{c})=\alpha_{n'}n'^4$. So $W^d_1(\mathscr{K}(m,\cdot),\mathscr{K}(m',\cdot))\le(1-\kappa)\delta(m_i,m'_i)$ with $\kappa=(1-\alpha_{n'})n'^4/N$, and the argument of the proof of Proposition 67(i) for the self-adjoint operator $0\le\mathscr{K}\le I$ on $L^2(\varsigma_V)$ gives $\langle h,(I-\mathscr{K})h\rangle\ge\kappa\,\mathrm{Var}_{\varsigma_V}(h)$, which is the claim. □

Theorem 119 (the exterior information at strong coupling)**.** *The information of the fine exterior of a cube on the block variables satisfies the following.*

(i) *For every x' and every real $g\in L^2(\mu^c)$ with $P^{\mu^c}_{T(x')}g=0$, $\|\Pi_{\mathscr{F}_{x'}}(g\circ c)\|\le q\|g\|$.*

(ii) *Let $n,n'\ge 2$ with $n'\ge bn$, $L\ge 2n+4$ and $bL\ge 2n'+4$ even. Every real $F\in L^2(\mu^c)$ satisfies*

$$(1-q^2)\,\Sigma(F)\le\mathscr{E}'_{n'}(F\circ c)\le\Sigma(F),$$
$$\Sigma(F)=\frac{1}{n'^4}\sum_{x'}\mu^c\big((F-P^{\mu^c}_{T(x')}F)^2\big),$$

and $\Sigma(F)\ge(bn/n')^4\mathscr{E}^c_n(F)$.

Proof. (i) Fix x' and write $T=T(x')$, $V'=V'_{x'}$, $\mathscr{F}=\mathscr{F}_{x'}$ and $P=P^{\mu^c}_T$. For bounded Borel g on G^E put

$$\mathscr{M}g(V)=\int\nu^\varphi_V(dU)\int\gamma^0_{V'}(d\sigma\mid U)\,g\big(c(\sigma,U_{E'\setminus V'})\big),$$

where $(\sigma,U_{E'\setminus V'})$ has the links σ on V' and those of U outside. By Lemma 116(i), $(\mathscr{M}g)\circ c=Q\Pi_{\mathscr{F}}(g\circ c)$, so $\mathscr{M}$ extends to a self-adjoint operator $0\le\mathscr{M}\le I$ on $L^2(\mu^c)$ with $\langle g,\mathscr{M}g\rangle=\|\Pi_{\mathscr{F}}(g\circ c)\|^2$. It fixes the functions of $c_{E\setminus T}$, which composed with c are $\mathscr{F}$-measurable, so $\mathscr{M}P=P$ and, taking adjoints, $P\mathscr{M}=P$. For $e\notin T$ the links of $\pi(e)$ lie outside V', so $\mathscr{M}(V,\cdot)$ is carried by $\{\tilde V:\tilde V_{E\setminus T}=V_{E\setminus T}\}$.

Let V, $\tilde V$ differ only at $e\in T$, and choose a marking l with $l(e)\in\pi(e)\cap V'$. Let h have the Haar law on the inner points and $(M,\tilde M)$, independent of h, be an optimal coupling of ς^l_V and $\varsigma^l_{\tilde V}$ for $d_{E'_0}$, and put $U=h^{-1}\Theta^l(V,M)$, $\tilde U=h^{-1}\Theta^l(\tilde V,\tilde M)$, which have the laws ν^φ_V, $\nu^\varphi_{\tilde V}$ by Lemma 116(ii). The two laws ς^l are $\gamma^0_{E'_0}$ at configurations that differ outside E'_0 only at $l(e)$, with the values V_e, $\tilde V_e$; by Lemma 70 and the eighteen partners of $l(e)$, $\mathbb{E}d_{E'_0}(M,\tilde M)\le q\,\delta(V_e,\tilde V_e)$. The path links of U and $\tilde U$ agree except at $l(e)\in V'$, and $\delta(U_{e'},\tilde U_{e'})=\delta(M_{e'},\tilde M_{e'})$ for $e'\in E'_0$ by bi-invariance. Given U, $\tilde U$, couple $\gamma^0_{V'}(\cdot\mid U)$ and $\gamma^0_{V'}(\cdot\mid\tilde U)$ optimally for $d_{V'}$, measurably [113, Cor. 5.22]; changing the links of $E'_0\setminus V'$ one at a time, Lemma 70 gives an expected distance at most $q\sum_{e'\in E'_0\setminus V'}\delta(U_{e'},\tilde U_{e'})$. The coarse values $c(\sigma,U_{E'\setminus V'})$ and $c(\tilde\sigma,\tilde U_{E'\setminus V'})$ agree outside T, and at $e''\in T$ they are products along $\pi(e'')$ of links that agree outside V'; as $\delta(XY,X'Y')\le\delta(X,X')+\delta(Y,Y')$ and the paths are disjoint, their distance for d_T is at most $d_{V'}(\sigma,\tilde\sigma)$. Hence $W^{d_T}_1(\mathscr{M}(V,\cdot),\mathscr{M}(\tilde V,\cdot))\le q^2\delta(V_e,\tilde V_e)$, and, as $\mathscr{M}(V,\cdot)$ does not depend on the marking, $W^{d_T}_1\le q^2 d_T(V,\tilde V)$ whenever $V_{E\setminus T}=\tilde V_{E\setminus T}$.

Let g be bounded with $|g(V)-g(\tilde V)|\le\Lambda d_T(V,\tilde V)$ whenever $V_{E\setminus T}=\tilde V_{E\setminus T}$. Then $\mathscr{M}^t g$ has this property with Λq^{2t}, its oscillation on each set $\{V_{E\setminus T}=\mathrm{const}\}$ is at most $\Lambda q^{2t}D|T|$, and since $P\mathscr{M}=\mathscr{M}P=P$,

$$\|\mathscr{M}^t(g-Pg)\|^2=\|\mathscr{M}^t g-P\mathscr{M}^t g\|^2\le(\Lambda q^{2t}D|T|)^2.$$

$\mathscr{M}$ maps $L^2(\mu^c)\ominus L^2(\mathscr{G}'_{x'})$ into itself, so the spectral measure of $g-Pg$ for $\mathscr{M}$ lives on $[0,q^2]$. The functions $g-Pg$ with g Lipschitz for d_E are dense in $L^2(\mu^c)\ominus L^2(\mathscr{G}'_{x'})$, by the Stone–Weierstrass theorem, so $\langle g,\mathscr{M}g\rangle\le q^2\|g\|^2$ whenever $Pg=0$.

(ii) The function $(P^{\mu^c}_{T(x')}F)\circ c$ is $\mathscr{F}_{x'}$-measurable, so the subtracted term of Proposition 109(ii) is $\|\Pi_{\mathscr{F}_{x'}}((F-P^{\mu^c}_{T(x')}F)\circ c)\|^2$, at most $q^2\|F-P^{\mu^c}_{T(x')}F\|^2$ by (i). The last inequality is the coarse term of the proof of Theorem 110. □

So on functions of the block variables the fine form lies between $1-q^2$ times that of the refinement of μ^c and that form itself, and (B) holds there with $C_B=(n'/(bn))^4/(1-q^2)$.

Theorem 120 (the split inequality at strong coupling)**.** *Let $n,n'\ge 2$ with $n'\ge bn$, $L\ge 2n+4$ and $bL\ge 2n'+4$ even, and $\alpha_{n'}<1$. Put*

$$R_{n'}=(2n'+3)^4+\frac{10(n'+2)^3\,\xi\,(8-5\xi+4\xi^2-\xi^3)}{(1-\xi)^5},$$
$$\eta=q\Big(\frac{R_{n'}}{(1-\alpha_{n'})n'^4}\Big)^{1/2}.$$

If $q^2+\eta<1$, every real $f\in\mathscr{A}'$ satisfies

$$\mathscr{E}'_{n'}(f)\ge(1-\eta)(1-\alpha_{n'})\,\mu'\big(\mathrm{Var}(f\mid c)\big)+(1-q^2-\eta)\Big(\frac{bn}{n'}\Big)^4\mathscr{E}^c_n(F_f),$$

and $\mathrm{S}(n',n;C_A,C_B)$ holds for (μ',c) with

$$C_A=\frac{1}{(1-\eta)(1-\alpha_{n'})},\qquad C_B=\frac{(n'/(bn))^4}{1-q^2-\eta}.$$

Proof. By Theorems 118 and 119(i), (F) holds with $C_F=1/(1-\alpha_{n'})$ and (X) with $\rho=q$. Then $\rho(RC_F)^{1/2}/n'^2=\eta$ for $R=R_{n'}$, and Theorem 114 gives the claim once (D) is proved with $R=R_{n'}$.

Use the marking by the last links, $\varsigma_V=\varsigma^l_V$. Let $g_{x'}\in\mathscr{A}^c$ be real with $P^{\mu^c}_{T(x')}g_{x'}=0$. Replacing $g_{x'}$ by its average over the gauge group of Λ_L, as in the proof of Lemma 108(iii), changes it on a μ^c-null set only, so we may take $g_{x'}(\hat g V)=g_{x'}(V)$ for all $\hat g$ and V. Then

$$\phi_{x'}(U)=\int\gamma^0_{V'_{x'}}(d\sigma\mid U)\,g_{x'}\big(c(\sigma,U_{E'\setminus V'_{x'}})\big)$$

is a version of $\Pi_{\mathscr{F}_{x'}}(g_{x'}\circ c)$. For a gauge transformation g of Λ_{bL}, $\gamma^0_{V'_{x'}}(\cdot\mid gU)$ is the image of $\gamma^0_{V'_{x'}}(\cdot\mid U)$ under g, and $c\circ g=\hat g\circ c$ by (B2); so $\phi_{x'}$ is gauge invariant and $\phi_{x'}(U)=\phi_{x'}(\Theta(c(U),M(U)))$. The law $\gamma^0_{V'_{x'}}(\cdot\mid\Theta(V,m))$ depends on m only through $m_{B(x')}$, with $B(x')=E'_0\cap\partial V'_{x'}$, and the coarse values $c(\sigma,\Theta(V,m)_{E'\setminus V'_{x'}})$ are products of links of σ and of path links of $\Theta(V,m)$; so $\phi_{x'}(\Theta(V,m))=\tilde\phi_{x'}(V,m_{B(x')})$. By Lemma 116(ii), $w_{x'}(g_{x'})=\tilde w_{x'}(c,M)$ with $\tilde w_{x'}(V,m)=\tilde\phi_{x'}(V,m_{B(x')})-\varsigma_V(\tilde\phi_{x'}(V,\cdot))$, a function of $m_{B(x')}$ of mean zero under ς_V, and

$$\langle w_{x'},w_{x''}\rangle=\int\mu^c(dV)\,\varsigma_V\big(\tilde w_{x'}(V,\cdot)\tilde w_{x''}(V,\cdot)\big).$$

Let $s=|x'-x''|_\infty$ and $r_{x'x''}=1$ for $s\le n'+1$, $r_{x'x''}=\xi^{s-n'-1}/(1-\xi)$ for $s\ge n'+2$. The base points of the links of $\partial V'_{x'}$ lie in $x'+[-1,n']^4$. For $s\ge n'+2$ choose a coordinate i in which the representative $t\in(-bL/2,bL/2]$ of $x''_i-x'_i$ has $|t|=s$; the i-th coordinates of base points of $B(x')$ and $B(x'')$ differ by $t+a$ with $|a|\le n'+1$, at distance at least $\min(s,bL-s)-n'-1=s-n'-1$ from $bL\mathbb{Z}$. The base points of partners differ by at most one in $|\cdot|_\infty$ (proof of Lemma 71), so $d(B(x'),B(x''))\ge s-n'-1\ge 1$. As $\varsigma_V=\gamma^0_{E'_0}(\cdot\mid\Theta(V,\cdot))$, Lemma 117(ii) with $V_0=E'_0$, or the Cauchy–Schwarz inequality for $s\le n'+1$, bounds $|\varsigma_V(\tilde w_{x'}\tilde w_{x''})|$ by $r_{x'x''}$ times the product of the norms in $L^2(\varsigma_V)$, and the Cauchy–Schwarz inequality in V gives $|\langle w_{x'},w_{x''}\rangle|\le r_{x'x''}\|w_{x'}\|\|w_{x''}\|$. Since r is symmetric,

$$\Big\|\sum_{x'}w_{x'}\Big\|^2\le\sum_{x'}\|w_{x'}\|^2\sup_{x'}\sum_{x''}r_{x'x''}.$$

At most $(2n'+3)^4$ points have $s\le n'+1$ and at most $(2s+1)^4-(2s-1)^4=64s^3+16s\le 80s^3$ have $|x''-x'|_\infty=s$. With $n'+1+t\le(n'+2)(1+t)/2$ for $t\ge 1$ and $\sum_{t\ge0}(t+1)^3\xi^t=(1+4\xi+\xi^2)/(1-\xi)^4$,

$$\sum_{s\ge n'+2}\frac{80s^3\xi^{s-n'-1}}{1-\xi}\le\frac{10(n'+2)^3}{1-\xi}\sum_{t\ge1}(t+1)^3\xi^t=\frac{10(n'+2)^3\,\xi\,(8-5\xi+4\xi^2-\xi^3)}{(1-\xi)^5},$$

so $\sup_{x'}\sum_{x''}r_{x'x''}\le R_{n'}$, which is (D). □

The numbers $\alpha_{n'}$ and $R_{n'}/n'^4$ decrease in n', so η decreases in n', and $R_{n'}/n'^4\to 16$, so $\eta\to 4q$ as $n'\to\infty$. As $\beta\to 0$, $\eta=O(\beta)$ uniformly in L and in $n'\ge 2$, since η is largest at $n'=2$; the constants are $C_A=1+O(\beta)$ and $C_B=(n'/(bn))^4(1+O(\beta))$, those of Theorem 110 up to factors $1+O(\beta)$.

Corollary 121 (explicit ranges)**.** *Theorem 120 applies, whenever $\alpha_{n'}<1$ and $q^2+\eta<1$, in the following cases.*

(i) *$G=\mathrm{SU}(2)$, $|\beta_W|<\frac{2}{9}$, with $\xi=\frac{9}{2}|\beta_W|$. For $|\beta_W|\le\frac{1}{100}$ and every $n'\ge 8$, $\eta<0.272$, $C_A<1.39$ and $C_B<1.39\,(n'/(bn))^4$.*

(ii) *$G=\mathrm{SU}(3)$, $\beta_W\ge 0$ with $\zeta(\beta_W)<1$, with $\xi=\zeta(\beta_W)$ as in Corollary 74(ii).*

(iii) *$G\subset\mathrm{U}(N)$ closed and connected with semisimple Lie algebra, $24N|\beta|<\kappa_G$, with $\xi=18N|\beta|/(\kappa_G-6N|\beta|)$.*

Proof. The bounds $C^0_{ee'}\le\mathfrak{c}$ with $18\mathfrak{c}=\xi$ are those of the proof of Corollary 74: $\mathfrak{c}=|\beta_W|/4$ for SU(2), $18\mathfrak{c}=\zeta(\beta_W)$ for SU(3), and $\mathfrak{c}=N|\beta|/(\kappa_G-6N|\beta|)$ from Lemma 73. The numbers q, $\alpha_{n'}$, $R_{n'}$ and η increase with ξ, so in (i) it suffices to take $\xi=\frac{9}{200}$ and $n'=8$. Then $q=\frac{9}{191}<0.04713$, $\mathfrak{c}=\frac{1}{400}$ and $\alpha_8=\frac{141}{24448}<0.0058$. As $\xi(8-5\xi+4\xi^2-\xi^3)\le 8\xi=0.36$ and $(1-\xi)^5\ge 1-5\xi=0.775$, $R_8\le 19^4+10^4\cdot 0.36/0.775<130321+4646=134967$, and $R_8/((1-\alpha_8)8^4)<134967/(0.9942\cdot 4096)<33.15$. So $\eta<0.04713\cdot 5.758<0.2714$, $q^2<0.0023$, $1-q^2-\eta>0.7263$ and $(1-\eta)(1-\alpha_8)>0.7286\cdot 0.9942>0.724$, which give the bounds. □

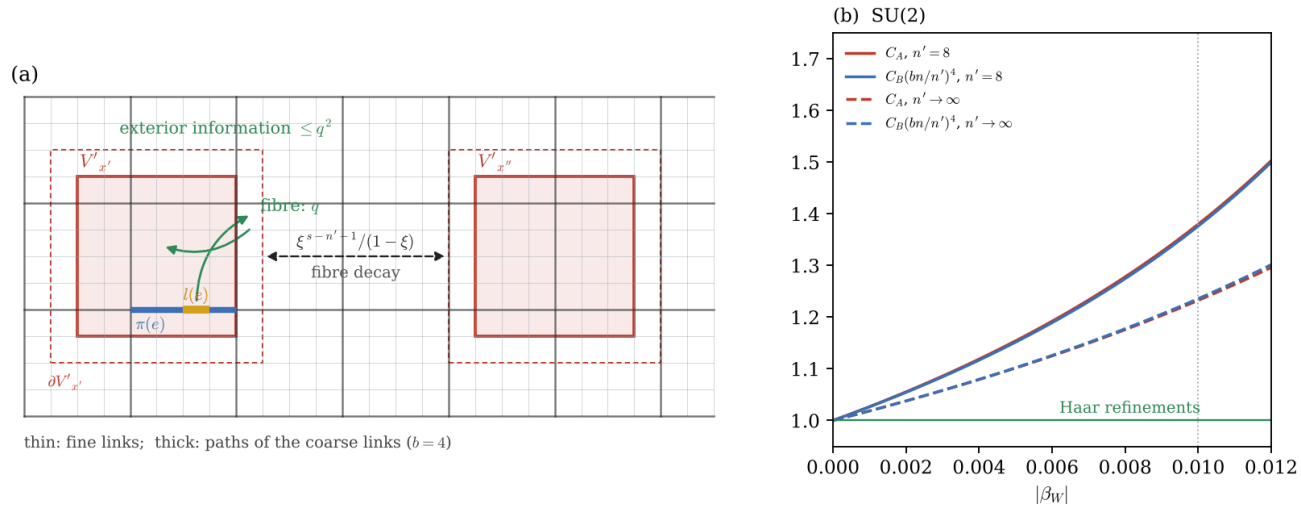


FIG. 17. The split inequality at strong coupling. (a) A coarse link e enters the fine field through its path $\pi(e)$; in a gauge whose marked link $l(e)$ lies in the cube $V'_{x'}$, a change of V_e moves the fibre law by at most q, and the cube, resampled given its exterior, feels that move through $\partial V'_{x'}$ by at most q again, so the exterior information is at most q^2 (Theorem 119); two cubes at distance s see each other through correlations of the fibre law that decay like $\xi^{s-n'-1}/(1-\xi)$ (Lemma 117). (b) The constants of Theorem 120 for SU(2) at $n'=8$ and in the limit $n'\to\infty$, where $\eta\to 4q$; both tend to the value 1 of the Haar refinements as $\beta_W\to 0$, and at $|\beta_W|=\frac{1}{100}$ they are below 1.39 (Corollary 121).

Figure 17 shows the mechanism and the constants. At strong coupling BP for the Wilson measure is already given by Corollary 74; Theorem 120 shows that the split mechanism of Theorem 110 survives a nonzero interaction at the finest scale, with constants close to the best ones. It is (H2) at the top step of a tower whose finest measure is a Wilson measure in the range above. The measures $\mu^{(j)}$, $j<K$, of the tower are laws of block variables, not Wilson measures, and the proofs above do not apply to them: they use that the conditional law of the fine field given the path links is the nearest-neighbour Wilson specification, with every link having eighteen partners.

The weak-coupling obstructions bound BP on the smallest tori. The following proposition shows where they act in the upward step. Let $X_n(\beta)=38400\,n^4(\log\beta+6.11)/\beta$ be the bound of Corollary 80.

Proposition 122 (the obstructions on small tori)**.** *Let $G=\mathrm{SU}(N)$, $N\ge 2$, $\beta\ge 1$, $b\ge 2$, $n\ge 2$, $L\ge 2n+4$ even, μ' the Wilson measure of Λ_{bL} at β and $\mu^c=c_*\mu'$.*

(i) *Every coarse plaquette satisfies $\mu^c(\|c_P-I\|^2_{\mathrm{HS}})\le b^4\mu'(\|U_p-I\|^2_{\mathrm{HS}})$, the right side being the same for every fine plaquette p. If $L=2n+4$ and $\mathrm{BP}_{\mu^c}(n,\varpi_c)$ holds, then $\varpi_c\le C'_N b^4 n^4\mu'(\|U_p-I\|^2_{\mathrm{HS}})$ with C'_N of Proposition 131(ii); for $N=2$, $\varpi_c\le b^4X_n(\beta)$.*

(ii) *Let $N=2$, $L=2n+4$ and $n'=bn+2b-2$, so that $bL=2n'+4$. If $\mathrm{BP}_{\mu^c}(n,\varpi_c)$ and $\mathrm{S}(n',n;C_A,C_B)$ hold with $C_B\ge 1$, then the constant $\varpi'=\min(1/C_A,\varpi_c/C_B)$ of Theorem 100(i) satisfies $\varpi'\le X_{n'}(\beta)$, the bound of Theorem 79(ii) for μ', for every $C_A>0$, by (i) alone.*

(iii) *Let $N=2$, $\beta\ge 2$, $bL\ge 8$ and $T=\mathrm{Tr}\,P_0$. If $\mathrm{BP}_{\mu^c}(n,\varpi_c)$ holds, then*

$$\mu'\big(T^2(4-T^2)\big)^2\le\frac{9\cdot 2^{19}\,b^{12}L^{11}n\,(\log\beta+6.11)^3}{\varpi_c\beta}.$$

Proof. (i) The coarse plaquette c_P is the holonomy of the boundary of a square of b^2 fine plaquettes, so by Lemma 76 and the Cauchy–Schwarz inequality $\|c_P-I\|^2_{\mathrm{HS}}\le b^2\sum_p\|U_p-I\|^2_{\mathrm{HS}}$ over these plaquettes, and the symmetries of Λ_{bL} act transitively on the plaquettes. The fine twist that multiplies the links $(x,0)$ with $x_0=0$ by z^{c_1} and the links $(x,1)$ with $x_1=0$ by z^{d_1} preserves μ'; on a path $\pi(y,0)$ it changes only the first link, and only when $y_0=0$, and likewise in direction 1, so it induces the coarse twist Z_{c_1,d_1} and μ^c is invariant under the coarse twists. μ^c has a continuous positive density (Lemma 116(i)), so $\mathrm{Var}_{\mu^c}(\mathrm{Re}\,\mathrm{Tr}\,c_P)>0$. Proposition 131(ii) gives the bound, and for $N=2$ Lemma 78 gives $\mu'(\|U_p-I\|^2_{\mathrm{HS}})=2\mu'(2-\mathrm{Tr}\,U_p)\le 2(\log\beta+6.11)/\beta$, with $2\cdot 19200=38400$.

(ii) $\varpi'\le\varpi_c/C_B\le\varpi_c\le b^4X_n(\beta)\le X_{n'}(\beta)$, since $bn\le n'$.

(iii) The coarse Polyakov loop P_0^c through 0 in direction 0 is the product of the bL fine links $(te_0,0)$, so $P_0^c\circ c=P_0$, and $h=(\mathrm{Tr}\,P_0^c)^2/4$ is a real element of $\mathscr{A}^c$ with $h\circ c=T^2/4$. The proof of Theorem 92(i) on Λ_L, with the coarse plaquettes and their largest mean ε^c of $\|c_P-I\|^2_{\mathrm{HS}}$, gives $\mathscr{E}^c_n(h)\le 4L^3n\varepsilon^c$, and $\varepsilon^c\le 2b^4(\log\beta+6.11)/\beta$ by (i). So $\mathrm{Var}_{\mu'}(h\circ c)=\mathrm{Var}_{\mu^c}(h)\le 8b^4L^3n(\log\beta+6.11)/(\varpi_c\beta)$. By (17) and Lemma 91(iii) on Λ_{bL}, $\mu'(T^2(4-T^2))^2\le 16\,\mathrm{Var}_{\mu'}(h\circ c)\cdot 36864(bL)^8(\log\beta+6.11)^2$, and $16\cdot 8\cdot 36864=9\cdot 2^{19}$. □

Through Theorem 100(i), Theorems 79(ii) and 92(i) for the Wilson measure μ' on Λ_{bL} bound $\varpi'=\min(1/C_A,\varpi_c/C_B)$

from above. By (ii) the first bound follows from the bound (i) on ϖ_c alone when $C_B \geq 1$; and since $(bL)^{11}n'C_B/\varpi_c \geq b^{12}L^{11}n/\varpi_c$ for $C_B \geq 1$, the second, $\mu'(T^2(4-T^2))^2 \leq 9 \cdot 2^{19}(bL)^{11}n'(\log\beta+6.11)^3/(\varpi'\beta)$, follows from (iii). So these obstructions give no lower bound on C_A and none on $C_B \geq 1$: they bound the condition for the law of the block variables on the coarse torus, the analogue of (H1) at that scale. On a tower of straight-path maps over Λ_{2n_0+4}, whose composite is the straight-path map with blocks of side b^K, (i) with b^K in place of b, equivalently Proposition 131(iii) with Lemma 76, bounds the constant of (H1) by $X_{n_K}(\beta)$ for $n_K = b^K n_0$ and $N = 2$; along the continuum tuning n_k^4 grows faster than β_k (Remark 81), and this bound exceeds one for large k. Corollary 102 uses an infinite set of tori for each k and does not need the smallest one.

Remark 123 (what is proved about the split inequality for Wilson measures). Proved: for every gauge-invariant measure, the split inequality follows from a fibre Poincaré inequality (F), a bound (X) on the information of the fine exterior of one cube on the block variables, and the almost orthogonality (D) of that information for different cubes, with explicit constants; the Haar refinements are the case $\rho = 0$ (Theorem 114); on one torus, every measure with a continuous positive density, in particular the Wilson measure at every coupling and the laws of the block variables of a tower, satisfies the split inequality with constants of order $e^{2\Omega}$ and $(L/n)^4$ (Proposition 115). For the Wilson measure with single-link coefficients at most $\mathfrak{c}$, $18\mathfrak{c} < 1$, on every torus: the fibre form satisfies $\Phi \geq (1-\alpha_{n'})\mu'(\mathrm{Var}(f \mid c))$ (Theorem 118); the exterior information is at most q^2 times the coarse form of the refinement, so the fine form of a function of the block variables lies between $1-q^2$ times that form and the form itself (Theorem 119); correlations in the fibres decay geometrically (Lemma 117); and $\mathrm{S}(n',n;1/((1-\eta)(1-\alpha_{n'})),(n'/(bn))^4/(1-q^2-\eta))$ holds whenever $q^2+\eta<1$, with constants within factors $1+O(\beta)$ of those of the refinements (Theorem 120), for $\mathrm{SU}(2)$, $\mathrm{SU}(3)$ and every compact connected group with semisimple Lie algebra at small β, for $\mathrm{SU}(2)$ at $|\beta_W| \leq \frac{1}{100}$ and $n' \geq 8$ with $C_A < 1.39$ and $C_B < 1.39(n'/(bn))^4$ (Corollary 121). The obstructions of Theorems 79 and 92 on small tori, carried through Theorem 100, follow from bounds on the condition for the law of the block variables and put no constraint on C_A or on $C_B \geq 1$ (Proposition 122).

Not proved: the split inequality with constants independent of the torus for the laws $\mu^{(j)}$ of the block variables below the finest scale of a tower, which are not Wilson measures and to which the proofs of Theorems 118–120 do not apply; the split inequality with constants independent of the torus for the Wilson measure outside the strong-coupling range, in particular along the tuning, where the range contains finitely many k, for $\mathrm{U}(1)$ at $\beta > 0$ and for $\mathrm{SU}(2)$ on calm fibres, where none of (F), (X), (D) is proved; the optimal constants at strong coupling; whether the split inequality holds or fails at weak coupling, on small tori or on large ones; (H1) along the tuning. Neither BP along the tuning nor the mass gap is proved here.

N. The condition at the coarsest scale

This subsection treats the hypothesis (H1) of Corollary 102, the condition $\mathrm{BP}(n_0,\varpi_0)$ for the laws $\mu^{(0)}_{k,L}$ of the block variables at the coarsest spacing a_0. These laws are not Wilson measures: for the straight-path maps they have smooth positive densities whose logarithms, the induced actions, couple every pair of coarse links. The first part proves a criterion for BP that uses only the single-link conditional laws of such a measure, with no product form of the action and no finite range: a Dobrushin condition on the Kantorovich coefficients of the single links, or a bound of Bakry–Émery type on the Hessian of the induced action. It recovers the leading term of Corollary 74(iii) for the Wilson action. The second part places (H1) among the other statements. It follows from $\mathrm{BP}(n_k,\varpi)$ along the tuning with the coarse cube enlarged by one, so that under (H2) it is equivalent to the conclusion of Corollary 102 up to one unit of cube size; it is independent of (H2); and it fails for the block laws of a measure whose local gauge-invariant covariances vanish beyond a fixed distance, the failure being carried by the Polyakov loops of the coarse torus.

Setting. Let $G \subset \mathrm{U}(N)$ be compact, δ a bi-invariant metric of diameter D on G, $n \geq 2$, Λ_L the torus with $L \geq 2n+4$ even and E its links, with V_x, P_x^ρ and $\mathscr{E}_n^\rho$ as in Sec. X J. Let ρ be a probability measure on G^E with a continuous positive density ρ with respect to λ_E. For $V \subset E$ and $U \in G^E$ put

$$\gamma_V^\rho(d\sigma \mid U) = \frac{\rho(\sigma, U_{E\setminus V})\,\lambda_V(d\sigma)}{\int \rho(\sigma', U_{E\setminus V})\,\lambda_V(d\sigma')}.$$

This law depends only on $U_{E\setminus V}$ and continuously on it, $P_x^\rho f(U) = \int f\, d\gamma^\rho_{V_x}(\cdot \mid U)$ is a version of the conditional expectation under ρ given the links outside V_x, and for $e \in V$ the conditional law of σ_e under $\gamma_V^\rho(\cdot \mid U)$ given the other links of V is $\gamma^\rho_{\{e\}}(\cdot \mid \sigma, U_{E\setminus V})$, for every U and σ. For $j \notin V$ let

$$C^\rho_{V,j} = \sup \frac{W_1^{d_V}\big(\gamma_V^\rho(\cdot \mid U), \gamma_V^\rho(\cdot \mid U')\big)}{\delta(U_j, U'_j)},$$

the supremum over the pairs U, U' that differ only at j, and $C^\rho_{ee'} = C^\rho_{\{e\},e'}$ for $e \neq e'$. For a Wilson measure these are the coefficients $C^0_{V,j}$, $C^0_{ee'}$ of Sec. X G; for a general ρ, $C^\rho_{ee'}$ need not vanish when e and e' are not partners. For links $e \neq j$ with base points $b(e)$, $b(j)$ let $\Delta = b(e) - b(j)$, with coordinates in $(-L/2, L/2]$, and

$$w_n(e,j) = \sum_{\nu=0}^{3} \min\big(|\Delta_\nu|, n\big).$$

Lemma 124. *Suppose that* $c_\rho = \max_{k\in E} \sum_{e\in E\setminus\{k\}} C^\rho_{ek} < 1$. *Then for every* $V \subset E$ *and every link* $j \notin V$,

$$C^\rho_{V,j} \leq \frac{1}{1-c_\rho} \sum_{e\in V} C^\rho_{ej}.$$

Proof. The proof of Lemma 70 applies with the kernels γ^ρ. Let U, U' differ only at j, let π^* be a coupling of $\gamma = \gamma_V^\rho(\cdot \mid U)$

and $\gamma' = \gamma^\rho_V(\cdot \mid U')$ that minimises $\sum_{e\in V}\mathbb{E}_\pi\delta(\sigma_e,\sigma'_e)$, and $\rho^*(e) = \mathbb{E}_{\pi^*}\delta(\sigma_e,\sigma'_e)$. Resampling the pair at $e\in V$ from an optimal coupling of $\gamma^\rho_{\{e\}}(\cdot\mid\sigma,U_{E\setminus V})$ and $\gamma^\rho_{\{e\}}(\cdot\mid\sigma',U'_{E\setminus V})$, chosen measurably [113, Cor. 5.22], gives a coupling of γ and γ'. The two conditioning configurations differ at the links of $V\setminus\{e\}$ and at j; changing them one at a time and using the definition of the coefficients bounds the new expected distance at e by $\sum_{k\in V\setminus\{e\}} C^\rho_{ek}\rho^*(k) + C^\rho_{ej}\delta(U_j,U'_j)$, and minimality gives

$$\rho^*(e) \le \sum_{k\in V\setminus\{e\}} C^\rho_{ek}\rho^*(k) + C^\rho_{ej}\,\delta(U_j,U'_j).$$

Summing over $e\in V$ and using $\sum_{e\in V\setminus\{k\}} C^\rho_{ek}\le c_\rho$ gives

$$(1-c_\rho)\sum_e \rho^*(e) \le \delta(U_j,U'_j)\sum_{e\in V} C^\rho_{ej},$$

and $W_1^{d_V}(\gamma,\gamma')\le\sum_e\rho^*(e)$. □

Theorem 125 (a single-link criterion for general measures)**.** *Let ρ be as in the setting, $n\ge 2$ and $L\ge 2n+4$ even.*

(i) *If $\sum_{x:\,j\notin V_x} C^\rho_{V_x,j}\le\alpha n^4$ for every link j, with $\alpha<1$, then $(1-\alpha)\operatorname{Var}_\rho(f)\le\mathscr{E}^\rho_n(f)$ for every real $f\in L^2(\rho)$; if ρ is gauge invariant,* $\mathrm{BP}_\rho(n,1-\alpha)$ *holds.*

(ii) *If $c_\rho<1$, the hypothesis of* (i) *holds with*

$$\alpha=\alpha_n(\rho)=\frac{1}{(1-c_\rho)n}\max_{j\in E}\sum_{e\neq j}C^\rho_{ej}w_n(e,j),$$

so $\mathrm{BP}_\rho(n,1-\alpha_n(\rho))$ *holds for gauge-invariant ρ whenever $\alpha_n(\rho)<1$. With $m_\rho=\max_j\sum_{e\neq j}C^\rho_{ej}|\Delta|_1$ and, for $R\ge1$, $t_\rho(R)=\max_j\sum_{e:\,|\Delta|_\infty>R}C^\rho_{ej}$,*

$$\alpha_n(\rho)\le\frac{m_\rho}{(1-c_\rho)n},\qquad \alpha_n(\rho)\le\frac{4}{1-c_\rho}\Big(\frac{Rc_\rho}{n}+t_\rho(R)\Big).$$

Proof. (i) Let U,U' differ only at a link i, and couple $\mathscr{K}^\rho(U,\cdot)$ and $\mathscr{K}^\rho(U',\cdot)$ by choosing the same x. If $i\in V_x$, the two exteriors agree, so do the laws of the resampled cube, and the identity coupling costs nothing. If $i\notin V_x$, an optimal coupling of the two cube laws costs at most $C^\rho_{V_x,i}\delta(U_i,U'_i)$, and the link i adds $\delta(U_i,U'_i)$. The x with $i\notin V_x$ number L^4-n^4, so

$$W_1^{d_E}\big(\mathscr{K}^\rho(U,\cdot),\mathscr{K}^\rho(U',\cdot)\big)\le\Big(1-\frac{(1-\alpha)n^4}{L^4}\Big)\delta(U_i,U'_i).$$

This is the contraction of the proof of Proposition 67(i) with $\kappa_0=(1-\alpha)n^4/L^4$; the rest of that proof uses only that the P^ρ_x are the orthogonal projections of $L^2(\rho)$ given by everywhere defined kernels, and it gives $\langle f,(I-\mathscr{K}^\rho)f\rangle\ge\kappa_0\operatorname{Var}_\rho(f)$, which is the claim after multiplication by L^4/n^4.

(ii) By Lemma 124,

$$\sum_{x:\,j\notin V_x}C^\rho_{V_x,j}\le\frac{1}{1-c_\rho}\sum_{e\neq j}C^\rho_{ej}\#\{x:e\in V_x,\ j\notin V_x\}.$$

A link with base point b' lies in V_x exactly when $b'-x\in[0,n)^4$ modulo L. Since $L\ge2n$, for $t\in[0,n)$ and $|d|\le L/2$ the residue of $t+d$ lies in $[0,n)$ only if $t+d\in[0,n)$, so the x with $e,j\in V_x$ number $\prod_\nu(n-|\Delta_\nu|)_+$. With $s_\nu=\min(|\Delta_\nu|,n)/n\in[0,1]$ and $\prod_\nu(1-s_\nu)\ge1-\sum_\nu s_\nu$,

$$\#\{x:e\in V_x,\ j\notin V_x\}=n^4-\prod_\nu(n-|\Delta_\nu|)_+\le n^3w_n(e,j),$$

which gives $\alpha_n(\rho)$. Finally $w_n(e,j)\le|\Delta|_1\le4|\Delta|_\infty$ and $w_n(e,j)\le4n$; splitting the sum at $|\Delta|_\infty=R$ gives the last bound. □

The criterion depends on ρ only through the column sums of its single-link coefficients and their first moment or tail; it uses no locality of the specification, and the bound holds on all of $L^2(\rho)$. For ρ given by an action the coefficients are bounded by its Hessian.

Lemma 126. *Let $G\subset\mathrm{U}(N)$ be closed and connected with semisimple Lie algebra, δ the geodesic distance of $\langle X,Y\rangle=-\operatorname{Tr}(XY)$, and ρ the probability measure with density proportional to e^{-H}, $H\in C^2(G^E)$. For a link e, a configuration U and $X\in\mathfrak{g}$ let $U^{e,X}$ be U with U_e replaced by U_ee^X. Suppose that there are $h_e<\kappa_G$ and $J_{ee'}\ge0$ with*

$$\frac{d^2}{dt^2}H(U^{e,tX})\Big|_{t=0}\ge-h_e|X|^2$$

and

$$\Big|\frac{\partial^2}{\partial s\,\partial t}H\big((U^{e,sX})^{e',tY}\big)\Big|_0\Big|\le J_{ee'}|X|\,|Y|$$

for all U, all $X,Y\in\mathfrak{g}$ and all links $e\neq e'$. Then $C^\rho_{ee'}\le J_{ee'}/(\kappa_G-h_e)$.

Proof. Given the other links, the law of U_e is $\nu\propto e^{-V}dU$ with $V(\sigma)=H(\sigma,U_{E\setminus e})$. The geodesics of the bi-invariant metric through σ are the curves σe^{tX}, so the first hypothesis says $\operatorname{Hess}V\ge-h_e$, and $\operatorname{Ric}_G+\operatorname{Hess}V\ge K_1=\kappa_G-h_e>0$. As in the proof of Lemma 73, $|\operatorname{Cov}_\nu(g,u)|\le\|\nabla g\|_\infty\|\nabla u\|_\infty/K_1$ for smooth g,u. Let U,U' differ only at e', with $U'_{e'}=U_{e'}e^{\delta X}$, $|X|=1$ and $\delta=\delta(U_{e'},U'_{e'})$, along a minimising geodesic, and let $\nu_s\propto e^{-V_s}$ be the law of U_e given the configuration $U^{e',sX}$ outside e. Then $\frac{d}{ds}\nu_s(g)=-\operatorname{Cov}_{\nu_s}(g,\partial_sV_s)$, and by the second hypothesis the gradient of $\sigma\mapsto\partial_sV_s(\sigma)$ has norm at most $J_{ee'}$. For smooth g that are 1-Lipschitz for δ this gives $|\nu_0(g)-\nu_\delta(g)|\le J_{ee'}\delta/K_1$, and the approximation and duality argument of that proof gives the bound. □

Corollary 127. *Under the hypotheses of Lemma 126 put $\theta_{ej}=J_{ej}/(\kappa_G-h_e)$ and $c_H=\max_k\sum_{e\neq k}\theta_{ek}$. If $c_H<1$ and $L\ge2n+4$, then every real $f\in L^2(\rho)$ satisfies $(1-\alpha)\operatorname{Var}_\rho(f)\le\mathscr{E}^\rho_n(f)$ with*

$$\alpha=\frac{1}{(1-c_H)n}\max_j\sum_{e\neq j}\theta_{ej}w_n(e,j)$$

whenever $\alpha<1$. For the Wilson action, $H=-N\beta\sum_p\operatorname{Re}\operatorname{Tr}U_p$, one may take $h_e=6N|\beta|$ and $J_{ee'}=N|\beta|$ for partners and 0 otherwise, and $\alpha=18N|\beta|/((\kappa_G-24N|\beta|)n)$, the leading term of α_n in Corollary 74(iii)*. The numbers h_e and $J_{ee'}$ may be taken subadditive in H: if $H=H_1+H_2$ and they are admissible for H_1 and for H_2, their sums are admissible for H.*

Proof. By Lemma 126, $c_\rho \le c_H$ and $\alpha_n(\rho) \le \alpha$, and Theorem 125 applies. For the Wilson action the proof of Lemma 73 gives the second derivative at least $-6N|\beta||X|^2$ and, by Lemma 22, the mixed derivatives of the one plaquette that contains two partners at most $N|\beta||X||Y|$; partners of j have $w_n = 1,1,0,1,1,2$ in each of the three planes through j, eighteen in all, and $1 - c_H = (\kappa_G - 24N|\beta|)/(\kappa_G - 6N|\beta|)$. Subadditivity is the triangle inequality for second derivatives. □

So the criterion holds for every C^2 perturbation of a Wilson action in the range of Corollary 74(iii) whose Hessian blocks are small in the norms above, local or not; this is the analogue of Proposition 261 without the gauge and without a finite range. The block laws are of this kind of object.

Lemma 128. *Let c be the straight-path map with blocks of side $b \ge 2$ from Λ_{bL} to Λ_L.*

(i) *If μ' has a continuous positive density r with respect to $\lambda_{E'}$, then $c_*\mu'$ has the continuous positive density $V \mapsto \nu_V(r)$ with respect to λ_E, of class C^m if r is.*

(ii) *Let c' be the straight-path map with blocks of side $b' \ge 2$ from $\Lambda_{b'bL}$ to Λ_{bL}, and R, R', R'' the Haar refinements along c, c' and the straight-path map $c \circ c'$ with blocks of side bb'. Then $R'(R(\rho)) = R''(\rho)$ for every probability measure ρ on G^E.*

(iii) *For $G = \mathrm{SU}(N)$, $c \circ Z_{c_1,d_1} = Z_{c_1,d_1} \circ c$ for the centre twists of Sec. X G on Λ_{bL} and on Λ_L, and c intertwines the translations by b of Λ_{bL} with the unit translations of Λ_L. So $c_*\mu'$ inherits invariance under these maps from μ'.*

In particular, for straight-path maps c_j in Corollary 102, $\mu^{(0)}_{k,L} = (c^{(k)})_*\mu_{k,L}$ with $c^{(k)} = c_1 \circ \cdots \circ c_k$ the straight-path map with blocks of side 2^k; it has the smooth positive density $V \mapsto \nu_V(\prod_p \varphi(U_p))/Z$, it is gauge invariant and invariant under the centre twists and the translations of Λ_{L_0}, and Theorem 125 and Lemma 126 apply to it, with the induced action $H^{(0)}_{k,L}(V) = -\log \nu_V(\prod_p \varphi(U_p))$ in place of H.

Proof. (i) By Lemma 108(i), $\lambda_{E'} = \int \lambda_E(dV)\nu_V$, so $\mu'(\psi \circ c) = \int \lambda_E(dV)\psi(V)\nu_V(r)$ for bounded Borel ψ. Under ν_V the first $b-1$ links of each path and the links of E'_0 are independent Haar variables u, and the last link of $\pi(e)$ is the inverse of the product of the others times V_e; so $\nu_V(r) = \int r(\Phi(V,u))\,du$ with Φ smooth. Continuity and differentiability under the integral follow from the uniform continuity of r and of its derivatives on the compact group, and $\nu_V(r) \ge \min r > 0$.

(ii) The map $c \circ c'$ multiplies the bb' links of the straight path from $b'by$ to $b'b(y + e_\nu)$, which consists of the paths of c' of the b links of the path $\pi(y,\nu)$ of c. It suffices to show $\int \nu_V(dW)\,\nu'_W = \nu''_V$ for every V, with the obvious notation. Under ν_V the links $W_1, \dots, W_{b-1}$ of $\pi(e)$ and the links of E'_0 are independent Haar variables and $W_b = (W_1 \cdots W_{b-1})^{-1}V_e$. If W_i is a Haar variable and the links of its c'-path have law $\nu'^{(b')}_{W_i}$ given W_i, then by Lemma 106(i) all b' of them are independent Haar variables. Hence under $\int \nu_V(dW)\nu'_W$ the first $bb'-1$ links of the path of $c \circ c'$ are independent Haar variables, the last is the inverse of their product times V_e, and every link on no path of $c \circ c'$, which lies on the c'-path of a link of E'_0 or is on no c'-path, is an independent Haar variable. This is ν''_V.

(iii) The coarse link $(y,0)$ is the product of the fine links $(by + te_0, 0)$, $0 \le t < b$, and exactly one of them has 0-coordinate 0 when $y_0 = 0$, none otherwise; the twist multiplies it by a central element. The links of the other directions are not affected, and likewise for the direction 1. The statement on translations is immediate, and the Wilson measure is invariant under the twists and the translations.

The composite statement is (ii)–(iii) and the identity $c_1 \circ \cdots \circ c_k = c^{(k)}$, and the density is (i) with $r \propto \prod_p \varphi(U_p)$. □

Through Theorem 125, (H1) reduces to bounds on single coarse links, uniform along the tuning, and the cube side n_0 is free.

Corollary 129 (what the criterion needs from the block laws)**.** *In Corollary 102 let the c_j be straight-path maps, and write $C^{(k,L)}_{ej}$ for the single-link coefficients of $\mu^{(0)}_{k,L}$ on Λ_{L_0}, $L = 2^k L_0$. Suppose that there are $c < 1$ and $R \ge 1$ with*

$$\sum_{e \ne j} C^{(k,L)}_{ej} \le c, \qquad \sum_{e:\,|\Delta|_\infty > R} C^{(k,L)}_{ej} \le \frac{1-c}{16}$$

for all k, all $L \in \mathscr{L}_k$ and all links j. Then (H1) holds with $\varpi_0 = \frac{1}{2}$ for every $n_0 \ge \max(2, 16Rc/(1-c))$ for which every $L \in \mathscr{L}_k$ has $L_0 \ge 2n_0 + 4$. The same holds if the induced actions $H^{(0)}_{k,L}$ satisfy the hypotheses of Lemma 126 with $\theta_{ej} = J_{ej}/(\kappa_G - h_e)$ in place of $C^{(k,L)}_{ej}$.

Proof. By Theorem 125(ii),

$$\alpha_{n_0} \le \frac{4Rc}{(1-c)\,n_0} + \frac{4t(R)}{1-c} \le \frac{1}{4} + \frac{1}{4}.$$

For the second statement use Lemma 126 and the monotonicity of these bounds in the coefficients. □

These are statements about the induced action of k steps at the spacing a_0, uniform in k and in the torus. The results of this paper do not supply them for any k with β_k outside the strong-coupling range: Corollary 74 bounds the coefficients of Wilson measures, Theorem 367 and Sec. XIII concern the law of the fine field given the block variables, and Proposition 565 concerns one step of the cut-off average at weak coupling. For the finitely many k in the strong-coupling range the criterion is not needed, since the descent of Sec. X J already gives (H1) there, with the cube enlarged by one.

Proposition 130 (the place of (H1))**.** (i) *Let μ' be a gauge-invariant probability measure on $G^{E'}$, c the straight-path map with blocks of side $b \ge 2$, $n \ge 2$, $L \ge 2n+4$ even, $2 \le n' \le b(n-1)+1$ with $bL \ge 2n'+4$, and $\varpi_c = \varpi'(n'/(bn))^4$. Then $\mathrm{BP}_{\mu'}(n', \varpi')$ implies $\mathrm{BP}_{c_*\mu'}(n, \varpi_c)$.*

(ii) *In Corollary 102 with straight-path maps, if $\mathrm{BP}(n_k, \varpi)$ holds for $\mu_{k,L}$ and $L_0 \ge 2n_0 + 6$, then $\mathrm{BP}(n_0+1, \varpi_1)$ holds for $\mu^{(0)}_{k,L}$, with $\varpi_1 = \varpi(n_0/(n_0+1))^4$. For $G = \mathrm{SU}(2)$ and $|\beta_{k,W}| < \frac{16}{105}$ this holds with $\varpi = 1 - \alpha_{n_k}$, α_{n_k} as in Corollary 74(i).*

(iii) *Under* (H2), (H1) *implies* $\mathrm{BP}(n_k,\varpi)$ *for all* k *with* $\varpi = \min(\varpi_0, 1/C_A)/\Gamma$, *which implies* (H1) *at the cube size* n_0+1, *with the constant* $\varpi(n_0/(n_0+1))^4$, *on the tori with* $L_0 \geq 2n_0+6$.

Proof. (i) The straight-path map satisfies (B1) with $r=0$, since $c_{(y,\nu)}$ depends only on the links with base points $by+te_\nu \in B_y$, and the proof of Theorem 94(i),(ii) holds with $r=0$: then $y(x')$ has coordinates $\lfloor x'_i/b \rfloor$, $Q_y = by+[0,bn)^4$, and $x'_i+n'-1 \leq by_i+b-2+n' \leq by_i+bn-1$ by the bound on n', so $V'_{x'}$ has its base points in Q_y; a coarse link (z,ν) with $z \notin y+[0,n)^4$ depends only on links with base points in B_z, which is disjoint from Q_y. The rest of that proof does not involve r.

(ii) Apply (i) to $c^{(k)}$, with $b=2^k$, $n'=n_k=bn_0$ and $n=n_0+1$, so that $n' \leq b(n-1)+1$; the coarse torus Λ_{L_0} has $L_0 \geq 2n+4$, and $L=2^kL_0 \geq 2n_k+4$. The constant is $\varpi(n_k/(bn))^4 = \varpi(n_0/(n_0+1))^4$. For SU(2), Corollary 74(i) gives $\mathrm{BP}(n_k, 1-\alpha_{n_k})$ on every torus, since the threshold $4n^2/(18n^2+18n-3)$ increases with n and equals $\frac{16}{105}$ at $n=2$.

(iii) Corollary 102 and (ii). □

Part (iii) says that, given (H2), the coarsest condition is necessary and sufficient for the conclusion of Corollary 102 up to one unit of cube size, so no proof of $\mathrm{BP}(n_k,\varpi)$ by the upward step avoids it. The condition also imposes two necessary conditions on the block laws, which hold for every gauge-invariant measure and use no locality.

Proposition 131 (what the condition forces)**.** *Let* ρ *be a gauge-invariant probability measure on the links of* Λ_L *for which* $\mathrm{BP}_\rho(n,\varpi)$ *holds, and* P_y *the Polyakov loops of Sec.* X G.

(i) *The averaged Polyakov loops satisfy*

$$\mathrm{Var}_\rho\Big(L^{-3}\sum_{y:\,y_0=0}\mathrm{Re}\,\mathrm{Tr}\,P_y\Big) \leq \frac{N^2n^2}{\varpi L^2},$$

and the same holds for the imaginary parts.

(ii) *Let* $G=\mathrm{SU}(N)$, $L=2n+4$, *let* ρ *be invariant under the centre twists* Z_{c_1,d_1}, *let* $\mathrm{Var}_\rho(\mathrm{Re}\,\mathrm{Tr}\,U_p)>0$ *for some plaquette* p, *and* $\varepsilon = \max_p \rho(\|U_p-I\|^2_{\mathrm{HS}})$. *Then* $\varpi \leq C'_N n^4\varepsilon$, *with* $C'_2=19200$ *and* $C'_N = \max(2048(N^2-1)N^2(N-1)^2/c_N, 2048/\eta_N^2)$, c_N,η_N *as in Lemma 77.*

(iii) *In Corollary 102 with straight-path maps,* (H1) *implies, for all* k *and* $L \in \mathscr{L}_k$,

$$\mathrm{Var}_{\mu_{k,L}}\Big(L_0^{-3}\sum_{y:\,y_0=0}\mathrm{Re}\,\mathrm{Tr}\,P_{2^ky}\Big) \leq \frac{N^2n_0^2}{\varpi_0 L_0^2},$$

the P_{2^ky} *being Polyakov loops of* Λ_L*; and if* $G=\mathrm{SU}(N)$ *and* $L=2^k(2n_0+4)\in\mathscr{L}_k$, *the holonomies* W *of the squares of side* 2^k *with corners in* $2^k\mathbb{Z}^4$ *satisfy* $\max_W \mu_{k,L}(\|W-I\|^2_{\mathrm{HS}}) \geq \varpi_0/(C'_Nn_0^4)$.

(iv) *If* $G=\mathrm{SU}(2)$ *and* ρ *on* Λ_{2n+4} *has a continuous positive density, is invariant under the centre twists and satisfies the hypothesis of Theorem 125*(i) *with* $\alpha \leq \frac12$, *then* $\varepsilon \geq 1/(38400n^4)$.

Proof. (i) This is the proof of Theorem 79(i) without the symmetry: $\|g_y - P_x^\rho g_y\|_2 \leq N$ for $g_y = \mathrm{Re}\,\mathrm{Tr}\,P_y$, so the right side of BP for the average is at most N^2n^2/L^2, and the left side is ϖ times its variance.

(ii) The proof of Theorem 79(ii),(iii), before the use of Lemma 78, uses the measure only through its invariance under the Z_{c_1,d_1}, which makes the traces F_{ab} centred, through the mean of $\|U_p-I\|^2_{\mathrm{HS}}$ at each plaquette, which enters as an upper bound and may be replaced by ε, and through $\varpi \leq 2$, which the proof of Proposition 75(ii) gives from the plaquette p. Lemmas 76 and 77 are deterministic.

(iii) By Lemma 128, $\mu^{(0)}_{k,L}$ is gauge invariant, invariant under the centre twists, and has a continuous positive density, so $\mathrm{Var}(\mathrm{Re}\,\mathrm{Tr}\,c_P)>0$. The Polyakov loops of Λ_{L_0} are those of Λ_L through the sites 2^ky, and the plaquette variables c_P are the holonomies W. Apply (i) and (ii).

(iv) Theorem 125(i) gives $\mathrm{BP}(n,\frac12)$, and (ii) gives $\frac12 \leq 19200n^4\varepsilon$. □

Part (iv) is an obstruction to the criterion on the smallest torus: a law of the block variables whose plaquettes are close to I in the mean cannot satisfy it there. The condition (H1) is posed on all $L\in\mathscr{L}_k$, and on large tori (i) asks that the spatially averaged Polyakov loops of the coarse torus decorrelate. That requirement is not implied by the decay of local correlations, and it is independent of (H2), as the following measure shows. For $G=\mathrm{SU}(N)$, $N\geq 2$, and $\theta\in[0,2\pi)$ let h_θ be the configuration of Λ_L with $(h_\theta)_{(y,0)} = \mathrm{diag}(e^{i\theta},e^{-i\theta},1,\dots,1)$ and $(h_\theta)_{(y,\nu)}=I$ for $\nu\neq 0$, and let τ_L be the law of the configuration $\hat g\cdot h_\theta$, $(\hat g h_\theta)_{(y,\nu)} = \hat g(y)(h_\theta)_{(y,\nu)}\hat g(y+e_\nu)^{-1}$, with θ uniform and $\hat g$ a Haar distributed gauge transformation independent of θ. It is a random flat connection with a uniformly distributed holonomy around the direction 0.

Proposition 132 (a flat coarse law)**.** *Let* $G=\mathrm{SU}(N)$, $N\geq 2$, $n\geq 2$ *and* $L\geq\max(2n+4,8)$ *even.*

(i) τ_L *is gauge invariant; every gauge-invariant function of the links with base points in a box of side at most* $L/2$ *is* τ_L*-almost surely constant; and* $\varpi\,\mathrm{Var}_{\tau_L}(f) \leq \mathscr{E}_n^{\tau_L}(f)$ *fails for* $\varpi > n^2/L^2$ *both for the charged function* $f=L^{-3}\sum_{y:\,y_0=0}\mathrm{Re}\,\mathrm{Tr}\,P_y$ *and for the neutral function* $f=L^{-3}\sum_{y:\,y_0=0}|\mathrm{Tr}\,P_y|^2$. *So* $\mathrm{BP}_{\tau_L}(n,\varpi)$ *and its restriction to* $\mathscr{A}_0$ *fail for* $\varpi>n^2/L^2$.

(ii) *Let* c *be the straight-path map with blocks of side* $b\geq 2$ *and* $\mu'=R(\tau_L)$ *on* Λ_{bL}. *Then* $c_*\mu'=\tau_L$, *and* $\mathrm{Cov}_{\mu'}(f,g)=0$ *for all continuous gauge-invariant* f,g *that depend only on the links with base points in boxes* B,B' *of side at most* $bL/2$ *with* $d(B,B')\geq b$.

(iii) *Let* $n_0\geq 2$, $L_0\geq\max(2n_0+4,8)$ *even, and for each* k *let* $\mu_k = R^{(k)}(\tau_{L_0})$ *be the Haar refinement along* $c^{(k)}$, *on* $\Lambda_{2^kL_0}$, *with the tower of straight-path maps* c_j *of side two. Then* $\mu^{(j)} = R_j\circ\cdots\circ R_1(\tau_{L_0})$, $\mathrm{S}(n_j,n_{j-1};1,1)$ *holds for* $(\mu^{(j)},c_j)$ *with* $n_j=2^jn_0$, *so (H2) holds with* $C_A=1$ *and* $\Gamma=1$; $\mu^{(0)}=\tau_{L_0}$, *so* $\mathrm{BP}(n_0,\varpi_0)$ *fails for it when* $\varpi_0>n_0^2/L_0^2$; *and the local gauge-invariant covariances*

of μ_k vanish at distance at least 2^k, that is, at physical distance a_0.

(iv) *Let $b \geq 2$, $L \geq 2n+4$ and $bL \geq 2n'+4$, let τ' be the law of $(\hat g(x)(h_\theta)_{(x,\mu)}\hat g(x+e_\mu)^{-1})_{(x,\mu)\in E'_0}$, with h_θ and $\hat g$ now on Λ_{bL}, and $\mu' = \lambda_{E'\setminus E'_0}\otimes\tau'$. Then μ' is gauge invariant, $c_*\mu' = \lambda_E$ satisfies $\mathrm{BP}(n,1)$, $\mathrm{BP}_{\mu'}(n',\varpi')$ fails for $\varpi' > \frac{64}{49}n'^2/(bL)^2$, and constants for which $\mathrm{S}(n',n;C_A,C_B)$ holds for (μ',c) satisfy $\max(C_A,C_B) \geq \frac{49}{64}(bL)^2/n'^2$.*

Proof. An averaging bound. Let ρ be a probability measure on the links of Λ_M, S a set of sites with $y_0 = 0$, and g_y, $y \in S$, a function of the links of the line through y in the direction 0, such that ρ-almost surely all g_y equal one random variable of variance $v > 0$. For $f = |S|^{-1}\sum_{y\in S} g_y$, $\mathrm{Var}_\rho(f) = v$; $P_x^\rho g_y = g_y$ unless the line meets V_x, which happens for at most n^3 points $y \in S$, and $\|g_y - P_x^\rho g_y\|_2 \leq \|g_y - \rho(g_y)\|_2 = \sqrt{v}$ because P_x^ρ is an orthogonal projection that fixes the constants. Hence $\mathscr{E}_n^\rho(f) \leq n^{-4}M^4(n^3\sqrt{v}/|S|)^2$, and $\varpi v \leq \mathscr{E}_n^\rho(f)$ forces $\varpi \leq n^2M^4/|S|^2$.

(i) A gauge transformation g maps $\hat g\cdot h_\theta$ to $(g\hat g)\cdot h_\theta$, and $g\hat g$ is again Haar distributed. Let f depend on the links with base points in a box of side at most $L/2$; their end points lie in a box Q of side at most $L/2+1 \leq L-1$, and with coordinates on Q taken in an interval of length less than L, $k(y) = (h_\theta)_{(0,0)}^{-y_0}$ satisfies $k(y)k(y+e_\nu)^{-1} = (h_\theta)_{(y,\nu)}$ on these links. So $f(\hat g\cdot h_\theta) = f(h_\theta) = f(k\cdot I) = f(I)$ almost surely. Almost surely $P_y = \hat g(y)(h_\theta)^L_{(0,0)}\hat g(y)^{-1}$ for every y, so $\mathrm{Re\,Tr}\,P_y = 2\cos(L\theta)+N-2$ and $|\mathrm{Tr}\,P_y|^2 = (2\cos(L\theta)+N-2)^2$, with variances 2 and $2+8(N-2)^2$; the second function is neutral by Proposition 82. The averaging bound with $M = L$, $|S| = L^3$ gives the claim.

(ii) $c_*\mu' = \tau_L$ by Lemma 108(i). By the proof of Lemma 108(i), the image of ν_V under the gauge transformation equal to $\hat g(y)$ at by and to I elsewhere is $\nu_{\hat g V}$; so μ' is the law of $g\cdot U$ with U distributed as $\int \nu_{h_\theta}\,d\theta$ and g a random gauge transformation, and $\mu'(fg) = \int \nu_{h_\theta}(fg)\,d\theta$, $\mu'(f) = \int \nu_{h_\theta}(f)\,d\theta$. Fix θ. Under ν_{h_θ} the links of distinct paths and of E'_0 are independent. The base points of a path lie in b consecutive sites of a line, so no path has links with base points in both B and B', and $\nu_{h_\theta}(fg) = \nu_{h_\theta}(f)\nu_{h_\theta}(g)$. The coarse links whose paths meet B have end points in a box of Λ_L of side at most $L/2+3 \leq L-1$, on which $h_\theta = k\cdot I$ as in (i). By Lemma 106(ii), path by path, and the invariance of Haar measure on E'_0, the links with base points in B have under ν_{h_θ} the law of $g_k\cdot U$ with U distributed as ν_I, g_k equal to $k(y)$ at by and to I elsewhere; so $\nu_{h_\theta}(f) = \nu_I(f)$, independent of θ, and likewise for g. Hence $\mathrm{Cov}_{\mu'}(f,g) = 0$.

(iii) By Lemma 128(ii) and induction, $R^{(k)} = R_k\circ R^{(k-1)}$, so $\mu_k = R_k\circ\cdots\circ R_1(\tau_{L_0})$, and by Lemma 108(i) the tower of μ_k is $\mu^{(j)} = R_j\circ\cdots\circ R_1(\tau_{L_0})$, with $\mu^{(0)} = \tau_{L_0}$. Theorem 110 with $n' = n_j = 2n_{j-1}$ gives $\mathrm{S}(n_j,n_{j-1};1,1)$. The rest is (i) and (ii) with $b = 2^k$.

(iv) The gauge group acts on the links of $E'\setminus E'_0$ and of E'_0 separately; Haar measure is invariant, and so is τ', as in (i). The map c depends only on the links of the paths, which are independent Haar variables, so $c_*\mu' = \lambda_E$, and Proposition 67(ii) gives $\mathrm{BP}(n,1)$. A link of direction 0 lies on a path exactly when the coordinates $1,2,3$ of its base point are divisible by b, so for the set S of the $(bL)^3 - L^3 \geq \frac{7}{8}(bL)^3$ sites y with $y_0 = 0$ and $(y_1,y_2,y_3) \notin (b\mathbb{Z})^3$ the lines through y lie in E'_0, and $\mathrm{Re\,Tr}\,P_y = 2\cos(bL\theta)+N-2$ for all of them, almost surely. The averaging bound with $M = bL$ gives $\varpi' \leq \frac{64}{49}n'^2/(bL)^2$. By Theorem 100(i), $\mathrm{S}(n',n;C_A,C_B)$ and $\mathrm{BP}(n,1)$ for $c_*\mu'$ give $\mathrm{BP}_{\mu'}(n',\min(1/C_A,1/C_B))$. □

Replacing τ_{L_0} in (iii) by the SU(2) Wilson measure at $|\beta_W| < \frac{16}{105}$ for finitely many k gives towers for which BP holds at every level (Corollary 111). So (H2) with the best constants, together with BP at every level of finitely many towers, does not imply (H1); and by (iv), (H1) does not imply (H2). The measures of (ii) and (iii) have local gauge-invariant covariances of finite range, uniformly in the torus, while the coarse law fails BP in both the charged and the neutral sector; the failure lies in functions spread over the coarse torus, on which the cube moves of a flat law act only through the curvature they create. The local form $\mathrm{BP}_{\mathrm{loc}}$ holds for τ_{L_0} trivially, but it does not enter the upward step as proved: Theorem 100(i) applies the coarse condition to $F_f = \mathbb{E}[f \mid c]$, which for a general measure depends on all block variables even when f is local.

On one torus the condition holds for every law of block variables, with a constant that decays exponentially in the volume; and single-link conditional laws close to Haar do not give a constant independent of the torus. For $G = \mathrm{SU}(N)$ let s_N^2 be the variance of $N^{-1}\,\mathrm{Re\,Tr}\,U$ under the Haar measure, $s_2^2 = \frac14$ and $s_N^2 = 1/(2N^2)$ for $N \geq 3$.

Proposition 133 (closeness to Haar link by link)**.** *Let $n \geq 2$ and $L \geq 2n+4$ even.*

(i) *Let c be a straight-path map or a composite of such maps onto Λ_L, and μ' a gauge-invariant probability measure on the fine links with a continuous positive density D with respect to the Haar measure, $\Omega = \log(\max D/\min D)$. Then $\mathrm{BP}_{c_*\mu'}(n,e^{-\Omega})$ holds.*

(ii) *Let $G = \mathrm{SU}(N)$, $N \geq 2$, and $0 < \varepsilon < 1$. Let $p_1,\dots,p_M$, $M = (L/2)^4$, be the plaquettes of the planes $\{0,1\}$ of Λ_L whose base points have four even coordinates, $X_i = N^{-1}\,\mathrm{Re\,Tr}\,U_{p_i}$, and*

$$\rho_L = \Big(\frac12\prod_{i=1}^M(1+\varepsilon X_i) + \frac12\prod_{i=1}^M(1-\varepsilon X_i)\Big)\lambda_E.$$

Then ρ_L is a gauge-invariant probability measure with a continuous positive density, the conditional law of each link given the others has a density with respect to the Haar measure with values in $[1-\varepsilon,1+\varepsilon]$, and every ϖ for which $\mathrm{BP}_{\rho_L}(n,\varpi)$ holds satisfies

$$\varpi \leq \frac{48\,(n+1)^4}{\varepsilon^2 s_N^4 L^4}.$$

Consequently, for $L^4 > 48(n+1)^4/(\varepsilon^2 s_N^4)$, some link j has $\sum_{x:\,j\notin V_x} C^{\rho_L}_{V_x,j} > (1-48(n+1)^4/(\varepsilon^2 s_N^4 L^4))\,n^4$.

Proof. (i) By Lemma 108(i), applied to each map of the composite, $c_*\lambda_{E'} = \lambda_E$ and $\lambda_{E'} = \int \lambda_E(dV)\,\nu_V$ with $c = V$ ν_V-almost surely, ν_V the conditional law; so $\rho = c_*\mu'$ has the density $r(V) = \nu_V(D)$ with respect to λ_E, and $\min D \le r \le \max D$. Under λ_E every $f \in L^2$ is the orthogonal sum of components f_A, $A \subset E$, where f_A depends only on U_A and has integral zero in each link of A; then $\lambda_E((f - \mathscr{P}_x f)^2)$ is the sum of $\|f_A\|^2$ over the A that meet V_x, and each nonempty A meets at least n^4 of the cubes, so $\mathrm{Var}_{\lambda_E}(f) \le n^{-4}\sum_x \lambda_E((f - \mathscr{P}_x f)^2)$. The variance and the conditional variances are infima of mean squares over constants and over functions of the links outside V_x, so

$$\mathrm{Var}_\rho(f) \le \max D\, \mathrm{Var}_{\lambda_E}(f),$$
$$\rho\big((f - P^\rho_x f)^2\big) \ge \min D\, \lambda_E\big((f - \mathscr{P}_x f)^2\big),$$

which give $e^{-\Omega}\mathrm{Var}_\rho(f) \le \mathscr{E}^\rho_n(f)$.

(ii) The base points of the links of p_i are y_i, $y_i + e_0$ and $y_i + e_1$, so different p_i have disjoint links, and $|X_i| \le 1$. Under λ_E the X_i are independent with mean zero and variance s_N^2, since $\int \mathrm{Re}\,\mathrm{Tr}\,U\,dU = 0$, $\int |\mathrm{Tr}\,U|^2 dU = 1$, and $\int (\mathrm{Tr}\,U)^2 dU$ is 1 for $N = 2$ and 0 for $N \ge 3$. So both products are densities of probability measures $\rho^\pm$, positive and continuous, and gauge invariant because X_i is a class function of a plaquette holonomy. Fix a link e of p_i and the other links. As a function of U_e, $X_i = N^{-1}\,\mathrm{Re}\,\mathrm{Tr}(AU_e^{\pm 1}B)$ with A, B fixed, which has Haar integral zero, and the density of ρ_L is $a(1 + \varepsilon X_i) + a'(1 - \varepsilon X_i)$ with $a, a' \ge 0$ not depending on U_e; normalised, it is $1 + \theta\varepsilon X_i$ with $\theta = (a - a')/(a + a') \in [-1, 1]$. For a link on no p_i the conditional law is Haar.

Let $F = M^{-1}\sum_i X_i$, a real gauge-invariant function. Under $\rho^\pm$ the X_i are independent with mean $\pm\varepsilon \int X_i^2\, d\lambda_E = \pm\varepsilon s_N^2$, so, as $\rho_L = \frac{1}{2}(\rho^+ + \rho^-)$, $\mathrm{Var}_{\rho_L}(F) \ge (\varepsilon s_N^2)^2$. Let r_x be the number of p_i with a link in V_x. Their base points lie in $x + [-1, n)^4$, so $r_x \le (n+1)^4$, and each p_i meets at most $3n^4$ cubes, one family for each of its three base points, so $\sum_x r_x \le 3n^4 M$. Given the links outside V_x, F varies only through those r_x terms, in an interval of length at most $2r_x/M$, so its conditional variance is at most r_x^2/M^2 and

$$\mathscr{E}^{\rho_L}_n(F) \le \frac{1}{n^4 M^2}\sum_x r_x^2 \le \frac{3(n+1)^4}{M} = \frac{48(n+1)^4}{L^4}.$$

Dividing by $\mathrm{Var}_{\rho_L}(F)$ gives the bound on ϖ. If the sums in the last statement were at most αn^4 for every j, Theorem 125(i) would give $\mathrm{BP}_{\rho_L}(n, 1 - \alpha)$, so $1 - \alpha \le 48(n+1)^4/(\varepsilon^2 s_N^4 L^4)$. □

The sign that chooses between the two products is drawn once for the whole torus.[25] The measures ρ_L are not laws of block variables of Wilson measures and nothing is claimed about the tuned laws from them; they show that the hypothesis of Theorem 125, a bound on the sums of the single-link coefficients over all links, cannot be replaced by the closeness of each single-link conditional law to the Haar measure, and that (H1) needs a collective bound. Part (i) holds for the laws of the block variables on every torus at every coupling; for the Wilson measure Ω is proportional to the volume, so its constant is not uniform in L. Figure 18 shows the measure of (ii).

Remark 134 (what is proved about the coarsest condition). Proved: for every measure with a continuous positive density on the links of a torus, a Dobrushin bound $c_\rho < 1$ on the column sums of the single-link Kantorovich coefficients gives $\mathrm{BP}(n, 1 - \alpha_n(\rho))$ on all of $L^2(\rho)$, with $\alpha_n(\rho) \le m_\rho/((1 - c_\rho)n)$ explicit and no locality or product form assumed; the coefficients are bounded by the Hessian of the action, which recovers the leading term of Corollary 74(iii) and is stable under perturbations small in the Hessian norms (Theorem 125, Lemma 126, Corollary 127); the laws of the block variables of the straight-path maps have smooth positive densities, so the criterion applies to them, and (H1) follows for every large n_0 from coefficient bounds uniform along the tuning (Lemma 128, Corollary 129); (H1) at the cube size $n_0 + 1$ follows from $\mathrm{BP}(n_k, \varpi)$, and given (H2) it is equivalent to the conclusion of Corollary 102 up to one unit of cube size (Proposition 130); it forces the decorrelation of the averaged coarse Polyakov loops on every torus and, on the smallest torus, a lower bound on the curvature of the coarse plaquettes, which also obstructs the criterion there (Proposition 131); it is independent of (H2), and it fails for the coarse law of a measure whose local covariances have range a_0 (Proposition 132); on each torus every law of block variables of a measure with a continuous positive density satisfies $\mathrm{BP}(n, e^{-\Omega})$, and single-link conditional laws within ε of the Haar measure do not give a constant independent of the torus (Proposition 133).

Not proved: (H1) along the tuning; the bounds of Corollary 129 for the induced actions $H^{(0)}_{k,L}$ for any k with β_k outside the strong-coupling range, that is, a single-link Hessian of the induced action of k steps below κ_G and summable mixed blocks at the scale where the running coupling is of order one, which is the large-field and effective-action problem of item (iv) of Sec. XIII V; whether $\mathrm{BP}(n_k, \varpi)$ gives (H1) at the cube size n_0 itself; the decorrelation of the coarse Polyakov loops of Proposition 131(iii) for the tuned measures. The measures of Proposition 132 are flat on the coarse scale and are not Wilson measures, nor are those of Proposition 133(ii), and nothing is claimed about the tuned measures from them. For the measures of Corollary 111, where BP holds at every scale, Lemma 126 gives no conclusion once K is large, since a fine link of a path enters six loops of $4b^K$ links; the criterion is a statement for the coarsest scale. Neither BP along the tuning nor the mass gap is proved here.

O. The single-link coefficients across scales

By Corollary 129, (H1) follows from bounds on the single-link coefficients of the laws $\mu^{(0)}_{k,L}$, uniform in k and in the torus. These laws are the image of a Wilson measure under k steps of the straight-path map, and bounds on them are a form of the effective-action problem of Bałaban's renormalisation

[25] Each cube carries a fraction of order $(n/L)^4$ of the evidence for the sign, so resampling cubes changes the spatial average F only slowly, while F itself reads the sign off with an error of order L^{-2}.

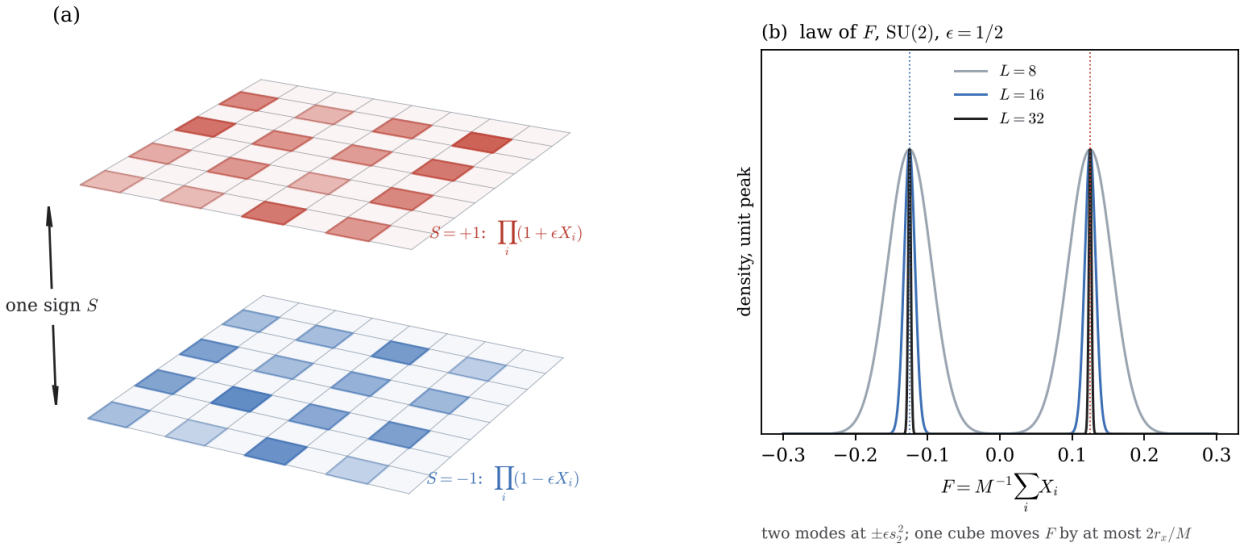


FIG. 18. The measure ρ_L of Proposition 133(ii). (a) One sign S, drawn once for the torus, chooses between two product tilts of the selected plaquettes; each link, given the others, has a law within ε of the Haar measure. (b) The law of the spatial average F for SU(2), $\varepsilon=\frac{1}{2}$ and $L=8,16,32$, drawn with normal approximations of its two components and scaled to unit peak: the modes at $\pm\varepsilon s_2^2$ stay apart as L grows, while resampling one cube moves F by at most $2r_x/M$, so the cube dynamics needs a time of order L^4 to change the sign.

group [6, 7, 12]. This subsection proves three things about them. First, one step of the straight-path map with blocks of side b contracts a weighted Dobrushin sum by the factor $e^{-\lambda b/2}$, up to a loss in the rate, on every torus and with no locality; along a tower this transfers a Dobrushin bound from any level to the coarsest one, so that (H1) follows from such a bound at one level per k. Second, gauge covariance forces a lower bound on the coefficients: every row sum of a gauge-invariant law is at least 2 minus four times the normalised spread of its single-link conditional laws, so the Dobrushin sum exceeds one for the SU(2) Wilson measure at $\beta_W \geq \frac{8}{3}$ and for every block law whose plaquette variables are close to I; the step can therefore only start at a level where the holonomies of the blocks are spread out, and it does not start at the finest level along the tuning. Third, the coarse conditional laws are spread out along the tuning for SU(2), so this obstruction does not apply at the coarsest scale, and the Hessian of the induced action is the curvature of the fine action minus the covariance of the forces in the fibre law. None of this bounds the coefficients of $\mu^{(0)}_{k,L}$ along the tuning.

Notation. Let S be a finite set and ρ a probability measure on G^S with a continuous positive density ρ with respect to λ_S. The kernels γ^ρ_V and the coefficients $C^\rho_{V,j}$ and C^ρ_{ej} are defined as in Sec. X N, with S in place of E, and the proof of Lemma 124 holds verbatim. For a pseudometric d on S and $\lambda \geq 0$ put

$$c_\lambda(\rho;d) = \max_{k\in S} \sum_{a\in S\setminus\{k\}} C^\rho_{ak}\, e^{\lambda d(a,k)},$$

so that $c_0(\rho;d) = c_\rho$.[26] For a measure on the links of a torus Λ_M, $d(e,j) = |\Delta|_\infty$ is the distance of the base points, coordinate by coordinate on the circle $\mathbb{Z}/M\mathbb{Z}$, and we write $c_\lambda(\rho)$. Since $x \leq e^{\lambda x}/(e\lambda)$ for $x \geq 0$ and $|\Delta|_1 \leq 4|\Delta|_\infty$, the quantities of Theorem 125(ii) satisfy

$$m_\rho \leq \frac{4\,c_\lambda(\rho)}{e\lambda}, \qquad t_\rho(R) \leq e^{-\lambda R}\, c_\lambda(\rho) \tag{18}$$

for $\lambda > 0$ and $R \geq 1$.

Lemma 135 (marginals)**.** *Let $\hat\rho$ be a probability measure on G^S with a continuous positive density, d a pseudometric on S with $c_\lambda = c_\lambda(\hat\rho;d) < 1$, $A \subset S$ with at least two points, ρ_A the marginal of $\hat\rho$ on G^A, and $d_A = \min\{d(a,a') : a \neq a' \in A\}$. For $e \in A$ put $S_e = (S\setminus A)\cup\{e\}$.*

(i) *For $e \neq j$ in A, with $C = C^{\hat\rho}$,*

$$C^{\rho_A}_{ej} \leq \sum_{m\geq 0} \sum_{k_1,\dots,k_m \in S_e} C_{ek_1} C_{k_1k_2} \cdots C_{k_mj},$$

the sum over the sequences with consecutive terms distinct.

(ii) *For $0 \leq \kappa \leq \lambda$ and every $j \in A$,*

$$\sum_{e\in A\setminus\{j\}} C^{\rho_A}_{ej}\, e^{\kappa d(e,j)} \leq e^{-(\lambda-\kappa)d_A}\, \frac{c_\lambda}{1-c_\lambda}.$$

Proof. (i) The density ρ_A of ρ_A is the integral of the density $\hat\rho$ of $\hat\rho$ over the coordinates in $S\setminus A$; it is continuous and positive,

[26] These are the sums of Dobrushin's uniqueness condition [42, 43] with exponential weights in the distance; Lemma 135 below is the expansion in walks that lies behind that condition, applied to a marginal.

and for every u the law $\gamma^{\rho_A}_{\{e\}}(\cdot \mid u)$, with density proportional to $\rho_A(\sigma, u_{A\setminus\{e\}})$, is the image of $\gamma^{\hat\rho}_{S_e}(\cdot \mid u)$ under the coordinate at e. Let u, u' differ only at $j \in A \setminus \{e\}$, put $\delta_j = \delta(u_j, u'_j)$, and let ρ^* be as in the proof of Lemma 124 with $V = S_e$. The Kantorovich distance of the two laws of σ_e is at most $\rho^*(e)$, and that proof gives, for every $k \in S_e$,

$$\rho^*(k) \le \sum_{i \in S_e \setminus \{k\}} C_{ki}\, \rho^*(i) + C_{kj}\, \delta_j .$$

Let K be the matrix $(C_{ki})_{k,i \in S_e}$ with zero diagonal and $w = (C_{kj})_{k \in S_e}$. Then $\rho^* \le K\rho^* + \delta_j w$ entrywise, and, K having nonnegative entries, $\rho^* \le \delta_j \sum_{m<M} K^m w + K^M \rho^*$ for every M. The column sums of K are at most c_λ, so $\|K^M x\|_1 \le c_\lambda^M \|x\|_1$ for $x \ge 0$, and $K^M \rho^* \to 0$. Hence $\rho^*(e) \le \delta_j \sum_{m \ge 0} (K^m w)(e)$, which is the sum in (i) times δ_j; divide by δ_j and take the supremum.

(ii) Write $k_0 = e$ and $k_{m+1} = j$, and $\ell = \sum_{i=0}^{m} d(k_i, k_{i+1})$. By the triangle inequality $\ell \ge d(e,j) \ge d_A$, so $e^{\kappa d(e,j)} \le e^{\kappa \ell} \le e^{-(\lambda-\kappa)d_A} e^{\lambda \ell}$. Insert (i) and enlarge the range of $k_0, \dots, k_m$ to all sequences in S with consecutive terms distinct and $k_m \ne j$. Summing first over k_0, then over k_1, and so on, each sum is a weighted column sum, at most c_λ; the sequences of length $m+1$ contribute at most c_λ^{m+1}, and $\sum_{m \ge 0} c_\lambda^{m+1} = c_\lambda/(1-c_\lambda)$. □

The law of the block variables is the marginal of a measure on fewer fine links, obtained by the axial gauge of Sec. X M. For a marking l of the straight-path map c from Λ_{bL} to Λ_L let $\hat E_l = l(E) \cup E'_0$, and let $\iota_l : G^{\hat E_l} \to G^{E'}$ extend a configuration by I on the unmarked links of the paths, so that $\Theta^l(V,m) = \iota_l(V,m)$ when V_e is placed at $l(e)$.

Lemma 136 (the axial gauge)**.** *Let μ' be a gauge-invariant probability measure on $G^{E'}$ with a continuous positive density r, and l a marking. Then r is gauge invariant, the law $\hat\mu_l$ of the configuration (c, M^l) placed on $\hat E_l$ has the continuous positive density $r \circ \iota_l$ with respect to $\lambda_{\hat E_l}$, its marginal on $l(E)$ is $c_*\mu'$, and $C^{\hat\mu_l}_{ab} \le C^{\mu'}_{ab}$ for all $a \ne b$ in $\hat E_l$.*

Proof. For every gauge transformation g, $r \circ g = r$ almost everywhere by the invariance of μ' and of $\lambda_{E'}$; both sides are continuous and $\lambda_{E'}$ has full support, so the equality holds everywhere. By Lemma 108(i) and the first paragraph of the proof of Lemma 116, which does not use φ, $\lambda_{E'}$ is the image of $\lambda_E \otimes \lambda \otimes \lambda_{E'_0}$ under $\Psi(V,h,m) = h^{-1}\Theta^l(V,m)$, where h runs over the values of a gauge transformation at the inner points of the paths and is I elsewhere. The links of Ψ along $\pi(e)$ are $h(s_{i-1})^{-1}h(s_i)$ off the mark and $h(s_{i-1})^{-1}V_e h(s_i)$ at it, with $h(s_0) = h(s_b) = I$; so $c(\Psi) = V$, the partial products give $g^l_\Psi = h$, and $M^l(\Psi) = m$. Hence under $\lambda_{E'}$ the triple (c, g^l, M^l) has the law $\lambda_E \otimes \lambda \otimes \lambda_{E'_0}$, and since $r(U) = r(g^l_U U) = r(\Theta^l(c(U), M^l(U)))$, the pair (c, M^l) has under μ' the density $r(\Theta^l(V,m))$. Its first component is c. For $a \in \hat E_l$ and $u \in G^{\hat E_l}$ the configuration $\iota_l(\sigma, u_{\hat E_l \setminus \{a\}})$ is $\iota_l(u)$ with the link a replaced by σ, so $\gamma^{\hat\mu_l}_{\{a\}}(\cdot \mid u) = \gamma^{\mu'}_{\{a\}}(\cdot \mid \iota_l(u))$; two configurations u, u' that differ only at b give $\iota_l(u), \iota_l(u')$ that differ only at b, by the same distance, and the supremum defining $C^{\hat\mu_l}_{ab}$ runs over a subset of the pairs for $C^{\mu'}_{ab}$. □

Theorem 137 (one step of the straight-path map)**.** *Let c be the straight-path map with blocks of side $b \ge 2$ from Λ_{bL} to Λ_L, $L \ge 2$, let μ' be a gauge-invariant probability measure on $G^{E'}$ with a continuous positive density, $\mu^c = c_*\mu'$, and $k_b = \lfloor b/2 \rfloor$. If $c_\lambda(\mu') < 1$, then for $0 \le \kappa \le \lambda$*

$$c_{b\kappa}(\mu^c) \le e^{-(\lambda-2\kappa)k_b}\, \frac{c_\lambda(\mu')}{1 - c_\lambda(\mu')} .$$

Proof. Mark each path $\pi(y,\nu)$ by its link with base point $by + k_b e_\nu$, and let d' be the distance $|\Delta|_\infty$ of Λ_{bL}, restricted to $\hat E_l$. By Lemma 136, $c_\lambda(\hat\mu_l; d') \le c_\lambda(\mu') < 1$, and the coefficients of μ^c are those of the marginal of $\hat\mu_l$ on $A = l(E)$. Let $e = (y,\nu)$ and $j = (y',\nu')$ be distinct coarse links and $\Delta = y - y'$, with coordinates in $(-L/2, L/2]$. The i-th coordinate of the difference of the base points of $l(e)$ and $l(j)$ is $b\Delta_i + t_i$ with $|t_i| \le k_b$, and the distance of $b\Delta_i$ to 0 on $\mathbb{Z}/bL\mathbb{Z}$ is $b|\Delta_i|$; so $d'(l(e), l(j)) \ge b|\Delta|_\infty - k_b$. If $y = y'$, then $\nu \ne \nu'$ and the difference is $k_b(e_\nu - e_{\nu'})$, at distance k_b since $k_b \le bL/2$; if $y \ne y'$, some coordinate is at distance at least $b - k_b \ge k_b$. Hence $d_A \ge k_b$, and Lemma 135(ii) gives

$$\sum_{e \ne j} C^{\mu^c}_{ej}\, e^{b\kappa|\Delta|_\infty} \le e^{\kappa k_b} \sum_{e \ne j} C^{\mu^c}_{ej}\, e^{\kappa d'(l(e), l(j))}$$

and the right side is at most $e^{\kappa k_b} e^{-(\lambda-\kappa)k_b} c_\lambda/(1-c_\lambda)$. The function $x \mapsto x/(1-x)$ is increasing. □

The rate $b\kappa$ on the coarse lattice is the rate κ on the fine one in physical units, and for $\kappa < \lambda/2$ the coarse sum is smaller than the fine one by $e^{-(\lambda-2\kappa)k_b}$. Along a tower the factors compound.

Corollary 138 (a tower)**.** *In Corollary 102 let the c_j be straight-path maps.*

(i) *For $0 \le i < j \le k$, $0 \le \kappa \le \lambda$ and $c_\lambda(\mu^{(j)}_{k,L}) < 1$,*

$$c_{2^{j-i}\kappa}\big(\mu^{(i)}_{k,L}\big) \le e^{-(\lambda-2\kappa)2^{j-i-1}}\, \frac{c_\lambda(\mu^{(j)}_{k,L})}{1 - c_\lambda(\mu^{(j)}_{k,L})} .$$

(ii) *Let $\lambda > 0$, $c < 1$ and $k \ge 1$, and suppose that for every $L \in \mathscr{L}_k$ some $j \in \{1, \dots, k\}$ satisfies*

$$c_\lambda\big(\mu^{(j)}_{k,L}\big) \le c, \qquad 2^j \lambda \ge 4 \log \frac{17c}{1-c} .$$

Then $\mathrm{BP}(n_0, \frac12)$ holds for $\mu^{(0)}_{k,L}$ for every $L \in \mathscr{L}_k$ and every $n_0 \ge 2$ with $L_0 \ge 2n_0 + 4$. If this holds for every $k \ge 1$ with the same λ and c, then (H1) holds with $\varpi_0 = \frac12$.

(iii) *If G is as in Lemma 126, δ is the geodesic distance of $\langle X, Y\rangle = -\mathrm{Tr}(XY)$, $(36e+6)N|\beta_k| \le \kappa_G$ and $k \ge 4$, the hypothesis of* (ii) *holds for this k with $j = k$, $\lambda = 1$ and $c = \frac12$.*

Proof. (i) By Lemma 128(ii) and induction, $c_{i+1}\circ\cdots\circ c_j$ is the straight-path map with blocks of side 2^{j-i} from Λ_{L_j} to Λ_{L_i}, and $\mu^{(i)}_{k,L}$ is its image of $\mu^{(j)}_{k,L}$, which is gauge invariant and has a continuous positive density by Lemma 128. Apply Theorem 137 with $b=2^{j-i}$, $k_b=2^{j-i-1}$.

(ii) By (i) with $i=0$ and $\kappa=\lambda/4$, $\lambda_0=2^{j-2}\lambda$ satisfies

$$c_{\lambda_0}\big(\mu^{(0)}_{k,L}\big)\le e^{-2^{j-2}\lambda}\,\frac{c}{1-c}\le\frac{1}{17}.$$

So the column sums and the tails of the coefficients of $\mu^{(0)}_{k,L}$ are at most $\frac{1}{17}=(1-\frac{1}{17})/16$, and the proof of Corollary 129, which treats one k at a time, applies with $c=\frac{1}{17}$ and $R=1$, for every $n_0\ge\max(2,1)$.

(iii) The Wilson measure $\mu_{k,L}=\mu^{(k)}_{k,L}$ has $C_{ej}=0$ unless e,j are partners, which have $|\Delta|_\infty\le 1$, and every link has eighteen partners. By Lemma 73, $c_1(\mu_{k,L})\le 18eN|\beta_k|/(\kappa_G-6N|\beta_k|)\le\frac12$, and $2^k\ge 16>4\log 17$. □

Part (ii) moves the problem by a bounded number of scales and does not remove it: the hypothesis is a Dobrushin bound at the spacing $a_j\le a_0\lambda/(4\log(17c/(1-c)))$, uniform in k, for the law of the block variables of $k-j$ steps. Part (iii) is the descent of Sec. XJ seen through the step; along the tuning $\beta_k\to\infty$ and the level $j=k$ is not available, by Corollary 141 below. The intermediate levels are restricted by gauge covariance.

Lemma 139 (gauge covariance)**.** *Let ρ be a gauge-invariant probability measure on the links of Λ_L, $L\ge 3$, with a continuous positive density, $e=(x,\nu)$ a link, $S_-(e)$ and $S_+(e)$ the sets of the seven other links with an end point at x and at $x+e_\nu$, and $\gamma=\gamma^\rho_{\{e\}}(\cdot\mid U)$. For $h\in G$ let $h\gamma$ and γh be the images of γ under $\sigma\mapsto h\sigma$ and $\sigma\mapsto\sigma h$, and*

$$s(\gamma)=\min_{\tau\in G}\int\delta(\sigma,\tau)\,\gamma(d\sigma)$$

the spread of γ. Then

$$\delta(h,I)-2s(\gamma)\le W_1(\gamma,h\gamma)\le\delta(h,I)\sum_{k\in S_-(e)}C^\rho_{ek},$$
$$\delta(h,I)-2s(\gamma)\le W_1(\gamma,\gamma h)\le\delta(h,I)\sum_{k\in S_+(e)}C^\rho_{ek}.$$

Proof. Let g be the gauge transformation equal to h at x and to I elsewhere. Then $(gU)_e=hU_e$, the other links at x are multiplied by h on the left or by h^{-1} on the right, and all others are unchanged. Since the density ρ is gauge invariant (proof of Lemma 136) and Haar measure is left invariant, $\gamma^\rho_{\{e\}}(\cdot\mid gU)=h\gamma$. The configurations U and gU differ outside e only on $S_-(e)$, each link by $\delta(h,I)$ by the bi-invariance of δ; changing them one at a time gives the upper bound. For γh take g equal to h^{-1} at $x+e_\nu$. For the lower bound let τ attain $s(\gamma)$; the function $\delta(\cdot,\tau)$ is 1-Lipschitz, and by the triangle inequality and bi-invariance

$$\int\delta(h\sigma,\tau)\,\gamma(d\sigma)\ge\delta(h\tau,\tau)-s(\gamma)=\delta(h,I)-s(\gamma),$$

so $W_1(\gamma,h\gamma)\ge\delta(h,I)-2s(\gamma)$; likewise with $\delta(\tau h,\tau)=\delta(h,I)$. □

Theorem 140 (a lower bound from gauge covariance)**.** *Let ρ be a gauge-invariant probability measure on the links of Λ_L, $L\ge 3$, with a continuous positive density, and for a link e let $s_e=\inf_U s(\gamma^\rho_{\{e\}}(\cdot\mid U))$ and $\bar s$ the mean of s_e over the $4L^4$ links.*

(i) *$\sum_{k\in S_-(e)\cup S_+(e)}C^\rho_{ek}\ge 2-4s_e/D$ for every link e.*

(ii) *$c_\rho\ge 2-4\bar s/D$; if ρ is invariant under the translations, with s_ν the common value of s_e for the links of direction ν, then $c_\rho\ge 2-(s_0+s_1+s_2+s_3)/D$.*

(iii) *$\rho(\delta(U_p,I)\mid U_{E\setminus\{e\}})\ge s_e$ almost surely for every plaquette p and every link e of p, and the mean of $\rho(\delta(U_p,I))$ over the $6L^4$ plaquettes is at least $\bar s\ge D(2-c_\rho)/4$.*

Proof. (i) The sets $S_\pm(e)$ are disjoint since $L\ge 3$. Take h with $\delta(h,I)=D$ in Lemma 139: each of the two sums is at least $1-2s(\gamma)/D$ for every U, hence at least $1-2s_e/D$.

(ii) The sum of all coefficients is the sum of the row sums and the sum of the column sums. So c_ρ, the largest column sum, is at least the mean row sum, which by (i) is at least the mean of $2-4s_e/D$.

(iii) The plaquette variable is $U_p=AU_eB$ or $AU_e^{-1}B$ with A,B products of the other links of p, and by bi-invariance $\delta(U_p,I)=\delta(U_e,T)$ with $T=A^{-1}B^{-1}$ or $T=BA$, a function of $U_{E\setminus\{e\}}$. Its conditional expectation is $\int\delta(\sigma,T)\,\gamma^\rho_{\{e\}}(d\sigma\mid U)\ge s_e$. So $\rho(\delta(U_p,I))$ is at least the mean of s_e over the four links of p; every link lies in six plaquettes, and averaging over p gives $\bar s$. The last inequality is (ii). □

For the product of the Haar measures $c_\rho=0$, and the bound is consistent with it: $s_e=\int\delta(\sigma,I)\,d\sigma\ge D/2$, since $\delta(\sigma,I)+\delta(\sigma,a)\ge D$ for a with $\delta(a,I)=D$ and the two terms have the same integral. The bound is informative when the single-link conditional laws are concentrated, which is the regime of the Wilson measure at weak coupling and of every block law in the small-field region.

Corollary 141 (the Wilson measure)**.** *Let $G=\mathrm{SU}(2)$, δ the geodesic distance of $\langle X,Y\rangle=-\mathrm{Tr}(XY)$, so that $\delta(Q,I)=\sqrt2\,\theta(Q)$ with $\theta(Q)\in[0,\pi]$ the angle of Q and $D=\sqrt2\pi$, and let μ be the Wilson measure of Λ_L with $B=6\beta_W\ge 1$. Then*

$$c_\mu\ge 2-2\sqrt{\frac{3}{B}}-\frac{3\pi^3}{4}e^{1/2}B^{3/2}e^{-B},$$

and $c_\mu>1.13$ for $\beta_W\ge\frac83$.

Proof. Given U, the law of U_e has density proportional to $\exp(\frac{\beta_W}{2}\,\mathrm{Re}\,\mathrm{Tr}(U_eS))$, S the sum of the six staples; at $U=I$, $S=6I$, and $\theta=\theta(U_e)$ has the density proportional to $\sin^2\theta\,e^{B\cos\theta}$ on $[0,\pi]$. So $s_e\le\sqrt2\,\mathbb{E}\theta$ for every e, and Theorem 140(ii) gives $c_\mu\ge 2-\frac{4}{\pi}\mathbb{E}\theta$. The derivative of $\sin^3\theta\,e^{B\cos\theta}$ is $(3\cos\theta-B\sin^2\theta)\sin^2\theta\,e^{B\cos\theta}$ and vanishes at both ends, so $3\mathbb{E}\cos\theta=B\mathbb{E}\sin^2\theta$ and $\mathbb{E}\sin^2\theta\le 3/B$. On $[0,\pi/2]$, $\theta\le\frac{\pi}{2}\sin\theta$, so

$$\mathbb{E}\theta\le\frac{\pi}{2}\sqrt{\mathbb{E}\sin^2\theta}+\pi P,\qquad P=\Pr(\theta>\pi/2).$$

The numerator of P is at most $\int_{\pi/2}^{\pi}\sin^2\theta\,d\theta=\pi/4$. With $t=B^{-1/2}\le 1$, on $[0,t]$ one has $\cos\theta\ge 1-t^2/2$ and $\sin\theta\ge 2\theta/\pi$, so the denominator is at least $\frac{4}{3\pi^2}B^{-3/2}e^{B-1/2}$, and $P\le\frac{3\pi^3}{16}e^{1/2}B^{3/2}e^{-B}$. This gives the bound. For $B\ge 16$ both subtracted terms decrease; at $B=16$ the first is $\sqrt{3}/2<0.8661$ and the second is less than $2455\,e^{-16}<3\cdot 10^{-4}$. □

So the Dobrushin route of Corollary 129 is closed at the finest level along the tuning, for every rate $\lambda\ge 0$, since $c_\lambda\ge c_0$, by gauge covariance alone; this is independent of the bound of Lemma 73, which is an upper bound. The same argument applies to every block law whose plaquette variables are concentrated. Figure 19 shows the step and the two lower bounds.

Corollary 142 (small fields)**.** *Let ρ be as in Theorem 140, $\varepsilon\ge 0$ and $\eta\ge 0$.*

(i) *If $\rho(\delta(U_p,I)>\varepsilon)\le\eta$ for every plaquette p, then $c_\rho\ge 2-4\varepsilon/D-4\eta$.*

(ii) *Under the hypotheses of Corollary 129 with $c<1$, the holonomies W of the squares of side 2^k with corners in $2^k\mathbb{Z}^4$ satisfy, for all k and $L\in\mathscr{L}_k$, the inequality $\langle\mu_{k,L}(\delta(W,I))\rangle\ge D(2-c)/4>D/4$, the mean taken over these squares.*

Proof. (i) The mean of $\rho(\delta(U_p,I))$ is at most $\varepsilon+D\eta$; use Theorem 140(iii). (ii) The plaquette variables of the coarse configuration are the holonomies W, and $c_\rho\le c$ for $\rho=\mu_{k,L}^{(0)}$; use Theorem 140(iii). □

A Dobrushin bound for the law of the block variables therefore requires that the holonomies of the blocks are, in the mean, at more than half the distance from I at which Haar measure puts them; for SU(2) the Haar mean of $\delta(\sigma,I)$ is $\sqrt{2}\int_0^\pi\theta\frac{2}{\pi}\sin^2\theta\,d\theta=D/2$. The small-field region of Bałaban's analysis [6, 7], in which every plaquette variable of every block lattice is close to I, lies outside the domain of the criterion. At the coarsest scale the obstruction does not apply along the tuning, for SU(2): the conditional laws of the coarse links are spread out.

Proposition 143 (randomisation along the paths)**.** *Let μ' be the probability measure on the links of Λ_{bL}, $b\ge 2$, $L\ge 3$, with density proportional to $\prod_p\varphi(U_p)$ for a positive continuous class function φ, c the straight-path map, $\mu^c=c_*\mu'$ and $\gamma_e^c=\gamma_{\{e\}}^{\mu^c}$.*

(i) *With $a_\varphi=(\min\varphi/\max\varphi)^6$, for every coarse link e and every V, the total variation distance of $\gamma_e^c(\cdot\mid V)$ to λ is at most $(1-a_\varphi)^b$, and $s(\gamma_e^c(\cdot\mid V))\ge D(\frac12-(1-a_\varphi)^b)$.*

(ii) *Let $G=\mathrm{SU}(2)$, $\varphi(Q)=\exp(\frac{\beta_W}{2}\operatorname{Re}\operatorname{Tr}Q)$, so that μ' is the Wilson measure, δ and D as in Corollary 141, and $\varepsilon_b=(1+1/(4|\beta_W|))^{-b}$. Then for every coarse link e and every V*

$$\Big\|\int\sigma\,\gamma_e^c(d\sigma\mid V)\Big\|_{\mathrm{op}}\le\varepsilon_b,\qquad s\big(\gamma_e^c(\cdot\mid V)\big)\ge\frac{D(1-\varepsilon_b)}{\pi},$$

and the mean of $\mu^c(\delta(V_P,I))$ over the coarse plaquettes P is at least $D(1-\varepsilon_b)/\pi$.

Proof. Fix a coarse link e. The coarse links other than e are functions of the fine links outside $\pi(e)$. Given the fine links outside $\pi(e)$, the b links of $\pi(e)$ are independent under μ': they are collinear, so no plaquette contains two of them, and their conditional density is a product of factors, the factor of a link being the product of φ over its six plaquettes, normalised.

(i) Each factor is at least a_φ times its integral,[27] so the law of each path link is $a_\varphi\lambda+(1-a_\varphi)q_i$ for probability measures q_i. Drawing the links independently in this form, with probability $1-(1-a_\varphi)^b$ at least one factor is drawn from λ, and then the product is a Haar variable by the invariance of λ; so the law of c_e given the fine links outside $\pi(e)$ is within total variation $(1-a_\varphi)^b$ of λ. The law of c_e given the other coarse links is a mixture of these laws, and the total variation distance is convex; so the bound holds for μ^c-almost every V. The set of laws within total variation t of λ is weakly closed, the kernel $\gamma_e^c(\cdot\mid V)$ is weakly continuous in V (Lemma 128(i)), and a set of full μ^c-measure is dense; so the bound holds for every V. Since $0\le\delta(\cdot,\tau)\le D$, $\int\delta(\cdot,\tau)\,d\gamma\ge\int\delta(\cdot,\tau)\,d\lambda-D(1-a_\varphi)^b$, and the Haar integral is at least $D/2$, as after Theorem 140.

(ii) Write $\frac12\operatorname{Re}\operatorname{Tr}$ as the real part of a unit quaternion. A path link u has, given the fine links outside $\pi(e)$, the density proportional to $\exp(\frac{\beta_W}{2}\operatorname{Re}\operatorname{Tr}(uS))$, where S, a sum of six elements of SU(2), equals sW with $0\le s\le 6$ and $W\in\mathrm{SU}(2)$. The law of uW is a class function times Haar measure, so its mean is the scalar $A=\mathbb{E}_B\cos\theta$, $B=s\beta_W$, and the mean of u is AW^{-1}. For $B\ge 0$ the identity of the proof of Corollary 141 and $\mathbb{E}\cos^2\theta\ge A^2$ give $3A+BA^2\le B$, while $x=2B/(2B+3)$ has $(3x+Bx^2-B)(2B+3)^2=9B^2\ge 0$; as $3x+Bx^2$ increases on $[0,\infty)$, $A\le 2B/(2B+3)$, and $B<0$ follows by $\theta\mapsto\pi-\theta$. Hence $|A|\le 2|B|/(2|B|+3)\le(1+1/(4|\beta_W|))^{-1}$. The mean of the product of the b independent path links is the product of their means, of operator norm at most ε_b; mixing and continuity as in (i) give the first bound for every V. For the second, $\theta\ge 1-\cos\theta$ on $[0,\pi]$ and $|\operatorname{Tr}X|\le 2\|X\|_{\mathrm{op}}$ give, for every τ and $M=\int\sigma\,d\gamma$,

$$\int\delta(\sigma,\tau)\,d\gamma\ge\sqrt{2}\Big(1-\frac12\operatorname{Re}\operatorname{Tr}(\tau^{-1}M)\Big)\ge\sqrt{2}\,(1-\varepsilon_b),$$

which is $D(1-\varepsilon_b)/\pi$. The last claim is Theorem 140(iii). □

Along the tuning $b=2^k$, $\varepsilon_{2^k}\le\exp(-2^k/(4|\beta_{W,k}|+1))$, which tends to zero when $2^{-k}\beta_{W,k}\to 0$, in particular under the growth $\beta_{W,k}\sim\frac{11}{3\pi^2}k\log 2$ that the renormalisation group gives for the tuning [63, 99].[28] Once $\varepsilon_{2^k}<1-\pi/4$ the mean coarse plaquette distance exceeds $D/4$, the condition of Corollary 142(ii) holds, and Theorem 140 excludes only

[27] This is Doeblin's minorisation: a law bounded below by $a\lambda$ is a mixture with weight a of λ and another law.

[28] With $\beta_W=2N/g^2$ and the one-loop running $g^{-2}=\frac{11N}{24\pi^2}\log a^{-1}+O(\log\log a^{-1})$, $\beta_W\sim\frac{11N^2}{12\pi^2}\log a^{-1}$, which is $\frac{11}{3\pi^2}\log a^{-1}$ for SU(2); with $a=2^{-k}a_0$ this is the growth quoted.

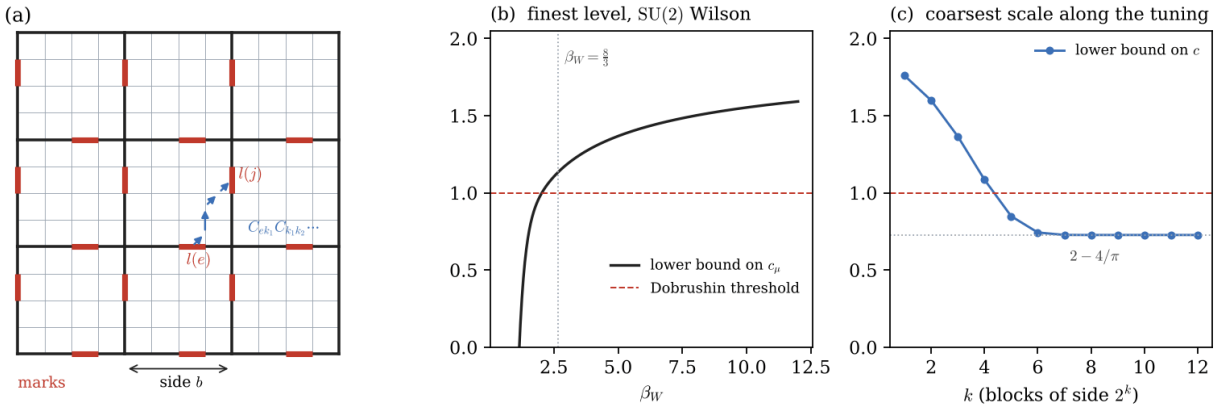


FIG. 19. The single-link coefficients across scales. (a) One step of the straight-path map with blocks of side $b=4$, in a plane: each coarse link is the product of the fine links of its path, marked at the middle link $l(e)$ (red); the law of the block variables is the marginal of the axial-gauge law on the marks, and the influence of $l(j)$ on $l(e)$ is a sum over walks of fine links weighted by products of coefficients (blue), which leave a factor $e^{-(\lambda-2\kappa)\lfloor b/2\rfloor}$ (Lemma 135, Theorem 137). (b) The lower bound on the Dobrushin sum of the SU(2) Wilson measure forced by gauge covariance, against β_W; it exceeds the threshold 1 for $\beta_W \geq \frac{8}{3}$ (Corollary 141). (c) The lower bound $2-4(1-\varepsilon_{2^k})/\pi$ for the law of the block variables of side 2^k, along $\beta_{W,k} = 2+\frac{11}{3\pi^2}k\log 2$, an illustration of the growth along the tuning; below 1 gauge covariance no longer excludes the Dobrushin condition (Proposition 143).

$c_\rho < 2-4(1-\varepsilon_{2^k})/\pi$, a bound that tends to $2-4/\pi < 0.73$. So gauge covariance does not obstruct a Dobrushin bound at the coarsest scale. Part (i) gives nothing along the tuning, since $a_\varphi = e^{-12|\beta_W|}$ for SU(2) and $2^k e^{-12\beta_{W,k}} \to 0$ under that growth. Neither part bounds the coefficients: the spread of the coarse conditional laws says nothing about their Lipschitz dependence on the exterior. That dependence is governed by the Hessian of the induced action.

Proposition 144 (the Hessian of the induced action)**.** *Let μ' have the density $e^{-\mathscr{S}}/Z$ with respect to $\lambda_{E'}$, $\mathscr{S}$ gauge invariant and of class C^2, let l be a marking, and for a fine link a and $X \in \mathfrak{g}$ let $D^a_X\mathscr{S}(U) = \frac{d}{dt}\mathscr{S}(U^{a,tX})|_{t=0}$. Then $\mu^c = c_*\mu'$ has the density e^{-H^c} with*

$$H^c(V) = -\log \int e^{-\mathscr{S}(\Theta^l(V,m))}\,\lambda_{E'_0}(dm) + \text{const.}$$

Let ς_V be the probability measure on $G^{E'_0}$ with density proportional to $e^{-\mathscr{S}(\Theta^l(V,m))}$, let $a = l(e)$ and $a' = l(e')$, and take the derivatives of $\mathscr{S}$ at $\Theta^l(V,m)$. Then

$$\frac{d^2}{dt^2}H^c(V^{e,tX})\Big|_0 = \varsigma_V(D^a_X D^a_X\mathscr{S}) - \mathrm{Var}_{\varsigma_V}(D^a_X\mathscr{S}),$$
$$\frac{\partial^2 H^c((V^{e,sX})^{e',tY})}{\partial s\,\partial t}\Big|_0 = \varsigma_V(D^{a'}_Y D^a_X\mathscr{S}) - \mathrm{Cov}_{\varsigma_V}(D^a_X\mathscr{S}, D^{a'}_Y\mathscr{S})$$

for $e \neq e'$. Let $-h_e$ be the infimum over V and unit X of the first right side and $J_{ee'}$ the supremum over V and unit X,Y of the absolute value of the second; if $h_e < \kappa_G$, Lemma 126 applies to μ^c with these numbers. For the Wilson action the term $\varsigma_V(D^{a'}_Y D^a_X\mathscr{S})$ vanishes unless a and a' are partners.

Proof. By the proof of Lemma 136, ν_V is the image of $\lambda \otimes \lambda_{E'_0}$ under $(h,m) \mapsto h^{-1}\Theta^l(V,m)$, and $\mathscr{S}$ is gauge invariant, so $\nu_V(e^{-\mathscr{S}}) = \int e^{-\mathscr{S}(\Theta^l(V,m))}\,dm$; this is the density of Lemma 128(i). Replacing V_e by $V_e e^{tX}$ replaces the link a of $\Theta^l(V,m)$ by the same product, so $\frac{d}{dt}$ of $\mathscr{S}(\Theta^l(V^{e,tX},m))$ is $D^a_X\mathscr{S}$ at the moved configuration. Differentiating $-\log Z(t)$ under the integral, which is allowed for C^2 functions on a compact group, gives $\frac{Z''}{Z} - (\frac{Z'}{Z})^2$ with the signs shown; the mixed derivative is the same computation in two variables. Distinct coarse links have distinct marks, since the paths are disjoint. For the Wilson action the mixed derivative of $\mathscr{S}$ at $a \neq a'$ comes from the plaquettes that contain both. □

The two terms of d^2H^c are of the same order. Under a single-link conditional law $\gamma \propto e^{-\mathscr{S}}\,d\sigma$ of the link a, integration by parts with the left-invariant vector field of X, which preserves Haar measure, gives $\gamma((D^a_X\mathscr{S})^2) = \gamma(D^a_X D^a_X\mathscr{S})$, so there the mean curvature equals the second moment of the force and is at least its variance. In the fibre law ς_V the link a is held fixed and the variance is that of the force on it produced by the fluctuations of the links of E'_0; the hypothesis $h_e < \kappa_G$ of Lemma 126 asks that this variance stay below the mean curvature plus κ_G, uniformly in the exterior, and the summability of $J_{ee'}$ asks that the forces on distant marks decorrelate in ς_V. For the Wilson action at large β both terms are of order $N|\beta|$ on the flat configuration, so the bound requires a cancellation between them to precision κ_G; this is the effective-action problem in the form posed by Corollary 129. For SU(2) and the straight-path map the cancellation holds along every tuning with $\beta_k = O(k)$ (Corollary 162), and the decorrelation is what remains.

Remark 145 (what is proved about the coefficients across scales). Proved: the marginal of a measure with a weighted Dobrushin sum $c_\lambda < 1$ on a set of points at mutual distance

at least d_A has weighted sums at most $e^{-(\lambda-\kappa)d_A}c_\lambda/(1-c_\lambda)$ (Lemma 135); the law of the block variables of a gauge-invariant measure is such a marginal, through the axial gauge, with coefficients bounded by the fine ones (Lemma 136); one step of the straight-path map of side b contracts the weighted sum by $e^{-(\lambda-2\kappa)\lfloor b/2\rfloor}$ on every torus (Theorem 137); along a tower of straight-path maps a bound $c_\lambda \le c < 1$ at one level j with $2^j\lambda \ge 4\log(17c/(1-c))$, for every k, gives (H1) with $\varpi_0 = \frac{1}{2}$ for every $n_0 \ge 2$, and the Wilson measure in the strong-coupling range supplies it with $j = k \ge 4$ (Corollary 138); gauge covariance bounds every row sum below by $2-4s_e/D$ and the Dobrushin sum below by $2-4\bar{s}/D$, and the mean plaquette distance by $D(2-c_\rho)/4$ (Lemma 139, Theorem 140); the SU(2) Wilson measure has $c_\mu > 1.13$ for $\beta_W \ge \frac{8}{3}$, and every law with concentrated plaquette variables fails the Dobrushin condition (Corollaries 141 and 142); for SU(2) the coarse conditional laws are spread out along the tuning, so that gauge covariance does not obstruct the condition at the coarsest scale (Proposition 143); the Hessian of the induced action is the mean curvature of the fine action minus the covariance of the forces in the fibre law (Proposition 144). Every numerical value is derived by hand in the proofs.

Not proved: a weighted Dobrushin bound $c_\lambda < 1$, uniform in k and the torus, for the law of the block variables at any level of the tower with β_k outside the strong-coupling range, and hence (H1) along the tuning; bounds on the fibre variances and covariances of Proposition 144 at weak coupling; any Lipschitz bound for the dependence of the coarse conditional laws on the exterior along the tuning; total variation randomisation of the coarse links along the tuning for groups other than SU(2), beyond the Doeblin bound of Proposition 143(i); the abelian and Villain cases, which are not treated. Neither BP along the tuning nor the mass gap is proved here.

P. The coarsest condition through the fibre law

This subsection approaches (H1) through the fibre law of Proposition 144. It proves four things. First, a single-link coefficient of every measure with a density e^{-H}, $H \in C^2$, is at most $D^2/4$ times the bound on the mixed derivatives of H in the two links, with no condition on the single-link Hessian and for every connected group, abelian or not; gauge invariance, conversely, bounds the single-link Hessian by the mixed derivatives into the seven links at one end point. Second, for straight-path maps with blocks of side at least three the marks of distinct coarse links are never partners, so the mixed derivatives of the induced action are minus the covariances of the forces on distinct marks in the fibre law; (H1) follows from a decorrelation of these forces, uniform in the coarse data and in the torus, the condition (FC) below, and (FC) in turn forces the force variance on a mark to equal its mean curvature up to $4c/D^2$. Third, in a Gaussian abelian model of the fibre the covariance of the forces on distinct marks is minus the coupling times the effective quadratic form of the straight-path map; this form is a conserved current on the coarse lattice, its diagonal lies between $1/b^2$ and $6/b$, its row sums are at least twice the diagonal and are bounded uniformly in the torus by an infinite-volume quantity γ_b, and a bound on γ_b gives the Gaussian form of (FC); Sec. X R shows that γ_b is of order $(\log b)/b$. Fourth, the strong-coupling range, the group U(1) and rough coarse data of SU(2) are placed with respect to (FC). (FC) along the tuning is not proved here.

Throughout, $G \subset \mathrm{U}(N)$ is closed and connected, δ is the geodesic distance of $\langle X,Y\rangle = -\operatorname{Tr}(XY)$ on G and D its diameter; for SU(2), $D = \sqrt{2}\pi$ (Corollary 141), and for U(1), where $\delta(e^{i\theta},1) = |\theta|$ for $|\theta| \le \pi$, $D = \pi$. For $H \in C^2(G^E)$ on the links of a torus and links $e \ne j$ put

$$J_{ej}(H) = \sup\Big|\frac{\partial^2}{\partial s\,\partial t}H\big((U^{e,sX})^{j,tY}\big)\Big|_0\Big|,$$

the supremum over all U and all unit $X,Y \in \mathfrak{g}$, with $U^{e,X}$ as in Lemma 126; then $J_{ej} = J_{je}$, and these are the smallest numbers admissible in the second hypothesis of that lemma.

Lemma 146 (oscillation)**.** *Let ρ be the probability measure on G^E with density proportional to e^{-H} with respect to λ_E, $H \in C^2(G^E)$. Then $C^\rho_{ej} \le \frac{D^2}{4}J_{ej}(H)$ for all links $e \ne j$.*

Proof. Let U,U' differ only at j, $\rho = \delta(U_j,U'_j) > 0$, $\gamma = \gamma^\rho_{\{e\}}(\cdot\,|\,U)$ and $\gamma' = \gamma^\rho_{\{e\}}(\cdot\,|\,U')$. Then γ' has the density $e^{-\Delta}/Z$ with respect to γ, where $\Delta(\sigma) = H(\sigma,U'_{E\setminus\{e\}}) - H(\sigma,U_{E\setminus\{e\}})$ and $Z = \int e^{-\Delta}d\gamma$.

The oscillation of Δ. Since G is compact and connected and the metric is bi-invariant, two points σ',σ are joined by a minimising geodesic $s \mapsto \sigma' e^{sX}$, $0 \le s \le r = \delta(\sigma,\sigma') \le D$, $|X| = 1$; likewise $U'_j = U_j e^{\rho Y}$ with $|Y| = 1$. Let $f(s,t)$ be the value of H at the configuration with $\sigma' e^{sX}$ at e, $U_j e^{tY}$ at j and the links of U elsewhere. Then

$$\begin{aligned}\Delta(\sigma) - \Delta(\sigma') &= f(r,\rho) - f(r,0) - f(0,\rho) + f(0,0)\\ &= \int_0^r\int_0^\rho \partial_s\partial_t f\,dt\,ds,\end{aligned}$$

and since $\sigma' e^{(s+s')X} = \sigma' e^{sX}e^{s'X}$, $\partial_s\partial_t f(s,t)$ is a mixed derivative as in the definition of $J_{ej}(H)$, at the configuration indexed by (s,t). Hence $\omega = \sup\Delta - \inf\Delta \le DJ_{ej}(H)\rho$.

Total variation. Let A be a Borel set with $p = \gamma(A) \in (0,1)$, and put $x = p^{-1}\int_A e^{-\Delta}d\gamma$, $y = (1-p)^{-1}\int_{G\setminus A} e^{-\Delta}d\gamma$, $t = x/y \in [e^{-\omega},e^{\omega}]$ and $q = e^{\omega}$. Then

$$\gamma'(A) - \gamma(A) = \frac{p(1-p)(t-1)}{pt+1-p} \le \frac{p(1-p)(q-1)}{pq+1-p},$$

because $t \mapsto (t-1)/(pt+1-p)$ has the derivative $(pt+1-p)^{-2} > 0$. Let $T = (\sqrt{q}-1)/(\sqrt{q}+1) = \tanh(\omega/4)$. Then $1-T = 2/(\sqrt{q}+1)$ and $(q-1)(1-T)^2 = 4T$, so

$$\begin{aligned}&T(pq+1-p) - p(1-p)(q-1)\\ &\quad = (q-1)p^2 - (q-1)(1-T)p + T\end{aligned}$$

is a quadratic polynomial in p with leading coefficient $q-1 \ge 0$ and discriminant $(q-1)\big((q-1)(1-T)^2 - 4T\big) = 0$, hence nonnegative. So $\gamma'(A) - \gamma(A) \le T$, and the same bound for $G\setminus A$ gives $\gamma(A) - \gamma'(A) \le T$; the cases $p \in \{0,1\}$ are

trivial, as γ and γ' are mutually absolutely continuous. Thus $|\gamma'(A)-\gamma(A)| \le \tanh(\omega/4)$ for every A.[29]

The Kantorovich distance. A 1-Lipschitz function g on G has oscillation at most D; with m the midpoint of its range, $|g-m| \le D/2$, so $|\int g\,d(\gamma'-\gamma)| = |\int (g-m)\,d(\gamma'-\gamma)|$ is at most $D/2$ times the total mass of $|\gamma'-\gamma|$, which is at most $2\tanh(\omega/4)$. By the Kantorovich–Rubinstein duality and $\tanh x \le x$,

$$W_1(\gamma,\gamma') \le D\tanh\Big(\frac{DJ_{ej}(H)\rho}{4}\Big) \le \frac{D^2}{4}J_{ej}(H)\,\rho. \qquad \square$$

Lemma 126 gives $J_{ej}/(\kappa_G - h_e)$; the two bounds agree when $\kappa_G - h_e = 4/D^2$, which for $\mathrm{SU}(2)$ is $2/\pi^2$, and Lemma 146 is the stronger one beyond that, in particular for $h_e \ge \kappa_G$, where Lemma 126 gives nothing, and for $\mathrm{U}(1)$, where $\kappa_G = 0$. The single-link Hessian does not disappear from the problem, however: for gauge-invariant H it is controlled by the mixed derivatives.

Lemma 147 (gauge covariance of the Hessian)**.** *Let $H \in C^2(G^E)$ be gauge invariant on the links of Λ_L, $L \ge 3$, let e be a link and $S_\pm(e)$ as in Lemma 139. Then for every U and $X \in \mathfrak{g}$*

$$\Big|\frac{d^2}{dt^2}H(U^{e,tX})\Big|_0\Big| \le |X|^2 \sum_{k\in S_+(e)} J_{ek}(H),$$

and if G is abelian the same holds with $S_-(e)$ in place of $S_+(e)$.

Proof. Write $D^k_Z H(U) = \frac{d}{dt}H(U^{k,tZ})|_{t=0}$. Let $e=(x,\nu)$, $y = x+e_\nu$, $Z \in \mathfrak{g}$, and let g_t be the gauge transformation equal to e^{tZ} at y and to I elsewhere. It replaces U_k by $U_k e^{-tZ}$ for the four links k that end at y, among them e, and U_k by $e^{tZ}U_k = U_k e^{tW_k}$, $W_k = U_k^{-1}ZU_k$, for the four links that start at y, and leaves the others unchanged. Differentiating $H(g_tU) = H(U)$ at $t=0$ gives, for every U,

$$D^e_Z H(U) = -\sum_k D^k_Z H(U) + \sum_{k'} D^{k'}_{W_{k'}} H(U),$$

the first sum over the three other links that end at y, the second over the four that start there; these seven are $S_+(e)$, $|W_{k'}| = |Z|$, and since $L \ge 3$ no link starting at y is e, so $W_{k'}$ does not depend on U_e. Replace U by $U^{e,sX}$ and differentiate at $s=0$. On the left, with $Z=X$, $\frac{d}{ds}D^e_X H(U^{e,sX})|_0 = \frac{d^2}{dt^2}H(U^{e,tX})|_0$, since $U_e e^{sX}e^{tX} = U_e e^{(s+t)X}$; on the right every term is a mixed derivative in e and a link of $S_+(e)$, bounded by $J_{ek}(H)|X|^2$. For abelian G the transformation at x multiplies every link at x by $e^{\pm tZ}$ with no dependence on the link, and the same computation applies with $S_-(e)$. $\square$

Combined with Proposition 144 and the marking of Theorem 137, these lemmas reduce the coefficients of the law of the block variables to covariances in the fibre law. For $b \ge 3$ let $k_b = \lfloor b/2 \rfloor$, so that $1 \le k_b \le b-2$, and let l_b be the marking with $l_b(y,\nu)$ the link of $\pi(y,\nu)$ with base point $by + k_b e_\nu$.

[29] The bound is sharp: for Δ equal to ω on A and to 0 elsewhere and $p = 1/(1+e^{-\omega/2})$, $\gamma(A) - \gamma'(A) = \tanh(\omega/4)$.

Proposition 148 (the fibre covariance)**.** *Let c be the straight-path map with blocks of side $b \ge 3$ from Λ_{bL} to Λ_L, $L \ge 3$, let μ' be the Wilson measure of Λ_{bL} with action $\mathscr{S} = -N\beta\sum_p \mathrm{Re}\,\mathrm{Tr}\,U_p$, $\mu^c = c_*\mu'$ with the induced action H^c, and let ς_V be the fibre law of Proposition 144 for the marking $l = l_b$. For a coarse link e write $a = l_b(e)$, and put*

$$K_{ee'} = \sup_V \sup_{|X|=|Y|=1} \big|\mathrm{Cov}_{\varsigma_V}\big(D^a_X\mathscr{S}, D^{a'}_Y\mathscr{S}\big)\big|,$$

with $a' = l_b(e')$.

(i) *For $e \ne e'$ the marks $l_b(e)$ and $l_b(e')$ are not partners.*

(ii) *$J_{ee'}(H^c) = K_{ee'}$ for $e \ne e'$.*

(iii) *$C^{\mu^c}_{ee'} \le \frac{D^2}{4}K_{ee'}$ for $e \ne e'$.*

(iv) *For every coarse link e, every V and every unit X,*

$$\big|\varsigma_V(D^a_X D^a_X \mathscr{S}) - \mathrm{Var}_{\varsigma_V}(D^a_X\mathscr{S})\big| \le \sum_{e'\in S_+(e)} K_{ee'},$$

and for abelian G also with $S_-(e)$ in place of $S_+(e)$.

Proof. (i) Both end points of $l_b(e)$, $by + k_b e_\nu$ and $by + (k_b+1)e_\nu$, are inner points of $\pi(e)$, since $1 \le k_b$ and $k_b + 1 \le b-1$, and an inner point lies on exactly one path. Two links of different directions in a common plaquette share an end point, so marks of different directions are not partners. Two links of the same direction in a common plaquette have base points that differ by $\pm e_\rho$ for a direction ρ, while the base points of $l_b(y,\nu)$ and $l_b(y',\nu)$ differ by $b(y-y')$, all of whose coordinates are divisible by b modulo bL.

(ii) The action $\mathscr{S}$ is a sum of functions of the four links of a plaquette, so $D^{a'}_Y D^a_X \mathscr{S} = 0$ unless a and a' are partners, and by (i) the mixed derivative of Proposition 144 is $-\mathrm{Cov}_{\varsigma_V}(D^a_X\mathscr{S}, D^{a'}_Y\mathscr{S})$.

(iii) By Lemma 128(i), μ^c has the density e^{-H^c} with H^c smooth; apply Lemma 146 and (ii).

(iv) The density of μ^c is gauge invariant (proof of Lemma 136), hence so is H^c. By Proposition 144 the left side is the second derivative of H^c along $V^{e,tX}$; apply Lemma 147 and (ii). $\square$

Corollary 149 (fibre decorrelation gives (H1))**.** *In Corollary 102 let the c_j be straight-path maps, let $H^{(0)}_{k,L}$ be the induced action of Lemma 128 and $J^{(k,L)}_{ej} = J_{ej}(H^{(0)}_{k,L})$. Suppose that the following condition* (FC) *holds: there are $c < 1$ and $R \ge 1$ with*

$$\frac{D^2}{4}\sum_{e\ne j} J^{(k,L)}_{ej} \le c, \qquad \frac{D^2}{4}\sum_{e:\,|\Delta|_\infty > R} J^{(k,L)}_{ej} \le \frac{1-c}{16}$$

for all k, all $L \in \mathscr{L}_k$ and all links j of Λ_{L_0}.

(i) *Condition* (H1) *holds with $\varpi_0 = \frac{1}{2}$ for every $n_0 \ge \max(2, 16Rc/(1-c))$ for which every $L \in \mathscr{L}_k$ has $L_0 \ge 2n_0+4$.*

(ii) *For $k \geq 2$, $J^{(k,L)}_{ej}$ is the number K_{ej} of Proposition 148 for the straight-path map with blocks of side 2^k from Λ_L to Λ_{L_0} and the Wilson measure $\mu_{k,L}$; for $k = 0$ it is at most $N|\beta_0|$ for partners and vanishes otherwise.*

(iii) *For every link e, every configuration V and every unit vector X, $|\frac{d^2}{dt^2} H^{(0)}_{k,L}(V^{e,tX})|_0| \leq 4c/D^2$; for $k \geq 2$ this is $|\varsigma_V(D^a_X D^a_X \mathscr{S}_k) - \mathrm{Var}_{\varsigma_V}(D^a_X \mathscr{S}_k)| \leq 4c/D^2$ for the marks a and the Wilson action $\mathscr{S}_k$ at β_k.*

Proof. (i) By Lemma 146 and Lemma 128, the coefficients of $\mu^{(0)}_{k,L}$ satisfy $C^{(k,L)}_{ej} \leq \frac{D^2}{4} J^{(k,L)}_{ej}$, and the first statement of Corollary 129 applies. (ii) By Lemma 128, $\mu^{(0)}_{k,L} = c^{(k)}_* \mu_{k,L}$ with $c^{(k)}$ the straight-path map with blocks of side $2^k \geq 4$; use Proposition 148(ii). For $k = 0$, $H^{(0)}_{0,L}$ is the Wilson action at β_0, and Lemma 22 bounds its mixed derivatives. (iii) Since $J^{(k,L)}$ is symmetric, its row sums are column sums, at most $4c/D^2$; apply Lemma 147 and, for $k \geq 2$, Proposition 148(iv). □

So (FC) is a decorrelation statement for the forces on distinct marks in the fibre law, uniform in the coarse data V, in k and in the torus. Part (iii) shows that the oscillation lemma removes the hypothesis $h_e < \kappa_G$ of Lemma 126 from the criterion but not from the problem: through gauge covariance, (FC) forces the variance of the force on every mark in the fibre law to equal its mean curvature, both of order $N|\beta_k|$, up to $4c/D^2$, which is $2c/\pi^2$ for SU(2), uniformly in V. This is the cancellation described after Proposition 144.

Remark 150 (strong coupling and Dobrushin bounds in the fibre). By Lemma 116(ii) the fibre law ς_V is the Wilson specification $\gamma^0_{E'_0}(\cdot \mid \Theta^l(V,\cdot))$, so its single-link coefficients are those of the Wilson measure. If these satisfy $C^0_{ee'} \leq \mathfrak{c}$ for all partners, with $\xi = 18\mathfrak{c} < 1$, Lemma 117(ii) applies to ς_V. The force $D^a_X \mathscr{S}$ depends only on the links of E'_0 in the six plaquettes of a, which are partners of a, and $|D^a_X \mathscr{S}| \leq 6N^{3/2}|\beta||X|$, since $|\mathrm{Re}\,\mathrm{Tr}(WX)| \leq \|W\|_{\mathrm{HS}}\|X\|_{\mathrm{HS}} = \sqrt{N}|X|$ for unitary W. Partners have base points at distance at most one in $|\cdot|_\infty$, so the two sets of links for marks $a = l_b(e)$, $a' = l_b(e')$ are at chain distance at least $d_{ee'} = \max(k_b, b|\Delta|_\infty - k_b) - 2$, by the distance bound in the proof of Theorem 137. Hence, for $b \geq 6$,

$$K_{ee'} \leq \frac{36N^3\beta^2}{1-\xi}\,\xi^{d_{ee'}}$$

uniformly in V and L, and the sums of (FC) for this b tend to zero as $b \to \infty$ at fixed β and ξ. This is the range of Corollary 138(iii) and adds nothing along the tuning. Outside it a Dobrushin argument for the fibre law itself is excluded by gauge covariance: a link of E'_0 whose two end points lie on no path has all the other links at those points in E'_0, the gauge transformations at those points preserve ς_V, and the proofs of Lemma 139 and Theorem 140(i) give it a row sum of single-link coefficients under ς_V of at least $2 - 4s/D$, with s the infimum of the spreads of its conditional laws; at $V = I$, $m = I$ these are the laws of the proof of Corollary 141, so for SU(2) at $\beta_W \geq \frac{8}{3}$ the row sum exceeds 1.13. The decay required by (FC) concerns the forces on the marks, not the single links of the fibre, and the Gaussian model below shows the difference.

A Gaussian model of the fibre. Let $b \geq 3$, $L \geq 3$ and $\beta > 0$. On Λ_{bL} consider real 1-forms $A \in \mathbb{R}^{E'}$, the plaquette variables $(dA)_p = A_{(x,\mu)} + A_{(x+e_\mu,\nu)} - A_{(x+e_\nu,\mu)} - A_{(x,\nu)}$ for the plaquette p at x in the plane $\mu < \nu$, $K = d^{\mathrm{T}}d$ on $\mathbb{R}^{E'}$ and the action $S(A) = \frac{\beta}{2}|dA|^2$, the quadratic part of the U(1) Wilson action $-\beta\sum_p \cos\theta_p$ in the angles. Let P be the set of path links, $l = l_b$, E'_0 as before, and Σ the set of sites on paths, the block corners and the inner points. For $v \in \mathbb{R}^E$ and $m \in \mathbb{R}^{E'_0}$ let $\Theta(v,m) \in \mathbb{R}^{E'}$ be v_e at $l(e)$, 0 on $P \setminus l(E)$ and m on E'_0; the force on a mark a is $F_a = \partial S/\partial A_a = \beta(K\Theta)_a$, and for a link a let δ_a be its indicator.

Lemma 151. *Let $\mathscr{N}$ be the set of restrictions to E'_0 of the forms $d\phi$ with ϕ a function on the sites of Λ_{bL} vanishing on Σ. Then every such $d\phi$ vanishes on P, and $d\Theta(0,m) = 0$ exactly when $m \in \mathscr{N}$.*

Proof. A path link joins two points of Σ, so $d\phi$ vanishes on it, and $\Theta(0, d\phi|_{E'_0}) = d\phi$ is closed. Conversely let $\omega = \Theta(0,m)$ be closed, and lift it to a bL-periodic closed form on $\mathbb{Z}^4$. On $\mathbb{Z}^4$ the sum of ω along a lattice path from 0 to x does not depend on the path, since two such paths differ by a sum of boundaries of plaquettes, so $\omega = d\phi$. The lift of Σ is connected by lifted path links, on which ω vanishes, so ϕ is constant there and may be taken to vanish there. For $z \in bL\mathbb{Z}^4$ the function $x \mapsto \phi(x+z) - \phi(x)$ has zero differential and vanishes on the lift of Σ, so ϕ is periodic and $m = d\phi|_{E'_0}$. □

Since $d\Theta(v, m+n) = d\Theta(v,m)$ for $n \in \mathscr{N}$, and $m \mapsto |d\Theta(0,m)|^2$ is positive definite on the orthogonal complement $\mathscr{N}^\perp$ of $\mathscr{N}$ in $\mathbb{R}^{E'_0}$, the *Gaussian fibre law* ς_v is defined as the probability measure on $\mathscr{N}^\perp$ with density proportional to $e^{-S(\Theta(v,m))}$ with respect to Lebesgue measure; the forces depend on m only through $d\Theta(v,m)$. Let Γ be the symmetric matrix on $\mathbb{R}^E$ with $\langle v, \Gamma v\rangle = \min_m |d\Theta(v,m)|^2$, the *effective quadratic form*: completing the square, $\frac{\beta}{2}\langle v, \Gamma v\rangle$ is, up to a constant, the induced action $-\log \int_{\mathscr{N}^\perp} e^{-S(\Theta(v,m))} dm$, and its mixed derivatives in v_e, $v_{e'}$ are $\beta\Gamma_{ee'}$.

Proposition 152 (the Gaussian fibre law)**.** *In the Gaussian model, with $a = l(e)$ and $a' = l(e')$:*

(i) *Under ς_v, $\mathrm{Var}(F_a) = \beta(6 - \Gamma_{ee})$ and $\mathrm{Cov}(F_a, F_{a'}) = -\beta\Gamma_{ee'}$ for $e \neq e'$, for every v. Under the Gaussian law on the orthogonal complement of $\ker K$ in $\mathbb{R}^{E'}$ with density proportional to e^{-S}, without a fixed skeleton, the forces $\beta(KA)_a$ have $\mathrm{Cov}(\beta(KA)_a, \beta(KA)_{a''}) = \beta K_{aa''}$ for all links a, a'', which vanishes unless $a = a''$ or they are partners.*

(ii) *Let m_v attain the minimum and $\mathscr{J} = K\Theta(v, m_v)$. Then $\mathscr{J}$ vanishes on E'_0, is constant on each path $\pi(e)$ with the value $(\Gamma v)_e$, and these values form a co-closed 1-form on the coarse torus: $\sum_\nu \big((\Gamma v)_{(y,\nu)} - (\Gamma v)_{(y-e_\nu,\nu)}\big) = 0$ for every coarse site y.*

(iii) *$1/b^2 \leq \Gamma_{ee} \leq 6/b$ and $|\Gamma_{ee'}| \leq 6/b$.*

(iv) *The sums of* $|\Gamma_{ee'}|$ *over* $e' \in S_+(e)$ *and over* $e' \in S_-(e)$ *are each at least* Γ_{ee}*, so* $\sum_{e' \neq e} |\Gamma_{ee'}| \geq 2\Gamma_{ee} \geq 2/b^2$.

Proof. (i) For a mark a let $\psi_a = (K\delta_a)|_{E_0'}$, let K_0 be the block of K on E_0', and $\Psi v = \sum_e v_e \psi_{l(e)}$. A link lies in six plaquettes and distinct marks are not partners (Proposition 148(i)), so the block of K on the marks is $6I$ and

$$|d\Theta(v,m)|^2 = 6|v|^2 + 2\langle m, \Psi v\rangle + \langle m, K_0 m\rangle.$$

By Lemma 151 the kernel of K_0 is $\mathscr{N}$, and ψ_a is orthogonal to $\mathscr{N}$, since $\langle \psi_a, d\phi|_{E_0'}\rangle = \langle K\delta_a, d\phi\rangle = \langle \delta_a, d^{\mathrm{T}} d d\phi\rangle = 0$. With K_0^+ the inverse of K_0 on $\mathscr{N}^\perp$, the minimiser is $m_v = -K_0^+ \Psi v$ and $\Gamma = 6I - \Psi^{\mathrm{T}} K_0^+ \Psi$. Under ζ_v, $m = m_v + n$ with n centred Gaussian on $\mathscr{N}^\perp$ of covariance $\beta^{-1} K_0^+$, and $F_a = \beta(K\Theta(v,m_v))_a + \beta\langle \psi_a, n\rangle$, as $(K\Theta(0,n))_a = \langle K\delta_a, \Theta(0,n)\rangle = \langle \psi_a, n\rangle$. Hence $\mathrm{Cov}(F_a, F_{a'}) = \beta\langle \psi_a, K_0^+ \psi_{a'}\rangle = \beta(6\delta_{ee'} - \Gamma_{ee'})$. Without the skeleton the law of A is the centred Gaussian law with covariance $\beta^{-1} K^+$, and then

$$\mathrm{Cov}\big(\beta(KA)_a, \beta(KA)_{a''}\big) = \beta\langle \delta_a, KK^+ K\delta_{a''}\rangle = \beta K_{aa''},$$

where $K_{aa''}$ is a signed count of the plaquettes containing a and a''.

(ii) The minimiser satisfies $(K\Theta(v,m_v))_{E_0'} = 0$, and $d^{\mathrm{T}} \mathscr{J} = d^{\mathrm{T}} d^{\mathrm{T}} d\Theta = 0$. At an inner point s of a path of direction ν the links with an end point at s are the two links of that path and six links of E_0': a link at s of direction $\rho \neq \nu$ lies on a path only if s lies on a line of direction ρ through block corners, whose points have their ν-coordinate divisible by b, which that of s is not. So $d^{\mathrm{T}} \mathscr{J}(s) = 0$ says that the values of $\mathscr{J}$ on the two path links at s agree, and $\mathscr{J}$ is constant on each path. At a block corner by the eight links are the first links of the paths $\pi(y,\nu)$ and the last links of the paths $\pi(y - e_\nu, \nu)$, and $d^{\mathrm{T}} \mathscr{J}(by) = 0$ is the stated relation. Finally $\mathscr{J}_{l(e)} = 6v_e + \langle \psi_{l(e)}, m_v\rangle = (\Gamma v)_e$.

(iii) First, $\langle v, \Gamma v\rangle$ is the minimum of $|dA|^2$ over all $A \in \mathbb{R}^{E'}$ whose sum along each path $\pi(e)$ is v_e. Every $\Theta(v,m)$ has these sums. Conversely, given such A, let $a_0, \dots, a_{b-1}$ be the links of $\pi(y,\nu)$ in order, from s_i to s_{i+1}, and let $\phi(s_t) = -\sum_{i<t} A_{a_i}$ for $t \leq k_b$, $\phi(s_t) = \sum_{i \geq t} A_{a_i}$ for $t > k_b$, and $\phi = 0$ at the sites on no path. This defines ϕ consistently, with $\phi = 0$ at the block corners, and $A + d\phi$ vanishes on a_i for $i \neq k_b$ and equals v_e on $a_{k_b} = l(e)$; so $A + d\phi = \Theta(v,m)$ for some m, and $|d(A+d\phi)| = |dA|$. For $v = \delta_e$ take $A = b^{-1} \sum_{a \in \pi(e)} \delta_a$: collinear links lie in no common plaquette, so the $6b$ plaquettes that contain a link of $\pi(e)$ are distinct and each has $(dA)_p = \pm 1/b$, and $|dA|^2 = 6/b$. For the lower bound let $\mathscr{P}$ be a coarse plaquette with e on its boundary and Φ the indicator of the b^2 plaquettes of the square of side b under it, which all lie in one plane. The interior links of the square cancel in $\langle dA, \Phi\rangle = \sum_p (dA)_p$, which is the signed sum of the path sums of the four coarse links of $\mathscr{P}$, that is $\pm v_e = \pm 1$. So $1 \leq |dA|\,|\Phi| = b|dA|$. As Γ is positive semidefinite, $|\Gamma_{ee'}| \leq (\Gamma_{ee}\Gamma_{e'e'})^{1/2}$.

(iv) Apply (ii) to $v = \delta_e$: the column $(\Gamma_{e'e})_{e'}$ is co-closed, and the relation at the end point $y + e_\nu$ of $e = (y,\nu)$ writes Γ_{ee} as a signed sum of the $\Gamma_{e'e}$, $e' \in S_+(e)$; likewise at y. The two sets are disjoint since $L \geq 3$, and Γ is symmetric. □

By (i), the decorrelation of forces asked for by (FC) is, in this model, the locality of the effective quadratic form of the straight-path map, and the decay of a free-field force covariance plays no role, since without the skeleton the forces are correlated only between partners. Part (i) and (iii) also exhibit the cancellation of Corollary 149(iii): the force variance on a mark is $6\beta(1 - \Gamma_{ee}/6) \geq 6\beta(1 - 1/b)$, equal to the mean curvature 6β up to $\beta\Gamma_{ee} \leq 6\beta/b$, because the coarse variable is spread over the b links of its path. Part (iv) is the analogue of Theorem 140 and of Lemma 147: gauge invariance forces the row sums of the mixed derivatives to be at least twice the diagonal.

The model is invariant under the translations by $b\mathbb{Z}^4$, which gives the dependence on the torus. For $\kappa \in \mathbb{R}^4$ and a set X of sites or links of the fine lattice $\mathbb{Z}^4$ invariant under $b\mathbb{Z}^4$, let $\mathscr{H}_\kappa(X)$ be the space of complex functions on X with $f(x + bz) = e^{i\kappa\cdot z} f(x)$ for $z \in \mathbb{Z}^4$, with the inner product summed over one fundamental domain.

Proposition 153 (the Gaussian model on all tori)**.** *Fix* $b \geq 3$. *There are* 4×4 *matrices* $\Gamma_b^\infty(w)$, $w \in \mathbb{Z}^4$, *and constants* $M_b, \kappa_b > 0$ *with* $\|\Gamma_b^\infty(w)\| \leq M_b e^{-\kappa_b |w|_1}$, *such that for every* $L \geq 3$ *the effective quadratic form on* Λ_L *is*

$$\Gamma_{(y,\mu),(y',\nu)} = \sum_{z \in \mathbb{Z}^4} \Gamma_b^\infty(y' - y + Lz)_{\mu\nu}.$$

In particular, for every $L \geq 3$ *and every coarse link* $e = (y,\mu)$,

$$\sum_{e' \neq e} |\Gamma_{ee'}| \leq \gamma_b = \max_\mu \sum_{(w,\nu) \neq (0,\mu)} \big|\Gamma_b^\infty(w)_{\mu\nu}\big| < \infty,$$

and $1/b^2 \leq \Gamma_b^\infty(0)_{\mu\mu} \leq 6/b$ *and* $\gamma_b \geq 2\min_\mu \Gamma_b^\infty(0)_{\mu\mu}$.

Proof. Gauge fixing. Let B map a function ϕ on the sites of Λ_{bL} off Σ to $d\phi|_{E_0'}$, ϕ extended by 0 on Σ, so that $\mathscr{N}$ is the range of B, and put $\mathscr{A} = K_0 + BB^{\mathrm{T}}$. If $\langle m, \mathscr{A} m\rangle = 0$, then $m \in \ker K_0 = \mathscr{N}$ and $B^{\mathrm{T}} m = 0$, so m lies in the range of B and in its orthogonal complement, and $m = 0$. Since K_0 maps into $\mathscr{N}^\perp$ and vanishes on $\mathscr{N}$, while BB^{T} maps into $\mathscr{N}$ and vanishes on $\mathscr{N}^\perp$, $\mathscr{A}^{-1}\psi = K_0^+ \psi$ for $\psi \in \mathscr{N}^\perp$, and by the proof of Proposition 152, $\Gamma = 6I - \Psi^{\mathrm{T}} \mathscr{A}^{-1} \Psi$.

Quasi-periodic functions.[30] The sets Σ, P, $l(E)$ and E_0', lifted to $\mathbb{Z}^4$, are invariant under $b\mathbb{Z}^4$, and d, K, the restrictions and B commute with these translations and have finite range. They map the spaces $\mathscr{H}_\kappa$ into one another, the adjoint of B on them is the restriction of B^{T}, and in bases indexed by a fundamental domain their matrices $K_0(\kappa)$, $B(\kappa)$, $\Psi(\kappa)$ have entries that are trigonometric polynomials in κ. The space of complex functions on the links of Λ_{bL} is the direct sum of the $\mathscr{H}_\kappa$ with $\kappa \in (2\pi/L)\mathbb{Z}^4$ modulo $2\pi\mathbb{Z}^4$, and the same holds for each invariant subset.

Positivity for every κ. Let $\mathscr{A}(\kappa) = K_0(\kappa) + B(\kappa)B(\kappa)^*$ on $\mathscr{H}_\kappa(E_0')$, and suppose that $\langle f, \mathscr{A}(\kappa) f\rangle = 0$. Then the form ω

[30] This is the Floquet–Bloch decomposition with respect to the translations by $b\mathbb{Z}^4$; the exponential decay of Γ_b^∞ below is the Paley–Wiener argument for a matrix symbol that is holomorphic and bounded in a strip.

equal to f on E_0' and to 0 on P is closed on $\mathbb{Z}^4$ and $B(\kappa)^* f = 0$. As in the proof of Lemma 151, $\omega = d\phi$ with $\phi = 0$ on the lift of Σ; for $z \in \mathbb{Z}^4$ the function $\phi(x+bz) - e^{i\kappa\cdot z}\phi(x)$ has zero differential and vanishes on the lift of Σ, so $\phi \in \mathscr{H}_\kappa$ and $f = B(\kappa)\phi$. Thus f lies in the range of $B(\kappa)$ and in the kernel of its adjoint, and $f = 0$. So $\mathscr{A}(\kappa)$ is positive definite for every real κ; its least eigenvalue is continuous and $2\pi\mathbb{Z}^4$-periodic, hence at least some $\lambda_b > 0$.

Analyticity and the torus. The entries of $\mathscr{A}(\kappa)$ are finite sums of terms $e^{\pm i\kappa_\rho}$, so there is $\kappa_b > 0$ with $\|\mathscr{A}(\kappa) - \mathscr{A}(\operatorname{Re}\kappa)\| \le \lambda_b/2$ for $|\operatorname{Im}\kappa_\rho| \le \kappa_b$, $\rho = 0,\dots,3$; there $\|\mathscr{A}(\kappa)f\| \ge \frac{\lambda_b}{2}\|f\|$, and

$$\hat\Gamma(\kappa) = 6I - \Psi(\kappa)^*\mathscr{A}(\kappa)^{-1}\Psi(\kappa),$$

with $\Psi(\kappa)^*$ continued as the transpose of $\Psi(-\bar\kappa)$ conjugated, is holomorphic, bounded and periodic. On Λ_{bL}, writing $\Gamma_{(y,\mu),(y',\nu)} = g_{\mu\nu}(y'-y)$ by translation invariance, Γ acts on $f \in \mathscr{H}_\kappa(l(E))$, $f(l(y,\nu)) = e^{i\kappa\cdot y}f_\nu$, by $(\Gamma f)(l(y,\mu)) = e^{i\kappa\cdot y}\sum_\nu \hat g_{\mu\nu}(\kappa)f_\nu$ with $\hat g_{\mu\nu}(\kappa) = \sum_{w\in(\mathbb{Z}/L\mathbb{Z})^4} g_{\mu\nu}(w)e^{i\kappa\cdot w}$, and by the gauge fixing applied on each $\mathscr{H}_\kappa$, $\hat g(\kappa) = \hat\Gamma(\kappa)$ for $\kappa \in (2\pi/L)\mathbb{Z}^4$. Let $\Gamma_b^\infty(w) = (2\pi)^{-4}\int_{[-\pi,\pi]^4}\hat\Gamma(\kappa)e^{-i\kappa\cdot w}d\kappa$. Shifting each variable κ_ρ by $\mp i\kappa_b$ according to the sign of w_ρ, which periodicity allows, gives $\|\Gamma_b^\infty(w)\| \le M_b e^{-\kappa_b|w|_1}$ with M_b the supremum of $\|\hat\Gamma\|$ on the strip. So $\hat\Gamma(\kappa) = \sum_w \Gamma_b^\infty(w)e^{i\kappa\cdot w}$ converges absolutely, and

$$\begin{aligned} g(w) &= L^{-4}\sum_\kappa \hat\Gamma(\kappa)e^{-i\kappa\cdot w} \\ &= \sum_{w'}\Gamma_b^\infty(w')L^{-4}\sum_\kappa e^{i\kappa\cdot(w'-w)} \\ &= \sum_{z\in\mathbb{Z}^4}\Gamma_b^\infty(w+Lz), \end{aligned}$$

the sums over $\kappa \in (2\pi/L)\mathbb{Z}^4$ modulo $2\pi\mathbb{Z}^4$. The pairs $(w,\nu) \ne (0,\mu)$ with w in a class modulo L fall into one term $\Gamma_{ee'}$, $e' \ne e$, or into Γ_{ee}, so the triangle inequality gives the bound by γ_b. As $L\to\infty$, $g(w) \to \Gamma_b^\infty(w)$ for each w, and the bounds and the co-closedness of Proposition 152(ii)–(iv) pass to the limit. □

So in the Gaussian model the row sums of the effective form are finite uniformly in the torus for each b, and their size is the single infinite-volume number γ_b; the proof gives no control of M_b and κ_b in b. The bound of the order of the cost $6/b$ of a thin flux tube is

$$\text{(G)} \qquad \gamma_* = \sup_{b\ge 3} b\,\gamma_b < \infty.$$

Proposition 153 gives only $b\gamma_b \ge 2/b$, and Theorem 173 shows that (G) fails: γ_b lies between absolute multiples of $(\log b)/b$ and $(1+\log b)/b$.

Corollary 154 (the Gaussian form of (FC))**.** *In the Gaussian model let $J^{\mathrm{G}}_{ee'} = \beta|\Gamma_{ee'}|$, the mixed derivatives of the induced action $\frac{\beta}{2}\langle v,\Gamma v\rangle$. For every $L \ge 3$ and every coarse link j,*

$$\frac{D^2}{2}\frac{\beta}{b^2} \le \frac{D^2}{2}\beta\,\Gamma_{jj} \le \frac{D^2}{4}\sum_{e\ne j} J^{\mathrm{G}}_{ej} \le \frac{D^2}{4}\beta\,\gamma_b.$$

The inequalities of (FC) *for J^{G} hold with $c = \frac{1}{17}$ and $R = 1$ on every torus as soon as $17D^2\beta\gamma_b \le 4$.*

Proof. The bounds are Propositions 152(iii),(iv) and 153. If $17D^2\beta\gamma_b \le 4$, the column sums are at most $\frac{D^2}{4}\beta\gamma_b \le \frac{1}{17}$, and the tails are at most the column sums, while $(1-\frac{1}{17})/16 = \frac{1}{17}$. □

The logarithmic growth of β_k along the tuning, after Proposition 143, satisfies $2^{-k}\beta_k \to 0$. So in the Gaussian model (FC) along the tuning reduces to a bound on γ_b, a statement on one explicit quadratic form for each b, which Corollary 174 proves along the tuning, uniform in the torus by Proposition 153 and in the coarse data because the covariances do not depend on v. The model is the quadratic part of the action, and for a non-abelian group it can describe the fibre law only where the fluctuations of the path-ordered products are small; the marks are b links apart, a distance at which the running coupling along the tuning is of order one, and nothing proved here relates the fibre covariances of SU(2) to Γ. Figure 20 shows the fibre, the current of Proposition 152(ii) and the bounds on Γ.

Remark 155 (the abelian group). For $G = \mathrm{U}(1)$ Lemma 126 does not apply, since $\kappa_G = 0$, while Lemmas 146 and 147, Proposition 148 and Corollary 149 do, with $D = \pi$, and the Gaussian model is the quadratic part of the Wilson action. Nothing in Propositions 152 and 153 distinguishes U(1) from a direction of the Lie algebra of SU(2). A proof of (FC) through the Gaussian model would therefore apply to U(1) in its Coulomb phase as well, at fixed β, and give (H1) there for large k. This does not contradict the massless photon of that phase [55, 64]. Condition (H1) concerns only the law of the block variables at the spacing a_0; it is independent of (H2) (Proposition 132), and the product of the Haar measures, the law towards which the straight-path maps randomise the coarse links, satisfies BP$(n,1)$ (Proposition 67(ii)). The averaged condition fails in the Coulomb phase of the Villain theory (Corollary 68, on boxes), and wherever BP(n_k,ϖ) fails along a tower of tori on which (H1) holds, Corollary 102 forces (H2) to fail. So the route of this subsection, completed, would prove (H1) and nothing about the gap: as for the averaged condition itself (Remark 69), the gap has to come from the non-abelian structure, and through Corollary 102 it has to enter (H2).

The uniformity in V in (FC) includes rough coarse data, whose plaquette variables are far from I. For them the fibre law carries a forced background, which the Gaussian model does not see.

Proposition 156 (rough coarse data)**.** *Let c, l_b, μ', ς_V and $K_{ee'}$ be as in Proposition 148.*

(i) *Let $\hat g$ be a gauge transformation of Λ_L and g the gauge transformation of Λ_{bL} equal to $\hat g(y)$ at by and at the inner points of $\pi(y,\nu)$ up to the start of the mark, to $\hat g(y+e_\nu)$ at the inner points after it, and to I at the sites on no path. Then $\varsigma_{\hat g V}$ is the image of ς_V under $m \mapsto (g\Theta^l(V,m))_{E_0'}$, and the covariances of the forces at $\hat g V$ with unit X,Y are those at V with unit vectors X',Y' conjugate to X,Y. So the supremum in $K_{ee'}$ is a supremum over the gauge classes of V.*

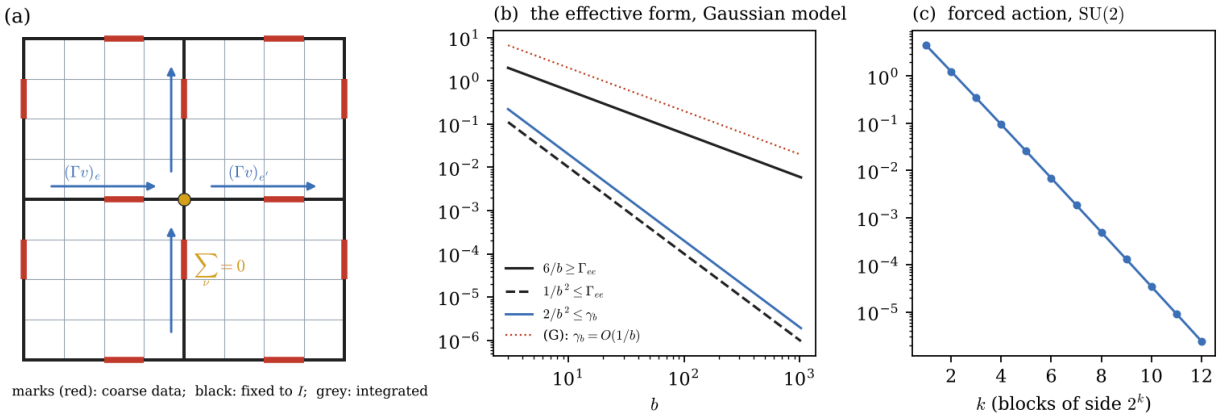


FIG. 20. The fibre route to the coarsest condition. (a) The fibre law of one step of the straight-path map with blocks of side $b = 4$, in a plane: the coarse data fix the marks (red), the other links of the paths are fixed to I (black) and the links off the paths (grey) are integrated. In the Gaussian model the minimiser carries a current that vanishes off the paths, is constant along each path with the value $(\Gamma v)_e$ and is conserved at every block corner (Proposition 152(ii)). (b) The proved bounds $1/b^2 \leq \Gamma_{ee} \leq 6/b$ and $\gamma_b \geq 2/b^2$ (Propositions 152 and 153), and the decay $\gamma_b = O(1/b)$ of (G), drawn with an arbitrary constant; (G) is false, and γ_b is of order $(\log b)/b$ (Theorem 173). (c) The lower bound $4\beta_k\delta^2/(\pi^2 b^2)$ on the action that rough coarse data of SU(2) force on the square of a coarse plaquette, for $\delta = \sqrt{2}\pi$, $b = 2^k$ and $\beta_k = 2 + \frac{11}{3\pi^2}k\log 2$, an illustration of the growth along the tuning (Proposition 156); it tends to zero and obstructs nothing.

(ii) *For every coarse plaquette $\mathscr{P}$ with plaquette variable $V_{\mathscr{P}}$, the set $Q_{\mathscr{P}}$ of the b^2 plaquettes of the square of side b under it, and every m,*

$$\sum_{p\in Q_{\mathscr{P}}} \delta\big(\Theta^l(V,m)_p, I\big) \geq \delta(V_{\mathscr{P}}, I).$$

For $G = \mathrm{SU}(2)$, $\beta \geq 0$ and $\mathscr{S}_0 = N\beta\sum_p(2 - \mathrm{Re}\,\mathrm{Tr}\,U_p)$,

$$\mathscr{S}_0(\Theta^l(V,m)) \geq \frac{4\beta}{\pi^2 b^2}\sum_{\mathscr{P}} \delta(V_{\mathscr{P}}, I)^2,$$

and hence

$$\int e^{-\mathscr{S}_0(\Theta^l(V,m))}\,\lambda_{E_0'}(dm) \leq \exp\Big(-\frac{4\beta}{\pi^2 b^2}\sum_{\mathscr{P}} \delta(V_{\mathscr{P}}, I)^2\Big).$$

Proof. (i) The values of g are consistent, since an inner point lies on one path and the corner *by* ends the paths $\pi(y - e_\nu, \nu)$, where g is $\hat{g}(y)$ after the mark. On $\pi(y,\nu)$ the links of $\Theta^l(V,m)$ before and after the mark become $\hat{g}(y)\hat{g}(y)^{-1} = I$ and $\hat{g}(y+e_\nu)\hat{g}(y+e_\nu)^{-1} = I$, and the mark becomes $\hat{g}(y)V_{(y,\nu)}\hat{g}(y+e_\nu)^{-1} = (\hat{g}V)_{(y,\nu)}$; so $g\Theta^l(V,m) = \Theta^l(\hat{g}V, m^g)$ with $m^g = (g\Theta^l(V,m))_{E_0'}$, a translation of each link of E_0' on both sides by elements that do not depend on m, which preserves $\lambda_{E_0'}$. As $\mathscr{S}$ is gauge invariant, the density of $\varsigma_{\hat{g}V}$ at m^g equals that of ς_V at m. For the mark a from s to s', $g(s)U_a g(s')^{-1}e^{tX} = g(s)U_a e^{tX'}g(s')^{-1}$ with $X' = g(s')^{-1}Xg(s')$, so $D^a_X\mathscr{S}(gU) = D^a_{X'}\mathscr{S}(U)$, with $g(s')$ independent of m.

(ii) The boundary of the square of $\mathscr{P}$, traversed from its corner, consists of the four paths of the coarse links of $\mathscr{P}$, whose products under $\Theta^l(V,m)$ are the coarse links, the unmarked links being I; so its holonomy is $V_{\mathscr{P}}$ up to orientation. By Lemma 76 it is an ordered product of conjugates of the $U_p^{\pm 1}$, $p \in Q_{\mathscr{P}}$, and for a bi-invariant metric $\delta(XY,I) \leq \delta(X,I) + \delta(Y,I)$, $\delta(hXh^{-1},I) = \delta(X^{-1},I) = \delta(X,I)$. For SU(2), with θ the angle of U, $2 - \mathrm{Re}\,\mathrm{Tr}\,U = 4\sin^2(\theta/2) \geq 4\theta^2/\pi^2 = 2\delta(U,I)^2/\pi^2$, so $\mathscr{S}_0 \geq \frac{4\beta}{\pi^2}\sum_p \delta(U_p,I)^2$; by the Cauchy–Schwarz inequality the sum over $Q_{\mathscr{P}}$ is at least $\delta(V_{\mathscr{P}},I)^2/b^2$, and distinct coarse plaquettes have disjoint squares. The integral is at most the supremum of the integrand. □

Along the tuning $\beta_k 4^{-k} \to 0$, so the background forced on the squares of the coarse plaquettes has vanishing action per coarse plaquette, and part (ii) gives no obstruction to the uniformity in V; part (i) reduces it to the gauge classes, which still contain every assignment of conjugacy classes to the coarse plaquettes. In the Gaussian model the covariances do not depend on v at all (Proposition 152(i)). The dependence of $K_{ee'}$ on V for SU(2) is a non-linear effect, and no bound on it is proved here.

Remark 157 (what is proved about the fibre route). Proved: every measure with a density e^{-H}, $H \in C^2$, on the links of a torus, for every closed connected $G \subset \mathrm{U}(N)$ including U(1), has single-link coefficients at most $D^2/4$ times the bounds on the mixed derivatives of H, with no condition on the single-link Hessian (Lemma 146), and gauge invariance bounds the single-link Hessian by the sum of the mixed-derivative bounds into the seven links at an end point, at both end points for abelian G (Lemma 147); for straight-path maps with blocks of side $b \geq 3$ and the middle marking the marks of distinct coarse links are not partners, the mixed derivatives of the induced action are minus the covariances of the forces on the marks in the fibre law, and the coefficients of the law of the block variables are at most $D^2/4$ times their suprema over the coarse data (Proposition 148); the condition (FC) on these covariances gives (H1)

with $\varpi_0 = \frac{1}{2}$, and it forces the force variance on every mark to equal its mean curvature up to $4c/D^2$, uniformly in the coarse data (Corollary 149); in the strong-coupling range the covariances decay exponentially in the distance of the marks, which covers the range of Corollary 138(iii), and the fibre law itself fails the Dobrushin condition for SU(2) at $\beta_W \geq \frac{8}{3}$ (Remark 150); in the Gaussian model the force covariances are $-\beta$ times the effective quadratic form of the straight-path map, independent of the coarse data, without the skeleton they vanish beyond partners, the form is a co-closed current on the coarse lattice with $1/b^2 \leq \Gamma_{ee} \leq 6/b$ and row sums at least $2\Gamma_{ee}$ (Proposition 152), and for each b it decays exponentially, with row sums bounded on every torus by one infinite-volume number γ_b (Proposition 153); the Gaussian form of (FC) holds on every torus when $17D^2\beta\gamma_b \leq 4$ (Corollary 154); a proof of (FC) through the Gaussian model would apply to U(1), which is consistent with its massless phase because (H1) carries no information on the gap (Remark 155); the covariances for rough coarse data depend only on their gauge class, and the background they force on the faces has action at least $4\beta\delta(V_{\mathscr{P}},I)^2/(\pi^2 b^2)$ per coarse plaquette for SU(2), which vanishes along the tuning (Proposition 156).

Not proved: (FC) for any k with β_k outside the strong-coupling range, and hence (H1) along the tuning; any control of the decay rate κ_b in b; any comparison of the fibre covariances of SU(2) with the Gaussian model at the separation of the marks; any bound on the covariances uniform in rough coarse data for SU(2); (H1) for U(1). The cancellation of Corollary 149(iii) holds for SU(2) along every tuning with $\beta_k = O(k)$ (Corollary 162). Neither BP along the tuning nor the mass gap is proved here.

Q. Smoothing along the paths

A block variable of the straight-path map is the product of the b fine links of its path, and given the links off the path these b links are independent. A product of many independent random elements of a compact group spreads out towards the Haar measure, and this subsection makes that quantitative for SU(2) and follows it through to the condition at the coarsest scale. Convolution with a one-link Wilson density contracts every nontrivial representation by the factor $q_W = 4\beta_W/(4\beta_W+1)$, so a product of m independent one-link variables has a density within $48(1+6\beta_W)^2 q_W^{m-2}$ of 1, together with its first two derivatives (Theorem 160); this extends to every representation the bound on the first moment in Proposition 143(ii). The single-link conditional laws of the block variables obey the same bounds, and along every tuning with $\beta_k = O(k)$ the single-link curvature of the induced action tends to zero uniformly in the torus and the coarse data, which is the cancellation asked for in Corollary 149(iii) (Theorem 161, Corollary 162). The interior links of all the paths lie in no common plaquette, so the law of the block variables is a mixture of product laws, and its mixed derivatives are covariances of the scores of the interior products under one posterior law, each at most K_b^2 with $K_b \to 0$ (Theorem 164). A column sum of single-link coefficients at most $\frac{1}{3}$ gives BP$(n, \frac{1}{2})$ for every cube side, with no condition on the tails (Corollary 165), and together these give BP$(n, \frac{1}{2})$ after $k \geq 12$ blocking steps with $\beta_k \leq k+2$ on every coarse torus of side at most $e^{2^k/(17k)}$ (Corollary 166). Two standard routes to a bound on every torus fail along the tuning, Dobrushin's condition for the posterior law and the Kotecký–Preiss condition for the unresummed cube surface (Proposition 167), and a bound on the sum of the posterior covariances, uniform in the torus, gives (H1) (Proposition 168).

Throughout, $G = \mathrm{SU}(2)$, with δ, $D = \sqrt{2}\pi$ and $\kappa_G = 1$ as in Corollary 141, λ is the Haar probability measure and $\theta(g) \in [0,\pi]$ is the angle of g, so that $\frac{1}{2}\mathrm{Re}\,\mathrm{Tr}\, g = \cos\theta(g)$ and θ has the density $\frac{2}{\pi}\sin^2\theta$ under λ. For a function F on G and a unit vector $X \in \mathfrak{g}$ write $D_X F(g) = \frac{d}{dt}F(ge^{tX})|_{t=0}$. For $B \geq 0$ let

$$f_B(g) = \frac{e^{B\cos\theta(g)}}{Z(B)},$$

$$Z(B) = \frac{2}{\pi}\int_0^\pi e^{B\cos\theta}\sin^2\theta\, d\theta = I_0(B) - I_2(B) = \frac{2I_1(B)}{B},$$

with $Z(0) = 1$, where I_ν is the modified Bessel function; the last two identities are the integral representation [94, (10.32.3)] and the recurrence [94, (10.29.1)]. For $\beta_W \geq 0$ and $m \geq 2$ put $B_W = 6\beta_W$,

$$q_W = \frac{4\beta_W}{4\beta_W+1}, \qquad \eta_m = 48(1+B_W)^2 q_W^{m-2},$$

and, when $\eta_m < 1$,

$$h_m = \frac{(B_W^2+B_W)\eta_m}{1-\eta_m} + \frac{B_W^2\eta_m^2}{(1-\eta_m)^2}.$$

At $B = 6\beta_W$ one has $2B/(2B+3) = q_W$.

Lemma 158 (every mode at once)**.** *Let $B > 0$. For every $d \geq 1$, convolution $h \mapsto h * f_B$ acts on the isotypic component of $L^2(G)$ of the irreducible representation of dimension d as multiplication by*

$$a_d(B) = \frac{I_d(B)}{I_1(B)},$$

and for $d \geq 2$

$$0 < a_d(B) \leq a_2(B) < \frac{2B}{2B+3}.$$

Consequently $\|h * f_B\|_2 \leq \frac{2B}{2B+3}\|h\|_2$ for every $h \in L^2(G)$ with $\int h\, d\lambda = 0$.

Proof. By the Peter–Weyl theorem $L^2(G)$ is the orthogonal sum of the spans of the matrix coefficients of the irreducible representations, one for each dimension $d \geq 1$ [30, Ch. III]. Since f_B is a class function, Schur's lemma shows that convolution with it acts on the span for dimension d as the scalar $\frac{1}{d}\int \chi_d f_B\, d\lambda$, where $\chi_d(g) = \sin(d\theta)/\sin\theta$ is the character. From $2\sin(d\theta)\sin\theta = \cos((d-1)\theta) - \cos((d+1)\theta)$ and [94, (10.32.3), (10.29.1)],

$$\frac{1}{d}\int \chi_d f_B\, d\lambda = \frac{I_{d-1}(B) - I_{d+1}(B)}{d\,Z(B)} = \frac{2dI_d(B)/B}{2dI_1(B)/B} = \frac{I_d(B)}{I_1(B)}.$$

For fixed $B>0$, $I_\nu(B)$ is positive and decreasing in $\nu>0$ [94, §10.37], so $0<a_d(B)\le a_2(B)$ for $d\ge 2$. Since $\chi_2=2\cos\theta$, $A=a_2(B)$ is the mean of $\cos\theta$ under $f_B\lambda$. Integrating the derivative of $\sin^3\theta\, e^{B\cos\theta}$ over $[0,\pi]$ gives $3\,\mathbb{E}\cos\theta = B\,\mathbb{E}\sin^2\theta$ for this law, and $\mathbb{E}\sin^2\theta = 1-\mathbb{E}\cos^2\theta \le 1-A^2$, so $BA^2+3A-B\le 0$. The polynomial Bx^2+3x-B increases on $x\ge 0$ and equals $9B/(2B+3)^2>0$ at $x=2B/(2B+3)$, hence $A<2B/(2B+3)$. A function of mean zero has no component for $d=1$, which gives the last statement. □

Lemma 159 (one link). *For every $B\ge 0$, $\|f_B\|_\infty \le 48(1+B)^{3/2}$, hence $\|f_B\|_2^2\le 48(1+B)^{3/2}$, and for every unit $X\in\mathfrak{g}$*

$$|D_X f_B|\le B f_B, \qquad |D_X D_X f_B|\le (B^2+B) f_B.$$

Proof. Let $t=(1+B)^{-1/2}\le 1$. On $[0,t]$ one has $\sin\theta\ge 2\theta/\pi$ and $\cos\theta\ge 1-\theta^2/2$, so

$$\begin{aligned} Z(B) &\ge \frac{2}{\pi}\int_0^t e^{B-B\theta^2/2}\,\frac{4\theta^2}{\pi^2}\,d\theta \ge \frac{8}{3\pi^3}\,e^{B-Bt^2/2}t^3 \\ &\ge \frac{8}{3\pi^3}\,e^{-1/2}\,\frac{e^B}{(1+B)^{3/2}}, \end{aligned}$$

and $\|f_B\|_\infty = e^B/Z(B) \le \frac{3\pi^3}{8}e^{1/2}(1+B)^{3/2} < 48(1+B)^{3/2}$, since $\pi<4$ and $e^{1/2}<2$. As f_B is a probability density, $\|f_B\|_2^2\le\|f_B\|_\infty$. Let $u(g)=\frac12\operatorname{Re}\operatorname{Tr} g$. A unit $X\in\mathfrak{g}$ satisfies $X^2=-\frac12 I$, so $D_X u(g)=\frac12\operatorname{Re}\operatorname{Tr}(gX)$ and $D_X D_X u(g) = -\frac14\operatorname{Re}\operatorname{Tr} g$; since $|\operatorname{Re}\operatorname{Tr}(gX)|\le\|g\|_{\mathrm{HS}}\|X\|_{\mathrm{HS}}=\sqrt2$, we get $|D_X u|\le 1$ and $|D_X D_X u|\le\frac12$. The bounds follow from $D_X f_B = B(D_X u)f_B$ and $D_X D_X f_B = (B D_X D_X u + B^2(D_X u)^2)f_B$. □

Theorem 160 (products of one-link variables). *Let $m\ge 2$, $B\ge 0$ and $A,C\in G$, and let $Y_1,\dots,Y_m$ be independent G-valued random variables such that Y_i has the density $g\mapsto f_{B_i}(gW_i)$ with respect to λ, for some $W_i\in G$ and $0\le B_i\le B$. Let p be the density of $AY_1\cdots Y_mC$ with respect to λ and $\eta = 48(1+B)^2(\frac{2B}{2B+3})^{m-2}$. Then for every $g\in G$ and every unit $X\in\mathfrak{g}$*

$$|p(g)-1|\le\eta, \qquad |D_X p(g)|\le B\eta,$$
$$|D_X D_X p(g)|\le (B^2+B)\eta.$$

Proof. If some $B_i=0$, then Y_i is Haar distributed, so is the product, and $p=1$. Let all $B_i>0$, and put $Z_i=Y_iW_i$, with the class density $f_i=f_{B_i}$. Since $W^{-1}Z_{i+1}=(W^{-1}Z_{i+1}W)W^{-1}$ and conjugation preserves the law of Z_{i+1}, moving the W_i^{-1} to the right one at a time gives $AY_1\cdots Y_mC = AZ_1'\cdots Z_m'C'$ with $C'=W_1^{-1}\cdots W_m^{-1}C$ and independent Z_i' with the densities f_i. Hence $p(g)=F(A^{-1}gC'^{-1})$ with $F=f_1*\cdots*f_m$. A two-sided translation is an isometry and replaces D_X by $D_{X'}$ with $X'=\operatorname{Ad}(C')X$, again a unit vector, so it suffices to bound F.

A class function commutes under convolution with every integrable function, and $D_X(k*f)=k*D_X f$. Since $1*f=1$ for a probability density f,

$$F-1=\big((f_1-1)*f_3*\cdots*f_m\big)*f_2,$$
$$D_X F=\big((D_X f_1)*f_3*\cdots*f_m\big)*f_2,$$

and likewise for $D_X D_X F$. The functions f_1-1, $D_X f_1$ and $D_X D_X f_1$ have mean zero, convolution with a probability density keeps the mean zero, and by Lemma 158 each of the $m-2$ convolutions with $f_3,\dots,f_m$ multiplies the L^2 norm by at most $2B/(2B+3)$, since $2B_i/(2B_i+3)\le 2B/(2B+3)$. With $\|k*f\|_\infty\le\|k\|_2\|f\|_2$, $\|f_1-1\|_2\le\|f_1\|_2$ and Lemma 159,

$$\begin{aligned}\|F-1\|_\infty &\le \Big(\frac{2B}{2B+3}\Big)^{m-2}\|f_1\|_2\|f_2\|_2 \\ &\le 48(1+B)^{3/2}\Big(\frac{2B}{2B+3}\Big)^{m-2}\le\eta,\end{aligned}$$

and the same argument gives the other two bounds, since $\|D_X f_1\|_2\le B\|f_1\|_2$ and $\|D_X D_X f_1\|_2\le(B^2+B)\|f_1\|_2$ by Lemma 159. □

Theorem 161 (the law of one block variable). *Let $b\ge 2$, $L\ge 3$ and $\beta_W\ge 0$, let μ' be the $\mathrm{SU}(2)$ Wilson measure of Λ_{bL}, with density proportional to $\exp(\frac{\beta_W}{2}\sum_p\operatorname{Re}\operatorname{Tr}U_p)$, let c be the straight-path map to Λ_L and $\mu^c=c_*\mu'$, with the induced action H^c of Lemma 128(i). For every coarse link e and every V the conditional law $\gamma^{\mu^c}_{\{e\}}(\cdot\mid V)$ has a density $r_e(\cdot\mid V)$ with respect to λ, and for every unit $X\in\mathfrak{g}$*

$$|r_e-1|\le\eta_b, \qquad |D_X r_e|\le B_W\eta_b,$$
$$|D_X D_X r_e|\le (B_W^2+B_W)\eta_b.$$

If $\eta_b<1$, then

$$\Big|\frac{d^2}{dt^2}H^c(V^{e,tX})\Big|_0\Big|\le h_b$$

for every e, V and unit X. Along every sequence $b_k=2^k$, $\beta_k\le C(1+k)$, the numbers η_{b_k} and h_{b_k} tend to zero faster than every power of 2^{-k}, uniformly in L and V.

Proof. Condition on the configuration W of the fine links off the path $\pi(e)$. Two links of a plaquette are either parallel and displaced by a unit vector transverse to them, or orthogonal; two links of $\pi(e)$ are collinear, so no plaquette contains two of them. Given W the b links of $\pi(e)$ are therefore independent, and each has the density proportional to $\exp(\frac{\beta_W}{2}\operatorname{Re}\operatorname{Tr}(US))$, where S is the sum of its six staples, a function of W. A sum of elements of $\mathrm{SU}(2)$ is a real multiple of an element of $\mathrm{SU}(2)$, so $S=sW_S$ with $0\le s\le 6$ and $W_S\in G$, and the density is $f_{s\beta_W}(UW_S)$. By Theorem 160 with $m=b$ and $B=B_W$, the density $p_e(\cdot\mid W)$ of c_e given W obeys the three bounds with η_b. The other coarse links are functions $\phi(W)$ of W, so for bounded measurable F

$$\mu^c(F)=\int\mu'(dW)\int F(g,\phi(W))\,p_e(g\mid W)\,\lambda(dg),$$

and the conditional density of V_e given $V_{E\setminus\{e\}}$ is the mean of $p_e(g\mid W)$ under the conditional law of W given $\phi(W)=V_{E\setminus\{e\}}$, a mixture whose weights do not depend on g. The three bounds pass to the mixture for μ^c-almost every V. By Lemma 128(i) the density of μ^c is smooth and positive, so r_e and its derivatives in g are continuous in V and the bounds hold

for every V. With the other coarse links fixed, $H^c(V^{e,tX}) = -\log r_e(V_e e^{tX} \mid V)$ up to a constant, and

$$\frac{d^2}{dt^2}\bigl(-\log r_e\bigr) = -\frac{D_X D_X r_e}{r_e} + \frac{(D_X r_e)^2}{r_e^2}$$

is at most h_b in absolute value. Finally $\log(1+x) \geq x/(1+x)$ gives $q_W^{b-2} \leq \exp(-(b-2)/(4\beta_W+1))$, and for $b = 2^k$, $\beta_W \leq C(1+k)$ the exponent $-(2^k-2)/(4C(1+k)+1)$ beats every multiple of k. □

Corollary 162 (the cancellation in the fibre law)**.** *In the setting of Theorem 161 let $b \geq 3$, $\eta_b < 1$, and let ς_V and $a = l_b(e)$ be as in Proposition 148. Then for every coarse link e, every V and every unit X*

$$\bigl|\varsigma_V(D_X^a D_X^a \mathscr{S}) - \mathrm{Var}_{\varsigma_V}(D_X^a \mathscr{S})\bigr| \leq h_b,$$

so along every tuning with $\beta_k = O(k)$ the variance of the force on each mark equals its mean curvature up to an error that tends to zero, uniformly in the torus and the coarse data, and the bound of Corollary 149(iii) *holds for all large k whatever $c > 0$.*

Proof. By Proposition 144 the left side is the second derivative of H^c along $V^{e,tX}$; apply Theorem 161. □

So the single-link part of the effective-action problem is settled for SU(2) along the tuning: both terms are of order β_k, and their difference is smaller than any power of the block size. What remains is the sum of the mixed derivatives over distant coarse links. To reach them, the interiors of all the paths are integrated at once. Let $b \geq 4$, let $\mathscr{I}$ be the set of fine links $(by + ie_\nu, \nu)$ with (y,ν) a coarse link and $1 \leq i \leq b-2$, the interiors of the paths, and let $\mathscr{R}$ be the set of all other fine links, which contains the first and the last link of every path.

Lemma 163 (the interiors)**.** *No plaquette of Λ_{bL} contains two links of $\mathscr{I}$. Consequently, given $U_{\mathscr{R}} = W$, the links of $\mathscr{I}$ are independent under μ', and the coarse links are independent with densities $p_e(\cdot \mid W)$ that obey the three bounds of Theorem 160 with η_{b-2} and $B = B_W$. The density of μ^c with respect to λ_E is*

$$\mathscr{D}(V) = \int \prod_{e \in E} p_e(V_e \mid W)\, \zeta(dW), \tag{19}$$

where ζ is the law of $U_{\mathscr{R}}$ under μ'.

Proof. Two parallel links of a plaquette have base points that differ by a unit vector transverse to them. For two links of $\mathscr{I}$ of direction ν the transverse coordinates of the base points are multiples of b, and so is their difference modulo bL, while ± 1 is not, since $b \geq 2$. A plaquette in the plane of $\mu \neq \nu$ with base point z has the links (z,μ), $(z+e_\mu,\nu)$, $(z+e_\nu,\mu)$ and (z,ν), so a μ-link with base point x and a ν-link with base point y in it satisfy $x - y \in \{0, -e_\mu, e_\nu, e_\nu - e_\mu\}$, and $x_\mu - y_\mu \in \{0,-1\}$. For links of $\mathscr{I}$, $x_\mu \equiv i \pmod b$ with $1 \leq i \leq b-2$ and $y_\mu \equiv 0 \pmod b$, so $x_\mu - y_\mu$ is congruent neither to 0 nor to -1 modulo b. Hence every Wilson factor contains at most one link of $\mathscr{I}$, and given W the density of $U_{\mathscr{I}}$ is a product. A link of $\mathscr{I}$ has the density $f_{s\beta_W}(UW_S)$ as in the proof of Theorem 161, with staples in W, and the coarse link e is $RY_1 \cdots Y_{b-2}R'$ with its first and last links R, R' in W and its interior links Y_i; Theorem 160 with $m = b-2$ gives the bounds. Distinct coarse links have disjoint interiors, which gives the independence and (19). □

Theorem 164 (mixed derivatives as posterior covariances)**.** *Let $b \geq 4$ and $\eta_{b-2} < 1$, put $K_b = B_W \eta_{b-2}/(1-\eta_{b-2})$, and for $V \in G^E$ let*

$$\pi_V(dW) = \mathscr{D}(V)^{-1} \prod_{e \in E} p_e(V_e \mid W)\, \zeta(dW),$$
$$a_e^X(W, V_e) = D_X \log p_e(V_e \mid W).$$

Then for coarse links $e \neq j$ and unit $X, Y \in \mathfrak{g}$

$$\frac{\partial^2}{\partial s\, \partial t} H^c\bigl((V^{e,sX})^{j,tY}\bigr)\Big|_0 = -\mathrm{Cov}_{\pi_V}\bigl(a_e^X, a_j^Y\bigr),$$

so $J_{ej}(H^c) \leq K_b^2$, and $|\frac{d^2}{dt^2} H^c(V^{e,tX})|_0| \leq h_{b-2}$ for every e, V and unit X.

Proof. By Lemma 163, $p_e \geq 1 - \eta_{b-2}$ and $|a_e^X| \leq K_b$. The densities p_e are smooth in V_e, with derivatives bounded uniformly in W, so (19) may be differentiated under the integral. With $H^c = -\log \mathscr{D}$ up to a constant, $\frac{d}{ds} H^c(V^{e,sX})|_0 = -\mathbb{E}_{\pi_V} a_e^X$. The score a_e^X does not depend on V_j, and differentiating π_V in V_j along Y multiplies it by $a_j^Y - \mathbb{E}_{\pi_V} a_j^Y$; this gives the covariance, which is at most K_b^2 in absolute value by the Cauchy–Schwarz inequality. Given the other coarse links, the law of V_e has the density proportional to $\int p_e(V_e \mid W) \prod_{e' \neq e} p_{e'}(V_{e'} \mid W)\, \zeta(dW)$, a mixture of the $p_e(\cdot \mid W)$ with weights independent of V_e, and the last statement follows as in the proof of Theorem 161. □

The bound on each mixed derivative carries no factor that decays with the distance between e and j; the posterior π_V couples all the paths through the fine links of $\mathscr{R}$. A bound on the sum of the coefficients still gives the cube bound for every cube side, and with no condition on the tails.

Corollary 165 (a column sum below one third)**.** *Let ρ be a gauge-invariant probability measure on G^E, E the links of Λ_L, with $c_\rho = \max_j \sum_{e \neq j} C_{ej}^\rho \leq \frac{1}{3}$. Then $\mathrm{BP}_\rho(n, \frac{1}{2})$ holds for every $n \geq 2$ with $L \geq 2n+4$ even. In particular, in Corollaries 129 and 149 the condition on the tails and the lower bound on n_0 can be dropped when $c \leq \frac{1}{3}$.*

Proof. A link with base point b' lies in V_x exactly when $b' - x \in [0,n)^4$ modulo L, so it lies in exactly n^4 cubes. By Lemma 124,

$$\sum_{x:\, j \notin V_x} C_{V_x,j}^\rho \leq \frac{1}{1-c_\rho} \sum_{e \neq j} C_{ej}^\rho \#\{x : e \in V_x\} \leq \frac{c_\rho}{1-c_\rho}\, n^4 \leq \frac{n^4}{2},$$

and Theorem 125(i) applies with $\alpha = \frac{1}{2}$. □

Corollary 166 (tori of side up to $e^{2^k/(17k)}$)**.** *Let $k \geq 12$, $b = 2^k$, $0 \leq \beta_W \leq k+2$, $n \geq 2$ and L even with $2n+4 \leq L \leq e^{2^k/(17k)}$. Then the law μ^c of the block variables of the* SU(2) *Wilson measure of Λ_{bL} under the straight-path map satisfies* $\mathrm{BP}(n, \frac{1}{2})$.

Proof. Put $N_k = 4k+9$, $B_k = 6(k+2)$, $M_k = 2^k/(17k)$ and $s_k = 48(6k+13)^2 e^{-(2^k-4)/N_k}$. Since $q_W \le 1 - 1/N_k$ and $\log(1-x) \le -x$,

$$\eta_{b-2} = 48(1+6\beta_W)^2 q_W^{b-4} \le s_k.$$

We first show that

$$96 B_k^2 s_k^2 e^{4M_k} \le 1. \tag{20}$$

In logarithms this reads $P_k \le Q_k$, with $P_k = \log(96\cdot 48^2) + 2\log B_k + 4\log(6k+13)$ and

$$Q_k = \frac{2(2^k-4)}{N_k} - 4M_k = 2^k \frac{18k-36}{17kN_k} - \frac{8}{N_k}.$$

For $k \ge 12$ one has $18k-36 \ge 15k$ and $N_k \le 5k$, so $Q_k \ge R_k - 1$ with $R_k = 3\cdot 2^k/(17k)$. At $k=12$, $\log 221184 < 12.31$, $\log 84 < 4.44$ and $\log 85 < 4.45$ give $P_{12}+1 < 40 < 60 < R_{12}$. From k to $k+1$, R grows by $R_k(k-1)/(k+1) \ge R_k/2 \ge 30$, while P grows by $2\log\frac{k+3}{k+2} + 4\log\frac{6k+19}{6k+13} < \frac{2}{k+2} + \frac{24}{6k+13} < 1$; so $P_k + 1 < R_k$ for every $k \ge 12$, which proves (20). Moreover $\log B_k \le M_k$ for $k \ge 12$: it holds at $k=12$, where $\log 84 < 5 < 20 < M_{12}$, and from k to $k+1$ the left side grows by less than 1 and the right side by $M_k(k-1)/(k+1) \ge 10$.

By (20), $B_k s_k \le e^{-2M_k}/9$, so $B_k^2 s_k \le e^{-M_k}/9 < 10^{-9}$ and $s_k \le \frac{1}{2}$. Hence $\eta_{b-2} \le \frac{1}{2}$, $K_b \le 2B_k s_k$ since $B_W \le B_k$, and

$$h_{b-2} \le 2(B_k^2 + B_k)s_k + 4B_k^2 s_k^2 \le 5B_k^2 s_k \le \frac{1}{2}.$$

By Theorem 164 and Lemma 126 with $\kappa_G = 1$, $C_{ej}^{\mu^c} \le K_b^2/(1-h_{b-2}) \le 2K_b^2$, and the $4L^4 - 1$ links $e \ne j$ give, with $L^4 \le e^{4M_k}$,

$$3c_{\mu^c} \le \frac{3(4L^4-1)K_b^2}{1-h_{b-2}} \le 96 L^4 B_k^2 s_k^2 \le 1.$$

So $c_{\mu^c} \le \frac{1}{3}$, μ^c is gauge invariant, and Corollary 165 applies. □

The side $e^{2^k/(17k)}$ exceeds 4^k from $k=12$ on and grows faster than every power of the block size; in physical units the coarse torus has side up to $a_0 e^{2^k/(17k)}$, which tends to infinity along the tuning. The range is still not (H1). Theorem 66 passes to infinite volume at fixed k, so (H1) asks for a bound on an infinite set of tori at each k, and the entrywise bound K_b^2 summed over $4L^4$ links fails once L is large at fixed k. If the volume and the cut-off are removed together, along coarse tori of side $L_k \le e^{2^k/(17k)}$ with $L_k \to \infty$, the corollary gives the condition at the coarsest scale on every torus of the sequence. That route needs, in place of Theorem 66, a lower bound on the gap of the transfer matrix of a finite torus that is uniform in its time extent and follows from BP on that torus, together with (H2) on the same tori; neither is proved here. Figure 21 shows the interiors, the multipliers of Lemma 158 and the numbers η_{b-2}, K_b^2 and h_{b-2} along the tuning. The next proposition shows that two standard routes to a bound on every torus fail in the range that matters.

Proposition 167 (two routes that fail)**.** (i) *Let $b \ge 256$ and $\beta_W \ge \frac{8}{3}$. For every V the largest column sum of the single-link coefficients of the posterior π_V of Theorem 164 on the links of $\mathscr{R}$ exceeds one.*

(ii) *Expand the plaquette weight $e^{\frac{\beta_W}{2}\operatorname{Re}\operatorname{Tr}U} / \int e^{\frac{\beta_W}{2}\operatorname{Re}\operatorname{Tr}U} d\lambda$ in characters, and keep the fundamental representation on the six faces of an elementary cube as one polymer with its unresummed activity $z_\square$. Then $z_\square = 4a_2(\beta_W)^6$, and for $\beta_W \ge 6$ the Kotecký–Preiss condition [78] fails for this polymer.*

Proof. (i) Take a link of $\mathscr{R}$ whose base point has all four coordinates in $\{3,\dots,b-4\}$ modulo b. A link of a path has at least three coordinates of its base point divisible by b, so neither this link nor the links of its six plaquettes nor the links at its end points lie on a path. The gauge transformations at its two end points therefore fix every coarse link and every factor $p_e(V_e \mid W)$, and they preserve ζ, the image of the gauge-invariant measure μ' under a map that commutes with them; so they preserve π_V. Its plaquettes contain no link of $\mathscr{I}$, so its conditional law under π_V is the one-link Wilson law, which at $W = I$ is $f_{6\beta_W}\lambda$. The proofs of Lemma 139, Theorem 140(i) and Corollary 141 then give it a row sum of single-link coefficients under π_V larger than 1.13. Such links number $4(b-6)^4L^4$ out of at most $4b^4L^4$ links of $\mathscr{R}$, and the coefficients are nonnegative, so the mean row sum, which equals the mean column sum, exceeds $1.13\,((b-6)/b)^4 \ge 1.13\,(125/128)^4 > 1$.

(ii) The coefficient of χ_d in the normalised plaquette weight is $d\,a_d(\beta_W)$, by Lemma 158 with $B = \beta_W$. On the surface of a cube each of the twelve edges lies in two faces with opposite orientations, Haar integration over it gives a factor $\frac{1}{2}$, and contracting the indices gives a factor 2 at each of the eight vertices, so $z_\square = (2a_2)^6 2^{8-12} = 4a_2^6$. Let $r = I_3(6)/I_2(6)$. By the integral representation [94, (10.32.2)], r is the mean of t under the density proportional to $e^{6t}(1-t^2)^{3/2}$ on $[-1,1]$, and integrating the derivative of $e^{6t}(1-t^2)^{5/2}$ gives $5r = 6\mathbb{E}(1-t^2) < 6(1-r^2)$, so $(3r-2)(2r+3) < 0$ and $r < \frac{2}{3}$. The recurrence $I_1(6) - I_3(6) = \frac{2}{3}I_2(6)$ then gives $a_2(6) > \frac{3}{4}$, and a_2 increases in β_W, its derivative being the variance of $\cos\theta$. So $z_\square > 4(\frac{3}{4})^6 = \frac{729}{1024} > \frac{1}{e}$ for $\beta_W \ge 6$. The condition of [78] asks for $A_\gamma \ge 0$ with $\sum_{\gamma' \not\sim \gamma} |z_{\gamma'}| e^{A_{\gamma'}} \le A_\gamma$, and a polymer is incompatible with itself, so it requires $|z_\square| \le Ae^{-A} \le \frac{1}{e}$. □

Part (i) concerns the posterior on single fine links and leaves block methods open; part (ii) concerns the unresummed expansion and leaves resummed and multiscale expansions open. What the cube bound on every torus needs is a statement about the scores of the interior products under the posterior, and the following condition is enough.

Proposition 168 (a sufficient covariance bound)**.** *In Corollary 102 let the c_j be straight-path maps, so that $\mu_{k,L}^{(0)}$ is the law of the block variables of the straight-path map with blocks of side $b = 2^k$, $k \ge 2$, and let π_V, a_e^X, K_b and h_{b-2} be as in Theorem 164 for $\mu_{k,L}$, with $\eta_{b-2} < 1$ and $h_{b-2} < 1$. Suppose*

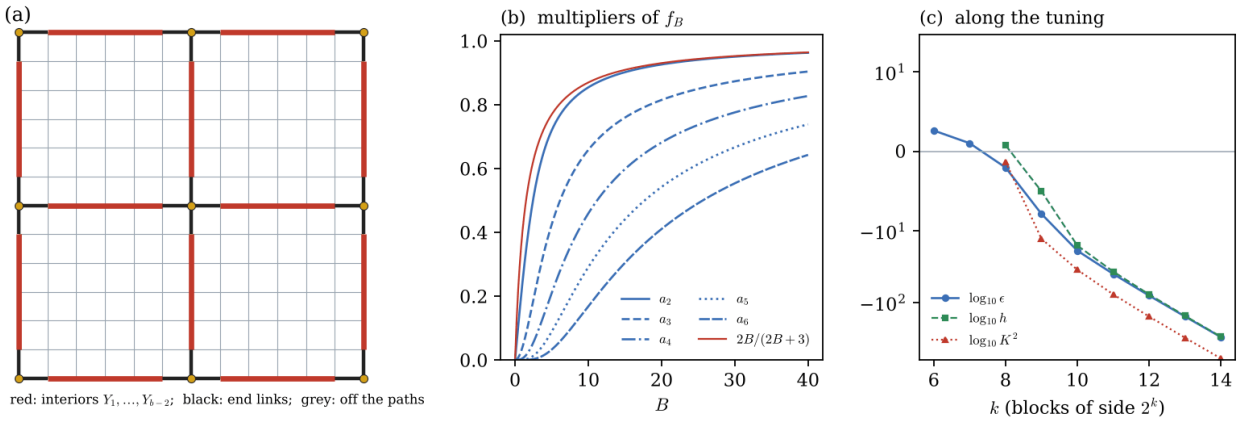


FIG. 21. Smoothing along the paths. (a) The paths of the straight-path map with blocks of side $b = 6$, in a plane: the interior links (red) of all the paths lie in no common plaquette and are independent given the end links (black) and the links off the paths (grey) (Lemma 163). (b) The multipliers $a_d(B) = I_d(B)/I_1(B)$ of convolution with the one-link density on the representations of dimension $d = 2,\dots,6$, all below $2B/(2B+3)$ (Lemma 158). (c) The numbers η_{b-2}, h_{b-2} and K_b^2 for $b = 2^k$ along the illustrative tuning $\beta_k = 2 + \frac{11}{3\pi^2} k \log 2$, on a symmetric logarithmic scale; h_{b-2} and K_b^2 are drawn where $\eta_{b-2} < \frac{1}{2}$. All three tend to zero faster than every power of 2^{-k} (Theorems 161 and 164).

that

$$\sum_{e \neq j} \sup_{V,X,Y} \left| \mathrm{Cov}_{\pi_V}\left(a_e^X, a_j^Y\right) \right| \leq \frac{1 - h_{b-2}}{3}$$

for all $L \in \mathscr{L}_k$ and all coarse links j. Then $\mathrm{BP}(n_0, \frac{1}{2})$ *holds for $\mu_{k,L}^{(0)}$ for every $n_0 \geq 2$ and every $L \in \mathscr{L}_k$ with $L_0 \geq 2n_0 + 4$. This holds in particular if $\beta_k = O(k)$ and there are $A < \infty$ and $p > 4$ with $|\mathrm{Cov}_{\pi_V}(a_e^X, a_j^Y)| \leq A K_b^2 (1 + |\Delta|_\infty)^{-p}$ for all L, V, unit X, Y and $e \neq j$, once k is so large that $A K_b^2 S_p \leq (1 - h_{b-2})/3$ with $S_p = 3 + 320(1 + 1/(p-4))$.*

Proof. By Theorem 164 and Lemma 126 the column sums of the coefficients are at most $\frac{1}{3}$, and Corollary 165 applies. For the second statement, three links share the base point of j, and at distance $r \geq 1$ in $|\cdot|_\infty$ there are $4((2r+1)^4 - (2r-1)^4) = 4(64r^3 + 16r) \leq 320 r^3$ links, so the sum is at most $A K_b^2 (3 + 320 \sum_{r \geq 1} r^{3-p}) \leq A K_b^2 S_p$; and K_b, $h_{b-2} \to 0$ by Theorem 161. □

The hypothesis is a decay of correlations under the posterior for the scores of distinct interior products, uniform in the torus and the coarse data. Proposition 133(ii) shows that it is a genuine collective input: its measure ρ_L is also a mixture of product laws with small scores and small single-link curvature, and there the sum of the mixed derivatives grows with the volume.

Remark 169 (what is proved by smoothing along the paths). Proved: convolution with a one-link SU(2) Wilson density contracts every nontrivial representation by at most $2B/(2B+3)$ (Lemma 158); a product of m independent one-link variables has a density within $48(1+B)^2 (2B/(2B+3))^{m-2}$ of 1 with its first two derivatives (Theorem 160); the single-link conditional densities of the law of the block variables of the straight-path map obey these bounds, the single-link curvature of the induced action and the difference between force variance and mean curvature in the fibre law tend to zero along every tuning with $\beta_k = O(k)$, uniformly in the torus and the coarse data (Theorem 161, Corollary 162); the law of the block variables is a mixture of product laws over the links off the path interiors, and its mixed derivatives are posterior covariances of the scores of the interior products, each at most K_b^2 (Lemma 163, Theorem 164); a column sum at most $\frac{1}{3}$ gives $\mathrm{BP}(n, \frac{1}{2})$ for every n with no condition on the tails (Corollary 165); for $k \geq 12$ and $\beta_k \leq k + 2$ the law of the block variables satisfies $\mathrm{BP}(n, \frac{1}{2})$ on every coarse torus of side at most $e^{2^k/(17k)}$ (Corollary 166); Dobrushin's condition for the posterior on single fine links fails for $b \geq 256$, $\beta_W \geq \frac{8}{3}$, and the Kotecký–Preiss condition for the unresummed cube surface fails for $\beta_W \geq 6$ (Proposition 167); a bound on the sums of the posterior covariances uniform in the torus gives (H1) with $\varpi_0 = \frac{1}{2}$ for every n_0 (Proposition 168).

Not proved: that bound for any k outside the strong-coupling range, and hence (H1) along the tuning; the split inequality (H2) along the tuning. Neither BP along the tuning nor the mass gap is proved here.

R. The row sums of the Gaussian model

This subsection determines the order of the row sums γ_b of the effective quadratic form of the Gaussian model of Sec. X P. The Bloch symbol of Γ is written in closed form through the sums of the lattice Green function along the paths, and its derivatives are controlled at the scale $b^{-1/2}$ of the coarse momenta, where the two bounds $1/b$ and $|\kappa|^2$ on the symbol meet. The result is that γ_b is of order $(\log b)/b$: condition (G) fails by a logarithm, and the Gaussian form of (FC) holds along every tuning with $\beta_k (\log b_k)/b_k \to 0$, in particular along the logarithmic tuning. The same numbers are the covariances of the scores in every representation of the coarse density as a mixture of products over the coarse links (Remark 175).

Throughout, $b \ge 3$, $|\kappa|$ is the Euclidean norm of $\kappa \in \mathbb{R}^4$, and for $p, \kappa \in \mathbb{R}^4$, $q \in \mathbb{R}$,

$$\varepsilon(p) = \sum_\rho 4\sin^2\frac{p_\rho}{2}, \qquad \sigma_\rho(\kappa) = e^{i\kappa_\rho} - 1,$$

$$F_b(q) = \Big|\sum_{t=0}^{b-1} e^{iqt}\Big|^2 = \sum_{|r|<b} (b-|r|)e^{iqr}.$$

Let Z_b be the set of $n \in \mathbb{Z}^4$ with $-\lfloor (b-1)/2\rfloor \le n_\rho \le \lfloor b/2\rfloor$ for every ρ, a set of representatives of $(\mathbb{Z}/b\mathbb{Z})^4$ with $|n_\rho| \le b/2$. For $\kappa \notin 2\pi\mathbb{Z}^4$ and a direction μ put

$$\mathscr{G}_\mu(\kappa) = \frac{1}{b^4}\sum_{n\in Z_b} \frac{F_b(p_\mu)}{\varepsilon(p)}, \qquad p = \frac{\kappa + 2\pi n}{b}.$$

No p lies in $2\pi\mathbb{Z}^4$, since $\kappa + 2\pi n \in 2\pi b\mathbb{Z}^4$ would put κ in $2\pi\mathbb{Z}^4$; so $\mathscr{G}_\mu$ is a finite sum of nonnegative real-analytic functions on $\mathbb{R}^4 \setminus 2\pi\mathbb{Z}^4$, and it is $2\pi\mathbb{Z}^4$-periodic, because each summand depends only on p modulo $2\pi\mathbb{Z}^4$, that is on n modulo b. Lemma 170(i) and the periodicity give $\mathscr{G}_\mu > 0$. Put

$$r_\mu = \frac{1}{\mathscr{G}_\mu}, \qquad \mathscr{T} = \sum_\rho r_\rho |\sigma_\rho|^2, \qquad \omega(\kappa) = \min\Big(\frac{1}{b}, |\kappa|^2\Big).$$

In the proofs, C denotes constants that depend only on the order of the derivatives involved and may change from line to line, and two elementary facts are used: for $|x| \le \pi$,

$$\frac{4}{\pi^2}x^2 \le 4\sin^2\frac{x}{2} \le x^2; \tag{21}$$

and for $\kappa \in [-\pi,\pi]^4$, $n \in Z_b$ and $p = (\kappa + 2\pi n)/b$, the distance δ_ρ of p_ρ to $2\pi\mathbb{Z}$ satisfies

$$\delta_\rho \ge \frac{\pi|n_\rho|}{b}, \tag{22}$$

since $|\kappa_\rho + 2\pi n_\rho| \ge \pi|n_\rho|$ and $|p_\rho| \le \pi + \pi/b$, and when $|p_\rho| > \pi$, $\delta_\rho = 2\pi - |p_\rho| \ge \pi/2 \ge \pi|n_\rho|/b$.

Lemma 170 (the line sums)**.** *For $\kappa \in [-\pi,\pi]^4 \setminus \{0\}$ and every direction μ:*

(i) *the two bounds $\mathscr{G}_\mu(\kappa) \ge 4/(\pi^2|\kappa|^2)$ and $\mathscr{G}_\mu(\kappa) \ge 4b/(7\pi^4)$ hold, and hence $\mathscr{G}_\mu(\kappa) \ge 4/(7\pi^4\omega(\kappa))$;*

(ii) *$\mathscr{G}_\mu(\kappa) \le \pi^2/(4|\kappa|^2) + 15b \le 18/\omega(\kappa)$;*

(iii) *for every multi-index α, $|\partial^\alpha \mathscr{G}_\mu(\kappa)| \le C\omega(\kappa)^{-1}|\kappa|^{-|\alpha|}$.*

Proof. (i) For $n = 0$, $F_b(\kappa_\mu/b) = \sin^2(\kappa_\mu/2)/\sin^2(\kappa_\mu/2b) \ge 4b^2/\pi^2$ by (21), and $F_b(0) = b^2$; and $\varepsilon(\kappa/b) \le |\kappa|^2/b^2$. This gives the first bound. For the second keep the terms with $n_\mu = 0$, for which $F_b(p_\mu) \ge 4b^2/\pi^2$ as before and, by (21), $\varepsilon(p) \le |p|^2 \le \pi^2 b^{-2}(1 + \sum_{\rho\ne\mu}(1+2|n_\rho|)^2)$. For $1 \le t \le \lfloor (b-1)/2\rfloor$ the vectors n with $n_\mu = 0$ and $\max_{\rho\ne\mu}|n_\rho| = t$ lie in Z_b, there are $(2t+1)^3 - (2t-1)^3 \ge 24t^2$ of them, and each has $1 + \sum_{\rho\ne\mu}(1+2|n_\rho|)^2 \le 1 + 27t^2 \le 28t^2$. So these terms of $\sum 1/\varepsilon(p)$ add up to at least $\frac{b^2}{\pi^2}\frac{6}{7}\lfloor\frac{b-1}{2}\rfloor \ge \frac{b^3}{7\pi^2}$, as $\lfloor (b-1)/2\rfloor \ge b/6$ for $b \ge 3$, and $\mathscr{G}_\mu \ge b^{-4}\frac{4b^2}{\pi^2}\frac{b^3}{7\pi^2}$.

(ii) For $n = 0$, $F_b \le b^2$ and $\varepsilon(\kappa/b) \ge 4|\kappa|^2/(\pi^2 b^2)$ by (21), which gives $\pi^2/(4|\kappa|^2)$. For $n \ne 0$, (21) and (22) give $\varepsilon(p) \ge \frac{4}{\pi^2}\sum_\rho \delta_\rho^2 \ge 4|n|^2/b^2$, and $F_b(p_\mu) \le b^2$, while for $n_\mu \ne 0$, $F_b(p_\mu) = \sin^2(\kappa_\mu/2)/\sin^2(p_\mu/2) \le \pi^2/\delta_\mu^2 \le b^2/n_\mu^2$. With $m(n) = \min(1, n_\mu^{-2})$ the term of n is at most $m(n)/(4|n|^2)$. The vectors of Z_b with $n_\mu = 0$ and $\max_\rho |n_\rho| = t \ge 1$ are at most $24t^2 + 2 \le 26t^2$, so these vectors contribute at most $26\cdot\frac{b}{2} = 13b$ to $\sum_{n\ne0}|n|^{-2}$, and

$$\sum_{n\in Z_b\setminus\{0\}} \frac{m(n)}{|n|^2} \le 13b + \sum_{n_\mu\ne0}\frac{1}{n_\mu^2}\Big(\frac{1}{n_\mu^2} + 13b\Big) \le 13b\Big(1 + \frac{\pi^2}{3}\Big) + \frac{\pi^4}{45} \le 60b.$$

So the terms with $n \ne 0$ add up to at most $15b$, and $\pi^2/4 + 15 \le 18$.

(iii) Since $\partial_\kappa = b^{-1}\partial_p$, the Leibniz rule writes ∂^α of the term of n as a sum of at most $2^{|\alpha|}$ products $b^{-j}F_b^{(j)}(p_\mu)\cdot b^{-|\alpha'|}(\partial^{\alpha'}\varepsilon^{-1})(p)$ with $j + |\alpha'| = |\alpha|$. Let $\delta(q)$ be the distance of $q \in \mathbb{R}$ to $2\pi\mathbb{Z}$ and $\delta(p)$ that of $p \in \mathbb{R}^4$ to $2\pi\mathbb{Z}^4$. First, $|F_b^{(j)}(q)| \le Cb^j\min(b^2, \delta(q)^{-2})$: the Fejér sum gives $|F_b^{(j)}| \le \sum_{|r|<b}(b-|r|)|r|^j \le b^{j+2}$; for $0 < |q| \le \pi$, $F_b(q) = (1-\cos bq)h(q)$ with $h(q) = 1/(1-\cos q)$, and $h - 2/q^2$ is analytic on $|q| < 2\pi$, so $|h^{(i)}(q)| \le C|q|^{-2-i}$ and $|F_b^{(j)}(q)| \le C\sum_{i\le j} b^i|q|^{-2-j+i} \le Cb^j|q|^{-2}$ for $|q| \ge 1/b$; the general case follows by periodicity. Second, $|\partial^{\alpha'}\varepsilon^{-1}(p)| \le C\delta(p)^{-2-|\alpha'|}$: by periodicity let $p \in [-\pi,\pi]^4$, so that $|p| = \delta(p)$; then $\varepsilon(p) \ge 4|p|^2/\pi^2$, $|\partial_\rho\varepsilon| = 2|\sin p_\rho| \le 2|p|$ and the derivatives of order $i \ge 2$ are at most $2 \le C|p|^{2-i}$, and $\partial^{\alpha'}(1/\varepsilon)$ is a sum of terms $\varepsilon^{-1-m}\prod_{i=1}^m \partial^{\alpha_i}\varepsilon$ with $\sum_i \alpha_i = \alpha'$. For $n = 0$, $\delta(p) = |\kappa|/b$, and a product is at most $Cb^2\cdot b^{-|\alpha'|}(b/|\kappa|)^{2+|\alpha'|} \le Cb^4|\kappa|^{-2-|\alpha|}$, as $|\kappa| \le 2\pi$. For $n \ne 0$, (22) gives $\delta(p) \ge \pi|n|/b$ and $\delta(p_\mu) \ge \pi|n_\mu|/b$, so a product is at most $Cb^2 m(n)\cdot b^2|n|^{-2}$, and by (ii) these add up, over $n \ne 0$, to at most Cb^5. With the factor b^{-4} of $\mathscr{G}_\mu$, $|\partial^\alpha\mathscr{G}_\mu| \le C(|\kappa|^{-2} + b)|\kappa|^{-|\alpha|} \le 2C\omega^{-1}|\kappa|^{-|\alpha|}$. □

Lemma 171 (the symbol)**.** *For $\kappa \in \mathbb{R}^4 \setminus 2\pi\mathbb{Z}^4$ the matrix $\hat\Gamma(\kappa)$ of the proof of Proposition 153 is*

$$\hat\Gamma(\kappa)_{\mu\nu} = r_\mu\delta_{\mu\nu} - \frac{r_\mu r_\nu\,\sigma_\mu\bar\sigma_\nu}{\mathscr{T}},$$

with r_ρ, σ_ρ and $\mathscr{T}$ taken at κ, and $\hat\Gamma(0) = 0$.

Proof. Both sides are continuous on $\mathbb{R}^4 \setminus 2\pi\mathbb{Z}^4$: the left side by Proposition 153, the right side because $\mathscr{G}_\rho > 0$ and $\mathscr{T} \ge \min_\rho r_\rho\,\varepsilon(\kappa) > 0$. The points of $(2\pi/L)\mathbb{Z}^4$, $L \ge 3$, are dense, so it suffices to fix $L \ge 3$ and $\kappa \in (2\pi/L)\mathbb{Z}^4 \setminus 2\pi\mathbb{Z}^4$ and to show that the right side is the matrix $\hat g(\kappa)$ of that proof.

For $A \in \mathbb{C}^{E'}$ let $\mathsf{S}A \in \mathbb{C}^E$ be its sums along the paths. By the proof of Proposition 152(iii), $\langle v, \Gamma v\rangle = \min\{|dA|^2 : \mathsf{S}A = v\}$ for

real v, and, splitting real and imaginary parts, also for complex v with the Hermitian forms. For $f \in \mathbb{C}^4$ let $v_{(y,\nu)} = e^{i\kappa\cdot y} f_\nu$; then $\langle v, \Gamma v\rangle = L^4 f^* \hat g(\kappa) f$ by the action of Γ on $\mathscr{H}_\kappa$ in that proof. The maps d and S commute with the translations by $b\mathbb{Z}^4$ and preserve the orthogonal decomposition into the spaces $\mathscr{H}_{\kappa'}$; the constraint concerns only the component of A in $\mathscr{H}_\kappa$, so the minimum is the minimum over A in $\mathscr{H}_\kappa$ of the links.

Write a form as $(A_\rho)_\rho$, $A_\rho(x) = A_{(x,\rho)}$, and $\partial_\rho \phi(x) = \phi(x+e_\rho) - \phi(x)$, with adjoint ∂_ρ^*. Then $(dA)_p = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)$ for the plaquette p at x in the plane $\mu < \nu$, and $d^{\mathrm T} A = \sum_\rho \partial_\rho^* A_\rho$. The differences commute, so $\langle \partial_\mu^* A_\mu, \partial_\nu^* A_\nu\rangle = \langle \partial_\nu A_\mu, \partial_\mu A_\nu\rangle$ and $\|\partial_\rho^* A_\rho\| = \|\partial_\rho A_\rho\|$, and the cross terms cancel in

$$|dA|^2 + |d^{\mathrm T} A|^2 = \sum_{\mu,\nu} \|\partial_\mu A_\nu\|^2 = \sum_\nu \langle A_\nu, \Delta A_\nu\rangle, \qquad (23)$$

where $\Delta = \sum_\mu \partial_\mu^* \partial_\mu$. The space $\mathscr{H}_\kappa$ of the sites of Λ_{bL} has dimension b^4 and is spanned by the plane waves $e_n(x) = e^{ip\cdot x}$, $p = (\kappa + 2\pi n)/b$, $n \in Z_b$: they satisfy $e_n(x + bz) = e^{i\kappa\cdot z} e_n(x)$, are periodic on Λ_{bL} because $L\kappa \in 2\pi\mathbb{Z}^4$, and are orthogonal, their p being distinct modulo $2\pi\mathbb{Z}^4$; and $\Delta e_n = \varepsilon(p) e_n$ with $\varepsilon(p) > 0$. The same holds for each component A_ρ of a form in $\mathscr{H}_\kappa$, the link (x,ρ) being identified with x.

If $A \in \mathscr{H}_\kappa$ and $\mathsf{S}A = v$, let $\phi \in \mathscr{H}_\kappa$ solve $\Delta\phi = -d^{\mathrm T} A$, so that $A' = A + d\phi$ has $d^{\mathrm T} A' = 0$ and $dA' = dA$; as $\phi(by) = e^{i\kappa\cdot y}\phi(0)$ and $\pi(y,\nu)$ runs from by to $b(y+e_\nu)$, $\mathsf{S}A'_{(y,\nu)} = e^{i\kappa\cdot y}(f_\nu + \phi(0)\sigma_\nu(\kappa))$. Conversely, if $A' \in \mathscr{H}_\kappa$ has $\mathsf{S}A'_{(y,\nu)} = e^{i\kappa\cdot y}(f_\nu + t\sigma_\nu)$, $t \in \mathbb{C}$, then $A' - d\phi'$, with ϕ' equal to $te^{i\kappa\cdot y}$ at by and to 0 at the other sites, lies in $\mathscr{H}_\kappa$, has the path sums v and the differential dA'. With (23) it follows that $L^4 f^* \hat g(\kappa) f$ is the minimum of $\sum_\nu \langle A_\nu, \Delta A_\nu\rangle$ over the $A \in \mathscr{H}_\kappa$ with $\mathsf{S}A_{(y,\nu)} = e^{i\kappa\cdot y} u_\nu$, $u \in f + \mathbb{C}\sigma(\kappa)$: the forms A' above attain the value $|dA|^2$, and every admissible A has $\sum_\nu \langle A_\nu, \Delta A_\nu\rangle \ge |dA|^2 = |d(A - d\phi')|^2$.

For fixed u this minimum splits over ν. For $A_\nu = \sum_n a_n e_n$ the constraint is the single condition $\sum_n a_n c_n = u_\nu$, $c_n = \sum_{t=0}^{b-1} e^{ip_\nu t}$, the sum along $\pi(0,\nu)$, the others following by quasi-periodicity, and $\langle A_\nu, \Delta A_\nu\rangle = (bL)^4 \sum_n |a_n|^2 \varepsilon(p)$. Since $|c_n|^2 = F_b(p_\nu)$, the Cauchy–Schwarz inequality gives the minimum

$$\frac{(bL)^4 |u_\nu|^2}{\sum_n |c_n|^2/\varepsilon(p)} = L^4 r_\nu |u_\nu|^2.$$

With $R = \mathrm{diag}(r_\nu)$, the minimum over $t \in \mathbb{C}$ of $(f + t\sigma)^* R (f + t\sigma)$ is $f^* R f - |\sigma^* R f|^2/\sigma^* R \sigma$, and $\sigma^* R \sigma = \mathscr{T}$. The matrix $\hat g(\kappa)$ is Hermitian, because Γ is real and symmetric, and so is the matrix of the statement; their quadratic forms agree, so they are equal. Finally, a form v that is constant on the coarse links of each direction has $\Gamma v = 0$, since A equal to $v_{(0,\nu)}/b$ on every fine link of direction ν has these path sums and $dA = 0$; so $\hat\Gamma(0) = 0$. □

Lemma 172 (the derivatives of the symbol)**.** *For $\kappa \in [-\pi,\pi]^4 \setminus \{0\}$, all μ,ν and every multi-index α with $|\alpha| \le 3$,*

$$\left|\partial^\alpha \hat\Gamma(\kappa)_{\mu\nu}\right| \le C\,\omega(\kappa)\,|\kappa|^{-|\alpha|}$$

with an absolute constant C.

Proof. Lemma 170(i),(ii) gives

$$\frac{\omega}{18} \le r_\mu \le \frac{7\pi^4\omega}{4}.$$

The derivative $\partial^\alpha(1/\mathscr{G}_\mu)$ is a sum of terms

$$\mathscr{G}_\mu^{-1-m} \prod_{i=1}^m \partial^{\alpha_i} \mathscr{G}_\mu,$$

so Lemma 170(iii) gives $|\partial^\alpha r_\mu| \le C\omega|\kappa|^{-|\alpha|}$. Since $|\sigma_\rho| \le |\kappa_\rho|$, $|\sigma_\rho|^2 = 2 - 2\cos\kappa_\rho$ and $|\kappa| \le 2\pi$,

$$|\partial^\alpha \sigma_\rho| \le C|\kappa|^{1-|\alpha|}, \qquad |\partial^\alpha |\sigma_\rho|^2| \le C|\kappa|^{2-|\alpha|}.$$

Hence

$$|\partial^\alpha (r_\mu r_\nu \sigma_\mu \bar\sigma_\nu)| \le C\omega^2 |\kappa|^{2-|\alpha|},$$
$$|\partial^\alpha \mathscr{T}| \le C\omega|\kappa|^{2-|\alpha|},$$

while $\mathscr{T} \ge \frac{\omega}{18}\varepsilon(\kappa) \ge \frac{2\omega}{9\pi^2}|\kappa|^2$ by (21); so $|\partial^\alpha \mathscr{T}^{-1}| \le C(\omega|\kappa|^2)^{-1}|\kappa|^{-|\alpha|}$, and the Leibniz rule applied to Lemma 171 gives the claim. □

Theorem 173 (the order of the row sums)**.** *There is an absolute constant C such that for every $b \ge 3$ and all directions $\mu \ne \nu$*

$$\frac{\log(1+60b)}{15\pi b} \le \sum_{w\in\mathbb{Z}^4} \left|\Gamma_b^\infty(w)_{\mu\nu}\right| \le \gamma_b \le \frac{C(1+\log b)}{b}.$$

In particular $b\gamma_b \to \infty$, so that (G) *fails, and $\gamma_b \log b \to 0$.*

Proof. The middle inequality holds because $(w,\nu) \ne (0,\mu)$.

Upper bound. Fix μ,ν, not necessarily distinct, and let $f(w) = \Gamma_b^\infty(w)_{\mu\nu}$, the Fourier coefficients of $\hat\Gamma_{\mu\nu}$: $\hat\Gamma(\kappa) = \sum_w \Gamma_b^\infty(w) e^{i\kappa\cdot w}$. By Lemmas 171 and 172, $\hat\Gamma_{\mu\nu}$ is smooth on $\mathbb{R}^4 \setminus 2\pi\mathbb{Z}^4$ with $|\partial^\alpha \hat\Gamma_{\mu\nu}| \le C|\kappa|^{2-|\alpha|}$ near 0 for $|\alpha| \le 3$. For a direction ρ and $j \le 3$ the function $\partial_\rho^j \hat\Gamma_{\mu\nu}$ is square integrable on $[-\pi,\pi]^4$, and its Fourier coefficients are $(iw_\rho)^j f(w)$: integrate by parts on the cube minus a ball of radius t about 0; the faces cancel by periodicity, and the terms on the sphere are $O(t^{3+2-(j-1)}) \to 0$. By the Parseval identity and $|w|^4 \le 4\sum_\rho w_\rho^4$, $|w|^6 \le 16\sum_\rho w_\rho^6$, the sums $A_0 = \sum_w |f|^2$, $A_4 = \sum_w |w|^4 |f|^2$ and $A_6 = \sum_w |w|^6 |f|^2$ satisfy $A_0 \le C\sup\omega^2 \le C/b^2$ and

$$A_4 \le C\int \frac{\omega^2}{|\kappa|^4}\,d\kappa \le \frac{C(1+\log b)}{b^2}, \qquad A_6 \le C\int \frac{\omega^2}{|\kappa|^6}\,d\kappa \le \frac{C}{b},$$

the integrals over $[-\pi,\pi]^4$, split at $|\kappa| = b^{-1/2}$ with $\omega \le |\kappa|^2$ inside and $\omega \le 1/b$ outside: the first is at most $\frac{\pi^2}{2b^2} + \frac{2\pi^2}{b^2}\log(2\pi b^{1/2})$ and the second at most $\frac{\pi^2}{b} + \frac{\pi^2}{b}$. Let J be the least integer with $2^J \ge b^{1/2}$. The shell S_j of the w with $2^j \le |w| < 2^{j+1}$ has at most 2^{4j+8} points, so by the Cauchy–Schwarz inequality $\sum_{S_j} |f| \le 2^{2j+4}(\sum_{S_j}|f|^2)^{1/2} \le 16(\sum_{S_j}|w|^4|f|^2)^{1/2}$, and once more

$$\sum_{1\le|w|<2^J} |f(w)| \le 16\sqrt{JA_4} \le \frac{C(1+\log b)}{b}.$$

Further $\sum_{|w|\ge 2^J}|w|^{-6}\le\sum_{j\ge J}2^{8-2j}\le 2^{10}b^{-1}$, so $\sum_{|w|\ge2^J}|f|\le(2^{10}A_6/b)^{1/2}\le C/b$, and $|f(0)|\le A_0^{1/2}$. Summing over ν gives the bound on γ_b.

Lower bound. Let $\mu\ne\nu$, $f(w)=\Gamma_b^\infty(w)_{\mu\nu}$, and let $k(w_\mu,w_\nu)$ be the sum of f over the other two coordinates; as f is summable, k is summable on $\mathbb{Z}^2$ with $\sum_{w_\mu,w_\nu}k\,e^{i(\kappa_\mu w_\mu+\kappa_\nu w_\nu)}=\hat\Gamma_{\mu\nu}(\kappa_\mu e_\mu+\kappa_\nu e_\nu)$. On the plane $\kappa=\kappa_\mu e_\mu+\kappa_\nu e_\nu$, Lemma 171 gives, with r_μ,r_ν at κ,

$$\hat\Gamma_{\mu\nu}(\kappa)=-\frac{4r_\mu r_\nu e^{i(\kappa_\mu-\kappa_\nu)/2}\sin\frac{\kappa_\mu}{2}\sin\frac{\kappa_\nu}{2}}{\mathscr{T}},$$

where $\mathscr{T}=4r_\mu\sin^2\frac{\kappa_\mu}{2}+4r_\nu\sin^2\frac{\kappa_\nu}{2}$. Let $\chi(t)=1$ for $t\ge1$ and -1 for $t\le0$, $\chi'(t)=\chi(t+1)$, and $h_N(w)=\chi(w_\mu)\chi'(w_\nu)$ for $-N\le w_\mu\le N+1$, $-N-1\le w_\nu\le N$ and $h_N(w)=0$ otherwise. For half-integers u, $\sum_{|u|\le N+1/2}\operatorname{sgn}(u)e^{-i\theta u}=-2i\sum_{l=0}^N\sin((l+\frac12)\theta)=-i(1-\cos((N+1)\theta))/\sin\frac\theta2$, and with $u=w_\mu-\frac12$ and $u=w_\nu+\frac12$,

$$\sum_w f(w)h_N(w)=\frac{1}{4\pi^2}\int_{[-\pi,\pi]^2}\frac{4r_\mu r_\nu}{\mathscr{T}}\big(1-\cos(N+1)\kappa_\mu\big)\times\big(1-\cos(N+1)\kappa_\nu\big)\,d\kappa_\mu d\kappa_\nu.$$

The function $4r_\mu r_\nu/\mathscr{T}$ is positive. By (21),

$$\mathscr{T}\ge\frac{4}{\pi^2}\min(r_\mu,r_\nu)|\kappa|^2,$$

so

$$\frac{4r_\mu r_\nu}{\mathscr{T}}\le\frac{\pi^2\max(r_\mu,r_\nu)}{|\kappa|^2}\le\frac{C\omega}{|\kappa|^2}\le C,$$

so it is integrable. As $N\to\infty$ the left side tends to $\sum_w f(w)h(w)$, $h=\lim h_N$, by dominated convergence, and the right side to the integral of $4r_\mu r_\nu/\mathscr{T}$, by the Riemann–Lebesgue lemma applied to the four products of cosines. Now $|h|\le1$, $\mathscr{T}\le\max(r_\mu,r_\nu)|\kappa|^2$ by (21), and Lemma 170(ii) gives

$$\min(r_\mu,r_\nu)\ge\big(\pi^2/(4|\kappa|^2)+15b\big)^{-1};$$

hence

$$\sum_w|f(w)|\ge\frac{1}{4\pi^2}\int_{|\kappa|\le\pi}\frac{4\,d\kappa_\mu d\kappa_\nu}{\pi^2/4+15b|\kappa|^2}=\frac{\log(1+60b)}{15\pi b}.$$

Finally $b\gamma_b\ge\log(1+60b)/(15\pi)\to\infty$ and $\gamma_b\log b\le C(1+\log b)\log b/b\to0$. □

In the lower bound the logarithm is the integral of $|\kappa|^{-2}$ over the coarse momenta with $b^{-1/2}\le|\kappa|\le\pi$ in the plane of two directions, where the entries of the symbol between these directions are of size $1/b$ and depend on the direction of κ through the projection onto the coarse gradient; the sign function h of the proof pairs with the kernel as the integral of a positive function, with no cancellation.

Corollary 174 (the Gaussian form of (FC) along the tuning)**.** *In the Gaussian model with blocks of side b and coupling β, let $J^{\mathrm G}_{ee'}=\beta|\Gamma_{ee'}|$ as in Corollary 154, and let C be the constant of Theorem 173.*

(i) *For every $L\ge3$ and every coarse link j, $\frac{D^2}{4}\sum_{e\ne j}J^{\mathrm G}_{ej}\le\frac{D^2}{4}\beta C(1+\log b)/b$, and the inequalities of* (FC) *for $J^{\mathrm G}$ hold with $c=\frac{1}{17}$ and $R=1$ on every torus as soon as $17D^2\beta C(1+\log b)\le4b$.*

(ii) *For every coarse link j,*

$$\sup_{L\ge3}\frac{D^2}{4}\sum_{e\ne j}J^{\mathrm G}_{ej}\ge\frac{D^2\beta\log(1+60b)}{60\pi b};$$

so the first inequality of (FC) *for $J^{\mathrm G}$ with some $c<1$ on all tori implies $\beta\log(1+60b)<60\pi b/D^2$.*

(iii) *Along every sequence $b=2^k$, β_k with $k\beta_k2^{-k}\to0$, in particular along $\beta_k=2+\frac{11}{3\pi^2}k\log2$, the inequalities of* (FC) *for $J^{\mathrm G}$ hold with $c=\frac{1}{17}$ and $R=1$ for all large k, uniformly in L_0, and the column sums tend to zero.*

Proof. (i) Proposition 153 and Theorem 173 bound the column sums, as in Corollary 154; the tails are at most the column sums, and $(1-\frac{1}{17})/16=\frac{1}{17}$. (ii) Let $j=(y,\mu)$ and $\nu\ne\mu$. For $|w|_\infty<L/2$, $\Gamma_{j,(y+w,\nu)}=\sum_z\Gamma_b^\infty(w+Lz)_{\mu\nu}$ differs from $\Gamma_b^\infty(w)_{\mu\nu}$ by at most $\sum_{|w'|_\infty\ge L/2}|\Gamma_b^\infty(w')|$, which tends to zero as $L\to\infty$, and distinct w give distinct links. So $\liminf_L\sum_{e\ne j}|\Gamma_{je}|\ge\sum_{w\in W}|\Gamma_b^\infty(w)_{\mu\nu}|$ for every finite $W\subset\mathbb{Z}^4$, and Theorem 173 applies. (iii) follows from (i), with $\log b=k\log2$. □

So the Gaussian model, for which Sec. X P reduced (FC) to a bound on γ_b, meets (FC) along the tuning without (G): the logarithm lost against (G) is absorbed by the factor 2^{-k}. By Corollary 174(i),(ii), (FC) fails in the model only if β_k exceeds an absolute multiple of $2^k/k$, and it fails on large tori once $\beta_k\log(1+60\cdot2^k)\ge60\pi2^k/D^2$. The model is not a lattice gauge measure, and Corollary 174 gives (H1) for none.

Remark 175 (the scores of the paths). Let a coarse density be written as $\mathsf D(v)=\int\zeta(dW)\prod_e p_e(v_e\mid W)$, a mixture over variables W of products of positive smooth densities of the single coarse links, as in Sec. X Q, and let $\pi_v(dW)$ be proportional to $\zeta(dW)\prod_e p_e(v_e\mid W)$ and $a_e=\partial_{v_e}\log p_e(v_e\mid W)$. Differentiation under the integral gives $\partial_{v_e}\mathsf D=\mathsf D\,\pi_v(a_e)$ and, for $e\ne j$, $\partial_{v_j}\partial_{v_e}\mathsf D=\mathsf D\,\pi_v(a_ea_j)$, so

$$-\partial_{v_j}\partial_{v_e}\log\mathsf D=-\operatorname{Cov}_{\pi_v}(a_e,a_j):$$

the score covariances are minus the mixed derivatives of the induced action, whatever the representation. In the Gaussian model the induced action is $\frac\beta2\langle v,\Gamma v\rangle$ up to a constant, so in every such representation the score covariances are $-\beta\Gamma_{ej}$, and their row sums are those of Theorem 173 and Corollary 174, of order $\beta(\log b)/b$. For W the links off the strict interiors of the paths, the interior links of a path lie in no common plaquette and each lies in six, so for fixed W the density e^{-S}

is, as a function of the interior links, proportional to a product of Gaussian densities of variance $1/(6\beta)$; $p_e(\cdot \mid W)$ is the Gaussian density of variance $(b-2)/(6\beta)$ centred at a function of W, and the variance of the path sum grows with b without randomising it. For SU(2) the interior links randomise the path product towards the Haar measure, which bounds each score covariance by a quantity that is super-exponentially small in k along the tuning (Sec. X Q), while in the Gaussian model every row has an entry $\beta|\Gamma_{ej}| \geq \beta/(7b^2)$ by Proposition 152(iii),(iv). So the Gaussian model is not the regime of the compact score covariances along the tuning. The input that remains for SU(2) is a decay of $\mathrm{Cov}_{\pi_V}(a_e, a_j)$ in the distance of e and j, uniform in V and in the torus; it is a property of the Wilson measure of the links off the interiors of the paths at β_k, and nothing proved here controls it.

Remark 176 (what is proved about the row sums). Proved: the Bloch symbol of the effective quadratic form of the Gaussian model is given in closed form by the sums of the lattice Green function along the paths (Lemma 171); these line sums lie between absolute multiples of $b+|\kappa|^{-2}$ and their derivatives are bounded at the same scale (Lemma 170); the derivatives of order at most three of the symbol are bounded by $\min(1/b, |\kappa|^2)|\kappa|^{-|\alpha|}$ (Lemma 172); the row sums satisfy $\log(1+60b)/(15\pi b) \leq \gamma_b \leq C(1+\log b)/b$, already for the entries between two given directions, so that (G) fails by a logarithm and $\gamma_b = o(1/\log b)$ (Theorem 173); the Gaussian form of (FC) holds on every torus when $17D^2\beta C(1+\log b) \leq 4b$, along every tuning with $k\beta_k 2^{-k} \to 0$, in particular the logarithmic one, uniformly in L_0, and it fails on large tori when $\beta \log(1+60b) \geq 60\pi b/D^2$ (Corollary 174); in every representation of a coarse density as a mixture of products over the coarse links the score covariances are minus the mixed derivatives of the induced action, so in the Gaussian model they are $-\beta\Gamma_{ej}$, while every row of the model has an entry at least $\beta/(7b^2)$ (Remark 175).

Not proved: an explicit value of C; the pointwise decay of $\Gamma_b^{\infty}(w)$ in $|w|$, and any control of the decay rate κ_b of Proposition 153 in b; any comparison of the fibre covariances or of the score covariances of SU(2) with the Gaussian model; (FC) for SU(2) along the tuning, in the form of Corollary 149 or in the form of Sec. X Q; any decay of the score covariances under π_V in the distance of the coarse links, uniform in V and in the torus; (H1) along the tuning; (H1) for U(1). Neither BP along the tuning nor the mass gap is proved here.

S. Metrics across scales

The constant Γ of (H2) bounds the product $\prod_{i\geq j} C_{B,i}$ of the split constants of all finer scales, so a proof of (H2) by a comparison at each scale has to carry a metric on the fields of one scale to the next through the reconstruction of the fine field from the coarse one, with losses whose product over the scales stays bounded. This subsection determines which metrics can do this when the reconstruction is linear, as it is at Gaussian order. A site-additive metric loses a fixed factor at every scale at which the reconstruction changes sign, and the conditional mean of a Gaussian field given its block averages always changes sign; a metric with a separate term for the deviation from the reconstructed field composes with a loss independent of the number of scales once one two-term inequality holds at every scale. Nothing in this subsection concerns the Wilson measure itself.

Let $X_0, X_1, \dots$ be finite sets, X_0 the finest, and for each k let R_k be a real $X_k \times X_{k+1}$ matrix, the reconstruction of a field on X_k from one on X_{k+1}. Put $T_0 = I$ and $T_k = R_0 R_1 \cdots R_{k-1}$, which maps fields on X_k to fields on X_0. For weights $p \in [0,\infty)^X$ let $d_p(x,x') = \sum_i p_i |x_i - x_i'|$. We say that d_q on X_{k+1} is *admissible* for d_p on X_k through R_k if $d_p(R_k z, R_k z') \leq d_q(z,z')$ for all z, z', and write $|R|$ for the matrix of the absolute values of the entries of R. The minimal weights are

$$m^{(0)} = \mathbf{1}, \qquad m^{(k+1)} = |R_k|^{\mathsf{T}} m^{(k)}.$$

Lemma 177 (admissible site-additive metrics)**.** *Let R be a real matrix and p, q weight vectors.*

(i) *If $p \geq 0$, then d_q is admissible for d_p through R if and only if $|R|^{\mathsf{T}} p \leq q$ entrywise; so $|R|^{\mathsf{T}} p$ is the least admissible weight vector.*

(ii) *If $p^{(0)} \geq \mathbf{1}$ and $d_{p^{(k+1)}}$ is admissible for $d_{p^{(k)}}$ through R_k for every $k < K$, then $p^{(k)} \geq m^{(k)}$ for $k \leq K$; the minimal weights themselves form such a chain.*

(iii) *$\|T_k x\|_1 \leq \langle m^{(k)}, |x| \rangle$ for every field x on X_k.*

Proof. (i) If $|R|^{\mathsf{T}} p \leq q$, then for $\xi = z - z'$

$$d_p(R\xi, 0) = \sum_i p_i \Big| \sum_j R_{ij} \xi_j \Big| \leq \sum_j \Big(\sum_i p_i |R_{ij}| \Big) |\xi_j| \leq d_q(\xi, 0).$$

Conversely, $\xi = e_j$ gives $\sum_i p_i |R_{ij}| \leq q_j$; this is the step that uses the additivity of d_q over the sites. (ii) By (i), $p^{(k+1)} \geq |R_k|^{\mathsf{T}} p^{(k)}$, and $|R_k|^{\mathsf{T}}$ preserves the order of nonnegative vectors, so induction from $p^{(0)} \geq m^{(0)}$ gives the claim; the minimal weights satisfy (i) with equality. (iii) Apply (i) k times with $p = m^{(0)}$. □

By (ii) and (iii), every admissible chain of site-additive metrics with $p^{(0)} \geq \mathbf{1}$ overestimates the ℓ^1 distance at the finest scale of two fields on X_k that differ by x at least by the factor

$$\omega_k(x) = \frac{\langle m^{(k)}, |x| \rangle}{\|T_k x\|_1} \geq 1.$$

Proposition 178 (the loss of site-additive metrics)**.** *Suppose that for each $i < k$ every column of R_i has the same ℓ^1 norm ν_i and the same sum σ_i, and that $R_i \mathbf{1} = \mathbf{1}$. Then $\sigma_i > 0$, the factors $\theta_i = \nu_i/\sigma_i$ satisfy $\theta_i \geq 1$,*

$$\omega_k(\mathbf{1}) = \prod_{i<k} \theta_i,$$

and $\theta_i > 1$ if and only if some column of R_i has a negative entry. If $R_i \geq 0$ for every $i < k$, then $\langle m^{(k)}, x \rangle = \|T_k x\|_1$ for every $x \geq 0$, so there is no loss on nonnegative fields.

Proof. Since the rows of R_i sum to one, the sum of all its entries is $|X_i| = \sigma_i |X_{i+1}|$, so $\sigma_i = |X_i|/|X_{i+1}| > 0$. The equal column norms give $|R_i|^{\mathsf{T}}\mathbf{1} = \nu_i \mathbf{1}$, hence $m^{(k)} = (\prod_{i<k} \nu_i)\mathbf{1}$ and $\langle m^{(k)}, \mathbf{1}\rangle = |X_k| \prod_{i<k} \nu_i$, while $T_k \mathbf{1} = \mathbf{1}$ gives $\|T_k \mathbf{1}\|_1 = |X_0| = |X_k| \prod_{i<k} \sigma_i$. The quotient is $\prod_{i<k} \theta_i$. For a column c, $\sigma_i = |\sum_j c_j| \le \sum_j |c_j| = \nu_i$, with equality if and only if the entries of c have one sign, which, as $\sigma_i > 0$, means that none is negative; all columns share ν_i and σ_i. If every $R_i \ge 0$, then $|R_i| = R_i$, so $(m^{(k)})^{\mathsf{T}} = \mathbf{1}^{\mathsf{T}} T_k$, and $T_k x \ge 0$ for $x \ge 0$ gives $\mathbf{1}^{\mathsf{T}} T_k x = \|T_k x\|_1$. □

The reconstructions that arise at Gaussian order are conditional means. A *positive block average* on a partition $X = \bigcup_{a \in \mathscr{B}} B(a)$ with $|\mathscr{B}| \ge 2$ is a matrix $P \in \mathbb{R}^{\mathscr{B} \times X}$ with $P_{ax} > 0$ for $x \in B(a)$ and $P_{ax} = 0$ otherwise. For a matrix A on X the blocks are *thick* if for $a \ne a'$ the block $B(a')$ contains a site x with $A_{xy} = 0$ for every $y \in B(a)$.

Proposition 179 (conditional means change sign)**.** *Let A be invertible with A^{-1} entrywise positive, let P be a positive block average whose blocks are thick for A, and suppose that $PA^{-1}P^{\mathsf{T}}$ is invertible. Then*

$$R = A^{-1}P^{\mathsf{T}}\left(PA^{-1}P^{\mathsf{T}}\right)^{-1}$$

has a negative entry. If A is symmetric and positive definite, Rz is the minimiser of $\langle \varphi, A\varphi\rangle$ under $P\varphi = z$, which is the conditional mean, given $P\varphi = z$, of the centred Gaussian field with covariance A^{-1}.

Proof. Put $\Lambda = (PA^{-1}P^{\mathsf{T}})^{-1}$, so that $PR = I$ and $AR = P^{\mathsf{T}}\Lambda$. Suppose that $R \ge 0$. For $a' \ne a$, $0 = (PR)_{a'a} = \sum_{x \in B(a')} P_{a'x} R_{xa}$ with positive $P_{a'x}$, so $R_{xa} = 0$ for $x \notin B(a)$: the column a of R is supported in $B(a)$. For a site $x \in B(a')$ as in the definition of thickness, $(AR)_{xa} = \sum_{y \in B(a)} A_{xy} R_{ya} = 0$, while $(P^{\mathsf{T}}\Lambda)_{xa} = P_{a'x}\Lambda_{a'a}$; hence $\Lambda_{a'a} = 0$ and Λ is diagonal. Then $PA^{-1}P^{\mathsf{T}} = \Lambda^{-1}$ is diagonal. But for $a \ne a'$ its entry $\sum_{x,y} P_{ax}(A^{-1})_{xy} P_{a'y}$ is positive, since every term is nonnegative and those with $x \in B(a)$, $y \in B(a')$ are positive. This contradiction shows that R has a negative entry. The last statement is the method of Lagrange multipliers: the minimiser satisfies $A\varphi = P^{\mathsf{T}}\lambda$ and $P\varphi = z$, so $\lambda = \Lambda z$ and $\varphi = Rz$. □

Remark 180 (thick blocks). On the periodic lattice $(\mathbb{Z}/L\mathbb{Z})^d$ let $A = \varepsilon + 2d - N$ with $\varepsilon > 0$ and N the adjacency matrix. Then $A^{-1} = \sum_{j \ge 0} N^j/(\varepsilon + 2d)^{j+1}$, since $\|N\| \le 2d$, and this is entrywise positive because the lattice is connected. With blocks of side $b \ge 2$ and at least three blocks in each direction the blocks are thick for A: if $B(a)$ and $B(a')$ are not neighbours in one direction, every site of $B(a')$ will do; if $B(a) = B(a') \pm be_\mu$, a site of $B(a')$ on its face opposite to $B(a)$ has its neighbours in $B(a')$, in $B(a') \mp be_\mu \ne B(a)$, or in blocks displaced in other directions. So the conditional mean of the block averages, with equal weights, of a massive Gaussian field has a negative entry in every column, the columns being translates of one another, and $|R|^{\mathsf{T}}\mathbf{1}$ exceeds $R^{\mathsf{T}}\mathbf{1}$ in every entry. Thickness cannot be dropped: on the cycle $\mathbb{Z}/4\mathbb{Z}$ with the two blocks $\{0,1\}$ and $\{2,3\}$, each site has a neighbour in the other block, and the reflection $x \mapsto 1 - x$, which preserves A, P and both blocks, shows that R is constant on each block; with $PR = I$ this makes R the matrix of the block indicators, which is nonnegative.

A metric need not lose at every scale. Let $\|\cdot\|_k$ be a norm on the fields on X_k, let Φ_k be a seminorm on them (a gradient term, say), and let V_k be linear from fields on X_k to fields on X_{k+1} with $V_k R_k = I$, so that $\Pi_k = R_k V_k$ is a projection onto the range of R_k. For $s > 0$ and $\beta', \delta \ge 0$ put

$$Q_k(y) = \|y\|_k + \beta' \Phi_k(y) + \delta \|(I - \Pi_k)y\|_k$$

and $N_0 = \|\cdot\|_0$, $N_k = s^k Q_k$ for $k \ge 1$. The chain (N_k) *composes* if $N_k(R_k z) \le N_{k+1}(z)$ for all k and z.

Theorem 181 (composition with a deviation term)**.** *Suppose that $\|R_k z\|_k \le c_1 \|z\|_{k+1}$, $\Phi_k(R_k z) \le c_2 \|z\|_{k+1}$ and $\Phi_k \le c_3 \|\cdot\|_k$ for all k, and that*

$$\delta \ge 1 + c_3\beta' + \frac{c_1 + \beta' c_2}{s}.$$

(i) *The chain composes if and only if $\|R_0 u\|_0 \le s(\|u\|_1 + \beta'\Phi_1(u))$ for every u in the range of R_1, and*

$$\|R_k u\|_k + \beta'\Phi_k(R_k u) \le s\big(\|u\|_{k+1} + \beta'\Phi_{k+1}(u)\big)$$

for every $k \ge 1$ and every u in the range of R_{k+1}.

(ii) *If moreover $s\|z\|_{k+1} \le \|R_k z\|_k$ and $\|V_k y\|_{k+1} \le c_4 \|y\|_k$ for all k, then*

$$N_k(y) \le \big(1 + c_3\beta' + \delta(1 + c_1 c_4)\big)\|T_k y\|_0$$

for every k and every field y on X_k.

Proof. (i) Since $V_k R_k = I$, $(I - \Pi_k)R_k = 0$, so $N_k(R_k z) = s^k(\|R_k z\|_k + \beta'\Phi_k(R_k z))$ for $k \ge 1$. If the chain composes, take $z = u = R_{k+1} w$: then $(I - \Pi_{k+1})z = 0$, and dividing by s^k gives the two conditions. Conversely, let $k \ge 1$ and write $z = u + e$ with $u = \Pi_{k+1} z$ in the range of R_{k+1} and $e = (I - \Pi_{k+1})z$. The triangle inequality, the condition at u and the bounds on R_k and Φ_k give

$$\begin{aligned}
&\|R_k z\|_k + \beta'\Phi_k(R_k z) \\
&\quad \le s\big(\|u\|_{k+1} + \beta'\Phi_{k+1}(u)\big) + (c_1 + \beta' c_2)\|e\|_{k+1} \\
&\quad \le s\big(\|z\|_{k+1} + \beta'\Phi_{k+1}(z)\big) \\
&\qquad + \big(s(1 + c_3\beta') + c_1 + \beta' c_2\big)\|e\|_{k+1},
\end{aligned}$$

using $\|u\| \le \|z\| + \|e\|$ and $\Phi_{k+1}(u) \le \Phi_{k+1}(z) + c_3\|e\|$; the last coefficient is at most $s\delta$, which is the step $N_k(R_k z) \le N_{k+1}(z)$. The step $k = 0$ is the same with $\beta' = 0$ on the left side.

(ii) Iterating $s\|z\|_{k+1} \le \|R_k z\|_k$ gives $s^k\|y\|_k \le \|T_k y\|_0$. Also $\|(I - \Pi_k)y\|_k \le (1 + c_1 c_4)\|y\|_k$ and $\Phi_k(y) \le c_3\|y\|_k$, so $Q_k(y) \le (1 + c_3\beta' + \delta(1 + c_1 c_4))\|y\|_k$, and the bound follows, for $k = 0$ because the constant is at least one. □

The theorem moves the whole difficulty into the two-term inequality with one factor s at every scale; the deviation term absorbs the part of the field that the reconstruction does not produce. A comparison argument with local costs needs, in addition, that the transport cost of a perturbation is controlled by finitely many local terms, and the next lemma says exactly when this is possible.

Lemma 182 (local response bounds)**.** *Let* $\mathtt{t}$ *be a seminorm on a finite-dimensional real vector space and* l_j, $j \in J$, *finitely many linear forms on it. There are* $K_j \geq 0$ *with* $\mathtt{t} \leq \sum_j K_j |l_j|$ *if and only if* $\mathtt{t}(\xi) = 0$ *for every* ξ *with* $l_j(\xi) = 0$ *for all* j. *Finiteness of* J *cannot be dropped.*

Proof. The condition is clearly necessary. Conversely, $n = \sum_j |l_j|$ is a seminorm whose kernel lies in that of $\mathtt{t}$, so both descend to the quotient by $\ker n$, on which n is a norm; all norms on a finite-dimensional space are equivalent and the seminorm $\mathtt{t}$ is continuous, so $\mathtt{t} \leq Kn$ on the unit sphere of n and everywhere. For the last statement take finitely supported real sequences, $\mathtt{t}(\xi) = |\xi_0|$ and $l_j(\xi) = \xi_j - \xi_{j+1}$ for $j \geq 0$. If every $l_j(\xi)$ vanishes, ξ is constant and finitely supported, hence zero; but for a finite set J and $M > \max J$ the indicator ξ of $\{0, \dots, M\}$ has $l_j(\xi) = 0$ for $j \in J$ and $\mathtt{t}(\xi) = 1$. □

Remark 183 (a chain in one dimension). Put

$$\rho_k = \sqrt{\frac{4 + 2 \cdot 4^{-k}}{3}}, \qquad \theta_k = 1 + \frac{2}{\rho_k (2 + \rho_k)},$$

$\zeta_0 = 0$, $\zeta_1 = (\sqrt{2} - 1)/2$, $\zeta_2 = (\theta_1 - 1 + \theta_1 \zeta_1)/2$ and

$$\zeta_{k+1} = \frac{1}{2}\Big(\frac{1}{2\rho_k} + \Big(1 + \frac{1}{\rho_k}\Big) \zeta_k \Big) \qquad (k \geq 2).$$

These are the closed forms, computed in the archive, of the per-scale loss of the site-additive metrics and of the gradient coefficient of the optimal value-and-gradient chain for the Gaussian chain of block averages of side two on the line; that identification is not proved here, and only the following arithmetic is. Since $\rho_k \leq \rho_0 = \sqrt{2}$, $\theta_k \geq 1 + 2/(\sqrt{2}(2 + \sqrt{2})) = \sqrt{2}$, so $\prod_{i<k} \theta_i \geq (\sqrt{2})^k$. The map $f_k(\zeta) = \frac{1}{2}(\frac{1}{2\rho_k} + (1 + \frac{1}{\rho_k})\zeta)$ is increasing, with slope less than one and fixed point $\zeta_k^* = 1/(2(\rho_k - 1))$, and $f_k(\zeta) - \zeta = (1 - \frac{1}{2}(1 + \frac{1}{\rho_k}))(\zeta_k^* - \zeta)$. As ρ_k decreases to $2/\sqrt{3}$, ζ_k^* increases to $\zeta^* = (3 + 2\sqrt{3})/2$, so $f_k(\zeta^*) \leq \zeta^*$; with $\zeta_2 < \zeta^*$ induction gives $\zeta_k < \zeta^*$, that is,

$$1 + 2\zeta_k < 4 + 2\sqrt{3} \qquad \text{for every } k,$$

and $4 + 2\sqrt{3}$ is the limit of $1 + 2\zeta_k$, since $\zeta_k - \zeta_k^*$ contracts by a factor tending to $(1 + \sqrt{3}/2)/2 < 1$ and $\zeta_k^* \to \zeta^*$. Figure 22 shows both sequences.

Remark 184 (what is proved about the metrics). Proved: the admissible site-additive weights through a linear reconstruction are exactly those above $|R|^{\mathsf{T}} p$, and every admissible chain lies above the minimal weights (Lemma 177); under equal column norms and sums the loss on constants is the product of the factors θ_i, which exceed one exactly when the reconstruction changes sign, and nonnegative reconstructions lose nothing (Proposition 178); the conditional means of block averages over thick blocks change sign (Proposition 179), in particular for massive Gaussian fields on tori with at least three blocks in each direction (Remark 180); a chain with a deviation term composes if and only if a two-term inequality holds at every scale, and then its loss is bounded independently of the number of scales (Theorem 181); local response bounds exist exactly when no perturbation is invisible to all local terms (Lemma 182). So a proof of (H2) through a scale-by-scale comparison cannot use site-additive metrics on the block variables at Gaussian order: their loss grows geometrically in the number of scales.

Not proved: a linear, or nearly linear, reconstruction of the block-spin maps of the Wilson measure along the tuning; the two-term inequality of Theorem 181 for them with one factor s at every scale; the identification in Remark 183; (H2) at weak coupling. The statements of this subsection and their Lean 4 formalisation are due to James Montgomery Lockwood.

XI. THE OPEN INPUTS

This section states the two unproved inputs of Sec. X, the cube estimate and the limit, and measures the first: what can be controlled uniformly in the coupling, and what any proof of the cube estimate has to produce.

A. The two open statements

Theorem 45 reduces a continuum mass gap to two statements, neither of which is proved here or, to our knowledge, anywhere else.

(D) *The cube estimate.* There are $\alpha < 1$ and $\ell > 0$ such that $D(n, \alpha)$ holds with $na \leq \ell$ along the continuum tuning $\beta = \beta(a) \to \infty$, uniformly in a.

(E) *The limit.* The renormalised Schwinger functions converge along $a_k = 2^{-k} a_0$ to Schwinger functions that satisfy the Osterwalder–Schrader axioms and are not those of the vacuum alone.

(D) and (E) together give the mass gap of the continuum theory; the remaining Clay requirements, Euclidean invariance and the local structure of the limit [73], belong to (E). (D) is a statement about a single cube of fixed physical size and a single coupling at each spacing; (E) is the construction problem, on which Bałaban's renormalisation group analysis [6, 7, 12] gives the ultraviolet part in finite volume. Theorem 66 draws the conclusion of (D) from an averaged condition, a spectral gap of the dynamics that resamples cubes of fixed physical size, on gauge-invariant functions of the Wilson measure on tori without gauge fixing; that condition is proved at strong coupling (Corollary 74) and is not proved along the tuning, nor is its local form $\mathrm{BP}_{\mathrm{loc}}$, which suffices (Sec. X H).

For unsmoothed local fields at separated points, Sec. XI H reduces (E), given the averaged condition along the tuning, to a moment bound on separated products whose constants grow at most like a power of the inverse separation and like $K^n (n!)^\beta$, one rotation of infinite order and uniqueness of the limit (Corollary 300), from which reflection positivity in every coordinate direction, clustering in every direction, the linear growth condition of [96] and, for constants fixed by one reflected correlation, nontriviality follow; without the rate in

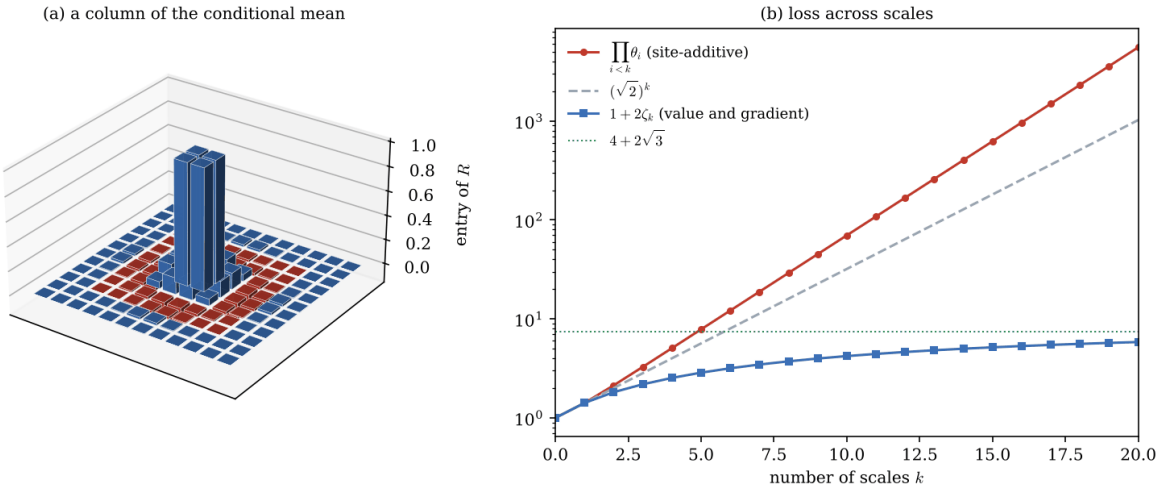


FIG. 22. Metrics across scales. (a) A column of the conditional mean $R = A^{-1}P^{\mathsf{T}}(PA^{-1}P^{\mathsf{T}})^{-1}$ of the block averages over blocks of side two on the periodic 18×18 lattice, for $A = \frac{1}{10}+4-N$: positive on its own block (blue), negative on the ring of neighbouring blocks (red), as Proposition 179 requires. (b) The site-additive loss $\prod_{i<k}\theta_i \geq (\sqrt{2})^k$ against $1+2\zeta_k < 4+2\sqrt{3}$ for the closed forms of Remark 183, on a logarithmic scale.

the separation the same inputs give these properties at separated points only (Corollary 275), and they do not imply temperedness (Proposition 302). For the two-point functions, with constants fixed by the largest reflected norm, that moment bound is equivalent to a comparison of reflected norms at short distance (Corollary 281), and its rate to a power rate in that comparison (Proposition 301); the comparison bounds the physical gap from above and fails in the strong-coupling range (Proposition 282). For $n \geq 3$ the moment bound does not follow from the two-point statements, reflection positivity and the gap, and its rate does not follow from bounds at each order (Propositions 310 and 312).

Remark 185. The following count relates (D) to a correlation length. Suppose that the gauge-fixed field responds to a change of one link on the boundary of a cube through a Poisson kernel with decay length ξ in lattice units. Then the response summed over the cube is of order ξ, so K_c is of order $n^3\xi$ and $D(n,\alpha)$ holds once $n \gtrsim \xi/\alpha$, that is, once $\ell \gtrsim \xi a/\alpha$, a fixed multiple of the physical correlation length. The hypothesis is that the gauge-fixed links decorrelate on the scale ξ, and it is a statement about the gauge, not about the gauge-invariant observables: the gauge here is built on straight lines, and in gauges of axial type the gauge-fixed variables can fluctuate on all scales while gauge-invariant observables cluster. Propositions 186 and 187 carry the count out exactly for Gaussian fields, and Proposition 187 shows that the quadratic approximation of the forest gauge fluctuates in this way. Where it does, (D) fails in this gauge and has to be posed for block-averaged variables, where it becomes part of the renormalisation group problem. Sec. XII A carries this out in the quadratic approximation.

The count of Remark 185 can be carried out exactly for a Gaussian field, the quadratic approximation of the gauge-fixed theory at weak coupling, and shows what (D) asks of the gauge-fixed variables.

Proposition 186. *Let $d \geq 1$, $b > 0$, $t \geq 0$, let $n \geq 2$ be even and $V = [0,n)^d \cap \mathbb{Z}^d$. Given $\eta \in \mathbb{R}^{\mathbb{Z}^d\setminus V}$, let $\gamma_V(\cdot \mid \eta)$ be the Gaussian law on $\mathbb{R}^V$ with density proportional to*

$$\exp\Big(-\frac{b}{2}\Big[\sum_{\{x,y\}\cap V\neq\emptyset}(\phi_x-\phi_y)^2 + t\sum_{x\in V}\phi_x^2\Big]\Big),$$

the first sum over nearest-neighbour pairs, with $\phi = \eta$ outside V. Define $C_{V,j}$, the classes c (residues modulo two) and K_c as above, with $d_V(\phi,\phi') = \sum_{x\in V}|\phi_x - \phi'_x|$ and $\delta(\eta_j,\eta'_j) = |\eta_j - \eta'_j|$, and put $N_n = (n/2)^d$. Then $K_c = \alpha_n(t)N_n$ for every class c, where

$$\alpha_n(t) = 1 - \frac{t}{n^d}\,\mathbf{1}^{\mathsf{T}}(t+L_n)^{-1}\mathbf{1}$$

and L_n is the Dirichlet Laplacian of V. In particular:

(i) *for $t = 0$, $K_c = N_n$ for every n and every b, so $D(n,\alpha)$ fails for every $\alpha < 1$;*

(ii) *for $t > 0$, $1-\alpha_n(t) \geq (5/6)^d\, tn^2/(tn^2+10d)$; for $d = 4$, $t = m^2a^2$ and $\ell/2 \leq na \leq \ell$ this gives $D(n,\alpha)$ with $\alpha = 1-(5/6)^4(m\ell)^2/((m\ell)^2+160)$, uniformly in a.*

Proof. Let $A = t + L_n$ and $G = A^{-1}$, and let B be the matrix with $B_{xj} = 1$ when $x \in V$ and $j \notin V$ are neighbours. The law $\gamma_V(\cdot \mid \eta)$ has covariance $b^{-1}G$ and mean $GB\eta$. The matrix A has nonpositive off-diagonal entries and is positive definite, so $G \geq 0$ entrywise. Changing η_j by Δ translates the law by ΔGB_j, where B_j is the column of j. The coupling by translation gives $W_1^{d_V} \leq |\Delta|\sum_x (GB_j)_x$, and the 1-Lipschitz function $\sum_x \phi_x$ gives the reverse inequality, so $C_{V,j} = \mathbf{1}^{\mathsf{T}}GB_j$. Since $B\mathbf{1} = L_n\mathbf{1}$ (both count the exterior neighbours of each site),

$$\sum_{j\notin V} C_{V,j} = \mathbf{1}^{\mathsf{T}}GL_n\mathbf{1} = \mathbf{1}^{\mathsf{T}}\mathbf{1} - t\,\mathbf{1}^{\mathsf{T}}G\mathbf{1} = n^d\alpha_n(t).$$

The reflection $x_i \mapsto n-1-x_i$ preserves V and the coefficients and, as n is even, exchanges the two residues of the coordinate x_i; so the 2^d class sums are equal, and each is $n^d \alpha_n(t)/2^d = \alpha_n(t) N_n$. This proves the formula and (i).

For (ii) let $v(i) = (i+1)(n-i)$ for $0 \le i < n$ and $\psi(x) = \prod_i v(x_i)$. The one-dimensional Dirichlet Laplacian maps v to the constant 2, so with $s_1 = \sum_i v(i) = n(n+1)(n+2)/6$ and $s_2 = \sum_i v(i)^2 = n(n+1)(n+2)(n^2+2n+2)/30$,

$$\mathbf{1}^{\mathsf{T}} \psi = s_1^d, \qquad \psi^{\mathsf{T}} A \psi = t s_2^d + 2d s_1 s_2^{d-1}.$$

By the Cauchy–Schwarz inequality for the form of A^{-1}, $\mathbf{1}^{\mathsf{T}} G \mathbf{1} \ge (\mathbf{1}^{\mathsf{T}}\psi)^2 / \psi^{\mathsf{T}} A \psi$, hence

$$1 - \alpha_n(t) \ge \Big(\frac{s_1^2}{n s_2}\Big)^d \frac{t}{t + 2d s_1/s_2} \ge \Big(\frac{5}{6}\Big)^d \frac{t n^2}{t n^2 + 10 d},$$

since $s_1^2/(n s_2) = \frac{5}{6}(n^2+3n+2)/(n^2+2n+2) \ge \frac{5}{6}$ and $2d s_1/s_2 = 10d/(n^2+2n+2) \le 10d/n^2$. With $d=4$ and $tn^2 = m^2 (na)^2 \ge (m\ell)^2/4$ the last fraction is at least $(m\ell)^2/((m\ell)^2 + 160)$. □

The Gaussian field is not Yang–Mills theory, and the proposition is a benchmark for the condition, not a statement about the Wilson measure. It shows that the averaged cube condition is met at a fixed physical scale by a field with a mass and fails exactly, at every size and every stiffness, for a massless one. Small local fluctuations alone therefore give nothing: in the massless Gaussian field on $\mathbb{Z}^d$, $d \ge 3$, the variance of $\phi_x - \phi_{x+e_1}$ is $1/(db)$, which tends to zero as $b \to \infty$, and still $K_c = N_n$. A proof of (D) along the tuning has to produce in the gauge-fixed variables the loss that the parameter t supplies here, from the Wilson action alone.

The quadratic approximation of the forest gauge itself, with the plaquette action in place of the Laplacian, is no better than the massless field of Proposition 186(i).

Proposition 187. *Let $b > 0$, let n be even and V the cube of Definition 42. For real functions φ on the links of $\mathbb{Z}^4$ that vanish on F, let $(d\varphi)_p$ be the sum of φ around the plaquette p with the signs of its orientation. Given $\eta \in \mathbb{R}^{A \setminus V}$, let $\gamma_V(\cdot \mid \eta)$ be the law on $\mathbb{R}^V$ with density proportional to*

$$\exp\Big(-\frac{b}{2} \sum_{p \cap V \neq \emptyset} (d\varphi)_p^2\Big),$$

with $\varphi = \eta$ on $A \setminus V$ and $\varphi = 0$ on F, and define $C_{V,j}$ and K_c as above with $d_V(\varphi, \varphi') = \sum_{i \in V} |\varphi_i - \varphi_i'|$ and $\delta(\eta_j, \eta_j') = |\eta_j - \eta_j'|$. Then

$$\sum_c K_c \ge |V| = 48 N_n,$$

so $K_c \ge N_n$ for some class c, and $D(n,\alpha)$ fails for every $\alpha < 1$, at every n and every b.

Proof. The laws. The quadratic form $\sum_{p \cap V \neq \emptyset} (d\varphi)_p^2$ with $\varphi = 0$ outside V vanishes only at $\varphi = 0$: the argument of Corollary 263, written additively, gives a function h on the sites with $\varphi_{(x,\mu)} = h(x) - h(x+e_\mu)$, constant outside the box and along the lines of F, hence constant. So $\gamma_V(\cdot \mid \eta)$ is a Gaussian law whose covariance does not depend on η and whose mean is $\sum_j \eta_j u_j$ for some vectors $u_j \in \mathbb{R}^V$. Two exteriors that differ in finitely many links give translates of one law by $t = \sum_j (\eta_j' - \eta_j) u_j$; the coupling by translation and the 1-Lipschitz function $\sum_i \operatorname{sign}(t_i)\varphi_i$ show that their distance is $\|t\|_1$. In particular $C_{V,j} = \|u_j\|_1$, and

$$W_1^{d_V}\big(\gamma_V(\cdot \mid \eta), \gamma_V(\cdot \mid \eta')\big) \le \sum_j C_{V,j} |\eta_j' - \eta_j|.$$

Lines. By Lemma 9(i) the lines of F are straight, disjoint and cover the sites, and the two ends of an active link (x,μ) lie on different lines, since the line through x has direction $\sigma(x) \neq \mu$. For a line ℓ let h_ℓ be the indicator of its sites and $(dh_\ell)_{(x,\mu)} = h_\ell(x) - h_\ell(x+e_\mu)$. Then dh_ℓ vanishes on F and around every plaquette, so the density is invariant under $\varphi \mapsto \varphi + dh_\ell$, and $\gamma_V(\cdot \mid dh_\ell)$ is $\gamma_V(\cdot \mid 0)$ translated by the restriction of dh_ℓ to V. Hence

$$\sum_{i \in V} |(dh_\ell)_i| \le \sum_{j \in A \setminus V} C_{V,j} |(dh_\ell)_j|.$$

Here $|(dh_\ell)_e| = 1$ when exactly one end of e lies on ℓ and 0 otherwise, so $\sum_\ell |(dh_\ell)_e| = 2$ for every active link e, the sum running over the finitely many lines that meet a link entering the inequality. Summing over these lines gives $2|V| \le 2\sum_j C_{V,j} = 2\sum_c K_c$. □

The proof uses only that gauge transformations constant on the lines of F move the whole cube while changing few boundary links, which is the behaviour of gauges of axial type anticipated in Remark 185. The class sums themselves, computed exactly for $n \le 8$, lie between 1.57 and 11.24 times N_n and increase with n. Below the correlation length the theory is weakly coupled, and the quadratic approximation is its leading order there. A proof of (D) along the tuning therefore has to show that the nonperturbative regime reduces the cube coefficients of the forest gauge below N_n, or (D) has to be posed for other variables.

The lines of F are not what makes the argument work. A constant field, corrected by a periodic gauge transformation so that it vanishes on the fixed links, costs no action and changes every exterior link by a bounded amount, and this happens in every gauge fixed on an invariant forest.

Proposition 188. *Let $b > 0$ and let F be a set of links of $\mathbb{Z}^4$, invariant under the translations by $2\mathbb{Z}^4$, that contains no closed path and whose trees are infinite and contain every site. Let A be the set of the other links, V the set of links $(x,\mu) \in A$ with $x \in [0,n)^4$, $n \ge 2$ even, and define the classes, $\gamma_V(\cdot \mid \eta)$, $C_{V,j}$ and K_c as in Proposition 187 with this F. Then there are numbers $a_c \ge 0$, not all zero, with*

$$\sum_c a_c K_c \ge N_n \sum_c a_c,$$

so $K_c \ge N_n$ for some class c, and $D(n,\alpha)$ fails for every $\alpha < 1$, at every n and every b.

Proof. The laws. Let φ vanish outside V and on F, with $(d\varphi)_p = 0$ for every plaquette p that meets V. The other plaquettes contain no link of V, so $d\varphi = 0$ on every plaquette, and $\varphi = dh$ for a function h on the sites. Since φ vanishes outside a box and the complement of a box is connected, h is constant there, and we take that constant to be 0. As φ vanishes on F, h is constant on every tree of F; each tree is infinite and meets the complement of the box, so $h = 0$ and $\varphi = 0$. Hence, as in the proof of Proposition 187, $\gamma_V(\cdot \mid \eta)$ is a nondegenerate Gaussian law with mean $\sum_j \eta_j u_j$, where only the links j that share a plaquette with V contribute, and $C_{V,j} = \|u_j\|_1$.

A flat field. For $c \in \mathbb{R}^4$ and a function h on $\mathbb{Z}^4$ of period two in each coordinate put $\psi_{(x,\mu)} = c_\mu + h(x) - h(x+e_\mu)$. Then $d\psi = 0$ on every plaquette and ψ is invariant under $2\mathbb{Z}^4$; if $\psi = 0$, adding ψ at (x,μ) and $(x+e_\mu,\mu)$ gives $2c_\mu = 0$, and then h is constant. These fields form a space of dimension $16 - 1 + 4 = 19$. Let f be the number of orbits of $2\mathbb{Z}^4$ in F. The box $[0,2m)^4$ has $16m^4$ sites, and the links of F with both ends in it form a forest, so there are at most $16m^4 - 1$ of them, while each orbit contributes at least $(m-1)^4$; letting $m \to \infty$ gives $f \le 16$. Vanishing on F is therefore at most 16 linear conditions, and some $\psi \neq 0$ of this form vanishes on F. It is nonzero on some link of A. Put $a_c = |\psi_j|$ for $j \in c$.

The inequality. Let η be the restriction of ψ to $A \setminus V$. The field equal to ψ on V and to η outside gives the quadratic form its least value 0, so by nondegeneracy the mean of $\gamma_V(\cdot \mid \eta)$ is the restriction of ψ to V, and $\psi_i = \sum_j \psi_j (u_j)_i$ for $i \in V$. Hence

$$\sum_{i \in V} |\psi_i| \le \sum_{j \in A \setminus V} C_{V,j} |\psi_j| = \sum_c a_c K_c .$$

A class consists of the active links with a given direction and a given residue of the base point modulo two, so it meets V in N_n links, and the left side is $N_n \sum_c a_c > 0$. So $K_c \ge N_n$ for some class with $a_c > 0$. □

For the direction field of Table VI the flat fields vanishing on F form a space of dimension 7, and for the axial field $\sigma = 0$ one of dimension 10. The proposition is weaker than Proposition 187 for the forest gauge, which bounds the sum over all classes, and it says nothing about two kinds of condition. A mass gives the constant fields an action, which is why Proposition 186 distinguishes $t > 0$ from $t = 0$. The constrained condition D_c of Theorem 334 holds the block averages fixed, and a constant field changes them. Within the quadratic approximation no gauge fixed on an invariant forest gives (D); the loss has to come from the Wilson action beyond its quadratic part, or (D) has to be posed for variables, such as the block averages, that a constant field moves.

Remark 189. Five other routes stop at (D) or (E), or earlier.

(a) *Disjoint blocks.* The block criterion of Lemma 667 with a partition into cubes of side n requires, for a link on a face, that its influence on the neighbouring cube be less than one. In the count of Remark 185 that influence is of order ξ, which grows like $1/a$. Averaging over the N_n translates divides it by the number of translates on whose boundary the link lies, a factor of order $1/n$, and this is why (D) is stated for averaged cubes.

(b) *Rescaling a fixed lattice gap.* Any bound uniform in lattice units, such as those of Theorems 1–7, gives trivial continuum two-point functions [21]; rescaling it produces no gap.

(c) *Assuming the gap.* Hypothesis R of [21] assumes the gap in physical units together with the limit. Theorem 45 replaces the gap part by the cube estimate (D), which concerns one box and one coupling at each spacing.

(d) *Computation.* At a fixed coupling and a fixed small n, $D(n,\alpha)$ is a finite set of Kantorovich bounds and could in principle be evaluated as in Remark 36; for $n = 2$ and SU(2) the block already has 48 links and 144 real variables. (D) asks for it at infinitely many couplings with $n \to \infty$, which no finite computation covers.

(e) *Axioms alone.* Proposition 41 shows that positivity, invariance and regularity do not force a gap, and Proposition 40 shows that the growth route to rotation invariance is closed for the action density. The infrared input therefore has to be an estimate such as (D).

B. The ratio, the strong-coupling range and the cube estimate

This subsection states the open input of Theorem 48 exactly, proves that it fails along every sequence of couplings in the range where the paper has a gap, shows that the hypothesis of Theorem 45 holds along some of these sequences and therefore cannot imply the input, and replaces the input by a condition that is necessary and sufficient and asks only for an upper bound on an energy and a lower bound on one overlap.

The notation is that of Theorem 48: translation invariant states ω_k at couplings β_k, with transfer matrices $\mathbb{T}_k$ and vacua Ω_k (Appendix F), the centred flowed packet A_k of a nonzero $f_0 \in C_c(\mathbb{R}^4)$ supported in $\{t > 0\}$, and $C_k(t) = \omega_k(\vartheta(\bar{A}_k)\tau_{t/a_k}A_k)$ for $t \in a_k\mathbb{Z}_{\ge 0}$, with $C_k(0) > 0$. For $u \ge 0$ and $\sigma > 0$ in $a_k\mathbb{Z}$ put

$$\rho_k(u,\sigma) = \frac{C_k(2u+\sigma)}{C_k(2u)},$$

so that $\rho(\sigma)$ of Theorem 48 is a limit of $\rho_k(0,\sigma)$. Let $f_{0,u} = f_0(\cdot - (u,0))$; for $u/a_k \in \mathbb{Z}$ its packet is $\tau_{u/a_k}A_k$, and $\Psi_{k,u} = C_k(2u)^{-1/2}\Psi_{\tau_{u/a_k}A_k}$ is the corresponding unit vector. For dyadic $u \ge 0$ consider

(N_u) $\liminf_{k\to\infty} \rho_k(u,\sigma) > 0$ for some dyadic $\sigma > 0$.

Proposition 190. (i) $C_k(t) > 0$ *for every* $t \in a_k\mathbb{Z}_{\ge 0}$, *and* $\log C_k$ *is convex and nonincreasing on* $a_k\mathbb{Z}_{\ge 0}$.

(ii) $0 < \rho_k(u,\sigma) \le 1$; $\rho_k(u,\sigma)$ *is nondecreasing in* u *and nonincreasing in* σ, *and* $\rho_k(u,\sigma') \ge \rho_k(u,\sigma)^{\sigma'/\sigma}$ *for* $\sigma' \ge \sigma$.

(iii) *For* $u \ge 0$ *dyadic, Theorem 48 applies to* $f_{0,u}$, *with reflected norm* $C_k(2u)$ *and ratio* $\rho_k(u,\sigma)$. *Every limit it produces along subsequences is nontrivial if and only if* (N_u) *holds, and some limit is nontrivial if and only if* $\limsup_k \rho_k(u,\sigma) > 0$. *If* ($\mathrm{N}_u$) *holds for one* σ, *it holds for every dyadic* $\sigma > 0$, *and* (N_u) *implies* ($\mathrm{N}_{u'}$) *for* $u' \ge u$.

(iv) *Let $\mu_{k,u}$ be the spectral measure, in energies $E = -a_k^{-1}\log\lambda$, of $\mathbb{T}_k$ at $\Psi_{k,u}$. If $\rho_k(u,\sigma) \geq c > 0$, then $\mu_{k,u}([0,\Lambda]) \geq c/2$ with $\Lambda = \sigma^{-1}\log(2/c)$; if $\mu_{k,u}([0,\Lambda]) \geq v$, then $\rho_k(u,\sigma) \geq ve^{-\sigma\Lambda}$. So (N_u) holds if and only if a fixed fraction of the spectral mass of $\Psi_{k,u}$ stays at bounded physical energy, uniformly in k.*

Proof. (i) The function f_0 is supported in $\{t > 0\}$, so the lattice points with $f_0(a_k x) \neq 0$ have $x_0 \geq 1$, and $A_k = \sum_{n\geq 1} \tau_n B_n$ is a finite sum in which B_n is a function of the spatial links of the slice at time 0, by Proposition 47(i). In the reconstruction [95–97], as in the proof of Theorem 48, $\Psi_{\tau_n B} = \mathbb{T}_k^n \Psi_B$, hence $\Psi_{A_k} = \mathbb{T}_k \xi$ with $\xi = \sum_n \mathbb{T}_k^{n-1}\Psi_{B_n}$. The transfer matrix is self-adjoint and positive [83, 97, 105], and its range is orthogonal to its kernel, so the spectral measure ν_k of $\mathbb{T}_k$ at $\Psi_k = C_k(0)^{-1/2}\Psi_{A_k}$ (Lemma 670) has no atom at 0. Therefore $C_k(t)/C_k(0) = \int \lambda^{t/a_k} d\nu_k > 0$. Convexity and monotonicity are Proposition 51(ii).

(ii) The first claims follow from (i). On a lattice, a convex function has chord slopes that increase in both end points; this gives monotonicity in u, and, comparing the chords $[2u, 2u+\sigma]$ and $[2u, 2u+\sigma']$, the power bound. Monotonicity in σ is that of C_k.

(iii) Translation invariance and $\vartheta\tau_u = \tau_{-u}\vartheta$ give, with times written in lattice steps,

$$\omega_k\big(\vartheta(\tau_u \bar{A}_k)\,\tau_{t+u}A_k\big) = \omega_k\big(\vartheta(\bar{A}_k)\,\tau_{t+2u}A_k\big),$$

which gives the reflected norm and the ratio. The function $f_{0,u}$ is nonzero and supported in $\{t > u\}$. If $\liminf_k \rho_k(u,\sigma) = 0$, a subsequence has $\rho_k(u,\sigma) \to 0$, and a refinement given by Theorem 48(ii) has a trivial limit by part (iv) of that theorem; if the $\liminf$ is at least $c > 0$, every limit has $\rho(\sigma) \geq c$ and is nontrivial. The statement with $\limsup$ is the last part of Theorem 48(iv). Independence of σ follows from (ii), by the power bound for $\sigma' \geq \sigma$ and by monotonicity for $\sigma' \leq \sigma$, and the last claim from monotonicity in u.

(iv) The vector $\Psi_{k,u}$ has no spectral weight at $\lambda = 0$, by the proof of (i), and $\rho_k(u,\sigma) = \int e^{-\sigma E} d\mu_{k,u}$. This integral is at most $\mu_{k,u}([0,\Lambda]) + e^{-\sigma\Lambda}$ and at least $e^{-\sigma\Lambda}\mu_{k,u}([0,\Lambda])$. □

The open input of Theorem 48 is therefore exactly (N_u) for one packet f_0 and one dyadic $u \geq 0$. It is a uniformity statement and nothing else: at each fixed k the ratio is positive by (i). The required uniformity is in k along the tuning, at fixed physical separations u and σ, so that $\sigma/a_k \to \infty$ lattice steps; the constant may depend on G, Φ, τ, f_0, u, σ and on the tuning, and nothing is required uniformly in f_0 or in τ. The ratio does not change when A_k is multiplied by a constant, so the renormalisation Z_k plays no role in it. The case $u = 0$ is the strongest and implies all the others. Proposition 49(i) assumes (N_0) and renormalises the packet shifted by $\sigma/2$. The hypothesis of Lemma 462 is $(\mathrm{N}_{s/4})$ at $\sigma = s/2$; it is weaker than (N_0), as the example after Proposition 51 shows, and it suffices for a nontrivial limit of the packet shifted by $s/4$. Granted the convergence asked for in (E), the clause of (E) that the limit is not that of the vacuum alone is, for the flowed packets, (N_u) for one f_0 and one u; the convergence of the whole sequence, Euclidean invariance and the higher correlations remain [73]. No bound is required at one lattice step: the first-step energy $-a_k^{-1}\log\rho_k(u,a_k)$ may diverge while (N_u) holds, as Remark 53 shows for the slice field of the unflowed magnetic energy.

The strong-coupling range. In the range where the paper proves a gap, the ratio has two-sided bounds in lattice units, and the upper one rules out every continuum tuning there. Put

$$m_0 = -\tfrac{1}{2}\log 0.9936,$$

so that $m_0 \geq 0.0032$ by $-\log(1-x) \geq x$.

Theorem 191. *Let $G = \mathrm{SU}(2)$, let $\beta_{W,k} \in [0, 0.348]$ be any couplings, and let ω_k be the states of Theorems 1 and 3. Then for every Φ, τ and f_0 with $C_k(0) > 0$, every dyadic $u \geq 0$ and $\sigma > 0$, and every k with $u/a_k, \sigma/a_k \in \mathbb{Z}$,*

$$\rho_k(0,a_k)^{\sigma/a_k} \leq \rho_k(u,\sigma) \leq e^{-m_0\sigma/a_k}.$$

Consequently every limit of Theorem 48 along these couplings, for every packet $f_{0,u}$, is trivial, and if (N_u) holds for some packet along a tuning $(\beta_{W,k})$ of $\mathrm{SU}(2)$, then $\beta_{W,k} > 0.348$ for all large k.

Proof. By Theorem 3 the transfer matrix of ω_k has gap $m_k \geq -\frac{1}{2}\log\alpha(\beta_{W,k})$ on $\Omega_k^\perp$, and α is increasing with $\alpha(0.348) \leq 0.9936$, so $m_k \geq m_0$. The vector $\Psi_{k,u}$ is a unit vector orthogonal to Ω_k, because A_k is centred, and $\mathbb{T}_k$ preserves $\Omega_k^\perp$, so $\rho_k(u,\sigma) = \langle \Psi_{k,u}, \mathbb{T}_k^{\sigma/a_k}\Psi_{k,u}\rangle \leq e^{-m_0\sigma/a_k}$. For the lower bound, $\rho_k(u,\sigma) \geq \rho_k(0,\sigma)$ by Proposition 190(ii), and Jensen's inequality for the convex function λ^{σ/a_k} and the probability measure ν_k of the proof of Proposition 190(i) gives $\rho_k(0,\sigma) = \int \lambda^{\sigma/a_k} d\nu_k \geq (\int \lambda\, d\nu_k)^{\sigma/a_k} = \rho_k(0,a_k)^{\sigma/a_k}$, with $\rho_k(0,a_k) > 0$ by Proposition 190(i). The upper bound tends to zero as $a_k \to 0$, which gives triviality by Proposition 190(iii). If (N_u) held along a tuning with $\beta_{W,k} \leq 0.348$ for infinitely many k, the upper bound along those k would contradict it. □

Writing $\rho_k(u,\sigma) = e^{-\varepsilon_k\sigma/a_k}$, the theorem says $m_0 \leq \varepsilon_k \leq -\log\rho_k(0,a_k)$: both ends are rates in lattice units, and (N_u) asks for $\varepsilon_k = O(a_k)$. Reflection positivity gives the positivity of the lower end, through the range of $\mathbb{T}_k$, and the transfer-matrix gap gives the upper bound; neither bears on the size of ε_k/a_k except to exclude the range. The size of $\rho_k(0,a_k)$ is not estimated here; any estimate of it at strong coupling is in lattice units, and it is at most e^{-m_0}. The same holds for every compact simple G and couplings in $[0,\beta_1]$ with $N\beta_1 < \frac{2}{29}\kappa_G$, with $m_0 = m_G(\beta_1)$ of Theorem 5; for fixed β this is Proposition 60(i). So the strong-coupling range bears on the continuum tuning only negatively: a tuning with a nontrivial limit leaves it for good. That the tuning has to go to $\beta_W \to \infty$ is the expectation from asymptotic freedom and is not proved here.

The cube estimate in the same range. The hypothesis of Theorem 45 holds at fixed strong coupling with cubes of fixed physical size, and with every α.

Proposition 192. (i) *Suppose $D(n_k,\alpha_k)$ holds at β_k with $n_k a_k \leq \ell$. Then $\rho_k(u,\sigma) \leq \alpha_k^{\sigma/(2\ell)}$ for every u and σ. Hence*

(N_u) *with constant c at σ forces $\alpha_k \ge c^{2\ell/\sigma}$ for large k, and the physical mass of Theorem 45 obeys $m_* \le \sigma^{-1}\log(1/c)$.*

(ii) *Let $G=\mathrm{SU}(2)$, $0 \le \beta_W < \frac{1}{3}$ fixed, $\ell > 0$, and $n_k = 2\lfloor \ell/(2a_k)\rfloor$. For every $\alpha \in (0,1)$, $D(n_k,\alpha)$ holds at β_W for all k with $n_k \ge 2$ and $5\beta_W \le \alpha(1-3\beta_W)n_k$, and $n_k a_k \le \ell$. So the hypothesis of Theorem 45 holds along the constant tuning, from some k on, with every $\alpha \in (0,1)$, while every limit of Theorem 48 is trivial.*

Proof. (i) By Theorem 43 the gap of $\mathbb{T}_k$ on $\Omega_k^\perp$ is at least $\log(1/\alpha_k)/(2n_k) \ge a_k \log(1/\alpha_k)/(2\ell)$, and $\rho_k(u,\sigma) \le e^{-m_k\sigma/a_k}$ as in the proof of Theorem 191. The bound on m_* is the case $\alpha_k = \alpha$, since then $\alpha^{\sigma/(2\ell)} = e^{-m_*\sigma}$.

(ii) Proposition 260 gives $D(n,\alpha)$ for every $\alpha \ge \hat\alpha_n$, and $\hat\alpha_n \le 5\beta_W/((1-3\beta_W)n)$. Triviality is Theorem 191. □

Part (i) shows that (D) is itself a triviality statement when its constant improves along the tuning: if $\alpha_k \to 0$ with $n_k a_k \le \ell$, every limit is trivial. Along the constant tuning of part (ii) with $\beta_W > 0$ this gives $\rho_k(u,\sigma) \le (5\beta_W/((1-3\beta_W)n_k))^{\sigma/(2\ell)}$, a power of a_k, from the cube estimate alone. Part (ii) shows that no argument that uses (D) only through the hypothesis of Theorem 45, or through Theorem 43, can give (N_u): that hypothesis is satisfied by Wilson measures whose continuum limits are trivial. Statement (D) of Sec. XI adds that $\beta \to \infty$ along the tuning, and an implication that uses this and the weak-coupling structure besides (D) is not excluded; it would use information that (D) does not contain.

Trial states. The input is equivalent to the existence of states of bounded physical energy with a fixed overlap with the packet.

Lemma 193. *Let $\mathbb{T}$ be self-adjoint with $0 \le \mathbb{T} \le 1$, let $a > 0$ and let Ψ be a unit vector.*

(a) *Let χ be a unit vector with $\langle \chi, (1-\mathbb{T})\chi\rangle \le \kappa a$ and $|\langle \chi, \Psi\rangle| \ge v > 0$. Then for $\sigma \ge 2a$,*

$$\langle \Psi, \mathbb{T}^{\sigma/a}\Psi\rangle \ge \Big(v - \sqrt{\kappa\sigma/2}\Big)_+^2,$$

and if $a\Lambda \le 1$ with $\Lambda = 8\kappa/v^2$, the spectral projection $P = \mathbf{1}_{[e^{-a\Lambda},1]}(\mathbb{T})$ has $\|P\Psi\| \ge v/2$.

(b) *Conversely, if $\langle \Psi, \mathbb{T}^{\sigma/a}\Psi\rangle \ge c > 0$ with $\sigma/a \in \mathbb{Z}_{>0}$, put $\Lambda = \sigma^{-1}\log(2/c)$ and $P = \mathbf{1}_{[e^{-a\Lambda},1]}(\mathbb{T})$. Then $\chi = P\Psi/\|P\Psi\|$ is a unit vector with $\langle \chi, (1-\mathbb{T})\chi\rangle \le a\Lambda$ and $\langle \chi, \Psi\rangle = \|P\Psi\| \ge (c/2)^{1/2}$. If $\mathbb{T}\Omega = \Omega$ and $\Psi \perp \Omega$, then $\chi \perp \Omega$.*

Proof. (a) Let $m = \sigma/(2a) \ge 1$ and let ν_χ be the spectral measure at χ. For $\lambda \in [0,1]$, $0 \le 1-\lambda^m \le m(1-\lambda)$ by Bernoulli's inequality, so

$$\|(1-\mathbb{T}^m)\chi\|^2 = \int (1-\lambda^m)^2 d\nu_\chi \le m\int (1-\lambda)\, d\nu_\chi \le \frac{\kappa\sigma}{2}.$$

Hence $\|\mathbb{T}^m\Psi\| \ge |\langle \mathbb{T}^m\chi, \Psi\rangle| \ge |\langle \chi, \Psi\rangle| - \|(1-\mathbb{T}^m)\chi\| \ge v - \sqrt{\kappa\sigma/2}$, and $\langle \Psi, \mathbb{T}^{\sigma/a}\Psi\rangle = \|\mathbb{T}^m\Psi\|^2$. For the projection, $1-\lambda > 1-e^{-a\Lambda} \ge a\Lambda/(1+a\Lambda) \ge a\Lambda/2$ off $[e^{-a\Lambda},1]$, so $\|(1-P)\chi\|^2 \le 2\kappa/\Lambda = v^2/4$, and $\|P\Psi\| \ge |\langle \chi, P\Psi\rangle| \ge |\langle \chi, \Psi\rangle| - \|(1-P)\chi\| \ge v/2$.

(b) With ν the spectral measure at Ψ, $c \le \int \lambda^{\sigma/a} d\nu \le \|P\Psi\|^2 + e^{-\sigma\Lambda} = \|P\Psi\|^2 + c/2$. On the range of P, $1-\mathbb{T} \le 1-e^{-a\Lambda} \le a\Lambda$. Since $P\Omega = \Omega$, $\langle \Omega, P\Psi\rangle = \langle \Omega, \Psi\rangle = 0$. □

Corollary 194. (N_u) *holds if and only if there are $\kappa, v > 0$ and unit vectors χ_k in the reconstructed spaces of the ω_k such that, for all large k,*

$$\langle \chi_k, (1-\mathbb{T}_k)\chi_k\rangle \le \kappa a_k, \qquad |\langle \chi_k, \Psi_{k,u}\rangle| \ge v.$$

Then $\rho_k(u,\sigma) \ge (v - \sqrt{\kappa\sigma/2})_+^2$ for $\sigma \ge 2a_k$, and the χ_k may be taken of the form $\Psi_{X_k}/\|\Psi_{X_k}\|$ for gauge-invariant local functions X_k at positive times, at the cost of replacing (κ, v) by $(2\kappa, v/2)$. If moreover $D(n_k,\alpha)$ holds with $n_k a_k \le \ell$, then $m_* = \log(1/\alpha)/(2\ell) \le 8\kappa/v^2$.

Proof. Sufficiency is Lemma 193(a) with $\Psi = \Psi_{k,u}$, $a = a_k$, and Proposition 190(ii) extends the bound from small σ to every σ. Necessity is Lemma 193(b) with $\kappa = \sigma^{-1}\log(2/c)$ and $v = (c/2)^{1/2}$. The vectors Ψ_X with X local are dense (Appendix F), and both conditions are continuous in χ_k for fixed k, $1-\mathbb{T}_k$ being bounded. Under $D(n_k,\alpha)$, Theorem 43 puts the spectral measure of $\Psi_{k,u} \perp \Omega_k$ in $[0, e^{-a_k m_*}]$; if $m_* > 8\kappa/v^2$, the projection of Lemma 193(a) annihilates $\Psi_{k,u}$ for large k, which is impossible. □

For $\chi_k = \Psi_{X_k}/\|\Psi_{X_k}\|$ the two conditions are statements about lattice correlations:

$$\omega_k\big(\vartheta(\bar X_k)X_k\big) - \omega_k\big(\vartheta(\bar X_k)\tau_1 X_k\big) \le \kappa a_k\, \omega_k\big(\vartheta(\bar X_k)X_k\big),$$
$$\big|\omega_k\big(\vartheta(\bar X_k)\tau_{u/a_k}A_k\big)\big|^2 \ge v^2\, \omega_k\big(\vartheta(\bar X_k)X_k\big)\, C_k(2u).$$

The first is an upper bound on a one-slab Dirichlet form, which is the direction of variational and perturbative estimates and the opposite of a mass gap; the second is a lower bound on one correlation. The choice $\chi_k = \Psi_{k,u}$ is Proposition 49(ii), and the order-two tree and saturation bounds give it by Proposition 61(iii); the corollary shows that the trial vector need not be the packet itself.

A soft component of the slice field. For product packets the trial vector can be built from a part of the slice field of Proposition 51, and the conditions become one first-step bound for that part and one bound for the rest at the earliest time of the packet. Let $f_0 = g \otimes h$ with $g \ge 0$ supported in $[t_0, t_1]$, and let u_k, n_0, n_1 be as in Proposition 51.

Theorem 195. *Suppose that for all large k there is a decomposition $u_k = u_k^{\mathrm{s}} + u_k^{\mathrm{h}}$ with $\mathbb{T}_k^{n_0} u_k^{\mathrm{s}} \ne 0$ and, for constants $\kappa > 0$ and $\theta \in [0,1)$ with $\kappa a_k \le \frac{1}{2}$,*

(S1) $\langle u_k^{\mathrm{s}}, (1-\mathbb{T}_k)u_k^{\mathrm{s}}\rangle \le \kappa a_k \|u_k^{\mathrm{s}}\|^2$;

(S2) $\|\mathbb{T}_k^{n_0} u_k^{\mathrm{h}}\| \le \theta e^{-2\kappa(t_1-t_0)} \|\mathbb{T}_k^{n_0} u_k^{\mathrm{s}}\|$.

Then the hypothesis of Corollary 194 holds for $u = 0$ with $\kappa e^{4\kappa(t_1-t_0)}$ in place of κ and $v = (1-\theta)/(1+\theta)$; in particular ($\mathrm{N}_0$) holds and

$$\frac{C_k(\sigma)}{C_k(0)} \ge \Big(\frac{1-\theta}{1+\theta} - \sqrt{\tfrac{1}{2}\kappa\sigma e^{4\kappa(t_1-t_0)}}\Big)_+^2$$

for $\sigma \ge 2a_k$.

Proof. By the proof of Proposition 51, $\Psi_{A_k} = W(\mathbb{T}_k)u_k$ with $W(\lambda) = \sum_n w_n \lambda^n$, $w_n \geq 0$, $w_n = 0$ unless $n_0 \leq n \leq n_1$, and $\|w\|_1 \lambda^{n_1} \leq W(\lambda) \leq \|w\|_1 \lambda^{n_0}$ on $[0,1]$, with $\|w\|_1 > 0$ because $C_k(0) > 0$. Write ν for the spectral measure at u_k^{s} and $y = \mathbb{T}_k^{n_0} u_k^{\mathrm{s}}$. For a probability measure, a nondecreasing and a nonincreasing function have $\int fg \leq \int f \int g$, because $\iint (f(x) - f(x'))(g(x) - g(x')) \leq 0$; with $f = \lambda^{2n_0}$, $g = 1 - \lambda$ and (S1),

$$\int \lambda^{2n_0}(1-\lambda)\, d\nu \leq \kappa a_k \|y\|^2.$$

So y satisfies (S1) as well, and Jensen's inequality for λ^{2j}, $j = n_1 - n_0$, gives $\|\mathbb{T}_k^j y\|^2 \geq \|y\|^2 (1 - \kappa a_k)^{2j} \geq \|y\|^2 e^{-4\kappa(t_1 - t_0)}$, since $\log(1-x) \geq -2x$ on $[0, \frac{1}{2}]$ and $a_k j \leq t_1 - t_0$. Hence

$$\|W(\mathbb{T}_k)u_k^{\mathrm{s}}\| \geq \|w\|_1 \|\mathbb{T}_k^{n_1} u_k^{\mathrm{s}}\| \geq \|w\|_1 e^{-2\kappa(t_1 - t_0)} \|y\|,$$

while $\|W(\mathbb{T}_k)u_k^{\mathrm{h}}\| \leq \|w\|_1 \|\mathbb{T}_k^{n_0} u_k^{\mathrm{h}}\| \leq \theta \|W(\mathbb{T}_k)u_k^{\mathrm{s}}\|$ by (S2). Let $\chi_k = W(\mathbb{T}_k)u_k^{\mathrm{s}} / \|W(\mathbb{T}_k)u_k^{\mathrm{s}}\|$. Its energy is

$$\frac{\int W^2 (1-\lambda)\, d\nu}{\int W^2 d\nu} \leq \frac{\|w\|_1^2 \kappa a_k \|y\|^2}{\|w\|_1^2 e^{-4\kappa(t_1 - t_0)} \|y\|^2} = \kappa a_k e^{4\kappa(t_1 - t_0)},$$

and with $r = \|W(\mathbb{T}_k)u_k^{\mathrm{h}}\| / \|W(\mathbb{T}_k)u_k^{\mathrm{s}}\| \leq \theta$ its overlap with $\Psi_k = \Psi_{A_k} / \|\Psi_{A_k}\|$ is at least $(1-r)/(1+r) \geq (1-\theta)/(1+\theta)$. Corollary 194 gives the rest. □

Condition (S1) asks for a soft component of the slice field whose first-step energy is bounded in physical units. Remark 53 shows why the whole slice field is not a candidate: at next order its first-step energy grows like $\log(\sqrt{\tau}/a_k)$, through a hard component of norm of order g^2. Condition (S2) asks that this hard component, propagated to the earliest time t_0 of the packet, be a fixed fraction of the soft part. For the free field of Remark 52, (S1) holds with $u_k^{\mathrm{h}} = 0$: since $(1 - e^{-a_k \omega})/a_k \leq \omega \leq a_k^{-1} \sinh(a_k \omega)$, the first-step energy $\langle u_k, (1 - \mathbb{T}_k)u_k \rangle / (a_k \|u_k\|^2)$ has a numerator whose integrand is at most a multiple of $e^{-8t_f |p|^2/\pi^2}$, and a denominator $D_k(0)$ with a positive limit, as in that remark. For the unflowed magnetic energy the whole slice field violates (S1) by a factor of order $1/a_k$, and the decomposition has to separate its hard modes. Both conditions concern energies of order $1/a_k$ at physical times below t_0; they are ultraviolet statements, the subject of a renormalisation group analysis between the scales a_k and t_0 [6, 7, 12], and they are not proved here. By Corollary 194 the trial-state condition is the least hypothesis that gives (N_u), since it is also necessary; by Proposition 192(ii) it is not a consequence of the cube estimate in the form used by Theorem 45; and (S1)–(S2) are its form for the slice field.

Remark 196. There is no circularity between (N_u) and (D). The proofs of Proposition 190, Lemma 193, Corollary 194 and Theorem 195 use reflection positivity, the transfer matrix and the spectral theorem only, and Theorem 48(i)–(iv), including the renormalisation $Z_k = C_k(0)^{-1/2}$, does not use (D). Conversely, Theorems 43 and 45 use no lower bound on any correlation. The two statements are opposite inequalities on the same spectral measures: (D) bounds ρ_k from above by $e^{-m_* \sigma}$, (N_u) from below. Their joint validity requires only the compatibility relations $\alpha \geq c^{2\ell/\sigma}$, $m_* \leq \sigma^{-1} \log(1/c)$ and $m_* \leq 8\kappa / \nu^2$ of Proposition 192 and Corollary 194, all of them statements in physical units. Neither implies the other. Strong-coupling Wilson measures satisfy the hypothesis of Theorem 45 and violate (N_u) (Proposition 192(ii)). In the free abelian theory the quadratic approximation of the forest gauge violates (D) at every size and stiffness (Proposition 187), while the zero-momentum packet of the magnetic energy has slice ratios with positive limits (Remark 53) and so satisfies (N_0) by Proposition 51(iii); (D) depends on the gauge and (N_u) does not. Theorem 57(iv) derives (N_0) from the order-two cases of (T1) and (T2), and Proposition 60 shows that those cases fail wherever the lattice gap is bounded below, in particular in the strong-coupling range, where the hypothesis of Theorem 45 holds.

C. Slab energies of local fields

This subsection tests the first condition of Corollary 194 against the most direct estimate for it: a pointwise bound on the change of a gauge-invariant function of one slice in one time step by the time-like plaquettes, combined with an upper bound on the plaquette energy. Both bounds are proved with explicit constants. The combination cannot give the condition along any tuning with $a_k \beta_k \to 0$, whatever the trial function, and the one-loop tuning is of this kind.

Let $G \subset \mathrm{U}(N)$ be a closed subgroup, let $|M|^2 = \mathrm{Tr}(M^* M)$ be the Frobenius norm, so that $|gMh| = |M|$ for $g, h \in G$, and put $s_p = 1 - N^{-1} \mathrm{Re}\,\mathrm{Tr}\, U_p$, so that $|U_p - I|^2 = 2N s_p$. Write $U_{(y,\mu)}$ for the link from y to $y + \hat{\mu}$, with $\mu = 0$ the time direction. Let ω be a state as in Appendix F, with transfer matrix $\mathbb{T}$, and let $\varepsilon_\omega = \sup_p \omega(s_p)$, which is $\omega(s_p)$ for every plaquette p when ω is invariant under translations and the hypercubic group. A *slice function* is a function X of the spatial links of the slice $x_0 = 0$ that depends only on the links of a finite set E, is invariant under the gauge transformations of that slice, and satisfies

$$|X(U) - X(V)| \leq \sum_{e \in E} L_e |U_e - V_e|$$

with constants $L_e \geq 0$; put $\Lambda(X) = \sum_{e \in E} L_e$. For a spatial link $e = ((0,x), i)$ let p_e be the time-like plaquette with lower edge e, with holonomy

$$U_{p_e} = U_{((0,x),i)} U_{((0,x+\hat{\imath}),0)} U^{-1}_{((1,x),i)} U^{-1}_{((0,x),0)}.$$

Lemma 197. *Let X be a bounded slice function.*

(i) $\langle \Psi_X, (1 - \mathbb{T})\Psi_X \rangle = \frac{1}{2}\omega(|\tau_1 X - X|^2)$, *and both sides are unchanged when a constant is added to X.*

(ii) *Pointwise,* $|\tau_1 X - X| \leq \sum_{e \in E} L_e |U_{p_e} - I|$.

(iii) $\langle \Psi_X, (1 - \mathbb{T})\Psi_X \rangle \leq N \Lambda(X)^2 \varepsilon_\omega$.

Proof. (i) The reflection ϑ in the plane $x_0 = 0$ fixes the spatial links of that slice, so $\vartheta(\bar{X}) = \bar{X}$ and $\|\Psi_X\|^2 = \omega(|X|^2)$. As in the proof of Proposition 51, $\Psi_{\tau_1 X} = \mathbb{T}\Psi_X$, so $\langle \Psi_X, \mathbb{T}\Psi_X \rangle = \omega(\bar{X}\tau_1 X)$, a real number because $\mathbb{T}$ is self-adjoint. Translation invariance gives $\omega(|\tau_1 X|^2) = \omega(|X|^2)$, hence

$$\omega(|\tau_1 X - X|^2) = 2\omega(|X|^2) - 2\omega(\bar{X}\tau_1 X) = 2\langle \Psi_X, (1 - \mathbb{T})\Psi_X \rangle.$$

The right side of the identity depends on X only through $\tau_1 X - X$.

(ii) For a site x of the slice put $g(x) = U_{((0,x),0)}$, and for $e = ((0,x),i)$ put $V_e = g(x) U_{((1,x),i)} g(x+\hat{\imath})^{-1}$. The function $\tau_1 X$ is X evaluated on the links $U_{((1,x),i)}$ in place of $U_{((0,x),i)}$, and gauge invariance under g gives $\tau_1 X(U) = X(V)$. Hence $|\tau_1 X - X| \le \sum_e L_e |V_e - U_e|$. Now $|V_e - U_e| = |U_e^{-1} V_e - I|$, and $U_e^{-1} V_e = U_e^{-1} U_{p_e}^{-1} U_e$, so $|V_e - U_e| = |U_{p_e}^{-1} - I| = |U_{p_e} - I|$.

(iii) By (i), (ii) and the Cauchy–Schwarz inequality,

$$\langle \Psi_X, (1-\mathbb{T})\Psi_X \rangle \le \frac{1}{2} \Lambda(X) \sum_e L_e \, \omega(|U_{p_e} - I|^2) \le N \Lambda(X)^2 \varepsilon_\omega .$$

□

The plaquette energy of Lemma 382 carries a factor $\log \beta_W$. It comes from the lower bound on the partition function alone; an upper bound in temporal gauge removes it.

Proposition 198. *Let $G = \mathrm{SU}(2)$ and let μ_L be the Wilson measure on $\Lambda_L = (\mathbb{Z}/L\mathbb{Z})^4$ of Lemma 382, with $L \ge 2$ and $\beta_W \ge 1$. Then $\varepsilon_L = \mathbb{E}_{\mu_L} s_p$ satisfies*

$$\varepsilon_L \le \frac{1}{\beta_W} \Big(K_1 + \frac{K_2}{L^4} + \frac{15}{16} \Big(\frac{1}{L} + \frac{1}{3L^4} \Big) \log \beta_W + \frac{K_3}{\beta_W} \Big)$$

with $K_1 = 3.7148$, $K_2 = 0.4699$ and $K_3 = 0.149$. In particular every limit as $L \to \infty$ of the measures μ_L along a subsequence, on the local functions, has $\varepsilon \le (K_1 + K_3/\beta_W)/\beta_W$.

Proof. Write $\beta = \beta_W$, $Z(\beta) = \int e^{-\beta S} dU$ with $S = \sum_p s_p$, and $\theta \in [0,\pi]$ for the angle of an element $\cos\theta\, I + \sin\theta\, J$ of $\mathrm{SU}(2)$; it has Haar density $\frac{2}{\pi}\sin^2\theta$ (Lemma 678), it is a bi-invariant distance from I, and $\frac{1}{2}\operatorname{Tr} U = \cos\theta$. The measure μ_L is invariant under the translations and the hypercubic group of the torus, so $\mathbb{E}_{\mu_L} S = 6L^4 \varepsilon_L$.

Convexity. The function $\log Z$ is convex with derivative $-\mathbb{E}_{\mu_L} S$ at β, so for $0 < \lambda < 1$

$$6L^4 (1-\lambda) \beta \, \varepsilon_L \le \log Z(\lambda\beta) - \log Z(\beta).$$

Upper bound. The integrand is gauge invariant, so the time links $U_{((t,x),0)}$ with $0 \le t \le L-2$, which form a forest, may be set to I without changing Z; this is the substitution $U \mapsto U^g$ with g built along each tree from its root, which preserves the Haar measure. Drop the spatial plaquettes and the time-like plaquettes between the times $L-1$ and 0, whose weights are at most 1. For each spatial link (x,i) the remaining weights are $\prod_{t=0}^{L-2} f(V_t V_{t+1}^{-1})$ with $V_t = U_{((t,x),i)}$ and $f(U) = e^{-\beta(1-\frac{1}{2}\operatorname{Tr} U)}$; integrating $V_0, \dots, V_{L-2}$ in turn, each integral is $z(\beta) = \int f \, dU$ by the invariance of the Haar measure. Hence $Z(\beta) \le z(\beta)^{3L^3(L-1)}$. With $y = \sin(\theta/2)$,

$$\begin{aligned} z(\beta) &= \frac{2}{\pi} \int_0^\pi e^{-\beta(1-\cos\theta)} \sin^2\theta \, d\theta \\ &= \frac{16}{\pi} \int_0^1 e^{-2\beta y^2} y^2 \sqrt{1-y^2} \, dy \le \sqrt{\frac{2}{\pi}} \, \beta^{-3/2}, \end{aligned}$$

by $\sqrt{1-y^2} \le 1$ and $\int_0^\infty e^{-2\beta y^2} y^2 dy = \frac{\sqrt{\pi}}{4} (2\beta)^{-3/2}$.

Lower bound. Fix a spanning tree of Λ_L and set its $L^4 - 1$ links to I, again without changing Z; $3L^4 + 1$ links remain. Let $0 < \delta \le \pi/2$ and let ν be the Haar measure restricted to $\{\theta \le \delta\}$ and normalised. Since $\sin t / t$ decreases on $(0,\pi)$,

$$\frac{2}{\pi} \int_0^\delta \sin^2 t \, dt \ge \frac{2\delta^3}{3\pi} \Big(\frac{\sin\delta}{\delta} \Big)^2 .$$

By Jensen's inequality for the product of the measures ν on the remaining links, $\log Z(\beta) \ge (3L^4+1) \log \frac{2\delta^3}{3\pi} (\frac{\sin\delta}{\delta})^2 - \beta \sum_p \mathbb{E}_\nu s_p$. The set $\{\theta \le \delta\}$ is invariant under conjugation and inversion, so $\mathbb{E}_\nu U = mI$ and $\mathbb{E}_\nu U^{-1} = mI$ with $m = \mathbb{E}_\nu \cos\theta \in [0,1]$. A plaquette has at most four links off the tree, independent under ν, so $\mathbb{E}_\nu \frac{1}{2} \operatorname{Tr} U_p = m^j$ with $j \le 4$ and $\mathbb{E}_\nu s_p = 1 - m^j \le 4(1-m) \le 2 \mathbb{E}_\nu \theta^2$. With $w(t) = \sin^2 t / t^2$ nonincreasing and t^2 increasing, Chebyshev's inequality for the measure proportional to $t^2 dt$ on $[0,\delta]$ gives $\mathbb{E}_\nu \theta^2 = \int_0^\delta t^2 w t^2 dt / \int_0^\delta w t^2 dt \le \frac{3}{5}\delta^2$. Hence $\mathbb{E}_\nu s_p \le \frac{6}{5}\delta^2$.

The constants. Take $\lambda = \frac{1}{5}$ and $\delta^2 = c/\beta$ with $c = \frac{5}{8}$, so that $\delta^2 \le \frac{5}{8} < \pi^2/4$. Dividing by $6L^4(1-\lambda)\beta$ and collecting the terms in $\log\beta$ gives the bound, with

$$K_1 = \tfrac{5}{4} \Big(\tfrac{1}{4} \log \tfrac{2}{\pi} - \tfrac{3}{4} \log\lambda - \tfrac{1}{2} \log \tfrac{2}{3\pi} - \tfrac{3}{4} \log c + \tfrac{6}{5} c \Big),$$
$$K_2 = \tfrac{5}{24} \Big(- \log \tfrac{2}{3\pi} - \tfrac{3}{2} \log c \Big),$$

after dropping the term $-\frac{5}{8L}(\frac{1}{2}\log\frac{2}{\pi} - \frac{3}{2}\log\lambda)$, which is negative. The factor $(\sin\delta/\delta)^2$ contributes at most $\frac{5}{4}(1+\frac{1}{48})\frac{y}{1-y}$ with $y = \delta^2/6 \le c/(6\beta)$, by $\sin\delta/\delta \ge 1-y$ and $-\log(1-y) \le y/(1-y)$, which is at most K_3/β inside the bracket. With $\lambda = \frac{1}{5}$ and $c = \frac{5}{8}$ the logarithms collect to

$$K_1 = \tfrac{5}{4} \big(2\log 2 + \tfrac{1}{2}\log 3 + \tfrac{1}{4}\log\pi + \tfrac{3}{4} \big),$$
$$K_2 = \tfrac{5}{24} \big(\tfrac{7}{2}\log 2 + \log 3 + \log\pi - \tfrac{3}{2}\log 5 \big),$$

and $\frac{5}{4} \cdot \frac{49}{48} \cdot \frac{c/6}{1-c/6} = \frac{1225}{8256} < 0.1484$. The series $2\operatorname{artanh} x = 2\sum_{k\ge0} x^{2k+1}/(2k+1)$, whose tail after any term is at most the next term divided by $1-x^2$, applied to $\log 2 = 2\operatorname{artanh}\frac{1}{3}$, $\log\frac{3}{2} = 2\operatorname{artanh}\frac{1}{5}$ and $\log\frac{5}{4} = 2\operatorname{artanh}\frac{1}{9}$, gives $\log 2 < 0.6931472$, $\log 3 < 1.0986123$ and $\log 5 > 1.6094379$. With $\pi < 3.1416$ and $\log(1+x) \le x - \frac{x^2}{2} + \frac{x^3}{3}$ at $x = 0.0472$, $\log\pi < \log 3 + 0.046122 < 1.144735$. Hence $K_1 < 3.71474$ and $K_2 < 0.46984$. At fixed λ the value $c = \frac{5}{8}$ minimises K_1. The last claim holds because s_p is a local function. □

For comparison, Lemma 382 gives the bound $(\log\beta_W + 4.9082)/\beta_W$ for ε_L, which is larger at every $\beta_W \ge 1$ in the limit $L \to \infty$. With Lemma 197, a slice function at a limit state of Proposition 198 has

$$\langle \Psi_X, (1-\mathbb{T})\Psi_X \rangle \le \frac{2\Lambda(X)^2}{\beta_W} \Big(K_1 + \frac{K_3}{\beta_W} \Big),$$

a bound of order one in lattice units at bounded $\Lambda(X)$. Whether it can serve as the first condition of Corollary 194 depends on the variance of X alone, and the next proposition shows that it cannot along the tunings of interest.

Proposition 199. *Let X be a real bounded slice function with $\mathrm{Var}_\omega X > 0$, and let $\chi = \Psi_{X-\omega(X)}/\|\Psi_{X-\omega(X)}\|$.*

(i) $\mathrm{Var}_\omega X \le N\Lambda(X)^2$. *Hence the bound of Lemma 197(iii) for $\langle \chi, (1-\mathbb{T})\chi\rangle$, which is $N\Lambda(X)^2\varepsilon_\omega/\mathrm{Var}_\omega X$, is at least ε_ω.*

(ii) *Let $G = \mathrm{SU}(2)$ and let ω_k be states as in Appendix F that are Wilson Gibbs states at β_k in the sense of Sec. XI F. If that bound gives the first condition of Corollary 194 with a constant κ, then $(4\beta_k+1)a_k \ge 1/\kappa$. Along a tuning with $a_k\beta_k \to 0$, no choice of slice functions X_k gives that condition through Lemma 197(iii).*

(iii) *Let the Lie algebra $\mathfrak{g}$ of G have trivial centre. If X, as a function on G^E, is differentiable at the configuration with every link equal to I, its differential there vanishes.*

Proof. (i) For any two configurations U and V,

$$|X(U)-X(V)| \le \Lambda(X)\max_e |U_e - V_e| \le 2\sqrt{N}\Lambda(X),$$

since $|U_e| = |V_e| = \sqrt{N}$. A real random variable with values in an interval of length ℓ has variance at most $\ell^2/4$. By Lemma 197(i), $\langle \chi, (1-\mathbb{T})\chi\rangle = \langle \Psi_X, (1-\mathbb{T})\Psi_X\rangle/\mathrm{Var}_\omega X$, since $\|\Psi_{X-\omega(X)}\|^2 = \omega(|X-\omega(X)|^2)$.

(ii) By (i) the bound is at least ε_{ω_k}, which is at least $1/(4\beta_k+1)$ by Proposition 257. If it is at most κa_k, then $1 \le \kappa a_k(4\beta_k+1)$.

(iii) A constant gauge transformation h on the slice gives $X(hUh^{-1}) = X(U)$. For $A = (A_e) \in \mathfrak{g}^E$ let $\phi(A)$ be the derivative of $t \mapsto X((e^{tA_e})_e)$ at $t=0$; ϕ is linear, and $\phi(\mathrm{Ad}_h A) = \phi(A)$ for $h \in G$. Averaging over the Haar measure, $\phi(A) = \phi(\bar A)$ with $\bar A_e = \int \mathrm{Ad}_h A_e\, dh$. Each $\bar A_e$ is invariant under Ad_G, hence commutes with $\mathfrak{g}$, and so vanishes. The curves $t \mapsto e^{tA_e}$ span the tangent space at I. □

The one-loop tuning has $\beta_W = \frac{11}{3\pi^2}\log(1/a) + O(1)$ for $\mathrm{SU}(2)$, and the two-loop tuning adds a term of order $\log\log(1/a)$ (Remark 247), so $a_k\beta_k \to 0$ along either, and Proposition 199(ii) applies. Part (i) shows where the loss lies: the bound of Lemma 197(iii) is first order in the time-like plaquettes, and it gives an energy at most κa only if the variance of X is at least the fraction $\varepsilon_\omega/(\kappa a)$ of its largest possible value $N\Lambda(X)^2$. Part (iii) shows that a gauge-invariant function has no first-order term at the configuration with all links I. If X is continuously differentiable, its local Lipschitz constants at configurations within distance r of that configuration, or of a gauge transform of it, tend to 0 with r, while the bound of Lemma 197(ii) uses the global constants L_e.

The bound does give the trial-state condition in the opposite regime, at the cost of a lower bound on a variance and on one correlation, which are the infrared content of (N_u).

Corollary 200. *Let $G = \mathrm{SU}(2)$, let ω_k be states as in Appendix F that are limits as in Proposition 198 at $\beta_k \ge 1$, and let X_k be real slice functions with $X_k^c = X_k - \omega_k(X_k)$. Suppose that there are $\nu, v > 0$ such that for all large k*

$$\mathrm{Var}_{\omega_k} X_k \ge 2\nu\Lambda(X_k)^2,$$
$$\omega_k\big(X_k^c\,\tau_{u/a_k}A_k\big)^2 \ge v^2\,\mathrm{Var}_{\omega_k} X_k C_k(2u).$$

If moreover $a_k\beta_k \ge \kappa_0 > 0$ for all large k, then (N_u) holds, and $\rho_k(u,\sigma) \ge (v - \sqrt{\kappa\sigma/2})_+^2$ for $\sigma \ge 2a_k$, with $\kappa = (K_1+K_3)/(\nu\kappa_0)$.

Proof. The vector $\chi_k = \Psi_{X_k^c}/\|\Psi_{X_k^c}\|$ is a unit vector with $\|\Psi_{X_k^c}\|^2 = \mathrm{Var}_{\omega_k} X_k$. By Lemma 197 and Proposition 198, since $\beta_k \ge 1$ and $a_k\beta_k \ge \kappa_0$,

$$\langle \chi_k, (1-\mathbb{T}_k)\chi_k\rangle \le \frac{2\Lambda(X_k)^2(K_1+K_3)}{\beta_k\,\mathrm{Var}_{\omega_k} X_k} \le \frac{K_1+K_3}{\nu\beta_k} \le \kappa a_k.$$

Since X_k is a real function of the slice $x_0 = 0$, $\vartheta(X_k^c) = X_k^c$ and $\langle \chi_k, \Psi_{k,u}\rangle = \omega_k(X_k^c\tau_{u/a_k}A_k)/(\mathrm{Var}_{\omega_k} X_k C_k(2u))^{1/2}$, whose modulus is at least v. Corollary 194 gives the rest. □

The hypotheses of Corollary 200 are not proved here. The first asks that X_k take values spread over a fixed fraction of the range allowed by its Lipschitz constants, at couplings where the plaquettes are of size $\beta_k^{-1/2}$; the second is a lower bound on one correlation at physical distance. Together with Proposition 199, the corollary locates the estimate: the upper bound on the slab energy that reflection positivity and the plaquette energy provide is of order β_k^{-1} in lattice units, it meets the scale a_k only where $a_k\beta_k$ is bounded below, and even there it transfers the whole of (N_u) to lower bounds on a variance and on an overlap.

D. The axioms for the flowed fields

Statement (E) asks for more than the nontriviality of Sec. XI B. This subsection determines, for the spatially flowed fields of Sec. X E, which of the Osterwalder–Schrader axioms follow from the hypothesis of Theorem 45 once the higher renormalised correlations are bounded, and which further statements are needed. Three are needed; together they suffice, and each is necessary: a uniform moment bound (M), which contains (N_0); invariance under one spatial rotation of infinite order (W); and coincidence of the subsequential limits (U). The spatial flow leaves at most the symmetry $\mathrm{O}(3)\times\mathrm{O}(1)$ of space and time reflection, and Gaussian examples show that neither (W) nor (U) follows from the gap, reflection positivity, the lattice symmetries and the tree bounds.

The notation is that of Sec. X E: Φ is a bounded real gauge-invariant function of finitely many spatial links of the slice $\{x_0 = 0\}$, continuous in them, $\varphi_k(x) = \Phi_x \circ \Phi_{s_k}$ with $s_k = \tau/a_k^2$, $A_k(f)$ is the centred packet of $f \in C_c(\mathbb{R}^4)$, f_0 is a fixed nonzero function supported in $\{t > 0\}$, $C_k(0) > 0$ and $Z_k = C_k(0)^{-1/2}$. Let W_3 be the group of the 48 signed permutations of the spatial coordinates, $\theta(t,\mathbf{x}) = (-t,\mathbf{x})$, and Γ the group of order 96 generated by W_3 and θ, the subgroup of the hyperoctahedral group B_4 that preserves the time axis. We assume that Φ is invariant under the action of W_3 on the links of the slice that fixes the site 0. Averaging Φ over W_3 achieves this; the sum of $N - \mathrm{Re}\,\mathrm{Tr}\,U_p$ over the twelve spatial plaquettes p with a corner at 0 is an example.

Throughout, the states ω_k at β_k satisfy the hypothesis of Theorem 45: $D(n_k,\alpha)$ holds with $n_k a_k \le \ell$. By Theorem 43

and Proposition 11, each ω_k is the restriction of the unique Gibbs measure of γ^F, is the free-boundary limit along every exhaustion of $\mathbb{Z}^4$, is invariant under translations and the hypercubic group and is reflection positive, and its transfer matrix has gap at least $\log(1/\alpha)/(2n_k) \geq m_* a_k$ on $\Omega_k^\perp$, with $m_* = \log(1/\alpha)/(2\ell)$. The infinite-volume state is therefore unique at each k, and the only non-uniqueness left is in the limit $k \to \infty$. The spatial flow commutes with Γ (Proposition 47(i) and the proof of Theorem 201(ii)) but not with the elements of B_4 that exchange time with a spatial axis, so Γ, not B_4, is the lattice symmetry of the flowed packets.

(M) *Moment bound.* For every $n \geq 1$ and every cube $Q \subset \mathbb{R}^4$ there are $M_n(Q) < \infty$ and k_0 such that

$$\Big|\omega_k\Big(\prod_{j=1}^n Z_k A_k(f_j)\Big)\Big| \leq M_n(Q) \prod_{j=1}^n \|f_j\|_\infty$$

for all $k \geq k_0$ and all $f_j \in C_c(\mathbb{R}^4)$ supported in Q.

By Theorem 57(i), (T1)–(T3) imply (M). The case $n = 2$ of (M) is not the bound of Theorem 48(i), which concerns reflected pairs of translates of f_0 and holds by reflection positivity alone; (M) concerns all test functions in Q, including those closer to the reflection plane than f_0, and it implies (N_0) (Theorem 201(v)). By Remark 59, no bound on the two-point functions gives (M) for $n = 4$.

A *polynomial* is a finite formal sum $P = \sum_\beta c_\beta \prod_{j=1}^{n_\beta} f_{\beta,j}$ with $c_\beta \in \mathbb{C}$ and $f_{\beta,j} \in C_c(\mathbb{R}^4)$; its lattice version is $P_k = \sum_\beta c_\beta \prod_j Z_k A_k(f_{\beta,j})$, and for a family $S = (S_n)$ with $S_0 = 1$ we put $S(P) = \sum_\beta c_\beta S_{n_\beta}(f_{\beta,1}, \dots, f_{\beta,n_\beta})$. The polynomials $\bar{P}$, θP, $\tau_y P$ and gP are obtained by replacing every $f_{\beta,j}$ by $\overline{f_{\beta,j}}$, $f_{\beta,j} \circ \theta$, $f_{\beta,j}(\cdot - y)$ and $f_{\beta,j} \circ g^{-1}$, products of polynomials are formal products, and P is *at positive times* if every $f_{\beta,j}$ is supported in $\{t > 0\}$. We write $\Psi(f_0)$ for $\Psi(P)$ with $P = f_0$.

Theorem 201. *Assume the hypothesis of Theorem 45 and* (M)*, with Φ real and W_3-invariant. Every sequence of k has a subsequence along which the following hold.*

(i) *For all n and all $f_j \in C_c(\mathbb{R}^4)$ the limit*

$$S_n(f_1, \dots, f_n) = \lim_k \omega_k\Big(\prod_{j=1}^n Z_k A_k(f_j)\Big)$$

exists. Each S_n is symmetric and multilinear, bounded by $M_n(Q)\prod_j \|f_j\|_\infty$ on functions supported in a cube Q, and a distribution on $(\mathbb{R}^4)^n$; $S_1 = 0$.

(ii) *Each S_n is invariant under the translations of $\mathbb{R}^4$ and under Γ: $S_n(f_1 \circ g^{-1}, \dots, f_n \circ g^{-1}) = S_n(f_1, \dots, f_n)$ for $g \in \Gamma$. Moreover $S(\bar{P}) = \overline{S(P)}$.*

(iii) *$S(\theta\bar{P} \cdot P) \geq 0$ for every polynomial P at positive times. The completion of the quotient of these polynomials by the null space of this form is a Hilbert space $\mathscr{H}$ with vectors $\Psi(P)$, $\langle \Psi(P), \Psi(P')\rangle = S(\theta\bar{P} \cdot P')$, and $\Omega = \Psi(1)$.*

(iv) *There are a self-adjoint $H \geq 0$ with $\Psi(\tau_{(t,0)}P) = e^{-tH}\Psi(P)$ for $t \geq 0$, a strongly continuous unitary representation U of $\mathbb{R}^3$ with $\Psi(\tau_{(0,x)}P) = U(x)\Psi(P)$, and a unitary representation V of W_3 with $\Psi(gP) = V(g)\Psi(P)$. They commute with H, $V(g)U(x)V(g)^{-1} = U(gx)$, $H\Omega = 0$, and U and V fix Ω.*

(v) *$\Psi(f_0)$ is a unit vector orthogonal to Ω, there is a dyadic $\sigma > 0$ with $C_k(\sigma) \geq \frac{1}{2}C_k(0)$ for all large k, and (N_0) holds.*

(vi) *$\sigma(H) \subset \{0\} \cup [m_*, \infty)$ and $\ker H = \mathbb{C}\Omega$. For polynomials P, P' at positive times and $t \geq 0$,*

$$\big|S(\theta\bar{P} \cdot \tau_{(t,0)}P') - \overline{S(P)}\,S(P')\big| \leq e^{-m_* t}\,\|\Psi(P) - S(P)\Omega\|\,\|\Psi(P') - S(P')\Omega\|.$$

The joint spectral measure of H and the generators of U is carried by $\{(0,0)\} \cup \big([m_*, \infty) \times \mathbb{R}^3\big)$.

(vii) *If $M_n(Q) \leq C_n(1+L)^{p_n}$ for every cube Q of side L centred at 0, each S_n extends to a tempered distribution on $\mathbb{R}^{4n}$.*

Conversely, if for every n and all $f_j \in C_c(\mathbb{R}^4)$ the sequence $\omega_k(\prod_j Z_k A_k(f_j))$ is bounded, then (M) *holds.*

Proof. (i) The proof of Theorem 57(ii) uses part (i) of that theorem only through the bound $M_n(Q)$, and applies verbatim. $S_1 = 0$ because the packets are centred.

(ii) Translation invariance is proved as in Theorem 57(ii). Let $g \in \Gamma$; it maps $\mathbb{Z}^4$ to itself and fixes the site 0. It acts on configurations by moving the links, a link whose orientation is reversed carrying the inverse group element. Inversion is an isometry of the bi-invariant metric, and g preserves the Wilson action of every time slice, so this action maps solutions of the flow equation to solutions and commutes with Φ_s; for $g = \theta$ this is Proposition 47(i). Write gF for the function $U \mapsto F(g^{-1}U)$. Since Φ is W_3-invariant and θ fixes the spatial links of the slice $\{x_0 = 0\}$, $g\Phi_x = \Phi_{gx}$, hence $g\varphi_k(x) = \varphi_k(gx)$, and substituting $y = g^{-1}x$,

$$A_k(f \circ g^{-1}) = \sum_y a_k^4 f(a_k y)\big(\varphi_k(gy) - c_k\big) = gA_k(f),$$

where $c_k = \omega_k(\varphi_k(x))$ does not depend on x. The state ω_k is invariant under the hypercubic group, so $\omega_k(\prod_j A_k(f_j \circ g^{-1})) = \omega_k(\prod_j A_k(f_j))$, and the limits agree. Since Φ is real, $A_k(\bar{f}) = \overline{A_k(f)}$, which gives the last claim.

(iii) This is the proof of Theorem 57(iii), which uses only the bound of (i).

(iv) Let P be at positive times and $t \geq 0$ dyadic. For large k, $t/a_k \in \mathbb{Z}$, and translation invariance of ω_k and of the flow gives $(\tau_{(t,0)}P)_k = \tau_{t/a_k}P_k$. In the reconstruction of Appendix F, $\Psi_{\tau_n F} = \mathbb{T}_k^n \Psi_F$ (proof of Theorem 48), so

$$\omega_k\big(\vartheta(\bar{P}_k)\tau_{2t/a_k}P_k\big) = \big\|\mathbb{T}_k^{t/a_k}\Psi_{P_k}\big\|^2 \leq \omega_k\big(\vartheta(\bar{P}_k)P_k\big).$$

By (i) and the approximation of θ by the lattice reflection in the proof of Theorem 57(iii), the left side tends to $S(\theta\bar{P} \cdot \tau_{(2t,0)}P)$, which equals $\|\Psi(\tau_{(t,0)}P)\|^2$ by translation invariance and $\theta\tau_{(t,0)} = \tau_{(-t,0)}\theta$, and the right side tends to $\|\Psi(P)\|^2$. The bound of (i) makes $S(\theta\bar{P}' \cdot P')$ continuous under uniform convergence of the test functions in a fixed cube, so $\|\Psi(\tau_{(t,0)}P)\| \leq \|\Psi(P)\|$ for all $t \geq 0$, and $\|\Psi(\tau_{(t,0)}P) - \Psi(P)\| \to 0$ as $t \to 0$, each monomial of $\tau_{(t,0)}P - P$ being a telescoping sum of monomials with one factor $f(\cdot - (t,0)) - f$. Hence $\tau_{(t,0)}$ maps null

vectors to null vectors and induces contractions $\mathsf{T}(t)$ of $\mathscr{H}$ with $\mathsf{T}(s)\mathsf{T}(t)=\mathsf{T}(s+t)$, symmetric by translation invariance and $\theta\tau_{(t,0)}=\tau_{(-t,0)}\theta$, and strongly continuous on a dense set and hence everywhere. A strongly continuous semigroup of symmetric contractions is e^{-tH} with $H\geq 0$ self-adjoint [95]. The spatial translations and the elements of W_3 preserve the positive times, commute with θ and preserve the form by (ii); they have inverses of the same kind, and so induce unitaries, strongly continuous in x by the argument just given. The relations follow from those of the maps of $\mathbb{R}^4$, and $\tau_y 1=g1=1$.

(v) As in the proof of Theorem 57(iv), $C_k(t)-C_k(0)=\omega_k(A_k(\bar{f}_0\circ\theta_k)A_k(f_{0,t}-f_0))$ for dyadic t and large k, with $f_{0,t}=f_0(\cdot-(t,0))$ and θ_k the lattice reflection. With Q a cube containing the supports of $\bar{f}_0\circ\theta_k$ and of $f_{0,t}$ for all k and $t\leq 1$, (M) for $n=2$ gives $Z_k^2|C_k(t)-C_k(0)|\leq M_2(Q)\|f_0\|_\infty\|f_{0,t}-f_0\|_\infty$, which is at most $\frac{1}{2}$ for a small dyadic t by uniform continuity. So $C_k(\sigma)/C_k(0)\geq\frac{1}{2}$ with $\sigma=t$, which is (N_0). Further $\|\Psi(f_0)\|^2=S_2(\bar{f}_0\circ\theta,f_0)=\lim_k Z_k^2C_k(0)=1$ by the same approximation of θ, and $\langle\Omega,\Psi(f_0)\rangle=S_1(f_0)=0$.

(vi) The proof of Theorem 57(vi) uses part (ii) of that theorem only through the convergence of the moments, which (i) provides. It gives $\langle v,e^{-tH}v\rangle\leq e^{-m_*t}\|v\|^2$ for $v=\Psi(P)-\langle\Omega,\Psi(P)\rangle\Omega$, P at positive times. These vectors are dense in $\Omega^\perp$, which e^{-tH} leaves invariant, so the restriction of the positive operator e^{-tH} to $\Omega^\perp$ has norm at most e^{-m_*t}; the spectral theorem gives the claims on $\sigma(H)$ and $\ker H$. Here $\langle\Omega,\Psi(P)\rangle=S(P)$ and $\langle\Psi(P),\Omega\rangle=S(\theta\bar{P})=\overline{S(P)}$ by (ii), and $\langle\Psi(P),e^{-tH}\Psi(P')\rangle-\langle\Psi(P),\Omega\rangle\langle\Omega,\Psi(P')\rangle=\langle v,e^{-tH}v'\rangle$ with v,v' the projections on $\Omega^\perp$, which gives the inequality. Since U commutes with H and fixes Ω, both reduce $\Omega^\perp$; the joint spectral measure at a vector of $\Omega^\perp$ is carried by $[m_*,\infty)\times\mathbb{R}^3$, and at Ω it is the point mass at $(0,0)$.

(vii) Let $\chi(x)=\prod_\mu(1-|x_\mu|)_+$, so that $\sum_{m\in\mathbb{Z}^4}\chi(x-m)=1$, and put $\chi_m=\chi(\cdot-m)$, $r=p_n+5$ and

$$N_r(f)=\sup_x(1+|x|_\infty)^r|f(x)|.$$

On the support of χ_m, $1+|x|_\infty\geq\max(1,|m|_\infty)\geq(2+|m|_\infty)/3$, so $\|\chi_m f\|_\infty\leq 3^r(2+|m|_\infty)^{-r}N_r(f)$. For $f_j\in C_c(\mathbb{R}^4)$, multilinearity gives the finite sum

$$S_n(f_1,\dots,f_n)=\sum_{m_1,\dots,m_n}S_n(\chi_{m_1}f_1,\dots,\chi_{m_n}f_n),$$

and the term of (m_j) has its supports in the cube of side $L=2+2\max_j|m_j|_\infty$, so it is at most

$$C_n\,2^{p_n}\prod_{j=1}^n(2+|m_j|_\infty)^{p_n}\,\|\chi_{m_j}f_j\|_\infty.$$

Summing, $|S_n(f_1,\dots,f_n)|\leq C_n2^{p_n}(3^rs)^n\prod_j N_r(f_j)$ with $s=\sum_{m\in\mathbb{Z}^4}(2+|m|_\infty)^{-5}<\infty$. Functions in $C_c(\mathbb{R}^4)$ are dense in $\mathscr{S}(\mathbb{R}^4)$ for N_r, so S_n extends to a jointly continuous multilinear form on $\mathscr{S}(\mathbb{R}^4)^n$, and the kernel theorem [101, Sect. V.3] gives a tempered distribution on $\mathbb{R}^{4n}$.

For the converse, let C_Q be the Banach space of continuous functions supported in Q with the supremum norm. For fixed k the form $B_k(f_1,\dots,f_n)=\omega_k(\prod_j Z_kA_k(f_j))$ is continuous on C_Q^n, since $|A_k(f)|\leq 2\|\Phi\|_\infty a_k^4\#\{x:a_kx\in Q\}\|f\|_\infty$. A family of continuous n-linear forms on a Banach space that is bounded at every point is bounded in norm: for $n=1$ this is the uniform boundedness principle [101], and for $n>1$, by induction, $\sup_k\|B_k(f_1,\cdot)\|<\infty$ for each f_1, so the linear maps $f_1\mapsto B_k(f_1,\cdot)$ into the Banach space of $(n-1)$-linear forms are bounded at every point, and hence in norm. The bound is $M_n(Q)$. □

Remark 202. If (M) is replaced by bounds on the dyadic packets only, namely $\sup_k Z_k^{n+n'}|\omega_k(\vartheta(\prod_{i\leq n}\bar{A}_k^{y_i})\prod_{j\leq n'}A_k^{y'_j})|<\infty$ for all $y_i,y'_j\in\mathscr{D}$, the construction of Theorem 48(iii) applies to polynomials in the vectors δ_y: every monomial is $\mathsf{T}(s)$ of a monomial at positive times for a small dyadic $s>0$, so the kernels of the positive contractions $\mathsf{T}(\sigma)$ are trivial, and $H=-\log\mathsf{T}(1)$ is self-adjoint and nonnegative, with unitary dyadic spatial translations; the proof of part (vi) gives the gap m_*. The limit is nontrivial when (N_u) holds for some u, but not only then: when $\rho\equiv 0$, the vectors δ_y vanish while products of them need not. Without continuity in the test functions, neither the distributions S_n nor continuous translations follow.

Under the tree bounds, the moment bound holds with polynomial constants, and the limit has two properties that (M) does not give: temperedness with an L^1 bound, and clustering in every direction. For $f\in\mathscr{S}(\mathbb{R}^4)$ put $|f|_{(1)}=\max(\|f\|_1,\|f\|_\infty)$; for a tree T on $\{1,\dots,n\}$, $\zeta\in(\mathbb{R}^4)^{E(T)}$ and $j\geq 2$, let ζ_j^T be the sum of ζ_e over the edges of the path from 1 to j in T.

Lemma 203. *Let* S *be a multilinear form on* $\mathscr{S}(\mathbb{R}^4)^n$ *with* $|\mathsf{S}(f_1,\dots,f_n)|\leq\sigma\prod_j|f_j|_{(1)}$. *Then* S *is given by a unique tempered distribution on* $\mathbb{R}^{4n}$, *and for* $F\in\mathscr{S}(\mathbb{R}^{4n})$, *with* $L_i=-\partial_i^2+x_i^2+4$ *in the* $4n$ *coordinates,*

$$|\mathsf{S}(F)|\leq\sigma\,\|L_1^3\cdots L_{4n}^3F\|_{L^2(\mathbb{R}^{4n})}\leq\sigma\,236^{4n}(\pi/2)^{2n}|F|_{32n},$$

where $|F|_m=\sup_{x,|\gamma|\leq m}(1+|x|^2)^{m/2}|\partial^\gamma F(x)|$ *are the norms of the linear growth condition of [96].*

Proof. The form is jointly continuous, so the kernel theorem [101, Sect. V.3] gives the distribution. Let h_k, $k\geq 0$, be the Hermite functions on $\mathbb{R}$, an orthonormal basis of $L^2(\mathbb{R})$ of real functions with $(-\partial^2+x^2)h_k=(2k+1)h_k$ and $xh_k=\sqrt{k/2}\,h_{k-1}+\sqrt{(k+1)/2}\,h_{k+1}$, whose tensor products h_κ, $\kappa\in\mathbb{N}^{4n}$, give an expansion of every $F\in\mathscr{S}(\mathbb{R}^{4n})$ that converges in $\mathscr{S}$ [101, Sect. V.3]. For real $g\in\mathscr{S}(\mathbb{R})$, $g(x)^2=\int_{-\infty}^x 2gg'=-\int_x^\infty 2gg'$, so $\|g\|_\infty^2\leq\|g\|_2\|g'\|_2$; with $\|h_k'\|_2^2\leq 2k+1$ this gives $\|h_k\|_\infty\leq(2k+1)^{1/4}$. Applying the recursion twice, $x^2h_k=\frac{1}{2}\big(\sqrt{k(k-1)}\,h_{k-2}+(2k+1)h_k+\sqrt{(k+1)(k+2)}\,h_{k+2}\big)$, so $\|(1+x^2)h_k\|_\infty\leq(2k+\frac{5}{2})(2k+5)^{1/4}\leq(2k+5)^{5/4}$. Since $\int_{\mathbb{R}}(1+x^2)^{-1}dx=\pi$, a function f on $\mathbb{R}^4$ has $|f|_{(1)}\leq\pi^4\|\prod_{i=1}^4(1+x_i^2)f\|_\infty$, and for $\alpha\in\mathbb{N}^4$ therefore $|h_\alpha|_{(1)}\leq\pi^4\prod_i(2\alpha_i+5)^{5/4}$. The operator $L=\prod_i L_i^3$ is symmetric with $Lh_\kappa=\prod_i(2\kappa_i+5)^3h_\kappa$, so the coefficient $c_\kappa=\langle h_\kappa,F\rangle$ has $|c_\kappa|\leq\prod_i(2\kappa_i+5)^{-3}\|LF\|_2$. Hence

$$|\mathsf{S}(F)|\leq\sum_\kappa|c_\kappa||\mathsf{S}(h_\kappa)|\leq\sigma\|LF\|_2\Big(\pi\sum_{k\geq 0}(2k+5)^{-7/4}\Big)^{4n},$$

and $\sum_k (2k+5)^{-7/4} \le 5^{-7/4} + \frac{2}{3}5^{-3/4} < 0.2592$, so the bracket is below 0.815. In one variable $L_i^3 = \sum c_{ab} x^a \partial^b$ with $a,b \le 6$ and integer coefficients, and expanding and normal ordering gives $\sum |c_{ab}| = 236$. Expanding the product, $\|LF\|_2 \le 236^{4n} \max \|x^a \partial^b F\|_2$ over multi-indices with entries at most 6, and since $\int_{\mathbb{R}^{4n}} \prod_i (1+x_i^2)^{-2} dx = (\pi/2)^{4n}$, $\|g\|_2 \le (\pi/2)^{2n} \|\prod_i (1+x_i^2) g\|_\infty$. Finally $\prod_i (1+x_i^2)|x_i|^{a_i} \le \prod_i (1+x_i^2)^4 \le (1+|x|^2)^{16n}$ and $|b| \le 24n$, so $\|\prod_i (1+x_i^2) x^a \partial^b F\|_\infty \le |F|_{32n}$. □

Theorem 204. *Let the ω_k be translation invariant and assume* (T1)–(T3). (i) (M) *holds, with $M_n(Q) \le C_n (1+L)^{2n}$ for cubes of side L centred at 0.*

(ii) *Along every subsequence on which the limits S_n of Theorem 201*(i) *exist there are a further subsequence, a number $b \in [0, B_*]$ and a positive measure ν on $\mathbb{R}^4$ with $\nu(\mathbb{R}^4) \le 1$ such that, for $n \ge 2$ and $f_j \in C_c(\mathbb{R}^4)$, the truncated functions S_n^T of the limit satisfy*

$$|S_n^T(f_1,\dots,f_n)| \le K_n c^{-n/2} b^{n/2-1} \sum_T \int |f_1(x)| \times \int \prod_{j=2}^n |f_j(x+\zeta_j^T)|\, d\nu^{\otimes E(T)}(\zeta)\, dx,$$

with $b^0 = 1$.

(iii) *For $n \ge 2$ and every i,*

$$|S_n^T(f_1,\dots,f_n)| \le K_n' \|f_i\|_1 \prod_{j\ne i} \|f_j\|_\infty$$

with $K_n' = K_n n^{n-2} c^{-n/2} B_*^{n/2-1}$. Hence S_n extends to $\mathscr{S}(\mathbb{R}^4)^n$ and

$$|S_n(f_1,\dots,f_n)| \le \sigma_n \prod_j |f_j|_{(1)},$$

where σ_n is the sum over the partitions of $\{1,\dots,n\}$ into blocks of size at least two of $\prod_B K'_{|B|}$, and Lemma 203 applies. If $K_n \le K^n (n!)^\beta$ for all n, then $\sigma_n \le C^n (n!)^{\beta+2}$ with $C = Ke\, c^{-1/2} \max(1, B_*)^{1/2}$, and the Schwinger functions satisfy the linear growth condition of [96].

(iv) *Let $\{1,\dots,n\} = I \cup J$ with I, J disjoint and nonempty, $y \in \mathbb{R}^4 \setminus \{0\}$, and $f_j^\lambda = f_j$ for $j \in I$, $f_j^\lambda = f_j(\cdot - \lambda y)$ for $j \in J$. Then*

$$\lim_{\lambda\to\infty} S_n(f_1^\lambda,\dots,f_n^\lambda) = S_{|I|}\big((f_j)_{j\in I}\big)\, S_{|J|}\big((f_j)_{j\in J}\big).$$

Under the hypotheses of Theorem 201, $U(\lambda x)$ converges weakly to the projection on $\mathbb{C}\Omega$ as $\lambda \to \infty$ for every $x \in \mathbb{R}^3 \setminus \{0\}$, and Ω is the only vector invariant under U.

Proof. (i) This is Theorem 57(i): a moment is a sum over partitions into blocks of size at least two, each contributing a factor $|Q|_k \le (L+a_0)^4$, and there are at most $n/2$ blocks.

(ii) By (T1) and (T2), as in the proof of Lemma 56, the truncated function of the renormalised packets at level k is at most $Z_k^n K_n r_k^n \sum_T a_k^{4n} \sum_x \prod_j |f_j(a_k x_j)| \prod_{\{i,j\}\in T} w_k(x_i - x_j)$. For fixed T change variables to x_1 and $z_e = x_{\text{child}} - x_{\text{parent}}$, the tree being rooted at 1; since w_k is even, the sum is $(a_k^4 W_k)^{n-1} \int h_k\, d\nu_k^{\otimes E(T)}$, where $\nu_k = W_k^{-1} \sum_z w_k(z) \delta_{a_k z}$ is a probability measure ($W_k > 0$, since $C_k(0) > 0$) and

$$h_k(\zeta) = a_k^4 \sum_{x\in\mathbb{Z}^4} |f_1(a_k x)| \prod_{j\ge 2} |f_j(a_k x + \zeta_j^T)|.$$

With $Z_k^n r_k^n \le (c\, a_k^4 W_k)^{-n/2}$ the bound becomes

$$K_n c^{-n/2} (a_k^4 W_k)^{n/2-1} \sum_T \int h_k\, d\nu_k^{\otimes E(T)}.$$

Choose a further subsequence along which $a_k^4 W_k \to b$ and $\nu_k \to \nu$ vaguely; then $\nu_k^{\otimes m} \to \nu^{\otimes m}$ vaguely, the masses being at most one. Let $h(\zeta) = \int |f_1(x)| \prod_{j\ge 2} |f_j(x+\zeta_j^T)|\, dx$. It is continuous, and it has compact support, because x and every $x + \zeta_j^T$ lie in compact sets when the integrand is nonzero, and ζ_e is the difference of two of these points. For ζ in $(a_k\mathbb{Z}^4)^{E(T)}$, $h_k(\zeta)$ is a Riemann sum for $h(\zeta)$ of a family of functions with supports in one compact set and a common modulus of continuity, so $\sup_\zeta |h_k(\zeta) - h(\zeta)| \to 0$. Hence $\int h_k d\nu_k^{\otimes E(T)} \to \int h\, d\nu^{\otimes E(T)}$. The truncated functions at level k are polynomials in finitely many moments, which converge, so the left side converges to $|S_n^T|$.

(iii) Relabel so that $i = 1$; the integral in (ii) is at most $\|f_1\|_1 \prod_{j\ge 2} \|f_j\|_\infty$, there are n^{n-2} trees, and $b^{n/2-1} \le B_*^{n/2-1}$. The moments are sums over partitions of products of truncated functions, which gives σ_n on $C_c(\mathbb{R}^4)$; the bound extends to $\mathscr{S}(\mathbb{R}^4)$, the closure of $C_c(\mathbb{R}^4)$ for $|\cdot|_{(1)}$ containing it. If $K_m \le K^m (m!)^\beta$, then, by $m^{m-2} \le e^m m!$, $K_m' \le C^m (m!)^{\beta+1}$; since $\prod_B |B|! \le n!$ and there are at most $n!$ partitions, $\sigma_n \le C^n (n!)^{\beta+2}$. With Lemma 203, $|S_n(F)| \le C^n (n!)^{\beta+2} 236^{4n} (\pi/2)^{2n} |F|_{32n}$, which is the condition of [96] with $s = 32$.

(iv) Assume $1 \in I$, by symmetry of S_n. In a partition of $\{1,\dots,n\}$ the blocks contained in I or in J do not change their truncated functions as λ varies, by translation invariance. For a block B meeting both, apply (ii) to $(f_j^\lambda)_{j\in B}$ with a root in $B \cap I$: the integrand is bounded by $|f_{\text{root}}(x)| \prod \|f_j\|_\infty$, integrable for $dx \otimes \nu^{\otimes E(T)}$, and tends to 0 pointwise as $\lambda \to \infty$, since for $j \in B \cap J$ the point $x + \zeta_j^T$ is fixed while the support of f_j^λ leaves every compact set. By dominated convergence the truncated function of B tends to 0, and the moment tends to the sum over the partitions into blocks in I and blocks in J, which is the product. For the last statement, let P, P' be at positive times; $\langle \Psi(P), U(\lambda x) \Psi(P') \rangle = S(\theta \bar{P} \cdot \tau_{(0,\lambda x)} P')$ tends to $S(\theta\bar{P}) S(P') = \langle \Psi(P), \Omega \rangle \langle \Omega, \Psi(P') \rangle$ by what was just proved, applied to each monomial. The operators are uniformly bounded and the vectors $\Psi(P)$ are dense, which gives weak convergence; a vector v with $U(x)v = v$ for all x then has $\|v\|^2 = |\langle \Omega, v \rangle|^2$, so $v \in \mathbb{C}\Omega$. □

Part (iv) is the cluster property of the axioms in every direction, for all n. The gap of Theorem 201(vi) gives it with the rate m_* in the time direction only. Spatial clustering does not follow from that gap and (M) without the tree bounds: the field of Proposition 207(b) below has the gap, satisfies (M), and does

not cluster in space. With the hypercubic group in place of Γ the exchange of time with a spatial axis would carry the time clustering to space; the spatial flow removes that exchange.

Rotations. For a family $S = (S_n)$ as in Theorem 201(i)–(ii) let

$$G_S = \big\{R \in \mathrm{O}(4) : S_n(f_1 \circ R^{-1}, \dots, f_n \circ R^{-1}) = S_n(f_1, \dots, f_n)\big\},$$

the condition holding for all n and $f_j \in C_c(\mathbb{R}^4)$. Let e_0 be the unit time vector, $\mathrm{SO}(3)_{\mathrm{sp}}$ the rotations that fix it and $\mathrm{O}(3) \times \mathrm{O}(1)$ the orthogonal maps that preserve the line $\mathbb{R}e_0$.

Proposition 205. (i) *G_S is a closed subgroup of* $\mathrm{O}(4)$ *that contains* Γ*, and its identity component is* $\{1\}$*,* $\mathrm{SO}(3)_{\mathrm{sp}}$ *or* $\mathrm{SO}(4)$.

(ii) *$G_S \supseteq \mathrm{O}(3) \times \mathrm{O}(1)$ if and only if G_S contains a rotation of the spatial coordinates of infinite order.*

(iii) *$G_S = \mathrm{O}(4)$ if and only if G_S contains a rotation R of infinite order with $Re_0 \notin \{\pm e_0\}$, and also if and only if G_S is infinite and contains some g with $ge_0 \notin \{\pm e_0\}$.*

(iv) *If $G_S \supseteq B_4$, then G_S is finite or equal to* $\mathrm{O}(4)$*, and it equals* $\mathrm{O}(4)$ *if and only if it contains an element of infinite order. One element of finite order outside B_4 does not suffice.*

Proof. (i) G_S is a group. If $R_m \to R$ with $R_m \in G_S$, the functions $f \circ R_m^{-1}$ converge uniformly to $f \circ R^{-1}$ with supports in one cube, and the bound of Theorem 201(i) gives $R \in G_S$. So G_S is a compact Lie group [32], and its Lie algebra $\mathfrak{g} \subset \mathfrak{so}(4)$ is invariant under $\mathrm{Ad}(g)$, $g \in \Gamma$. Let $E_{ij} = e_i e_j^{\mathsf{T}} - e_j e_i^{\mathsf{T}}$. The diagonal sign changes D_ε, $\varepsilon \in \{\pm 1\}^4$, lie in Γ, and $\mathrm{Ad}(D_\varepsilon)E_{ij} = \varepsilon_i\varepsilon_j E_{ij}$. The six characters $\varepsilon \mapsto \varepsilon_i\varepsilon_j$ are distinct, so the operators $2^{-4}\sum_\varepsilon \varepsilon_i\varepsilon_j \mathrm{Ad}(D_\varepsilon)$ project $\mathfrak{g}$ onto $\mathfrak{g} \cap \mathbb{R}E_{ij}$, and $\mathfrak{g}$ is spanned by the E_{ij} it contains. The permutations of the spatial axes act transitively on the pairs $\{i,j\} \subset \{1,2,3\}$ and on the pairs $\{0,i\}$. So $\mathfrak{g}$ is 0, $\mathfrak{so}(3)_{\mathrm{sp}} = \mathrm{span}\{E_{ij} : i,j \geq 1\}$, $\mathrm{span}\{E_{0i}\}$ or $\mathfrak{so}(4)$, and the third is not a Lie algebra, since $[E_{01}, E_{02}] = -E_{12}$. A connected Lie subgroup is determined by its Lie algebra.

(ii) If $R \in \mathrm{SO}(3)_{\mathrm{sp}}$ has infinite order, G_S is infinite, hence of positive dimension, and $G_S \cap (\mathrm{O}(3) \times \mathrm{O}(1))$ has the same property, so its identity component is a connected subgroup of $\mathrm{SO}(3)_{\mathrm{sp}}$ of positive dimension normalised by the signed permutations, and by the argument of (i), applied within $\mathfrak{so}(3)_{\mathrm{sp}}$, it is $\mathrm{SO}(3)_{\mathrm{sp}}$. With Γ this gives $\mathrm{O}(3) \times \mathrm{O}(1)$. The converse is clear.

(iii) If G_S is infinite, its identity component is $\mathrm{SO}(3)_{\mathrm{sp}}$ or $\mathrm{SO}(4)$ by (i). In the first case it is normal in G_S, and every $g \in G_S$ maps the vectors fixed by $\mathrm{SO}(3)_{\mathrm{sp}}$, the line $\mathbb{R}e_0$, to the vectors fixed by $g\mathrm{SO}(3)_{\mathrm{sp}}g^{-1} = \mathrm{SO}(3)_{\mathrm{sp}}$, so $ge_0 = \pm e_0$. Hence an infinite G_S with some $ge_0 \neq \pm e_0$ contains $\mathrm{SO}(4)$, and with θ it is $\mathrm{O}(4)$. A rotation R of infinite order with $Re_0 \neq \pm e_0$ makes G_S infinite and supplies g; the converses are clear.

(iv) The sign changes and all permutations of the four axes act on the pairs transitively, so $\mathfrak{g}$ is 0 or $\mathfrak{so}(4)$. The reflection s in the hyperplane orthogonal to $u = \frac{1}{2}(1,1,1,1)$, with B_4, generates a group that permutes the 24 vectors $\pm e_\mu$ and $\frac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1)$: s maps e_μ to $e_\mu - u$, and v to $v - 2\langle v,u\rangle u$ with $\langle v,u\rangle \in \{0, \pm\frac{1}{2}, \pm 1\}$ for the other vectors. These vectors span $\mathbb{R}^4$, so the group is finite. □

The spatial flow commutes with spatial rotations in the continuum, and in the free field it leaves exactly $\mathrm{O}(3) \times \mathrm{O}(1)$ (Proposition 207(c)). So for the spatially flowed fields at a positive flow time the symmetry to ask for is $\mathrm{O}(3) \times \mathrm{O}(1)$, and by part (ii) one rotation suffices:

(W) *One spatial rotation.* There is a rotation R of the spatial coordinates, of infinite order, with

$$\lim_{k\to\infty}\Big[\omega_k\Big(\prod_j Z_k A_k(f_j \circ R^{-1})\Big) - \omega_k\Big(\prod_j Z_k A_k(f_j)\Big)\Big] = 0$$

for all n and $f_j \in C_c(\mathbb{R}^4)$.

Under the hypotheses of Theorem 201, (W) holds if and only if every subsequential limit is invariant under $\mathrm{O}(3) \times \mathrm{O}(1)$: (W) puts R in each G_S; conversely, if the difference in (W) stayed above ε along a subsequence for some f_j, a further subsequence would give a limit S with $R \notin G_S$. Full Euclidean invariance, as (E) asks, is not available for these fields at a positive flow time; it has to be posed for observables whose regularisation is $\mathrm{O}(4)$-covariant, such as the four-dimensional flow, which at positive flow time mixes the two sides of the reflection plane (Sec. X D), or the limit of vanishing flow time. For families invariant under B_4 part (iv) applies, and one rotation of infinite order gives $\mathrm{O}(4)$. In either case the rotation enters as a lattice statement: a Ward identity in which the breaking of the rotation is an insertion that vanishes in the limit, as the symmetry restoration of Symanzik's improvement programme predicts. It is not proved here.

Convergence of the whole sequence.

(U) *One limit.* Any two subsequential limits of Theorem 201(i) coincide.

Proposition 206. *Assume the hypotheses of Theorem 201.*

(i) *The moments $\omega_k(\prod_j Z_k A_k(f_j))$ converge for all n and $f_j \in C_c(\mathbb{R}^4)$ if and only if* (U) *holds.*

(ii) *Let (β_k) and (β'_k) satisfy the hypothesis of Theorem 45 with the same α, ℓ, and* (M) *and* (W) *with the same $M_n(Q)$ and R, and let β''_k be β_k for even k and β'_k for odd k. Then (β''_k) satisfies the same hypotheses, and if the moments along (β_k) and along (β'_k) converge to different limits, those along (β''_k) do not converge.*

(iii) *Suppose that every subsequential limit lies in a set $\mathfrak{F}$ of families on which $S \mapsto S_2(\bar{f}_0 \circ \theta, f_0(\cdot - (\sigma_0, 0)))$ is injective, for a dyadic $\sigma_0 > 0$, and that $\rho_k(0,\sigma_0) = C_k(\sigma_0)/C_k(0)$ converges. Then* (U) *holds.*

Proof. (i) If a moment does not converge, it has two limit points along two subsequences, and refining each by Theorem 201(i) gives two different limits. The converse is clear. (ii) The hypotheses are conditions at each k with constants uniform in k, and the even and the odd terms have different limits. (iii) Along a subsequence with limit S, $S_2(\bar{f}_0 \circ \theta, f_0(\cdot - (\sigma_0, 0))) = \lim Z_k^2 C_k(\sigma_0) = \lim \rho_k(0,\sigma_0)$ by the approximation of θ in the proof of Theorem 57(iii), the same number for every subsequence. □

Part (ii) shows that (U) is a property of the tuning that the other hypotheses do not contain. Part (iii) states it in the form

that Yang–Mills theory requires: if every limit is determined up to the choice of one physical scale, a single renormalisation condition fixes that scale. For Yang–Mills theory this is the statement that the continuum theory has no free dimensionless parameter; it is not proved here.

What these inputs cannot give. Gaussian fields show the limits of the inputs. They are not lattice gauge theories, and the hypothesis of Theorem 45 concerns the Wilson measure in the forest gauge; but the proofs of Theorems 201 and 204 use it only through the conclusions of Theorem 43, namely translation and lattice-symmetry invariance, reflection positivity and the transfer-matrix gap $m_* a_k$, and the fields below have these.

Proposition 207. *Let $m > 0$, $q_k = e^{-m a_k}$, and $f_0 \geq 0$.*

(a) *Let ϕ_k be the centred Gaussian field on $\mathbb{Z}^4$ with covariance $\prod_{\mu=0}^{3} q_k^{|x_\mu - y_\mu|}$, and let $A_k(f) = \sum_x a_k^4 f(a_k x)\phi_k(x)$, with C_k, Z_k formed from f_0 as before. Its law is invariant under translations and B_4 and reflection positive for the reflections in coordinate hyperplanes through sites and bisecting links; its transfer matrix in each coordinate direction has spectrum $\{q_k^N : N \geq 0\}$ with 1 a simple eigenvalue, so its gap is $m a_k$. For large k,* (T1)–(T3) *hold with $r_k = 1$, $w_k(z) = \prod_\mu q_k^{|z_\mu|}$, $K_2 = 1$ and $K_n = 0$ for $n \geq 3$. The moments converge along the whole sequence to the Gaussian family with $S_2(f,g) = \kappa \iint f(x) g(y) e^{-m\|x-y\|_1} dx\, dy$, $\kappa > 0$. Its group G_S is exactly B_4, its Hamiltonian has $\sigma(H) \subset \{0\} \cup [m, \infty)$ with $m \in \sigma(H)$, and the joint spectrum of H and the spatial momentum contains $\{m\} \times \mathbb{R}^3$.*

(b) *Let ψ_k be the centred Gaussian field on $\mathbb{Z}^4$ with covariance $q_k^{|x_0 - y_0|}$, and form A_k, C_k, Z_k in the same way. Its law is invariant under translations and Γ, reflection positive in time, and its transfer matrix has gap $m a_k$.* (M)*, the bound of Theorem* 201(vii) *and* (N_0) *hold, and the moments converge along the whole sequence; the limit has $U(x) = 1$ for every $x \in \mathbb{R}^3$, so it does not cluster in space.* (T3) *fails.*

(c) *For $t_f > 0$ and $\mu > 0$, the centred Gaussian field on $\mathbb{R}^4$ with covariance $\widehat{C}(p) = e^{-2t_f |\mathbf{p}|^2} (|p|^2 + \mu^2)^{-1}$, the massive free field flowed by the spatial heat flow for the time t_f, is reflection positive and invariant under translations, and its group G_S is exactly $\mathrm{O}(3) \times \mathrm{O}(1)$.*

Proof. (a) The covariance is a product of the one-dimensional covariances $q^{|n|}$, which are positive definite, their Fourier transforms being $(1-q^2)/(1-2q\cos p + q^2) > 0$; invariance under translations and B_4 is clear. Fix a coordinate direction as time. The field at time 0 has a Gaussian law γ with covariance $\prod_{i \geq 1} q^{|x_i - y_i|}$, and the stationary chain $\phi_{t+1} = q\phi_t + (1-q^2)^{1/2} \xi_{t+1}$ with ξ_t independent of law γ has covariance $q^{|t-s|}$ times that of γ, so it is the field. It is a reversible Markov chain with transition operator $Pg(\phi) = \mathbb{E}\, g(q\phi + (1-q^2)^{1/2}\xi)$ on $L^2(\gamma)$. For h of finite support and $\mathscr{E}_h(\phi) = \exp(\phi(h) - \frac{1}{2}\mathbb{E}\phi(h)^2)$ one computes $P\mathscr{E}_h = \mathscr{E}_{qh}$; the part of $\mathscr{E}_h$ homogeneous of degree N in h spans the N-th Wiener chaos, so P is multiplication by q^N on it, and these chaoses span $L^2(\gamma)$ [60, Sect. 6.3]. So $0 \leq P \leq 1$ with spectrum $\{q^N\}$ and 1 simple. For F, F' functions of the field at times ≥ 0, the Markov property and reversibility give $\mathbb{E}[\vartheta(\bar{F}) F'] = \langle g_F, g_{F'} \rangle_{L^2(\gamma)}$ with $g_F = \mathbb{E}[F \mid \phi_0 = \cdot]$, and for the reflection bisecting the links between times 0 and 1, with F, F' at times ≥ 1, $\langle g_F, P g_{F'} \rangle$ with $g_F = \mathbb{E}[F \mid \phi_1 = \cdot]$; both are positive for $F = F'$, and the transfer matrix is P. Gaussian fields have vanishing cumulants of order $n \geq 3$, and the covariance is $w_k(x-y)$, which is (T1). Further $a_k^4 W_k = a_k^4 ((1+q_k)/(1-q_k))^4 \to (2/m)^4$, which is (T3), and since $q_k^{|n|} = e^{-m a_k |n|}$, $C_k(0)$ is a Riemann sum converging to $\iint f_0(x) f_0(y) e^{-m\|\theta x - y\|_1} dx\, dy > 0$, which gives (T2) for large k and $Z_k \to \kappa^{1/2}$ with κ the inverse of this integral. The moments are sums of products of covariances, Riemann sums converging to the stated ones. Invariance of a Gaussian family under R is invariance of S_2, that is $e^{-m\|Rz\|_1} = e^{-m\|z\|_1}$ for all z; a linear isometry of the norm $\|\cdot\|_1$ maps the extreme points $\pm e_\mu$ of its unit ball to themselves, so $R \in B_4$. The spectral bound follows from the lattice gap $m a_k$ as in Theorem 201(vi). For $f = h \otimes g$ with h supported in $\{t > 0\}$ and $\int h e^{-mt} dt \neq 0$, $\langle \Psi(f), e^{-sH} U(x) \Psi(f) \rangle$ equals e^{-ms} times a function of x whose Fourier transform is a positive multiple of $|\hat{g}(\mathbf{p})|^2 \prod_{i=1}^3 2m/(p_i^2 + m^2)$. So the joint spectral measure at $\Psi(f)$ is carried by $\{m\} \times \mathbb{R}^3$, with support $\{m\} \times \mathbb{R}^3$ when $\hat{g}$ has no zeros.

(b) The field is $\psi_k(x) = X(x_0)$ for the chain of (a) with $\gamma = N(0,1)$, which gives the symmetry, reflection positivity and the gap. Write $F_f(t) = \int f(t, \mathbf{x})\, d\mathbf{x}$. The covariance of two packets converges to $\iint F_f(t) F_g(s) e^{-m|t-s|} dt\, ds$, which is positive for $f = \bar{f}_0 \circ \theta$, $g = f_0$, and bounded by $|Q|^2 \|f\|_\infty \|g\|_\infty$ for f, g supported in Q; with the Gaussian moment formula this gives (M) with $M_n(Q)$ a constant times $|Q|^n$, and convergence. The limiting covariance does not change when the test functions of one factor are translated in space, so neither does any moment, and $\langle \Psi(P), U(x) \Psi(P') \rangle = \langle \Psi(P), \Psi(P') \rangle$. The ratio $C_k(\sigma)/C_k(0)$ converges to a positive number, which is (N_0), and $\sum_z q_k^{|z_0|} = \infty$.

(c) For f supported in $\{t > 0\}$, the p_0 integral gives

$$\iint \bar{f}(\theta x) C(x-y) f(y)\, dx\, dy = \int e^{-2t_f |\mathbf{p}|^2} (2\omega)^{-1} \left| \int \hat{f}(t, \mathbf{p}) e^{-\omega t} dt \right|^2 d\mathbf{p} \geq 0,$$

with $\omega = (|\mathbf{p}|^2 + \mu^2)^{1/2}$ and $\hat{f}$ the spatial Fourier transform; reflection positivity of the Gaussian measure follows [60, Sect. 6.2]. An orthogonal R preserves $\widehat{C}$ if and only if $|(Rp)^{\mathrm{sp}}| = |p^{\mathrm{sp}}|$ for every p, the spatial parts, because $|Rp| = |p|$ and $s \mapsto e^{-2t_f s}$ is injective; that is, R preserves $|p_0|$, so $R^{\mathsf{T}} e_0 = \pm e_0$. □

Example (a) is reflection positive, has the gap in every direction and the lattice symmetry, satisfies the tree bounds with the best constants, and converges; its limit is invariant under no rotation outside B_4, so (W) does not follow from these properties. Its one-particle energy m does not depend on the momentum, so the spectral condition of the Wightman axioms, energy at least $|\mathbf{p}|$, fails; the gap transfers to $H \geq m_*$ on $\Omega^\perp$ and to the joint spectrum of Theorem 201(vi), and the spectral condition needs Lorentz covariance, which only O(4) invariance supplies. Example (b) shows that the time gap and (M) do not give spatial clustering. By Proposition 206(ii), example (a) at two

masses, interleaved, satisfies the hypotheses of Theorems 201 and 204 with uniform constants and has two limits.

Corollary 208. *Assume the hypothesis of Theorem 45, with Φ real and W_3-invariant. The moments $\omega_k(\prod_j Z_k A_k(f_j))$ converge for every n and all $f_j \in C_c(\mathbb{R}^4)$ to a family invariant under translations and $\mathrm{O}(3)\times\mathrm{O}(1)$ if and only if* (M), (W) *and* (U) *hold. The limit then satisfies* (i)–(vi) *of Theorem 201 with Γ replaced by $\mathrm{O}(3)\times\mathrm{O}(1)$: symmetric, reflection positive distributions, a Hamiltonian with $\sigma(H)\subset\{0\}\cup[m_*,\infty)$, $\ker H=\mathbb{C}\Omega$ and $\Omega^\perp\neq 0$, and unitary representations of the spatial translations and rotations commuting with H. If* (T1)–(T3) *hold as well, the limit is tempered and clusters in every direction, and it satisfies the linear growth condition of [96] when $K_n\le K^n(n!)^\beta$.*

Proof. Convergence gives bounded moments, hence (M) by Theorem 201; it gives (U), and with the invariance it gives (W) for every spatial rotation. Conversely, (M) and (U) give convergence by Theorem 201(i) and Proposition 206(i), and (W) gives the invariance by Proposition 205(ii). The representation of the spatial rotations is constructed as V in Theorem 201(iv), and its continuity follows as there. The rest is Theorems 201 and 204. □

So, given (D) along the tuning, the part of (E) that concerns the spatially flowed fields consists of three statements: (M), an ultraviolet bound on the products of renormalised packets that contains (N_0) and therefore fails along every tuning that stays in the strong-coupling range (Theorem 191); (W), the restoration of the spatial rotations; and (U), the uniqueness of the limit along the tuning. Each is necessary. Example (a) shows that (W) does not follow from (M), (U), the tree bounds and the conclusions of Theorem 43, and the same example at two interleaved masses shows that (U) does not follow from (M), the tree bounds and those conclusions. None of the three is proved. Euclidean invariance and the Wightman spectral condition are beyond these fields at a positive flow time.

Three further routes. The trial condition of Corollary 194 can be approached in three other ways: through upper bounds on correlations, through trial states smeared in time, and through a comparison with the Gaussian theory at weak coupling. The first never implies (N_u). The second removes the lattice scale from the first condition and leaves a lower bound on a ratio of correlations at physical times. The third gives (N_u) from a relative comparison at two physical separations; when that comparison is built scale by scale along the one-loop tuning, it must be exact to first order in the running coupling at every scale.

Proposition 209. *Let $G=\mathrm{SU}(2)$, let $\beta_{W,k}\in[0,0.348]$ be any couplings, let ω_k be the states of Theorems 1 and 3, and let m_0 be as in Theorem 191.*

(i) *Let F,H be bounded functions at positive times that are uniform limits of gauge-invariant local functions, with $\omega_k(F)=\omega_k(H)=0$. Then for every integer $n\ge 0$*

$$\big|\omega_k\big(\vartheta(\bar F)\tau_n H\big)\big|\le e^{-m_0 n}\|\Psi_F\|\,\|\Psi_H\|\le e^{-m_0 n}\|F\|_\infty\|H\|_\infty.$$

(ii) *Let* (H) *be a set of inequalities indexed by $i\in I$, each of the form*

$$\big|\omega_k\big(\vartheta(\bar F_{i,k})\tau_{n_{i,k}}H_{i,k}\big)\big|\le B_{i,k}\|\Psi_{F_{i,k}}\|\,\|\Psi_{H_{i,k}}\|\qquad(k\ge k_i),$$

with $F_{i,k},H_{i,k}$ as in (i), *integers $n_{i,k}\ge t_i/a_k$ for some $t_i>0$, and $B_{i,k}>0$ with $a_k\log B_{i,k}\to 0$ as $k\to\infty$. Then each inequality of* (H) *holds for the states ω_k for all large k, while (N_u) fails for every packet with $C_k(0)>0$ and every u, and every limit of Theorem 48 along these states is trivial. The same holds with $\|F_{i,k}\|_\infty\|H_{i,k}\|_\infty$ in place of $\|\Psi_{F_{i,k}}\|\|\Psi_{H_{i,k}}\|$.*

Proof. (i) Since $\langle\Omega,\Psi_F\rangle=\omega_k(F)=0$ and likewise for H, both vectors lie in $\Omega_k^\perp$, on which $\mathbb{T}_k$ has norm at most e^{-m_0}, as in the proof of Theorem 191. Hence $|\langle\Psi_F,\mathbb{T}_k^n\Psi_H\rangle|\le e^{-m_0 n}\|\Psi_F\|\|\Psi_H\|$, and $\|\Psi_F\|\le\|F\|_\infty$ by the footnote of Appendix F.

(ii) Since $a_k\log B_{i,k}\to 0$, for large k we have $a_k\log B_{i,k}\ge -m_0 t_i$, that is $B_{i,k}\ge e^{-m_0 t_i/a_k}\ge e^{-m_0 n_{i,k}}$, and (i) gives the inequality. The statements on (N_u) and on the limits are Theorem 191. □

The class (H) contains uniform bounds $B_{i,k}=B_i$, decay at a physical rate $B_{i,k}=B_i e^{-Mt_i}$ with any M, and bounds of order a_k^p, for connected correlations of any centred observables at physical separations, flowed or not. The same holds for every compact simple G at couplings in $[0,\beta_1]$ with $N\beta_1<\frac{2}{29}\kappa_G$, with $m_0=m_G(\beta_1)$ of Theorem 5. So a proof of (N_u) must contain a lower bound. In Corollary 194 it is $\langle\chi_k,\mathbb{T}_k\chi_k\rangle\ge 1-\kappa a_k$, a lower bound on a normalised correlation at one lattice step that is accurate to order a_k, together with a lower bound on one overlap. The spectral arguments of Lemmas 193 and 463 turn such a lower bound into a bound on the ratio, but they cannot produce one from upper bounds: the states above satisfy every inequality of (H), and the spectral measures of their centred vectors are carried by energies at least m_0/a_k.

Trial states smeared in time. A slice function enters Lemma 197 through its change in one lattice step, which Proposition 199 shows to be too large. Smearing the same function in time over a physical interval replaces that change by a telescoping sum, in which a lattice step costs a_k times the total variation of the time profile. Let ω be a state as in Appendix F, with transfer matrix $\mathbb{T}$ and vacuum Ω. Let X be a real bounded function of the spatial links of the slice $x_0=0$, invariant under the gauge transformations of that slice and a uniform limit of local such functions; by Proposition 47(ii) this includes $X=N^{-1}\operatorname{Re}\operatorname{Tr}W_\gamma\circ\Phi_s$ for the holonomy W_γ of a closed path γ in the slice and every flow time s. Put $X^c=X-\omega(X)$ and

$$D_X(n)=\omega\big(X^c\,\tau_n X^c\big)\qquad(n\in\mathbb{Z}_{\ge 0}).$$

Let $w=(w_n)_{n\in\mathbb{Z}}$ be nonnegative, with $w_n=0$ unless $n_0\le n\le n_1$, where $1\le n_0\le n_1$, and with $\|w\|_1=\sum_n w_n>0$; put $V(w)=\sum_{n\in\mathbb{Z}}|w_n-w_{n-1}|$ and

$$B=\sum_n w_n\,\tau_n X^c.$$

Proposition 210. (i) *D_X is nonnegative, nonincreasing and log-convex on $\mathbb{Z}_{\geq 0}$, and $\Psi_B \perp \Omega$. If $D_X(2n_1) > 0$, then $\|\Psi_B\|^2 \geq \|w\|_1^2 D_X(2n_1) > 0$, and the unit vector $\chi = \Psi_B/\|\Psi_B\|$ satisfies*

$$\langle \chi, (1-\mathbb{T})\chi\rangle \leq \frac{V(w)}{\|w\|_1}\,\frac{D_X(2n_0)}{D_X(2n_1)}.$$

(ii) *If $D_X(2n_1) > 0$ and $\langle \chi, (1-\mathbb{T})\chi\rangle \leq \eta \leq \frac{1}{2}$, then $D_X(2n_0+m) \geq e^{-2\eta m} D_X(2n_1)$ for every integer $m \geq 0$.*

(iii) *For a function H at positive times as in Appendix F, $\langle \Psi_B, \Psi_H\rangle = \omega(\vartheta(B)H)$.*

Proof. Let $x = \Psi_{X^c}$. As in the proof of Lemma 197(i), $\vartheta(X^c) = X^c$ and $\Psi_{\tau_n X^c} = \mathbb{T}^n x$, so $D_X(n) = \langle x, \mathbb{T}^n x\rangle = \int \lambda^n d\nu$ for the spectral measure ν of $\mathbb{T}$ at x, a finite positive measure on $[0,1]$. Hence $D_X \geq 0$, D_X is nonincreasing, and it is log-convex by Hölder's inequality. Since $\langle \Omega, x\rangle = \omega(X^c) = 0$ and $\mathbb{T}\Omega = \Omega$, $x \perp \Omega$ and so $\Psi_B = W(\mathbb{T})x \perp \Omega$, with $W(\lambda) = \sum_n w_n \lambda^n$.

(i) On $[0,1]$, $\|w\|_1 \lambda^{n_1} \leq W(\lambda) \leq \|w\|_1 \lambda^{n_0}$, so $\|\Psi_B\|^2 = \int W^2 d\nu \geq \|w\|_1^2 D_X(2n_1)$. Moreover

$$W(\lambda)(1-\lambda) = \sum_{n=n_0}^{n_1+1} (w_n - w_{n-1})\lambda^n,$$

which is nonnegative and at most $V(w)\lambda^{n_0}$ on $[0,1]$. Therefore

$$\begin{aligned}\langle \Psi_B, (1-\mathbb{T})\Psi_B\rangle &= \int W(\lambda)\cdot W(\lambda)(1-\lambda)\,d\nu \\ &\leq \|w\|_1 V(w) D_X(2n_0),\end{aligned}$$

and dividing by $\|\Psi_B\|^2$ gives the bound.

(ii) Let $\nu_\chi = W^2\nu/\int W^2 d\nu$ be the spectral measure at χ. By Jensen's inequality for λ^m, $\int \lambda^m d\nu_\chi \geq (1-\eta)^m \geq e^{-2\eta m}$, since $\log(1-y) \geq -2y$ on $[0, \frac{1}{2}]$. By the bounds on W,

$$\int \lambda^m d\nu_\chi = \frac{\int W^2 \lambda^m d\nu}{\int W^2 d\nu} \leq \frac{D_X(2n_0+m)}{D_X(2n_1)}.$$

(iii) This is the identity of Appendix F, with $\vartheta(\bar{B}) = \vartheta(B)$ because B is real. □

Along a tuning, take X_k as above for ω_k, a time profile $g \geq 0$ of bounded variation supported in $[t_0, t_1]$ with $0 < t_0 < t_1$ dyadic and $\int g > 0$, and $w_n = a_k g(a_k n)$, so that $B_k = \sum_n a_k g(a_k n)\,\tau_n X_k^c$ is the packet of X_k with time profile g; let χ_k be its unit vector, $\mathrm{Var}(g)$ the total variation of g on $\mathbb{R}$, and write $D_{X_k}(r)$ for $D_{X_k}(r/a_k)$ when $r \in a_k\mathbb{Z}_{\geq 0}$ is a physical time.

Corollary 211. (i) *If $D_{X_k}(2t_1) \geq c\, D_{X_k}(2t_0)$ with $c > 0$ for all large k, then $\langle \chi_k, (1-\mathbb{T}_k)\chi_k\rangle \leq \kappa a_k$ for all large k, with $\kappa = 2\,\mathrm{Var}(g)/(c\int g)$. The hypothesis holds with $c = c_1^{\max(1,(2t_1-2t_0)/(r'-r))}$ if $D_{X_k}(r') \geq c_1 D_{X_k}(r)$ for dyadic $0 < r < r'$ with $r \leq 2t_0$, a $c_1 \in (0,1]$ and all large k.*

(ii) *Conversely, if $\langle \chi_k, (1-\mathbb{T}_k)\chi_k\rangle \leq \kappa a_k$ for all large k, then $D_{X_k}(2t_0+s) \geq e^{-2\kappa s} D_{X_k}(2t_1)$ for all large k and every $s \in a_k\mathbb{Z}_{\geq 0}$; for $s > 2(t_1-t_0)$ this is a lower bound on the ratio of two correlations of X_k at the physical times $2t_1 < 2t_0 + s$.*

(iii) *If the hypothesis of* (i) *holds and*

$$\left|\omega_k\big(\vartheta(B_k)\,\tau_{u/a_k} A_k\big)\right| \geq v\,\|\Psi_{B_k}\|\,C_k(2u)^{1/2}$$

with $v > 0$ for all large k, then (N_u) *holds and $\rho_k(u,\sigma) \geq (v - \sqrt{\kappa\sigma/2})_+^2$ for $\sigma \geq 2a_k$.*

Proof. For large k, t_0/a_k and t_1/a_k are integers, and Proposition 210 applies with $n_0 = t_0/a_k$ and $n_1 = t_1/a_k$; $n_0 \geq 1$. The points $a_k n$ partition $\mathbb{R}$, so $V(w) = a_k \sum_n |g(a_k n) - g(a_k(n-1))| \leq a_k\,\mathrm{Var}(g)$, while $\|w\|_1 = a_k \sum_n g(a_k n)$ is a Riemann sum of g, which is Riemann integrable, and so is at least $\frac{1}{2}\int g$ for large k. Then (i) follows from Proposition 210(i); the condition $D_{X_k}(2t_1) > 0$ holds because $D_{X_k}(2t_0) > 0$ is implicit in the hypothesis. Its second sentence is the log-convexity argument of Proposition 51(ii) with $\alpha = 2t_0$, $\beta = 2t_1$, which uses only that D_{X_k} is nonincreasing and log-convex. For (ii), $\kappa a_k \leq \frac{1}{2}$ for large k, and Proposition 210(ii) with $m = s/a_k$ gives the claim. For (iii), Proposition 210(iii) gives $|\langle \chi_k, \Psi_{k,u}\rangle| \geq v$, and Corollary 194 gives the rest. □

For trial states smeared in time, the first condition of Corollary 194 therefore holds along a tuning if and only if, up to the choice of the physical times, a ratio of two correlations of the smeared field at physical times is bounded below. No lattice-scale quantity enters: no Lipschitz constant, no plaquette energy, and no first step. Proposition 199 concerns single slices and does not apply to B_k. When Φ is real, X_k is the slice field U_k of a product packet $f_0 = g \otimes h$ and $u = 0$, $B_k = A_k$, the overlap is 1, and part (i) gives the hypothesis of Proposition 49(ii) from the slice ratio of Proposition 51(iii), without the comparison of u_k with the packet that was left open after Proposition 49. For a flowed loop $X_k = N^{-1}\,\mathrm{Re}\,\mathrm{Tr}\,W_{\gamma_k} \circ \Phi_{\tau/a_k^2}$, with γ_k the lattice approximation of a closed curve of physical size in the slice, the input that the route needs is the pair of lower bounds of Corollary 211(i) and (iii): a ratio of two flowed-loop correlations at physical times, and the overlap of the smeared loop with the flowed packet at physical distance. Both are lower bounds at physical separations, of the same kind as (N_u), and by Proposition 209 no upper bound supplies them. The overlap vanishes for every k if Ψ_{B_k} and Ψ_{A_k} lie in different eigenspaces of a unitary that commutes with $\mathbb{T}_k$; for instance, if $G = \mathrm{SU}(N)$ with $N \geq 3$ and ω_k is invariant under charge conjugation $U \mapsto \bar{U}$, then $X_k = N^{-1}\,\mathrm{Im}\,\mathrm{Tr}\,W_{\gamma_k} \circ \Phi_{\tau/a_k^2}$ is odd and the flowed energy packet is even, so X_k has to be chosen in the symmetry sector of the packet.

Comparison with the Gaussian theory. Let $C_k^{\mathrm{G}}(t) > 0$ be the correlations of the corresponding packet in a comparison theory, such as the Gaussian theory at tree level, with ratio $\rho_k^{\mathrm{G}} = C_k^{\mathrm{G}}(2u+\sigma)/C_k^{\mathrm{G}}(2u)$. If there are $Z_k > 0$ and $\delta_k \in [0,1)$ such that for $t = 2u$ and $t = 2u+\sigma$

$$(1-\delta_k) Z_k C_k^{\mathrm{G}}(t) \leq C_k(t) \leq (1+\delta_k) Z_k C_k^{\mathrm{G}}(t),$$

then dividing the lower bound at $2u+\sigma$ by the upper bound at $2u$ gives

$$\rho_k(u,\sigma) \geq \frac{1-\delta_k}{1+\delta_k}\,\rho_k^{\mathrm{G}}.$$

The normalisation Z_k cancels, and only the two physical separations $2u$ and $2u+\sigma$ enter. For the free fields of Remarks 52 and 53, the slice ratios have positive limits, so $\liminf_k \rho_k^{\mathrm{G}} > 0$

by the argument of Proposition 51(iii), which uses only reflection positivity and the transfer matrix, and (N_u) follows from $\limsup_k \delta_k < 1$. This relative comparison at two physical separations, uniform in k, is what a perturbative lower bound has to provide. It is not a statement of fixed order in the bare coupling: along the one-loop tuning $b\log(\sigma/a_k)/\beta_k \to 1$, so the terms $(\log(\sigma/a_k)/\beta_k)^n$ of bare perturbation theory are of order one at every n. The comparison has to be organised by scales, as in Proposition 460, where repeating a comparison over scales multiplies the factors $1\pm\varepsilon$. The next proposition determines which losses per scale the one-loop tuning allows.

Proposition 212. *Let $b>0$, $h=b\log 2$, $a_k=2^{-k}a_0$, and let the couplings satisfy $|\beta_k - b\log(1/a_k)| \le M$ for all k. Let $\sigma>0$ be dyadic with $\beta_\sigma^- = b\log(1/\sigma)-M>0$, put $\beta_\sigma^+ = b\log(1/\sigma)+M$, and for k with $\sigma/a_k = 2^{J_k}$, $J_k\in\mathbb{Z}_{\ge 0}$, put*

$$\beta_k^{(j)} = \beta_k - hj \qquad (0\le j\le J_k),$$

the one-loop coupling at the scale $2^j a_k$. Let $c_{k,j}\in[0,\frac12]$.

(i) $\beta_\sigma^- + h(J_k-j) \le \beta_k^{(j)} \le \beta_\sigma^+ + h(J_k-j)$.

(ii) *If $c_{k,j}\ge c/\beta_k^{(j)}$ for all j, with $c>0$, then*

$$\prod_{j=0}^{J_k}(1-c_{k,j}) \le \Big(\frac{\beta_\sigma^+}{\beta_\sigma^+ + b\log(2\sigma/a_k)}\Big)^{c/h},$$

which tends to 0 as $k\to\infty$.

(iii) *If $c_{k,j}\le c/(\beta_k^{(j)})^2$ for all j, then for every k*

$$\prod_{j=0}^{J_k}(1-c_{k,j}) \ge \exp\Big(-2c\Big(\frac{1}{(\beta_\sigma^-)^2}+\frac{1}{h\beta_\sigma^-}\Big)\Big).$$

More generally, $\prod_j(1-c_{k,j}) \ge e^{-2\sum_j c_{k,j}}$, and the products are bounded below uniformly in k if and only if the sums $\sum_j c_{k,j}$ are bounded.

For SU(2) *and the one-loop tuning, $b=11/(3\pi^2)$ and $h\in[0.25751,0.25752]$.*

Proof. (i) Since $\log(1/a_k)=\log(1/\sigma)+J_k\log 2$, the hypothesis gives $\beta_\sigma^- + hJ_k \le \beta_k \le \beta_\sigma^+ + hJ_k$; subtract hj.

(ii) By $1-y\le e^{-y}$ and (i), with $i=J_k-j$,

$$\sum_{j=0}^{J_k} c_{k,j} \ge c\sum_{i=0}^{J_k}\frac{1}{\beta_\sigma^+ + hi} \ge c\int_0^{J_k+1}\frac{dt}{\beta_\sigma^+ + ht} = \frac{c}{h}\log\frac{\beta_\sigma^+ + h(J_k+1)}{\beta_\sigma^+},$$

since the integrand decreases, and $h(J_k+1)=b\log(2\sigma/a_k)\to\infty$.

(iii) Since $\log(1-y)\ge -2y$ on $[0,\frac12]$, $\prod_j(1-c_{k,j}) \ge e^{-2\sum_j c_{k,j}}$, and $\prod_j(1-c_{k,j}) \le e^{-\sum_j c_{k,j}}$ gives the converse of the last statement. By (i),

$$\sum_{j=0}^{J_k}\frac{1}{(\beta_k^{(j)})^2} \le \sum_{i\ge 0}\frac{1}{(\beta_\sigma^- + hi)^2} \le \frac{1}{(\beta_\sigma^-)^2} + \int_0^\infty \frac{dt}{(\beta_\sigma^- + ht)^2},$$

and the integral is $1/(h\beta_\sigma^-)$. For SU(2), $h = b\log 2 = 11\log 2/(3\pi^2)$, and $0.6931471805 < \log 2 < 0.6931471806$, from the series of $2\operatorname{artanh}\frac13$, with $3.1415926535 < \pi < 3.1415926536$ give $h\in[0.2575118, 0.2575119]$, so $4/h < 15.534$. □

With $g_j^2 = 4/\beta_k^{(j)}$, part (ii) says that a scale-by-scale comparison that loses a factor $1-cg_j^2$ at every scale between a_k and σ, however small $c>0$, gives a lower bound on $\rho_k(u,\sigma)$ that tends to zero, at the rate $(\log(1/a_k))^{-4c/h}$, with $4/h<15.534$;[31] such a bound carries no information in the limit. Part (iii) says that losses of second order in the running coupling give $\rho_k(u,\sigma) \ge \rho_k^{\mathrm{G}} e^{-2c(1/(\beta_\sigma^-)^2 + 1/(h\beta_\sigma^-))}$ uniformly in k, a bound that approaches ρ_k^{G} as σ decreases. So a perturbative lower bound on the ratio along the one-loop tuning has to account exactly for the first-order effect of every scale on the ratio, leaving errors whose sum over the scales is bounded. A first-order effect common to $C_k(2u)$ and $C_k(2u+\sigma)$ cancels in the ratio, and the proof has to exhibit that cancellation; bounding each correlation separately to first order is not enough. This is the summability of the one-step errors in Proposition 460, now for the ratio, and the effective actions it requires are item (iv) of Sec. XIII V, the renormalisation group analysis of Bałaban [6, 7, 12], which is not carried out here. None of the three routes gives (N_u) along the one-loop tuning without an input of this kind: a lower bound on correlations at physical separations, for the packet, for a smeared field that overlaps it, or for the Gaussian comparison.

Reductions of (M), (W) and (U). The rest of this subsection reduces each of the three statements to a lattice statement and determines what the bounds of this paper give for it. The constant Z_k is fixed by the reflected norm $C_k(0)$, while the moments in (M) are not reflected. Let

$$V_k = \omega_k\big(|A_k|^2\big), \qquad \Upsilon_k = Z_k^2 V_k = \frac{V_k}{C_k(0)},$$

the variance of the packet of f_0 and its ratio to the reflected norm, and consider:

$(\mathrm{M}_{\mathrm{loc}})$ For every $n\ge 1$ and every cube Q there are $M_n^{\mathrm{loc}}(Q)<\infty$ and k_0 with

$$\Big|\omega_k\Big(\prod_{j=1}^n A_k(f_j)\Big)\Big| \le M_n^{\mathrm{loc}}(Q)\, V_k^{n/2}\prod_{j=1}^n \|f_j\|_\infty$$

for $k\ge k_0$ and $f_j\in C_c(\mathbb{R}^4)$ supported in Q.

(M_Z) $\sup_k \Upsilon_k < \infty$.

Statement $(\mathrm{M}_{\mathrm{loc}})$ is (M) with Z_k replaced by the local normalisation $V_k^{-1/2}$; it does not change when Φ is multiplied by a constant, and it does not involve the reflection.

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[31] Since $\beta_k = b\log(1/a_k)+O(1)$, this rate is a constant times $(g_0^2)^{4c/h}$ with $g_0^2 = 4/\beta_k$ the bare coupling: a power of the coupling at the lattice scale, which is not small at any fixed order of perturbation theory in g_0^2 but tends to zero along the tuning.

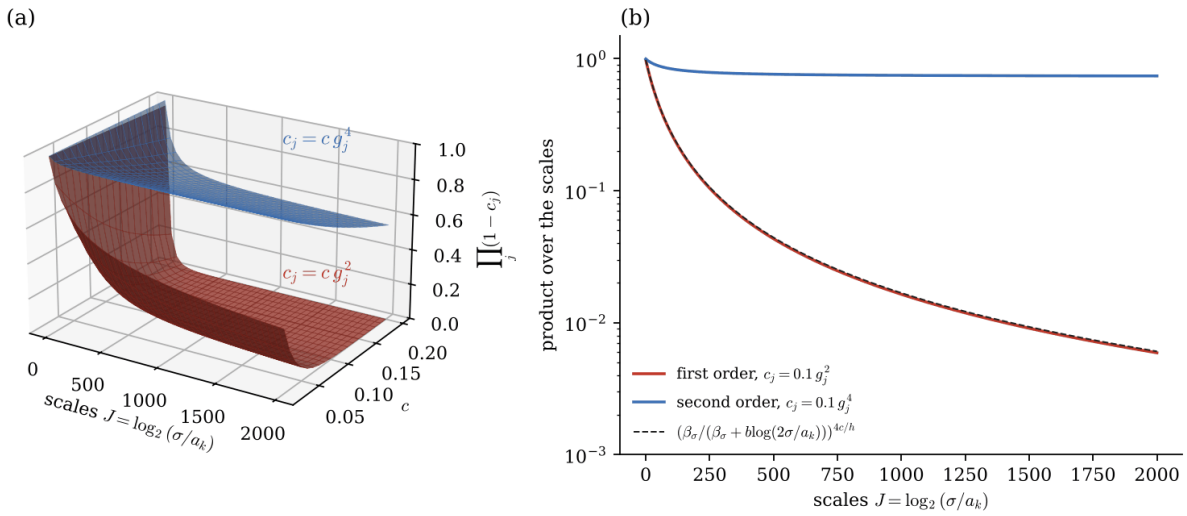


FIG. 23. Products of the factors $1-c_j$ over the $J+1$ scales between a_k and σ along the one-loop SU(2) tuning, with $\beta_W = 20$ at the scale σ, as in Proposition 212. (a) Losses of first order in the running coupling, $c_j = c\,g_j^2$, drive the product to zero for every $c > 0$; losses of second order, $c_j = c\,g_j^4$, keep it bounded below. (b) The section $c = 0.1$, with the bound of part (ii) dashed.

Proposition 213. *Let the states ω_k be invariant under translations and under ϑ and reflection positive, and let Φ be real.*

(i) $\Upsilon_k \geq 1$, *and for f_j supported in a cube Q*

$$\Big|\omega_k\Big(\prod_{j=1}^n Z_k A_k(f_j)\Big)\Big| \leq \big(2\|\Phi\|_\infty Z_k |Q|_k\big)^n \prod_{j=1}^n \|f_j\|_\infty.$$

In particular (M) *holds if* $\sup_k Z_k < \infty$.

(ii) (M) *holds if and only if* ($\mathrm{M}_{\mathrm{loc}}$) *and* ($\mathrm{M}_Z$) *hold. Precisely,* (M) *gives* ($\mathrm{M}_{\mathrm{loc}}$) *with* $M_n^{\mathrm{loc}} = M_n$ *and* $\Upsilon_k \leq M_2(Q_0)\|f_0\|_\infty^2$ *for a cube Q_0 containing the support of f_0, and* ($\mathrm{M}_{\mathrm{loc}}$) *and* ($\mathrm{M}_Z$) *give* (M) *with* $M_n = M_n^{\mathrm{loc}} \sup_k \Upsilon_k^{n/2}$.

(iii) *Assume* (T1). *Then* (T2) *holds if and only if* (M_Z) *holds and* $V_k \geq c' r_k^2 a_k^4 W_k$ *for a $c' > 0$ and all k. If* (T1), (T3) *and this lower bound on V_k hold, then* ($\mathrm{M}_{\mathrm{loc}}$) *holds.*

(iv) *Let $f_0 = g \otimes h$ be a product packet as in Proposition 51, choose an interval I of length $\ell_I > 0$ and $c_g > 0$ with $g \geq c_g$ on I, and put*

$$\kappa_g = \tfrac{1}{2} c_g^2 \ell_I (1 - e^{-1}), \quad \gamma_g = (t_1 - t_0 + \ell_I)\|g\|_\infty,$$
$$\Lambda_g = \max\Big(\frac{4}{\ell_I}, \frac{1}{2t_0}\Big), \quad \varepsilon_g(\Lambda) = \frac{\gamma_g^2 \Lambda e^{-2\Lambda t_0}}{\kappa_g}.$$

Assume that the states have the transfer matrices of Appendix F. If $a_k \leq \ell_I/4$, then for every $\Lambda \geq \Lambda_g$ and every $\sigma \in a_k \mathbb{Z}_{\geq 0}$

$$\rho_k(0,\sigma) \geq e^{-\Lambda\sigma}\big(1 - \varepsilon_g(\Lambda)\Upsilon_k\big).$$

Hence (M_Z) *implies* (N_u) *for every dyadic $u \geq 0$.*

(v) *Let $G = \mathrm{SU}(2)$, $\beta_{W,k} \in [0, 0.348]$ and ω_k the states of Theorem 191, and let f_0 be as in* (iv). *Then for $a_k \leq \min(\ell_I/4, m_0/(2\Lambda_g))$*

$$\Upsilon_k \geq \big(1 - e^{-m_0/2}\big)\frac{2 a_k \kappa_g}{m_0 \gamma_g^2} e^{m_0 t_0/a_k}.$$

(vi) *Let $m > 0$, $q_k = e^{-m a_k}$, let ψ_k be the field of Proposition 207*(a), *ξ_k a family of independent standard normal variables on $\mathbb{Z}^4$ independent of ψ_k, $\phi_k = \psi_k + a_k^{-5/2}\xi_k$, $A_k(f) = \sum_x a_k^4 f(a_k x)\phi_k(x)$, and $f_0 \geq 0$ nonzero. The law of ϕ_k is invariant under translations and B_4 and reflection positive for the reflections in coordinate hyperplanes through sites and bisecting links, its transfer matrix in each coordinate direction has gap $m a_k$, $\rho_k(u,\sigma) = e^{-m\sigma}$ for all u, σ, and Z_k converges; but $\Upsilon_k \to \infty$, so* (M) *fails for $n = 2$.*

Proof. (i) Since ϑ is an automorphism with $\overline{\vartheta(F)} = \vartheta(\bar F)$ and ω_k is ϑ-invariant, $\omega_k(|\vartheta(\bar A_k)|^2) = \omega_k(\vartheta(|A_k|^2)) = V_k$, and the Cauchy–Schwarz inequality gives $C_k(0) \leq V_k$. Further $|A_k(f)| \leq 2\|\Phi\|_\infty a_k^4 \#\{x : a_k x \in Q\}\|f\|_\infty$.

(ii) Since Φ is real, $A_k(\bar f_0) = \overline{A_k(f_0)}$, and (M) with $n = 2$, $f_1 = \bar f_0$ and $f_2 = f_0$ is the bound on Υ_k. By (i), $Z_k^{-n} = C_k(0)^{n/2} \leq V_k^{n/2}$, which gives ($\mathrm{M}_{\mathrm{loc}}$). Conversely, since $Z_k^n V_k^{n/2} = \Upsilon_k^{n/2}$,

$$\Big|\omega_k\Big(\prod_j Z_k A_k(f_j)\Big)\Big| \leq M_n^{\mathrm{loc}} \Upsilon_k^{n/2} \prod_j \|f_j\|_\infty.$$

(iii) The packets are centred, so V_k is the joint cumulant of $A_k(\bar f_0)$ and $A_k(f_0)$, and Lemma 56 gives $V_k \leq K_2 r_k^2 |Q_0|_k a_k^4 W_k \|f_0\|_\infty^2$. If (T2) holds, then $V_k \geq C_k(0) \geq c\, r_k^2 a_k^4 W_k$ and $\Upsilon_k \leq K_2 |Q_0|_k \|f_0\|_\infty^2/c$, which is bounded since $|Q_0|_k \leq (L + a_0)^4$ for a cube of side L. Conversely $C_k(0) = V_k/\Upsilon_k \geq (c'/\sup_j \Upsilon_j) r_k^2 a_k^4 W_k$. For ($\mathrm{M}_{\mathrm{loc}}$), Lemma 56 and the lower bound give for a cumulant of order $p \geq 2$

$$\big|\omega_k^T\big(A_k(f_1); \ldots; A_k(f_p)\big)\big|$$
$$\leq K_p p^{p-2} c'^{-p/2} B_*^{p/2-1} |Q|_k V_k^{p/2} \prod_{j=1}^p \|f_j\|_\infty,$$

using (T3), and a moment is the sum over the partitions of $\{1,\dots,n\}$ without singletons of the products of the cumulants of the blocks, each partition contributing a factor $V_k^{n/2}$.

(iv) We use the notation of the proof of Proposition 51: $A_k=\sum_n w_n\tau_n U_k$ with $w_n=a_k g(a_k n)$, ν the spectral measure of $\mathbb{T}_k$ at u_k, $W(\lambda)=\sum_n w_n\lambda^n$, and $C_k(\sigma)=\int W^2\lambda^{\sigma/a_k}d\nu$. The function U_k depends only on the spatial links of the slice at time 0 (Proposition 47(i)), which ϑ fixes, so for $j\ge 0$, by translation invariance,

$$\omega_k\big(\bar U_k\,\tau_j U_k\big)=\omega_k\big(\vartheta(\bar U_k)\tau_j U_k\big)=\int\lambda^j d\nu,$$

and the same holds for $j<0$ with $|j|$ in place of j, the integral being real. Hence

$$V_k=\int\mathscr{Q}\,d\nu,\qquad \mathscr{Q}(\lambda)=\sum_{n,n'}w_n w_{n'}\lambda^{|n-n'|}.$$

Write $\lambda=e^{-a_k E}$ with $E\in[0,\infty]$. First, $W(\lambda)\le\|w\|_1\lambda^{n_0}\le\gamma_g e^{-Et_0}$, since $n_0\ge t_0/a_k$ and $\|w\|_1\le(t_1-t_0+a_k)\|g\|_\infty$. Second, let J be the set of n with $a_k n\in I$; it consists of $M\ge\ell_I/a_k-1\ge\frac34\ell_I/a_k$ consecutive integers, and $w_n\ge a_k c_g$ on J. Every $n\in J$ has on one side at least $p=\lceil M/2\rceil-1$ further elements of J, so, the terms of $\mathscr{Q}$ being nonnegative,

$$\mathscr{Q}(\lambda)\ge a_k^2c_g^2M\sum_{j=0}^{p}\lambda^j\ge a_k^2c_g^2M\,\frac{1-\lambda^{M/2}}{1-\lambda}.$$

For $E\ge 4/\ell_I$, $\lambda^{M/2}\le e^{-3E\ell_I/8}\le e^{-1}$ and $1-\lambda\le a_kE$, so $\mathscr{Q}(\lambda)\ge\frac34\ell_I c_g^2(1-e^{-1})/E\ge\kappa_g/E$. The function Ee^{-2Et_0} decreases for $E\ge 1/(2t_0)$, so for $\Lambda\ge\Lambda_g$, on the set $\{E>\Lambda\}$,

$$W^2\le\frac{\gamma_g^2}{\kappa_g}Ee^{-2Et_0}\mathscr{Q}\le\varepsilon_g(\Lambda)\mathscr{Q},$$

which also holds at $E=\infty$, where $W=0$. Therefore $\int_{\{E>\Lambda\}}W^2d\nu\le\varepsilon_g(\Lambda)V_k=\varepsilon_g(\Lambda)\Upsilon_k C_k(0)$, and

$$C_k(\sigma)\ge e^{-\Lambda\sigma}\int_{\{E\le\Lambda\}}W^2d\nu\ge e^{-\Lambda\sigma}\big(1-\varepsilon_g(\Lambda)\Upsilon_k\big)C_k(0).$$

If $\Upsilon_k\le\Upsilon$ for all k, choose $\Lambda\ge\Lambda_g$ with $\varepsilon_g(\Lambda)\Upsilon\le\frac12$; then $\rho_k(0,\sigma)\ge\frac12e^{-\Lambda\sigma}$, which is (N_0), and Proposition 190(iii) gives (N_u).

(v) By Theorem 191 with $\sigma=a_k$, $\rho_k(0,a_k)\le e^{-m_0}$. Part (iv) with $\sigma=a_k$ and $\Lambda=m_0/(2a_k)\ge\Lambda_g$ gives $e^{-m_0}\ge e^{-m_0/2}(1-\varepsilon_g(\Lambda)\Upsilon_k)$, so $\Upsilon_k\ge(1-e^{-m_0/2})/\varepsilon_g(m_0/(2a_k))$, which is the claim.

(vi) The covariance of ϕ_k is $\prod_\mu q_k^{|x_\mu-y_\mu|}+a_k^{-5}\delta_{xy}$, which is invariant under translations and B_4. Fix a coordinate direction as time. The process of the time slices of (ψ_k,ξ_k) is the stationary reversible Markov chain of the proof of Proposition 207(a) for ψ_k, together with independent slices of ξ_k; its transition operator is $P\otimes\Pi$, with P as there and Π the projection onto the functions that do not depend on the slice of ξ_k. It is positive with spectrum $\{q_k^N:N\ge0\}\cup\{0\}$ and 1 simple, and the argument given there gives reflection positivity and the gap. The packets in $C_k(\sigma)$, $\sigma\ge0$, are supported at times $x_0\le-1$ and $x_0\ge1$, where the variables ξ_k are independent and centred, so $C_k(\sigma)$ is that of example (a): with the time factor $q_k^{y_0-x_0}$, the substitution $y\mapsto y+(\sigma/a_k)e_0$ gives $C_k(2u+\sigma)=e^{-m\sigma}C_k(2u)$, and Z_k converges as in (a). The variances add, so $V_k\ge a_k^{-5}a_k^8\sum_x f_0(a_kx)^2$, which is a_k^{-1} times a Riemann sum converging to $\|f_0\|_2^2>0$. Hence $\Upsilon_k\to\infty$, and (M) for $n=2$ fails by (ii). □

By (ii), the renormalisation enters (M) only through (M_Z), a comparison of the variance of the packet with its reflected norm. By the proof of (iv), for a product packet V_k weighs the spectral measure of the slice field by $\mathscr{Q}\ge\kappa_g/E$ at energies $E\ge4/\ell_I$, while $C_k(0)$ weighs it by $W^2\le\gamma_g^2e^{-2Et_0}$: (M_Z) says that the part of the variance carried at high energy, which the reflection suppresses over the distance $2t_0$, is bounded by a multiple of the reflected norm, and it implies (N_0). In the strong-coupling range Υ_k grows exponentially in $1/a_k$ by (v). Example (vi) has the lattice symmetry, reflection positivity, the gap ma_k and ratios $\rho_k(u,\sigma)=e^{-m\sigma}$ independent of k, and (M_Z) fails for it; so (M_Z) does not follow from (N_u) and these properties. By Remark 59, $(\mathrm{M}_{\mathrm{loc}})$ does not follow from bounds on two-point functions. Every inequality that holds, with constants independent of k, for the states of Theorem 191 along all sequences of couplings in $[0,0.348]$ holds along sequences on which (M_Z) fails by (v), and (M) fails with it; so no such inequality implies (M_Z) or (M). This applies to every consequence of reflection positivity at all $\beta_W\ge0$, such as the chessboard estimates [54], and to the bounds of class (H) of Proposition 209; a proof of (M) has to use that the couplings leave the strong-coupling range. A sufficient condition without smearing follows from Hölder's and Minkowski's inequalities and translation invariance: if

$$\sup_k Z_k\big\|\varphi_k(0)-\omega_k(\varphi_k(0))\big\|_{L^n(\omega_k)}<\infty\qquad(n\ge1),$$

then $\|Z_kA_k(f)\|_{L^n(\omega_k)}$ is at most $|Q|_k\|f\|_\infty$ times this supremum, and (M) holds. This pointwise bound on the renormalised flowed field is not proved.

The flow does not bound the field. Part (i) of Proposition 213 is the bound that uses only $\|\Phi\|_\infty$. The spatial flow at a positive flow time does not improve it configuration by configuration.

Proposition 214. *Let $G=\mathrm{SU}(N)$, $N\ge2$, $\zeta=e^{2\pi i/N}$, and let U^* be the configuration with all temporal links equal to 1 and spatial links*

$$U^*_{(x,x+e_1)}=1,\quad U^*_{(x,x+e_2)}=\zeta^{x_1},\quad U^*_{(x,x+e_3)}=\zeta^{x_1+x_2},$$

multiples of the identity.

(i) *Every spatial plaquette of U^* is $\zeta^{\pm1}$, and $\Phi_s(gU^*)=gU^*$ and $\Phi_s(g1)=g1$ for every gauge transformation g and every $s\in\mathbb{R}$.*

(ii) *Let Φ be the sum of $N-\mathrm{Re}\,\mathrm{Tr}\,U_p$ over the twelve spatial plaquettes with a corner at 0. Then for all x and s*

$$\Phi_x\circ\Phi_s(U^*)=12N\big(1-\cos(2\pi/N)\big),\qquad\Phi_x\circ\Phi_s(1)=0.$$

(iii) *Let $\omega(O)=\mu(O^F)$ for gauge-invariant O, with μ a Gibbs measure of γ^F at $\beta\ge 0$, as for the states of Theorem 201, let Φ be as in* (ii) *and $f\ge 0$. Then the ω-essential supremum and infimum of $A_k(f)$ differ by at least $12N(1-\cos(2\pi/N))\,a_k^4\sum_x f(a_kx)$, at every flow time, and*

$$\|Z_kA_k(f)\|_{L^\infty(\omega)}\ge 6N\Big(1-\cos\frac{2\pi}{N}\Big)Z_k\,a_k^4\sum_x f(a_kx).$$

Proof. (i) Let $1\le i<j\le 3$. The plaquette at x in the plane of e_i and e_j is

$$U_{(x,x+e_i)}U_{(x+e_i,x+e_i+e_j)}U^{-1}_{(x+e_j,x+e_i+e_j)}U^{-1}_{(x,x+e_j)}.$$

For U^* and $(i,j)=(1,2),(1,3),(2,3)$ it is

$$\zeta^{x_1+1}\zeta^{-x_1},\quad \zeta^{x_1+x_2+1}\zeta^{-x_1-x_2},\quad \zeta^{x_1}\zeta^{x_1+x_2+1}\zeta^{-x_1}\zeta^{-x_1-x_2},$$

that is ζ; with the opposite orientation it is ζ^{-1}. Let e be a spatial link and X in the Lie algebra of $\mathrm{SU}(N)$. All links of U^* are central, so replacing U^*_e by $U^*_e e^{\varepsilon X}$ changes a plaquette $p\ni e$ into $c\,e^{\pm\varepsilon X}$ with $c=\zeta^{\pm1}$, and

$$\frac{d}{d\varepsilon}\operatorname{Re}\operatorname{Tr}\big(c\,e^{\pm\varepsilon X}\big)\Big|_{\varepsilon=0}=\pm\operatorname{Re}\big(c\operatorname{Tr}X\big)=0,$$

since $\operatorname{Tr}X=0$. So the gradient in the flow equation vanishes at U^*, and $\Phi_s(U^*)=U^*$ by the uniqueness in Proposition 47(i). The same computation with $c=1$ applies to the configuration 1, and the flow commutes with gauge transformations.

(ii) $N-\operatorname{Re}\operatorname{Tr}(\zeta^{\pm1}1)=N(1-\cos(2\pi/N))$ for each of the twelve plaquettes, and $N-\operatorname{Re}\operatorname{Tr}1=0$; by (i) the flow does not move either configuration.

(iii) As in the proof of Lemma 10, choosing a base point on each component of F gives a gauge transformation g with $(gU^*)_e=1$ on F; let Q^* be the restriction of gU^* to the active links, and $Q^1\equiv 1$. The functions $F_x=(\Phi_x\circ\Phi_{s_k})^F$ are uniform limits of continuous functions of finitely many active links (Proposition 47(ii)), and gauge invariance gives $F_x(Q^*)=\Phi_x\circ\Phi_{s_k}(U^*)$. Given $\eta>0$ and the finite set X_f of x with $f(a_kx)\ne 0$, there is a neighbourhood of Q^*, fixed by conditions on finitely many links B, on which $|F_x-F_x(Q^*)|<\eta$ for $x\in X_f$, and similarly for Q^1. Given the other links, the conditional law under μ of the links of B has density proportional to $\exp(N\beta\sum_p\operatorname{Re}\operatorname{Tr}Q_p^F)$ with respect to Haar measure, the sum running over the plaquettes meeting B, and this density is at least $e^{-2N^2\beta|P_B|}$, with $|P_B|$ the number of these plaquettes. So both neighbourhoods have positive probability. On them $A_k(f)$ is within $\eta a_k^4\sum_x f(a_kx)$ of its values at Q^* and Q^1, which differ by $12N(1-\cos(2\pi/N))a_k^4\sum_x f(a_kx)$ by (ii). Since η is arbitrary, this gives the first claim, and the L^∞ norm is at least half the difference. □

So the bound of Proposition 213(i) is attained up to a constant factor at every flow time, and it gives (M) only when Z_k is bounded. Away from the configurations of least action the flow has stationary points at which the flowed energy density keeps its lattice size; a proof of (M) by way of the flow has to bound the probability of the configurations near such points, which is a property of the measure and not of the flow.

Rotations as Ward identities. For $X\in\mathfrak{so}(4)$ and $f\in C^1_c(\mathbb{R}^4)$ let $L_Xf(x)=\nabla f(x)\cdot Xx$, the derivative of $f\circ e^{\varepsilon X}$ at $\varepsilon=0$, and $E_{\mu\nu}$ as in the proof of Proposition 205.

Lemma 215. *Let S be a family as in Theorem 201*(i)–(ii)*, $X\in\mathfrak{so}(4)$ and $n\ge 1$. Then $S_n(f_1\circ e^{-\varepsilon X},\dots,f_n\circ e^{-\varepsilon X})=S_n(f_1,\dots,f_n)$ for all $\varepsilon\in\mathbb{R}$ and $f_j\in C_c(\mathbb{R}^4)$ if and only if*

$$\sum_{j=1}^n S_n(f_1,\dots,L_Xf_j,\dots,f_n)=0$$

for all $f_j\in C_c^\infty(\mathbb{R}^4)$.

Proof. Let $R_\varepsilon=e^{\varepsilon X}$ and $F(\varepsilon)=S_n(f_1\circ R_\varepsilon,\dots,f_n\circ R_\varepsilon)$ for $f_j\in C_c^\infty(\mathbb{R}^4)$. Since R_ε commutes with X and $\nabla(f\circ R)(x)=R^{\mathsf T}\nabla f(Rx)$,

$$L_X(f\circ R_\varepsilon)(x)=\nabla f(R_\varepsilon x)\cdot R_\varepsilon Xx=(L_Xf)(R_\varepsilon x).$$

By Taylor's formula $h^{-1}(f\circ R_{\varepsilon+h}-f\circ R_\varepsilon)$ converges uniformly to $(L_Xf)\circ R_\varepsilon$ as $h\to 0$, with supports in a fixed cube for $|h|\le 1$, so the bound of Theorem 201(i) shows that F is differentiable with $F'(\varepsilon)$ equal to the sum of the lemma at the functions $f_j\circ R_\varepsilon$. If the sum vanishes, F is constant, which is the invariance for smooth f_j, and the bound and the density of C_c^∞ in C_c in the supremum norm on a cube give it for all f_j. Conversely invariance gives $F'(0)=0$. □

For $X\in\mathfrak{so}(4)$ and $f_j\in C_c^\infty(\mathbb{R}^4)$ put

$$\Delta_k^X(f_1,\dots,f_n)=\sum_{j=1}^n\omega_k\Big(Z_kA_k(L_Xf_j)\prod_{i\ne j}Z_kA_k(f_i)\Big),$$

the derivative at $\varepsilon=0$ of the lattice moment $\omega_k(\prod_j Z_kA_k(f_j\circ e^{\varepsilon X}))$, a finite sum. It measures the breaking of the rotation by the lattice and by the flow.

Proposition 216. *Assume the hypotheses of Theorem 201.*

(i) *$\Delta_k^X(f_1,\dots,f_n)\to 0$ for all n and $f_j\in C_c^\infty(\mathbb{R}^4)$ if and only if every subsequential limit is invariant under the group $\{e^{\varepsilon X}\}$.*

(ii) *Let $X\ne 0$ lie in $\mathfrak{so}(3)_{\mathrm{sp}}$, for instance $X=E_{12}$. Then* (W) *holds if and only if $\Delta_k^X(f_1,\dots,f_n)\to 0$ for all n and $f_j\in C_c^\infty(\mathbb{R}^4)$.*

(iii) *Let $X=E_{01}$. Then $\Delta_k^X(f_1,\dots,f_n)\to 0$ for all n and $f_j\in C_c^\infty(\mathbb{R}^4)$ if and only if every subsequential limit is invariant under* O(4)*. For the family of Proposition 207*(c) *the identity of Lemma 215 with $X=E_{01}$ fails.*

(iv) *If* (T1)–(T3) *hold and $a_k^4W_k\to 0$, the conditions in* (ii) *and* (iii) *need only be imposed for $n=2$.*

Proof. (i) Let S be the limit along a subsequence. By Theorem 201(i), applied to the functions $L_Xf_j\in C_c^\infty$, Δ_k^X converges along it to the sum of Lemma 215; if $\Delta_k^X\to 0$, the lemma gives the invariance. If $|\Delta_k^X(f_1,\dots,f_n)|\ge\eta>0$ along a subsequence, a refinement given by Theorem 201(i) has a limit for which the sum of Lemma 215 is at least η in absolute value, and that limit is not invariant.

(ii) $e^{\varepsilon X}$ is the rotation about the line $\ker X$ through an angle proportional to ε, of infinite order for all but countably many

ε. If (W) holds, every limit is invariant under $\mathrm{O}(3)\times\mathrm{O}(1)$, as stated after (W), and (i) gives $\Delta_k^X\to 0$. Conversely, by (i) each G_S contains a spatial rotation of infinite order, so $G_S\supseteq \mathrm{O}(3)\times\mathrm{O}(1)$ by Proposition 205(ii), and (W) follows.

(iii) $e^{\varepsilon E_{01}}$ rotates the $(0,1)$ plane, has infinite order for $\varepsilon\notin\pi\mathbb{Q}$ and moves e_0 off $\pm e_0$ for $\varepsilon\notin\pi\mathbb{Z}$; Proposition 205(iii) and (i) give the equivalence. The family of Proposition 207(c) has $G_S=\mathrm{O}(3)\times\mathrm{O}(1)$, which does not contain $e^{\varepsilon E_{01}}$ for $\varepsilon\notin\pi\mathbb{Z}$, so by Lemma 215 the identity fails for some f_j.

(iv) By Theorem 57(v) every limit is Gaussian, so it is invariant under a rotation if and only if S_2 is, and the proofs of (i) and of Lemma 215 apply to $n=2$ alone. □

So the lattice estimate that gives the spatial rotations is $\Delta_k^X\to 0$ for one $X\in\mathfrak{so}(3)_{\mathrm{sp}}$, with no rate; a bound $|\Delta_k^X(f_1,\dots,f_n)|\le C(f_1,\dots,f_n)a_k^p$ with $p>0$ would suffice and is not proved. For $X=E_{12}$ the rotation by $\pi/2$ lies in W_3, so $\varepsilon\mapsto\omega_k(\prod_j Z_kA_k(f_j\circ e^{\varepsilon X}))$ has period $\pi/2$, and the integral of $\Delta_k^X(f_1\circ e^{\varepsilon X},\dots,f_n\circ e^{\varepsilon X})$ over a period vanishes at every k: the lattice symmetry cancels the breaking on average over the angle, and (ii) asks for the cancellation at each angle in the limit. For the time–space generators the same identity is the condition for $\mathrm{O}(4)$, and part (iii) shows that it fails for the spatially flowed free field at a positive flow time.

The four-dimensional flow. The flow in all four directions preserves $\mathrm{O}(4)$, and it loses reflection positivity already for the free field.

Proposition 217. (i) *Let $t_f>0$ and $\mu>0$. The centred Gaussian field on $\mathbb{R}^4$ with covariance*

$$\widehat{C}(p)=e^{-2t_f|p|^2}(|p|^2+\mu^2)^{-1},$$

the massive free field flowed by the four-dimensional heat flow for the time t_f, is invariant under translations and $\mathrm{O}(4)$ and is not reflection positive: there is a real $f\in C_c^\infty(\mathbb{R}^4)$ supported in $\{t>0\}$ with $\iint f(\theta x)C(x-y)f(y)\,dx\,dy<0$.

(ii) *Let S^j, $j\ge 1$, be families as in Theorem 201(i)–(ii), each invariant under $\mathrm{O}(3)\times\mathrm{O}(1)$, bounded by $M_n(Q)\prod_j\|f_j\|_\infty$ uniformly in j, and with $S_n^j(f_1,\dots,f_n)\to S_n^0(f_1,\dots,f_n)$ for all n and $f_i\in C_c(\mathbb{R}^4)$. Then $G_{S^0}=\mathrm{O}(4)$ if and only if S^0 is invariant under the exchange of e_0 and e_1.*

Proof. (i) Invariance is clear. Put

$$H_{\mathbf{p}}(u)=\frac{1}{2\pi}\int e^{ip_0u}\widehat{C}(p_0,\mathbf{p})\,dp_0,$$

an even real function, smooth in u and continuous in $(\mathbf{p},u)$, with $|H_{\mathbf{p}}(u)|\le H_{\mathbf{p}}(0)\le H_0(0)$. For $f=g\otimes h$ with g supported in $(0,\infty)$ and g,h real, the spatial Fourier transform gives

$$\iint f(\theta x)C(x-y)f(y)\,dx\,dy=(2\pi)^{-3}\int|\hat h(\mathbf{p})|^2G_{\mathbf{p}}\,d\mathbf{p},$$
$$G_{\mathbf{p}}=\iint g(s)g(t)H_{\mathbf{p}}(s+t)\,ds\,dt.$$

Since $H_0(0)>0$ and $H_0''(0)=-(2\pi)^{-1}\int p_0^2\widehat{C}(p_0,0)\,dp_0<0$, $\log H_0$ is strictly concave on an interval $[0,2\delta]$. For $0<u_1<u_2<\delta$ this gives $H_0(u_1+u_2)^2>H_0(2u_1)H_0(2u_2)$, so the matrix $(H_0(u_i+u_j))_{i,j\le 2}$ has negative determinant and there is a real c with $\sum_{i,j}c_ic_jH_0(u_i+u_j)<0$. Let $j_\eta\ge 0$ be a mollifier with integral 1 supported in $[-\eta,\eta]$, $\eta<u_1$, and $g=c_1j_\eta(\cdot-u_1)+c_2j_\eta(\cdot-u_2)$; for small η, $G_0<0$ by continuity of H_0. Fix such g; then $\mathbf{p}\mapsto G_{\mathbf{p}}$ is continuous and bounded by $\|g\|_1^2H_0(0)$. For a real nonzero $h_1\in C_c^\infty(\mathbb{R}^3)$ and $h_L(\mathbf{x})=L^{-3/2}h_1(\mathbf{x}/L)$, the right side for $h=h_L$ is $(2\pi)^{-3}\int|\hat h_1(\mathbf{q})|^2G_{\mathbf{q}/L}\,d\mathbf{q}$, which tends to $\|h_1\|_2^2G_0<0$ as $L\to\infty$ by dominated convergence.

(ii) The bound makes each condition $S_n^0(f_1\circ R^{-1},\dots)=S_n^0(f_1,\dots)$ a limit of the same conditions for S^j, so $G_{S^0}\supseteq\mathrm{O}(3)\times\mathrm{O}(1)$, which is infinite. The exchange g has $ge_0=e_1$, and Proposition 205(iii) gives the claim; the converse is clear. □

At $t_f=0$ the function $H_{\mathbf{p}}(u)=e^{-\omega|u|}/(2\omega)$ is log-linear on $(0,\infty)$, and the field is reflection positive. The proof of (i) shows that reflection positivity needs the matrices $(H_{\mathbf{p}}(u_i+u_j))$ with $u_i>0$ to be positive semidefinite, which makes $H_{\mathbf{p}}$ log-convex on $(0,\infty)$; a smooth even function with $H''(0)<0$ is not log-convex near 0. So at a positive flow time the spatial flow keeps reflection positivity and loses $\mathrm{O}(4)$ (Proposition 207(c)), and the four-dimensional flow keeps $\mathrm{O}(4)$ and loses reflection positivity. By part (ii), at vanishing flow time, for limits S^0 of families that satisfy (W) at positive flow times with uniform bounds, $\mathrm{O}(4)$ reduces to one discrete symmetry, the exchange of time with a spatial axis, which the unflowed lattice theory has; the interchange of the limits $k\to\infty$ and $t_f\to 0$ that would carry it to S^0, and the renormalisation of the fields as $t_f\to 0$, are not treated here.

One limit under a renormalisation condition. Proposition 206(iii) uses the convergence of one ratio. Choosing the couplings so that it holds is a renormalisation condition:

(RC) There are a dyadic $\sigma_0>0$ and $\rho_*\in(0,1)$ with $\rho_k(0,\sigma_0)\to\rho_*$.

That a tuning with (RC) exists is not proved. Since (RC) contains (N_0), it is not weaker than the open input of Theorem 48.

Under (M), let $\mathfrak{X}$ be the set of families $S=(S_n)$ of multilinear forms on $C_c(\mathbb{R}^4)$ with $|S_n(f_1,\dots,f_n)|\le M_n(Q)\prod_j\|f_j\|_\infty$ for f_j supported in a cube Q. For $L\in\mathbb{N}$ let $\mathscr{F}_L$ be a countable $\mathbb{Q}(i)$-linear set of functions supported in $[-L,L]^4$, dense there in the supremum norm, let $P_1,P_2,\dots$ enumerate the tuples of elements of $\bigcup_L\mathscr{F}_L$, and put $d(S,S')=\sum_i 2^{-i}\min(1,|S(P_i)-S'(P_i)|)$. By the bounds and density, $d(S^m,S)\to 0$ if and only if $S_n^m(f_1,\dots,f_n)\to S_n(f_1,\dots,f_n)$ for all n and $f_j\in C_c(\mathbb{R}^4)$, and by the diagonal argument of Theorem 57(ii) $(\mathfrak{X},d)$ is a compact metric space. For $k\ge k_0$ the lattice moments $x_k=(\omega_k(\prod_jZ_kA_k(f_j)))_n$ lie in $\mathfrak{X}$; let $\mathfrak{L}$ be their set of limit points, the set of subsequential limits, and put

$$\delta_k(f_1,\dots,f_n)=\omega_{k+1}\Big(\prod_jZ_{k+1}A_{k+1}(f_j)\Big)-\omega_k\Big(\prod_jZ_kA_k(f_j)\Big).$$

Lemma 218. *If (x_k) is a sequence in a compact metric space with $d(x_k,x_{k+1})\to 0$, its set of limit points is nonempty and connected.*

Proof. The set L of limit points is nonempty and closed. If $L=A\cup B$ with A,B disjoint, closed and nonempty, let 3η be their distance and A_η,B_η their open η-neighbourhoods, at distance at least η. The sequence enters both infinitely often, and once $d(x_k,x_{k+1})<\eta$ it cannot pass from one to the other in one step, so infinitely many x_k lie in the compact complement of $A_\eta\cup B_\eta$, which then contains a limit point. That contradicts $L\subseteq A\cup B$. □

Proposition 219. (i) (RC) *implies* (N_u) *for every dyadic $u\ge 0$, and every limit of Theorem 48 along a subsequence is nontrivial and has $\rho(\sigma_0)=\rho_*$. Under the hypothesis of Theorem 45, $m_*\le\sigma_0^{-1}\log(1/\rho_*)$. For* SU(2), (RC) *implies $\beta_{W,k}>0.348$ for all large k. Under* (M), *every $S\in\mathfrak{L}$ has $S_2(\bar f_0\circ\theta,f_0(\cdot-(\sigma_0,0)))=\rho_*$.*

(ii) *Assume the hypotheses of Theorem 201. Then* (U) *holds if and only if the following hold:*

(U1) *$\delta_k(f_1,\dots,f_n)\to 0$ for all n and $f_j\in C_c(\mathbb{R}^4)$;*

(U2) *$\mathfrak{L}$ is totally disconnected.*

(iii) *Under the same hypotheses, if $\sum_k|\delta_k(f_1,\dots,f_n)|<\infty$ for all n and all f_j in a set whose linear span is dense in the continuous functions supported in each cube, then* (U) *holds.*

(iv) *Let $m>0$, $q_k=e^{-ma_k}$, $\varepsilon_k=2+\sin(\log k)$, $K_\varepsilon(\mathbf{z})=e^{-|\mathbf{z}|^2/\varepsilon}$ on $\mathbb{R}^3$, let ϕ_k be the centred Gaussian field on $\mathbb{Z}^4$ with covariance*

$$q_k^{|x_0-y_0|}K_{\varepsilon_k}\big(a_k(\mathbf{x}-\mathbf{y})\big),$$

$A_k(f)=\sum_x a_k^4 f(a_kx)\phi_k(x)$, and $f_0\ge 0$ nonzero. The law of ϕ_k is invariant under translations and Γ and reflection positive for the time reflections through sites and bisecting links, and its transfer matrix has gap ma_k. Moreover $\rho_k(u,\sigma)=e^{-m\sigma}$ for all u,σ, so (RC) *holds for every σ_0;* (T1)–(T3) *hold with $r_k=1$ for large k, and* (M), (W) *and* (U1) *hold, with $|\delta_k(f_1,\dots,f_n)|\le C/k$ for $f_j\in C_c^1(\mathbb{R}^4)$. But $\mathfrak{L}$ is homeomorphic to $[1,3]$, and* (U) *fails.*

Proof. (i) The first claims are Proposition 190(iii) and Theorem 48(ii). Under the hypothesis of Theorem 45 the gap is at least m_*a_k on $\Omega_k^\perp$, and as in the proof of Theorem 191 $\rho_k(0,\sigma_0)\le e^{-m_*\sigma_0}$, so $\rho_*\le e^{-m_*\sigma_0}$. The statement for SU(2) is Theorem 191. The last claim is the computation in the proof of Proposition 206(iii).

(ii) If (U) holds, x_k converges in $\mathfrak{X}$ by Proposition 206(i), so $d(x_k,x_{k+1})\to 0$, which is (U1), and $\mathfrak{L}$ is a point. Conversely (U1) is $d(x_k,x_{k+1})\to 0$, so $\mathfrak{L}$ is connected by Lemma 218; a nonempty, connected and totally disconnected set is a point, and (U) follows.

(iii) The moments of tuples from the set converge, being Cauchy; by multilinearity so do those of tuples from its span, and by (M) and density all moments converge. Proposition 206(i) gives (U).

(iv) K_ε is positive definite on $\mathbb{R}^3$, its Fourier transform being positive, hence on $a_k\mathbb{Z}^3$, and the field is the stationary Markov chain of the proof of Proposition 207(a) with γ the Gaussian law of covariance $K_{\varepsilon_k}(a_k(\mathbf{x}-\mathbf{y}))$; this gives reflection positivity, the gap and, since K_ε is radial, the invariance. As in Proposition 213(vi), the time factor of the covariance gives $C_k(2u+\sigma)=e^{-m\sigma}C_k(2u)$. The kernel $k_\varepsilon(z)=e^{-m|z_0|}K_\varepsilon(\mathbf{z})$ is bounded by 1 and Lipschitz, uniformly in $\varepsilon\in[1,3]$, and the covariance of $A_k(f)$ and $A_k(f')$ is a Riemann sum of mesh a_k for $\iint f(x)f'(y)k_{\varepsilon_k}(x-y)\,dx\,dy$, with error at most $C(f,f')a_k$ for $f,f'\in C_c^1$ and tending to 0 uniformly in ε for $f,f'\in C_c$. In particular $C_k(0)$ differs by $o(1)$ from $c(\varepsilon_k)$, where $c(\varepsilon)=\iint f_0(\theta x)f_0(y)k_\varepsilon(x-y)\,dx\,dy$ is positive and continuous on $[1,3]$. With $w_k(z)=q_k^{|z_0|}e^{-a_k^2|\mathbf{z}|^2/3}$, which bounds the covariance since $\varepsilon_k\le 3$, $a_k^4W_k\to(2/m)(3\pi)^{3/2}$, and (T1) holds with $K_2=1$, $K_n=0$ for $n\ge 3$, (T2) for large k, and (T3); Theorem 204(i) gives (M). Let S^ε be the Gaussian family with

$$S_2^\varepsilon(f,f')=c(\varepsilon)^{-1}\iint f(x)f'(y)k_\varepsilon(x-y)\,dx\,dy.$$

By Wick's rule, $|x_k(f_1,\dots,f_n)-S^{\varepsilon_k}(f_1,\dots,f_n)|$ tends to 0, and is at most Ca_k for $f_j\in C_c^1$. The derivative of k_ε in ε is bounded on compact sets uniformly in $\varepsilon\in[1,3]$ and $c>0$ there, so $|S^\varepsilon(f_1,\dots,f_n)-S^{\varepsilon'}(f_1,\dots,f_n)|\le C|\varepsilon-\varepsilon'|$, and $|\varepsilon_{k+1}-\varepsilon_k|\le\log(1+1/k)\le 1/k$; this gives the bound on δ_k, and (U1) follows by approximation and (M). Since $\log k\to\infty$ with increments tending to 0, the limit points of (ε_k) form $[1,3]$, so $\mathfrak{L}=\{S^\varepsilon:1\le\varepsilon\le 3\}$. Each S^ε is invariant under $\mathrm{O}(3)\times\mathrm{O}(1)$, since k_ε is, which gives (W). If $S_2^\varepsilon=S_2^{\varepsilon'}$, the continuous kernels $k_\varepsilon/c(\varepsilon)$ and $k_{\varepsilon'}/c(\varepsilon')$ coincide; at $z=0$ this gives $c(\varepsilon)=c(\varepsilon')$, and then $K_\varepsilon=K_{\varepsilon'}$, so $\varepsilon=\varepsilon'$. So $\varepsilon\mapsto S^\varepsilon$ is a continuous injection of $[1,3]$ onto $\mathfrak{L}$, a homeomorphism. □

So (RC) fixes the value of one functional on every limit and leaves (U1) and (U2) open: example (iv) satisfies (RC), (M), (W), the tree bounds and (U1), and its limits form an interval; the same field with ε_k alternating between 1 and 3 satisfies (RC), (M), (W), the tree bounds and (U2), and has two limits. Part (iii) is the form that a scale-by-scale analysis would supply. Along the one-loop tuning of Proposition 212, $\beta_k=hk+O(1)$, so a bound $|\delta_k(f_1,\dots,f_n)|\le C/\beta_k$, of first order in the running coupling, is of the size C/k of example (iv) and does not imply (U), while a bound $C\beta_k^{-p}$ with $p>1$ is summable and gives (U) by (iii). As for the ratio in Proposition 212, a bound on the change of the moments from one scale to the next gives (U) only if it is better than first order in the running coupling, or if the changes cancel between scales.

Remark 220. Let f_0 be invariant under the spatial rotations and $R\in\mathrm{SO}(3)_{\mathrm{sp}}$. The moments $x_k^R=(\omega_k(\prod_j Z_kA_k(f_j\circ R^{-1})))_n$ are those of the theory on the rotated lattice $R(a_k\mathbb{Z}^4)$, whose packet of f_0 is A_k; it has the same C_k, Z_k and ρ_k, and satisfies (RC) when the original theory does. The sequence $x_1,x_1^R,x_2,x_2^R,\dots$ converges if and only if (U) holds and the difference in (W) for R tends to 0: if it converges, both subsequences have the same limit, and conversely. For R of infinite order this is (U) and (W) together, the statement that the limit does not depend on which of the lattices $R(a_k\mathbb{Z}^4)$ is used. Example (a) of Proposition 207 satisfies (RC), since its time dependence factorises as in Proposition 213(vi), as well as (M), (U) and

the tree bounds, and not (W); example (iv) of Proposition 219 satisfies (RC), (M), (W) and the tree bounds, and not (U).

The time direction and the slice field. The two parts of (M) can be compared with correlations of slice fields, as the ratio was in Proposition 51. In the time direction the comparison is exact, because the transfer matrix is positive. In the spatial directions no such comparison is available: the spatial flow acts within each time slice and mixes the two sides of every spatial reflection plane. For $h' \in C_c(\mathbb{R}^3)$ put

$$U_k^{h'} = \sum_{x\in\mathbb{Z}^3} a_k^3 h'(a_k x)\big(\varphi_k(0,x) - \omega_k(\varphi_k(0,x))\big),$$

and, for $r \in a_k\mathbb{Z}_{\ge 0}$ and $\delta > 0$,

$$D_k^{h'}(r) = \omega_k\big(\bar U_k^{h'}\,\tau_{r/a_k} U_k^{h'}\big),$$
$$I_k^{h'}(\delta) = a_k \sum_{r\in a_k\mathbb{Z},\,0\le r<\delta} D_k^{h'}(r).$$

For the spatial factor h of a product packet, $U_k^h = U_k$ and $D_k^h = D_k$ are those of Proposition 51; write $I_k = I_k^h$. As in the proof of Proposition 213(iv), $D_k^{h'}(r) = \langle u, \mathbb{T}_k^{r/a_k} u\rangle$ with $u = \Psi_{U_k^{h'}}$.

Lemma 221. *Let $\mathbb{T}$ be self-adjoint with $0 \le \mathbb{T} \le 1$ on a Hilbert space, let v_n, $n \in \mathbb{Z}$, be vectors, nonzero only for n in a set of N consecutive integers, put $D_n(j) = \langle v_n, \mathbb{T}^j v_n\rangle$ and*

$$B = \sum_{n,n'\in\mathbb{Z}} \langle v_n, \mathbb{T}^{|n-n'|} v_{n'}\rangle.$$

(i) $|B| \le 2N \max_n \sum_{j=0}^{N-1} D_n(j)$.

(ii) *If $v_n = w_n v$ with $w_n \ge 0$, and $w_n \ge \eta$ on a set of $M \ge 1$ consecutive integers, then, with $D(j) = \langle v, \mathbb{T}^j v\rangle$,*

$$B \ge \eta^2 M \sum_{j=0}^{\lceil M/2\rceil - 1} D(j).$$

(iii) *$D(j)$ is nonnegative and nonincreasing in j, and $\sum_{j<pq} D(j) \le q \sum_{j<p} D(j)$ for integers $p, q \ge 1$.*

Proof. (i) Since $\mathbb{T}^j \ge 0$, the Cauchy–Schwarz inequality for the form $\langle \cdot, \mathbb{T}^j \cdot\rangle$ gives $|\langle v_n, \mathbb{T}^j v_{n'}\rangle| \le \frac12 (D_n(j) + D_{n'}(j))$. Summing over n, n' and using the symmetry, $|B| \le \sum_n \sum_{n'} D_n(|n-n'|)$, and for fixed n the integers n' with $v_{n'} \ne 0$ give each value of $|n-n'|$ at most twice, all values below N.

(ii) All terms of B are nonnegative, since $D \ge 0$. Keep the pairs in the set J of M consecutive integers. An element of J has at least $\lceil M/2\rceil - 1$ further elements of J on one of its sides, so its row contributes at least $\eta^2 \sum_{j<\lceil M/2\rceil} D(j)$.

(iii) $D(j) = \int \lambda^j d\nu$ for the spectral measure ν of $\mathbb{T}$ at v, carried by $[0,1]$. Each of the q blocks of p consecutive terms is at most the first block. □

By (iii), $D_k^{h'}$ is nonincreasing and $I_k^{h'}(p\delta) \le p I_k^{h'}(\delta)$ for integers $p \ge 1$ and $\delta \in a_k\mathbb{Z}$. By (i) and (ii), with $v_n = \Psi_{U_k^{h'}}$ for $0 \le a_k n < \delta$, the number $\delta^{-1} I_k^{h'}(\delta)$ lies between one half and twice the variance of the average of $\tau_n U_k^{h'}$ over these n: $I_k^{h'}(\delta)$ is the variance of the slice field averaged over a physical time δ, times δ. In the rest of this paragraph the states are invariant under translations and ϑ and reflection positive, with the transfer matrices of Appendix F, Φ is real, and $f_0 = g \otimes h$ is a product packet as in Proposition 51, with an interval I of length $\ell_I > 0$ and $c_g > 0$ such that $g \ge c_g$ on I, as in Proposition 213(iv).

Proposition 222. *Let $\delta > 0$ be dyadic and let k be so large that $\delta/a_k \in \mathbb{Z}$ and $a_k \le \min(\ell_I/4, \delta)$.*

(i) *Put $T_g = t_1 - t_0 + \delta$ and*

$$c_-(\delta) = \frac{3c_g^2 \ell_I}{4\lceil 8\delta/(3\ell_I)\rceil}, \qquad c_+(\delta) = 2\|g\|_\infty^2 T_g \Big\lceil \frac{T_g}{\delta}\Big\rceil.$$

Then $c_-(\delta) I_k(\delta) \le V_k \le c_+(\delta) I_k(\delta)$ and

$$\big(\tfrac34 c_g \ell_I\big)^2 D_k(2a_k n_1) \le C_k(0) \le \|g\|_\infty^2 T_g^2 D_k(2a_k n_0).$$

(ii) *Consider*

($\mathrm{Z_{sl}}$) *There are a dyadic $\delta > 0$ and $C < \infty$ with $I_k(\delta) \le C D_k(\delta)$ for all large k.*

If ($\mathrm{Z_{sl}}$) *holds for one dyadic δ, it holds for every dyadic $\delta > 0$, and then $D_k(\delta) \ge (\delta/(2C)) D_k(\delta/2)$ and $a_k D_k(0) \le C D_k(\delta)$ for all large k.*

(iii) ($\mathrm{M_Z}$) *holds if and only if* ($\mathrm{Z_{sl}}$) *holds. In particular* ($\mathrm{M_Z}$) *holds for one product packet with spatial factor h if and only if it holds for every product packet $g' \otimes h$ with g' as above, and it implies* ($\mathrm{N_u}$) *for every dyadic $u \ge 0$.*

Proof. (i) By time translation invariance, $A_k = \sum_n w_n \tau_n U_k$ with $w_n = a_k g(a_k n)$, and $V_k = \sum_{n,n'} w_n w_{n'} D_k(a_k|n-n'|)$ by the identity of the proof of Proposition 213(iv). The n with $w_n \ne 0$ lie in $[n_0, n_1]$, a set of $N \le (t_1 - t_0)/a_k + 1$ integers, so $a_k N \le T_g$. Lemma 221(i) with $v_n = w_n u_k$ and $|w_n| \le a_k \|g\|_\infty$ gives

$$V_k \le 2\|g\|_\infty^2 (a_k N) I_k(a_k N) \le 2\|g\|_\infty^2 T_g \Big\lceil \frac{T_g}{\delta}\Big\rceil I_k(\delta),$$

by (iii) of the lemma. The set J of n with $a_k n \in I$ consists of $M \ge \ell_I/a_k - 1 \ge \frac34 \ell_I/a_k$ consecutive integers, and $w_n \ge a_k c_g$ there. Lemma 221(ii) gives, with $b = a_k \lceil M/2\rceil \ge \frac38 \ell_I$,

$$V_k \ge a_k^2 c_g^2 M \sum_{j<\lceil M/2\rceil} D_k(a_k j) = c_g^2 (a_k M) I_k(b) \ge \tfrac34 c_g^2 \ell_I I_k(b),$$

and $I_k(\delta) \le \lceil \delta/b\rceil I_k(b) \le \lceil 8\delta/(3\ell_I)\rceil I_k(b)$. The bounds on $C_k(0)$ are those of the proof of Proposition 51(i), $C_k(0) = \int W^2 d\nu$ with $\|w\|_1 \lambda^{n_1} \le W(\lambda) \le \|w\|_1 \lambda^{n_0}$, together with $\|w\|_1 \ge a_k c_g M \ge \frac34 c_g \ell_I$ and $\|w\|_1 \le a_k N \|g\|_\infty \le T_g \|g\|_\infty$.

(ii) Since D_k is nonincreasing, $I_k(\delta) \ge \frac{\delta}{2} D_k(\delta/2)$ and $I_k(\delta) \ge a_k D_k(0)$, which gives the two bounds; in particular $c = \delta/(2C) \le 1$, because $D_k(\delta) \le D_k(\delta/2)$ and $D_k(\delta/2) > 0$ by Proposition 51(i). Let δ' be dyadic. If $\delta' < \delta$, then $I_k(\delta') \le I_k(\delta) \le C D_k(\delta) \le C D_k(\delta')$. If $\delta' > \delta$, Proposition 51(ii) with $r = \delta/2$, $r' = \delta$ gives $D_k(\delta') \ge c^e D_k(\delta)$ with $e = \max(1, 2(\delta' - \delta)/\delta)$, and

$$I_k(\delta') \le I_k(\delta) + (\delta' - \delta) D_k(\delta) \le (C + \delta')\, c^{-e} D_k(\delta').$$

(iii) Suppose $\Upsilon_k \le \Upsilon$ for all k, and take a dyadic $\delta \le 2t_0$. Then $2a_k n_0 \ge 2t_0 \ge \delta$, and by (i)

$$I_k(\delta) \le \frac{V_k}{c_-(\delta)} \le \frac{\Upsilon C_k(0)}{c_-(\delta)} \le \frac{\Upsilon \|g\|_\infty^2 T_g^2}{c_-(\delta)} D_k(\delta),$$

which is (Z_{sl}). Conversely, assume (Z_{sl}) at δ with the constant C, and take a dyadic $R \ge \max(2t_1, \delta)$. Then $D_k(2a_k n_1) \ge D_k(R) \ge c^e D_k(\delta)$ with $c = \delta/(2C)$ and $e = \max(1, 2(R-\delta)/\delta)$, as in (ii), and by (i)

$$\Upsilon_k = \frac{V_k}{C_k(0)} \le \frac{c_+(\delta) C c^{-e}}{(\frac{3}{4} c_g \ell_I)^2}$$

for all large k; the finitely many other Υ_k are finite, since $C_k(0) > 0$. Condition (Z_{sl}) involves h and not g, and (M_Z) implies (N_u) by Proposition 213(iv). □

So for product packets (M_Z) is a statement about the slice field alone: the variance of U_k averaged over a physical time δ is at most a constant times $D_k(\delta)/\delta$. Since $I_k(\delta) \ge a_k D_k(0)$, it contains the bound $a_k \operatorname{Var} U_k \le C D_k(\delta)$: the equal-time variance of the slice field may exceed its correlation at a physical separation by at most a factor of order $1/a_k$. In terms of the spectral measure μ_k of $\mathbb{T}_k$ at u_k, in energies $E = -a_k^{-1} \log \lambda$, $I_k(\delta)$ is the integral of $a_k(1-e^{-\delta E})/(1-e^{-a_k E})$, a weight between $(1-e^{-\delta E})/E$ and that plus a_k, and (Z_{sl}) bounds it by $\int e^{-\delta E} d\mu_k$; it bounds the spectral weight at high energy, counted with the factor $1/E$, by the weight at bounded physical energy. By (ii) it contains a ratio bound for D_k, hence (N_0), and it is strictly stronger than (N_u): for the field of Proposition 213(vi), $\rho_k(u,\sigma) = e^{-m\sigma}$ while $a_k D_k(0)$, which contains $a_k \cdot a_k^{-5} a_k^6 \sum_x h(a_k x)^2$, grows like a_k^{-1}, and $D_k(\delta)$ converges. Along the constant tunings of Proposition 192(ii) the hypothesis of Theorem 45 holds and (Z_{sl}) fails, since (N_u) fails there. So (M_Z) reduces to (N_u) in one direction only, and no argument that uses (D) only through the hypothesis of Theorem 45 gives it.

Proposition 223. *Let δ and k be as in Proposition 222.*

(i) *Let $g_1, g_2 \in C_c(\mathbb{R})$ be supported in an interval of length L, $y_1, y_2 \in a_k \mathbb{Z}^3$ and $f_i = g_i \otimes h(\cdot - y_i)$. Then*

$$\big|\omega_k\big(A_k(f_1)A_k(f_2)\big)\big| \le K_L V_k \|g_1\|_\infty \|g_2\|_\infty,$$

with $K_L = 2(L+\delta)\lceil (L+\delta)/\delta \rceil / c_-(\delta)$. So ($M_{loc}$) holds for $n = 2$ on these products, along every sequence of couplings.

(ii) (M_{loc}) *holds for $n = 2$ if and only if the following holds for one dyadic $\delta > 0$, and then it holds for every dyadic $\delta > 0$:*

(M_{sp}) *For every cube $Q_3 \subset \mathbb{R}^3$ there is $C < \infty$ with $I_k^{h'}(\delta) \le C I_k(\delta)$ for all large k and all $h' \in C_c(\mathbb{R}^3)$ supported in Q_3 with $\|h'\|_\infty \le 1$.*

(iii) (M_{loc}) *holds if and only if for every even $n \ge 2$ and every cube Q there are $C_n(Q)$ and k_0 with $\omega_k(|A_k(f)|^n) \le C_n(Q) V_k^{n/2} \|f\|_\infty^n$ for $k \ge k_0$ and $f \in C_c(\mathbb{R}^4)$ supported in Q. The case $n = 2$ is equivalent to (M_{sp}), and the case n implies the cases below it.*

(iv) *Let $m > 0$, $q_k = e^{-m a_k}$, $\kappa(\mathbf{z}) = e^{-|\mathbf{z}|^2}$ on $\mathbb{R}^3$, and let ψ_k be the centred Gaussian field on $\mathbb{Z}^4$ with covariance*

$$q_k^{|x_0 - y_0|} \Big(\kappa\big(a_k(\mathbf{x}-\mathbf{y})\big) + a_k^{-1} (-1)^{\sum_{i=1}^3 (x_i - y_i)} \Big),$$

$A_k(f) = \sum_x a_k^4 f(a_k x) \psi_k(x)$, and $f_0 = g \otimes h$ a product packet as in Proposition 51 with $g, h \ge 0$ nonzero and continuously differentiable. The law of ψ_k is invariant under translations and Γ and reflection positive for the time reflections through sites and bisecting links, its transfer matrix has gap $m a_k$, $\rho_k(u,\sigma) = e^{-m\sigma}$ for all u, σ, so (RC) *holds for every σ_0, and* (M_Z) *holds. But* (M_{sp}) *fails, and* (M_{loc}) *and* (M) *fail for $n = 2$.*

(v) *Let $m > 0$, $q_k = e^{-m a_k}$, $\varepsilon_k = a_k$, let $(\xi_n)_{n \in \mathbb{Z}}$ be the stationary Markov chain on $\{-1, 0, 1\}$ with the law π of Remark 59 at $\varepsilon = \varepsilon_k$ and transition operator $P = q_k I + (1 - q_k)\Pi$, and $X_n = \xi_n / \sqrt{\varepsilon_k}$. Let $\psi_k(x) = X_{x_0}$, and A_k, f_0 as in* (iv). *This field has the properties listed in* (iv), *and* (M_{sp}) *holds; but $\omega_k(A_k^4)/V_k^2 \to \infty$, so* ($M_{loc}$) *and* (M) *fail for $n = 4$.*

Proof. (i) Since Φ is real, $A_k(\bar f) = \overline{A_k(f)}$, so by the Cauchy–Schwarz inequality

$$\big|\omega_k\big(A_k(f_1)A_k(f_2)\big)\big| \le \prod_{i=1}^2 \omega_k\big(|A_k(f_i)|^2\big)^{1/2},$$

and it suffices to bound $\omega_k(|A_k(f)|^2)$ for $f = g \otimes h(\cdot - y)$ by $K_L V_k \|g\|_\infty^2$. By spatial translation invariance y may be taken to be 0. As in the proof of Proposition 222(i), $\omega_k(|A_k(f)|^2)$ is B of Lemma 221 with $v_n = a_k g(a_k n) u_k$, nonzero for n in a set of $N \le L/a_k + 1$ consecutive integers, so

$$\begin{aligned}\omega_k(|A_k(f)|^2) &\le 2\|g\|_\infty^2 (a_k N) I_k(a_k N) \\ &\le 2\|g\|_\infty^2 (L+\delta) \Big\lceil \frac{L+\delta}{\delta} \Big\rceil I_k(\delta),\end{aligned}$$

and Proposition 222(i) gives the claim.

(ii) Independence of δ: for dyadic $\delta' > \delta$, $I_k^{h'}(\delta') \le \lceil \delta'/\delta \rceil I_k^{h'}(\delta) \le \lceil \delta'/\delta \rceil C I_k(\delta')$, and for $\delta' < \delta$, $I_k^{h'}(\delta') \le I_k^{h'}(\delta) \le C I_k(\delta) \le C \lceil \delta/\delta' \rceil I_k(\delta')$. Assume ($M_{sp}$), and let $f \in C_c(\mathbb{R}^4)$ be supported in a cube $Q = [t_Q, t_Q + L] \times Q_3$. As in (i) it suffices to bound $\omega_k(|A_k(f)|^2)$. By time translation invariance $A_k(f) = \sum_n \tau_n(a_k U_k^{h_n})$ with $h_n = f(a_k n, \cdot)$, supported in Q_3 with $\|h_n\|_\infty \le \|f\|_\infty$, and nonzero for n in a set of $N \le L/a_k + 1$ consecutive integers. The identity of the proof of Proposition 213(iv) and translation invariance give $\omega_k(\bar U_k^{h_n} \tau_j U_k^{h_{n'}}) = \langle \Psi_{U_k^{h_n}}, \mathbb{T}_k^j \Psi_{U_k^{h_{n'}}} \rangle$ for $j \ge 0$, so $\omega_k(|A_k(f)|^2)$ is B of Lemma 221 with $v_n = a_k \Psi_{U_k^{h_n}}$, and

$$\begin{aligned}\omega_k(|A_k(f)|^2) &\le 2(a_k N) \max_n I_k^{h_n}(a_k N) \\ &\le \frac{2(L+\delta)C}{c_-(\delta)} \Big\lceil \frac{L+\delta}{\delta} \Big\rceil \|f\|_\infty^2 V_k,\end{aligned}$$

by (iii) of the lemma, (M_{sp}) and Proposition 222(i). Conversely, let $g_* \in C_c(\mathbb{R})$ satisfy $0 \le g_* \le 1$, $g_* = 0$ outside $[0,1]$ and $g_* = 1$ on $[\frac{1}{4}, \frac{3}{4}]$, and let h' be as in (M_{sp}). For $a_k \le \frac{1}{4}$ the n with $a_k n \in [\frac{1}{4}, \frac{3}{4}]$ are $M' \ge 1/(4a_k)$ consecutive integers, and

Lemma 221(ii) with $v_n = a_k g_*(a_k n)\Psi_{U_k^{h'}}$ and $\eta = a_k$ gives, with $b = a_k\lceil M'/2\rceil \geq \frac{1}{8}$,

$$\omega_k\big(|A_k(g_*\otimes h')|^2\big) \geq a_k M' I_k^{h'}(b) \geq \frac{I_k^{h'}(\delta)}{4\lceil 8\delta\rceil}.$$

Since $A_k(g_*\otimes\bar h') = \overline{A_k(g_*\otimes h')}$, $(\mathrm{M_{loc}})$ for $n=2$ on a cube $Q \supseteq [0,1]\times Q_3$ bounds the left side by $M_2^{\mathrm{loc}}(Q)V_k \leq M_2^{\mathrm{loc}}(Q)c_+(\delta)I_k(\delta)$, which is $(\mathrm{M_{sp}})$.

(iii) The bounds are $(\mathrm{M_{loc}})$ with f_j equal to $\bar f$ and f alternately, since $A_k(\bar f) = \overline{A_k(f)}$. Conversely, for $f_1,\dots,f_n$ supported in Q and an even $n' \geq n$, Hölder's inequality gives $|\omega_k(\prod_j A_k(f_j))| \leq \prod_j \omega_k(|A_k(f_j)|^{n'})^{1/n'}$. The case $n=2$ is $(\mathrm{M_{loc}})$ for $n=2$ by the Cauchy–Schwarz inequality, hence $(\mathrm{M_{sp}})$ by (ii), and the last claim is the monotonicity of L^n norms for the probability measure ω_k.

(iv) The spatial kernel is the sum of two positive definite functions on $\mathbb{Z}^3$, the first because the Fourier transform of κ is positive, the second because it is $\varepsilon_{\mathbf{x}}\varepsilon_{\mathbf{y}}$ with $\varepsilon_{\mathbf{x}} = (-1)^{x_1+x_2+x_3}$; both are invariant under the signed permutations of the spatial coordinates. The field is the stationary Markov chain of the proof of Proposition 207(a), with γ the Gaussian law of this spatial covariance, which gives the invariance, reflection positivity and the gap, and as in Proposition 213(vi) the time factor gives $\rho_k(u,\sigma) = e^{-m\sigma}$. Put $S_k = \sum_{\mathbf{x}} a_k^3 h(a_k\mathbf{x})\varepsilon_{\mathbf{x}}$. Grouping $\mathbf{x}$ with x_1 even with $\mathbf{x}+e_1$, $|S_k| \leq a_k\|\partial_1 h\|_\infty (L_h+2a_k)^3$, where L_h is the side of a cube containing the support of h. The part of V_k from the second term of the kernel is

$$a_k^{-1}\sum_{n,n'} a_k^2 g(a_k n)g(a_k n')q_k^{|n-n'|}S_k^2 \leq a_k^{-1}\Big(a_k\sum_n g(a_k n)\Big)^2 S_k^2,$$

which tends to 0, and the same holds for $C_k(0)$. The parts from κ are Riemann sums converging to

$$\iint f_0(x)f_0(y)e^{-m|x_0-y_0|}\kappa(\mathbf{x}-\mathbf{y})\,dx\,dy > 0,$$
$$\iint g(t)g(s)e^{-m(t+s)}dt\,ds \iint h(\mathbf{x})h(\mathbf{y})\kappa(\mathbf{x}-\mathbf{y})\,d\mathbf{x}\,d\mathbf{y} > 0.$$

So Υ_k converges, which is $(\mathrm{M_Z})$. Let

$$f_k(t,\mathbf{x}) = g(t)h(\mathbf{x})\cos\big(\pi(x_1+x_2+x_3)/a_k\big),$$

a continuous function supported in the support of f_0, with $\|f_k\|_\infty \leq \|g\|_\infty\|h\|_\infty$, equal to $g(a_k x_0)h(a_k\mathbf{x})\varepsilon_{\mathbf{x}}$ at the point $a_k x$. Both terms of the kernel are positive definite, so

$$\omega_k\big(A_k(f_k)^2\big) \geq a_k^{-1}\sum_{n,n'} a_k^2 g(a_k n)g(a_k n')q_k^{|n-n'|}\Big(\sum_{\mathbf{x}} a_k^3 h(a_k\mathbf{x})\Big)^2,$$

and the factor of a_k^{-1} on the right converges to

$$\iint g(t)g(s)e^{-m|t-s|}dt\,ds\Big(\int h\Big)^2 > 0,$$

while V_k converges. So $(\mathrm{M_{loc}})$ fails for $n=2$, and with it $(\mathrm{M_{sp}})$, by (ii), and (M), by Proposition 213(ii).

(v) $P = \Pi + q_k(I-\Pi)$ is symmetric on $L^2(\pi)$ with spectrum $\{1,q_k\}$ and 1 simple, and $P \geq 0$; the argument of the proof of Proposition 207(a) gives reflection positivity, and the gap is ma_k. The field does not depend on the spatial coordinates, which gives the invariance. Since $X \perp 1$ and $\pi(X^2)=1$, $P^jX = q_k^jX$ and the covariance is $q_k^{|x_0-y_0|}$, that of Proposition 207(b); so $\rho_k(u,\sigma) = e^{-m\sigma}$, and V_k and $C_k(0)$ converge to positive limits, which gives $(\mathrm{M_Z})$. Further $U_k^{h'} = (\sum_{\mathbf{x}} a_k^3 h'(a_k\mathbf{x}))X_0$, so $I_k^{h'}(\delta)/I_k(\delta) = |\sum a_k^3 h'(a_k\mathbf{x})|^2/(\sum a_k^3 h(a_k\mathbf{x}))^2$, which is bounded for h' as in $(\mathrm{M_{sp}})$, since the denominator converges to $(\int h)^2 > 0$. For times $s_1 \leq s_2 \leq s_3 \leq s_4$ with $j_i = s_{i+1}-s_i$, the Markov property and $P^j = \Pi + q_k^j(I-\Pi)$ give, as in Remark 59 and with $X^3 = X/\varepsilon$,

$$\mathbb{E}\Big[\prod_{i=1}^4 X_{s_i}\Big] = q_k^{j_1+j_3}\big(q_k^{j_2}/\varepsilon_k + 1 - q_k^{j_2}\big) \geq \varepsilon_k^{-1}q_k^{s_4-s_1}.$$

With $F_n = a_k^3\sum_{\mathbf{x}} f_0(a_k n, a_k\mathbf{x}) \geq 0$, $A_k = \sum_n a_k F_n X_n$, and $F_n = 0$ unless $a_k n \in [t_0,t_1]$, so

$$\omega_k(A_k^4) \geq \varepsilon_k^{-1}e^{-m(t_1-t_0)}\Big(a_k\sum_n F_n\Big)^4,$$

and $a_k\sum_n F_n \to \int f_0 > 0$. Since $\varepsilon_k = a_k \to 0$ and V_k converges, the ratio diverges. □

Part (i) is the part of $(\mathrm{M_{loc}})$ that the positivity of the transfer matrix gives: test functions in time, on both sides of the reflection plane and of either sign, cost no more than the packet f_0, for every state with a positive transfer matrix. What remains for $n=2$ is $(\mathrm{M_{sp}})$, a comparison between spatial profiles at one spacing, which involves neither the reflection nor the tuning. Example (iv) has a spatial mode at the lattice scale whose variance diverges, and the time structure of an exact Markov field; example (v) has no spatial structure and four-point functions of order $1/\varepsilon_k$, as in Remark 59. For a Gaussian field the linear heat flow in space at the flow time τ/a_k^2 multiplies the variance of the lattice-scale mode of (iv) by $e^{-c\tau/a_k^2}$, with $c>0$ fixed by the normalisation of the flow, and removes the divergence there. For the gauge field the flow does not do this configuration by configuration.

Proposition 224. *Let $G = \mathrm{SU}(N)$, $N \geq 2$, $\zeta = e^{2\pi i/N}$ and $c_N = N(1-\cos(2\pi/N))$. Let $m : \mathbb{Z}\to\mathbb{Z}$ satisfy $m(j+1)-m(j) = 1$ for $j \equiv 0,1$ and $m(j+1)-m(j) = 0$ for $j \equiv 2,3$ modulo 4, and let $U^\sharp$ be the configuration with all temporal links equal to 1 and spatial links*

$$U^\sharp_{(x,x+e_1)} = U^\sharp_{(x,x+e_3)} = 1, \qquad U^\sharp_{(x,x+e_2)} = \zeta^{m(x_1)}.$$

(i) *The spatial plaquettes of $U^\sharp$ in the plane of e_1,e_2 with base point x are $\zeta^{\pm1}$ for $x_1 \equiv 0,1$ and 1 for $x_1 \equiv 2,3$ modulo 4, all other spatial plaquettes are 1, and $\Phi_s(gU^\sharp) = gU^\sharp$ for every gauge transformation g and every $s\in\mathbb{R}$.*

(ii) *Let Φ be the sum of $N - \mathrm{Re}\,\mathrm{Tr}\,U_p$ over the twelve spatial plaquettes with a corner at 0. Then for all x and s*

$$\Phi_x\circ\Phi_s(U^\sharp) = 2c_N\Big(1+\cos\frac{\pi(x_1-1)}{2}\Big).$$

(iii) *Let $g \in C_c(\mathbb{R})$ and $h \in C_c(\mathbb{R}^3)$ be nonnegative and $h_k^\sharp(\mathbf{x}) = h(\mathbf{x})\cos(\pi(x_1/a_k - 1)/2)$, and let ω be as in Proposition 214*(iii)*. At every flow time the ω-essential supremum and infimum of $A_k(g \otimes h_k^\sharp)$ differ by at least*

$$4c_N a_k^4 \sum_n g(a_k n) \sum_{\mathbf{x}: x_1 \equiv 1} h(a_k \mathbf{x}), \tag{4}$$

which converges to $c_N \int g \int h$. For $A_k(g \otimes h)$ the difference is at least

$$12 c_N a_k^4 \sum_n g(a_k n) \sum_{\mathbf{x}} h(a_k \mathbf{x}),$$

which converges to $12 c_N \int g \int h$.

Proof. (i) With the plaquette formula of the proof of Proposition 214(i), the plaquette in the plane of e_1, e_2 at x is $\zeta^{m(x_1+1)-m(x_1)}$; in the plane of e_1, e_3 all four links are 1, and in the plane of e_2, e_3 it is $\zeta^{m(x_1)}\zeta^{-m(x_1)} = 1$, because x and $x + e_3$ have the same first coordinate. All links are central, and the computation of the proof of Proposition 214(i) shows that the gradient in the flow equation vanishes; uniqueness and gauge covariance give the rest.

(ii) Only the four plaquettes of the plane of e_1, e_2 with a corner at x can be nontrivial; their base points have first coordinates x_1 and $x_1 - 1$, twice each. So $\Phi_x(U^\sharp) = 2c_N(d(x_1) + d(x_1 - 1))$ with $d(j) = m(j+1) - m(j)$, which is $2c_N$ times $1, 2, 1, 0$ for $x_1 \equiv 0, 1, 2, 3$, and the flow does not move $U^\sharp$.

(iii) With $c_x = \cos(\pi(x_1 - 1)/2) \in \{1, 0, -1, 0\}$ for $x_1 \equiv 1, 2, 3, 0$,

$$\sum_x a_k^4 g(a_k x_0) h_k^\sharp(a_k \mathbf{x})\, \Phi_x \circ \Phi_s(U^\sharp) = 2c_N \sum_x a_k^4 g(a_k x_0) h(a_k \mathbf{x})(c_x + c_x^2),$$

and $c_x + c_x^2$ is 2 for $x_1 \equiv 1$ and 0 otherwise; at the configuration 1 the sum is 0. The proof of Proposition 214(iii) applies with $U^\sharp$ in place of U^*; the centring constant cancels in the difference. The points with $x_1 \equiv 1$ modulo 4 form a sublattice of index four, which gives the limit. The second statement is Proposition 214(iii). □

At every flow time the packet with a spatial profile oscillating at the lattice scale has, at the stationary configurations, a range of values of the same order as the smooth packet, the ratio of the two lower bounds tending to $1/12$. A proof of ($\mathrm{M_{sp}}$) by way of the flow has therefore to bound the probability of the configurations near those with nontrivial central plaquettes, uniformly along the tuning; this is an estimate on large fields of the Wilson measure at the couplings β_k, of the kind of Sec. XII E, compared with the size of $I_k(\delta)$, which is not known.

Remark 225. For the flowed fields along a tuning, with a product packet $f_0 = g \otimes h$, the five statements stand as follows.

($\mathrm{M_{loc}}$): the case $n = 1$ holds since the packets are centred, and the case $n = 2$ holds on the products of time profiles with the lattice translates of h (Proposition 223(i)), for every state with a positive transfer matrix. Not proved: ($\mathrm{M_{sp}}$), equivalent to the case $n = 2$, and the bounds $\omega_k(|A_k(f)|^n) \le C_n(Q) V_k^{n/2} \|f\|_\infty^n$ for even $n \ge 4$, which with it are equivalent to ($\mathrm{M_{loc}}$). Both concern one spacing at a time, without reflection; they are not consequences of the gap, (N_u), (RC), (M_Z), reflection positivity and the symmetry Γ, by Proposition 223(iv), (v), so no argument that uses (D) only through the transfer-matrix gap, together with (N_u), gives them.

(M_Z): equivalent to ($\mathrm{Z_{sl}}$), $I_k(\delta) \le C D_k(\delta)$ for the slice field, and independent of g (Proposition 222). Not proved. It implies (N_u) for every u; it is not implied by (N_u), the gap and the lattice symmetries (Proposition 213(vi)), nor by the hypothesis of Theorem 45 (Proposition 192(ii)). The missing estimate is a bound on the high-energy spectral weight of the slice field, with the factor $1/E$, and on $a_k \operatorname{Var} U_k$, by its correlation at a physical separation.

(W): equivalent to $\Delta_k^{E_{12}}(f_1, \dots, f_n) \to 0$ for all n and $f_j \in C_c^\infty(\mathbb{R}^4)$ (Proposition 216(ii)); a bound $C(f_1, \dots, f_n) a_k^p$ with $p > 0$ suffices. Not proved. Example (a) of Proposition 207 has the gap in every direction, (RC), (M), hence ($\mathrm{M_{sp}}$) and ($\mathrm{Z_{sl}}$), and (U), and not (W); so no argument that uses (D) only through the gap, together with (N_u) and (M), gives (W). The lattice symmetry gives the cancellation of $\Delta_k^{E_{12}}$ only on average over the angle.

(U1) and (U2), given (RC): (U1) is $\delta_k \to 0$, for which $\sum_k |\delta_k| < \infty$ on a set with dense span suffices (Proposition 219(iii)), and (U2) is the total disconnectedness of $\mathfrak{L}$. Not proved. Example (iv) of Proposition 219 and its alternating variant satisfy (RC), (M) and (W), each with one of (U1), (U2) and not the other, so neither follows from the gap, (N_u), (RC), (M) and (W); (U1) compares two spacings, which no statement at one spacing contains.

The examples are Gaussian and Markov fields, not lattice gauge theories. They show that a proof of any of the four open statements has to use properties of the Wilson measure along the tuning beyond translation and Γ invariance, reflection positivity, the transfer-matrix gap, the normalisation (RC) and the parts of (M) proved or assumed here.

The spatial spectral measure of the slice field. The comparison ($\mathrm{M_{sp}}$) is a statement about the distribution in spatial momentum of the time-integrated correlations of the slice field. In this paragraph the states are invariant under translations and ϑ and reflection positive, with the transfer matrices of Appendix F, Φ is real, $\delta > 0$ is dyadic and k is so large that $n_\delta = \delta/a_k$ is an even integer. Write $\mathring{\phi}_k(x) = \varphi_k(x) - \omega_k(\varphi_k(x))$,

$$K_k(z) = a_k \sum_{j=0}^{n_\delta - 1} \omega_k\big(\mathring{\phi}_k(0,0)\, \mathring{\phi}_k(j,z)\big),$$

$$K_k^V(z) = a_k^2 \sum_{n,n'=0}^{n_\delta - 1} \omega_k\big(\mathring{\phi}_k(n,0)\, \mathring{\phi}_k(n',z)\big)$$

for $z \in \mathbb{Z}^3$. For a finitely supported $c : \mathbb{Z}^3 \to \mathbb{C}$ put $\hat{c}(p) = \sum_x c_x e^{ip\cdot x}$ on $\mathbf{T}^3 = (\mathbb{R}/2\pi\mathbb{Z})^3$, $U_k^c = \sum_x c_x \mathring{\phi}_k(0,x)$, and

$$I_k^c = a_k \sum_{j=0}^{n_\delta - 1} \omega_k\big(\bar{U}_k^c\, \tau_j U_k^c\big), \qquad V_k^c = \omega_k\Big(\Big|a_k \sum_{n=0}^{n_\delta - 1} \tau_n U_k^c\Big|^2\Big).$$

For $c_x = a_k^3 h'(a_k x)$, $U_k^c = U_k^{h'}$ and $I_k^c = I_k^{h'}(\delta)$. By translation invariance $I_k^c = \sum_{x,y} \bar{c}_x c_y K_k(y-x)$ and $V_k^c = \sum_{x,y} \bar{c}_x c_y K_k^V(y-x)$. For a closed box $\mathscr{B}_r(p_0) = \{p : |p_i - p_{0,i}| \le r,\ i = 1,2,3\}$ in $\mathbf{T}^3$, $r > 0$, and a finite positive measure ϖ on $\mathbf{T}^3$, put $\bar{\varpi}(r) = \sup_{p_0} \varpi(\mathscr{B}_r(p_0))$. A box of half-side r' is covered by $\lceil r'/r \rceil^3$ boxes of half-side r, so $\bar{\varpi}(r') \le \lceil r'/r \rceil^3 \bar{\varpi}(r)$.

Lemma 226. (i) *There are unique finite positive measures ϖ_k and ϖ_k^V on $\mathbf{T}^3$ with $K_k(z) = \int e^{ip\cdot z} d\varpi_k(p)$ and $K_k^V(z) = \int e^{ip\cdot z} d\varpi_k^V(p)$ for $z \in \mathbb{Z}^3$. For every finitely supported c*

$$I_k^c = \int |\hat{c}|^2 d\varpi_k, \qquad V_k^c = \int |\hat{c}|^2 d\varpi_k^V, \qquad \tfrac{\delta}{2} I_k^c \le V_k^c \le 2\delta I_k^c,$$

and $\varpi_k(\mathbf{T}^3) = K_k(0) \le \delta \|\Phi\|_\infty^2$.

(ii) *Let ϖ be a finite positive measure on $\mathbf{T}^3$, $L \ge 4a_k$, and $c_x = a_k^3 h'(a_k x)$ with h' supported in a cube of side L and $|h'| \le 1$. Then*

$$\int |\hat{c}|^2 d\varpi \le 7 \cdot 10^6 L^6 \bar{\varpi}(\pi a_k / L).$$

(iii) *Let $\chi \in C_c(\mathbb{R}^3)$ take values in $[0,1]$, be supported in a cube Q' of side $\ell \ge 4a_k$ and equal 1 on the cube of side $\ell/2$ with the same centre, and for $p_0 \in \mathbf{T}^3$ let $h'(\mathbf{y}) = \chi(\mathbf{y}) e^{-ip_0\cdot \mathbf{y}/a_k}$. Then for every finite positive measure ϖ*

$$\varpi\big(\mathscr{B}_{a_k/(3\ell)}(p_0)\big) \le 16384\, \ell^{-6} \int |\hat{c}|^2 d\varpi, \qquad c_x = a_k^3 h'(a_k x).$$

(iv) *Let $h \in C_c(\mathbb{R}^3)$ be real with $\int h \ne 0$, supported in a cube of side L_h, and $\rho_h = L_h^{-1}(|\int h|/(72 \int |h|))^{1/2}$. For all large k, with $c_x = a_k^3 h(a_k x)$, $|\hat{c}| \ge \frac{3}{4}|\int h|$ on $\mathscr{B}_{a_k\rho_h}(0)$, so $\int |\hat{c}|^2 d\varpi \ge \frac{1}{2}(\int h)^2 \varpi(\mathscr{B}_{a_k\rho_h}(0))$ for every finite positive measure ϖ.*

(v) *Let $\varkappa(p) = \sum_{i=1}^3 4\sin^2(p_i/2)$, $m \ge 1$, $h \in C_c^{2m}(\mathbb{R}^3)$ real, supported in a cube of side L_h, $\|h\|_{2m}$ the largest supremum of a derivative of order $2m$ of h, and $\Theta_m = 3^m \|h\|_{2m}(L_h + 2m + 1)^3$. For $a_k \le 1$ and $c_x = a_k^3 h(a_k x)$, $\varkappa^m |\hat{c}| \le \Theta_m a_k^{2m}$ on $\mathbf{T}^3$.*

Proof. (i) With $u_c = \Psi_{U_k^c}$, the identity of the proof of Proposition 213(iv) gives $\omega_k(\bar{U}_k^c \tau_j U_k^c) = \langle u_c, \mathbb{T}_k^j u_c \rangle \ge 0$ for $j \ge 0$, so $I_k^c \ge 0$ and K_k is positive definite; $V_k^c \ge 0$ shows the same for K_k^V. For a positive definite bounded K on $\mathbb{Z}^3$ and an integer $M \ge 1$, the function

$$f_M(p) = M^{-3} \sum_{x,y \in \{0,\dots,M-1\}^3} K(y-x) e^{-ip\cdot(y-x)}$$

is nonnegative, and $\int e^{ip\cdot z} f_M(p) \frac{dp}{(2\pi)^3} = K(z) \prod_i (1 - |z_i|/M)_+$. The measures $f_M\, dp/(2\pi)^3$ have mass $K(0)$, so by Lemma 671 a subsequence converges weakly to a measure with Fourier coefficients $K(z)$; trigonometric polynomials are dense in $C(\mathbf{T}^3)$, which gives uniqueness. Then $\sum_{x,y} \bar{c}_x c_y \int e^{ip\cdot(y-x)} d\varpi = \int |\hat{c}|^2 d\varpi$. The numbers $\langle u_c, \mathbb{T}_k^j u_c \rangle$ are real, so translation invariance and complex conjugation give $\omega_k(\overline{\tau_n U_k^c}\, \tau_{n'} U_k^c) = \langle u_c, \mathbb{T}_k^{|n-n'|} u_c \rangle$ for all n, n', and V_k^c is B of Lemma 221 with $v_n = a_k u_c$ for $0 \le n < n_\delta$. Part (i) of that lemma gives $V_k^c \le 2 n_\delta a_k I_k^c = 2\delta I_k^c$; part (ii) with $\eta = a_k$, $M = n_\delta$ and part (iii) give $V_k^c \ge \delta\, a_k \sum_{j < n_\delta/2} \langle u_c, \mathbb{T}_k^j u_c \rangle \ge \frac{\delta}{2} I_k^c$. Finally $|\omega_k(\mathring{\varphi}_k(0,0)\mathring{\varphi}_k(j,0))| \le \|\Phi\|_\infty^2$, since φ_k takes values in $[-\|\Phi\|_\infty, \|\Phi\|_\infty]$.

(ii) Translating c by a lattice vector multiplies $\hat{c}$ by a phase. With $n = \lceil L/(2a_k) \rceil$ the support of c may therefore be taken in $\{|x_i| \le n\}$; it has at most $(L/a_k + 1)^3$ points. Let $F_n(t) = \sum_{|j| \le n} (1 - \frac{|j|}{n+1}) e^{ijt} = \frac{1}{n+1} \big(\sin \frac{(n+1)t}{2} / \sin \frac{t}{2}\big)^2$ be the Fejér kernel and $W_n = 2F_{2n+1} - F_n$. Its coefficients equal 1 for $|j| \le n$, $\int |W_n| \frac{dt}{2\pi} \le 3$, and from $F_n \le n+1$ and $\sin \frac{t}{2} \ge \frac{t}{\pi}$ on $[0,\pi]$, $F_n(t) \le 2(n+1)/(1 + ((n+1)t/\pi)^2)$ for $|t| \le \pi$, so

$$|W_n(t)| \le \frac{10(n+1)}{1 + ((n+1)t/\pi)^2} \qquad (|t| \le \pi).$$

With $W(q) = \prod_i W_n(q_i)$, $\hat{c}(p) = \int \hat{c}(q) W(p-q) dq'$, where $dq' = dq/(2\pi)^3$, and the Cauchy–Schwarz inequality gives $|\hat{c}(p)|^2 \le 27 \int |\hat{c}(q)|^2 |W(p-q)| dq'$. Divide $\mathbf{T}^3$ into the $(2n+2)^3$ half-open boxes of side $\pi/(n+1)$ centred at the points $q + \pi j/(n+1)$, $j \in \{-n, \dots, n+1\}^3$; each lies in a closed box of half-side $\pi/(2(n+1)) \le \pi a_k/L$, and on the box with index j the i-th factor of $|W(p-q)|$ is at most $10(n+1)$ if $j_i = 0$ and at most $10(n+1)/(1 + (|j_i| - \frac{1}{2})^2)$ otherwise. Since $(1 + (x - \frac{1}{2})^2)^{-1}$ decreases on $[\frac{1}{2}, \infty)$, $\sum_{j \ge 2} (1 + (j - \frac{1}{2})^2)^{-1} \le \frac{\pi}{2} - \arctan \frac{1}{2} \le 1.1125$, using $\arctan \frac{1}{2} \ge \frac{1}{2} - \frac{1}{24}$. The bounds of the i-th factor therefore add up to at most $10(n+1)(1 + 2(0.8 + 1.1125)) \le 50(n+1)$, the term $j = 1$ being 0.8, and

$$\int |W(p-q)| d\varpi(p) \le \big(50(n+1)\big)^3 \bar{\varpi}(\pi a_k / L).$$

Integrating against $|\hat{c}(q)|^2 dq'$ and using $\int |\hat{c}|^2 dq' = \sum_x |c_x|^2 \le a_k^3 (L + a_k)^3$ and $n + 1 \le (L + 4a_k)/(2a_k)$,

$$\int |\hat{c}|^2 d\varpi \le \frac{27 \cdot 50^3}{8} (L + a_k)^3 (L + 4a_k)^3\, \bar{\varpi}(\pi a_k / L),$$

and $\frac{27 \cdot 50^3}{8} (L + a_k)^3 (L + 4a_k)^3 \le 421875 (\frac{5}{4})^3\, 8 L^6 < 7 \cdot 10^6 L^6$ for $a_k \le L/4$.

(iii) Here $\hat{c}(p) = \sum_x a_k^3 \chi(a_k x) e^{i(p - p_0)\cdot x}$. Let $\mathbf{y}_0$ be the centre of Q' and $x_0 \in \mathbb{Z}^3$ a nearest lattice point to $\mathbf{y}_0/a_k$. For $a_k x \in Q'$, $|x_i - x_{0,i}| \le \ell/(2a_k) + \frac{1}{2} \le \frac{5}{8} \ell/a_k$, so for $p \in \mathscr{B}_{a_k/(3\ell)}(p_0)$ the phase $(p - p_0)\cdot(x - x_0)$ is at most $\frac{5}{8}$ in absolute value and $|\hat{c}(p)| \ge \cos(1) \sum_x a_k^3 \chi(a_k x) \ge \frac{1}{2} a_k^3 \#\{x : a_k x \in Q''\}$, Q'' the middle cube. It has at least $\lfloor \ell/(2a_k) \rfloor \ge \ell/(4a_k)$ lattice points in each direction, so $|\hat{c}(p)| \ge \ell^3/128$ on the box, and $\varpi(\mathscr{B}_{a_k/(3\ell)}(p_0))(\ell^3/128)^2 \le \int |\hat{c}|^2 d\varpi$.

(iv) Let $\mathbf{x}_*$ be the centre of the cube divided by a_k. For $p \in \mathscr{B}_{a_k\rho_h}(0)$ and $a_k x$ in the cube, $\theta_x = p \cdot (x - \mathbf{x}_*)$ satisfies $|\theta_x| \le 3 a_k \rho_h L_h/(2a_k)$, so $\theta_x^2/2 \le \frac{9}{8} \rho_h^2 L_h^2 = |\int h|/(64 \int |h|)$, and

$$\begin{aligned} |\hat{c}(p)| &\ge \Big| \sum_x a_k^3 h(a_k x) \cos \theta_x \Big| \\ &\ge \Big| \sum_x a_k^3 h(a_k x) \Big| - \frac{|\int h|}{64 \int |h|} \sum_x a_k^3 |h(a_k x)|. \end{aligned}$$

For large k the Riemann sums are within $\frac{1}{8}|\int h|$ of $\int h$ and below $2 \int |h|$, which gives $|\hat{c}| \ge (\frac{7}{8} - \frac{1}{32})|\int h|$.

(v) With $(\Delta c)_x = \sum_i (c_{x+e_i} + c_{x-e_i} - 2c_x)$, $\widehat{\Delta c} = -\varkappa \hat{c}$, so $\varkappa^m |\hat{c}| \le \sum_x |(\Delta^m c)_x|$. The operator Δ^m is a sum of 3^m products of m second differences $D_i\phi(x) = \phi(x+e_i) + \phi(x-e_i) - 2\phi(x) = \int_{-1}^{1} (1-|t|)\partial_t^2 \phi(x+te_i)\,dt$, so $|(\Delta^m c)_x| \le 3^m a_k^{3+2m} \|h\|_{2m}$, and $(\Delta^m c)_x = 0$ unless x is within distance m of $a_k^{-1}\operatorname{supp} h$ in each coordinate, which leaves at most $(L_h/a_k + 2m + 1)^3$ points. □

Proposition 227. *Let $f_0 = g \otimes h$ be a product packet as in Proposition 51, so that $I_k = \int |\hat{c}_h|^2 d\varpi_k$ with $(c_h)_x = a_k^3 h(a_k x)$.*

(i) ($\mathrm{M_{sp}}$) *holds at δ if and only if there are $\rho > 0$ and $C < \infty$ with*

$$\bar{\varpi}_k(a_k\rho) \le C I_k(\delta) \quad \text{for all large } k;$$

if it holds for ρ' with C, it holds for every $\rho > 0$ with $\lceil \rho/\rho' \rceil^3 C$. If ($\mathrm{M_{sp}}$) *holds with the constant C for the unit cube, the bound holds with $\rho = \frac{1}{3}$ and $16384C$; conversely the bound gives* ($\mathrm{M_{sp}}$) *with the constant $7 \cdot 10^6 L^6 \lceil \pi/(L\rho) \rceil^3 C$ for cubes of side L.*

(ii) *If the bound of* (i) *holds with ρ and C, then $K_k(0) \le C \lceil \pi/(a_k\rho) \rceil^3 I_k(\delta)$ for all large k.*

(iii) *If $\int h \neq 0$,* ($\mathrm{M_{sp}}$) *holds as soon as $\bar{\varpi}_k(a_k\rho_h) \le C\varpi_k(\mathscr{B}_{a_k\rho_h}(0))$ for all large k, with ρ_h as in Lemma 226*(iv)*.*

Proof. (i) Since $\bar{\varpi}(r') \le \lceil r'/r \rceil^3 \bar{\varpi}(r)$ and $\bar{\varpi}$ is nondecreasing, the bound for ρ' gives it for every $\rho > 0$. If ($\mathrm{M_{sp}}$) holds for the unit cube, apply Lemma 226(iii) with $\ell = 1$ to a fixed χ and every p_0: $\varpi_k(\mathscr{B}_{a_k/3}(p_0)) \le 16384 I_k^{h'}(\delta) \le 16384 C I_k(\delta)$; if only real h' are admitted, the real and imaginary parts of the probe give the same with $4 \cdot 16384C$, since $|\hat{c}|^2 \le 2(|\widehat{\mathrm{Re}\, c}|^2 + |\widehat{\mathrm{Im}\, c}|^2)$. Conversely, for h' supported in a cube of side L with $|h'| \le 1$, Lemma 226(ii) gives $I_k^{h'}(\delta) \le 7 \cdot 10^6 L^6 \bar{\varpi}_k(\pi a_k/L) \le 7 \cdot 10^6 L^6 \lceil \pi/(L\rho) \rceil^3 \bar{\varpi}_k(a_k\rho)$ once $a_k \le L/4$.

(ii) $K_k(0) = \varpi_k(\mathbf{T}^3)$, and $\mathbf{T}^3$ is covered by $\lceil \pi/(a_k\rho) \rceil^3$ boxes of half-side $a_k\rho$.

(iii) Lemma 226(iv) gives $\varpi_k(\mathscr{B}_{a_k\rho_h}(0)) \le 2I_k(\delta)/(\int h)^2$, and (i) applies with $\rho = \rho_h$. □

So ($\mathrm{M_{sp}}$) says that no box of lattice momenta of side of order a_k carries more of the time-integrated spectral weight than a constant times $I_k(\delta)$; by (ii) it contains the bound $K_k(0) \le C' a_k^{-3} I_k(\delta)$ on the time-integrated autocorrelation at one site. Part (iii) reduces it, for $\int h \neq 0$, to a comparison of the weight of every box with that of the box at zero momentum. Reflection positivity in the spatial directions gives that comparison when it holds exactly. For $r \ge 0$, $i \in \{1,2,3\}$, let ι_i^{s} and ι_i^{b} be the reflections $x_i \mapsto -x_i$ and $x_i \mapsto -1 - x_i$ of $\mathbb{Z}^4$, and consider

(SR_r) Let $i \in \{1,2,3\}$ and $\iota \in \{\iota_i^{\mathrm{s}}, \iota_i^{\mathrm{b}}\}$. For every $c : \mathbb{Z}^4 \to \mathbb{C}$ of finite support in the half-space $\{x_i \ge r\}$,

$$\sum_{x,y} \bar{c}_x c_y\, \omega_k\big(\mathring{\phi}_k(\iota x)\, \mathring{\phi}_k(y)\big) \ge 0.$$

For a field $\varphi_k(x) = \Phi_x$ without flow, with Φ a function of the links within distance r of 0 in the slice, and a state reflection positive for the spatial reflections, (SR_r) holds, since Φ is W_3-invariant; for the flowed fields no r is known.

Theorem 228. *Assume* (SR_0) *for the slice fields at the spacing a_k.*

(i) *$\varpi_k^V(\mathscr{B}_r(p_0)) \le \varpi_k^V(\mathscr{B}_r(0))$ for every $p_0 \in \mathbf{T}^3$ and $r > 0$.*

(ii) *Let $h \in C_c(\mathbb{R}^3)$ be real with $\int h \neq 0$, and ρ_h as in Lemma 226*(iv)*. If* (SR_0) *holds for all large k, then for every cube of side L, all large k and all h' supported in it with $|h'| \le 1$,*

$$I_k^{h'}(\delta) \le \frac{6 \cdot 10^7 L^6}{(\int h)^2} \left\lceil \frac{\pi}{L\rho_h} \right\rceil^3 I_k^h(\delta),$$

so ($\mathrm{M_{sp}}$) *holds for the spatial factor h.*

Proof. (i) Fix i, write $x = (x_i, x_\perp)$, and let b be a finitely supported function of $x_\perp$. Put $k_b(n) = \sum_{x_\perp, y_\perp} \bar{b}_{x_\perp} b_{y_\perp} K_k^V(ne_i + y_\perp - x_\perp)$ for $n \in \mathbb{Z}$. For α finitely supported in $\mathbb{Z}_{\ge 0}$ let $c_{(t,x)} = a_k \mathbf{1}_{0 \le t < n_\delta} \alpha_{x_i} b_{x_\perp}$, supported in $\{x_i \ge 0\}$. By translation invariance the sum of (SR_0) for ι_i^{s} is $\sum_{j,l \ge 0} \bar{\alpha}_j \alpha_l k_b(j+l)$, and for ι_i^{b} it is $\sum_{j,l \ge 0} \bar{\alpha}_j \alpha_l k_b(j+l+1)$; both are nonnegative. On finitely supported sequences put $\langle \alpha, \beta \rangle_b = \sum_{j,l} \bar{\alpha}_j \beta_l k_b(j+l)$ and $(S\alpha)_j = \alpha_{j-1}$, with $\alpha_{-1} = 0$. Then $\langle \cdot, \cdot \rangle_b$ is positive semidefinite, S is symmetric with $\langle \alpha, S\alpha \rangle_b \ge 0$, and $\|S^m \alpha\|_b^2 \le (\sum_j |\alpha_j|)^2 \sup_n |k_b(n)|$ is bounded in m. Since S is symmetric, the Cauchy–Schwarz inequality gives $\|S^j \alpha\|_b^2 = \langle \alpha, S^{2j}\alpha \rangle_b \le \|\alpha\|_b \|S^{2j}\alpha\|_b$ for $j \ge 1$, and with $j = 1, 2, 4, \dots$, by induction

$$\|S\alpha\|_b \le \|\alpha\|_b^{1-2^{-q}} \|S^{2^q}\alpha\|_b^{2^{-q}} \qquad (q \ge 1),$$

so $\|S\alpha\|_b \le \|\alpha\|_b$. Hence S passes to the completion of the quotient by the null vectors as a self-adjoint operator with $0 \le S \le 1$, and $k_b(n) = \langle e_0, S^n e_0 \rangle_b$ for $n \ge 0$. By Lemma 670 there is a finite positive measure μ_b on $[0,1]$ with $k_b(n) = \int \lambda^n d\mu_b$, real, and $k_b(-n) = \overline{k_b(n)}$ because K_k^V is positive definite. So $k_b(n) = \int \lambda^{|n|} d\mu_b(\lambda)$ for all $n \in \mathbb{Z}$.

The measure ϖ_b on the circle with Fourier coefficients $k_b(n)$ is the image of $|\hat{b}(p_\perp)|^2 d\varpi_k^V(p)$ under $p \mapsto p_i$. Since $\int e^{int} P_\lambda(t) \frac{dt}{2\pi} = \lambda^{|n|}$ for the Poisson kernel $P_\lambda(t) = (1-\lambda^2)/(1 - 2\lambda\cos t + \lambda^2)$, $0 \le \lambda < 1$, uniqueness of Fourier coefficients gives $\varpi_b = \int P_\lambda \frac{dt}{2\pi} d\mu_b(\lambda)$, with $P_1 \frac{dt}{2\pi}$ read as the point mass at 0. Each P_λ is even and nonincreasing in $|t|$ on $[-\pi, \pi]$, so its superlevel sets are arcs centred at 0, and for an arc J of length $|J| < 2\pi$ and the arc J_0 of the same length centred at 0,

$$\begin{aligned} \int_J P_\lambda\, dt &= \int_0^\infty |J \cap \{P_\lambda > s\}|\, ds \\ &\le \int_0^\infty |J_0 \cap \{P_\lambda > s\}|\, ds = \int_{J_0} P_\lambda\, dt; \end{aligned}$$

the point mass also gives J_0 at least the weight of J. So $\varpi_b(J) \le \varpi_b(J_0)$, that is $\int (\mathbf{1}_{J_0} - \mathbf{1}_J)(p_i) |\hat{b}(p_\perp)|^2 d\varpi_k^V \ge 0$. Every continuous $g \ge 0$ on the transverse torus is a uniform limit of functions $|\hat{b}|^2$, since $\sqrt{g}$ is a uniform limit of trigonometric polynomials, so the signed measure $A \mapsto \varpi_k^V(\{p_i \in J_0,\ p_\perp \in A\}) - \varpi_k^V(\{p_i \in J,\ p_\perp \in A\})$ is nonnegative. Applying this with A a product of two closed arcs, successively for $i = 1, 2, 3$, moves each coordinate of the centre of $\mathscr{B}_r(p_0)$ to 0 without decreasing the weight; for $r \ge \pi$ there is nothing to prove.

(ii) Let k be so large that $a_k \le L/4$ and Lemma 226(iv) applies. Then Lemma 226(i), (ii), (iv) and part (i) give

$$I_k^{h'}(\delta) \le \tfrac{2}{\delta} V_k^{h'} \le \tfrac{2}{\delta} 7\cdot 10^6 L^6 \Big\lceil \frac{\pi}{L\rho_h}\Big\rceil^3 \varpi_k^V\big(\mathscr{B}_{a_k\rho_h}(0)\big)$$
$$\le \tfrac{2}{\delta} 7\cdot 10^6 L^6 \Big\lceil \frac{\pi}{L\rho_h}\Big\rceil^3 \frac{2V_k^h}{(\int h)^2}$$
$$\le \frac{56\cdot 10^6 L^6}{(\int h)^2}\Big\lceil \frac{\pi}{L\rho_h}\Big\rceil^3 I_k^h(\delta),$$

using $V_k^h \le 2\delta I_k^h(\delta)$. □

Proposition 229. *Let $m>0$ and $q_k = e^{-ma_k}$, let $\varkappa$ and Θ_2 be as in Lemma 226(v), and let Δ be the spatial lattice Laplacian, $(\Delta\phi)(x) = \sum_{i=1}^3 (\phi(x+e_i)+\phi(x-e_i)-2\phi(x))$.*

(a) *Let ψ_k be the field of Proposition 207(a) and $\varphi_k = \Delta^2\psi_k$. Its law is invariant under translations and Γ and reflection positive for the time reflections through sites and bisecting links, its transfer matrix has gap ma_k, $\rho_k(u,\sigma) = e^{-m\sigma}$, ($Z_{sl}$) and ($M_Z$) hold for every product packet, and (SR_2) holds. For every real $h \in C_c^4(\mathbb{R}^3)$ and every cube Q_3, (M_{sp}) fails.*

(b) *The field $\phi_k = \psi_k + a_k^{-5/2}\xi_k$ of Proposition 213(vi) satisfies (SR_0) and, for every nonzero real $h \in C_c(\mathbb{R}^3)$, (M_{sp}); and (Z_{sl}) fails.*

Proof. (a) The covariance of φ_k is $q_k^{|x_0-y_0|}$ times the spatial kernel with Fourier transform $\varkappa(p)^4\prod_i P_{q_k}(p_i) \ge 0$, $P_q(t) = (1-q^2)/(1-2q\cos t+q^2)$, as in the proof of Proposition 207(a). That proof, with γ the centred Gaussian law with this spatial covariance, gives reflection positivity for the time reflections and a transfer matrix with spectrum $\{q_k^N : N\ge 0\}$ and 1 simple; invariance under translations and Γ holds because Δ^2 commutes with translations and W_3. The slice field is linear, so $D_k^{h'}(r) = q_k^{r/a_k} D_k^{h'}(0)$, which gives the ratio and $I_k(\delta) = T_k D_k(0) \le \delta e^{m\delta} D_k(\delta)$ with $T_k = a_k\sum_{j<n_\delta} q_k^j$; ($M_Z$) follows from Proposition 222(iii). The field $\varphi_k(y)$ is a linear combination of ψ_k at points $y+z$ with $|z|_1 \le 2$ in the slice of y, with coefficients invariant under $z_i \mapsto -z_i$, so for c supported in $\{x_i \ge 2\}$ the sum of (SR_2) is $\omega_k(\overline{F\circ\iota}\,F)$ for a linear function F of ψ_k in $\{x_i\ge 0\}$, nonnegative by the reflection positivity of ψ_k in the spatial directions.

In the notation of Lemma 226, $d\varpi_k = T_k\varkappa^4\prod_i P_{q_k}(p_i)\frac{dp}{(2\pi)^3}$. For $c_x = a_k^3 h(a_k x)$, Lemma 226(v) gives $\varkappa^2|\hat c| \le \Theta_2 a_k^4$ for $a_k \le 1$. Since $\int P_q \frac{dt}{2\pi} = 1$,

$$I_k(\delta) \le T_k\Theta_2^2 a_k^8.$$

Let Q_3 have side ℓ, χ as in Lemma 226(iii) for $Q' = Q_3$, and $h_k'(\mathbf{y}) = \chi(\mathbf{y})\cos(\pi(y_1+y_2+y_3)/a_k)$, real, supported in Q_3, with $|h_k'| \le 1$; this is the h' of the lemma with $p_0 = (\pi,\pi,\pi)$. On $\mathscr{B}_{a_k/(3\ell)}(p_0)$, of half-side at most $\frac{1}{12}$, each $\sin^2(p_i/2) \ge \cos^2\frac{1}{24}$, so $\varkappa \ge 11$, and $P_{q_k}(p_i) \ge (1-q_k)/(1+q_k) \ge ma_k/4$ for $ma_k \le 1$. The lower bound of the proof of Lemma 226(iii), $|\hat c'| \ge \ell^3/128$ on that box, gives

$$I_k^{h_k'}(\delta) \ge T_k\, 11^4 \Big(\frac{\ell^3}{128}\Big)^2\Big(\frac{ma_k}{4}\Big)^3\Big(\frac{2a_k}{3\ell}\Big)^3\frac{1}{(2\pi)^3}$$
$$\ge 10^{-5}\, T_k\,\ell^3 m^3 a_k^6.$$

So $I_k^{h_k'}(\delta)/I_k(\delta) \ge 10^{-5}\ell^3 m^3\Theta_2^{-2}a_k^{-2} \to \infty$.

(b) The covariance is $\prod_\mu q_k^{|x_\mu - y_\mu|} + a_k^{-5}\mathbf{1}_{x=y}$. For c supported in $\{x_i \ge 0\}$ the second term contributes $a_k^{-5}\sum|c_x|^2$ over $x_i = 0$ for ι_i^{s} and 0 for ι_i^{b}, and the first is reflection positive by Proposition 207(a); this is (SR_0). The noise is independent in time, so $I_k^{h'}(\delta) = I_{\psi,k}^{h'}(\delta) + a_k^2\sum_x |h'(a_k x)|^2$, where the first term, for the field ψ_k, is at most $T_k(\sum_x a_k^3|h'(a_k x)|)^2 \le \delta(L+1)^6$ for h' supported in a cube of side L with $|h'|\le 1$ and $a_k \le 1$, and the second is at most $a_k^{-1}(L+1)^3$. For $h \ne 0$, $I_k(\delta) \ge a_k^2\sum_x h(a_k x)^2 \ge (2a_k)^{-1}\int h^2$ for large k, so $I_k^{h'}(\delta) \le 2(\delta+1)(L+1)^6 I_k(\delta)/\int h^2$, which is ($M_{sp}$). By Proposition 213(vi), $\Upsilon_k \to \infty$, so (M_Z) and, by Proposition 222(iii), (Z_{sl}) fail. □

So exact spatial reflection positivity gives (M_{sp}) for every spatial factor with $\int h \ne 0$, while positivity beyond a fixed lattice distance, as for local functions of a reflection positive field, does not; and neither gives (Z_{sl}). The flow enters through the size of the lattice-scale part of ϖ_k. For a function Φ as in Sec. XI D put $\Delta\Phi = \sum_{i=1}^3(\Phi_{e_i}+\Phi_{-e_i}) - 6\Phi$. It is again a bounded real gauge-invariant W_3-invariant function of finitely many spatial links of the slice, since W_3 permutes the six translates, and since the flow commutes with translations, the field of $\Delta\Phi$ is $\Delta\varphi_k$.

Proposition 230. *Let Φ be as above, and m, h, Θ_m as in Lemma 226(v). Let $\varpi_k^{(m)}$ be the measure of Lemma 226 for $\Delta^m\Phi$.*

(i) *$\varpi_k^{(m)} = \varkappa^{2m}\varpi_k$, and for $a_k \le 1$ the integral $I_k(\delta)$ formed with $\Delta^m\Phi$ is at most $\Theta_m^2 a_k^{4m} K_k(0) \le \Theta_m^2\,\delta\,\|\Phi\|_\infty^2 a_k^{4m}$.*

(ii) *If (M_{sp}) holds for $\Delta^m\Phi$ and the spatial factor h, with the constant C for a cube of side ℓ, then for all large k*

$$\varpi_k\big(\{p : |p|_\infty \ge 2\}\big) \le 1.4\cdot 10^8\,\frac{C\Theta_m^2\delta\|\Phi\|_\infty^2}{\ell^3}\,a_k^{4m-3}.$$

This holds at every flow time τ, with constants that do not depend on τ.

Proof. (i) Summation by parts gives U_k^c for $\Delta^m\Phi$ equal to $U_k^{\Delta^m c}$ for Φ, and $\widehat{\Delta c} = -\varkappa\hat c$; so $\int|\hat c|^2 d\varpi_k^{(m)} = \int \varkappa^{2m}|\hat c|^2 d\varpi_k$ for all finitely supported c, which determines the measure. For $c_x = a_k^3 h(a_k x)$, Lemma 226(v) gives $\int\varkappa^{2m}|\hat c|^2 d\varpi_k \le \Theta_m^2 a_k^{4m}\varpi_k(\mathbf{T}^3)$, with $\varpi_k(\mathbf{T}^3) = K_k(0) \le \delta\|\Phi\|_\infty^2$ by Lemma 226(i).

(ii) Let $a_k \le \min(1,\ell/4)$ and $r = a_k/(3\ell) \le \frac{1}{12}$. Divide $\mathbf{T}^3$ into $\lceil 3\pi\ell/a_k\rceil^3 \le (4\pi\ell/a_k)^3$ half-open boxes of side at most $2r$. A box meeting $\{|p|_\infty \ge 2\}$ lies in $\mathscr{B}_r(p_0)$ for its centre p_0, which has a coordinate $|p_{0,i}| \ge 2-r \ge \frac{\pi}{2}$; on $\mathscr{B}_r(p_0)$ that coordinate lies at distance at least $\frac{\pi}{2}-\frac{1}{12}$ from 0 on the circle, and $4\sin^2(\frac{\pi}{4}-\varepsilon) = 2(\cos\varepsilon-\sin\varepsilon)^2 \ge 2(1-\varepsilon-\frac{\varepsilon^2}{2})^2 \ge 1$ for $\varepsilon = \frac{1}{24}$, so $\varkappa \ge 1$ there. By Lemma 226(iii) with Q' the cube of side ℓ, (M_{sp}) for $\Delta^m\Phi$ and (i),

$$\varpi_k(\mathscr{B}_r(p_0)) \le \varpi_k^{(m)}(\mathscr{B}_r(p_0)) \le 16384\,\ell^{-6}C\,\Theta_m^2\,\delta\,\|\Phi\|_\infty^2 a_k^{4m},$$

and summing over the boxes gives the bound, since $4\cdot 16384\cdot 64\pi^3 \le 1.4\cdot 10^8$; the factor 4 covers the case in which only

real h' are admitted, as in the proof of Proposition 227(i). The constants involve Φ only through $\|\Phi\|_\infty$, which the flow does not change. □

So a proof of (M_{sp}) for all the fields $\Delta^m\Phi$, which are admissible observables, must show that the time-integrated spectral weight of φ_k at lattice momenta tends to zero faster than every power of a_k. Such suppression is what the linear heat flow does to a Gaussian field at flow time τ/a_k^2; for the gauge field the flow leaves the central configurations unchanged, and with them their lattice-scale patterns (Proposition 224 and Proposition 231(iv) below).

Proposition 231. *Let $G=\mathrm{SU}(N)$, $\zeta=e^{2\pi i/N}$ and $c_N=N(1-\cos(2\pi/N))$. Let U° be a configuration of the spatial links of a slice with values in the centre $\{\zeta^j\}$, X a finitely supported assignment of elements of $\mathfrak{su}(N)$ to the spatial links, and $U^\varepsilon_e=U^\circ_e e^{\varepsilon X_e}$. For a plaquette p with links e_1,e_2 traversed forwards and e_3,e_4 backwards put $Y_p=X_{e_1}+X_{e_2}-X_{e_3}-X_{e_4}$, $|Y|^2=-\mathrm{Tr}\,Y^2$, and let $z_p\in\{\zeta^j\}$ be the plaquette of U°.*

(i) *The finite sum $S(\varepsilon)=\sum_p\big(\mathrm{Re\,Tr}\,U^\circ_p-\mathrm{Re\,Tr}\,U^\varepsilon_p\big)$ over the spatial plaquettes of the slice satisfies $S'(0)=0$ and $S''(0)=\sum_p\mathrm{Re}(z_p)\,|Y_p|^2$.*

(ii) *At $U^\circ=1$, $S''(0)=0$ if and only if there is a finitely supported $\Lambda:\mathbb{Z}^3\to\mathfrak{su}(N)$ with $X_{(x,y)}=\Lambda(x)-\Lambda(y)$ for every link, that is, X is an infinitesimal gauge transformation.*

(iii) *Let U^ζ have $U^\zeta_{e_*}=\zeta$ on one spatial link e_* and all other links* 1*, and let S_0'' be the form of* (ii)*. For $N\geq 5$, $S''(0)\geq\cos(2\pi/N)S_0''(0)\geq 0$, with equality to* 0 *only at infinitesimal gauge transformations; for $N=4$, $S''(0)\geq 0$; for $N=2,3$ and X supported on e_*, $S''(0)=4\cos(2\pi/N)|X_{e_*}|^2<0$.*

(iv) *For Φ the sum of $N-\mathrm{Re\,Tr}\,U_p$ over the twelve spatial plaquettes with a corner at* 0 *and every $s\in\mathbb{R}$, $\Phi_x\circ\Phi_s(U^\zeta)=c_Nn_x$, where $n_x=4$ at the two endpoints of e_*, $n_x=1$ at the eight other corners of the four plaquettes containing e_*, and $n_x=0$ elsewhere.*

Proof. (i) Central elements commute with everything, so $U^\varepsilon_p=z_pe^{\varepsilon X_{e_1}}e^{\varepsilon X_{e_2}}e^{-\varepsilon X_{e_3}}e^{-\varepsilon X_{e_4}}=z_pe^{Z_p(\varepsilon)}$ with $Z_p(\varepsilon)\in\mathfrak{su}(N)$ and, by the Baker–Campbell–Hausdorff formula, $Z_p(\varepsilon)=\varepsilon Y_p+O(\varepsilon^2)$. Since $\mathrm{Tr}\,Z_p=0$, $\mathrm{Tr}\,e^{Z_p}=N+\frac12\mathrm{Tr}\,Z_p^2+O(\varepsilon^3)=N-\frac{\varepsilon^2}{2}|Y_p|^2+O(\varepsilon^3)$, and only the finitely many plaquettes meeting the support of X contribute.

(ii) The vectors $\Lambda(x)-\Lambda(y)$ give $Y_p=0$ by telescoping. Conversely, $S''(0)=\sum_p|Y_p|^2=0$ means that X is a closed $\mathfrak{su}(N)$-valued cochain on the cubical complex of $\mathbb{Z}^3$. Choose a box B containing the endpoints of the support of X and a point $x_\infty\notin B$, and let $\Lambda(x)$ be minus the sum of $\pm X_e$ along a lattice path from x_∞ to x, the sign recording the orientation. Two paths with the same ends differ by a sum of boundaries of plaquettes, so Λ is well defined, and $X_{(x,y)}=\Lambda(x)-\Lambda(y)$. The complement of B in $\mathbb{Z}^3$ is connected and X vanishes on its links, so $\Lambda=0$ there.

(iii) The plaquettes of U^ζ are 1 except the four containing e_*, which are $\zeta^{\pm1}$, so $S''(0)=S_0''(0)-(1-\cos\frac{2\pi}{N})\sum_{p\ni e_*}|Y_p|^2$, and $\sum_{p\ni e_*}|Y_p|^2\leq S_0''(0)$. For $N\geq5$, $\cos\frac{2\pi}{N}>0$, which with (ii) gives the first claim, and $\cos\frac{\pi}{2}=0$ gives the case $N=4$. For X supported on e_*, $Y_p=\pm X_{e_*}$ for the four plaquettes containing e_* and $Y_p=0$ otherwise, and $\cos\pi=-1$, $\cos\frac{2\pi}{3}=-\frac12$.

(iv) By (i) the first variation of the action of the slice vanishes in every direction, so the right side of the flow equation vanishes at U^ζ and $\Phi_s(U^\zeta)=U^\zeta$ by uniqueness. Each of the four plaquettes containing e_* contributes $N-\mathrm{Re\,Tr}\,\zeta^{\pm1}=c_N$ to Φ_x for each of its four corners x; the two endpoints of e_* are corners of all four, and each of the other eight corners of exactly one. □

Spatial reflection positivity of a field and of its Laplacian. Theorem 228 turns (SR_0) for one field into (M_{sp}) for that field. The admissible fields are closed under $\Phi\mapsto\Delta\Phi$, and by Proposition 230 a proof of (M_{sp}) for all of them has to suppress the lattice-scale spectral weight. This paragraph shows that (SR_0) cannot hold for Φ and $\Delta\Phi$ together in a state that clusters in space, that the linear heat flow destroys (SR_0) at every positive flow time, and that (SR_0) for one field forces spectral weight at the lattice scale.

Write $\mathbf{T}=\mathbb{R}/2\pi\mathbb{Z}$, identified with $(-\pi,\pi]$. A finite positive measure ν on $\mathbf{T}$ is *of Poisson type* if there is a finite positive measure μ on $[0,1]$ with

$$\hat\nu(n)=\int e^{int}d\nu(t)=\int\lambda^{|n|}d\mu(\lambda)\qquad(n\in\mathbb{Z}),$$

with $0^0=1$. The moments determine μ, as in the proof of Lemma 670. Since $\int e^{int}P_\lambda(t)\frac{dt}{2\pi}=\lambda^{|n|}$ for $P_\lambda(t)=(1-\lambda^2)/(1+\lambda^2-2\lambda\cos t)$, $0\leq\lambda<1$, uniqueness of Fourier coefficients gives

$$\nu=\mu(\{1\})\,\delta_0+f\,\frac{dt}{2\pi},\qquad f=\int_{[0,1)}P_\lambda\,d\mu(\lambda).\tag{24}$$

For $i\in\{1,2,3\}$ write $p=(p_i,p_\perp)$ for $p\in\mathbf{T}^3$, with $p_\perp$ in the torus $\mathbf{T}^2$ of the other two coordinates, and put $\varkappa_\perp(p_\perp)=\sum_{j\neq i}4\sin^2(p_j/2)$, so that $\varkappa(p)=4\sin^2(p_i/2)+\varkappa_\perp(p_\perp)$. For a finite positive measure ϖ on $\mathbf{T}^3$ and a continuous $g\geq0$ on $\mathbf{T}^2$ let ϖ^i_g be the image of $g(p_\perp)\,d\varpi(p)$ under $p\mapsto p_i$. Call ϖ *of Poisson type in direction i* if every ϖ^i_g is of Poisson type.

Lemma 232. (i) *Let ν_r be measures of Poisson type on $\mathbf{T}$ with $\sup_r\nu_r(\mathbf{T})<\infty$, and let ν be a finite positive measure with $\hat\nu_r(n)\to\hat\nu(n)$ for every n. Then ν is of Poisson type.*

(ii) *If (SR_0) holds for the slice field at the spacing a_k, then ϖ^V_k is of Poisson type in every direction $i\in\{1,2,3\}$. The same holds for every real field on $\mathbb{Z}^4$ with translation invariant, bounded covariance that satisfies the inequalities of (SR_0), with K^V_k formed from its covariance.*

(iii) *If ν is of Poisson type, then for every integer $n\geq1$*

$$\hat\nu(0)-\hat\nu(n)\leq n\big(\hat\nu(0)-\hat\nu(1)\big),$$
$$\nu\big(\{t:2\leq|t|\leq\pi\}\big)\geq\frac{\pi-2}{2\pi}\big(\hat\nu(0)-\hat\nu(1)\big).$$

Proof. (i) Let μ_r be the measures of ν_r. Their masses $\mu_r([0,1])=\hat\nu_r(0)$ converge to $\hat\nu(0)$. If $\hat\nu(0)=0$, then $\nu=0$ and $\mu=0$ will do. Otherwise Lemma 671, applied to the probability measures $\mu_r/\hat\nu_r(0)$ for large r, gives a subsequence

along which μ_r converges weakly to a finite positive measure μ of mass $\hat\nu(0)$. Since $\lambda\mapsto\lambda^{|n|}$ is continuous on $[0,1]$, $\hat\nu(n)=\lim\int\lambda^{|n|}d\mu_r=\int\lambda^{|n|}d\mu$.

(ii) For a finitely supported b on the transverse lattice, the proof of Theorem 228(i) shows that $k_b(n)=\int e^{inp_i}|\hat b(p_\perp)|^2 d\varpi_k^V(p)$ equals $\int\lambda^{|n|}d\mu_b$ for all $n\in\mathbb{Z}$, so $\varpi_{k,|\hat b|^2}^{V,i}$ is of Poisson type. That proof uses only translation invariance, the boundedness and positive definiteness of K_k^V, and (SR_0). For a continuous $g\ge0$ on $\mathbf{T}^2$, $\sqrt g$ is a uniform limit of trigonometric polynomials $\hat b_r$, so $|\hat b_r|^2\to g$ uniformly, the Fourier coefficients converge, the masses are bounded, and (i) applies.

(iii) By the representation, $\hat\nu(0)-\hat\nu(n)=\int(1-\lambda^n)d\mu$ and $1-\lambda^n\le n(1-\lambda)$ on $[0,1]$. Further $P_\lambda(t)\ge(1-\lambda^2)/(1+\lambda)^2=(1-\lambda)/(1+\lambda)\ge(1-\lambda)/2$ for $0\le\lambda<1$, and the set $\{2\le|t|\le\pi\}$ has measure $2(\pi-2)$, so by (24) its ν-measure is at least $\frac{\pi-2}{2\pi}\int(1-\lambda)\,d\mu=\frac{\pi-2}{2\pi}(\hat\nu(0)-\hat\nu(1))$; the atom at $\lambda=1$ contributes 0 to both sides. □

The measures of $\Delta\Phi$ are those of Φ times $\varkappa^2$: the summation by parts of the proof of Proposition 230(i) gives V_k^c for $\Delta\Phi$ equal to $V_k^{\Delta c}$ for Φ, so the measure ϖ_k^V of $\Delta\Phi$ is $\varkappa^2\varpi_k^V$, and that of K_k is $\varkappa^2\varpi_k$.

Theorem 233. *Let Φ be as in Proposition 230, and assume (SR_0) at the spacing a_k for the slice fields of both Φ and $\Delta\Phi$.*

(i) *$\varpi_k^V=K_k^V(0)\,\delta_0$ and $\varpi_k=K_k(0)\,\delta_0$. So K_k^V and K_k are constant on $\mathbb{Z}^3$, and the time average $Y_k(\mathbf{x})=a_k\sum_{n=0}^{n_\delta^{-1}}\mathring\phi_k(n,\mathbf{x})$ satisfies $Y_k(\mathbf{x})=Y_k(0)$ almost surely for every $\mathbf{x}$.*

(ii) *The slice field of $\Delta\Phi$ equals its mean almost surely at every point: $\Delta\mathring\phi_k(x)=0$ for every $x\in\mathbb{Z}^4$.*

(iii) *If $\omega_k(\mathring\phi_k(0,0)\,\mathring\phi_k(j,\mathbf{z}))\to0$ as $|\mathbf{z}|\to\infty$ for every integer $j\ge0$, then $\mathring\phi_k(x)=0$ almost surely for every x, and $C_k(0)=0$ for every packet f_0.*

Proof. (i) Fix i. By Lemma 232(ii), applied to Φ with $g=1,\varkappa_\perp,\varkappa_\perp^2$ and to $\Delta\Phi$ with $g=1$, the images ν_0,ν_1,ν_2 of ϖ_k^V, $\varkappa_\perp\varpi_k^V$, $\varkappa_\perp^2\varpi_k^V$ and the image ν' of $\varkappa^2\varpi_k^V$ under $p\mapsto p_i$ are of Poisson type, with measures μ_0,μ_1,μ_2,μ' on $[0,1]$. Put $k_l=\hat\nu_l$ and $F=\hat\nu'$. With $4\sin^2(t/2)=2-e^{it}-e^{-it}$,

$$\varkappa^2=(2-e^{it}-e^{-it})^2+2(2-e^{it}-e^{-it})\varkappa_\perp+\varkappa_\perp^2,\qquad t=p_i,$$

so ν' is ν_0, ν_1, ν_2 multiplied by these trigonometric polynomials in t and added, and

$$\begin{aligned}F(n)=\;&6k_0(n)-4k_0(n-1)-4k_0(n+1)\\&+k_0(n-2)+k_0(n+2)+4k_1(n)\\&-2k_1(n-1)-2k_1(n+1)+k_2(n).\end{aligned}$$

Insert $k_l(m)=\int\lambda^{|m|}d\mu_l$. For $n\ge2$ all the exponents $|n-j|$, $|j|\le2$, equal $n-j$, and the integrands combine to

$$\lambda^{n-2}(1-\lambda)^4,\qquad-2\lambda^{n-1}(1-\lambda)^2,\qquad\lambda^n$$

against μ_0, μ_1, μ_2; these are polynomial identities, valid at $\lambda=0$ with $0^0=1$. For $n=1$ the integrand against μ_0 is $6\lambda-4-4\lambda^2+\lambda+\lambda^3$, which for $\lambda>0$ equals $\lambda^{-1}(1-\lambda)^4-\lambda^{-1}+\lambda$, and the integrand against μ_1 is $4\lambda-2-2\lambda^2=-2(1-\lambda)^2$.

For $n\ge3$ the three integrands vanish at $\lambda=0$, and $F(n)=\int\lambda^n d\mu'$. So the finite signed measure

$$\lambda^3\mu'-\lambda(1-\lambda)^4\mu_0+2\lambda^2(1-\lambda)^2\mu_1-\lambda^3\mu_2$$

on $[0,1]$ annihilates every polynomial, hence every continuous function, and it is zero. Dividing by λ and by λ^2 on $(0,1]$,

$$\begin{aligned}\lambda^2\mu'&=(1-\lambda)^4\mu_0-2\lambda(1-\lambda)^2\mu_1+\lambda^2\mu_2,\\ \lambda\mu'&=\lambda^{-1}(1-\lambda)^4\mu_0-2(1-\lambda)^2\mu_1+\lambda\mu_2\end{aligned}\tag{25}$$

as measures on $(0,1]$. For $n=2$, $F(2)=\int\lambda^2d\mu'=\int_{(0,1]}\lambda^2d\mu'$ and $F(2)=\int_{[0,1]}((1-\lambda)^4d\mu_0-2\lambda(1-\lambda)^2d\mu_1+\lambda^2d\mu_2)$; integrating the first line of (25) over $(0,1]$ and subtracting leaves $\mu_0(\{0\})=0$. Integrating the second line over $[\varepsilon,1]$,

$$\int_{[\varepsilon,1]}\frac{(1-\lambda)^4}{\lambda}\,d\mu_0\le\nu'(\mathbf{T})+2\nu_1(\mathbf{T})$$

for every $\varepsilon>0$, so $\int_{(0,1]}\lambda^{-1}(1-\lambda)^4d\mu_0<\infty$, and since $(1-\lambda)^4\ge\frac{1}{16}$ on $(0,\frac12]$, $\int_{(0,1]}\lambda^{-1}d\mu_0<\infty$. Now, with $\mu_0(\{0\})=0$,

$$\begin{aligned}F(1)&=\int_{(0,1]}\Big(\frac{(1-\lambda)^4}{\lambda}-\frac{1}{\lambda}+\lambda\Big)d\mu_0\\&\quad-2\int_{[0,1]}(1-\lambda)^2d\mu_1+\int\lambda\,d\mu_2,\\F(1)&=\int\lambda\,d\mu'=\int_{(0,1]}\frac{(1-\lambda)^4}{\lambda}\,d\mu_0\\&\quad-2\int_{(0,1]}(1-\lambda)^2d\mu_1+\int\lambda\,d\mu_2,\end{aligned}$$

the second by the second line of (25). Subtracting,

$$\int_{(0,1]}\Big(\frac{1}{\lambda}-\lambda\Big)d\mu_0+2\mu_1(\{0\})=0.$$

Both terms are nonnegative and $1/\lambda-\lambda>0$ on $(0,1)$, so μ_0 is carried by $\{1\}$, k_0 is constant, and ν_0 is a point mass at 0. Thus $\varpi_k^V(\{p:p_i\ne0\})=0$ for $i=1,2,3$, and $\varpi_k^V=\varpi_k^V(\mathbf{T}^3)\delta_0=K_k^V(0)\delta_0$. By Lemma 226(i), $\frac{\delta}{2}\int|\hat c|^2d\varpi_k\le\int|\hat c|^2d\varpi_k^V\le2\delta\int|\hat c|^2d\varpi_k$ for every finitely supported c. As in the proof of Theorem 228(i) the functions $|\hat c|^2$ approximate every continuous $g\ge0$ on $\mathbf{T}^3$ uniformly, so $\frac{2}{\delta}\varpi_k^V-\varpi_k$ and $2\delta\varpi_k-\varpi_k^V$ are positive functionals on $C(\mathbf{T}^3)$, hence positive measures:

$$\tfrac{\delta}{2}\varpi_k\le\varpi_k^V\le2\delta\varpi_k.\tag{26}$$

So $\varpi_k(\{p\ne0\})=0$. Finally Y_k is real with $\omega_k(Y_k(0)Y_k(\mathbf{z}))=K_k^V(\mathbf{z})$, so $\omega_k((Y_k(\mathbf{x})-Y_k(0))^2)=2K_k^V(0)-2K_k^V(\mathbf{x})=0$.

(ii) The measure of K_k for $\Delta\Phi$ is $\varkappa^2\varpi_k=K_k(0)\varkappa(0)^2\delta_0=0$. So, with U_k^c formed from $\Delta\Phi$, $0=I_k^c=a_k\sum_{j<n_\delta}\langle u_c,\mathbb{T}_k^ju_c\rangle$ for every c, and each term is nonnegative because $\mathbb{T}_k\ge0$; the term

$j=0$ is $a_k\,\omega_k(|U_k^c|^2)$. With $c=\delta_{\mathbf{x}}$ this gives $\Delta\mathring\phi_k(0,\mathbf{x})=0$ almost surely, and time translation invariance gives every time.

(iii) $K_k^V(\mathbf{z})$ is a finite sum of the correlations

$$a_k^2\,\omega_k\big(\mathring\phi_k(n,0)\,\mathring\phi_k(n',\mathbf{z})\big),\qquad 0\le n,n'<n_\delta,$$

and by translation invariance each of them equals

$$a_k^2\,\omega_k\big(\mathring\phi_k(0,0)\,\mathring\phi_k(j,\pm\mathbf{z})\big),\qquad j=|n'-n|.$$

Under the hypothesis $K_k^V(\mathbf{z})\to 0$, and since it is constant by (i), $K_k^V=0$, so $\varpi_k^V=0$ and $\varpi_k=0$. As in (ii), now for Φ, $\omega_k(|\mathring\phi_k(0,\mathbf{x})|^2)=0$ for every $\mathbf{x}$, hence $\mathring\phi_k(x)=0$ almost surely for every $x\in\mathbb{Z}^4$. Every A_k is then 0 almost surely, and $C_k(0)=\omega_k(\vartheta(\bar A_k)A_k)=0$. □

Corollary 234. *Suppose that $D(n_k,\alpha)$ holds at β_k, as in Theorem 45, and that $C_k(0)>0$ for the packet f_0 of Φ. Then at every spacing a_k,* (SR_0) *fails for the slice field of Φ or for that of $\Delta\Phi$. So along such a sequence no class of admissible fields closed under $\Phi\mapsto\Delta\Phi$ satisfies* (SR_0)*, and Theorem 228 applies at the spacing a_k to at most one of Φ and $\Delta\Phi$.*

Proof. By Theorem 43, part (v) of Proposition 11 holds at β_k: covariances of gauge-invariant local functions, one of them Lipschitz in each link, decay exponentially in the distance. A continuous gauge-invariant function of finitely many links is a uniform limit of Lipschitz ones, which may be averaged over the gauge transformations at the sites they involve, since these act by isometries. By Proposition 47(ii), $\varphi_k(x)$ is a uniform limit of continuous gauge-invariant local functions at the time of x, and $|\mathrm{Cov}(F,F')-\mathrm{Cov}(G,G')|\le 2\|F-G\|_\infty\|F'\|_\infty+2\|F'-G'\|_\infty\|G\|_\infty$. So $\omega_k(\mathring\phi_k(0,0)\mathring\phi_k(j,\mathbf{z}))\to 0$ as $|\mathbf{z}|\to\infty$. If (SR_0) held for both fields, Theorem 233(iii) would give $C_k(0)=0$. □

The linear heat flow, the model of the smoothing that the flow is meant to provide, removes (SR_0) at every positive flow time. For $s>0$ let $p_s(\mathbf{z})$ be the probability that the continuous-time random walk on $\mathbb{Z}^3$ with generator Δ is at $\mathbf{z}$ at time s, starting from 0, so that $\sum_{\mathbf{z}}p_s(\mathbf{z})=1$ and $\sum_{\mathbf{z}}p_s(\mathbf{z})e^{ip\cdot\mathbf{z}}=e^{-s\varkappa(p)}$, and let $e^{s\Delta}$ act on the slices of a field on $\mathbb{Z}^4$ by $(e^{s\Delta}\psi)(t,\mathbf{x})=\sum_{\mathbf{y}}p_s(\mathbf{x}-\mathbf{y})\psi(t,\mathbf{y})$, a series that converges in L^2 for fields with bounded second moments.

Proposition 235. (i) *Let ϖ be a finite positive measure on $\mathbf{T}^3$, $s>0$ and $i\in\{1,2,3\}$. If ϖ and $e^{-2s\varkappa}\varpi$ are both of Poisson type in direction i, then $\varpi(\{p:p_i\ne 0\})=0$.*

(ii) *Let ψ_k be the field of Proposition 207*(a) *and $s>0$. Then* (SR_0) *holds for ψ_k and fails for $e^{s\Delta}\psi_k$, at every k.*

Proof. (i) Let $g\ge 0$ be continuous on $\mathbf{T}^2$ and $g'=ge^{-2s\varkappa_\perp}$. Since $e^{-2s\varkappa}=e^{-4s(1-\cos p_i)}e^{-2s\varkappa_\perp}$, the measure $\nu=\varpi^i_{g'}$ and the image $e^{-4s(1-\cos t)}\nu$ of $g\,e^{-2s\varkappa}\varpi$ are of Poisson type, with measures μ and μ^s. Write ν as in (24) and put

$$F(c)=\int_{[0,1)}\frac{1-\lambda^2}{1+\lambda^2-2\lambda c}\,d\mu(\lambda)$$

for c in $\Omega=\mathbb{C}\setminus[1,\infty)$, and F^s likewise with μ^s, so that $f(t)=F(\cos t)$ for $t\ne 0$. The denominator vanishes only at $c=(1+\lambda^2)/(2\lambda)\ge 1$. On a compact $K\subset\Omega$ with $|c|\le R$ and distance $d>0$ to $[1,\infty)$, it is at least $\frac12$ for $\lambda\le 1/(4R)$ and at least $2\lambda d\ge d/(2R)$ for larger λ, so the integrand is bounded on K uniformly in λ, and F is analytic on Ω. For $\mathrm{Im}\,c>0$,

$$\mathrm{Im}\,F(c)=\int_{[0,1)}\frac{2\lambda(1-\lambda^2)\,\mathrm{Im}\,c}{|1+\lambda^2-2\lambda c|^2}\,d\mu(\lambda)\ge 0,$$

and the same holds for F^s. The absolutely continuous parts of $e^{-4s(1-\cos t)}\nu$ in its two representations agree, so $F^s(\cos t)=e^{-4s(1-\cos t)}F(\cos t)$ for almost every t, hence for all $c\in(-1,1)$ by continuity, and on Ω, which is connected, by analyticity: $F^s(c)=e^{-4s(1-c)}F(c)$.

If $\mu((0,1))>0$, then $\mathrm{Im}\,F>0$ on the upper half plane, and $\theta(y)=\arg F(iy)\in(0,\pi)$ is continuous in $y>0$. Since $e^{-4s(1-iy)}=e^{-4s}e^{4isy}$,

$$\mathrm{Im}\,F^s(iy)=e^{-4s}|F(iy)|\sin\big(4sy+\theta(y)\big)\ge 0$$

and $F^s(iy)\ne 0$, so the continuous function $4sy+\theta(y)$ takes its values in the union of the disjoint closed intervals $[2\pi j,2\pi j+\pi]$, $j\in\mathbb{Z}$; on the connected set $(0,\infty)$ it stays in one of them, which is impossible since it exceeds $4sy$. So $\mu((0,1))=0$ and F is the constant $B=\mu(\{0\})\ge 0$. Then $\mathrm{Im}\,F^s(iy)=Be^{-4s}\sin(4sy)$, which is $-Be^{-4s}$ at $y=3\pi/(8s)$, so $B=0$. Hence $\nu=\mu(\{1\})\delta_0$. For $g=1$ this says $\int_{\{p_i\ne 0\}}e^{-2s\varkappa_\perp}d\varpi=0$, and $e^{-2s\varkappa_\perp}>0$.

(ii) The covariance of ψ_k is $q_k^{|x_0-y_0|}\prod_{i=1}^3 q_k^{|x_i-y_i|}$, so its measure ϖ_k^V is $T_k\prod_i P_{q_k}(p_i)\frac{dp}{(2\pi)^3}$ with $T_k=a_k^2\sum_{n,n'<n_\delta}q_k^{|n-n'|}>0$, and that of $e^{s\Delta}\psi_k$, whose spatial covariance kernel has Fourier transform $e^{-2s\varkappa}\prod_i P_{q_k}(p_i)$, is $e^{-2s\varkappa}\varpi_k^V$. Both covariances are bounded and translation invariant. The law of ψ_k is reflection positive for the spatial reflections through sites and bisecting links (Proposition 207(a)); for c supported in $\{x_i\ge 0\}$ the sum of (SR_0) is $\mathbb{E}[\overline{\psi_k(\imath c)}\,\psi_k(c)]\ge 0$ with $\psi_k(c)=\sum c_x\psi_k(x)$ and $\psi_k(\imath c)=\sum c_x\psi_k(\imath x)$, which is ($\mathrm{SR}_0$). If ($\mathrm{SR}_0$) held for $e^{s\Delta}\psi_k$, Lemma 232(ii) for both fields and (i) would give $\varpi_k^V(\{p_i\ne 0\})=0$, but ϖ_k^V has a positive density. □

The continuum analogue is the spatially flowed free field of Proposition 207(c). It is reflection positive in time, and the proof of Proposition 217(i), with x_1 in the role of time, shows that it is not reflection positive for the reflection of x_1: it uses only $H_0(0)>0$ and $H_0''(0)<0$, and here $H_0''(0)=-(2\pi)^{-1}\int p_1^2e^{-2t_fp_1^2}(p_1^2+\mu^2)^{-1}dp_1$. For one field, ($\mathrm{SR}_0$) forces spectral weight at the lattice scale: by part (i) below with $n=L/a_k$, $\varpi_k(\{|p|_\infty\ge 2\})$ is at least $a_k/(12\delta L)$ times $K_k^V(0)-K_k^V(ne_i)$, the decorrelation of the time-averaged field over the physical distance L.

Proposition 236. *Assume* (SR_0) *at the spacing a_k for the slice field of Φ, and let $i\in\{1,2,3\}$ and $n\ge 1$.*

(i)

$$\begin{aligned}K_k^V(0)-K_k^V(ne_i)&\le n\big(K_k^V(0)-K_k^V(e_i)\big)\\&\le\frac{2\pi n}{\pi-2}\,\varpi_k^V\big(\{|p_i|\ge 2\}\big)\\&\le 12\,\delta\,n\,\varpi_k\big(\{|p|_\infty\ge 2\}\big).\end{aligned}$$

(ii) *For h' supported in a cube of side L' with $|h'| \le 1$, $a_k \le 1$ and $h'_n(\mathbf{y}) = h'(\mathbf{y} - n a_k e_i)$,*

$$I_k^{h'-h'_n}(\delta) \le \frac{4}{\delta}(L'+1)^6 \big(K_k^V(0) - K_k^V(ne_i)\big).$$

(iii) *If in addition* ($\mathrm{M_{sp}}$) *holds for the slice field of $\Delta^m\Phi$, $m \ge 1$, and the spatial factor h, with the constant C for a cube of side ℓ, then for all large k*

$$K_k^V(0) - K_k^V(ne_i) \le 1.7 \cdot 10^9 \, \frac{C\Theta_m^2 \delta^2 \|\Phi\|_\infty^2}{\ell^3} \, n a_k^{4m-3}.$$

Proof. (i) By Lemma 232(ii) with $g = 1$, $\nu = \varpi_{k,1}^{V,i}$ is of Poisson type, and $\hat\nu(n) = K_k^V(ne_i)$. Lemma 232 (iii) gives the first two inequalities. The third follows from (26), which uses only Lemma 226(i), and from $4\pi/(\pi-2) < 12$, since $\pi - 2 > 1.14$ and $4\pi < 12.57 < 13.68$.

(ii) With $c_{\mathbf{x}} = a_k^3 h'(a_k \mathbf{x})$ the coefficients of $h' - h'_n$ are $c - c(\cdot - ne_i)$, with Fourier transform $\hat c\,(1 - e^{inp_i})$. So, by (26) and $|1 - e^{inp_i}|^2 = 2(1 - \cos np_i)$,

$$\begin{aligned} I_k^{h'-h'_n}(\delta) &\le \|\hat c\|_\infty^2 \, \frac{2}{\delta} \int 2(1-\cos np_i)\, d\varpi_k^V \\ &= \frac{4}{\delta} \|\hat c\|_\infty^2 \big(K_k^V(0) - K_k^V(ne_i)\big), \end{aligned}$$

because K_k^V is real and even. Finally $\|\hat c\|_\infty \le \sum_{\mathbf{x}} a_k^3 |h'(a_k\mathbf{x})| \le a_k^3 (L'/a_k + 1)^3 \le (L'+1)^3$.

(iii) Proposition 230(ii) bounds $\varpi_k(\{|p|_\infty \ge 2\})$, and $12 \cdot 1.4 \cdot 10^8 \le 1.7 \cdot 10^9$. □

So ($\mathrm{SR_0}$) for Φ together with ($\mathrm{M_{sp}}$) for $\Delta^2\Phi$ gives, by (ii) and (iii) with $n = L/a_k$, $I_k^{h'-h'_n}(\delta) \le C' L a_k^4$ for spatial translates by a physical distance L. Hence $I_k^{h'-h'_n}(\delta)/I_k(\delta) \to 0$ for every bounded L unless $\liminf_k a_k^{-4} I_k(\delta) < \infty$: the time-averaged slice field becomes invariant, relative to $I_k(\delta)$, under spatial translations by bounded physical distances. No lower bound on $I_k(\delta)$ along a tuning is proved here.

Remark 237. For the flowed fields of the Wilson measure along a tuning, the two statements stand as follows.

($\mathrm{M_{sp}}$) is equivalent to the box bound of Proposition 227(i) and, for $\int h \ne 0$, follows from the comparison of Proposition 227(iii). Exact spatial reflection positivity ($\mathrm{SR_0}$) gives that comparison (Theorem 228). The Wilson action is reflection positive in every coordinate hyperplane [97], so for states whose restrictions to gauge-invariant functions are reflection positive for the spatial reflections, a field Φ_x without flow, depending on the links within distance r of x, satisfies (SR_r); the flowed field $\Phi_x \circ \Phi_{s_k}$ depends on all links of its slice, and Proposition 47(i) bounds the influence of links at distance R only for R beyond a multiple of $s_k = \tau/a_k^2$, so no (SR_r) is known for it. Positivity beyond a fixed distance does not suffice (Proposition 229(a)). Nor can ($\mathrm{SR_0}$) hold for a field and its Laplacian together: in a state that clusters in space, in particular under the hypothesis of Theorem 45, it forces $C_k(0) = 0$ (Theorem 233, Corollary 234), and the linear heat flow destroys it at every positive flow time (Proposition 235). So Theorem 228 cannot give ($\mathrm{M_{sp}}$) for a class of fields closed under $\Phi \mapsto \Delta\Phi$, and for one field ($\mathrm{SR_0}$) forces spectral weight at the lattice scale (Proposition 236). Compactness of G gives $\bar\varpi_k \le \delta \|\Phi\|_\infty^2$, an upper bound of order one, and no lower bound on $I_k(\delta)$; by Proposition 230 a proof valid for the admissible fields $\Delta^m\Phi$ must suppress the lattice-scale spectral weight of φ_k faster than every power of a_k. For $G = \mathrm{SU}(2)$ the estimates of Sec. XII E give each plaquette a conditional probability at least $v_* e^{-3\beta_W}$ of being bad (Proposition 386), a power of a_k along couplings of order $\log(1/a_k)$, and the single-link defects of Proposition 231 are fixed by the flow at every flow time, with nonnegative second variation for $N \ge 4$ and a direction of negative second variation for $N = 2, 3$. Whether the flow over the time τ/a_k^2 moves the configurations near such defects to configurations with lattice-scale weight below every power of a_k is not known; this is the missing estimate.

($\mathrm{Z_{sl}}$) is equivalent to ($\mathrm{M_Z}$) and implies (N_u) (Proposition 222); for $G = \mathrm{SU}(2)$ it fails along every sequence with $\beta_{W,k} \le 0.348$ (Proposition 213(v)) and along the constant tunings (Proposition 192(ii)). Compactness bounds $D_k(0)$ above and gives no lower bound on $D_k(\delta)$, which is what ($\mathrm{Z_{sl}}$) requires; exact spatial reflection positivity together with ($\mathrm{M_{sp}}$) does not give it (Proposition 229(b)), and there the term that breaks it is constant in space, so that no spatial smoothing removes it. The missing estimate is that of Remark 225: a bound on the high-energy spectral weight of the slice field by its weight at bounded physical energy.

At a fixed k both inequalities hold with a constant depending on k whenever $C_k(0) > 0$, since then $D_k(\delta) > 0$ and $I_k(\delta) > 0$ by Proposition 51(i); their content is the uniformity in k, and neither is proved along any sequence of couplings with $a_k \to 0$.

Status of (M), (W) and (U). None of the three statements is proved. Proposition 213(i)–(iii) uses translation and ϑ-invariance and reflection positivity of the states and the reality of Φ, part (iv) also the transfer matrix, and part (v) Theorem 191; none of them uses the hypothesis of Theorem 45 or (N_u). Proposition 214 is deterministic except for the positivity of the conditional densities in part (iii). Proposition 216 uses (M) and the invariance of the states under translations and Γ, through Theorem 201(i), (ii), and part (iv) the tree bounds. Proposition 217 concerns the free field and limits of families. Proposition 219(i) assumes (RC), whose existence along a tuning is not proved and which contains ($\mathrm{N_0}$), and the hypothesis of Theorem 45 for the bound on m_*; parts (ii) and (iii) assume (M). Lemma 221 and Propositions 222 and 223(i)–(iii) use translation and ϑ-invariance, reflection positivity and the positivity of the transfer matrix, and no hypothesis on the tuning; so do Lemma 226 and Propositions 227 and 230, while Theorem 228 assumes the exact spatial reflection positivity ($\mathrm{SR_0}$) of the slice field, which is not known for the flowed fields and, under the hypothesis of Theorem 45, fails for Φ or for $\Delta\Phi$ (Corollary 234). Proposition 231 is deterministic. What remains open is ($\mathrm{M_{loc}}$), whose time direction for $n = 2$ is Proposition 223(i) and whose remainder is the spatial comparison ($\mathrm{M_{sp}}$) and the bounds of Proposition 223(iii) for even $n \ge 4$; ($\mathrm{M_Z}$), which for product packets is the short-time bound ($\mathrm{Z_{sl}}$) on the slice field and is strictly stronger than ($\mathrm{N_0}$); the Ward identity $\Delta_k^X \to 0$ for one $X \in \mathfrak{so}(3)_{\mathrm{sp}}$; and, given (RC),

(U1) and (U2) (Remark 225).

E. Tuning by the ratio

This subsection uses the ratio of Sec. XI B as the renormalisation condition: the couplings β_k are chosen so that $\rho_k(u,\sigma) = r$ for a fixed $r \in (0,1)$. It determines how the ratio depends on the coupling, when the value r is attained, what is known about the location of the couplings and their relation to the one-loop tuning of Proposition 212, and what the choice gives for (U). The tuning exists if and only if (N_u) holds along some sequence of couplings. Along it, (N_u) and the convergence of the ratio required in Proposition 206(iii) hold by construction. Whether $\beta_k \to \infty$, and whether (D) holds along it, remain open, and the joint input of (D) and (N_u) is the same as before.

Admissible states. Throughout, $\beta_W \geq 0$ denotes the Wilson coupling, $G = \mathrm{SU}(N)$, and the weight of a configuration is $\exp(\beta_W \sum_p N^{-1} \operatorname{Re}\operatorname{Tr} Q_p)$. Let $\mathscr{A}$ be the closure in the supremum norm of the continuous gauge-invariant local functions of the links of $\mathbb{Z}^4$. It is a commutative C^*-algebra with unit, and it contains the flowed fields $\Phi_x \circ \Phi_s$ and the packets A_k by Proposition 47(ii). For a finite set Λ of links, let $\gamma_\Lambda^{\beta_W}$ resample the links of Λ from the law whose density with respect to Haar measure on G^Λ is proportional to $\exp(\beta_W \sum_{p \cap \Lambda \neq \emptyset} N^{-1} \operatorname{Re}\operatorname{Tr} Q_p)$, given the other links. The weight is gauge invariant and Haar measure is invariant under translations of the group, so $\gamma_\Lambda^{\beta_W}$ commutes with gauge transformations; it preserves locality and continuity, and maps $\mathscr{A}$ into itself. Let P_Λ be the number of plaquettes that meet Λ.

Definition 238. For $\beta_W \geq 0$, $\mathscr{S}(\beta_W)$ is the set of states ω on $\mathscr{A}$ that are invariant under translations and the hypercubic group, reflection positive for reflections in hyperplanes through sites and in hyperplanes bisecting links, and satisfy $\omega(\gamma_\Lambda^{\beta_W} H) = \omega(H)$ for every finite set Λ of links and every $H \in \mathscr{A}$.

Every $\omega \in \mathscr{S}(\beta_W)$ is a state as in Appendix F. For the data Φ, τ, f_0 of Sec. XI B we write $C_k^\omega(t)$ and $\rho_k^\omega(u,\sigma)$ for the quantities $C_k(t)$ and $\rho_k(u,\sigma)$ computed in ω. If $\tilde{A}_k$ is the packet before centring, then $A_k = \tilde{A}_k - \omega(\tilde{A}_k)$, and invariance of ω under ϑ and τ gives

$$C_k^\omega(t) = \omega\big(\vartheta(\bar{\tilde{A}}_k)\, \tau_{t/a_k} \tilde{A}_k\big) - |\omega(\tilde{A}_k)|^2. \tag{27}$$

Lemma 239. (i) *For every finite set Λ of links, $H \in \mathscr{A}$ and $\beta_W, \beta_W' \geq 0$,*

$$\|\gamma_\Lambda^{\beta_W} H - \gamma_\Lambda^{\beta_W'} H\|_\infty \leq \big(e^{2P_\Lambda |\beta_W - \beta_W'|} - 1\big) \|H\|_\infty.$$

(ii) *$\mathscr{S}(\beta_W)$ is convex and weak-$*$ compact, and the set of pairs (β_W, ω) with $\omega \in \mathscr{S}(\beta_W)$ is closed.*

(iii) *$\mathscr{S}(\beta_W)$ is not empty: it contains every limit point, as $L \to \infty$ through even values, of the Wilson measures on the tori $(\mathbb{Z}/L\mathbb{Z})^4$, and the states of Propositions 11 and 17, of Theorems 1, 3 and 5, and of Theorem 43, whenever their hypotheses hold at β_W.*

(iv) *$\mathscr{S}(0)$ consists of the restriction of the product of Haar measures, and there $C_k(t) = 0$ for all k and t.*

(v) *For a compact interval $I \subset [0,\infty)$ the set Γ_I of pairs (β_W, ω) with $\beta_W \in I$ and $\omega \in \mathscr{S}(\beta_W)$ is compact and connected.*

Proof. (i) Write the density of $\gamma_\Lambda^{\beta_W}$ as $e^{\beta_W V}/Z$, where V is the sum of $N^{-1} \operatorname{Re}\operatorname{Tr} Q_p$ over the plaquettes that meet Λ, so that $|V| \leq P_\Lambda$. Changing β_W by δ multiplies the numerator and the normalisation by factors in $[e^{-P_\Lambda|\delta|}, e^{P_\Lambda|\delta|}]$, so the ratio q of the two densities lies in $[e^{-2P_\Lambda|\delta|}, e^{2P_\Lambda|\delta|}]$ and $|q-1| \leq e^{2P_\Lambda|\delta|} - 1$. Integrate $|H||q-1|$ against the second law.

(ii) Invariance, the inequalities $\omega(\vartheta(\bar{H})H) \geq 0$ and, at fixed β_W, the identities $\omega(\gamma_\Lambda^{\beta_W} H) = \omega(H)$ are preserved by convex combinations and weak-$*$ limits, and the states of $\mathscr{A}$ form a weak-$*$ compact set. If $\beta_n \to \beta_W$, $\omega_n \in \mathscr{S}(\beta_n)$ and $\omega_n \to \omega$, then by (i)

$$|\omega(\gamma_\Lambda^{\beta_W} H) - \omega(H)| = \lim_n |\omega_n(\gamma_\Lambda^{\beta_W} H - \gamma_\Lambda^{\beta_n} H)| = 0.$$

(iii) For even L the Wilson measure μ_L on $(\mathbb{Z}/L\mathbb{Z})^4$ is invariant under the translations and the hypercubic group of the torus and reflection positive for both kinds of planes [97, 105]. A gauge-invariant local function of $\mathbb{Z}^4$ is a gauge-invariant function on the torus once L exceeds the diameter of its support by two, and for such H and Λ with L large, $\mu_L(\gamma_\Lambda^{\beta_W} H) = \mu_L(H)$ because the plaquettes that meet Λ are plaquettes of the torus. Since $\mathscr{A}$ is separable, a diagonal sequence L_n makes $\mu_{L_n}(H)$ converge for all local H; the limit is positive, normalised and bounded by $\|H\|_\infty$, extends to a state on $\mathscr{A}$, and inherits the three properties. The free-boundary states are limits of $W_{\Lambda'}$ along boxes, $W_{\Lambda'}(\gamma_\Lambda^{\beta_W} H) = W_{\Lambda'}(H)$ once every plaquette that meets Λ lies in $P(\Lambda')$, and invariance and reflection positivity are part of each of the cited statements for $\beta_W \geq 0$.

(iv) At $\beta_W = 0$ the law of γ_Λ^0 is Haar measure on G^Λ, so $\omega(H) = \omega(\gamma_\Lambda^0 H)$ is the Haar integral of H for every H depending on the links of Λ. The packet A_k is a uniform limit of functions of the spatial links at times $x_0 \geq 1$, and $\vartheta(\bar{A}_k)$ of those at times $x_0 \leq -1$ (Proposition 47(i)); under the product measure the two are independent, so $C_k(t) = \omega(\vartheta(\bar{A}_k))\, \omega(\tau_{t/a_k} A_k) = 0$.

(v) Γ_I is closed in the product of I with the compact set of states, by (ii). Suppose $\Gamma_I = U \cup V$ with U, V disjoint, nonempty and closed. Each fibre $\{\beta_W\} \times \mathscr{S}(\beta_W)$ is convex, hence connected, and lies in U or in V. The projections of U and V to I are then disjoint, compact, nonempty by (iii), and cover I, which contradicts the connectedness of I. □

Lemma 240. *Let $\beta = \beta_W/N^2 \geq 0$ be such that γ^F has exactly one Gibbs measure μ_F, and let $\omega_\beta(O) = \mu_F(O^F)$ for gauge-invariant local O. Then every state ω on $\mathscr{A}$ with $\omega(\gamma_\Lambda^{\beta_W} H) = \omega(H)$ for all finite Λ and $H \in \mathscr{A}$ equals ω_β. In particular $\mathscr{S}(\beta_W)$ has no element other than ω_β, and if $\beta_n \to \beta_W$ and $\omega_n \in \mathscr{S}(\beta_n)$, then $\omega_n \to \omega_\beta$ weak-$*$. This applies under the hypotheses of Propositions 11 and 17 and of Theorem 43, where $\mathscr{S}(\beta_W) = \{\omega_\beta\}$ by Lemma 239(iii), in particular for $G = \mathrm{SU}(2)$ and $0 \leq \beta_W \leq 0.348$.*

Proof. Let E be the set of links and $\mathscr{G} = G^{\mathbb{Z}^4}$ the compact gauge group with its Haar measure dg. For $H \in C(G^E)$ put $(\mathscr{P}H)(Q) = \int H(Q^g)\,dg$. If H is local, $\mathscr{P}H$ is local and gauge invariant, and $\mathscr{P}$ is a contraction, so $\mathscr{P}$ maps $C(G^E)$ onto $\mathscr{A}$ and is the identity on $\mathscr{A}$. Since $\gamma_\Lambda^{\beta_W}$ commutes with gauge transformations, $\mathscr{P}\gamma_\Lambda^{\beta_W} = \gamma_\Lambda^{\beta_W}\mathscr{P}$. Hence $\tilde\omega = \omega\circ\mathscr{P}$ is a probability measure on G^E with $\tilde\omega\gamma_\Lambda^{\beta_W} = \tilde\omega$ for every finite Λ, as measures.

By Lemma 9(i), the line field F is a disjoint union of straight bi-infinite lines. On a line in the direction μ put $g = 1$ at its site with $x_\mu = 0$, and extend by $g(y+e_\mu) = g(y)Q_{(y,\mu)}$ in both directions. This defines a continuous map $Q_F \mapsto g$, and the gauge transform $T(Q) = (Q_e^g)_{e\in A}$, $Q_e^g = g_x Q_e g_y^{-1}$ for $e = (x,y)$, is a continuous map $G^E \to G^A$; Q^g equals 1 on F. Let $V \subset A$ be finite and h bounded and measurable on G^A. Given the links outside V, which fix g and $T(Q)_{A\setminus V}$, the map $Q_V \mapsto T(Q)_V$ is a product of left and right translations and preserves Haar measure, and $\operatorname{Re}\operatorname{Tr} Q_p = \operatorname{Re}\operatorname{Tr} Q_p^g$ for every plaquette. With $\beta_W N^{-1} = N\beta$ this gives $\gamma_V^{\beta_W}(h\circ T) = (\gamma_V^F h)\circ T$, where γ_V^F is the kernel of the specification γ^F on V. The law $\nu = \tilde\omega\circ T^{-1}$ therefore satisfies

$$\nu(\gamma_V^F h) = \tilde\omega(\gamma_V^{\beta_W}(h\circ T)) = \tilde\omega(h\circ T) = \nu(h),$$

so ν is a Gibbs measure of γ^F and $\nu = \mu_F$. For gauge-invariant local O, $O(Q) = O(Q^g) = O^F(T(Q))$, because O depends on finitely many links. Hence $\omega(O) = \tilde\omega(O^F\circ T) = \mu_F(O^F) = \omega_\beta(O)$, and two states that agree on a dense set are equal.

For the last statement, every limit point of (ω_n) lies in $\mathscr{S}(\beta_W)$ by Lemma 239(ii). The three cited statements give (i) and (ii) of Proposition 11, and Theorems 1 and 3 are proved through Propositions 11 and 17. □

For SU(2) and $\beta_W \le 0.348$ the upper bound of Theorem 191 therefore holds for every admissible state. For the other groups in the range of Theorem 5 the bound holds for the state of that theorem; uniqueness of the Gibbs measure of γ^F is not established there, so it is not known for all of $\mathscr{S}(\beta_W)$.

Proposition 241. *Fix Φ, τ, f_0, dyadic $u \ge 0$ and $\sigma > 0$, and k with $u/a_k, \sigma/a_k \in \mathbb{Z}$.*

(i) *For each t, $\omega \mapsto C_k^\omega(t)$ is weak-$*$ continuous on the states of $\mathscr{A}$. For $\omega \in \mathscr{S}(\beta_W)$, $0 \le C_k^\omega(t') \le C_k^\omega(t)$ for $t' \ge t$, and $C_k^\omega(t) > 0$ for all t if and only if $C_k^\omega(0) > 0$.*

(ii) *Let I be a compact interval with*

(P$_k$) *$C_k^\omega(0) > 0$ for every $(\beta_W,\omega) \in \Gamma_I$.*

Then $(\beta_W,\omega) \mapsto \rho_k^\omega(u,\sigma)$ is continuous on Γ_I with values in $(0,1]$; its set of values on Γ_I, and on each fibre $\mathscr{S}(\beta_W)$, is a compact interval.

(iii) *If $\mathscr{S}(\beta_*) = \{\omega_*\}$ and $C_k^{\omega_*}(0) > 0$, then for every $\varepsilon > 0$ there is $\delta > 0$ such that $C_k^\omega(0) > 0$ and $|\rho_k^\omega(u,\sigma) - \rho_k^{\omega_*}(u,\sigma)| < \varepsilon$ for all $\omega \in \mathscr{S}(\beta_W)$ with $|\beta_W - \beta_*| < \delta$.*

Proof. (i) In (27), $\tilde A_k$ and $\vartheta(\tilde{\bar A}_k)\tau_{t/a_k}\tilde A_k$ are finite sums of elements of $\mathscr{A}$, which gives continuity. By Appendix F, $C_k^\omega(t) = \langle\Psi_{A_k}, \mathbb{T}^{t/a_k}\Psi_{A_k}\rangle$ with $0 \le \mathbb{T} \le 1$, which is nonnegative and nonincreasing in t; the last statement is Proposition 190(i).

(ii) By (i) the ratio is a quotient of continuous functions with a positive denominator, with values in $(0,1]$ by Proposition 190(ii). The image of the compact connected set Γ_I (Lemma 239(v)) and of each convex fibre is compact and connected.

(iii) Otherwise there are $\beta_n \to \beta_*$ and $\omega_n \in \mathscr{S}(\beta_n)$ violating one of the two properties. A subsequence converges, its limit lies in $\mathscr{S}(\beta_*)$ by Lemma 239(ii) and equals ω_*, and by (i) $C_k^{\omega_n}(t) \to C_k^{\omega_*}(t)$ for $t = 0, 2u, 2u+\sigma$, a contradiction. □

So the ratio depends continuously on the coupling wherever the admissible state is unique, and in general its values at one coupling form a compact interval. The hypothesis (P$_k$) is qualitative, a nonvanishing at each fixed k, and it is open in infinite volume: Theorem 48 assumes it, and in a periodic box it is Proposition 50. It fails at $\beta_W = 0$ by Lemma 239(iv). In the box the ratio is real-analytic.

Proposition 242. *In the box $\mathbb{Z}\times(\mathbb{Z}/L_k\mathbb{Z})^3$ of Proposition 50, with its $f_0 = g\otimes h$ and Φ, let $\rho_k(\beta_W)$ be the ratio $\rho_k(u,\sigma)$ of the infinite-time Wilson state of the box at $\beta_W > 0$.*

(i) *ρ_k is real-analytic on $(0,\infty)$, takes values in $(0,1)$, and $\rho_k(\beta_W) \to 0$ as $\beta_W \to 0$.*

(ii) *For $r \in (0,1)$ the set $\{\beta_W > 0 : \rho_k(\beta_W) = r\}$ has no accumulation point in $(0,\infty)$, and it is nonempty if and only if $\rho_k(\beta_W) \ge r$ for some $\beta_W > 0$.*

Proof. (i) In the notation of the proof of Proposition 50, the transfer matrix is $T_{\beta_W} = MKM$ on the gauge-invariant part $\mathscr{K}$ of $L^2(G^E)$, E the finite set of spatial links of a slice. The function $e^{\beta_W N^{-1}\operatorname{Re}\chi}$ is an entire function of β_W with values in $C(G)$, and convolution with a continuous function has operator norm at most its supremum, so K is entire in β_W in operator norm; so is M, multiplication by the exponential of β_W times a bounded function. At $\beta_W = 0$, K is the projection onto the constants and $M = 1$, so T_0 is the projection onto the constant function 1. For $\beta_W \ge 0$ the largest eigenvalue λ_0 of the compact positive operator T_{β_W} is simple and isolated, by the proof of Proposition 50 for $\beta_W > 0$ and because T_0 is a projection of rank one for $\beta_W = 0$. For β_W near a point $\beta_* \ge 0$ the Riesz projection $P_0 = (2\pi i)^{-1}\oint(z - T_{\beta_W})^{-1}dz$, over a fixed circle around $\lambda_0(\beta_*)$ that separates it from the rest of the spectrum, is analytic in β_W, has rank one because $\|P_0 - P_0(\beta_*)\| < 1$, and projects onto the eigenspace of $\lambda_0(\beta_W) = \operatorname{Tr}(T_{\beta_W}P_0)$; by Weyl's inequality the other eigenvalues stay outside the circle, so this is the largest eigenvalue. For real β_W, P_0 is the orthogonal projection onto $\psi_0 > 0$, so $\langle 1, P_0 1\rangle = \langle 1,\psi_0\rangle^2 > 0$ and $\psi_0 = P_0 1/\langle 1, P_0 1\rangle^{1/2}$ is real-analytic. The flowed fields do not depend on β_W. Hence, with $A = T_{\beta_W}/\lambda_0$, $F = \sum_x h(a_k x)\Phi_x\circ\Phi_{s_k}$ and $F^c = F - \langle\psi_0, F\psi_0\rangle$,

$$C_k(t) = a_k^8\langle p(A)F^c\psi_0, A^{t/a_k}p(A)F^c\psi_0\rangle$$

is real-analytic in $\beta_W \ge 0$, and positive for $\beta_W > 0$ by Proposition 50, so ρ_k is real-analytic on $(0,\infty)$. The vector $\Psi_u = A^{u/a_k}p(A)F^c\psi_0$ is nonzero and orthogonal to ψ_0, so with λ_1 the largest eigenvalue of T_{β_W} on $\psi_0^\perp$,

$$\rho_k(\beta_W) = \frac{\langle\Psi_u, A^{\sigma/a_k}\Psi_u\rangle}{\|\Psi_u\|^2} \le \Big(\frac{\lambda_1}{\lambda_0}\Big)^{\sigma/a_k} < 1.$$

By the minimax principle with the vector 1,

$$\lambda_1 \le \max_{w\perp 1,\ \|w\|=1} \langle w, T_{\beta_W} w\rangle \le \|T_{\beta_W} - T_0\| \to 0,$$

while $\lambda_0 \ge \langle 1, T_{\beta_W} 1\rangle \to 1$ as $\beta_W \to 0$.

(ii) The function $\rho_k - r$ is real-analytic on the connected set $(0,\infty)$ and not identically zero, since it tends to $-r$ at 0, so its zeros do not accumulate in $(0,\infty)$. If $\rho_k(\beta'_W) \ge r$, then $\rho_k < r$ near 0 and the intermediate value theorem gives a zero in $(0,\beta'_W]$. □

At $\beta_W = 0$ in the box, $\Psi_u \to 0$: both $C_k(2u)$ and $C_k(2u+\sigma)$ vanish there, as in Lemma 239(iv). For the packet of Proposition 50, whether $\rho_k(\beta_W) \ge r$ for some β_W, at fixed k and L_k, is not known; Proposition 249 below shows that concentration on flat configurations does not decide it. For packets odd under the centre of SU(2) in a box of fixed size it holds for every $r \in (0,1)$ (Corollary 253).

Definition 243. Fix Φ, τ, f_0, dyadic $u \ge 0$ and $\sigma > 0$, and $r \in (0,1)$. A *ratio tuning at level r* is a sequence (β_k, ω_k) with $\omega_k \in \mathscr{S}(\beta_k)$, $C_k^{\omega_k}(0) > 0$ and $\rho_k^{\omega_k}(u,\sigma) = r$ for all $k \ge k_0$. In the box of Proposition 242, ω_k is the state of the box.

Theorem 244. *Let $G = \mathrm{SU}(2)$, $r \in (0,1)$ and $\beta_- \in (0, 0.348]$, and assume* (P_k) *on every compact interval in $[\beta_-,\infty)$, for all large k.*

(i) *A ratio tuning at level r with $\beta_k \ge \beta_-$ exists if and only if, for every large k, some $\beta_W \ge \beta_-$ and $\omega \in \mathscr{S}(\beta_W)$ have $\rho_k^{\omega}(u,\sigma) \ge r$. If $\liminf_k \rho_k^{\omega_k}(u,\sigma) = c > 0$ for some $\omega_k \in \mathscr{S}(\beta_k)$ with $\beta_k \ge \beta_-$, ratio tunings exist at every level $r \in (0,c)$; conversely, a ratio tuning at level r satisfies* (N_u). *In the box the same holds without* (P_k) *and without the condition $\beta_k \ge \beta_-$.*

(ii) *Every ratio tuning at level r in infinite volume has $\beta_k > 0.348$ for all large k, and along every ratio tuning, in infinite volume or in the box:*

(a) *$\rho_k(u',\sigma) \ge r$ for $u' \ge u$; for dyadic σ' with $\sigma'/a_k \in \mathbb{Z}$, $r^{\sigma'/\sigma} \le \rho_k(u,\sigma') \le r$ if $\sigma' \ge \sigma$, and $r \le \rho_k(u,\sigma') \le r^{\sigma'/\sigma}$ if $\sigma' \le \sigma$;*

(b) *every limit of Theorem 48 for $f_{0,u}$ is nontrivial and has $\rho(\sigma) = r$;*

(c) *the spectral measures of Proposition 190*(iv) *satisfy, with $\Lambda = \sigma^{-1}\log(2/r)$ and every $\lambda \ge 0$,*

$$\mu_{k,u}([0,\Lambda]) \ge \tfrac{r}{2}, \qquad \mu_{k,u}([0,\lambda]) \le r e^{\sigma\lambda};$$

(d) *the norm e^{-m_k} of $\mathbb{T}_k$ on $\Omega_k^\perp$ satisfies $m_k \le a_k \sigma^{-1}\log(1/r)$.*

(iii) *Let $u = 0$, let Φ be real and W_3-invariant as in Sec. XI D, and suppose that $D(n_k,\alpha)$ holds at β_k with $n_k a_k \le \ell$ and that* (M) *holds. Then ω_k is the state of Theorem 43, $\alpha \ge r^{2\ell/\sigma}$ and $m_* \le \sigma^{-1}\log(1/r)$. If every subsequential limit of Theorem 201*(i) *lies in a set $\mathfrak{F}$ as in Proposition 206*(iii) *with $\sigma_0 = \sigma$, then* (U) *holds and the moments $\omega_k(\prod_j Z_k A_k(f_j))$ converge for all n and $f_j \in C_c(\mathbb{R}^4)$.*

Proof. (i) Let k be so large that $e^{-m_0\sigma/a_k} < r$, with m_0 of Theorem 191, and let $\rho_k^{\omega'}(u,\sigma) \ge r$ with $\omega' \in \mathscr{S}(\beta'_W)$, $\beta'_W \ge \beta_-$. By Lemma 240, $\mathscr{S}(\beta_-)$ consists of the state of Theorem 3, where $\rho_k(u,\sigma) \le e^{-m_0\sigma/a_k} < r$ by Theorem 191. By Proposition 241(ii) on $I = [\beta_-,\beta'_W]$ the values of the ratio on Γ_I form an interval that contains r. The other implication is clear. If the $\liminf$ is c and $r < c$, then $\rho_k^{\omega_k} \ge r$ for large k. In the box, Proposition 242(ii) replaces this argument.

(ii) For large k every admissible state at $\beta_W \le 0.348$ has $\rho_k(u,\sigma) \le e^{-m_0\sigma/a_k} < r$, by Lemma 240 and Theorem 191. Part (a) is Proposition 190(ii), the power bound applied once to the pair (σ,σ') and once to (σ',σ), and (b) is Proposition 190(iii). In (c) the first bound is Proposition 190(iv) with $c = r$, and the second follows from $r = \int e^{-\sigma E} d\mu_{k,u} \ge e^{-\sigma\lambda}\mu_{k,u}([0,\lambda])$. For (d), $\Psi_{k,u}$ is a unit vector in $\Omega_k^\perp$, so $r = \langle \Psi_{k,u}, \mathbb{T}_k^{\sigma/a_k}\Psi_{k,u}\rangle \le e^{-m_k\sigma/a_k}$.

(iii) Lemma 240 under the hypothesis of Theorem 43 identifies ω_k, and Proposition 192(i) with $c = r$ gives the bounds on α and m_*. The sequence $\rho_k(0,\sigma) = r$ converges, so Proposition 206(iii) gives (U), and Proposition 206(i) the convergence of the moments. □

The same holds for $\mathrm{SU}(N)$ with β_- in the range of Theorem 5 and $m_0 = m_G(\beta_-/N^2)$, provided the state of that theorem is the only admissible state at β_-; in (ii) the bound $\beta_k > 0.348$ is then replaced by the exclusion of the couplings at which the admissible state is that of Theorem 5. Part (d) is the precise sense in which the tuning has to leave every region of couplings with a lattice gap bounded below: the lattice gap at β_k is at most $a_k\sigma^{-1}\log(1/r)$.

Circularity with (D). The ratio is defined for every admissible state without (D), and (D) is a property of the specification at one coupling, so the definition of the ratio tuning does not use (D), and (D) can be posed along it, as in Theorem 244(iii). The next statement shows that this does not reduce the joint input.

Proposition 245. *Fix k and a compact interval $I = [\beta_-, B]$ such that, at every $\beta_W \in I$, $D(n,\alpha)$ holds with $na_k \le \ell$ (n may depend on β_W) and $C_k(0) > 0$ for the state of Theorem 43. Then $\mathscr{S}(\beta_W)$ is that state for $\beta_W \in I$, the function $\beta_W \mapsto \rho_k(u,\sigma)$ is continuous on I and bounded by $\alpha^{\sigma/(2\ell)}$, and for $r > \rho_k(u,\sigma)|_{\beta_-}$ the value r is attained on I if and only if $\max_I \rho_k(u,\sigma) \ge r$. Consequently, given such intervals I_k for all large k, with $\rho_k(u,\sigma) < r$ at β_- (for* $\mathrm{SU}(2)$ *this holds for large k when $\beta_- \le 0.348$), a ratio tuning at level r with $\beta_k \in I_k$ exists if and only if* (N_u) *holds with $\rho_k(u,\sigma) \ge r$ along some sequence $\beta_k \in I_k$, and then $r \le \alpha^{\sigma/(2\ell)}$.*

Proof. Lemma 240 under Theorem 43 gives uniqueness, Proposition 241(iii) continuity, Proposition 192(i) the bound, and the intermediate value theorem the rest. □

So the ratio tuning moves no difficulty: (D) at the couplings where the ratio equals r is the same requirement as (D) and (N_u) at common couplings, with the compatibility $\alpha \ge r^{2\ell/\sigma}$ of Proposition 192. What it adds is that the normalisation in Proposition 206(iii) is built into the choice of the couplings,

and that the couplings are defined by a quantity of the theory, not by perturbation theory.

Location of the couplings. For $B \geq \beta_-$ let $R_k(B)$ be the supremum of $\rho_k^\omega(u,\sigma)$ over $\omega \in \mathscr{S}(\beta_W)$ and $\beta_W \in [\beta_-, B]$. The rescaling below uses $f_0^{(2)} = f_0(\cdot/2)$, which is supported in $\{t > 0\}$, and the coefficient $b = \frac{11N^2}{12\pi^2}$ of Proposition 212, with $h = b\log 2$.

Proposition 246. *Assume the hypotheses of Theorem 244.*

(i) *If for every B, $R_k(B) < r$ for all large k, then every ratio tuning at level r has $\beta_k \to \infty$. If $R_k(B) \geq r$ for infinitely many k and a ratio tuning at level r exists, then one exists with $\beta_k \leq B$ for infinitely many k. If for every B there is $m(B) > 0$ such that the transfer matrix of every $\omega \in \mathscr{S}(\beta_W)$, $\beta_W \leq B$, has norm at most $e^{-m(B)}$ on $\Omega^\perp$, then $R_k(B) \leq e^{-m(B)\sigma/a_k}$ and every ratio tuning has $\beta_k \to \infty$.*

(ii) *For every state ω as in Appendix F and every k,*

$$\rho_{k+1}^\omega(u,\sigma;\tau,f_0) = \rho_k^\omega(2u,2\sigma;4\tau,f_0^{(2)}). \tag{28}$$

So β_{k+1} of a ratio tuning for (u,σ,τ,f_0) solves at level k the equation for $(2u,2\sigma,4\tau,f_0^{(2)})$, and with $\Delta_k = \beta_{k+1} - \beta_k$ and $D_k = \beta_k - b\log(1/a_k)$,

$$D_K - D_{k_0} = \sum_{j=k_0}^{K-1} (\Delta_j - h).$$

(iii) *If $\beta_j > 0$ and $\Delta_j \geq h + c/\beta_{j+1}$ for $j \geq k_0$, with $c > 0$, then D_K is nondecreasing for $K \geq k_0$ and $D_K \to \infty$. If $\sum_j |\Delta_j - h| < \infty$, then D_K converges.*

(iv) *Suppose that $\rho_k^\omega(u,\sigma) \leq \rho_k^{\omega'}(u,\sigma)$ whenever $\omega \in \mathscr{S}(\beta_W)$, $\omega' \in \mathscr{S}(\beta_W')$, $\beta_W < \beta_W'$ and k is large, and that for every $r' \in (0,1)$ there is a ratio tuning at level r' with $\Delta_j \geq h + c(r')/\beta_{j+1}$, $c(r') > 0$. Then (N_u) fails along every sequence of admissible states at couplings $\beta_k \leq b\log(1/a_k) + M$.*

Proof. (i) In the first case, for every B the couplings β_k exceed B for large k. In the second, the intermediate value theorem of Proposition 241(ii) on $[\beta_-, B]$ gives solutions $\beta_k \leq B$ at those k, which replace the given terms. In the third, $\rho_k^\omega(u,\sigma) = \langle \Psi_{k,u}, \mathbb{T}^{\sigma/a_k}\Psi_{k,u}\rangle \leq e^{-m(B)\sigma/a_k}$, and the first case applies.

(ii) Since $a_k = 2a_{k+1}$, $\tau/a_{k+1}^2 = 4\tau/a_k^2$ and $a_k^4 f_0^{(2)}(a_k x) = 2^4 a_{k+1}^4 f_0(a_{k+1}x)$, the packet of $(4\tau, f_0^{(2)})$ at level k is 2^4 times that of (τ, f_0) at level $k+1$, and a time t at level $k+1$ is $t/a_{k+1} = 2t/a_k$ lattice steps. Hence $C_k^\omega(2t;4\tau,f_0^{(2)}) = 2^8 C_{k+1}^\omega(t;\tau,f_0)$, which gives (28). The identity for D_K follows from $b\log(1/a_{j+1}) - b\log(1/a_j) = h$.

(iii) The terms of the sum are positive. If $D_K \leq M'$ for all K, then $0 < \beta_{j+1} \leq b\log(1/a_{j+1}) + M'$, which grows linearly in j, and $\sum_j c/\beta_{j+1} = \infty$, a contradiction; a nondecreasing sequence that is not bounded above tends to ∞. The second statement is clear.

(iv) If (N_u) holds along admissible states at couplings β_k, choose r' with $\rho_k^{\omega_k}(u,\sigma) \geq 2r'$ for large k, and let (β_k', ω_k') be a ratio tuning at level r' as assumed. If $\beta_k < \beta_k'$, monotonicity would give $2r' \leq r'$; hence $\beta_k \geq \beta_k'$ and $\beta_k - b\log(1/a_k) \geq D_k' \to \infty$ by (iii). □

By (28), Δ_k is the change of coupling that compensates, at a fixed lattice level, a doubling of all physical scales of the renormalisation condition: a step-scaling function of the ratio. Neither the monotonicity in (iv) nor any bound on Δ_k is proved here.

Remark 247. Perturbation theory to two loops [34, 63, 75, 99] gives, with $b_0 = \frac{11N}{48\pi^2}$, $b_1 = \frac{34N^2}{3(16\pi^2)^2}$ and $L = \log\frac{1}{a\Lambda}$,

$$g(a)^{-2} = 2b_0 L + \frac{b_1}{b_0}\log L + O(1),$$

so the coupling of fixed physics is $\beta_W = bL + \gamma\log L + O(1)$ with $b = 4Nb_0$ and $\gamma = \frac{b_1}{2b_0^2}\,b = \frac{51}{121}\,b$, the ratio $\frac{51}{121}$ being the same for every N. Along $a_k = 2^{-k}a_0$ its increments are $\Delta_j = h + \gamma h/\beta_{j+1} + O(\log\beta_j/\beta_j^2)$, which is the hypothesis of Proposition 246(iii) for every $c < \gamma h$. For SU(2), $b = 11/(3\pi^2)$ and $h = b\log 2$, and $9.8696 < \pi^2 < 9.8697$, $0.69314 < \log 2 < 0.69315$ give $b \in [0.37150, 0.37152]$, $h \in [0.25750, 0.25752]$, $\gamma \in [0.15658, 0.15659]$ and $\gamma h \in [0.04032, 0.04033]$. By the same expansion, along a tuning with $\beta_k = b\log(1/a_k) + M$ the invariant scale is $\Lambda \propto (\log(1/a_k))^{\gamma/b}$, so masses grow like $(\log(1/a_k))^{51/121}$ and the limit is trivial. This is the perturbative prediction, not a result: it says that the tuning to fixed physics differs from the one-loop tuning of Proposition 212 by $\gamma\log\log(1/a_k)$, as the footnote in Sec. IX records, and that a tuning with bounded M is, by this prediction, a tuning to a trivial limit.

The estimates of Proposition 212 do not use the one-loop form of the tuning, only the size of its increments.

Corollary 248. *Let (β_k) be couplings with $\beta_{k+1} - \beta_k \in [h_-, h_+]$ for $k \geq k_0$, where $0 < h_- \leq h_+$, let $\sigma = a_i$ with $i \geq k_0$ and $\beta_i > 0$, and for $k \geq i$ put $J_k = k - i$ and $\beta_k^{(j)} = \beta_{k-j}$, the coupling at the scale $2^j a_k$. Let $c_{k,j} \in [0, \frac{1}{2}]$.*

(i) *If $c_{k,j} \geq c/\beta_k^{(j)}$ for $0 \leq j \leq J_k$, with $c > 0$, then*

$$\prod_{j=0}^{J_k}(1 - c_{k,j}) \leq \Big(\frac{\beta_i}{\beta_i + h_+(J_k+1)}\Big)^{c/h_+} \to 0.$$

(ii) *If $c_{k,j} \leq c/(\beta_k^{(j)})^2$ for $0 \leq j \leq J_k$, then for every $k \geq i$*

$$\prod_{j=0}^{J_k}(1 - c_{k,j}) \geq \exp\Big(-2c\Big(\frac{1}{\beta_i^2} + \frac{1}{h_-\beta_i}\Big)\Big).$$

Proof. With $l = J_k - j$, $\beta_i + h_- l \leq \beta_k^{(j)} \leq \beta_i + h_+ l$. The proofs of Proposition 212(ii) and (iii) apply with β_i in place of $\beta_\sigma^\pm$ and $h_\pm$ in place of h. □

The two-loop tuning of Remark 247 has increments in $[h, h + \gamma h/\beta_{k_0}]$, so the conclusion drawn from Proposition 212, that a scale-by-scale comparison has to be exact to first order in the running coupling, holds for it as well, and for a ratio tuning whose increments are bounded above and below.

Large couplings. Concentration on flat configurations does not determine the ratio. Let δ_1 be the evaluation at the configuration with every link equal to 1, a character of $\mathscr{A}$.

Proposition 249. *Let ω_n be states on $\mathscr{A}$, invariant under translations and the hypercubic group, with $\varepsilon_{\omega_n} \to 0$. Then $\omega_n \to \delta_1$ weak-$*$, and $C_k^{\omega_n}(t) \to 0$ for every k and t. For* SU(2) *this applies to limit points of the torus measures at couplings $\beta_n \to \infty$, which lie in $\mathscr{S}(\beta_n)$.*

Proof. Let O be a continuous gauge-invariant function of the links of a box B, and $\eta > 0$. There is $\delta > 0$ such that $|O(U) - O(1)| < \eta$ whenever $s_p(U) < \delta$ for all $p \in P(B)$. Otherwise, by compactness, some U with $U_p = 1$ for all $p \in P(B)$ would have $|O(U) - O(1)| \geq \eta$; but a flat configuration of a box is a gauge transform of the trivial one (in the axial gauge, which sets the links of a maximal tree to 1, the plaquettes determine the remaining links inductively over the coordinates), and O is gauge invariant. Hence, pointwise,

$$|O - O(1)| \leq \eta + \frac{2\|O\|_\infty}{\delta} \sum_{p \in P(B)} s_p,$$

and $|\omega_n(O) - O(1)| \leq \eta + 2\|O\|_\infty |P(B)| \varepsilon_{\omega_n}/\delta$. This gives $\omega_n(O) \to O(1)$ for local O, and for all of $\mathscr{A}$ by density. In (27), δ_1 is multiplicative and fixed by ϑ and τ, so $C_k^{\omega_n}(t) \to |\tilde{A}_k(1)|^2 - |\tilde{A}_k(1)|^2 = 0$. For SU(2), Proposition 198 gives $\varepsilon \leq (K_1 + K_3/\beta_n)/\beta_n$ for the limit points, and Lemma 239(iii) places them in $\mathscr{S}(\beta_n)$. □

At fixed k both $C_k(2u)$ and $C_k(2u+\sigma)$ therefore tend to zero as $\beta_W \to \infty$ along these states, and the ratio is a quotient of two vanishing quantities, on which concentration gives no information. In particular (P_k) holds at no uniform level: $\inf C_k(0)$ over these states is zero.

Centre-odd packets in a fixed box. In the box, the question left open after Proposition 242 has an answer for packets that change sign under the centre of SU(2). For these packets the ratio tends to 1 as $\beta_W \to \infty$ at every fixed k. Ratio tunings exist at every level, and (N_u) holds along every sequence with $a_k^2 \beta_k / \log(1/a_k) \to \infty$. The limits are degenerate, and the mechanism has no counterpart in infinite volume.

Let $G = \mathrm{SU}(2)$, fix $L \geq 3$, and take the box $\mathbb{Z} \times (\mathbb{Z}/L\mathbb{Z})^3$ of Proposition 50 with $L_k = L$ for all k. Let E be the set of $3L^3$ spatial links of a slice and dU the Haar probability measure on G^E. For $V \in G$ put $s(V) = 1 - \frac{1}{2}\operatorname{Tr} V$, so that $|V - I|^2 = 4s(V)$. Write $s_p = s(U_p)$ and

$$e_s(U) = \sum_p s_p(U), \qquad e_t(U,U') = \sum_{e \in E} s(U_e U_e'^{-1}),$$

where the first sum runs over the $3L^3$ plaquettes of the slice. With $b = \beta_W$, the transfer matrix $\mathbb{T}_b = MKM$ of the proof of Proposition 50 is the integral operator with the continuous positive kernel

$$t_b(U,U') = \exp\Big(b\big(6L^3 - \tfrac{1}{2}e_s(U) - e_t(U,U') - \tfrac{1}{2}e_s(U')\big)\Big).$$

Here M contributes half of the weight $b(1 - s_p)$ of every plaquette of the two slices, and K contributes the weight $b(1 - s(U_e U_e'^{-1}))$ of the time-like plaquette above e in the temporal gauge. The kernel, the largest eigenvalue $\lambda_0 = \lambda_0(b)$, the eigenvector $\psi_0 > 0$ and $A = \mathbb{T}_b/\lambda_0$ do not depend on k. Put $\omega(F) = \langle \psi_0, F\psi_0 \rangle$ for functions F of the slice and

$$\varepsilon_s(b) = \omega(s_p),$$
$$\varepsilon_t(b) = \frac{1}{\lambda_0} \iint s(U_e U_e'^{-1})\, \psi_0(U) t_b(U,U') \psi_0(U')\, dU\, dU'.$$

These are the energies of a spatial and of a time-like plaquette in the state of the box. The function $\psi_0(U) t_b(U,U') \psi_0(U')/\lambda_0$ is a probability density on $G^E \times G^E$.

The following maps act on G^E: the translations of the slice, the permutations of the three axes, and, for $\mu = 1,2,3$, the map ζ_μ that multiplies by $-I$ the links (x,μ) with $x_\mu = L-1$. Each of them preserves the Haar measure, commutes with the gauge transformations, and leaves t_b invariant. For ζ_μ this holds because a plaquette contains either none or two of the links it changes, $-I$ is central, and $U_e U_e'^{-1}$ is unchanged. Hence ψ_0 composed with any of these maps is again a positive normalised eigenvector for λ_0, and so it equals ψ_0. The translations and permutations act transitively on the plaquettes and on the links, and $s(V^{-1}) = s(gVg^{-1}) = s(V)$, so ε_s and ε_t do not depend on p and e.

Lemma 250. *For $b \geq 1$,*

$$\varepsilon_s(b) + \varepsilon_t(b) \leq \varepsilon(b) := \frac{3(\log b + 5)}{2b}.$$

The function ε decreases on $[1,\infty)$, and for $0 < \varepsilon' \leq 3$ the inequality $\varepsilon(b) \leq \varepsilon'$ holds for every $b \geq B(\varepsilon')$, where $B(\varepsilon') = (3/\varepsilon')(\log(3/\varepsilon') + 4)$.

Proof. The eigenvalue. The operator $\mathbb{T}_b$ on $L^2(G^E)$ is positive and compact, and its kernel is continuous and positive. By Jentzsch's theorem its largest eigenvalue is simple, with a positive eigenfunction. The gauge transformations commute with $\mathbb{T}_b$ and preserve positivity, so that eigenfunction is gauge invariant. Hence it is ψ_0, and λ_0 is also the largest eigenvalue on $L^2(G^E)$. In particular $\int t_b(U,U')\psi_0(U')\,dU' = \lambda_0 \psi_0(U)$ for every U, and $\lambda_0 = \sup_f \langle f, \mathbb{T}_b f\rangle / \|f\|^2$ over $f \in L^2(G^E)$.

Convexity. Since $\langle 1, \psi_0 \rangle > 0$, we have $\lambda_0^n \langle 1, \psi_0 \rangle^2 \leq \langle 1, \mathbb{T}_b^n 1 \rangle \leq \lambda_0^n$. Hence $\log \lambda_0 = \lim_n n^{-1} \log \langle 1, \mathbb{T}_b^n 1 \rangle$. Here $\langle 1, \mathbb{T}_b^n 1 \rangle$ is the integral over $(G^E)^{n+1}$ of $\prod_{i<n} t_b(U_i, U_{i+1})$, which is the exponential of b times a function of the configuration. By Hölder's inequality its logarithm is convex in $b \in \mathbb{R}$. So $\log \lambda_0$ is convex on $[0,\infty)$.

Derivative. The kernel is smooth in b, uniformly on the compact set $G^E \times G^E$, so $\mathbb{T}_b$ is differentiable in the Hilbert–Schmidt norm. Its derivative has the kernel $(6L^3 - \frac{1}{2}e_s(U) - e_t(U,U') - \frac{1}{2}e_s(U'))t_b(U,U')$. By the proof of Proposition 242, the eigenvector ψ_0 is real-analytic in b. Differentiating $\lambda_0 = \langle \psi_0, \mathbb{T}_b \psi_0 \rangle$ and using $\|\psi_0\| = 1$ gives $\lambda_0' = \langle \psi_0, \mathbb{T}_b' \psi_0 \rangle$. The eigenvector equation then gives

$$(\log \lambda_0)' = 6L^3 - \omega(e_s) - \frac{1}{\lambda_0} \iint e_t\, \psi_0 t_b \psi_0$$
$$= 6L^3 - 3L^3(\varepsilon_s + \varepsilon_t).$$

Concavity. The function $F(b) = 6L^3 b - \log \lambda_0(b)$ is concave, and $F(0) = 0$, since $t_0 = 1$ makes $\mathbb{T}_0$ the projection onto the

constants. It also satisfies $F' = 3L^3(\varepsilon_s+\varepsilon_t)$. Concavity gives $F(0) \le F(b) - bF'(b)$, that is,

$$\varepsilon_s+\varepsilon_t \le \frac{F(b)}{3L^3 b}.$$

Upper bound on F. Let $\theta(V)\in[0,\pi]$ be the angle of $V\in \mathrm{SU}(2)$. It is a bi-invariant distance from I, so $\theta(VW)\le\theta(V)+\theta(W)$, and $s(V) = 1-\cos\theta(V) \le \frac12\theta(V)^2$. Let $0<r\le\pi/2$ and $B_r = \{U : \theta(U_e)\le r\ \forall e\in E\}$. On $B_r\times B_r$ every spatial plaquette has $\theta(U_p)\le 4r$ and every $U_eU_e'^{-1}$ has angle at most $2r$, so $\frac12 e_s(U)+e_t(U,U')+\frac12 e_s(U') \le 3L^3\cdot 8r^2 + 3L^3\cdot 2r^2 = 30L^3r^2$. The Haar measure of $\{\theta\le r\}$ is $\frac{2}{\pi}\int_0^r \sin^2\theta\,d\theta \ge 8r^3/(3\pi^3)$, by $\sin\theta\ge 2\theta/\pi$. Testing with the indicator f of B_r, whose squared norm is $|B_r|$, gives

$$\lambda_0 \ge \frac{\langle f, \mathbb{T}_b f\rangle}{|B_r|} \ge |B_r|\, e^{b(6L^3-30L^3r^2)}, \qquad |B_r| \ge \Big(\frac{8r^3}{3\pi^3}\Big)^{3L^3}.$$

With $r^2 = 3/(20b) \le \frac{3}{20}$ this gives

$$\begin{aligned} F(b) &\le 30L^3 b r^2 + 3L^3 \log\frac{3\pi^3}{8r^3} \\ &= \frac{9}{2}L^3\Big(\log b + 1 + \log\frac{20}{3} + \frac23\log\frac{3\pi^3}{8}\Big). \end{aligned}$$

The constant in parentheses beside $\log b$ is less than 5. Indeed $e^2 > 2.718^2 > 7.38$, so $\log\frac{20}{3} < \log 7 < 2$. Also $\pi < 3.15$ gives $\pi^3 < 31.3 < 32$ and $\frac{3\pi^3}{8} < 12$, while $e^5 > 2.718^5 > 148 > 144 = 12^2$ gives $\frac23\log 12 < \frac53$. So the constant is less than $1+2+\frac53 < 5$, and $\varepsilon_s+\varepsilon_t \le \frac{3}{2b}(\log b + 5)$.

The coupling. The derivative of $(\log b+5)/b$ is $(-4-\log b)/b^2 < 0$. Let $x = 3/\varepsilon' \ge 1$ and $b \ge x(\log x+4) \ge 1$. Since $\log y \le y-1$,

$$\begin{aligned} \log b + 5 &\le \log x + \log(\log x+4) + 5 \\ &\le 2\log x + 8 = \frac{2}{x}x(\log x+4). \end{aligned}$$

So $(\log b+5)/b \le 2/x$ at $b = x(\log x+4)$, hence for all larger b, and then $\varepsilon(b)\le 3/x = \varepsilon'$. □

For a site x of the slice and $\mu\in\{1,2,3\}$, let $W_\mu(x) = U_{(x,\mu)}U_{(x+\hat\mu,\mu)}\cdots U_{(x+(L-1)\hat\mu,\mu)}$ be the holonomy of the straight loop of L links through x in direction μ. Put

$$\Phi = \mathrm{Tr}\,W_1(0) + \mathrm{Tr}\big(W_1(0)W_2(0)\big), \tag{29}$$

a real, continuous, gauge-invariant function of $2L$ spatial links of the slice. The second term is the trace of the holonomy of the closed path that goes once around direction 1 and then once around direction 2. The loop $W_1(x)$ contains exactly one link changed by ζ_1 and none changed by ζ_2, and $W_2(x)$ contains one link changed by ζ_2. So $\Phi_x\circ\zeta_1 = -\Phi_x$ for every translate Φ_x, while ζ_2 changes the sign of the second term only. The packets of Φ are therefore odd under the centre transformation ζ_1. In infinite volume no nonzero element of $\mathscr{A}$ is odd under the corresponding map (Remark 255).

Lemma 251. (a) *For $V,W\in\mathrm{SU}(2)$ put $x=\mathrm{Tr}V$, $y=\mathrm{Tr}W$, $z=\mathrm{Tr}VW$ and $q = 2-\mathrm{Tr}(VWV^{-1}W^{-1})\in[0,4]$. Then*

$$x^2+y^2+z^2-xyz = 4-q, \qquad x^2+y^2+z^2 \ge 3-\tfrac34 q.$$

The identity is the trace identity of Fricke for SL(2)*, which expresses the trace of a commutator through* TrV, TrW *and* TrVW.

(b) *For unitary $g_1,\dots,g_n$, $|g_1\cdots g_n - I| \le \sum_i |g_i - I|$. For $U_i,V_i\in\mathrm{SU}(2)$, $|\mathrm{Tr}(U_1\cdots U_m) - \mathrm{Tr}(V_1\cdots V_m)| \le \sqrt2\sum_i|U_i - V_i|$.*

(c) *Let γ be a closed path of links of the slice, γ' its translate by $\pm\hat\mu$, and h_γ, $h_{\gamma'}$ their holonomies. Each link of γ not parallel to μ spans, with its translate, a plaquette. Let $R(\gamma,\mu)$ be the list of these plaquettes, one entry for each such link. Then*

$$|\mathrm{Tr}\,h_\gamma - \mathrm{Tr}\,h_{\gamma'}| \le \sqrt2 \sum_{p\in R(\gamma,\mu)} |U_p - I|.$$

(d) *Let P_{12} be the set of L^2 plaquettes of the plane $x_3=0$ in the directions $1,2$. Then*

$$|W_2(0)W_1(0)W_2(0)^{-1}W_1(0)^{-1} - I| \le \sum_{p\in P_{12}} |U_p - I|.$$

Proof. (a) By the Cayley–Hamilton theorem, $V^{-1} = x - V$, $W^{-1} = y - W$, $W^2 = yW - I$, $V^2 = xV - I$ and $(VW)^2 = zVW - I$. So $\mathrm{Tr}(VW^2) = yz - x$, $\mathrm{Tr}(VWV) = \mathrm{Tr}(V^2W) = xz - y$ and $\mathrm{Tr}((VW)^2) = z^2 - 2$. Hence

$$\begin{aligned} \mathrm{Tr}(VWV^{-1}W^{-1}) &= \mathrm{Tr}\big(VW(x-V)(y-W)\big) \\ &= xyz - x(yz-x) - y(xz-y) + z^2 - 2, \end{aligned}$$

which is the identity. Put $S = x^2+y^2+z^2$. If $S\ge 3$, the inequality holds because $q\ge 0$. If $S<3$, then $|xyz| \le (S/3)^{3/2} \le S/3$ by the inequality of the arithmetic and geometric means for x^2,y^2,z^2. So $4-q \le S + S/3$, and $S \ge 3 - \frac34 q$.

(b) Write $g_1\cdots g_n - I = \sum_i g_1\cdots g_{i-1}(g_i - I)$, and use $|gM| = |M|$ for unitary g. Similarly, $U_1\cdots U_m - V_1\cdots V_m = \sum_i U_1\cdots U_{i-1}(U_i - V_i)V_{i+1}\cdots V_m$. Finally, $|\mathrm{Tr}\,M| \le \sqrt2|M|$ for 2×2 matrices, by the Cauchy–Schwarz inequality.

(c) Let γ visit $x_0, x_1, \dots, x_m = x_0$, and let $U_{x_i\to x_{i+1}}$ be the link variable of its i-th step, inverted when the step runs against the orientation of the link. Let $x_i' = x_i \pm \hat\mu$, let c_i be the link variable of the step from x_i to x_i', and let $V_i = U_{x_0\to x_1}\cdots U_{x_{i-1}\to x_i}$, with $V_m = h_\gamma$. Put

$$D_i = V_i\, c_i \left(U_{x_0'\to x_1'}\cdots U_{x_{i-1}'\to x_i'}\right)^{-1} c_0^{-1},$$

so that $D_0 = I$. A direct computation gives $D_{i+1} = V_iP_iV_i^{-1}D_i$, with $P_i = U_{x_i\to x_{i+1}}c_{i+1}U^{-1}_{x_i'\to x_{i+1}'}c_i^{-1}$. If the i-th step is not parallel to μ, then P_i is the holonomy of the plaquette spanned by that step and its translate. If it is parallel to μ, then $P_i = I$: for a step $+\hat\mu$ the two step variables are c_i and c_{i+1}, and for a step $-\hat\mu$ they are c_{i+1}^{-1} and c_i^{-1}, in the case of the translate by $+\hat\mu$, and symmetrically for $-\hat\mu$. Since $c_m = c_0$, $D_m = h_\gamma c_0 h_{\gamma'}^{-1} c_0^{-1}$, where $h_{\gamma'}$ is based at x_0'. So $\mathrm{Tr}\,h_{\gamma'} = \mathrm{Tr}(D_m^{-1}h_\gamma)$, and

$$\begin{aligned} |\mathrm{Tr}\,h_\gamma - \mathrm{Tr}\,h_{\gamma'}| &= |\mathrm{Tr}((D_m^{-1}-I)h_\gamma)| \le \sqrt2\,|D_m - I| \\ &\le \sqrt2 \sum_{p\in R(\gamma,\mu)} |U_p - I|, \end{aligned}$$

by (b) and $|gU_pg^{-1}-I| = |U_p-I|$.

(d) Let $y_j=(j,0,0)$, let γ_j be the loop in direction 2 through y_j, with holonomy $W_2(y_j)$, and let $c_j = U_{(y_j,1)}$. The proof of (c), for γ_j and $\mu=1$, gives $W_2(y_j)c_j = D^{(j)}c_jW_2(y_{j+1})$. Here $D^{(j)}$ is a product of conjugates of the holonomies of the L plaquettes between the columns j and $j+1$. Let $H_j = c_0\cdots c_{j-1}$, so that $H_L = W_1(0)$. By induction on j, $W_2(0)H_j = E_jH_jW_2(y_j)$ with $E_0=I$ and $E_{j+1} = E_jH_jD^{(j)}H_j^{-1}$. For $j=L$, periodicity gives $W_2(y_L)=W_2(0)$, so $E_L = W_2(0)W_1(0)W_2(0)^{-1}W_1(0)^{-1}$. This is a product of conjugates of the holonomies of the L^2 plaquettes of P_{12}, and (b) gives the bound. □

The spatial flow Φ_s of the box is the gradient flow, on G^E with the product of the bi-invariant metric of Proposition 47, of $\sum_p(2-\mathrm{Tr}U_p) = 2e_s$. Let c_G be a constant with $|V-W| \le c_G\,\delta(V,W)$ for the geodesic distance δ of that metric. For the metric $\mathrm{Re}\,\mathrm{Tr}(X^*Y)$, $c_G=1$, because $\delta(V,W)$ is the length of a curve in $M_2(\mathbb{C})$ with the Frobenius norm. For a packet $f_0 = g\otimes h$ with g supported in $[t_0,t_1]$, let $n_1 = \lfloor t_1/a_k\rfloor$ and $p(z)=\sum_{n\ge1} g(a_kn)z^n$, as in the proof of Proposition 50.

Theorem 252. *In this box, let Φ be given by (29) and let $\tau>0$. Let $f_0 = g\otimes h$ with $g\ge0$ supported in $[t_0,t_1]\subset(0,\infty)$, and with $h\ge0$ and $h(0)>0$. Let k be such that $g(a_kn)>0$ for some integer $n\ge1$.*

(i) *$C_k(0)>0$ for every $\beta_W>0$.*

(ii) *Let $\kappa>0$, $\eta\in(0,1)$ and dyadic $u\ge0$ with $u/a_k\in\mathbb{Z}$, and put*

$$\varepsilon_k(\kappa,\eta) = \min\Big(\frac{1}{1600L^4}, \frac{\kappa a_k}{36L^2}, \frac{\eta^2a_k^2}{72c_G^2L^4\tau}\Big),$$
$$v = \frac{1-(t_1+u)\kappa-\eta}{1+\eta}.$$

If $\beta_W\ge1$ and $\varepsilon(\beta_W)\le\varepsilon_k(\kappa,\eta)$, which holds for every $\beta_W\ge B(\varepsilon_k(\kappa,\eta))$, then for every $\sigma\ge2a_k$ with $\sigma/a_k\in\mathbb{Z}$

$$\rho_k(u,\sigma) \ge \Big(v-\sqrt{\kappa\sigma/2}\Big)_+^2.$$

(iii) *Let $e^{-m(\beta_W)}$ be the norm of A on $\psi_0^\perp$. If $\beta_W\ge1$ and $\varepsilon(\beta_W)\le1/(1600L^4)$, then $1-e^{-m(\beta_W)}\le36L^2\varepsilon(\beta_W)$. In particular $m(\beta_W)\to0$ as $\beta_W\to\infty$.*

Proof. Write $a=a_k$, $s=\tau/a^2$, $S_h=\sum_x h(ax)$ and $w_x = h(ax)/S_h$, the sum running over the sites of the slice. The weights $w_x\ge0$ are well defined, since $h\ge0$ and $S_h\ge h(0)>0$. Put $G_0=\sum_x w_x\Phi_x$, $\varphi=G_0\psi_0$ and $\psi=(G_0\circ\Phi_s)\psi_0$. The flow commutes with ζ_1, because the slice action is invariant under it and its gradient is equivariant. Hence $F=S_hG_0\circ\Phi_s$ is odd under ζ_1, and $\omega(F)=0$. By the proof of Proposition 50, $\Psi_{A_k} = a^4S_hp(A)\psi$. With $R=A^{u/a}p(A)$, therefore, $\Psi_{k,u} = R\psi/\|R\psi\|$.

(i) The configuration 1 with every link equal to I minimises the slice action, so $\Phi_s(1)=1$ and $F(1)=S_hG_0(1)=4S_h>0$. Since F is continuous and $\psi_0>0$, $F\psi_0\ne0$. The operator $p(A)$ is injective by the proof of Proposition 50. So $C_k(0) = a^8S_h^2\|p(A)\psi\|^2>0$.

Lipschitz constants. By Lemma 251(b), $\mathrm{Tr}W_1(x)$ has the constant $\sqrt2$ on each of its L links, and $\mathrm{Tr}(W_1(x)W_2(x))$ has $\sqrt2$ on each of its $2L$ links. So Φ_x has constants $L_e\le2\sqrt2$, with $\sum_eL_e = 3\sqrt2L$. The same bounds hold for the convex combination G_0. Hence $\Lambda(G_0)\le3\sqrt2L$ and $\sum_eL_e^2\le2\sqrt2\cdot3\sqrt2L = 12L$.

Norm. Put $y_1=\mathrm{Tr}W_1(0)$ and $y_2=\mathrm{Tr}W_2(0)$, and let $y_{12}=\mathrm{Tr}(W_1(0)W_2(0))$. Under ζ_2 the product y_1y_{12} is odd, so $\omega(y_1y_{12})=0$. The exchange of the axes 1 and 2 maps $W_1(0)$ to $W_2(0)$, so $\omega(y_1^2)=\omega(y_2^2)$. By Lemma 251(a),

$$\begin{aligned}\|\Phi\psi_0\|^2 &= \tfrac12\omega(y_1^2+y_2^2)+\omega(y_{12}^2)\\ &\ge \tfrac12\omega(y_1^2+y_2^2+y_{12}^2) \ge \tfrac32-\tfrac38\omega(q).\end{aligned}$$

By Lemma 251(d), with $q=\frac12|C-I|^2$ for the commutator C there, which has the same trace as its inverse, we get $q\le\frac12L^2\sum_{p\in P_{12}}|U_p-I|^2 = 2L^2\sum_{p\in P_{12}}s_p$. So $\omega(q)\le2L^4\varepsilon_s$. Next, move the site 0 to a site x by at most $\lfloor L/2\rfloor$ steps $\pm\hat\mu$ in each direction. Steps in direction 1 do not change $\mathrm{Tr}W_1$, and the other steps cost L plaquettes each in Lemma 251(c), at most L^2 in all. For the path of W_1W_2 the steps cost L, L and $2L$ plaquettes in the directions 1, 2 and 3, at most $2L^2$ in all. Since $\||U_p-I|\,\psi_0\| = (4\varepsilon_s)^{1/2}$,

$$\|(\Phi_x-\Phi)\psi_0\| \le \sqrt2\cdot3L^2\cdot2\varepsilon_s^{1/2} = 6\sqrt2L^2\varepsilon_s^{1/2},$$

and the same bound holds for $\|\varphi-\Phi\psi_0\|$. If $L^4\varepsilon_s\le1/1600$, then $(\frac32-\frac{3}{6400})^{1/2}\ge1.22$, because $1.22^2=1.4884$. Also $6\sqrt2/40<0.22$, so $\|\varphi\|\ge1$ and $\|\Phi\psi_0\|\ge1$.

Energy. For a real slice function X, the eigenvector equation and the symmetry of t_b give

$$\begin{aligned}\langle X\psi_0,(1-A)X\psi_0\rangle = \frac{1}{2\lambda_0}\iint\big(X(U)-X(U')\big)^2\\ \times\psi_0(U)t_b(U,U')\psi_0(U')\,dU\,dU'.\end{aligned}$$

With $(\sum_eL_e|U_e-U_e'|)^2\le\Lambda\sum_eL_e|U_e-U_e'|^2$ and $|U_e-U_e'|^2 = 4s(U_eU_e'^{-1})$, this is at most $2\Lambda(X)^2\varepsilon_t\le36L^2\varepsilon_t$ for $X=G_0$ and for $X=\Phi$.

(iii) The vector $\Phi\psi_0$ is orthogonal to ψ_0, because Φ is odd under ζ_1, and A preserves $\psi_0^\perp$. So the spectral measure of A at $\Phi\psi_0$ lives on $[0,e^{-m}]$. Hence $1-e^{-m}\le\langle\Phi\psi_0,(1-A)\Phi\psi_0\rangle/\|\Phi\psi_0\|^2\le36L^2\varepsilon_t$, by the two previous steps and Lemma 250.

(ii) By Lemma 250, ε_s and ε_t are at most $\varepsilon_k(\kappa,\eta)$. Let ν be the spectral measure of A at φ, and $\kappa_\varphi a = \int(1-\lambda)\,d\nu/\|\varphi\|^2\le36L^2\varepsilon_t\le\kappa a$; also $\kappa_\varphi a\le1$.

Flow. The energy identity $\frac{d}{ds'}2e_s = -|\nabla2e_s|^2$ and the Cauchy–Schwarz inequality bound the length of the path $s'\mapsto\Phi_{s'}(U)$, $0\le s'\le s$, by $(2se_s(U))^{1/2}$. By Minkowski's inequality, the lengths ℓ_e of its components have $\sum_e\ell_e^2\le2se_s(U)$. Hence

$$|G_0(\Phi_sU)-G_0(U)| \le c_G\sum_eL_e\ell_e \le c_G(12L)^{1/2}(2se_s(U))^{1/2},$$

and $\|\psi-\varphi\|^2\le24c_G^2Ls\,\omega(e_s) = 72c_G^2L^4s\varepsilon_s\le\eta^2\le\eta^2\|\varphi\|^2$.

Overlap. If $v \le 0$ there is nothing to prove, so assume $v > 0$. Then $\theta = (t_1+u)\kappa < 1-\eta$. Since p has nonnegative coefficients supported in $[1,n_1]$, we have $p(1)\lambda^{n_1} \le p(\lambda) \le p(1)$ on $[0,1]$, and $\|R\| \le p(1)$. Jensen's inequality for λ^{2j} with $j = u/a + n_1$, and Bernoulli's inequality, give

$$\|R\varphi\|^2 \ge p(1)^2 \int \lambda^{2j} d\nu \ge p(1)^2 \|\varphi\|^2 (1-\kappa_\varphi a)^{2j},$$
$$\|R\varphi\| \ge p(1)\|\varphi\|(1-\theta),$$

since $ja \le u + t_1$. So $\rho = \|R(\psi-\varphi)\|/\|R\varphi\| \le \eta/(1-\theta) < 1$. Let $\chi = R\varphi/\|R\varphi\|$. Then

$$|\langle \chi, \Psi_{k,u}\rangle| \ge \frac{\|R\varphi\|^2 - \|R\varphi\|\,\|R(\psi-\varphi)\|}{\|R\varphi\|(\|R\varphi\| + \|R(\psi-\varphi)\|)}$$
$$= \frac{1-\rho}{1+\rho} \ge \frac{1-\theta-\eta}{1-\theta+\eta} \ge v.$$

The function $\lambda^{2u/a} p(\lambda)^2$ is nondecreasing on $[0,1]$ and $1-\lambda$ is nonincreasing. As in the proof of Theorem 195, this gives $\int \lambda^{2u/a} p^2 (1-\lambda)\, d\nu \le \|R\varphi\|^2 \kappa_\varphi a$, so $\langle \chi, (1-A)\chi\rangle \le \kappa a$. Lemma 193(a) with $\mathbb{T} = A$ gives the bound. The last statement of (ii) is Lemma 250, since $\varepsilon_k(\kappa,\eta) \le 3$. □

Fig. 24 draws the box, the observable Φ and the energy bound.

Corollary 253. *Under the hypotheses of Theorem 252, let $u \ge 0$ and $\sigma > 0$ be dyadic, and let $\rho_k(\beta_W) = \rho_k(u,\sigma)$ at the coupling β_W.*

(i) *At every fixed large k, ρ_k is real-analytic on $(0,\infty)$, with $\rho_k(\beta_W) \to 0$ as $\beta_W \to 0$ and $\rho_k(\beta_W) \to 1$ as $\beta_W \to \infty$. For every $r \in (0,1)$ the set $\{\beta_W > 0 : \rho_k(\beta_W) = r\}$ is nonempty and has no accumulation point in $(0,\infty)$. So ratio tunings at every level $r \in (0,1)$ exist in this box.*

(ii) *For every $B > 0$ there is $q_B < 1$ such that, for every packet in this box with $C_k(0) > 0$, $\rho_k(\beta_W) \le q_B^{\sigma/a_k}$ for $0 < \beta_W \le B$ and all k. So every ratio tuning in this box has $\beta_k \to \infty$.*

(iii) *For $r \in (0,1)$ let $\beta_k(r)$ be the smallest solution of $\rho_k(\beta_W) = r$. Then $\beta_k(r) \to \infty$, and $\beta_k(r) \le C a_k^{-2} \log(1/a_k)$ for large k, with C depending on r, L, τ, t_1, u, σ and c_G.*

(iv) *Let β_k be couplings with $a_k^2 \beta_k / \log(1/a_k) \to \infty$. Then $\rho_k(u,\sigma) \to 1$ for all dyadic $u \ge 0$ and $\sigma > 0$, so (N_u) holds for every u. Every limit of Theorem 48 in the box along these couplings is two-dimensional. In it, $\psi(f_0(\cdot - (t,0))) = \psi_*$ is the same unit vector for every dyadic $t > 0$, $\psi_* \perp \Omega$, and $H = 0$. The inequality of $(\mathrm{Z_{sl}})$ in Proposition 222 holds in the box at every dyadic $\delta > 0$: $I_k(\delta) \le 2\delta D_k(\delta)$ for large k.*

Proof. (i) The proof of Proposition 242 uses the packet only through $C_k(0) > 0$ at every $\beta_W > 0$. That positivity is Theorem 252(i), so analyticity, the limit at 0 and the discreteness follow as there. Given $\varepsilon' > 0$, choose $\eta \in (0,1)$ and $\kappa > 0$ so small that $(v - \sqrt{\kappa\sigma/2})^2 \ge 1 - \varepsilon'$. Theorem 252(ii) then gives $\rho_k \ge 1 - \varepsilon'$ for $\beta_W \ge B(\varepsilon_k(\kappa,\eta))$. Existence follows from the intermediate value theorem, as in Proposition 242(ii).

(ii) By the proof of Proposition 242, the Riesz projection P_0 onto ψ_0 is continuous in $\beta_W \in [0,B]$ in operator norm. Hence so is $\mathbb{T}_{\beta_W}(1-P_0)$, whose norm is the largest eigenvalue λ_1 of $\mathbb{T}_{\beta_W}$ on $\psi_0^\perp$. By simplicity, $\lambda_1 < \lambda_0$ for $\beta_W > 0$, and at $\beta_W = 0$ we have $\lambda_1 = 0 < 1 = \lambda_0$. So $q_B = \max_{[0,B]} \lambda_1/\lambda_0 < 1$, and $\rho_k \le (\lambda_1/\lambda_0)^{\sigma/a_k}$ as in that proof. The transfer matrix does not depend on k, so q_B does not either. If a ratio tuning at level r had $\beta_k \le B$ for infinitely many k, then $r \le q_B^{\sigma/a_k} \to 0$ along them.

(iii) By (i), $\rho_k < r$ on some $(0,\delta_k]$, and ρ_k takes the value r. So the solutions form a nonempty closed subset of $[\delta_k,\infty)$, and the smallest one exists. Fix $\eta \in (0,1)$ and $\kappa > 0$ with $(v - \sqrt{\kappa\sigma/2})_+^2 > r$. For large k, $\varepsilon_k = \varepsilon_k(\kappa,\eta) = \eta^2 a_k^2/(72 c_G^2 L^4 \tau)$, and at $\beta_k' = B(\varepsilon_k)$ Theorem 252(ii) gives $\rho_k > r$. By the intermediate value theorem, $\beta_k(r) \le \beta_k' = (3/\varepsilon_k)(\log(3/\varepsilon_k) + 4)$, which is at most $C a_k^{-2} \log(1/a_k)$. By (ii), $\beta_k(r) \to \infty$.

(iv) Write $\beta_k = M_k a_k^{-2} \ell_k$ with $\ell_k = \log(1/a_k)$, so that $M_k \to \infty$. Then

$$\frac{\log \beta_k + 5}{a_k^2 \beta_k} = \frac{\log M_k + 2\ell_k + \log \ell_k + 5}{M_k \ell_k} \to 0.$$

So $\varepsilon(\beta_k)/a_k^2 \to 0$, hence also $\varepsilon(\beta_k)/a_k \to 0$ and $\varepsilon(\beta_k) \to 0$. For fixed κ and η, $\varepsilon(\beta_k) \le \varepsilon_k(\kappa,\eta)$ for large k, and Theorem 252(ii) gives $\liminf_k \rho_k(u,\sigma) \ge (v - \sqrt{\kappa\sigma/2})_+^2$. Letting $\kappa, \eta \to 0$ and using $\rho_k \le 1$ gives $\rho_k(u,\sigma) \to 1$. In the limit of Theorem 48(iii) in the box, put $y = (t,0)$ and $y' = (t',0)$. The vectors $\psi(f_0(\cdot - y))$ have inner products $\lim_k \rho_k(0, t+t') = 1$ and are orthogonal to Ω, since the packets are centred. So they coincide with one unit vector ψ_*. Moreover $\psi(f_*) = \Omega$, since $\|\psi(f_*)\| = \langle \Omega, \psi(f_*)\rangle = 1$, and $\langle \psi_*, e^{-\tau' H}\psi_*\rangle = 1$ for dyadic τ'. So $H\psi_* = 0$, the range of ψ is spanned by Ω and ψ_*, and $H = 0$. For the slice bound, $D_k(r) = a_k^6 S_h^2 \langle \psi, A^{r/a_k}\psi\rangle$, in the notation of the proof of Theorem 252, and $\langle \psi, A^n \psi\rangle = \|A^{n/2}\psi\|^2$. As in the overlap step, for $na_k = \delta$ and k so large that $n \ge 2$,

$$\|A^{n/2}\psi\| \ge \|\varphi\|(1-\kappa_\varphi a_k)^{n/2} - \|\psi - \varphi\|$$
$$\ge \|\varphi\|(1 - \tfrac{1}{2}\kappa\delta - \eta),$$

while $\|\psi\| \le (1+\eta)\|\varphi\|$. Hence $D_k(\delta) \ge \big((1 - \frac{1}{2}\kappa\delta - \eta)/(1+\eta)\big)^2 D_k(0)$. For small κ, η the factor exceeds $\frac{1}{2}$, and $I_k(\delta) \le \delta D_k(0) \le 2\delta D_k(\delta)$. □

Proposition 254. *Let $G \subset \mathrm{U}(N)$ be closed, let ω be a state as in Appendix F, and let X be a real bounded slice function whose links join sites of the cube $B_R = \{0,\dots,R\}^3$ of the slice. Then*

$$\mathrm{Var}_\omega X \le 8NR^2 \Lambda(X)^2 \varepsilon_\omega.$$

Let χ be the unit vector along $\Psi_{X-\omega(X)}$. The bound of Lemma 197(iii) for $\langle \chi, (1-\mathbb{T})\chi\rangle$, which is $N\Lambda(X)^2\varepsilon_\omega/\mathrm{Var}_\omega X$, is then at least $1/(8R^2)$. If it is at most κa_k, then $R \ge (8\kappa a_k)^{-1/2}$.

Proof. Set to I, by a gauge transformation of the slice that is the identity off B_R, the links of the comb tree of B_R. This tree consists of the links in direction 3, the links in direction 2 with $x_3 = 0$, and the links in direction 1 with $x_2 = x_3 = 0$.

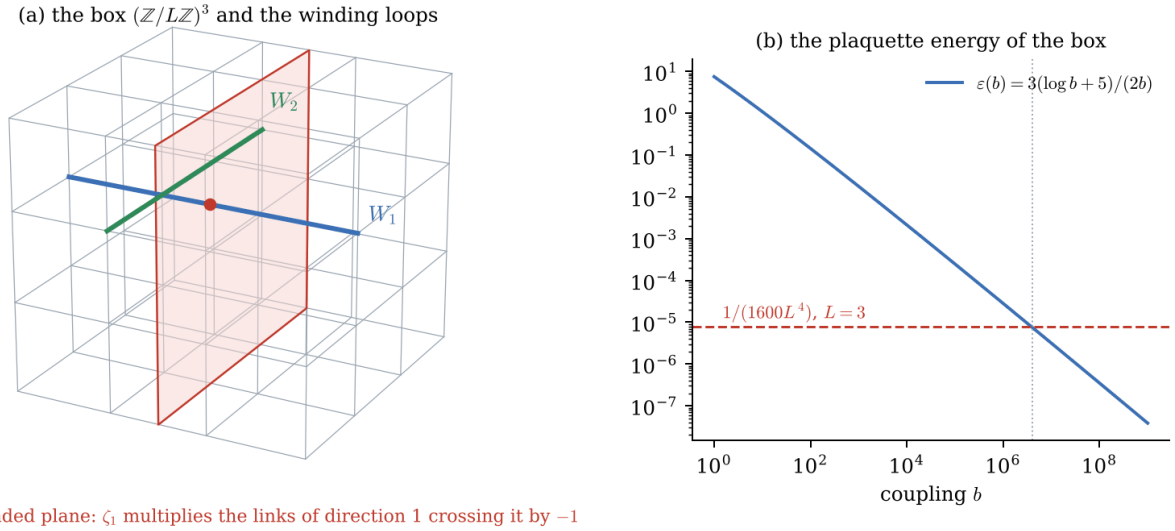


FIG. 24. The box of Theorem 252. (a) The spatial torus $(\mathbb{Z}/L\mathbb{Z})^3$ as a cube with identified faces, the winding loops W_1 and W_2 of (29), and a plane $x_1 = \text{const}$ across which the centre transformation multiplies the links of direction 1 by -1; $\operatorname{Tr} W_1$ and $\operatorname{Tr} W_1 W_2$ change sign. (b) The energy bound $\varepsilon(b) = 3(\log b + 5)/(2b)$ of Lemma 250 against the threshold $1/(1600L^4)$ of Theorem 252(ii) at $L = 3$.

This changes neither X nor the plaquette holonomies, up to conjugation. In this gauge, a link in direction 2 at height x_3 is the product of the x_3 plaquettes of the $(2,3)$-plane below it, inverted where needed. A link in direction 1 at (x_2, x_3) is a product of at most $x_3 + x_2 \le 2R$ plaquettes, first in the $(1,3)$-plane down to $x_3 = 0$ and then in the $(1,2)$-plane. Let $P(e)$ be the plaquettes used for the link e, so $|P(e)| \le 2R$. With $X(1)$ the value at the configuration with all links equal to I, Lemma 251(b) gives

$$|X - X(1)| \le \sum_e L_e \sum_{p \in P(e)} |U_p - I|.$$

By the Cauchy–Schwarz inequality twice and $|U_p - I|^2 = 2N s_p$,

$$\begin{aligned} \operatorname{Var}_\omega X &\le \omega\big((X - X(1))^2\big) \\ &\le \Lambda(X) \sum_e L_e |P(e)| \sum_{p \in P(e)} 2N\omega(s_p) \\ &\le 8NR^2 \Lambda(X)^2 \varepsilon_\omega . \end{aligned}$$

□

Remark 255. Theorem 252 and Corollary 253 prove (N_u) without hypotheses, but only for a degenerate limit. The couplings grow like $a_k^{-2} \log(1/a_k)$, not like $\log(1/a_k)$. The box has physical size $La_k \to 0$. The limit vector ψ_* has energy zero and is orthogonal to Ω, so 0 is not a simple eigenvalue of H, and the gap of the box closes as $\beta_W \to \infty$ (Theorem 252(iii)). The mechanism is the centre degeneracy of a periodic box. The packet is odd under ζ_1. Its unflowed slice field has norm at least 1 once $\varepsilon(\beta_W) \le 1/(1600L^4)$, by the trace identity of Lemma 251(a), its time-step energy is at most $36L^2\varepsilon(\beta_W)$, and the flow moves it by at most $(72c_G^2 L^4 s\varepsilon(\beta_W))^{1/2}$ in norm.

None of this carries over to infinite volume. There, the map that multiplies by $-I$ the links (x,μ) with $x_\mu = c$ is the gauge transformation equal to $-I$ at the sites with $x_\mu > c$ and to I elsewhere. So every element of $\mathscr{A}$ is invariant under it, and no centre-odd packet exists. Proposition 254 shows the loss for slice functions of a cube. The bound of Lemma 197(iii) reaches κa_k only for cubes of side at least $(8\kappa a_k)^{-1/2}$ lattice spacings, at every coupling. Together with Proposition 199(ii), the slab-energy route needs both $a_k\beta_k$ bounded below and supports of diverging lattice size. In the box the winding loops W_μ are not functions of a cube, and the comb gauge cannot remove their holonomy.

Three points are not proved. First, the infinite-volume condition, which is (N_u) along some sequence of couplings, equivalently the existence of a ratio tuning (Theorem 244). Second, the question of Proposition 242 for the centre-even packet of Proposition 50. Third, the case $\mathrm{SU}(N)$ with $N \ge 3$. For the last, the proof uses $N = 2$ in Lemma 251(a). For $N \ge 3$, the commuting pair $W_1 = W_2 = \operatorname{diag}(1, e^{2\pi i/N}, \dots, e^{2\pi i(N-1)/N})$, with $W_3 = I$, comes from a flat configuration of the slice, with the links $(x,1)$ with $x_1 = L-1$ and $(x,2)$ with $x_2 = L-1$ equal to that matrix and all others equal to I. It has $\operatorname{Tr} W_1 = \operatorname{Tr} W_2 = \operatorname{Tr} W_1 W_2 = 0$, so the inequality of (a) has no analogue for these three words. With $L_k = \ell/a_k$ and the loops of length L_k, the proof gives the bounds of Theorem 252(ii) with L replaced by L_k, at couplings with $\varepsilon(\beta_k) \le \eta^2 a_k^6/(72c_G^2\ell^4\tau)$ and a_k small. These packets are not of the form of Theorem 48, because Φ then depends on k.

What is proved. In infinite volume: the admissible states form compact convex sets with closed graph; the ratio is continuous in the state, continuous in the coupling wherever the admissible state is unique (for $\mathrm{SU}(2)$ at $\beta_W \le 0.348$, and wherever (D) holds), and its values over a compact interval of couplings form a compact interval under (P_k). In the box: (P_k) holds, the ratio is real-analytic in $\beta_W > 0$ and tends to zero at $\beta_W = 0$, and the couplings at which it equals r form a

discrete set, nonempty exactly when the ratio reaches r. In a box of fixed size, for packets odd under the centre of SU(2): (N_u) holds along every sequence with $a_k^2\beta_k/\log(1/a_k)\to\infty$, ratio tunings exist at every level, at couplings at most of order $a_k^{-2}\log(1/a_k)$ and tending to infinity, and the limits are two-dimensional with $H=0$ (Theorem 252, Corollary 253). A ratio tuning exists exactly when (N_u) holds along some sequence of couplings; it leaves the strong-coupling range, has lattice gap at most $a_k\sigma^{-1}\log(1/r)$, satisfies (N_u) with two-sided bounds, and, with (D), (M) and the family $\mathfrak{F}$, gives (U). Its increments are a step-scaling function, and if they exceed h by c/β_{k+1}, as two loops predict, it is not a one-loop tuning of bounded M.

What is moved, not proved. In infinite volume the existence of a ratio tuning is equivalent to (N_u) along some sequence, and no packet is odd under the centre (Remark 255); (P_k) is open in infinite volume; $\beta_k\to\infty$ is equivalent to $\lim_k R_k(B)<r$ for every B and follows from a lattice gap bounded below on compact sets of couplings, proved only for $\beta_W\le 0.348$; (D), (M) and $\mathfrak{F}$ along the ratio tuning remain hypotheses, and Proposition 245 shows that (D) along it is the joint input of Sec. XI B unchanged. Monotonicity of the ratio in β_W is not proved and is used only in Proposition 246(iv). The Ward-identity route is not pursued here.

F. Estimates uniform in the coupling

Some local quantities of the Wilson measure can be controlled uniformly in β_W, the volume and the boundary condition, by integrating by parts in a single link. Such estimates hold along every tuning, but they concern local fields in lattice units, and the three results below show how far they are from the open inputs: the force has bounded moments but its centred form is a null vector of the reflected form, the plaquette has fluctuations of order β_W^{-2} but this is a single-site variance, and bare links over a fixed physical distance become independent Haar variables.

Let $G=\mathrm{SU}(N)$, write $b=\beta_W$, so that the Wilson weight is e^{bS} with $S=\sum_p N^{-1}\operatorname{Re}\operatorname{Tr}Q_p$, and give $\mathfrak{g}$ the metric $\langle T,R\rangle=-\operatorname{Tr}(TR)$. For a link e let S_e be the sum of the six plaquette terms containing e, X_T the derivative along $U_e\mapsto e^{uT}U_e$ at $u=0$, ∇_e and Δ_e the gradient and Laplacian in U_e, $q=N^2-1$, $c_N=(N^2-1)/N$, and

$$E_e=b|\nabla_e S_e|^2,\qquad W_e=E_e-c_N S_e=b|\nabla_e S_e|^2+\Delta_e S_e.$$

Below, ω is a Wilson Gibbs state at $b>0$: a finite-volume measure in which e is not fixed by the boundary condition, or an infinite-volume state whose conditional law of U_e given the other links is proportional to $e^{bS_e}dU_e$ with dU_e the Haar measure.

Proposition 256. (i) *For every unit $T\in\mathfrak{g}$, $t\in\mathbb{R}$ and $0\le\eta<N/12$,*

$$\mathbb{E}_\omega\big[e^{t\sqrt{b}X_TS_e}\,\big|\,U_{e^c}\big]\le e^{3t^2/N},$$
$$\mathbb{E}_\omega\big[e^{\eta E_e}\,\big|\,U_{e^c}\big]\le(1-12\eta/N)^{-q/2},$$

so $\mathbb{E}_\omega E_e^m\le m!\,(24/N)^m2^{q/2}$ for $m\ge1$, uniformly in b, the volume and the boundary condition. Moreover $\mathbb{E}_\omega E_e=c_N\mathbb{E}_\omega S_e$.

(ii) *For a bounded function F of finitely many links, continuously differentiable in U_e, $\mathbb{E}_\omega[W_eF]=-\mathbb{E}_\omega\langle\nabla_eS_e,\nabla_eF\rangle$. In particular $\mathbb{E}_\omega[W_eF]=0$ if F does not depend on U_e, and $\mathbb{E}_\omega[W_eW_f]=0$ if no plaquette contains both e and f.*

(iii) *Let ω be reflection positive as in Appendix F, and let $A=\sum_e h_eW_e$ be a finite sum over links whose endpoints have times at least 2. Then $\omega(\vartheta(\bar A)A)=0$, so $\Psi_A=0$: in the reconstructed Hilbert space the packets $\sum_e h_eE_e$ and $c_N\sum_e h_eS_e$ define the same vector.*

Proof. (i) Along $u\mapsto e^{uT}U_e$ a plaquette term containing e is $N^{-1}\operatorname{Re}\operatorname{Tr}(e^{uT}V)$ or $N^{-1}\operatorname{Re}\operatorname{Tr}(Ve^{-uT})$ with V unitary, whose second derivative is $N^{-1}\operatorname{Re}\operatorname{Tr}(T^2e^{\pm uT}V)$. Since $-T^2\ge0$ and $\operatorname{Tr}(-T^2)=1$, $|\operatorname{Tr}(T^2M)|\le1$ for unitary M, so the second derivative of S_e along the curve is at most $6/N$ in absolute value. Taylor's formula with $u=t/\sqrt{b}$ gives $b\big(S_e(e^{tT/\sqrt{b}}U_e)-S_e(U_e)\big)\ge t\sqrt{b}X_TS_e(U_e)-3t^2/N$. The conditional density is e^{bS_e}/Z with respect to Haar measure, and the left invariance of Haar measure gives $\mathbb{E}_\omega[e^{b(S_e(e^{tT/\sqrt{b}}U_e)-S_e(U_e))}\mid U_{e^c}]=1$, which with the Taylor bound proves the first inequality. For the second let G_0 be a standard Gaussian vector in $\mathfrak{g}$, independent of ω; then $e^{\eta|Y|^2}=\mathbb{E}e^{\sqrt{2\eta}\langle G_0,Y\rangle}$ for $Y=\sqrt{b}\nabla_eS_e$, and the first inequality in the direction of G_0 with $t=\sqrt{2\eta}|G_0|$ gives the bound $\mathbb{E}e^{6\eta|G_0|^2/N}=(1-12\eta/N)^{-q/2}$. With $\eta=N/24$ and $x^m\le m!\,\eta^{-m}e^{\eta x}$ the moment bound follows. For the mean, $\sum_A T_A^2=-c_NI$ in an orthonormal basis gives $\Delta_eS_e=-c_NS_e$, and the integral of the divergence of $e^{bS_e}\nabla_eS_e$ over G vanishes.

(ii) The integral over U_e of the divergence of $e^{bS_e}F\nabla_eS_e$ vanishes, and that divergence is $e^{bS_e}(W_eF+\langle\nabla_eS_e,\nabla_eF\rangle)$; integrate the conditional identity. The function W_f depends only on the links of the plaquettes containing f.

(iii) Each W_e in the sum depends on links with times at least 1, so $\vartheta(\bar A)$ depends only on links with times at most 0, none of which is a link e of the sum. By (ii), $\omega(\vartheta(\bar A)W_e)=0$ for each e. □

Proposition 257. *Let $G=\mathrm{SU}(2)$, $b\ge0$, ω a Wilson Gibbs state at b and p a plaquette with a link that is not fixed by the boundary condition. Then*

$$\operatorname{Var}_\omega\big(\tfrac12\operatorname{Tr}Q_p\big)\ge\frac{1}{2^{19}(1+6b)^2},\qquad \mathbb{E}_\omega\big[1-\tfrac12\operatorname{Tr}Q_p\big]\ge\frac{1}{4b+1}.$$

Proof. Let e be the link and identify SU(2) with the unit quaternions. Given the other links, $U_e=x$ has density proportional to $e^{rx\cdot u}$ with $|u|=1$ and $0\le r\le6b$, since the six staples have norm one, and $\frac12\operatorname{Tr}Q_p=x\cdot v$ for a unit vector v fixed by the conditioning. Let $X=x\cdot u$ and $T=1-X$; the law of T has density proportional to $e^{-rt}\sqrt{t(2-t)}$ on $[0,2]$. Put $n=1+r$. Since $e^{-rt}\le e^2e^{-nt}$, the normalisation is at most $e^2\int_0^\infty\sqrt{2t}e^{-nt}dt=e^2\sqrt{\pi/2}\,n^{-3/2}<14n^{-3/2}$. On $I=[\frac{1}{4n},\frac{1}{2n}]$ the density is at least $\frac12n^{-1/2}e^{-1/2}$ and on $J=[\frac1n,\frac{3}{2n}]$ at least $\frac12n^{-1/2}e^{-3/2}$, so $P(T\in I)\ge\frac{1}{224}$ and $P(T\in J)\ge\frac{1}{336}$. With an independent copy T', $\operatorname{Var}T=\frac12\mathbb{E}(T-T')^2\ge P(T\in I)P(T\in J)(2n)^{-2}>2^{-19}n^{-2}$. By symmetry the conditional covariance of x is diagonal in u and $u^\perp$, with eigenvalues $\operatorname{Var}X$ and $\mathbb{E}(1-X^2)/3$; on I, $1-X^2=T(2-T)\ge\frac{3}{8n}$, so $\mathbb{E}(1-X^2)/3\ge$

$\frac{1}{1792n} \geq 2^{-19}n^{-2}$. Hence the conditional variance of $x\cdot v$ is at least $2^{-19}(1+6b)^{-2}$, and so is the variance, by the law of total variance. For the mean, integrating $\frac{d}{dX}\big(e^{rX}(1-X^2)^{3/2}\big)$ over $[-1,1]$ gives $3\mathbb{E}X = r\mathbb{E}(1-X^2) \leq r(1-(\mathbb{E}X)^2)$, whence $\mathbb{E}X \leq (\sqrt{9+4r^2}-3)/(2r) \leq 2r/(2r+3)$ for $r>0$, and $\mathbb{E}[1-x\cdot v] \geq 1-\mathbb{E}X \geq 3/(2r+3) \geq 1/(4b+1)$. □

For the blocking statement let $G=\mathrm{SU}(2)$, $B\geq 4$ an integer and, for a coarse edge (x,μ), $V_{(x,\mu)} = \prod_{r=0}^{B-1} U_{(Bx+r\hat{\mu},\mu)}$ the ordered product of the fine links along it.

Proposition 258. *For every Wilson Gibbs state ω of* $\mathrm{SU}(2)$ *at $b>0$ and every set $\mathscr{E}$ of K coarse edges,*

$$\big\|\mathrm{Law}_\omega\big((V_e)_{e\in\mathscr{E}}\big) - m^{\otimes K}\big\|_{\mathrm{TV}} \leq \frac{K}{2}e^{3b}\Big(\frac{4b}{4b+1}\Big)^{B-3} \leq \frac{K}{2}\exp\Big(3b-\frac{B-3}{4b+1}\Big),$$

where m is the Haar measure. If F is a function of the V_e of a set of coarse edges at positive times, $|F|\leq 1$, and ε is this bound for those edges and their reflections, then $0\leq \omega(\vartheta(\bar{F}^c)F^c) \leq 6\varepsilon$ with $F^c = F-\omega(F)$.

Proof. Select on each coarse edge the fine links with $1\leq r\leq B-2$. No plaquette contains two selected links: collinear links do not share a plaquette, parallel coarse edges are B apart, and a plaquette containing a selected link of (x,μ) has its links in direction $\nu\neq\mu$ at μ-offsets $r, r+1\in[1,B-1]$ from Bx, while coarse edges in direction ν lie at offsets in $B\mathbb{Z}$. Given the other links, the selected ones are therefore independent, each with density $e^{r_i x\cdot u_i}/Z(r_i)$, $0\leq r_i\leq 6b$. Convolution with such a density acts on the representation of spin $l\geq 1$ by $D^{(l)}(u_i)$ times $I_{l+1}(r_i)/I_1(r_i)$, whose absolute value is at most $I_2(r_i)/I_1(r_i)$ because $I_\nu(r)$ decreases in $\nu\geq 0$, and $I_2(r_i)/I_1(r_i) = \mathbb{E}X \leq 4b/(4b+1)$ by the proof of Proposition 257, so it contracts the mean-zero part of L^2 by that factor. The density is at most e^{6b} because $Z\geq 1$ by Jensen's inequality, so its distance to 1 in L^2 is at most e^{3b}. The conditional law of V_e is that of a product of $B-2$ such independent variables between two fixed ones, so its total variation distance to m is at most $\frac{1}{2}e^{3b}(4b/(4b+1))^{B-3}$, and the coarse edges are conditionally independent. Total variation distance from a product is at most the sum over the factors, and averaging over the conditioning does not increase it. For the last statement, under $m^{\otimes K}$ the covariance vanishes because F and its reflection depend on disjoint edges; replacing ω by $m^{\otimes K}$ changes $\omega(\vartheta(\bar{F})F)$ by at most 2ε and $|\omega(F)|^2$ by at most 4ε. □

With $B=\ell/a_k$ for a fixed physical length ℓ and $b_k = O(\log(1/a_k))$ the bound tends to zero: the bare coarse links of a fixed coarse graph become independent Haar variables, their reflected form has rank one, and a renormalisation Z_k that keeps the reflected norm of $Z_k F^c$ at least $c>0$ must grow at least like $(c/6\varepsilon)^{1/2}$. The renormalised flowed fields of Theorem 48 are not bounded functions of bare coarse links, and none of the three propositions bears on the slice ratio, the tree bounds or (D).

G. What the cube estimate requires

This subsection proves seven facts about $D(n,\alpha)$. It holds at small coupling for every even n above an explicit size (Proposition 260), and it survives small perturbations of the action (Proposition 261), although gauge-invariant perturbations still need strong coupling at the scale of the cube (Proposition 268). It forces the gauge-fixed variables of a cube with trivial exterior to stay a fixed distance from the identity on average (Theorem 262, and Proposition 264 for $\mathrm{SU}(2)$), so that for every fixed n it fails once β is large (Corollary 263). And it cannot be obtained from bounds on the conditional law of a single line of links that are uniform in the configuration around the line (Proposition 272). For the compact abelian theory with the Villain action it fails at every size once β is large (Proposition 269). In Sec. XII A the estimate is posed for block averages, where its quadratic approximation holds along the tuning. None of these decides (D) along the continuum tuning, where n grows while β grows.[32]

For a finite $V\subset A$ and $j\in A\setminus V$ let $b(V,j)$ be the number of links of V that share a plaquette with j, and write $k\sim i$ when two active links share a plaquette.

Lemma 259. *Suppose that $C_{ee'}\leq c_0$ for all active e,e', with $12c_0<1$. Then $C_{V,j}\leq c_0\, b(V,j)/(1-12c_0)$ for every finite $V\subset A$ and every $j\in A\setminus V$.*

Proof. Fix Q,Q' that differ only at j, put $d=\delta(Q_j,Q'_j)$, $\mu=\gamma_V(\cdot\,|\,Q)$ and $\nu=\gamma_V(\cdot\,|\,Q')$. The couplings of μ and ν form a weakly compact set on which $\pi\mapsto\int d_V\,d\pi$ is continuous, so one of them, π, minimises it; let $\rho_i=\int\delta(X_i,Y_i)\,d\pi$. Fix $i\in V$. Under μ the conditional law of X_i given the other links of V is $\gamma_i^F(\cdot\,|\,X)$, and likewise for ν. Replace (X_i,Y_i) by a pair drawn from an optimal coupling of $\gamma_i^F(\cdot\,|\,X)$ and $\gamma_i^F(\cdot\,|\,Y)$, chosen measurably [113, Cor. 5.22], and keep the other coordinates. The new coupling still has marginals μ and ν, and changing the links in which X and Y differ one at a time gives

$$W_1\big(\gamma_i^F(\cdot\,|\,X),\gamma_i^F(\cdot\,|\,Y)\big) \leq \sum_{k\in V,k\neq i} C_{ik}\,\delta(X_k,Y_k) + C_{ij}\,d.$$

Minimality of π therefore gives $\rho_i\leq c_0\sum_{k\in V,k\sim i}\rho_k + c_0[i\sim j]\,d$. Sum over $i\in V$: every ρ_k occurs at most twelve times by Lemma 9(iv), so $(1-12c_0)\sum_i\rho_i\leq c_0\,b(V,j)\,d$. Since $W_1^{d_V}(\mu,\nu)\leq\sum_i\rho_i$, dividing by d proves the lemma. □

Proposition 260. *Let $n=2m$ and let V be the cube of Definition 42. For every class c,*

$$\sum_{j\in c\setminus V} b(V,j) \leq 10m^3-2m^2,$$

[32] The four statements answer the question of which parts of (D) can be settled with the methods of this paper. The first places (D) in the range where the single-link method already works, the second and third show what any proof of (D) at weak coupling has to produce, and the fourth excludes the most direct one-dimensional route.

with equality for three classes. Consequently, if $C_{ee'} \le c_0$ for all e, e' with $12c_0 < 1$, then $D(n,\alpha)$ holds for every $\alpha \in (0,1)$ with

$$\alpha \ge \hat\alpha_n = \frac{c_0}{1-12c_0}\Big(\frac{20}{n} - \frac{8}{n^2}\Big).$$

For SU(2) *and $|\beta_W| < \frac{1}{3}$ this reads $\hat\alpha_n = \frac{|\beta_W|}{1-3|\beta_W|}\big(\frac{5}{n} - \frac{2}{n^2}\big)$, which is below one for every even $n > 5|\beta_W|/(1-3|\beta_W|)$.*

Proof. Let $j = (y,\mu)$ lie in the class c, with $y \equiv r$ modulo two. A link of V that shares a plaquette with j has base point $y+d$ with $d \in \{-1,0,1\}^4$ and at most two nonzero coordinates, and the twelve offsets d depend only on c. For a fixed offset we count the $y \equiv r$ with $y+d \in [0,n)^4$ and $y \notin [0,n)^4$. In the coordinate l there are m values of $y_l \equiv r_l$ with $y_l + d_l \in [0,n)$, and one of them lies outside $[0,n)$ exactly when $d_l = 1$ and $r_l = 1$ (the value $y_l = -1$) or $d_l = -1$ and $r_l = 0$ (the value $y_l = n$). If q is the number of such coordinates, the count is $m^4 - m^{4-q}(m-1)^q$, that is 0, m^3 or $2m^3 - m^2$ for $q = 0,1,2$. Summing over the twelve offsets, the class sum is $a_c m^3 - b_c m^2$, where b_c is the number of offsets with $q = 2$ and a_c is the number with $q = 1$ plus twice b_c. Reading the offsets from Table VI, the pairs (a_c, b_c) over the 48 classes are $(2,0)$, $(7,0)$, $(7,1)$, $(9,1)$, $(9,2)$ and $(10,2)$ three times each and $(3,0)$, $(4,0)$, $(5,0)$, $(6,0)$ and $(8,1)$ six times each. In every case $a_c \le 10$ and $8 - a_c + b_c \ge 0$, so

$$\begin{aligned}(10m^3 - 2m^2) - (a_c m^3 - b_c m^2) &= m^2\big((10-a_c)m + b_c - 2\big)\\ &\ge m^2(8 - a_c + b_c) \ge 0,\end{aligned}$$

with equality for the three classes with $(a_c, b_c) = (10,2)$. By Lemma 259, $K_c \le c_0(10m^3 - 2m^2)/(1-12c_0)$, and $N_n = m^4$ gives $K_c/N_n \le \hat\alpha_n$. For SU(2) the proof of Theorem 1 gives $c_0 = |\beta| = |\beta_W|/4$. □

At $\beta_W = \frac{3}{10}$ and $n = 16$, for instance, $\hat\alpha_{16} = \frac{117}{128}$, and Theorem 43 gives the gap $\log\frac{128}{117}/32$, much weaker than Theorem 1. The proposition shows that the cube condition is not empty; it does not extend the range, because the absorption $12c_0 < 1$ is the single-link condition itself.

The cube condition is also stable under small changes of the action, which is what a block renormalisation would need at its last scale, where the effective action is no longer of Wilson form. Let $r \ge 0$ and $\varepsilon \ge 0$, and let (w_X) be real functions on G^A indexed by finite sets $X \subset A$ of diameter at most r, with w_X depending only on the links of X, such that the family is invariant under the translations by $2\mathbb{Z}^4$, every active link lies in at most M of the sets X, every X has at most s links, and

$$|w_X(Q) - w_X(Q')| \le \varepsilon \sum_{i \in X} \delta(Q_i, Q_i')$$

for all Q, Q'. Let $\gamma_V^W(\cdot \mid Q)$ be the probability measure with density proportional to $\exp\sum_{X\cap V\neq\emptyset} w_X$ with respect to $\gamma_V(\cdot \mid Q)$, and define $C_{V,j}^W$, K_c^W and the condition $D^W(n,\alpha)$ as in Definition 42, with γ_V^W in place of γ_V.

Proposition 261. *Let n be even and suppose that $D(n,\alpha)$ holds at β. Then $D^W(n,\alpha')$ holds with*

$$\alpha' = \alpha\, e^{\Omega}(1+2\Omega) + 96\, s\,\Omega, \qquad \Omega = 48\, M\, D\, N_n\, \varepsilon.$$

In particular $\alpha' < 1$ for all ε below an explicit $\varepsilon_0(n,\alpha,M,s) > 0$.

Proof. The cube has $|V| = 48N_n$ links, so d_V has diameter $\Delta_V = 48N_nD$. For fixed exterior links put $u = \sum_{X\cap V\neq\emptyset} w_X$ as a function on G^V. Each link of V lies in at most M sets X, so u is $M\varepsilon$-Lipschitz for d_V and its oscillation is at most $M\varepsilon\Delta_V = \Omega$; we normalise u so that $\sup u = 0$, which does not change the tilted measures, and put $h = e^u$, so that $e^{-\Omega} \le h \le 1$ and h is $M\varepsilon$-Lipschitz.

A common tilt. Let μ, ν be probability measures on G^V and $\mu_h = h\mu/\mu(h)$, $\nu_h = h\nu/\nu(h)$. Let f be 1-Lipschitz for d_V; subtracting a constant we may take $|f| \le \Delta_V$. Then fh is $(1+\Delta_V M\varepsilon)$-Lipschitz, $|\nu(fh)| \le \Delta_V \nu(h)$, $|\mu(h) - \nu(h)| \le M\varepsilon W_1^{d_V}(\mu,\nu)$ and $\mu(h) \ge e^{-\Omega}$, so

$$\begin{aligned}\mu_h(f) - \nu_h(f) &= \frac{\mu(fh) - \nu(fh)}{\mu(h)} + \nu(fh)\,\frac{\nu(h) - \mu(h)}{\mu(h)\nu(h)}\\ &\le e^{\Omega}(1+2\Omega)\, W_1^{d_V}(\mu,\nu),\end{aligned}$$

since $\Delta_V M\varepsilon = \Omega$. By Kantorovich duality $W_1^{d_V}(\mu_h,\nu_h) \le e^{\Omega}(1+2\Omega)W_1^{d_V}(\mu,\nu)$.

Two tilts. If u' is another weight with $|u - u'| \le \eta$, the density of $\nu_{h'}$ with respect to ν_h lies in $[e^{-2\eta}, e^{2\eta}]$, so the total variation distance of the two is at most $1 - e^{-2\eta} \le 2\eta$, and $W_1^{d_V}(\nu_h, \nu_{h'}) \le 2\eta\Delta_V$.

The coefficients. Let Q, Q' differ only at $j \notin V$, with $\delta = \delta(Q_j, Q_j')$, and let u, u' be the weights for the two exteriors. They differ only through the sets X that contain j and meet V; if there are m_j of them, $|u - u'| \le m_j\varepsilon\delta$. With $\mu = \gamma_V(\cdot \mid Q)$ and $\nu = \gamma_V(\cdot \mid Q')$, the triangle inequality and the two steps give

$$C_{V,j}^W \le e^{\Omega}(1+2\Omega)\, C_{V,j} + 2\Delta_V\, m_j\, \varepsilon.$$

The pairs (X, j) with $j \in X \setminus V$ and $X \cap V \neq \emptyset$ number at most s times the number of sets meeting V, hence at most $sM|V|$. Summing over $j \in c \setminus V$,

$$\begin{aligned}K_c^W &\le e^{\Omega}(1+2\Omega)\,\alpha N_n + 2\Delta_V\, sM|V|\varepsilon\\ &= \big(\alpha e^{\Omega}(1+2\Omega) + 96\, s\,\Omega\big)N_n.\end{aligned}$$

The family is invariant under $2\mathbb{Z}^4$, so the coefficients are, and this is $D^W(n,\alpha')$. □

The admissible perturbation is of order $1/N_n = (2/n)^4$, so the proposition is useful in a box of a fixed number of lattice spacings, which is how a renormalisation argument would use it: after enough block steps that the cube of physical size ℓ has become a cube of fixed side, the effective action has to lie within ε_0 of an action that satisfies the condition there. Whether the effective actions along the tuning do so is not decided here, and that is where the problem lies: at the scale where the

blocked theory is strongly coupled enough for a condition of this kind, the running coupling is of order one, which is outside the range of the existing ultraviolet analysis. The proposition shows only that the condition itself is not the obstacle.

The next theorem uses the configuration in which every link outside V is the identity. Write $\gamma_V^1 = \gamma_V(\cdot \mid 1)$ and

$$\varepsilon_V(\beta) = \frac{1}{|V|} \sum_{i \in V} \int \delta(1, Q_i)\, d\gamma_V^1(Q)$$

for the mean distance of the gauge-fixed variables of the cube from the identity. Recall that δ is bi-invariant of diameter D.

Theorem 262. *Let n be even, V the cube of Definition 42, and suppose that $D(n,\alpha)$ holds at β. Then*

$$\varepsilon_V(\beta) \geq \frac{(1-\alpha)D}{3}.$$

Proof. A flat configuration. By Lemma 9(i) and the period two of σ, the sixteen residues modulo two fall into eight pairs $\{r, r+e_{\sigma(r)}\}$, the residues met by one line of F. The eight residues 0010, 0011, 0100, 0110, 1001, 1011, 1100, 1101 form four of these pairs. Choose $z, w \in G$ with $\delta(z,w) = D$, put $g(x) = w$ if x is congruent to one of those eight residues and $g(x) = z$ otherwise, and let $Q^g_{(x,\mu)} = g(x)g(x+e_\mu)^{-1}$ for every link of $\mathbb{Z}^4$. Then g is constant on every line of F, so Q^g equals 1 on F, every plaquette variable of Q^g is 1, and Q^g has period two. By bi-invariance $\delta(1, Q^g_{(x,\mu)}) = \delta(g(x+e_\mu), g(x))$, which is D when x and $x+e_\mu$ receive different values and 0 otherwise; from Table VI the first case occurs for exactly 32 of the 48 classes. Let δ_c be the common value of $\delta(1, Q^g_e)$ for e in the class c, so that $\sum_c \delta_c = 32D$.

Gauge covariance. Let $\Phi(Q)_{(x,\mu)} = g(x) Q_{(x,\mu)} g(x+e_\mu)^{-1}$ on G^V. The gauge transformation by g maps the configuration equal to Q_V on V and to 1 elsewhere to the one equal to $\Phi(Q_V)$ on V and to Q^g elsewhere; it leaves every plaquette trace unchanged and every link of F equal to 1, and Φ preserves Haar measure. Hence $\gamma_V(\cdot \mid Q^g) = \Phi_* \gamma_V^1$.

Upper bound. Only the finitely many links outside V that share a plaquette with V enter γ_V. Changing them one at a time from 1 to Q^g and applying the definition of $C_{V,j}$ at each step,

$$\begin{aligned} W_1^{d_V}\left(\gamma_V^1, \gamma_V(\cdot \mid Q^g)\right) &\leq \sum_{j \in A \setminus V} C_{V,j}\, \delta(1, Q^g_j) = \sum_c \delta_c K_c \\ &\leq \alpha N_n \sum_c \delta_c = 32\,\alpha D N_n. \end{aligned}$$

Lower bound. The function $f(Q) = -\sum_{i \in V} \delta(Q_i, Q^g_i)$ is 1-Lipschitz for d_V. By gauge covariance and bi-invariance, $\int f\, d\gamma_V(\cdot \mid Q^g) = -\int \sum_i \delta(Q_i, 1)\, d\gamma_V^1 = -|V|\varepsilon_V$, and by the triangle inequality $\int f\, d\gamma_V^1 \leq -\sum_i \delta(1, Q^g_i) + |V|\varepsilon_V$. Every class has N_n links in V, so $\sum_i \delta(1, Q^g_i) = 32 D N_n$ and

$$W_1^{d_V}\left(\gamma_V^1, \gamma_V(\cdot \mid Q^g)\right) \geq 32 D N_n - 2|V|\varepsilon_V.$$

With $|V| = 48 N_n$ the two bounds give $96 N_n \varepsilon_V \geq 32(1-\alpha) D N_n$. □

Corollary 263. *For every even n and every $\alpha < 1$ there is $\beta_0(n,\alpha)$ such that $D(n,\alpha)$ fails at every $\beta \geq \beta_0(n,\alpha)$. In particular, if $D(n_k,\alpha)$ holds at couplings $\beta_k \to \infty$, then $n_k \to \infty$.*

Proof. For $\beta > 0$, γ_V^1 has density proportional to $e^{-N\beta S}$ with respect to Haar measure, where $S = \sum_{p \cap V \neq \emptyset} (N - \operatorname{Re}\operatorname{Tr} Q_p^F)$ is continuous and nonnegative on G^V and vanishes exactly when $Q_p^F = 1$ for every plaquette p that meets V. We show that this forces $Q_V = 1$. The other plaquettes have all their links equal to 1, so every plaquette variable is 1; since the plaquettes of $\mathbb{Z}^4$ fill every closed lattice path, there is a function h on the sites with $Q_{(x,\mu)} = h(x)h(x+e_\mu)^{-1}$ on every link, and h takes equal values at the two ends of every link on which $Q = 1$. Every link with base point outside $[0,n)^4$ is such a link. From a site with a coordinate at least n one can move in every positive direction along links of this kind, and from a site with a negative coordinate x_ν one can move in the positive directions other than ν until another coordinate reaches n; so h is constant on the sites outside the box. Every line of F leaves the box, and h is constant along it, so h is constant and $Q_V = 1$.

Let U be a neighbourhood of 1 in G^V, $s = \min_{G^V \setminus U} S > 0$, and $U' \subset U$ a neighbourhood of 1 on which $S < s/2$. Then $\gamma_V^1(G^V \setminus U) \leq e^{-N\beta s}/(|U'| e^{-N\beta s/2})$, which tends to zero, so $\varepsilon_V(\beta) \to 0$ as $\beta \to \infty$, and Theorem 262 excludes $D(n,\alpha)$ once $\varepsilon_V(\beta) < (1-\alpha)D/3$. □

For SU(2) the constant of Theorem 262 can be improved: a flat configuration with four values instead of two moves every class of active links, and the means of the gauge-fixed links are scalar.

Proposition 264. *Let $G = \mathrm{SU}(2)$ with the chordal metric, let n be even, V the cube of Definition 42, and suppose that $D(n,\alpha)$ holds at β. Then*

$$\frac{1}{|V|} \sum_{i \in V} \left| \int \operatorname{Re}\operatorname{Tr} Q_i\, d\gamma_V^1 \right| \leq 2\alpha,$$

$$\frac{1}{|V|} \sum_{i \in V} \int \delta(1, Q_i)^2\, d\gamma_V^1 \geq 2(1-\alpha),$$

and $\varepsilon_V(\beta) \geq 1 - \alpha$.

Proof. Write $\langle \cdot, \cdot \rangle$ for the Euclidean product of quaternions; then $\operatorname{Re}\operatorname{Tr} q = 2\langle q, 1\rangle$ and $\delta(p,q) = |p-q|$.

Four colours. Split the sixteen residues modulo two into the sets

$$\begin{aligned} X_1 &= \{0000, 1000, 0111, 1111\}, \\ X_2 &= \{0001, 0101, 1010, 1110\}, \\ X_3 &= \{0010, 0011, 1100, 1101\}, \\ X_4 &= \{0100, 0110, 1001, 1011\}. \end{aligned}$$

Each is the union of two of the eight pairs $\{r, r+e_{\sigma(r)}\}$, and Table VI shows that r and $r+e_\mu$ lie in different sets for every residue r and every $\mu \neq \sigma(r)$. (The eight residues used in the proof of Theorem 262 form $X_3 \cup X_4$.) Let

$$\begin{aligned} z_1 &= (\mathbf{i}+\mathbf{j}+\mathbf{k})/\sqrt{3}, \quad z_2 = (\mathbf{i}-\mathbf{j}-\mathbf{k})/\sqrt{3}, \\ z_3 &= (-\mathbf{i}+\mathbf{j}-\mathbf{k})/\sqrt{3}, \quad z_4 = (-\mathbf{i}-\mathbf{j}+\mathbf{k})/\sqrt{3}, \end{aligned}$$

unit quaternions with $\langle z_k, z_l\rangle = -\frac{1}{3}$ for $k \neq l$, so that $\delta(z_k, z_l) = d_* = (8/3)^{1/2}$. Put $g(x) = z_k$ when the residue of x lies in X_k, and $Q^g_{(x,\mu)} = g(x)g(x+e_\mu)^{-1}$. As in the proof of Theorem 262, Q^g equals 1 on F, every plaquette variable of Q^g is 1, and $\gamma_V(\cdot \mid Q^g) = \Phi_* \gamma_V^1$ with $\Phi(Q)_{(x,\mu)} = g(x)Q_{(x,\mu)}g(x+e_\mu)^{-1}$; now $\delta(1, Q^g_e) = d_*$ for every active link e.

Upper bound. Changing the exterior links one at a time,

$$W_1^{d_V}\big(\gamma_V^1, \gamma_V(\cdot \mid Q^g)\big) \le d_* \sum_c K_c \le 48\,\alpha\, d_* N_n = \alpha\, d_* |V|.$$

Lower bound. Conjugating every link of V by one unit quaternion leaves the exterior 1, the links of F, Haar measure and every plaquette trace unchanged, so it preserves γ_V^1. The mean $m_i = \int Q_i \, d\gamma_V^1$, taken in the space of quaternions, therefore commutes with every unit quaternion, and $m_i = a_i 1$ with $a_i = \frac{1}{2}\int \mathrm{Re}\,\mathrm{Tr}\, Q_i \, d\gamma_V^1$ real. Since Φ acts on each link by a fixed left and right multiplication, the mean of Q_i under $\gamma_V(\cdot \mid Q^g)$ is $a_i Q_i^g$. Choose unit quaternions w_i with $\langle w_i, a_i Q_i^g - a_i 1\rangle = |a_i|\,|Q_i^g - 1|$. The function $f(Q) = \sum_{i\in V}\langle w_i, Q_i\rangle$ is 1-Lipschitz for d_V, and Kantorovich duality gives

$$W_1^{d_V}\big(\gamma_V^1, \gamma_V(\cdot \mid Q^g)\big) \ge \sum_{i\in V} |a_i|\,\delta(1, Q_i^g) = d_* \sum_{i\in V} |a_i|.$$

Conclusion. Comparing the two bounds, $\sum_i |a_i| \le \alpha |V|$, which is the first inequality. Since $\delta(1,q)^2 = 2 - 2\langle q, 1\rangle$, $\int \delta(1,Q_i)^2\, d\gamma_V^1 = 2(1-a_i) \ge 2(1-|a_i|)$, and averaging gives the second. As $\delta \le 2$, $\delta \ge \delta^2/2$, and the bound on ε_V follows. □

The four colours need nothing from SU(2) except four elements whose pairwise quotients are conjugate, and every compact simple group contains such elements. For a metric that comes from a representation the bound then compares the cube with Haar measure.

Proposition 265. *Let G be a compact simple group, ρ a nontrivial irreducible unitary representation of G of dimension d, and $\delta(p,q) = \kappa^{-1/2}\|\rho(p) - \rho(q)\|_{\mathrm{HS}}$ for some $\kappa > 0$. Let n be even, V the cube of Definition 42, and suppose that $D(n,\alpha)$ holds at β. Then*

$$\frac{1}{|V|}\sum_{i\in V}\Big|\int \mathrm{Tr}\,\rho(Q_i)\, d\gamma_V^1\Big| \le \alpha\, d,$$

$$\frac{1}{|V|}\sum_{i\in V}\int \delta(1,Q_i)^2\, d\gamma_V^1 \ge (1-\alpha)\int_G \delta(1,q)^2\, dq,$$

where dq is Haar measure, and $\varepsilon_V(\beta) \ge (1-\alpha)D/2$.

Proof. Four elements. Let $\varphi\colon \mathrm{SU}(2) \to G$ be the homomorphism of the SU(2) subgroup of a root, and $z_1, \dots, z_4$ the tetrahedral quaternions of Proposition 264. For $k \neq l$ the quaternion $z_k z_l^{-1}$ has real part $\langle z_k, z_l\rangle = -\frac{1}{3}$; unit quaternions with equal real parts are conjugate in SU(2), so the twelve elements $\varphi(z_k z_l^{-1})$ are conjugate in G, and by bi-invariance $\delta(\varphi(z_k), \varphi(z_l)) = d_*$ for all $k \neq l$, with one number d_*. If $d_* = 0$, then $\varphi(z_k z_l^{-1})$ lies in $\ker\rho$, a proper normal subgroup of the simple group G and therefore central; it then commutes with $\varphi(\mathrm{SU}(2))$, so it is central there, and $z_k z_l^{-1}$ lies in $\{\pm 1\}\ker\varphi \subset \{\pm 1\}$, which its real part $-\frac{1}{3}$ excludes. So $d_* > 0$.

The two bounds. Put $g(x) = \varphi(z_k)$ when the residue of x lies in X_k. The proof of Proposition 264 applies word for word: Q^g equals 1 on F and on every plaquette, $\delta(1, Q^g_e) = d_*$ on every active link, and $W_1^{d_V}(\gamma_V^1, \gamma_V(\cdot \mid Q^g)) \le \alpha d_* |V|$. Simultaneous conjugation of the links of V preserves γ_V^1, so the mean $\int \rho(Q_i)\, d\gamma_V^1$ commutes with $\rho(G)$, and by Schur's lemma it is $c_i I$ with $c_i = d^{-1}\int \mathrm{Tr}\,\rho(Q_i)\, d\gamma_V^1$. Its mean under $\gamma_V(\cdot \mid Q^g)$ is $c_i \rho(Q_i^g)$. With matrices W_i of Hilbert–Schmidt norm $\kappa^{-1/2}$ chosen so that $\mathrm{Re}\,\mathrm{Tr}(W_i^* c_i(\rho(Q_i^g) - I)) = |c_i|\,\delta(1, Q_i^g)$, the function $f(Q) = \sum_i \mathrm{Re}\,\mathrm{Tr}(W_i^* \rho(Q_i))$ is 1-Lipschitz for d_V, and Kantorovich duality gives $W_1^{d_V} \ge d_* \sum_i |c_i|$. Hence $\sum_i |c_i| \le \alpha |V|$, the first inequality.

Comparison with Haar measure. Since $\kappa\,\delta(1,q)^2 = 2d - 2\,\mathrm{Re}\,\mathrm{Tr}\,\rho(q)$ and $\int_G \mathrm{Tr}\,\rho(q)\, dq = 0$ for a nontrivial irreducible ρ, $\int_G \delta(1,q)^2 dq = 2d/\kappa$, while $\int \delta(1,Q_i)^2 d\gamma_V^1 = 2d(1 - \mathrm{Re}\, c_i)/\kappa \ge 2d(1-|c_i|)/\kappa$. Averaging gives the second inequality. Finally $\kappa D^2 \le 4d$, because $\|\rho(p) - \rho(q)\|_{\mathrm{HS}} \le 2\sqrt{d}$, so $\delta \ge \delta^2/D$ gives $\varepsilon_V \ge 2d(1-\alpha)/(\kappa D) \ge (1-\alpha)D/2$. □

For SU(N) with the Hilbert–Schmidt distance this improves the constant $\frac{1}{3}$ of Theorem 262 to $\frac{1}{2}$, and for SU(3), where $D^2 = 9$ and the Haar mean of δ^2 is 6, to $\frac{2}{3}$: in a cube with the identity outside, $D(n,\alpha)$ leaves the gauge-fixed links at least the fraction $1-\alpha$ of the disorder of independent Haar-distributed links.

The size $\beta_0(n,\alpha)$ in Corollary 263 comes from a compactness argument. For SU(2) it can be made explicit, and then reads as a lower bound on the side of any cube that satisfies the condition.

Proposition 266. *Let $G = \mathrm{SU}(2)$ with the chordal metric, so that $D = 2$, let $b = \beta_W \ge \frac{1}{9}$, let $n \ge 2$ be even and put $L(b) = 1 + \log 24 + \frac{3}{2}\log(9b)$. Then*

$$\frac{1}{|V|}\sum_{e\in V}\int \delta(1,Q_e)^2\, d\gamma_V^1 \le \frac{n^2(n+2)(2n+1)\,L(b)}{b},$$

so $\varepsilon_V(\beta)^2$ is at most the same quantity. Consequently, if $D(n,\alpha)$ holds at b, then

$$n^4 \ge \frac{2(1-\alpha)\,b}{5\,L(b)}.$$

Proof. Put $m = n/2$. Under γ_V^1 every link outside V equals 1: the links of F always, the others by the choice of exterior. A plaquette that does not meet V therefore has plaquette variable 1.

Links as loops. Let $e = (x,\mu) \in V$, $y = x + e_\mu$, $\sigma = \sigma(x)$ and $\tau = \sigma(y)$. The directions σ, τ, μ are distinct: $\sigma \neq \mu$ because e is active; $\tau \neq \mu$ because otherwise the line of F through y would contain x, and the lines of F are disjoint; and $\tau \neq \sigma$ by Table VI, in which $\sigma(x+e_\mu) \neq \sigma(x)$ whenever $\mu \neq \sigma(x)$. Since $0 \le x_\sigma < n$, one of $n - x_\sigma$ and $x_\sigma + 1$ is at most m; call it s and let $\varepsilon = \pm 1$ be the sign for which $x' = x + s\varepsilon e_\sigma$ has

σ-coordinate outside $[0,n)$. Choose $t \le m$ and ε' in the same way for y and τ (note $y_\tau = x_\tau$). Let R_1 be the s plaquettes of the rectangle with corners x, x', $x'+e_\mu$, y in the plane $\{\sigma,\mu\}$, and R_2 the st plaquettes of the rectangle with corners y, $x'+e_\mu$, $x'+e_\mu+t\varepsilon' e_\tau$, $y+t\varepsilon' e_\tau$ in the plane $\{\sigma,\tau\}$. The union $R(e) = R_1 \cup R_2$ is a disc, and its boundary runs from x along the line of F through x to x', then to $x'+e_\mu$, along e_τ to $x'+e_\mu+t\varepsilon' e_\tau$, along e_σ to $y+t\varepsilon' e_\tau$, back along the line of F through y to y, and over e to x. The links of the first and fifth pieces lie in F. The links of the second and third pieces have base points with σ-coordinate $x'_\sigma \notin [0,n)$, and those of the fourth have base points with τ-coordinate outside $[0,n)$; none of them lies in V. So the holonomy of the boundary of $R(e)$ is $Q_e^{\pm 1}$. Adding the plaquettes of a disc one at a time along its boundary shows that this holonomy is a product of conjugates of the plaquette variables $(Q_p^F)^{\pm 1}$, $p \in R(e)$, one for each plaquette. By bi-invariance and the triangle inequality,

$$\delta(1,Q_e) \le \sum_{p \in R(e)} \delta(1,Q_p^F), \qquad |R(e)| = s(t+1) \le m(m+1).$$

Multiplicity. If a plaquette p lies in $R_2(e)$, its plane is $\{\sigma,\tau\}$, μ is one of the two other directions, and the base point of p is $y + ae_\sigma + ce_\tau$ with $-m \le a,c \le m-1$. Given p, there are two ways to name the directions of its plane, two choices of μ and $(2m)^2$ of (a,c), and they determine y, hence e; so p lies in at most $16m^2$ of the sets $R_2(e)$. Likewise $p \in R_1(e)$ forces its plane to be $\{\sigma,\mu\}$ and its base point to be $x + ae_\sigma$ with $-m \le a \le m-1$, which leaves at most $4m$ links e. Every plaquette therefore lies in at most $16m^2 + 4m = 4n^2 + 2n$ of the discs $R(e)$.

The energy. For the chordal metric $\delta(1,q)^2 = 2 - \operatorname{Re}\operatorname{Tr} q$. Let $S = \sum_{p \cap V \neq \emptyset} \delta(1,Q_p^F)^2$; then γ_V^1 has density $e^{-bS/2}/Z$ with $Z = \int e^{-bS/2}$, the integral over $\mathrm{SU}(2)^V$ with Haar measure. The function $t \mapsto \log \int e^{-tS}$ is convex and vanishes at $t = 0$, so its derivative at $t = b/2$, which is $-\int S\, d\gamma_V^1$, is at least $2\log Z/b$. A plaquette that meets V contains at most three links of V and the other links are 1, so $\delta(1,Q_p^F)^2 \le 3\sum_{e \in p \cap V} \delta(1,Q_e)^2$; every link lies in six plaquettes, so $S \le 18\sum_{e \in V} \delta(1,Q_e)^2$ and $Z \ge I^{|V|}$ with $I = \int e^{-9b\,\delta(1,q)^2} dq$. The chordal distance is at most the geodesic one, and the geodesic ball of radius $r = (9b)^{-1/2} \le 1$ has measure greater than $r^3/24$ (proof of Proposition 272), so $I \ge e^{-1}(9b)^{-3/2}/24$ and

$$\int S\, d\gamma_V^1 \le \frac{2|V|\,L(b)}{b}.$$

Conclusion. The Cauchy–Schwarz inequality inside each disc squares the loop bound:

$$\begin{aligned}\delta(1,Q_e)^2 &\le |R(e)| \sum_{p \in R(e)} \delta(1,Q_p^F)^2 \\ &\le m(m+1) \sum_{p \in R(e)} \delta(1,Q_p^F)^2.\end{aligned}$$

Summing over $e \in V$ and using the multiplicity,

$$\sum_{e \in V} \delta(1,Q_e)^2 \le m(m+1)(4n^2+2n)\,S = \frac{n^2(n+2)(2n+1)}{2}\,S,$$

and integrating against γ_V^1 gives the first bound; the bound on ε_V follows by the Cauchy–Schwarz inequality. If $D(n,\alpha)$ holds, Proposition 264 gives $2(1-\alpha) \le n^2(n+2)(2n+1)L(b)/b$, and $(n+2)(2n+1) \le 5n^2$ for $n \ge 2$. □

The loops and their multiplicity use nothing of $\mathrm{SU}(2)$ but the bi-invariance of δ, and the energy uses only the form of the action and the measure of small balls. The bound therefore holds for every $\mathrm{SU}(N)$, with the same power of n.

Proposition 267. *Let $G = \mathrm{SU}(N)$, $N \ge 2$, with the weight $\exp(\frac{b}{N}\sum_p \operatorname{Re}\operatorname{Tr} Q_p)$, $b = \beta_W \ge N/9$, and the Hilbert–Schmidt distance $\delta(p,q) = \|p-q\|_{\mathrm{HS}}$. Let $n \ge 2$ be even and put*

$$L_N(b) = 1 + \frac{N^2-1}{2}\log(81\pi^2 b) + N(N-1)\log\frac{12}{11}.$$

Then

$$\frac{1}{|V|}\sum_{e \in V} \int \delta(1,Q_e)^2\, d\gamma_V^1 \le \frac{N n^2 (n+2)(2n+1)\,L_N(b)}{b}.$$

Consequently, if $D(n,\alpha)$ holds at b, then

$$n^4 \ge \frac{2(1-\alpha)\,b}{5\,L_N(b)}.$$

Proof. Loops. The discs $R(e)$ of the proof of Proposition 266 and the bound $4n^2+2n$ on their multiplicity do not depend on the group, and the loop bound uses only bi-invariance and the triangle inequality. With $S = \sum_{p \cap V \neq \emptyset} \delta(1,Q_p^F)^2$ the same Cauchy–Schwarz step gives $\sum_{e \in V} \delta(1,Q_e)^2 \le \frac{1}{2} n^2 (n+2)(2n+1)\,S$.

The energy. Since $\delta(1,q)^2 = 2N - 2\operatorname{Re}\operatorname{Tr} q$, the measure γ_V^1 has density $e^{-bS/(2N)}/Z$ with respect to Haar measure. As in the proof of Proposition 266, the convexity of $t \mapsto \log \int e^{-tS}$ gives $\int S\, d\gamma_V^1 \le 2N \log(1/Z)/b$, and $S \le 18\sum_{e \in V} \delta(1,Q_e)^2$ gives $Z \ge I^{|V|}$ with $I = \int e^{-9b\,\delta(1,q)^2/N} dq$.

Small balls. Give $\mathfrak{su}(N)$ the inner product $\operatorname{Tr}(X^* Y)$ and $\mathrm{SU}(N)$ the corresponding bi-invariant metric, of volume v_N. In exponential coordinates the Riemannian measure is $j(X)\,dX$, where dX is Lebesgue measure and

$$j(X) = \left|\det \frac{1 - e^{-\operatorname{ad} X}}{\operatorname{ad} X}\right| = \prod_{k<l} \left(\frac{\sin(\varphi_{kl}/2)}{\varphi_{kl}/2}\right)^2$$

if X has eigenvalues $i\theta_1, \dots, i\theta_N$ and $\varphi_{kl} = \theta_k - \theta_l$ [66, Ch. II, Thm. 1.7]; in particular $j \le 1$. Every $q \in \mathrm{SU}(N)$ is conjugate to a diagonal matrix with entries $e^{i\psi_k}$, $-\pi \le \psi_k < \pi$; then $\sum_k \psi_k = 2\pi k_0$ with $|k_0| \le N/2$, and changing $|k_0|$ of the ψ_k by $\mp 2\pi$ gives angles θ_k with sum zero and $|\theta_k| \le 3\pi$. So $q = e^X$ with $X \in \mathfrak{su}(N)$ and $\|X\|_{\mathrm{HS}} \le 3\pi\sqrt{N}$, and $v_N \le \omega_{N^2-1}(3\pi\sqrt{N})^{N^2-1}$, where ω_k is the volume of the unit ball of $\mathbb{R}^k$. Now let $\|X\|_{\mathrm{HS}} \le r \le 1$. The eigenvalues have $|\theta_k| \le 1 < \pi$, so exp is injective on this ball; $\delta(1,e^X)^2 = \sum_k |e^{i\theta_k} - 1|^2 \le \|X\|_{\mathrm{HS}}^2$; and $|\varphi_{kl}|/2 \le (\theta_k^2 + \theta_l^2)^{1/2}/\sqrt{2} \le 2^{-1/2}$, so each factor of $j(X)$ is at least $(1 - \frac{1}{12})^2$ because $\sin x/x \ge 1 - x^2/6$.

Dividing by v_N, the Haar measure of $\{q : \delta(1,q) \le r\}$ is at least

$$\Big(\frac{11}{12}\Big)^{N(N-1)} \frac{\omega_{N^2-1} r^{N^2-1}}{v_N} \ge \Big(\frac{11}{12}\Big)^{N(N-1)} \Big(\frac{r}{3\pi\sqrt{N}}\Big)^{N^2-1}.$$

With $r = (N/9b)^{1/2} \le 1$ this gives $I \ge e^{-1}$ times the right-hand side, and since $3\pi\sqrt{N}/r = 9\pi\sqrt{b}$, $\log(1/I) \le L_N(b)$. Hence $\int S\, d\gamma_V^1 \le 2N|V|L_N(b)/b$, and the first bound follows from the loops.

Conclusion. Proposition 265 with the defining representation and $\kappa = 1$, for which the Haar mean of $\delta(1,q)^2$ is $2N$, gives $2N(1-\alpha) \le N n^2 (n+2)(2n+1) L_N(b)/b$ when $D(n,\alpha)$ holds, and $(n+2)(2n+1) \le 5n^2$. □

For every N, a cube on which the condition holds therefore has side at least of order $(\beta_W / \log \beta_W)^{1/4}$, with a constant that depends on N only through L_N, which grows like $\frac{N^2}{2} \log \beta_W$.

The cube condition is a statement about strong coupling at its own scale, and this persists under the perturbations of Proposition 261 when they respect the gauge. Call (w_X) *gauge invariant* if $w_X(Q^h) = w_X(Q)$ for every function h on the sites that is constant on every line of F, where $Q^h_{(x,\mu)} = h(x) Q_{(x,\mu)} h(x+e_\mu)^{-1}$. Such transformations keep the links of F equal to 1, so they act on G^A, and the restriction to the gauge of an action invariant under all gauge transformations is gauge invariant in this sense. Write ε_V^W for ε_V with $\gamma_V^W(\cdot \mid 1)$ in place of γ_V^1.

Proposition 268. *Let (w_X) be as in Proposition 261 and gauge invariant, let $n = 2m$, and suppose that $D^W(n,\alpha)$ holds.*

(i) $\varepsilon_V^W \ge (1-\alpha) D/3$.

(ii) *If $\eta \ge 0$, $0 \le \kappa \le 1$, and the event that $\delta(1, Q_p^F) \le \eta$ for every plaquette p that meets V has probability at least $1-\kappa$ under $\gamma_V^W(\cdot \mid 1)$, then*

$$m(m+1)\,\eta + \kappa D \ge \frac{(1-\alpha)D}{3}.$$

Proof. (i) The proof of Theorem 262 uses the action only through the identity $\gamma_V(\cdot \mid Q^g) = \Phi_* \gamma_V^1$ and through the coefficients $C_{V,j}$. The function g of that proof is constant on every line of F, so the gauge transformation by g leaves every w_X unchanged, as it leaves every plaquette trace, and the same identity holds for γ_V^W. Only finitely many links outside V enter γ_V^W, because the sets X have bounded diameter, so the upper bound holds with $C_{V,j}^W$ and K_c^W in place of $C_{V,j}$ and K_c; the lower bound is unchanged.

(ii) Under $\gamma_V^W(\cdot \mid 1)$ every link outside V equals 1, so a plaquette that does not meet V has variable 1. The loop bound of the proof of Proposition 266, which uses only the bi-invariance of δ, gives $\delta(1,Q_e) \le \sum_{p \in R(e)} \delta(1, Q_p^F) \le m(m+1)\eta$ for every $e \in V$ on the event of the statement, and $\delta(1,Q_e) \le D$ off it. Averaging over $e \in V$ and integrating, $\varepsilon_V^W \le m(m+1)\eta + \kappa D$, and (i) completes the proof. □

In the small-field region of a gauge-covariant block renormalisation the blocked plaquette variables lie within a distance η that tends to zero with the running coupling, and large fields have small probability κ. If $\kappa < (1-\alpha)/3$, part (ii) allows the cube condition on the blocked lattice only when $m(m+1) \ge x$, where $x = ((1-\alpha)/3 - \kappa) D/\eta$, so the side must grow like $\eta^{-1/2}$. On such a cube $N_n = m^4 \ge x^2/4$, and since $\alpha' \ge \alpha + 96 s \Omega$, Proposition 261 gives $\alpha' < 1$ only for

$$\varepsilon < \frac{1-\alpha}{4608\, sMDN_n} \le \frac{(1-\alpha)\,\eta^2}{1152\, sMD^3 \big((1-\alpha)/3 - \kappa\big)^2},$$

which is of order η^2. At a fixed side the condition is therefore a statement about strong coupling, and a renormalisation argument has to carry the blocked theory out of the small-field regime before the condition can be checked. That passage, at a running coupling of order one, is the step that remains open.

The next statement shows that the cube condition separates a theory with a massless photon from the expected behaviour of $\mathrm{SU}(N)$: for the compact abelian theory in its Coulomb phase it fails at every scale. Let $G = \mathrm{U}(1)$ with the chordal metric, $D = 2$, and replace the Wilson weight by the Villain weight

$$\varphi_\beta(\theta) = \sum_{k \in \mathbb{Z}} e^{-\beta(\theta + 2\pi k)^2/2},$$

so that the measure of a box Λ has density proportional to $\prod_{p \in P(\Lambda)} \varphi_\beta(\theta_p)$, where $e^{i\theta_p}$ is the plaquette variable. Define γ^F, $C_{V,j}$, K_c and $D(n,\alpha)$ as before with φ_β in place of the Wilson weight.

Proposition 269. *There is β_0 such that for the Villain theory at every $\beta > \beta_0$ the condition $D(n,\alpha)$ fails for every even $n \ge 2$ and every $\alpha < 1$. In particular, for couplings $\beta_k \to \infty$ and any even n_k, the hypothesis of Theorem 45 fails for this theory.*

Proof. Lemma 10, Proposition 11(i)–(iii) and (v), Lemma 665, Corollary 666 and Lemma 668 use the Wilson weight only as a positive continuous class function of the plaquette variable, with one exception: the Doeblin bound in Lemma 668(a), which for the Villain weight reads that each single-link density is at least $(\min \varphi_\beta / \max \varphi_\beta)^6 > 0$ times its maximum. Suppose that $D(n,\alpha)$ holds. As in the proof of Theorem 43, the limit $\omega(O) = \lim W_\Lambda(O)$ of the measures of boxes with free boundary conditions then exists along every sequence of boxes exhausting $\mathbb{Z}^4$, and for gauge-invariant local O, O' with O Lipschitz in each link,

$$|\mathrm{Cov}_\omega(O,O')| \le \|O'\|_\infty L_O \frac{D(1 + c_n e^{\lambda \rho_n})}{1-\theta} |S_{O'}|\, e^{-\lambda\, \mathrm{dist}(O,O')}$$

for every $\lambda > 0$ with $\theta = \alpha e^{\lambda \rho_n} < 1$, where L_O is a Lipschitz constant of O and $S_{O'}$ the support of O'.

Put $\Psi_p = \varphi_\beta'(\theta_p)/\varphi_\beta(\theta_p)$; up to the factor $-\mathbf{i}$ this is the lattice field strength of Fröhlich and Spencer [55, Sect. 2] for the Villain action. It is real, odd and smooth in θ_p, so it is gauge invariant, local and Lipschitz in each of its four links, with

constants that do not depend on p, and $\omega(\Psi_p)=0$ by the symmetry $\theta\mapsto-\theta$. The bound above gives $\sum_{p'}|\omega(\Psi_p\Psi_{p'})|<\infty$, the sum over the plaquettes p' parallel to p. The state of [55] is the limit of the same free-boundary measures over boxes, which is ω. Fröhlich and Spencer prove, for β larger than some β_0, that the Fourier transform of the two-plaquette function of the field strength is discontinuous at zero momentum, so that this sum is infinite [55, Sect. 2]; see also [64]. This contradiction proves the first statement, and the second follows because $\beta_k>\beta_0$ for all large k. □

At large β the plaquette variables of the abelian theory are Gaussian to leading order, as are those of SU(N), and every property of the action used in this paper (compactness, locality, positivity and smoothness of the weight, reflection positivity, the forest gauge) is shared by the two theories. A proof of (D) along the continuum tuning therefore has to use the non-abelian structure of G; Proposition 269 holds at every cube size, so letting the side grow with β, which Corollary 263 already requires, does not remove the need. Fröhlich and Spencer indicate how their result extends to the Wilson action without giving the details [55, Sect. 2], so the proposition is stated for the Villain action only.

Remark 270. The proof loses the area of a disc, of order n^2, in passing from plaquettes to links, and on loops that loss is of the right order. On the periodic lattice $(\mathbb{Z}/L\mathbb{Z})^4$ with L even and coordinates $0\le x_j<L$, let $\phi=4\pi/L^2$, $q(\theta)=\cos\theta+\mathbf{i}\sin\theta$, and put $U_{(x,2)}=q(\phi x_1)$, $U_{(x,1)}=q(-\phi L x_2)$ when $x_1=L-1$, and $U=1$ on every other link. Every plaquette of the plane $\{1,2\}$ equals $q(\phi)$, including those on the seam $x_1=L-1$ and at the corner $x_1=x_2=L-1$, where the angle is $\phi-4\pi$; all other plaquettes equal 1. So every plaquette variable lies within ϕ of the identity, while the boundary of the square $[0,L/2]^2$ of the plane $\{1,2\}$ has holonomy $q(\phi L^2/4)=-1$, at the largest possible distance $D=2$, against the bound $\phi L^2/4=\pi$ given by the disc. Every staple sum of this configuration is parallel to its link, so it is a stationary point of the Wilson action, and gradient flow [84] does not change it. Plaquette variables within $O(n^{-2})$ of the identity therefore do not force loops of length n near the identity, and an argument of the kind above that places gauge-fixed links near the identity needs plaquette bounds of order $o(n^{-2})$.

Up to the factor $L(b)^{1/2}$ the rate $b^{-1/2}$ of Proposition 266 is the right one, and the lower bound below holds for every n and every exterior.

Proposition 271. *Let $G=\mathrm{SU}(2)$ with the chordal metric and $b=\beta_W>0$. For every Gibbs specification of the gauge-fixed theory, every active link e and every configuration of the other links, the conditional law of Q_e satisfies*

$$\int\delta(1,Q_e)\,d\mu_e\ge\frac{1}{9\max(6b,1)^{1/2}}.$$

In particular $\varepsilon_V(\beta)\ge\frac19\max(6b,1)^{-1/2}$ for every n; at every fixed n, $\varepsilon_V(\beta)$ lies between constant multiples of $b^{-1/2}$ and $(L(b)/b)^{1/2}$.

Proof. Write $\langle\cdot,\cdot\rangle$ for the Euclidean product of quaternions, so that $\mathrm{Re}\,\mathrm{Tr}\,q=2\langle q,1\rangle$. The six plaquettes that contain e contribute $\exp(b\langle Q_e,W_p\rangle)$ to the conditional density, with $|W_p|=1$; so μ_e has density $e^{\langle u,v\rangle}/Z(|v|)$ with respect to normalised Haar measure, for some v with $a=|v|\le 6b$. Put $a'=\max(a,1)$ and $\eta=a'^{-1/2}\le1$. On the geodesic ball of radius η about $v/|v|$ the exponent is at least $a(1-\eta^2/2)\ge a-\frac12$, and that ball has measure greater than $\eta^3/24$ (proof of Proposition 272); hence $Z(a)\ge e^{a-1/2}\eta^3/24$ and the density is at most $24e^{1/2}a'^{3/2}$ (for $a<1$ directly, since $Z\ge1$ by Jensen's inequality). A chordal ball of radius r is a geodesic ball of radius θ with $2\sin(\theta/2)=r$, so $\theta\le\pi r/2$, and its measure is $(\theta-\sin\theta\cos\theta)/\pi\le2\theta^3/(3\pi)\le\pi^2r^3/12$ (Lemma 678). With $r=\frac29a'^{-1/2}$,

$$\mu_e\big(\delta(1,Q_e)<r\big)\le24e^{1/2}a'^{3/2}\,\frac{\pi^2r^3}{12}=2e^{1/2}\pi^2\Big(\frac29\Big)^3<\frac12,$$

so the mean of $\delta(1,Q_e)$ is at least $r/2=\frac19a'^{-1/2}$. Averaging over $e\in V$ and over the configuration gives the bound on ε_V. □

Along the continuum tuning Theorem 262 says the following. If (D) holds, then in a cube of fixed physical size ℓ with the identity outside, the forest-gauge variables stay on average at distance at least $(1-\alpha)D/3$ from the identity, and at least $1-\alpha$ for SU(2), at every spacing. This is not a contradiction: $n=\ell/a$ grows like $1/a$, while asymptotic freedom makes $\beta(a)$ grow only like $\log(1/a)$, and the corollary concerns fixed n. The size bound of Proposition 266 is likewise met along the tuning, since its right side grows only like $\log(1/a)$. It turns the hypothesis of Remark 185, that the gauge-fixed links decorrelate on the scale of the cube, into a necessary condition: (D) requires the forest-gauge variables to be disordered at a fixed physical scale. For gauge-invariant observables of a massive theory that disorder is clustering; for the gauge-fixed variables it is not proved.

The last statement concerns the following strategy: control the conditional law of one line of links, given everything else, uniformly in the configuration around it, and assemble the cube from lines. Fix a site x, let $s=\sigma(x)$ and $\mu\ne s$, and let $c_k=(x+ke_s,\mu)$, $k\in\mathbb{Z}$. All c_k are active, since σ is constant along the line through x.

Proposition 272. *Let $G=\mathrm{SU}(2)$, identified with the unit quaternions, and $b=\beta_W\ge1$. There is a configuration of the active links outside $\{c_k\}$, independent of b, for which the conditional law of $(Q_{c_k})_k$ has a ten-step transfer operator, similar to a positive self-adjoint operator K_b on $L^2(S^3)$ with simple top eigenvalue Λ_b, whose gap satisfies*

$$0<1-\frac{\Lambda_{b,1}}{\Lambda_b}\le e^{20}\,24^{10}\,b^{15}\,e^{-b/1280},$$

where $\Lambda_{b,1}$ is the next point of the spectrum.

Proof. The interactions. By Lemma 9(iii), c_k lies in two plaquettes of the plane $\{\mu,s\}$, which contain c_{k-1} or c_{k+1}, the link $(x+ke_s,s)$ or $(x+(k-1)e_s,s)$ of F, and an active twist link

of direction s; and in four plaquettes of the planes $\{\mu,t\}$ and $\{\mu,\rho\}$, each containing c_k, one link of F and two active links not on the line. The four plaquettes at c_k lie in the hyperplane of s-coordinate x_s+k, so these plaquettes at different k share no link; two plaquettes through c_k share no other link; and their links have directions other than s, so they are not twists. Set every twist to 1; then the plaquette at $x+ke_s$ in the plane $\{\mu,s\}$ contributes $b\langle q_k,q_{k+1}\rangle$ to the exponent, with $q_k=Q_{c_k}$. In each of the four other plaquettes at c_k set one of its two links to 1 and the other to X; after a cyclic rotation and possibly an inversion the plaquette variable is $q_kX^{\pm1}$ or $q_k^{-1}X^{\pm1}$, and since $\mathrm{Re}(q\bar u)=\mathrm{Re}(\bar q u)=\langle q,u\rangle$ for unit quaternions, X can be chosen so that the plaquette contributes $b\langle q_k,u\rangle$ for any prescribed unit vector u. Every other active link is set to 1. Four unit vectors add up to any vector of length at most 4 (two copies each of $t\pm(1-|t|^2)^{1/2}\nu$ with $t=v/4$ and $\nu\perp t$ a unit vector), so the conditional density of the line is proportional to $\exp\bigl(b\sum_k\langle q_k,q_{k+1}\rangle+\sum_k\langle h_k,q_k\rangle\bigr)$ for any fields h_k with $|h_k|\le 4b$.

The fields. Let $a=\frac12$, $c=\frac{\sqrt3}{2}$, let $e_1,\dots,e_4$ be the standard basis of $\mathbb{R}^4$, and let r_k have period ten with $(r_0,\dots,r_9)=a(e_1,-e_1,e_2,-e_2,e_3,-e_3,e_3,-e_2,e_2,-e_1)$. Put $h_k=b(2r_k-r_{k-1}-r_{k+1})$, so $|h_k|\le 4ab=2b$, and the configuration is independent of b because h_k/b is. With $v_k=q_k-r_k$ and $|q_k|=1$, $|r_k|=a$, summation by parts gives, over $0\le k\le 9$,

$$\frac12\sum_k|v_{k+1}-v_k|^2=\sum_k\bigl(1-\langle q_k,q_{k+1}\rangle\bigr)-\frac1b\sum_k\langle h_k,q_k\rangle+\frac12\sum_k|r_{k+1}-r_k|^2+B(q_0)-B(q_{10}),$$

with $B(q)=\langle q,r_0-r_9\rangle$. The boundary terms only conjugate the ten-step transfer kernel by a multiplication operator, so up to similarity it is

$$K_b(x,y)=\int_{(S^3)^9}\exp\Bigl(-\frac{b}{2}\sum_{k=0}^{9}|v_{k+1}-v_k|^2\Bigr)dq_1\cdots dq_9,$$

with $q_0=x$, $q_{10}=y$ and normalised Haar measure. The sequence $r_0,\dots,r_{10}$ is a palindrome, so $K_b=B_bB_b^*$ for the five-step kernel B_b; K_b is compact, positive and self-adjoint with a continuous positive kernel, its top eigenvalue Λ_b is simple, and its eigenfunction $\psi_b>0$ is even under the reflection $x_4\mapsto -x_4$, which commutes with K_b.

A barrier. Let $E=\frac12\sum_{k=0}^{9}|v_{k+1}-v_k|^2$ and suppose that $(q_0)_4\ge 0\ge(q_{10})_4$ and $E\le\frac{1}{1280}$. Put $\Delta=(20E)^{1/2}\le\frac18$ and $w=v_0$; by Cauchy–Schwarz $|w-v_k|\le\Delta$ for all k, and $0\le w_4\le\Delta$ because the r_k have no fourth coordinate. From $|v_k+r_k|=1$, $\bigl||w+r_k|^2-1\bigr|\le\Delta(2+\Delta)=:\eta$ for $0\le k\le 5$. The pairs $r_k=\pm ae_l$ give $|w_l|\le\eta/(2a)=\eta$ for $l=1,2,3$ and $|w|^2\ge\frac34-\eta$, so $\frac34\le 3\eta^2+\Delta^2+\eta\le 3(\frac{17}{64})^2+\frac{1}{64}+\frac{17}{64}=\frac{2019}{4096}$, a contradiction. Hence $K_b(x,y)\le e^{-b/1280}$ whenever $x_4\ge 0\ge y_4$.

The top eigenvalue. Let $\varepsilon=b^{-1/2}$. The geodesic ball of radius $\varepsilon\le 1$ in S^3 has normalised measure $\frac{2}{\pi}\int_0^{\varepsilon}\sin^2\theta\,d\theta\ge\frac{8\varepsilon^3}{3\pi^3}>\frac{\varepsilon^3}{24}$. If each q_k lies in the ball of radius ε about r_k+ce_4, then $|v_k-ce_4|\le\varepsilon$ and $E\le 20\varepsilon^2$, so the integrand is at least e^{-20}. Testing K_b on the normalised indicator of the ball about r_0+ce_4 gives $\Lambda_b\ge e^{-20}(\varepsilon^3/24)^{10}=e^{-20}24^{-10}b^{-15}$.

The gap. Let $\phi(x)=\mathrm{sign}(x_4)\psi_b(x)$, a unit vector orthogonal to ψ_b. Then

$$\Lambda_b-\langle\phi,K_b\phi\rangle=4\int_{x_4>0>y_4}\psi_b(x)K_b(x,y)\psi_b(y)\,dx\,dy\le e^{-b/1280},$$

because $\int_{x_4>0}\psi_b\le\frac12$ by Cauchy–Schwarz. By the variational principle $\Lambda_{b,1}\ge\langle\phi,K_b\phi\rangle$, and dividing by the lower bound on Λ_b proves the estimate; the gap is positive because Λ_b is simple. □

The exterior of Proposition 272 is legal at every coupling, so a bound on the conditional law of single lines that is uniform in the exterior cannot be better than $e^{-b/1280}$ up to powers of b, and an argument of this kind cannot give (D) along the tuning. A proof of (D) has to use what such a bound discards: the joint resampling of the transverse links inside the cube, which the averaged coefficients $C_{V,j}$ retain, or the smallness of the probability of such exteriors under the Wilson measure.

H. The limit at separated points

This subsection treats (E) for local fields without flow, under the averaged condition of Theorem 66. Two features separate it from Sec. XI D. A local field without flow is covariant under the whole hypercubic group B_4, so reflection positivity and the gap pass from the time direction to every coordinate direction; the limit then clusters in every direction at the rate m_* without the tree bounds (Theorem 274(v)), and by Proposition 205(iv) one rotation of infinite order gives $\mathrm{O}(4)$. On the other hand, asymptotic freedom predicts for the action density a two-point function of order $|x-y|^{-8}$ at short distance, up to logarithms (the discussion after Proposition 40), and such a two-point function is incompatible with the moment bound (M) at coinciding supports (Proposition 278). The moment bound is therefore posed for separated supports, on which it determines distributions on the complement of the coinciding points (Theorem 274(i)). The Schwinger functions of [95] are defined on test functions that vanish, with all their derivatives, where two arguments coincide.

Throughout, the hypothesis of Theorem 66 holds, ω_k is the torus limit there and $m_*=16\varpi/(625e^2\ell)$. By Corollary 65, ω_k is invariant under translations and B_4, reflection positive for the reflections in coordinate hyperplanes through sites and bisecting links, and its transfer matrix $\mathbb{T}_k$ has spectrum on $\Omega_k^{\perp}$ in $[0,e^{-m_*a_k}]$, since $m_{n_k}\ge 16\varpi/(625e^2n_k)\ge m_*a_k$. Let $r\ge 1$ be an integer and Φ a real continuous gauge-invariant function of the links with both endpoints in $[-r,r]^4$, invariant under the action of B_4 on these links described in the proof of Theorem 201(ii); the sum of $N-\mathrm{Re}\,\mathrm{Tr}\,U_p$ over the 24 plaquettes p with a corner at 0 is an example. Write Φ_x for its translate by $x\in\mathbb{Z}^4$, so that $g\Phi_x=\Phi_{gx}$ for $g\in B_4$, and $\tau_v\Phi_x=\Phi_{x+v}$ for

the translation τ_v, $v \in \mathbb{Z}^4$. In this subsection

$$A_k(f) = \sum_{x\in\mathbb{Z}^4} a_k^4 f(a_k x)\big(\Phi_x - c_k\big), \qquad c_k = \omega_k(\Phi_0),$$

for $f \in C_c(\mathbb{R}^4)$, and $Z_k > 0$ are renormalisation constants.

Let e_μ be the unit vectors, θ_μ the reflection $x_\mu \mapsto -x_\mu$ of $\mathbb{R}^4$, $\pi_\mu \in B_4$ the transposition of the axes 0 and μ ($\pi_0 = 1$), ϑ the lattice reflection of Appendix F, $\vartheta_\mu = \pi_\mu \vartheta \pi_\mu$, and $\tau_t^\mu = \tau_{te_\mu}$. Let $\theta_{\mu,k}$ be the reflection of $\mathbb{R}^4$ with $\theta_{\mu,k}(a_k x) = a_k \vartheta_\mu(x)$ for $x \in \mathbb{Z}^4$; as in the proof of Theorem 57(iii) it is θ_μ followed by a translation along e_μ of length at most a_k, so the plane fixed by ϑ_μ is $\{x_\mu = c\}$ with $|c| \le \frac12$. For a cube Q and $\delta > 0$ let $\mathscr{C}_n(Q,\delta)$ be the set of $(f_1,\dots,f_n) \in C_c(\mathbb{R}^4)^n$ with supports in Q and $\operatorname{dist}_\infty(\operatorname{supp} f_i, \operatorname{supp} f_j) \ge \delta$ for $i \ne j$, $\operatorname{dist}_\infty$ being the ℓ^∞ distance; a tuple is *separated* if it lies in some $\mathscr{C}_n(Q,\delta)$. A *separated polynomial* is a polynomial in the sense of Sec. XI D each of whose monomials is a separated tuple; it is *at positive μ-times* if all its functions are supported in $\{x_\mu > 0\}$. Put $\mathfrak{D}_n = \{(x_1,\dots,x_n) \in (\mathbb{R}^4)^n : x_i \ne x_j \text{ for } i \ne j\}$ and $\mathsf{K}_L = (2L+3)^4$.

(M$_\mathrm{s}$) *Separated moment bound.* For every $n \ge 1$, every cube Q and every $\delta > 0$ there are $M_n(Q,\delta) < \infty$ and k_0 such that

$$\Big|\omega_k\Big(\prod_{j=1}^n Z_k A_k(f_j)\Big)\Big| \le M_n(Q,\delta)\prod_{j=1}^n \|f_j\|_\infty$$

for all $k \ge k_0$ and $(f_1,\dots,f_n) \in \mathscr{C}_n(Q,\delta)$.

Posed for these packets, (M) contains the tuples with overlapping supports and implies (M$_\mathrm{s}$).

Lemma 273. (i) *For $f \in C_c(\mathbb{R}^4)$, $g \in B_4$ and $v \in \mathbb{Z}^4$,*

$$A_k(f\circ g^{-1}) = gA_k(f), \quad A_k(f(\cdot - a_k v)) = \tau_v A_k(f),$$
$$A_k(\bar f) = \overline{A_k(f)}, \quad \vartheta_\mu A_k(f) = A_k(f\circ\theta_{\mu,k}).$$

(ii) *If f is supported in $\{x_\mu \ge \varepsilon\}$ and $(r+1)a_k \le \varepsilon$, then $A_k(f)$ is a function of the links with both endpoints in $\{x_\mu \ge 1\}$, which lie on the positive side of the plane of ϑ_μ.*

(iii) *Let F, F' be continuous gauge-invariant functions of finitely many links with both endpoints in $\{x_\mu \ge 1\}$, and $t \ge 0$ an integer. Then $\omega_k(\vartheta_\mu(\bar F)F) \ge 0$,*

$$0 \le \omega_k\big(\vartheta_\mu(\bar F)\tau_{2t}^\mu F\big) \le \omega_k\big(\vartheta_\mu(\bar F)F\big),$$

and, with $N_\mu(F) = \omega_k(\vartheta_\mu(\bar F)F)^{1/2}$,

$$\big|\omega_k\big(\vartheta_\mu(\bar F)\tau_t^\mu F'\big) - \overline{\omega_k(F)}\,\omega_k(F')\big| \le e^{-m_* a_k t} N_\mu(F) N_\mu(F').$$

For $v \in \mathbb{Z}^4$ with $v_\mu = 0$, $N_\mu(\tau_v F) = N_\mu(F)$.

Proof. (i) Substitute $y = g^{-1}x$, $y = x - v$ and $y = \vartheta_\mu x$ in the sums, using $g\Phi_x = \Phi_{gx}$, $\tau_v\Phi_x = \Phi_{x+v}$ and $\vartheta_\mu\Phi_x = \Phi_{\vartheta_\mu x}$; the last holds because ϑ_μ is an element of B_4 followed by a lattice translation. The constant c_k does not depend on x by translation invariance, and it is real because Φ is real and ω_k is a state.

(ii) $f(a_k x) \ne 0$ forces $x_\mu \ge \varepsilon/a_k \ge r+1$, and Φ_x depends on the links with both endpoints in $x + [-r,r]^4$.

(iii) For $\mu = 0$ this is the reconstruction of Appendix F: with $\langle \Psi_F, \mathbb{T}_k^t \Psi_{F'}\rangle = \omega_k(\vartheta(\bar F)\tau_t F')$ and the density of the vectors Ψ_G, $\Psi_{\tau_t F} = \mathbb{T}_k^t \Psi_F$; then $\omega_k(\vartheta(\bar F)\tau_{2t}F) = \|\mathbb{T}_k^t\Psi_F\|^2 \le \|\Psi_F\|^2$ by $0 \le \mathbb{T}_k \le 1$, and with $\Psi'_F = \Psi_F - \omega_k(F)\Omega_k$ the left side of the last inequality is $|\langle \Psi'_F, \mathbb{T}_k^t \Psi'_{F'}\rangle| \le e^{-m_* a_k t}\|\Psi'_F\|\|\Psi'_{F'}\|$, where $\langle \Omega_k, \Psi_F\rangle = \omega_k(F)$ and $\|\Psi'_F\| \le \|\Psi_F\| = N_0(F)$. Since ϑ is θ_0 followed by a translation along e_0, it commutes with τ_v for $v_0 = 0$, and $N_0(\tau_v F)^2 = \omega_k(\tau_v(\vartheta(\bar F)F)) = N_0(F)^2$. For general μ apply this to $\pi_\mu F$ and $\pi_\mu F'$, using $\vartheta_\mu = \pi_\mu\vartheta\pi_\mu$, $\tau_t^\mu = \pi_\mu\tau_t\pi_\mu$ and the invariance of ω_k under π_μ. □

Theorem 274. *Assume the hypothesis of Theorem 66 and* (M$_\mathrm{s}$)*. Every sequence of k has a subsequence along which the following hold.*

(i) *For every separated tuple the limit*

$$S_n(f_1,\dots,f_n) = \lim_k \omega_k\Big(\prod_{j=1}^n Z_k A_k(f_j)\Big)$$

exists. It is symmetric and multilinear, $S_1 = 0$, and on $\mathscr{C}_n(Q,\delta)$ it is bounded by $M_n(Q,\delta)\prod_j\|f_j\|_\infty$. There is a unique distribution on $\mathfrak{D}_n$ whose value at $f_1\otimes\cdots\otimes f_n$ is $S_n(f_1,\dots,f_n)$ for every separated tuple of smooth functions.

(ii) *Each S_n is invariant under the translations of $\mathbb{R}^4$ and under B_4, and $S(\bar P) = \overline{S(P)}$ for separated polynomials.*

(iii) *For every μ and every separated polynomial P at positive μ-times, $S(\theta_\mu \bar P\cdot P) \ge 0$. The completion of the quotient of these polynomials by the null space of this form is a Hilbert space $\mathscr{H}_\mu$ with vectors $\Psi_\mu(P)$, $\langle\Psi_\mu(P),\Psi_\mu(P')\rangle = S(\theta_\mu\bar P\cdot P')$, and $\Omega_\mu = \Psi_\mu(1)$; $P \mapsto \pi_\mu P$ induces a unitary map of $\mathscr{H}_0$ onto $\mathscr{H}_\mu$ that maps Ω_0 to Ω_μ.*

(iv) *There are a self-adjoint $H_\mu \ge 0$ with $\Psi_\mu(\tau_{te_\mu}P) = e^{-tH_\mu}\Psi_\mu(P)$ for $t \ge 0$ and a strongly continuous unitary representation U_μ of the hyperplane $e_\mu^\perp$ with $\Psi_\mu(\tau_y P) = U_\mu(y)\Psi_\mu(P)$, commuting with H_μ; $H_\mu\Omega_\mu = 0$ and U_μ fixes Ω_μ. Further $\sigma(H_\mu) \subset \{0\}\cup[m_*,\infty)$, $\ker H_\mu = \mathbb{C}\Omega_\mu$, and for P, P' at positive μ-times and $t \ge 0$*

$$\big|S(\theta_\mu\bar P\cdot\tau_{te_\mu}P') - \overline{S(P)}\,S(P')\big| \le e^{-m_* t}\,\|\Psi_\mu(P)\|\,\|\Psi_\mu(P')\|.$$

(v) *Let $(f_1,\dots,f_n) \in \mathscr{C}_n(Q,\delta)$, Q of side L, let $\{1,\dots,n\} = I\cup J$ with I, J disjoint and nonempty, $y \in \mathbb{R}^4\setminus\{0\}$, and $f_j^\lambda = f_j$ for $j \in I$, $f_j^\lambda = f_j(\cdot - \lambda y)$ for $j \in J$. There is $K < \infty$ independent of λ such that, for $\lambda|y|_\infty \ge L+2$,*

$$\Big|S_n(f_1^\lambda,\dots,f_n^\lambda) - S_{|I|}\big((f_j)_{j\in I}\big)S_{|J|}\big((f_j)_{j\in J}\big)\Big| \le Ke^{-m_*\lambda|y|_\infty}.$$

(vi) *$\mathscr{H}_0 \ne \mathbb{C}\Omega_0$ if and only if S_n does not vanish identically on the separated tuples for some $n \ge 2$. In that case $\sigma(H_\mu)\cap[m_*,\infty) \ne \emptyset$ for every μ: H_μ has a mass gap of at least m_*.*

(vii) *The set G_S of $R \in \mathrm{O}(4)$ with $S_n(f_1\circ R^{-1},\dots,f_n\circ R^{-1}) = S_n(f_1,\dots,f_n)$ for all n and all separated tuples is a closed subgroup of $\mathrm{O}(4)$ that contains B_4. It is finite or equal*

to $\mathrm{O}(4)$*, and it equals* $\mathrm{O}(4)$ *if and only if it contains an element of infinite order.*

Conversely, if for every n and every separated tuple the sequence $\omega_k(\prod_j Z_k A_k(f_j))$ is bounded, then $(\mathrm{M_s})$ *holds.*

Proof. (i) Let $\chi_{m,p}(x)=\prod_\nu(1-2^m|x_\nu-p_\nu|)_+$ for $m\in\mathbb{N}$ and $p\in 2^{-m}\mathbb{Z}^4$, and let $\mathscr{G}$ be the countable set of the functions $\sum_{p\in F}c_p\chi_{m,p}$ with F finite and $c_p\in\mathbb{Q}+i\mathbb{Q}$. Since $\sum_p\chi_{m,p}=1$ and $\chi_{m,p}(x)\neq 0$ only for $|x-p|_\infty<2^{-m}$, for $f\in C_c(\mathbb{R}^4)$ and $\eta>0$ there is $g\in\mathscr{G}$ with $\|f-g\|_\infty<\eta$ and support within ℓ^∞ distance η of $\operatorname{supp} f$: take $2^{-m}\le\eta$ with $|f(x)-f(p)|<\eta/2$ for $|x-p|_\infty\le 2^{-m}$, $F=\{p:f(p)\neq 0\}$ and $|c_p-f(p)|<\eta/2$. By $(\mathrm{M_s})$ the moment sequence of each of the countably many separated tuples of elements of $\mathscr{G}$ is bounded, and a diagonal subsequence makes all of them converge. Let $(f_j)\in\mathscr{C}_n(Q,\delta)$ and choose $g_j\in\mathscr{G}$ as above with $\eta\le\delta/4$. With Q' the cube Q enlarged by $\delta/4$ on every side, the tuples $(g_1,\dots,g_{i-1},f_i-g_i,f_{i+1},\dots,f_n)$ and (g_j) lie in $\mathscr{C}_n(Q',\delta/2)$, so for $k\ge k_0$ the moments B_k satisfy

$$|B_k(f_1,\dots,f_n)-B_k(g_1,\dots,g_n)|\le M_n(Q',\tfrac{\delta}{2})\sum_i\|f_i-g_i\|_\infty\prod_{j\neq i}\max(\|f_j\|_\infty,\|g_j\|_\infty).$$

Hence $B_k(f_1,\dots,f_n)$ is a Cauchy sequence. Symmetry and multilinearity pass to the limit, and so does the bound; $S_1=0$ because the packets are centred.

For the distribution, let $U_1,\dots,U_n$ be bounded open sets with pairwise disjoint closures, $U=U_1\times\cdots\times U_n$, and $M=M_n(Q,\delta)$ with $Q\supset\bigcup_j U_j$ and δ the least distance of the closures; $|S_n(f_1,\dots,f_n)|\le M\prod_j\|f_j\|_\infty$ for $f_j\in C_c(U_j)$. Fix a compact $K\subset U$, R with $U_j\subset(-R,R)^4$, and $\varphi_j\in C_c^\infty(U_j)$ equal to 1 near the projection of K. For $F\in C^\infty(\mathbb{R}^{4n})$ supported in K let c_κ, $\kappa\in\mathbb{Z}^{4n}$, be the Fourier coefficients of its $2R$-periodic extension, so that $F=\sum_\kappa c_\kappa e_\kappa$ on $(-R,R)^{4n}$ with $e_\kappa(x)=\prod_j e^{i\pi\kappa_j\cdot x_j/R}$. Integrating by parts against $1-(R/\pi)^2\Delta$, which multiplies e_κ by $1+|\kappa|^2$,

$$|c_\kappa|\le(2R)^{-4n}(1+|\kappa|^2)^{-2n-1}\big\|(1-(R/\pi)^2\Delta)^{2n+1}F\big\|_{L^1},$$

and $\sum_\kappa(1+|\kappa|^2)^{-2n-1}<\infty$. Put

$$T(F)=\sum_\kappa c_\kappa S_n\big(\varphi_1e_{\kappa_1},\dots,\varphi_ne_{\kappa_n}\big).$$

The series converges absolutely, each term being at most $M\prod_j\|\varphi_j\|_\infty|c_\kappa|$, so T is continuous on the smooth functions supported in K. For $F=f_1\otimes\cdots\otimes f_n$ the expansions $f_j=\varphi_j\sum_{\kappa_j}c^{(j)}_{\kappa_j}e_{\kappa_j}$ converge uniformly with supports in $\operatorname{supp}\varphi_j$, so multilinearity and the bound give $T(F)=S_n(f_1,\dots,f_n)$. Conversely, for every F supported in K the partial sums $\sum_{|\kappa|_\infty\le N}c_\kappa\prod_j\varphi_j(x_j)e_{\kappa_j}(x_j)$ are finite sums of products and converge to $\prod_j\varphi_j(x_j)F=F$ in $C_c^\infty(U)$, since $\sum_\kappa|\kappa|^s|c_\kappa|<\infty$ for every s by the same bound with a higher power. So every distribution on U with the stated values on products agrees with T on the functions supported in K; in particular T does not depend on K, R and φ_j, and defines the unique such distribution T_U on U. Every point of $\mathfrak{D}_n$ has a neighbourhood of this form, namely a product of small open cubes around its n distinct entries, and the intersection of two such products is again one, on which T_U and $T_{U'}$ agree by uniqueness. By [71, Sect. 2.2] the T_U are the restrictions of one distribution on $\mathfrak{D}_n$, and it is unique by the same localisation.

(ii) For dyadic y and large k, $y/a_k\in\mathbb{Z}^4$, and Lemma 273(i) and translation invariance of ω_k give invariance of the moments under the translation by y. For real y take dyadic $y_m\to y$: the translated tuples lie in one $\mathscr{C}_n(Q,\delta)$ and converge uniformly, so the bound of (i) carries the invariance to y. For $g\in B_4$, $A_k(f\circ g^{-1})=gA_k(f)$, ω_k is B_4-invariant, and g preserves ℓ^∞ distances. The last claim follows from $A_k(\bar f)=\overline{A_k(f)}$ and $\omega_k(\bar G)=\overline{\omega_k(G)}$.

(iii) Let $P=\sum_\beta c_\beta\prod_j f_{\beta,j}$ have all its functions supported in $Q\cap\{x_\mu\ge\varepsilon\}$, $\varepsilon>0$, and its monomials in $\mathscr{C}(Q,\delta)$, and put $P_k=\sum_\beta c_\beta\prod_j Z_kA_k(f_{\beta,j})$. For $(r+1)a_k\le\varepsilon$, Lemma 273(ii),(iii) gives $\omega_k(\vartheta_\mu(\bar P_k)P_k)\ge 0$, and by Lemma 273(i) $\vartheta_\mu(\bar P_k)$ is the lattice version of $\theta_\mu\bar P$ with θ_μ replaced by $\theta_{\mu,k}$. For $a_k\le\min(\varepsilon,\delta/4)$ the monomials of $\theta_\mu\bar P\cdot P$ and of the polynomials obtained by replacing θ_μ by $\theta_{\mu,k}$ in some factors lie in $\mathscr{C}(Q',\min(\varepsilon,\delta/2))$ for a cube Q' independent of k, because the reflected factors are supported in $\{x_\mu\le-\varepsilon+a_k\}$. By (i) and multilinearity the difference of the two moments is at most a constant times $\max_{\beta,j}\|f_{\beta,j}\circ\theta_{\mu,k}-f_{\beta,j}\circ\theta_\mu\|_\infty$, which tends to 0 by uniform continuity. So $S(\theta_\mu\bar P\cdot P)\ge 0$. The form is Hermitian by (ii) and the symmetry of S_n, and the construction is that of Theorem 201(iii). Since $\pi_\mu\theta_0\pi_\mu=\theta_\mu$ and S is π_μ-invariant, $S(\theta_\mu\overline{\pi_\mu P}\cdot\pi_\mu P')=S(\theta_0\bar P\cdot P')$ for P,P' at positive 0-times, which gives the unitary map.

(iv) By the unitary map of (iii) it suffices to take $\mu=0$, with H_μ and U_μ the conjugates of H_0 and U_0. Let P be at positive 0-times and $t\ge 0$ dyadic. For large k, $(\tau_{te_0}P)_k=\tau_{t/a_k}P_k$ by Lemma 273(i), and Lemma 273(iii) gives $\omega_k(\vartheta(\bar P_k)\tau_{2t/a_k}P_k)\le\omega_k(\vartheta(\bar P_k)P_k)$. By the approximation of $\theta_{0,k}$ by θ_0 in (iii), the left side tends to $S(\theta_0\bar P\cdot\tau_{2te_0}P)=\|\Psi_0(\tau_{te_0}P)\|^2$, by translation invariance and $\theta_0\tau_{te_0}=\tau_{-te_0}\theta_0$, and the right side tends to $\|\Psi_0(P)\|^2$. The translates $\tau_{te_0}P$, $0\le t\le\delta/2$, have their monomials in one $\mathscr{C}(Q',\delta/2)$, and each monomial of $\tau_{te_0}P-P$ is a telescoping sum of monomials with one factor $f(\cdot-te_0)-f$; so the bound of (i) gives $\|\Psi_0(\tau_{te_0}P)\|\le\|\Psi_0(P)\|$ for all $t\ge 0$ and $\|\Psi_0(\tau_{te_0}P)-\Psi_0(P)\|\to 0$ as $t\to 0$. From here the proof of Theorem 201(iv) applies verbatim and gives H_0, and U_0 from the translations orthogonal to e_0, which commute with θ_0. For the gap, the proof of Lemma 273(iii) with $F=F'=P_k$ and t/a_k steps gives

$$\omega_k\big(\vartheta(\bar P_k)\tau_{t/a_k}P_k\big)-|\omega_k(P_k)|^2\le e^{-m_*t}\big(\omega_k(\vartheta(\bar P_k)P_k)-|\omega_k(P_k)|^2\big),$$

the right side being $e^{-m_*t}\|\Psi'_{P_k}\|^2$ in the notation of the proof of the lemma. In the limit, with $v=\Psi_0(P)-S(P)\Omega_0$ and $\langle\Omega_0,\Psi_0(P)\rangle=S(P)$, this reads $\langle v,e^{-tH_0}v\rangle\le e^{-m_*t}\|v\|^2$. As in the proofs of Theorems 45 and 201(vi), it extends to all $t\ge 0$ and to the closure $\Omega_0^\perp$ of these vectors, and gives the claims on $\sigma(H_0)$, on $\ker H_0$ and the inequality, using $\langle\Psi_0(P),\Omega_0\rangle=\overline{S(P)}$ by (ii).

(v) Choose μ with $|y_\mu| = |y|_\infty = \eta$. Replacing every f_j by $f_j \circ \theta_\mu$ and y by $\theta_\mu y$, which changes no term by (ii), we may take $y_\mu = \eta > 0$, and after a common translation of all f_j, $Q \subset \{-L-1 \le x_\mu \le -1\}$. Let $P'' = \prod_{j\in I} \bar f_j \circ \theta_\mu$ and $P_0' = \prod_{j\in J} f_j(\cdot - (L+2)e_\mu)$, both at positive μ-times, write $y = \eta e_\mu + y_\perp$ with $y_\perp \perp e_\mu$ and put $s = \lambda\eta - L - 2 \ge 0$. Then $\prod_{j\in J} f_j(\cdot - \lambda y) = \tau_{se_\mu}\tau_{\lambda y_\perp} P_0'$ and $\theta_\mu \bar P'' = \prod_{j\in I} f_j$, so by (iv)

$$S_n(f_1^\lambda, \dots, f_n^\lambda) = \langle \Psi_\mu(P''), e^{-sH_\mu} U_\mu(\lambda y_\perp) \Psi_\mu(P_0')\rangle,$$

while, by translation invariance,

$$S_{|I|}\big((f_j)_{j\in I}\big) = \langle \Psi_\mu(P''), \Omega_\mu\rangle,$$
$$S_{|J|}\big((f_j)_{j\in J}\big) = \langle \Omega_\mu, \Psi_\mu(P_0')\rangle.$$

The difference is the same matrix element between the projections of the two vectors on $\Omega_\mu^\perp$, which e^{-sH_μ} and U_μ preserve, so it is at most $e^{-m_* s}\|\Psi_\mu(P'')\|\|\Psi_\mu(P_0')\|$. This is the claim with $K = e^{m_*(L+2)}\|\Psi_\mu(P'')\|\|\Psi_\mu(P_0')\|$.

(vi) If all S_n with $n \ge 1$ vanish, $\langle \Psi_0(P), \Psi_0(P')\rangle$ is the product of the constant terms, and $\mathscr{H}_0 = \mathbb{C}\Omega_0$. Conversely, let $\mathscr{H}_0 = \mathbb{C}\Omega_0$; by (iii) then $\mathscr{H}_\mu = \mathbb{C}\Omega_\mu$, so $S(\theta_\mu \bar P \cdot P') = \overline{S(P)} S(P')$ for P, P' at positive μ-times, and by (ii) $S(\theta_\mu \bar P) = \overline{S(P)}$. With translation invariance this says: if a separated tuple splits into $(f_j)_{j\in I}$ supported in $\{x_\mu < c\}$ and $(f_j)_{j\in J}$ supported in $\{x_\mu > c\}$, then $S_n = S_{|I|} S_{|J|}$. Suppose $S_m = 0$ for $1 \le m < n$, let $(f_j) \in \mathscr{C}_n(Q,\delta)$, $n \ge 2$, and put $s = \delta/(2n)$ and $\chi_m^s(x) = \prod_\nu (1 - |2x_\nu/s - m_\nu|)_+$, $m \in \mathbb{Z}^4$, which have sum 1 and supports in closed cubes of side s. By multilinearity $S_n(f_1, \dots, f_n)$ is the finite sum of the terms $S_n(\chi_{m_1}^s f_1, \dots, \chi_{m_n}^s f_n)$. In a nonzero term pick $p \in \operatorname{supp} \chi_{m_1}^s f_1$ and $q \in \operatorname{supp} \chi_{m_2}^s f_2$, and μ with $|p_\mu - q_\mu| \ge \delta$. The projections of the n cubes on the μ axis have total length at most $ns < \delta$, so some c strictly between p_μ and q_μ lies in none of them; it splits the term into two nonempty groups as above, and the term is 0. Hence $S_n = 0$. The last claim follows from (iv), since $\Omega_\mu^\perp \ne 0$ by (iii).

(vii) For $R \in \mathrm{O}(4)$, $|Rz|_\infty \ge \frac12 |Rz|_2 = \frac12 |z|_2 \ge \frac12 |z|_\infty$, so R maps $\mathscr{C}_n(Q,\delta)$ into some $\mathscr{C}_n(Q', \delta/2)$ and G_S is well defined; it is a group, and it contains B_4 by (ii). If $R_m \to R$ with $R_m \in G_S$, the functions $f_j \circ R_m^{-1}$ converge uniformly to $f_j \circ R^{-1}$, and for large m the telescoping tuples lie in one $\mathscr{C}_n(Q', \delta/4)$, so the bound of (i) gives $R \in G_S$. The proof of Proposition 205(iv) uses only that the group is a closed subgroup of $\mathrm{O}(4)$ containing B_4.

For the converse, cover Q by the closed cubes of side $s = \delta/4$ that carry the functions χ_m^s, at most $(2L/s + 3)^4$ of them. For $(f_j) \in \mathscr{C}_n(Q,\delta)$, every nonzero term $\omega_k(\prod_j Z_k A_k(\chi_{m_j}^s f_j))$ has the n cubes of $m_1, \dots, m_n$ at pairwise distance at least $\delta - 2s = \delta/2$. For each such choice of cubes $C_1, \dots, C_n$ the forms $(h_j) \mapsto \omega_k(\prod_j Z_k A_k(h_j))$ on the Banach spaces of continuous functions supported in C_j are bounded at every point by hypothesis, hence bounded in norm by the uniform boundedness argument of the proof of Theorem 201. Summing over the finitely many choices gives $M_n(Q,\delta)$ with $k_0 = 1$. □

For the flowed fields of Sec. XI D the gap gives clustering in the time direction only (Theorem 201(vi)), and spatial clustering needs the tree bounds; part (v) obtains it in every direction from the hypercubic covariance alone. The symmetry of the S_n is the Euclidean form of locality [95]. With

($\mathrm{W_s}$) there is $R \in \mathrm{O}(4)$ of infinite order with

$$\lim_k \Big[\omega_k\Big(\prod_j Z_k A_k(f_j \circ R^{-1})\Big) - \omega_k\Big(\prod_j Z_k A_k(f_j)\Big)\Big] = 0$$

for every separated tuple,

($\mathrm{U_s}$) any two subsequential limits of Theorem 274(i) coincide,

the argument of Corollary 208 gives the following.

Corollary 275. *Assume the hypothesis of Theorem 66. The moments $\omega_k(\prod_j Z_k A_k(f_j))$ converge for every separated tuple to a family invariant under the translations and* $\mathrm{O}(4)$ *if and only if* ($\mathrm{M_s}$), ($\mathrm{W_s}$) *and* ($\mathrm{U_s}$) *hold. The limit then has properties* (i)–(vi) *of Theorem 274 with B_4 replaced by* $\mathrm{O}(4)$.

Proof. Convergence gives bounded moments, hence ($\mathrm{M_s}$) by the converse in Theorem 274, and it gives ($\mathrm{U_s}$) and ($\mathrm{W_s}$) for every R. Conversely, ($\mathrm{M_s}$) and ($\mathrm{U_s}$) give convergence of the whole sequence, since a nonconvergent moment would have two limit points and Theorem 274(i) would refine them to two different limits; ($\mathrm{W_s}$) puts R in G_S, and Theorem 274(vii) gives $G_S = \mathrm{O}(4)$. □

Nontriviality is not a further input once the constants are fixed by one reflected correlation. Fix $f_0 \in C_c(\mathbb{R}^4)$, not zero, supported in $\{x_0 \ge t_0\}$ with $t_0 > 0$, and for $(r+2)a_k \le t_0$ and $t \in a_k \mathbb{Z}_{\ge 0}$ put

$$C_k(t) = \omega_k\big(\vartheta(\overline{A_k(f_0)})\, \tau_{t/a_k} A_k(f_0)\big).$$

Proposition 276. (i) *$C_k(0) \ge 0$, and if $C_k(0) > 0$, then $0 < C_k(t) \le e^{-m_* t} C_k(0)$ for every $t \in a_k\mathbb{Z}_{\ge 0}$.*

(ii) *Let $\sigma_0 > 0$ be dyadic, suppose $C_k(0) > 0$ for large k, put $Z_k = C_k(\sigma_0)^{-1/2}$ and assume* ($\mathrm{M_s}$). *Every limit of Theorem 274 has $\langle \Psi_0(f_0), e^{-\sigma_0 H_0}\Psi_0(f_0)\rangle = 1$ and $\|\Psi_0(f_0)\|^2 \ge e^{m_*\sigma_0}$, and is nontrivial. Moreover, with Q a cube that contains* $\operatorname{supp} f_0$ *and $\theta_{0,k}(\operatorname{supp} f_0)$ for all k,*

$$\frac{C_k(\sigma_0)}{C_k(0)} \ge \frac{1}{M_2(Q,t_0)\|f_0\|_\infty^2}$$

for all large k.

(iii) *Suppose that the transfer matrices $\mathbb{T}_k$ have spectrum on $\Omega_k^\perp$ in $[0, e^{-m_{\mathrm{lat}}}]$ for a fixed $m_{\mathrm{lat}} > 0$ and all k. Then for every choice of Z_k satisfying* ($\mathrm{M_s}$), *every limit of Theorem 274 is trivial, and, when $C_k(0) > 0$ for large k,* ($\mathrm{M_s}$) *fails for $Z_k = C_k(\sigma_0)^{-1/2}$.*

Proof. (i) Let $A = A_k(f_0)$. By Lemma 273(ii) and its proof, A and $A' = A_k(f_0(\cdot + a_k e_0))$ are at positive times, and $A = \tau_1 A'$, so $\Psi_A = \mathbb{T}_k \Psi_{A'}$ lies in the range of $\mathbb{T}_k$, which is orthogonal to $\ker \mathbb{T}_k$. Also $\langle \Omega_k, \Psi_A\rangle = \omega_k(A) = 0$. So the spectral measure ν of $\mathbb{T}_k$ at Ψ_A has no atom at 0 and is carried by $(0, e^{-m_* a_k}]$, and $C_k(t) = \int \lambda^{t/a_k} d\nu(\lambda)$. If $C_k(0) = \nu((0,1]) > 0$, the integrand is positive on a set of positive measure and at most $e^{-m_* t}$.

(ii) The pairs $(\bar f_0\circ\theta_0, f_0(\cdot - te_0))$, $t\ge 0$, are separated by at least $2t_0$, and $\omega_k(\vartheta(\bar A)\tau_{t/a_k}A) = \omega_k(A_k(\bar f_0\circ\theta_{0,k})A_k(f_0(\cdot - te_0)))$ by Lemma 273(i). By the approximation of $\theta_{0,k}$ by θ_0 in the proof of Theorem 274(iii), $\langle\Psi_0(f_0), e^{-\sigma_0 H_0}\Psi_0(f_0)\rangle = \lim_k Z_k^2 C_k(\sigma_0) = 1$ and $\|\Psi_0(f_0)\|^2 = \lim_k C_k(0)/C_k(\sigma_0) \ge e^{m_*\sigma_0}$ by (i). Since $\langle\Omega_0,\Psi_0(f_0)\rangle = S_1(f_0) = 0$, $\mathscr{H}_0\ne\mathbb{C}\Omega_0$. For $a_k\le t_0$ the pair $(\bar f_0\circ\theta_{0,k}, f_0)$ lies in $\mathscr{C}_2(Q,t_0)$, so $(\mathrm{M_s})$ gives $Z_k^2C_k(0)\le M_2(Q,t_0)\|f_0\|_\infty^2$, which is the last claim.

(iii) For a separated polynomial P at positive 0-times and dyadic $t>0$, the proof of Theorem 274(iv) with $e^{-m_{\mathrm{lat}}t/a_k}$ in place of $e^{-m_* t}$ gives $\langle v, e^{-tH_0}v\rangle \le \lim_k e^{-m_{\mathrm{lat}}t/a_k}\|\Psi'_{P_k}\|^2$, $v=\Psi_0(P)-S(P)\Omega_0$, where $\|\Psi'_{P_k}\|^2$ converges to $\|v\|^2$. So $\|e^{-tH_0/2}v\|^2=0$, and $v=0$ because $e^{-tH_0/2}$ is injective. Hence $\mathscr{H}_0=\mathbb{C}\Omega_0$. For $Z_k = C_k(\sigma_0)^{-1/2}$, $C_k(\sigma_0)/C_k(0)\le e^{-m_{\mathrm{lat}}\sigma_0/a_k}$ by the argument of (i), which contradicts (ii) for large k. □

The lower bound in (ii) is the analogue of $(\mathrm{N_0})$ for the packets without flow, and (iii) is the analogue of Theorem 191. The averaged condition itself does not exclude a lattice gap: it holds at strong coupling for every cube size.

Corollary 277. *Let $G=\mathrm{SU}(2)$ and $\beta_{W,k}\in[0,\frac18]$ for every k. Then the hypothesis of Theorem 66 holds with $\varpi=0.41$, $n_k=2$ and $\ell=2a_0$, every torus limit ω_k has transfer-matrix gap at least $m_{\mathrm{lat}}=3.28/(625e^2)>7\cdot10^{-4}$ in lattice units, and for every Φ and every choice of Z_k satisfying $(\mathrm{M_s})$ every limit of Theorem 274 is trivial.*

Proof. By Corollary 74(i), $\alpha_n\le\alpha_2\le 0.59$ for $|\beta_W|\le\frac18$ and every $n\ge2$, so $\mathrm{BP}(2,0.41)$ holds on Λ_L for every even $L\ge8$, and $n_ka_k=2a_k\le 2a_0$. By Corollary 65 with $n=2$ the gap is at least $\varpi/(2e^2q_2)\ge 16\varpi/(1250e^2)=3.28/(625e^2)$, and $625e^2<4619$. Proposition 276(iii) applies. □

So the content of Theorem 274 beyond the averaged condition lies in $(\mathrm{M_s})$ for constants Z_k that keep the reflected correlation $C_k(\sigma_0)$ fixed. The bound (M) at coinciding supports cannot serve for the action density.

Proposition 278. *Let S_2 be a limit along a subsequence of $\omega_k(Z_k^2A_k(f)A_k(g))$ for all separated pairs, and suppose that $S_2(f,g)=\iint f(x)g(y)K(x-y)\,dx\,dy$ for separated pairs, with $K\ge0$ measurable on $\mathbb{R}^4\setminus\{0\}$ and*

$$K(z)\ge c\,|z|^{-8}\Big(\log\frac{1}{|z|}\Big)^{-p}\qquad \text{for } 0<|z|\le r_0,$$

where $c>0$, $p\ge0$ and $0<r_0<1$. Then the moment bound (M) *of Sec. XI D, posed for the packets $Z_kA_k(f)$, fails for $n=2$ and $Q=[0,1]^4$.*

Proof. If (M) held for $n=2$, the limit would satisfy $|S_2(f,g)|\le M_2(Q)\|f\|_\infty\|g\|_\infty$ for separated f,g supported in Q. Let $0<\delta\le\min(\frac18,\frac{r_0}{5})$, let f_δ be continuous with values in $[0,1]$, equal to 1 on $[\frac12-2\delta,\frac12-\delta]\times[\frac14,\frac34]^3$ and supported in $[0,\frac12-\frac{\delta}{2}]\times[0,1]^3$, and let g_δ be continuous with values in $[0,1]$, equal to 1 on $[\frac12+\delta,\frac12+2\delta]\times[\frac18,\frac78]^3$ and supported in $[\frac12+\frac{\delta}{2},1]\times[0,1]^3$. The supports are at distance at least δ. For x in the first box and y in the second with $|y_i-x_i|\le\delta$ for $i=1,2,3$, $2\delta\le|x-y|\le\sqrt{19}\,\delta<5\delta\le r_0$, so $K(x-y)\ge c(5\delta)^{-8}(\log\frac{1}{2\delta})^{-p}$; these pairs have measure $\frac{\delta}{8}\cdot8\delta^4=\delta^5$. Hence $S_2(f_\delta,g_\delta)\ge c5^{-8}\delta^{-3}(\log\frac{1}{2\delta})^{-p}$, which is unbounded as $\delta\to0$, while $\|f_\delta\|_\infty\|g_\delta\|_\infty=1$. □

The law of the proposition, with $p=2$, is the one that asymptotic freedom predicts for the action density. It places no restriction on $(\mathrm{M_s})$ when K is bounded on $\{|z|_\infty\ge\delta\}$ for every $\delta>0$.

The two-point functions. For $n=2$ the separated moments reduce to one quantity of the lattice theory. For $\varepsilon>0$ let $B_\varepsilon=[\varepsilon,\varepsilon+1+a_0]\times[0,1+a_0]^3$ and, for $(r+1)a_k\le\varepsilon$,

$$\mathscr{N}_k(\varepsilon)=\sup\big\{\omega_k\big(\vartheta(\overline{A_k(h)})A_k(h)\big):\operatorname{supp}h\subset B_\varepsilon,\ \|h\|_\infty\le1\big\},$$

which is finite because $|A_k(h)|$ is at most $2\|\Phi\|_\infty a_k^4\|h\|_\infty$ times the number of sites x with $a_kx\in B_\varepsilon$. Let $\gamma_k\le e^{-m_*a_k}$ be the norm of $\mathbb{T}_k$ on $\Omega_k^\perp$.

Lemma 279. *Let $(r+1)a_k\le\varepsilon$ and let $g\in C_c(\mathbb{R}^4)$ be supported in a cube of side L and in $\{x_0\ge\varepsilon\}$. Then*

$$\omega_k\big(\vartheta(\overline{A_k(g)})A_k(g)\big)\le \mathsf{K}_L^2\,\mathscr{N}_k(\varepsilon)\,\|g\|_\infty^2.$$

The same holds with ϑ_μ for g supported in $\{x_\mu\ge\varepsilon\}$.

Proof. Write $g=\sum_m h_m$ with $h_m=\chi^1_m g$, χ^1_m as in the proof of Theorem 274(vi); χ^1_m is supported in the closed cube of side 1 centred at $m/2$, and $\chi^1_m g\ne0$ requires $m_\nu\in(2q_\nu-1,2q_\nu+2L+1)$ for the cube $\prod_\nu[q_\nu,q_\nu+L]$, so at most K_L pieces are nonzero, each with $\|h_m\|_\infty\le\|g\|_\infty$. Fix one, supported in $[s,s+1]\times\prod_i[u_i,u_i+1]$ and in $\{x_0\ge s'\}$, $s'=\max(s,\varepsilon)$. Let $n_0=\lfloor(s'-\varepsilon)/a_k\rfloor\ge0$, $v_i=\lfloor u_i/a_k\rfloor$, $v=(n_0,v_1,v_2,v_3)$ and $h'=h_m(\cdot+a_kv)$. Then h' is supported in $[s'-n_0a_k,s+1-n_0a_k]\times\prod_i[u_i-v_ia_k,u_i-v_ia_k+1]\subset B_\varepsilon$, since $s'-n_0a_k\in[\varepsilon,\varepsilon+a_k)$ and $u_i-v_ia_k\in[0,a_k)$. By Lemma 273(i), $A_k(h_m)=\tau_{n_0e_0}\tau_{v_\perp}A_k(h')$ with $v_\perp=(0,v_1,v_2,v_3)$, and by Lemma 273(ii),(iii) and its proof, $\|\Psi_{A_k(h_m)}\|=\|\mathbb{T}_k^{n_0}\Psi_{\tau_{v_\perp}A_k(h')}\|\le\|\Psi_{A_k(h')}\|\le\mathscr{N}_k(\varepsilon)^{1/2}\|g\|_\infty$. The triangle inequality for the linear map $F\mapsto\Psi_F$ gives $\|\Psi_{A_k(g)}\|\le\mathsf{K}_L\mathscr{N}_k(\varepsilon)^{1/2}\|g\|_\infty$. For ϑ_μ apply this to $g\circ\pi_\mu$. □

Theorem 280. *Assume the hypothesis of Theorem 66, let $\varepsilon>0$, suppose $\mathscr{N}_k(\varepsilon)>0$ for large k, and put $Z_k=\mathscr{N}_k(\varepsilon)^{-1/2}$.*

(i) *Let $c\in\mathbb{R}$, $d\ge0$, f supported in a cube of side L in $\{x_\mu\le c-2\varepsilon\}$ and g in a cube of side L' in $\{x_\mu\ge c+2\varepsilon+d\}$. For $a_k\le\varepsilon$ and $(r+1)a_k\le\varepsilon$,*

$$Z_k^2\big|\omega_k\big(A_k(f)A_k(g)\big)\big|\le\mathsf{K}_L\mathsf{K}_{L'}\,\gamma_k^{\lfloor d/a_k\rfloor}\,\|f\|_\infty\|g\|_\infty.$$

(ii) *For $(f,g)\in\mathscr{C}_2(Q,\delta)$ with Q of side L and $\delta\ge8\varepsilon$, and k as in* (i),

$$Z_k^2\big|\omega_k\big(A_k(f)A_k(g)\big)\big|\le\mathsf{K}_{4L/\delta}^2\mathsf{K}_{\delta/4}^2\,\|f\|_\infty\|g\|_\infty.$$

(iii) *Every sequence of k has a subsequence along which $S_2(f,g)=\lim_k Z_k^2\omega_k(A_k(f)A_k(g))$ exists for all f,g with $\operatorname{dist}_\infty(\operatorname{supp}f,\operatorname{supp}g)>8\varepsilon$. It is symmetric, bilinear, bounded as in* (ii), *invariant under the translations and B_4, $S_2(\bar f,\bar g)=$*

$\overline{S_2(f,g)}$, *and* $\sum_{i,j}\bar c_i c_j S_2(\bar g_i\circ\theta_\mu, g_j)\ge 0$ *for all* $c_i\in\mathbb{C}$ *and* g_i *supported in* $\{x_\mu>4\varepsilon\}$.

(iv) *For* f,g *supported in a cube of side* L, $y\neq 0$ *and* $\lambda|y|_\infty > L+8\varepsilon$,

$$\big|S_2(f,g(\cdot-\lambda y))\big|\le \mathsf{K}_L^2\|f\|_\infty\|g\|_\infty e^{-m_*(\lambda|y|_\infty-L-4\varepsilon)}.$$

(v) *If* $\gamma_k\le e^{-m_{\mathrm{lat}}}$ *for a fixed* $m_{\mathrm{lat}}>0$ *and all* k, *every* S_2 *of* (iii) *vanishes.*

Parts (i)–(iii) *use only the invariance of* ω_k *under translations and* B_4 *and its reflection positivity.*

Proof. (i) By the invariance of ω_k under π_μ and Lemma 273(i) we may take $\mu=0$. Let $b_k\in a_k\mathbb{Z}$ with $c\le b_k<c+a_k$; by the invariance under the translation by $-b_k e_0/a_k$ we may take $b_k=0$, so that f is supported in $\{x_0\le -2\varepsilon\}$ and g in $\{x_0\ge 2\varepsilon+d-a_k\}$. Let $h=\bar f\circ\theta_{0,k}$, supported in $\{x_0\ge 2\varepsilon-a_k\}\subset\{x_0\ge\varepsilon\}$; then $A_k(f)=\vartheta(\overline{A_k(h)})$. Let $n=\lfloor d/a_k\rfloor$ and $g'=g(\cdot+na_ke_0)$, supported in $\{x_0\ge 2\varepsilon-a_k\}$ because $na_k\le d$. Then $A_k(g)=\tau_nA_k(g')$, and

$$\omega_k\big(A_k(f)A_k(g)\big)=\langle\Psi_{A_k(h)},\mathbb{T}_k^n\Psi_{A_k(g')}\rangle.$$

Both vectors are orthogonal to Ω_k, the packets being centred, so the modulus is at most $\gamma_k^n\|\Psi_{A_k(h)}\|\|\Psi_{A_k(g')}\|$, and Lemma 279 bounds the norms by $\mathsf{K}_L\mathscr{N}_k(\varepsilon)^{1/2}\|f\|_\infty$ and $\mathsf{K}_{L'}\mathscr{N}_k(\varepsilon)^{1/2}\|g\|_\infty$.

(ii) Let $s=\delta/4$ and write $f=\sum_m\chi_m^s f$, $g=\sum_{m'}\chi_{m'}^s g$; at most $\mathsf{K}_{L/s}=\mathsf{K}_{4L/\delta}$ terms of each sum are nonzero. For a pair of nonzero pieces pick p,q in their supports, and μ with $|p_\mu-q_\mu|\ge\delta$. The projections of the two cubes of side s on the μ axis contain p_μ and q_μ, so they are at distance at least $\delta-2s=\delta/2\ge 4\varepsilon$, and (i) applies to the pair, possibly with f and g exchanged, with $L=L'=\delta/4$ and $\gamma_k^{\lfloor d/a_k\rfloor}\le 1$. Summing over the pairs gives the bound.

(iii) The diagonal argument of the proof of Theorem 274(i), with (ii) in place of ($\mathrm{M_s}$) and approximants within $\eta<(\delta-8\varepsilon)/4$, gives the limit for pairs at distance $\delta>8\varepsilon$. Invariance and the conjugation identity follow as in Theorem 274(ii). For $g=\sum_i c_ig_i$ supported in $\{x_\mu\ge 4\varepsilon+\eta'\}$, $\eta'>0$, $Z_k^2\omega_k(\vartheta_\mu(\overline{A_k(g)})A_k(g))\ge 0$ by Lemma 273, and it equals $Z_k^2\omega_k(A_k(\bar g\circ\theta_{\mu,k})A_k(g))$; the pair $(\bar g\circ\theta_\mu,g)$ is at distance at least $8\varepsilon+2\eta'$, and (ii) carries the approximation of $\theta_{\mu,k}$ by θ_μ as in Theorem 274(iii).

(iv) By B_4-invariance we may take $y_\mu=|y|_\infty=\eta>0$, and by translation invariance the cube to be $\prod_\nu[q_\nu,q_\nu+L]$. Then f is supported in $\{x_\mu\le c-2\varepsilon\}$ with $c=q_\mu+L+2\varepsilon$, and $g(\cdot-\lambda y)$ in $\{x_\mu\ge c+2\varepsilon+d\}$ with $d=\lambda\eta-L-4\varepsilon>0$. By (i) and $\gamma_k^{\lfloor d/a_k\rfloor}\le e^{-m_*(d-a_k)}$ the lattice quantities obey the bound with $e^{-m_*(d-a_k)}$, and the limit gives the claim.

(v) In the proof of (ii), now with $\delta>8\varepsilon$, each pair of pieces satisfies (i) with $d=\delta/2-4\varepsilon>0$, so the lattice quantity is at most $\mathsf{K}_{4L/\delta}^2\mathsf{K}_{\delta/4}^2\|f\|_\infty\|g\|_\infty e^{-m_{\mathrm{lat}}(d/a_k-1)}$, which tends to 0. □

Two statements about the lattice two-point function control the small separations:

(SD) $\limsup_k\mathscr{N}_k(\varepsilon')/\mathscr{N}_k(\varepsilon)<\infty$ for every $\varepsilon'\in(0,\varepsilon)$;

(NE) there is $g_0\in C_c(\mathbb{R}^4)$ supported in $\{x_0>0\}$ with

$$\liminf_k\frac{\omega_k\big(\vartheta(\overline{A_k(g_0)})A_k(g_0)\big)}{\mathscr{N}_k(\varepsilon)}>0.$$

(SD) bounds the growth of the largest reflected norm of a packet of unit size as it approaches the reflection plane from the distance ε to any smaller distance, uniformly in the spacing; (NE) says that a fixed packet carries a fixed fraction of that largest norm.

Corollary 281. *Assume the hypothesis of Theorem 280.*

(a) (SD) *holds if and only if* ($\mathrm{M_s}$) *holds for* $n=2$.

(b) *Under* (SD), *every sequence of* k *has a subsequence along which* $S_2(f,g)$ *exists for every separated pair. It is given by a distribution on* $\mathfrak{D}_2$, *invariant under the translations and* B_4, *satisfies* $S_2(\bar f,\bar g)=\overline{S_2(f,g)}$ *and* $S_2(\bar g\circ\theta_\mu,g)\ge 0$ *for* g *supported in* $\{x_\mu>0\}$, *and for* f,g *supported in a cube of side* L, $y\neq 0$, $\varepsilon'\in(0,\varepsilon]$ *and* $\lambda|y|_\infty>L+4\varepsilon'$,

$$\big|S_2(f,g(\cdot-\lambda y))\big|\le C_{\varepsilon'}\mathsf{K}_L^2\|f\|_\infty\|g\|_\infty e^{-m_*(\lambda|y|_\infty-L-4\varepsilon')}$$

with $C_{\varepsilon'}=\limsup_k\mathscr{N}_k(\varepsilon')/\mathscr{N}_k(\varepsilon)$. *The time translations define on the completion* $\mathscr{H}^{(1)}$ *of the functions supported in* $\{x_0>0\}$ *for the form* $S_2(\bar g\circ\theta_0,g')$ *a semigroup* $e^{-tH^{(1)}}$ *with* $\sigma(H^{(1)})\subset[m_*,\infty)$. *Some limit* S_2 *is not identically zero if and only if* $\limsup_k\omega_k(\vartheta(\overline{A_k(g)})A_k(g))/\mathscr{N}_k(\varepsilon)>0$ *for some* g *supported in* $\{x_0>0\}$; *under* (NE) *every limit is not identically zero, and then* $\sigma(H^{(1)})$ *is a nonempty subset of* $[m_*,\infty)$.

Proof. (a) If ($\mathrm{M_s}$) holds for $n=2$, let h be supported in $B_{\varepsilon'}$ with $\|h\|_\infty\le 1$. For $a_k\le\varepsilon'$ the pair $(\bar h\circ\theta_{0,k},h)$ lies in $\mathscr{C}_2(Q,\varepsilon')$ for a cube Q independent of k and h, and $\omega_k(\vartheta(\overline{A_k(h)})A_k(h))=\omega_k(A_k(\bar h\circ\theta_{0,k})A_k(h))$, so $\mathscr{N}_k(\varepsilon')\le M_2(Q,\varepsilon')\mathscr{N}_k(\varepsilon)$ for large k. Conversely, let $(f,g)\in\mathscr{C}_2(Q,\delta)$ and $\varepsilon'=\min(\varepsilon,\delta/8)$. The proof of Theorem 280(i), (ii) with ε' in place of ε gives $|\omega_k(A_k(f)A_k(g))|\le\mathsf{K}_{4L/\delta}^2\mathsf{K}_{\delta/4}^2\mathscr{N}_k(\varepsilon')\|f\|_\infty\|g\|_\infty$ for large k, and dividing by $\mathscr{N}_k(\varepsilon)$ and using (SD) gives ($\mathrm{M_s}$) for $n=2$.

(b) By (a), the proof of Theorem 274(i) for $n=2$ gives the limits and the distribution, and that of Theorem 274(ii), (iii) the invariance, the conjugation identity and the positivity. The decay follows from Theorem 280(i) with ε' in place of ε, multiplied by $\mathscr{N}_k(\varepsilon')/\mathscr{N}_k(\varepsilon)$, as in the proof of Theorem 280(iv). The semigroup comes from the proof of Theorem 274(iv) restricted to single functions. Since $\langle\Omega_k,\Psi_{A_k(g)}\rangle=0$, Lemma 273(iii) gives $\omega_k(\vartheta(\overline{A_k(g)})\tau_{t/a_k}A_k(g))\le e^{-m_*t}\omega_k(\vartheta(\overline{A_k(g)})A_k(g))$, hence $\langle w,e^{-tH^{(1)}}w\rangle\le e^{-m_*t}\|w\|^2$ on a dense set of w, and $\|e^{-tH^{(1)}}\|\le e^{-m_*t}$. If the lim sup is positive for g, a subsequence along which the ratio converges to it has a further subsequence with $S_2(\bar g\circ\theta_0,g)$ equal to that limit, by the approximation of $\theta_{0,k}$ by θ_0. Conversely, if $S_2(\bar g\circ\theta_\mu,g)=0$ for all μ and all g supported in $\{x_\mu>0\}$, the Cauchy–Schwarz inequality for these positive forms and translation invariance give $S_2(f,g)=0$ whenever f and g lie on opposite sides of a coordinate hyperplane, and the decomposition in the proof of Theorem 280(ii) splits every separated pair into such pairs; by B_4-invariance

$S_2(\bar g\circ\theta_\mu,g)=S_2(\overline{g\circ\pi_\mu}\circ\theta_0,g\circ\pi_\mu)$, so a nonzero S_2 has $S_2(\bar g\circ\theta_0,g)>0$ for some g, and the ratio for g converges to it along the subsequence. Under (NE), $S_2(\bar g_0\circ\theta_0,g_0)$ is at least the lim inf in (NE), so $\mathscr{H}^{(1)}\neq 0$. □

The short-distance comparison. The comparison (SD) is a statement about the spectrum of the transfer matrices. For $\Lambda>0$ let $\mathbb{E}_k(\Lambda)$ be the spectral projection of $\mathbb{T}_k$ on $[e^{-\Lambda a_k},1]$, the states of energy at most Λ in physical units, put $\mathsf{K}=\mathsf{K}_{1+a_0}=(5+2a_0)^4$ and, for $(r+1)a_k\le\varepsilon$,

$$\mathscr{N}_k(\varepsilon;\Lambda)=\sup\big\{\langle\Psi_{A_k(h)},\mathbb{E}_k(\Lambda)\Psi_{A_k(h)}\rangle:\ \operatorname{supp}h\subset B_\varepsilon,\ \|h\|_\infty\le 1\big\}.$$

(UV) For every $\varepsilon'\in(0,\varepsilon)$ there are $\Lambda<\infty$ and $c>0$ with $\mathscr{N}_k(\varepsilon';\Lambda)\ge c\,\mathscr{N}_k(\varepsilon')$ for all large k.

(UV) says that the largest reflected norm of a unit packet near the reflection plane is carried, up to a fixed fraction, by energies bounded in physical units.

Proposition 282. *Assume the hypothesis of Theorem 280.*
(i) *For* $(r+1)a_k\le\varepsilon'<\varepsilon$,

$$\mathscr{N}_k(\varepsilon)\le\mathsf{K}^2\gamma_k^{2\lfloor(\varepsilon-\varepsilon')/a_k\rfloor}\mathscr{N}_k(\varepsilon')\le\mathsf{K}^2e^{-2m_*(\varepsilon-\varepsilon'-a_k)}\mathscr{N}_k(\varepsilon').$$

(ii) (SD) *holds if and only if* (UV) *holds.*
(iii) *If* (SD) *holds, then for every* $\varepsilon'\in(0,\varepsilon)$

$$\limsup_k\frac{1}{a_k}\log\frac{1}{\gamma_k}\le\frac{\log(\mathsf{K}^2C_{\varepsilon'})}{2(\varepsilon-\varepsilon')},\qquad C_{\varepsilon'}=\limsup_k\frac{\mathscr{N}_k(\varepsilon')}{\mathscr{N}_k(\varepsilon)}.$$

In particular, if $\gamma_k\le e^{-m_{\rm lat}}$ *for a fixed* $m_{\rm lat}>0$ *and all* k, *then* $\mathscr{N}_k(\varepsilon')/\mathscr{N}_k(\varepsilon)\to\infty$ *for every* $\varepsilon'\in(0,\varepsilon)$ *and* (SD) *fails; this is the case under the hypotheses of Corollary 277 whenever* $\mathscr{N}_k(\varepsilon)>0$ *for large* k.

Proof. (i) Let h be supported in B_ε with $\|h\|_\infty\le 1$, put $n=\lfloor(\varepsilon-\varepsilon')/a_k\rfloor$ and $h'=h(\cdot+na_ke_0)$, supported in a cube of side $1+a_0$ in $\{x_0\ge\varepsilon'\}$. By Lemma 273(i) and the proof of Lemma 273(iii), $\Psi_{A_k(h)}=\mathbb{T}_k^n\Psi_{A_k(h')}$, and $\Psi_{A_k(h')}\perp\Omega_k$ because the packet is centred. So $\|\Psi_{A_k(h)}\|\le\gamma_k^n\|\Psi_{A_k(h')}\|\le\gamma_k^n\mathsf{K}\mathscr{N}_k(\varepsilon')^{1/2}$ by Lemma 279, and $\gamma_k^n\le e^{-m_*a_kn}\le e^{-m_*(\varepsilon-\varepsilon'-a_k)}$.

(ii) First a version of Lemma 279 for $\mathbb{E}=\mathbb{E}_k(\Lambda)$. For a function F of links in $\{x_0\ge1\}$ and $v\in\mathbb{Z}^4$ with $v_0=0$, the spectral measures of $\mathbb{T}_k$ at Ψ_F and at Ψ_{τ_vF} are measures on $[0,1]$ with the same moments $\omega_k(\vartheta(\bar F)\tau_jF)=\omega_k(\tau_v(\vartheta(\bar F)\tau_jF))$, $j\ge0$, so they coincide and $\|\mathbb{E}\Psi_{\tau_vF}\|=\|\mathbb{E}\Psi_F\|$. If g is supported in a cube of side L in $\{x_0\ge\varepsilon\}$, the proof of Lemma 279 writes $\Psi_{A_k(g)}$ as a sum of at most K_L vectors $\mathbb{T}_k^{n_m}\Psi_{\tau_{v_m}A_k(h'_m)}$ with $(v_m)_0=0$, h'_m supported in B_ε and $\|h'_m\|_\infty\le\|g\|_\infty$; since $\mathbb{E}$ commutes with $\mathbb{T}_k$ and $\|\mathbb{T}_k\|\le1$,

$$\|\mathbb{E}\Psi_{A_k(g)}\|\le\mathsf{K}_L\mathscr{N}_k(\varepsilon;\Lambda)^{1/2}\|g\|_\infty.$$

Assume (UV) and let $\varepsilon'\in(0,\varepsilon)$, with Λ,c as in (UV). Take h supported in $B_{\varepsilon'}$ with $\|h\|_\infty\le1$ and $\langle\Psi,\mathbb{E}\Psi\rangle\ge\frac{c}{2}\mathscr{N}_k(\varepsilon')$, $\Psi=\Psi_{A_k(h)}$, let $n=\lceil(\varepsilon-\varepsilon')/a_k\rceil$ and $h_n=h(\cdot-na_ke_0)$, supported in a cube of side $1+a_0$ in $\{x_0\ge\varepsilon\}$. With ν the spectral measure of $\mathbb{T}_k$ at Ψ,

$$\|\Psi_{A_k(h_n)}\|^2=\int\lambda^{2n}d\nu(\lambda)\ge e^{-2\Lambda a_kn}\nu\big([e^{-\Lambda a_k},1]\big)\ge e^{-2\Lambda(\varepsilon-\varepsilon'+a_k)}\tfrac{c}{2}\mathscr{N}_k(\varepsilon'),$$

and Lemma 279 bounds the left side by $\mathsf{K}^2\mathscr{N}_k(\varepsilon)$. So $\mathscr{N}_k(\varepsilon')/\mathscr{N}_k(\varepsilon)\le2c^{-1}\mathsf{K}^2e^{2\Lambda(\varepsilon-\varepsilon'+a_k)}$, which is (SD).

Conversely, let (SD) hold and $\varepsilon'\in(0,\varepsilon)$, and fix a number C greater than $\limsup_k\mathscr{N}_k(\varepsilon')/\mathscr{N}_k(\varepsilon)$, and put $\Lambda=\log(4C\mathsf{K}^2)/(\varepsilon-\varepsilon')$. For h, n and h' as in the proof of (i), $\|\mathbb{T}_k(1-\mathbb{E})\|\le e^{-\Lambda a_k}$, so

$$\|\Psi_{A_k(h)}\|=\|\mathbb{T}_k^n\Psi_{A_k(h')}\|\le\|\mathbb{E}\Psi_{A_k(h')}\|+e^{-\Lambda a_kn}\|\Psi_{A_k(h')}\|,$$

and the two forms of Lemma 279 give

$$\mathscr{N}_k(\varepsilon)\le2\mathsf{K}^2\big(\mathscr{N}_k(\varepsilon';\Lambda)+e^{-2\Lambda a_kn}\mathscr{N}_k(\varepsilon')\big).$$

For large k, $a_k\le\frac12(\varepsilon-\varepsilon')$, so $a_kn\ge\frac12(\varepsilon-\varepsilon')$ and $2C\mathsf{K}^2e^{-2\Lambda a_kn}\le\frac12$; with $\mathscr{N}_k(\varepsilon')\le C\mathscr{N}_k(\varepsilon)$ this gives $\mathscr{N}_k(\varepsilon')\le4C\mathsf{K}^2\mathscr{N}_k(\varepsilon';\Lambda)$, which is (UV) at ε'.

(iii) By (i), $\mathscr{N}_k(\varepsilon')/\mathscr{N}_k(\varepsilon)\ge\mathsf{K}^{-2}\gamma_k^{-2n}$ with $n\ge(\varepsilon-\varepsilon')/a_k-1$; in particular $\mathsf{K}^2C_{\varepsilon'}\ge1$. For $C>C_{\varepsilon'}$ and large k this gives $2n\log(1/\gamma_k)\le\log(\mathsf{K}^2C)$, that is $a_k^{-1}\log(1/\gamma_k)\le\log(\mathsf{K}^2C)/(2(\varepsilon-\varepsilon'-a_k))$, and $C\downarrow C_{\varepsilon'}$ gives the bound. Under a lattice gap the same inequality gives $\mathscr{N}_k(\varepsilon')/\mathscr{N}_k(\varepsilon)\ge\mathsf{K}^{-2}e^{2m_{\rm lat}((\varepsilon-\varepsilon')/a_k-1)}\to\infty$. Corollary 277 provides the lattice gap $m_{\rm lat}=3.28/(625e^2)$. □

So (SD) bounds the physical gap from above, and it fails in the strong-coupling range of Corollary 277. It does not follow from invariance, reflection positivity and a gap bounded above and below in physical units: a heavy species with a growing weight removes it.

Proposition 283. *Let* $m>0$, $\rho\in(0,1)$, $q_k=e^{-ma_k}$, *let* $\phi_k^{(1)}$ *and* $\phi_k^{(2)}$ *be independent centred Gaussian fields on* $\mathbb{Z}^4$ *with covariances* $\prod_\mu q_k^{|x_\mu-y_\mu|}$ *and* $\prod_\mu\rho^{|x_\mu-y_\mu|}$, *let* $\lambda_k>0$, $\phi_k=\phi_k^{(1)}+\lambda_k\phi_k^{(2)}$, $A_k(f)=\sum_x a_k^4f(a_kx)\phi_k(x)$, *and form* $\mathscr{N}_k(\varepsilon)$ *with these packets and* $r=0$.

(a) *The law of* ϕ_k *is invariant under translations and* B_4 *and reflection positive for the reflections in coordinate hyperplanes through sites and bisecting links. For* $q_k\ge\rho$ *the spectrum of its transfer matrix in each coordinate direction on* $\Omega_k^\perp$ *lies in* $[0,q_k]$ *and contains* q_k: *the gap is* m *in physical units.*

(b) *Given* $\varepsilon>0$, *there are* λ_k *with* $\mathscr{N}_k(\varepsilon')/\mathscr{N}_k(\varepsilon)\to\infty$ *for every* $\varepsilon'\in(0,\varepsilon)$, *so that* (SD) *fails.*

(c) *For these* λ_k *and every* Z_k *for which* (M$_s$) *holds at* $n=2$, $Z_k^2\omega_k(A_k(f)A_k(g))\to0$ *for every separated pair: every limit of the two-point functions vanishes.*

Proof. (a) Each $\phi_k^{(i)}$ is the field of Proposition 207(a), with q_k replaced by ρ for $i=2$, which has the invariance and the reflection positivity, with the transfer operators P_1, P_2 of spectra $\{q_k^N\}$ and $\{\rho^N\}$, 1 simple, given in its proof. The pair

$(\phi_k^{(1)},\phi_k^{(2)})$ is then a reversible Markov chain in each coordinate direction with transfer operator $P_1\otimes P_2$, of spectrum $\{q_k^N\rho^{N'}\}$ with 1 simple, and the argument of that proof gives $\mathbb{E}[\vartheta(\bar F)F']=\langle g_F,g_{F'}\rangle$ for functions F,F' of the pair at positive times. The functions of ϕ_k at positive times are functions of the pair there, so ϕ_k is reflection positive, its transfer matrix is the restriction of $P_1\otimes P_2$ to an invariant subspace, and its spectrum on $\Omega_k^\perp$ lies in $[0,\max(q_k,\rho)]=[0,q_k]$. For a site x at positive time, $g_{\phi_k(x)}$ is a sum of two orthogonal eigenvectors, of eigenvalues q_k and ρ, the first one nonzero, so q_k is in the spectrum.

(b) Let ϑ fix the plane $\{x_0=c\}$, $|c|\le\frac12$, so that $\vartheta x=(2c-x_0,\mathbf{x})$. By independence and centring, $\omega_k(\vartheta(\overline{A_k(h)})A_k(h))=\mathscr{N}^{(1)}(h)+\lambda_k^2\mathscr{N}^{(2)}(h)$, with $\mathscr{N}^{(i)}(h)\ge0$ formed with $\phi_k^{(i)}$. For h supported in $\{x_0\ge a_k\}$,

$$\mathscr{N}^{(2)}(h)=a_k^8\sum_{\mathbf{x},\mathbf{y}}\overline{H(\mathbf{x})}H(\mathbf{y})\prod_{i=1}^3\rho^{|x_i-y_i|},$$
$$H(\mathbf{x})=\sum_{x_0}h(a_kx)\rho^{x_0-c}.$$

If h is supported in B_ε with $\|h\|_\infty\le1$ and $a_k\le1$, then $|H|\le\rho^{\varepsilon/a_k-1/2}/(1-\rho)$, at most $((2+a_0)/a_k)^3$ points $\mathbf{x}$ have $H(\mathbf{x})\ne0$, and $\sum_{\mathbf{y}}\prod_i\rho^{|x_i-y_i|}=((1+\rho)/(1-\rho))^3$, so

$$\mathscr{N}^{(2)}(h)\le U_k(\varepsilon)=c_\rho a_k^5\rho^{2\varepsilon/a_k},\qquad c_\rho=\frac{(2+a_0)^3(1+\rho)^3}{\rho(1-\rho)^5}.$$

For $\varepsilon'>0$ and $\tau\in(0,\frac12)$ let h_* be continuous with values in $[0,1]$, supported in $B_{\varepsilon'}$ and equal to 1 on $D=[\varepsilon'+\tau,\varepsilon'+1]\times[\tau,1]^3$. All terms of $\mathscr{N}^{(2)}(h_*)$ are nonnegative; keeping $x=y$ with $x_0=\lceil(\varepsilon'+\tau)/a_k\rceil$ and $a_k\mathbf{x}\in[\tau,1]^3$, for $a_k\le\frac12(1-\tau)$,

$$\mathscr{N}^{(2)}(h_*)\ge L_k(\varepsilon')=\Big(\frac{1-\tau}{2}\Big)^3a_k^5\rho^{2(\varepsilon'+\tau)/a_k+3}.$$

Let $\mathscr{N}_k^{(1)}(\varepsilon)$ be the supremum of $\mathscr{N}^{(1)}(h)$ over h supported in B_ε with $\|h\|_\infty\le1$. For $a_kx,a_ky\in D_\varepsilon=[\varepsilon+\frac14,\varepsilon+1]\times[\frac14,1]^3$ the covariance of $\phi_k^{(1)}(\vartheta x)$ and $\phi_k^{(1)}(y)$ is at least $e^{-m(2\varepsilon+6)}$; with the function h_* for ε and $\tau=\frac14$ this gives $\mathscr{N}_k^{(1)}(\varepsilon)\ge\ell_0=(\frac38)^8e^{-m(2\varepsilon+6)}>0$ for $a_k\le\frac38$, and it is finite. Put $\lambda_k^2=\mathscr{N}_k^{(1)}(\varepsilon)/U_k(\varepsilon)$. Then $\mathscr{N}_k(\varepsilon)\le\mathscr{N}_k^{(1)}(\varepsilon)+\lambda_k^2U_k(\varepsilon)=2\mathscr{N}_k^{(1)}(\varepsilon)$ and $\mathscr{N}_k(\varepsilon')\ge\lambda_k^2L_k(\varepsilon')$, so for $\tau<\varepsilon-\varepsilon'$

$$\frac{\mathscr{N}_k(\varepsilon')}{\mathscr{N}_k(\varepsilon)}\ge\frac{L_k(\varepsilon')}{2U_k(\varepsilon)}=\Big(\frac{1-\tau}{2}\Big)^3\frac{\rho^4(1-\rho)^5\rho^{-2(\varepsilon-\varepsilon'-\tau)/a_k}}{2(2+a_0)^3(1+\rho)^3}\to\infty.$$

(c) Let $(f,g)\in\mathscr{C}_2(Q_0,\delta)$, Q_0 of side L_0, and fix $\varepsilon',\tau>0$ with $\varepsilon'+\tau<\min(\varepsilon,\delta/2)$ and $\tau<\frac12$. For $a_k\le\varepsilon'$ the pair $(\bar h_*\circ\theta_{0,k},h_*)$ lies in $\mathscr{C}_2(Q',\varepsilon')$ for a cube Q' independent of k, so (M$_s$) gives $Z_k^2\lambda_k^2L_k(\varepsilon')\le Z_k^2\omega_k(\vartheta(A_k(h_*))A_k(h_*))\le M=M_2(Q',\varepsilon')$, that is $Z_k^2\mathscr{N}_k^{(1)}(\varepsilon)\le MU_k(\varepsilon)/L_k(\varepsilon')$. The covariance of $\phi_k^{(1)}$ is at most 1 and at most $(L_0/a_k+1)^4$ sites carry f or g, so $|\mathbb{E}[A_k^{(1)}(f)A_k^{(1)}(g)]|\le(L_0+1)^8\|f\|_\infty\|g\|_\infty$ for $a_k\le1$, and

$$Z_k^2\big|\mathbb{E}[A_k^{(1)}(f)A_k^{(1)}(g)]\big|\le\frac{M(L_0+1)^8}{\ell_0}\,\frac{U_k(\varepsilon)}{L_k(\varepsilon')}\|f\|_\infty\|g\|_\infty\to0.$$

For the second field the sites carrying f and g are at ℓ^∞ distance at least δ/a_k, and $\sum_{|z|_\infty\ge d}\prod_\mu\rho^{|z_\mu|}\le8\rho^d(1+\rho)^3/(1-\rho)^4$, so $\lambda_k^2|\mathbb{E}[A_k^{(2)}(f)A_k^{(2)}(g)]|\le\lambda_k^2C_1a_k^4\rho^{\delta/a_k}\|f\|_\infty\|g\|_\infty$ with $C_1=8(L_0+1)^4(1+\rho)^3/(1-\rho)^4$. Since $Z_k^2\lambda_k^2\le M/L_k(\varepsilon')$,

$$Z_k^2\lambda_k^2\big|\mathbb{E}[A_k^{(2)}(f)A_k^{(2)}(g)]\big|\le MC_1\Big(\frac{2}{1-\tau}\Big)^3\frac{\rho^{(\delta-2\varepsilon'-2\tau)/a_k-3}}{a_k}\|f\|_\infty\|g\|_\infty\to0,$$

because $\delta>2(\varepsilon'+\tau)$. The cross terms vanish. □

In the example the part $\lambda_k^2\mathscr{N}^{(2)}$ of the reflected norm, which produces the growth, is carried by energies at least $a_k^{-1}\log(1/\rho)\to\infty$, and (UV) fails, by the proof of Proposition 282(ii), which uses only the invariance, reflection positivity and the gap. For $\lambda_k=0$ it is the field of Proposition 207(a), for which (SD) and (M$_s$) hold. Like that field, it is not a lattice gauge theory; it shows that (SD), equivalently (UV), is an input on the ultraviolet behaviour of the Wilson measure that reflection positivity and the gap do not supply.

The nonvanishing condition. Under (SD) the condition (NE) is exactly the nontriviality of the limits. Write $R_k(g)=\omega_k(\vartheta(\overline{A_k(g)})A_k(g))/\mathscr{N}_k(\varepsilon)$ for g supported in $\{x_0\ge(r+1)a_k\}$.

Proposition 284. *Assume the hypothesis of Theorem 280 and* (SD).

(i) *The following are equivalent:* (NE)*; every limit S_2 of Corollary 281*(b) *is not identically zero; every such limit has $\mathscr{H}^{(1)}\ne0$.*

(ii) *If $\omega_k(\Phi_0\Phi_z)\ge c_k^2$ for all large k and all $z\in\mathbb{Z}^4$ with $|z|_\infty\ge2r+1$, then $R_k(g_0)\ge1$ for large k and every continuous g_0 with values in $[0,1]$, equal to 1 on B_ε and supported in $\{x_0\ge\varepsilon/2\}$; in particular* (NE) *holds.*

Proof. (i) The first statement implies the second by Corollary 281(b). The second and the third are equivalent: $\mathscr{H}^{(1)}\ne0$ means $S_2(\bar g\circ\theta_0,g)>0$ for some g supported in $\{x_0>0\}$, and the proof of Corollary 281(b) shows that a limit with this property for no g vanishes identically.

Suppose (NE) fails, so that $\liminf_kR_k(g)=0$ for every $g\in C_c(\mathbb{R}^4)$ supported in $\{x_0>0\}$. Let $g_1,\dots,g_J$ be supported in a cube of side L and in $\{x_0\ge\eta\}$, $\eta>0$, let $D>0$ be dyadic and $G=\sum_jg_j(\cdot-jDe_1)$. For large k, $jD/a_k\in\mathbb{Z}$ and $R_k(g_j(\cdot-jDe_1))=R_k(g_j)$ by Lemma 273(i),(iii). For $i\ne j$ the cross term $\langle\Psi_{A_k(g_i(\cdot-iDe_1))},\Psi_{A_k(g_j(\cdot-jDe_1))}\rangle$ equals $\omega_k(A_k(\bar g_i(\cdot-iDe_1)\circ\theta_{0,k})A_k(g_j(\cdot-jDe_1)))$, two packets in cubes of side L whose projections on the e_1 axis are at distance at least $|i-j|D-L$; by Theorem 280(i) in the direction e_1 with $d=D-L-4\varepsilon$, its modulus is at most $\mathscr{N}_k(\varepsilon)\mathsf{K}_L^2e^{-m_*(D-L-4\varepsilon-1)}\max_j\|g_j\|_\infty^2$. Hence $\sum_jR_k(g_j)\le R_k(G)+J^2\mathsf{K}_L^2e^{-m_*(D-L-4\varepsilon-1)}\max_j\|g_j\|_\infty^2$ for large k, and

since $\liminf_k R_k(G)=0$ and D is arbitrary, $\liminf_k \sum_j R_k(g_j)=0$.

Enumerate as $g_1,g_2,\dots$ the elements of the countable set $\mathscr{G}$ of the proof of Theorem 274(i) that are supported in $\{x_0>0\}$, and choose $k_1<k_2<\cdots$ with $\sum_{j\le J} R_{k_J}(g_j)<1/J$. Then $R_{k_J}(g_j)\to 0$ for every j. Let g be supported in a cube of side L and in $\{x_0\ge\eta\}$, and let $\tau\in(0,\min(\eta,1)/2]$. That proof gives $g_j\in\mathscr{G}$ with $\|g-g_j\|_\infty<\tau$ and support within τ of $\operatorname{supp} g$, so $g-g_j$ is supported in a cube of side $L+1$ in $\{x_0\ge\eta/2\}$, and by Lemma 279

$$R_k(g)^{1/2}\le R_k(g_j)^{1/2}+\mathsf{K}_{L+1}\Big(\frac{\mathscr{N}_k(\eta/2)}{\mathscr{N}_k(\varepsilon)}\Big)^{1/2}\tau.$$

The ratio has finite lim sup, by (SD) if $\eta/2<\varepsilon$ and by Proposition 282(i) otherwise. As τ is arbitrary, $R_{k_J}(g)\to 0$. By Corollary 281(b) a subsequence of (k_J) has a limit S_2, and by the approximation of $\theta_{0,k}$ by θ_0 in the proof of Theorem 274(iii), $S_2(\bar g\circ\theta_0,g)=\lim_J R_{k_J}(g)=0$ for every g supported in $\{x_0>0\}$. So this limit vanishes identically, and the second statement fails.

(ii) Let h be supported in B_ε with $\|h\|_\infty\le 1$, and $(r+1)a_k\le\varepsilon/2$. By Lemma 273(i), $\omega_k(\vartheta(\overline{A_k(h)})A_k(h))=\sum_{x,y}a_k^8\overline{h(a_kx)}h(a_ky)K(x,y)$ with $K(x,y)=\omega_k(\Phi_{\vartheta x}\Phi_y)-c_k^2=\omega_k(\Phi_0\Phi_{y-\vartheta x})-c_k^2$. For $a_kx_0,a_ky_0\ge\varepsilon/2$, $x_0,y_0\ge r+1$ and the time component of $y-\vartheta x$ is $x_0+y_0-2c\ge 2r+1$, so $K(x,y)\ge 0$ and the sum is at most $\sum_{x,y}a_k^8 g_0(a_kx)g_0(a_ky)K(x,y)=\omega_k(\vartheta(A_k(g_0))A_k(g_0))$, because $|h|\le g_0$. Taking the supremum over h gives $\mathscr{N}_k(\varepsilon)\le\omega_k(\vartheta(A_k(g_0))A_k(g_0))$. □

The condition of (ii) is a correlation inequality of Griffiths type for the connected two-point function of Φ; it is not proved here for the Wilson measure.

Products of packets. For $n\ge 3$ the moment bound reduces to reflected norms of products. Consider

(RN) For every $m\ge 1$, every cube Q and every $\delta>0$ there are $R_m(Q,\delta)<\infty$ and k_0 such that

$$Z_k^{2m}\omega_k\big(\vartheta(\bar P_k)P_k\big)\le R_m(Q,\delta)\prod_{j=1}^m\|f_j\|_\infty^2,\qquad P_k=\prod_{j=1}^m A_k(f_j),$$

for all $k\ge k_0$ and $(f_1,\dots,f_m)\in\mathscr{C}_m(Q,\delta)$ with supports in $\{x_0\ge\delta\}$.

We write (RN_m) for the statement at a fixed m, and $(\mathrm{M}_{\mathrm{s},n})$ for (M_s) at a fixed n.

Proposition 285. *Assume the hypothesis of Theorem 66.*

(i) $(\mathrm{M}_{\mathrm{s},2m})$ *implies* (RN_m).

(ii) *Let $n\ge 2$. If (RN_i) holds for $1\le i\le n-1$, then $(\mathrm{M}_{\mathrm{s},n})$ holds, with*

$$M_n(Q,\delta)=\mathsf{K}_{2nL/\delta}^n\max_{1\le i\le n-1}\big(R_i(Q^*,\delta_n)R_{n-i}(Q^*,\delta_n)\big)^{1/2}$$

for Q of side L, $Q^*=[-L-2,L+2]^4$ and $\delta_n=\delta/(16(n+1))$.

(iii) (M_s) *holds if and only if* (RN) *holds, and* $(\mathrm{M}_{\mathrm{s},2})$ *if and only if* (RN_1). *With $Z_k=\mathscr{N}_k(\varepsilon)^{-1/2}$ as in Theorem 280,* (RN_1) *is equivalent to* (SD)*, and* $(\mathrm{M}_{\mathrm{s},3})$ *follows from* (SD) *and* (RN_2).

Proof. (i) Let $(f_j)\in\mathscr{C}_m(Q,\delta)$ be supported in $\{x_0\ge\delta\}$ and $a_k\le\delta$. The functions $\bar f_j\circ\theta_{0,k}$ are supported in $\{x_0\le a_k-\delta\}$, so the $2m$-tuple of these and the f_j lies in $\mathscr{C}_{2m}(Q',\delta)$ for a cube Q' containing Q and all $\theta_{0,k}(Q)$, and $\omega_k(\vartheta(\bar P_k)P_k)=\omega_k(\prod_j A_k(\bar f_j\circ\theta_{0,k})\prod_j A_k(f_j))$ by Lemma 273(i). $(\mathrm{M}_{\mathrm{s},2m})$ gives (RN_m) with $R_m(Q,\delta)=M_{2m}(Q',\delta)$.

(ii) Let $(f_j)\in\mathscr{C}_n(Q,\delta)$, $s=\delta/(2n)$, and decompose every f_j with the functions χ_κ^s of the proof of Theorem 274(vi); at most $\mathsf{K}_{L/s}=\mathsf{K}_{2nL/\delta}$ pieces of each are nonzero, each with norm at most $\|f_j\|_\infty$. Fix a nonzero term, with pieces in closed cubes $C_1,\dots,C_n$ of side s, points $p\in C_1$ and $q\in C_2$ in the supports and μ with $q_\mu-p_\mu\ge\delta$ (after exchanging f_1 and f_2). The projections of the C_j on the μ axis have total length $\delta/2$, so their complement in $[p_\mu,q_\mu]$ contains an interval of length at least $\delta/(2(n+1))$; let c be its midpoint. Every projection is at distance at least $\delta/(4(n+1))=4\delta_n$ from c; let I and J be the indices of the cubes below and above c, nonempty since $1\in I$ and $2\in J$. Translate all pieces by one vector of $a_k\mathbb{Z}^4$ so that c moves within $a_k/2$ of the plane $\{x_\mu=a_kc_\mu\}$ fixed by $\theta_{\mu,k}$, $|c_\mu|\le\frac12$, and so that the pieces lie in Q^*; the moment does not change. For $a_k\le 2\delta_n$ the pieces of I lie in $\{x_\mu\le a_kc_\mu-3\delta_n\}$ and those of J in $\{x_\mu\ge a_kc_\mu+3\delta_n\}$. With $F=\prod_{j\in I}A_k(\bar f_j'\circ\theta_{\mu,k})$ and $G=\prod_{j\in J}A_k(f_j')$, f_j' the translated pieces, Lemma 273(i) gives $\prod_{j\in I}A_k(f_j')=\vartheta_\mu(\bar F)$, and by the invariance under π_μ and the Cauchy–Schwarz inequality for the positive form of Lemma 273(iii),

$$\Big|\omega_k\Big(\prod_{j=1}^n A_k(f_j')\Big)\Big|=|\omega_k(\vartheta(\overline{\pi_\mu F})\pi_\mu G)|\le N_0(\pi_\mu F)N_0(\pi_\mu G).$$

The factors of $\pi_\mu F$ and $\pi_\mu G$ are supported in Q^* and in $\{x_0\ge 3\delta_n-a_k\}\subset\{x_0\ge\delta_n\}$, pairwise at distance at least δ, so $(\mathrm{RN}_{|I|})$ and $(\mathrm{RN}_{|J|})$ bound $Z_k^nN_0(\pi_\mu F)N_0(\pi_\mu G)$ by $(R_{|I|}R_{|J|})^{1/2}\prod_j\|f_j\|_\infty$ for large k. Summing over the at most $\mathsf{K}_{2nL/\delta}^n$ terms gives the claim.

(iii) The first two statements follow from (i) and (ii). For $Z_k=\mathscr{N}_k(\varepsilon)^{-1/2}$, (RN_1) at $Q\supset B_{\varepsilon'}$ and $\delta=\varepsilon'$ gives $\mathscr{N}_k(\varepsilon')\le R_1\mathscr{N}_k(\varepsilon)$, which is (SD); conversely Lemma 279 bounds the left side of (RN_1) by $\mathsf{K}_L^2\|f\|_\infty^2$ times $\mathscr{N}_k(\delta)/\mathscr{N}_k(\varepsilon)$, which is bounded by (SD) for $\delta<\varepsilon$ and by Proposition 282(i) for $\delta\ge\varepsilon$. The last claim is (ii) for $n=3$. □

The Cauchy–Schwarz inequality of reflection positivity bounds a moment by the reflected norms of the products of the two groups of packets on the two sides of a plane, and for $n\ge 3$ one group contains at least two packets; the reduction therefore stops at (RN_m) with $m\ge 2$, the reflected norms of products of separated packets. If the packets are jointly Gaussian, Wick's formula writes $\omega_k(\vartheta(\bar P_k)P_k)$ as a sum of $(2m-1)!!$ products of covariances of separated pairs, and (RN) follows from (RN_1) by (ii) with $n=2$; for the Wilson measure (RN_m) with $m\ge 2$ is not proved.

Rotations at separated points. The hypercubic group fixes the lowest anisotropy that a separated two-point function can carry. Let $\mathscr{P}_l$ be the space of homogeneous polynomials of degree l on $\mathbb{R}^4$, $\mathscr{Y}_l\subset\mathscr{P}_l$ the harmonic ones, and $\mathscr{Y}_l^{B_4}$ the

elements of $\mathscr{Y}_l$ invariant under B_4. Put

$$Y_4(z) = \sum_\mu z_\mu^4 - \tfrac{1}{2}|z|^4,$$
$$Y_6(z) = \sum_\mu z_\mu^6 - \tfrac{5}{4}|z|^2 \sum_\mu z_\mu^4 + \tfrac{5}{16}|z|^6.$$

Lemma 286. *$\mathscr{Y}_l^{B_4} = 0$ for odd l, and $\dim \mathscr{Y}_{2j}^{B_4} = p(j) - p(j-1)$ for $j \ge 1$, where $p(j)$ is the number of partitions of j into at most four parts. In particular $\dim \mathscr{Y}_l^{B_4}$ is 0, 1, 1, 2 for $l = 2,4,6,8$, and $\mathscr{Y}_4^{B_4} = \mathbb{R}Y_4$, $\mathscr{Y}_6^{B_4} = \mathbb{R}Y_6$.*

Proof. A sign change $z_\mu \mapsto -z_\mu$ multiplies a monomial by -1 to the power of its degree in z_μ, so a polynomial invariant under the sign changes contains only monomials of even degree in every variable; it is a polynomial in $w_\mu = z_\mu^2$, and invariance under the permutations makes it symmetric in $w_0, \dots, w_3$. The monomial symmetric polynomials $m_\kappa(w)$, κ a partition of j into at most four parts, form a basis of the symmetric polynomials of degree j in four variables, so $\dim \mathscr{P}_{2j}^{B_4} = p(j)$ and $\mathscr{P}_l^{B_4} = 0$ for odd l. By [110, Ch. IV], $\mathscr{P}_l = \mathscr{Y}_l \oplus |z|^2 \mathscr{P}_{l-2}$, and both summands are invariant under $\mathrm{O}(4)$, so the projections on them commute with B_4 and $\mathscr{P}_l^{B_4} = \mathscr{Y}_l^{B_4} \oplus |z|^2 \mathscr{P}_{l-2}^{B_4}$; multiplication by $|z|^2$ is injective. This gives the dimension, and $p(0), \dots, p(4)$ are $1,1,2,3,5$. In $\mathbb{R}^4$, $\Delta |z|^{2i} = 2i(2i+2)|z|^{2i-2}$, $\Delta \sum_\mu z_\mu^{2i} = 2i(2i-1)\sum_\mu z_\mu^{2i-2}$ and $\Delta(|z|^2 \sum_\mu z_\mu^4) = 24 \sum_\mu z_\mu^4 + 12|z|^4$, so $\Delta Y_4 = 12|z|^2 - \frac{1}{2}\cdot 24|z|^2 = 0$ and $\Delta Y_6 = 30\sum_\mu z_\mu^4 - \frac{5}{4}(24\sum_\mu z_\mu^4 + 12|z|^4) + \frac{5}{16}\cdot 48|z|^4 = 0$. Both are invariant and nonzero, since $Y_4(e_0) = \frac{1}{2}$ and $Y_6(e_0) = \frac{1}{16}$. □

Proposition 287. *Let S_2 be a bilinear form on the separated pairs, invariant under the translations and B_4, with $S_2(f,g) = \iint f(x)g(y)K(x-y)\,dx\,dy$ for separated pairs, where K is square integrable on every compact subset of $\mathbb{R}^4 \setminus \{0\}$.*

(i) *For $\varphi \in C_c(\mathbb{R}^4 \setminus \{0\})$ and $g_\varepsilon = \varepsilon^{-4}\chi(\cdot/\varepsilon)$, $\chi \ge 0$ continuous with $\int \chi = 1$ and support in $[-1,1]^4$,*

$$\int K\varphi\,dz = \lim_{\varepsilon\to 0} S_2(\varphi, g_\varepsilon),$$

so K is determined almost everywhere by S_2. For $R \in \mathrm{O}(4)$, $S_2(f\circ R^{-1}, g\circ R^{-1}) = S_2(f,g)$ for all separated pairs if and only if $K \circ R = K$ almost everywhere; in particular $K \circ g = K$ for $g \in B_4$.

(ii) *For almost every $\rho > 0$, $K(\rho\,\cdot) \in L^2(S^3)$ and $K(\rho\omega) = \sum_{j\ge 0} K_{2j}(\rho,\omega)$ in $L^2(S^3)$, with $K_{2j}(\rho,\cdot)$ the restriction of an element of $\mathscr{Y}_{2j}^{B_4}$. Hence $K_2 = 0$, $K_0(\rho)$ is the mean of K on the sphere of radius ρ, and there are measurable k_4, k_6 with*

$$K_4(\rho,\omega) = k_4(\rho)Y_4(\omega), \qquad K_6(\rho,\omega) = k_6(\rho)Y_6(\omega).$$

(iii) *The following are equivalent:* (a) *S_2 is invariant under $\mathrm{O}(4)$ on the separated pairs;* (b) *it is invariant under one $R \in \mathrm{O}(4)$ of infinite order;* (c) *$K_{2j}(\rho,\cdot) = 0$ for every $j \ge 2$ and almost every ρ;* (d) *$K(z) = K_0(|z|)$ for almost every z.*

Proof. (i) Let φ be supported in $\{|z|_\infty \ge 3\eta\}$ and $\varepsilon \le \eta$; the pair (φ, g_ε) is separated by at least 2η, and

$$S_2(\varphi, g_\varepsilon) = \int K(z)\varphi_\varepsilon(z)\,dz, \qquad \varphi_\varepsilon(z) = \int \varphi(z+y)g_\varepsilon(y)\,dy.$$

The functions φ_ε are supported in one compact subset of $\mathbb{R}^4 \setminus \{0\}$ and converge uniformly to φ, and K is integrable there. For $R \in \mathrm{O}(4)$ the substitution $x = Ru$, $y = Rv$ gives $S_2(f\circ R^{-1}, g\circ R^{-1}) = \iint f(u)g(v)K(R(u-v))\,du\,dv$, so invariance holds if and only if the form with kernel $K\circ R - K$ vanishes on the separated pairs, and by the formula just proved this happens if and only if $K \circ R = K$ almost everywhere.

(ii) By Fubini's theorem in polar coordinates, $K(\rho\,\cdot) \in L^2(S^3)$ for almost every ρ, and for each of the finitely many $g \in B_4$, $K(\rho g\,\cdot) = K(\rho\,\cdot)$ for almost every ρ. The restrictions of the spaces $\mathscr{Y}_l$ to S^3 are orthogonal in $L^2(S^3)$ and span it [110, Ch. IV]; each is invariant under $\mathrm{O}(4)$, so the orthogonal projections on them commute with composition by elements of $\mathrm{O}(4)$. The component of degree l of $K(\rho\,\cdot)$ is therefore invariant under B_4, and Lemma 286 gives the claims.

(iii) Let G_K be the set of $R \in \mathrm{O}(4)$ with $K \circ R = K$ almost everywhere. On every closed annulus $A = \{r \le |z| \le r'\}$, $0 < r < r'$, the map $R \mapsto K \circ R$ into $L^2(A)$ is continuous, as one sees by approximating K in $L^2(A)$ by continuous functions; so G_K is a closed subgroup of $\mathrm{O}(4)$, and it contains B_4 by (i). By the proof of Proposition 205(iv), which uses only these two properties, G_K is finite or equal to $\mathrm{O}(4)$, so (a) and (b) are equivalent by (i). If (a) holds, let $R_1, R_2, \dots$ be dense in $\mathrm{O}(4)$; for almost every ρ, $K(\rho R_i\,\cdot) = K(\rho\,\cdot)$ in $L^2(S^3)$ for every i, hence, by continuity of $R \mapsto u \circ R$ in $L^2(S^3)$, for every R, and each component $K_l(\rho,\cdot)$ is the restriction of an $\mathrm{O}(4)$-invariant element of $\mathscr{Y}_l$. Such an element is constant on spheres, hence equal to $c|z|^l$, and $\Delta|z|^l = l(l+2)|z|^{l-2}$ vanishes only for $l = 0$; so (d) holds. (d) implies (c), (c) implies (d) because $K_2 = 0$, and (d) implies (a) by (i). □

So the part of a separated two-point kernel that is invariant under B_4 and not under $\mathrm{O}(4)$ has no components of degree below four, and its component of degree four has the fixed angular profile Y_4, with one radial coefficient k_4; by (i), $\int K(z)Y_4(z)\psi(|z|)\,dz = \lim_{\varepsilon\to 0} S_2(Y_4\psi(|\cdot|), g_\varepsilon)$ for $\psi \in C_c((0,\infty))$, a separated two-point function of packets. By (iii), $(\mathrm{W_s})$ for $n = 2$ is the vanishing of all K_{2j} with $j \ge 2$. Reflection positivity and clustering do not remove the component of degree four, by Proposition 289(iv) below.

Two statements about the lattice moments

$$M_k(x) = \omega_k\Big(\prod_{j=1}^n (\Phi_{x_j} - c_k)\Big), \qquad x = (x_1, \dots, x_n) \in (\mathbb{Z}^4)^n,$$

give $(\mathrm{W_s})$. For $0 < \delta < \Lambda$ let $\mathscr{W}_k(\delta,\Lambda)$ be the set of $x \in (\mathbb{Z}^4)^n$ with $\delta/a_k \le |x_i - x_j| \le \Lambda/a_k$ for $i \ne j$, $|\cdot|$ being the Euclidean norm. For $n \ge 2$:

(SB_n) for $0 < \delta < \Lambda$,

$$b_n(\delta,\Lambda) = \limsup_k \sup_{x\in\mathscr{W}_k(\delta,\Lambda)} Z_k^n |M_k(x)| < \infty;$$

(RI_n) for $0 < \delta < \Lambda$, $\limsup_k Z_k^n |M_k(x) - M_k(x')| = 0$, the supremum over the pairs $x, x' \in \mathscr{W}_k(\delta,\Lambda)$ with $\big||x_i - x_j| - |x_i' - x_j'|\big| \le 4$ for all $i < j$.

(SB_n) is a pointwise bound of Symanzik type on the renormalised lattice moments at separated points. (RI_n) says that

at separations of physical size, that is of order $1/a_k$ lattice units, the renormalised moments depend on the configuration only through its mutual Euclidean distances, up to an error that tends to zero, when these distances are changed by at most four lattice units. This is rotation invariance at large lattice distance, the form in which the restoration of rotation invariance predicted by Symanzik's improvement programme enters; neither statement is proved here for the Wilson measure.

Theorem 288. *Let the states ω_k be invariant under the translations of $\mathbb{Z}^4$, let Φ be as above and $Z_k > 0$, and let $n \geq 2$.*

(i) (SB_n) *implies* ($\mathrm{M_s}$) *for this n.*

(ii) (SB_n) *and* (RI_n) *imply that for every $R \in \mathrm{O}(4)$ and every separated n-tuple*

$$\lim_k \Big[\omega_k\Big(\prod_j Z_k A_k(f_j \circ R^{-1})\Big) - \omega_k\Big(\prod_j Z_k A_k(f_j)\Big)\Big] = 0.$$

(iii) *If* (SB_n) *and* (RI_n) *hold for every $n \geq 2$, then* ($\mathrm{M_s}$) *holds and* ($\mathrm{W_s}$) *holds for every R. Under the hypothesis of Theorem 66 every limit of Theorem 274 then has $G_S = \mathrm{O}(4)$, and with* ($\mathrm{U_s}$) *the conclusions of Corollary 275 hold.*

(iv) *Under* (SB_2) *and* (RI_2)*, every subsequence has a further subsequence along which $\lim_k Z_k^2 \omega_k(A_k(f)A_k(g)) = \iint f(x)g(y)K(x-y)\,dx\,dy$ for every separated pair, with $K(z) = K_0(|z|)$ and K_0 bounded on every compact subset of $(0,\infty)$.*

Proof. For $u \in \mathbb{R}^4$ let $\rho_k(u)$ be the point of $\mathbb{Z}^4$ nearest to u/a_k, ties broken by a fixed rule. Then $|u/a_k - \rho_k(u)| \leq 1$, the half diagonal of the unit cube of $\mathbb{Z}^4$, and each set $\rho_k^{-1}(x)$ is, up to a null set, the cube of side a_k centred at $a_k x$. Let $(f_j) \in \mathscr{C}_n(Q,\delta)$, Q of side L, so that all supports lie in a ball B of radius L, points of different supports are at Euclidean distance between δ and $2L$, and the same holds for the rotated functions $f_j \circ R^{-1}$ with the ball RB about the rotated centre. Let $a_k \leq \min(\delta/4, L/2)$ and put $F(u) = \prod_j f_j(u_j)$, $\mathscr{B}_k(F) = \omega_k(\prod_j Z_k A_k(f_j))$ and

$$I_k(F) = \int F(u)\, Z_k^n M_k\big(\rho_k(u_1), \dots, \rho_k(u_n)\big)\, du.$$

(i) $\mathscr{B}_k(F) = Z_k^n a_k^{4n} \sum_x \prod_j f_j(a_k x_j) M_k(x)$, the nonzero terms have $x \in \mathscr{W}_k(\delta, 2L)$, and at most $(3L/a_k)^4$ sites x_j have $a_k x_j \in B$, because the cubes of side a_k about them are disjoint and lie in a ball of radius $L + a_k \leq \frac{3}{2}L$ and volume at most $\frac{\pi^2}{2}(\frac{3}{2}L)^4 < (3L)^4$. So $|\mathscr{B}_k(F)| \leq (b_n(\delta, 2L)+1)(3L)^{4n} \prod_j \|f_j\|_\infty$ for large k.

(ii) Since ρ_k is constant on the cubes,

$$I_k(F) - \mathscr{B}_k(F) = \sum_x Z_k^n M_k(x) \int_{\prod_j \rho_k^{-1}(x_j)} \big(F(u) - F(a_k x)\big)\, du.$$

A nonzero term has $|u_j - a_k x_j| \leq a_k$ for some u_j in the support of f_j, hence $x \in \mathscr{W}_k(\delta/2, 3L)$, and the number of such x is at most $(3L/a_k)^{4n}$, the cubes about the x_j lying in a ball of radius $L + 2a_k \leq 2L$ and volume $8\pi^2 L^4 < (3L)^4$. So $|I_k(F) - \mathscr{B}_k(F)| \leq (b_n(\delta/2, 3L)+1)(3L)^{4n} \varpi_F(a_k)$ for large k, with $\varpi_F(s)$ the largest change of F when every u_j moves by at most s; it tends to 0, and the same holds for $F_R(u) = \prod_j f_j(R^{-1}u_j)$. By the substitution $u = Rv$,

$$I_k(F_R) - I_k(F) = \int F(v) Z_k^n \big[M_k(x'(v)) - M_k(x(v))\big]\, dv,$$

with $x_j(v) = \rho_k(v_j)$ and $x'_j(v) = \rho_k(Rv_j)$. For v in the support of F, $|x_i - x_j|$ and $|x'_i - x'_j|$ differ from $|v_i - v_j|/a_k = |Rv_i - Rv_j|/a_k$ by at most 2, hence from each other by at most 4, and both configurations lie in $\mathscr{W}_k(\delta/2, 3L)$. By ($\mathrm{RI}_n$) the bracket is at most $\varepsilon_k \to 0$, and $|I_k(F_R) - I_k(F)| \leq \varepsilon_k \prod_j \|f_j\|_{L^1}$.

(iii) The first claims are (i) and (ii) for every n, and the rest is Theorem 274(vii) and Corollary 275.

(iv) By translation invariance $M_k(x_1, x_2) = G_k(x_2 - x_1)$ with $G_k(y) = M_k(0, y)$; put $\kappa_k(z) = Z_k^2 G_k(\rho_k(z))$. For separated f, g as above, $y = \rho_k(v) - \rho_k(u)$ and $y' = \rho_k(v-u)$ have $|y|$ and $|y'|$ within 2 and 1 of $|v-u|/a_k$, so (RI_2) and the proof of (ii) give $Z_k^2 \omega_k(A_k(f)A_k(g)) - \iint f(u)g(v)\kappa_k(v-u)\,du\,dv \to 0$. For $r \leq |z| \leq r'$ and large k, $|\rho_k(z)| \in [r/(2a_k), 2r'/a_k]$, so $|\kappa_k| \leq b_2(r/2, 2r') + 1$ there, and $|\kappa_k(Rz) - \kappa_k(z)| \to 0$ uniformly there for every $R \in \mathrm{O}(4)$ by (RI_2), since $|\rho_k(Rz)|$ and $|\rho_k(z)|$ differ by at most 2. A diagonal subsequence converges weakly-$*$ in L^∞ of every annulus to some K, and then $K \circ R = K$ almost everywhere for every R. For separated f, g, $\iint f(u)g(v)\kappa_k(v-u)\,du\,dv = \int \kappa_k(z)\Theta(z)\,dz$ with $\Theta(z) = \int f(u)g(u+z)\,du$ integrable and supported in an annulus, so the limit is $\iint f(u)g(v)K(v-u)\,du\,dv$, and $K(-z) = K(z)$. Proposition 287(iii) gives $K(z) = K_0(|z|)$ almost everywhere. □

Part (i) does not use the averaged condition, and parts (i), (ii), (iv) use of the states only their translation invariance. (SB_n) alone does not give ($\mathrm{W_s}$), and neither do reflection positivity, the gap in every coordinate direction and non-Gaussian moments, by the following family; it is not a lattice gauge theory, and, as in Remark 319, the proofs of this subsection use the states only through the properties in (i) below.

Proposition 289. *Let $m > 0$, $q_k = e^{-m a_k}$, $p \geq 1$, let $\phi^{(1)}, \dots, \phi^{(p)}$ be independent copies of the Gaussian field of Proposition 207(a), $\Phi_x = \prod_i \phi_x^{(i)}$, $A_k(f) = \sum_x a_k^4 f(a_k x)\Phi_x$ and $Z_k = 1$.*

(i) *The law of $(\phi^{(1)}, \dots, \phi^{(p)})$ is invariant under translations and B_4, reflection positive for the reflections in coordinate hyperplanes through sites and bisecting links, and its transfer matrix in each coordinate direction has spectrum $\{q_k^N : N \geq 0\}$ with 1 a simple eigenvalue.*

(ii) *For all n and $f_j \in C_c(\mathbb{R}^4)$ the moments converge,*

$$\lim_k \omega_k\Big(\prod_j A_k(f_j)\Big) = \int \prod_j f_j(u_j) \Big(\sum_\pi \prod_{\{a,b\} \in \pi} e^{-m\|u_a - u_b\|_1}\Big)^p du,$$

the sum over the perfect matchings π of $\{1, \dots, n\}$; for f_j supported in a cube of side $L \geq a_k$ they are at most $(n!)^p (2L)^{4n} \prod_j \|f_j\|_\infty$. So (M), ($\mathrm{M_s}$), ($\mathrm{SB}_n$) *and* ($\mathrm{U_s}$) *hold.*

(iii) *For $p \geq 2$ the limit is not Gaussian.*

(iv) *$G_S = B_4$,* (RI_2) *fails, and the kernel $K(z) = e^{-pm\|z\|_1}$ of S_2 has $k_4 \neq 0$ on a set of positive measure.*

Proof. (i) In each coordinate direction the field $(\phi^{(1)},\dots,\phi^{(p)})$ is the stationary reversible Markov chain with transition operator $P^{\otimes p}$, P that of the proof of Proposition 207(a); the spectrum of $P^{\otimes p}$ consists of the products $q^{N_1+\cdots+N_p}$, and 1 is simple. That proof uses only these properties.

(ii) By independence and Wick's formula, $\omega_k(\prod_j \Phi_{x_j}) = \big(\sum_\pi \prod_{\{a,b\}\in\pi} w_k(x_a-x_b)\big)^p$ with $w_k(z)=\prod_\mu q_k^{|z_\mu|}=e^{-ma_k\|z\|_1}$, so $c_k=0$; this is at most the number of matchings to the power p when n is even and 0 otherwise, and the moments are Riemann sums of a bounded continuous function. At most $(L/a_k+1)^4\le(2L/a_k)^4$ sites x have a_kx in a cube of side $L\ge a_k$, which gives the bound, hence (M), ($\mathrm{M_s}$) and (SB_n), and the convergence of the whole sequence gives ($\mathrm{U_s}$).

(iii) For $n=4$ write $a=w_{12}w_{34}$, $b=w_{13}w_{24}$, $c=w_{14}w_{23}$ with $w_{ij}=e^{-m\|u_i-u_j\|_1}$. The Gaussian family with the same S_2 has the integrand $a^p+b^p+c^p$, and $(a+b+c)^p>a^p+b^p+c^p$ for $p\ge2$ and $a,b,c>0$; so S_4 differs from it for nonnegative nonzero f_j.

(iv) $B_4\subset G_S\subset G_{S_2}$, and $G_{S_2}=B_4$ by the argument of Proposition 207(a) with pm in place of m. For $y=(2N,0,0,0)$ and $y'=(N,N,N,N)$, $|y|=|y'|=2N$ and $M_k(0,y)-M_k(0,y')=e^{-2pma_kN}-e^{-4pma_kN}$; with $N=\lceil\delta/a_k\rceil$ both lie in $\mathscr{W}_k(\delta,3\delta)$ for large k, and the difference tends to $e^{-2pm\delta}-e^{-4pm\delta}>0$, so ($\mathrm{RI}_2$) fails. Finally $Y_4(z)=\frac12\sum_\mu z_\mu^4-\sum_{\mu<\nu}z_\mu^2z_\nu^2$, and with $I_i=\int_{\mathbb R}|t|^ie^{-M|t|}dt=2\,i!/M^{i+1}$, $M=pm$,

$$\int_{\mathbb R^4}K(z)Y_4(z)\,dz=2I_4I_0^3-6I_2^2I_0^2=\frac{768-384}{M^8}>0.$$

In polar coordinates the left side is $\|Y_4\|^2_{L^2(S^3)}\int_0^\infty\rho^7k_4(\rho)\,d\rho$, by the orthogonality of the components in Proposition 287(ii), so k_4 is not almost everywhere zero. □

For a product kernel $K(z)=\prod_\mu k(z_\mu)$ with $k\ge0$ and moments $I_i=\int|t|^ik(t)\,dt$ the same computation gives $\int KY_4\,dz=2I_0^2I_2^2(I_4I_0/I_2^2-3)$: the weighted coefficient $\int\rho^7k_4\,d\rho$ has the sign of the excess kurtosis of k, which is 3 for $e^{-M|t|}$ and 0 for a Gaussian k. The family satisfies every hypothesis of Theorem 288 except (RI_2), so among the lattice statements used here it is (RI_n) that excludes it.

The scale. For a family $S=(S_n)$ on the separated tuples and $\lambda,c>0$ put

$$(D_{\lambda,c}S)_n(f_1,\dots,f_n)=c^nS_n\big(f_1(\lambda\,\cdot),\dots,f_n(\lambda\,\cdot)\big);$$

this is a family on the separated tuples, because the map taking f to $f(\lambda\,\cdot)$ takes $\mathscr{C}_n(Q,\delta)$ onto $\mathscr{C}_n(\lambda^{-1}Q,\delta/\lambda)$. Moreover $D_{\lambda,c}D_{\lambda',c'}=D_{\lambda\lambda',cc'}$. For a nontrivial limit S of Theorem 274 let $m(S)=\min(\sigma(H_0)\setminus\{0\})$, which exists and is at least m_* by Theorem 274(iv), (vi).

Lemma 290. *Let S be a limit of Theorem 274 and $\lambda,c>0$. For P at positive 0-times let $P^{\lambda,c}$ be obtained by replacing every $f_{\beta,j}$ by $f_{\beta,j}(\lambda\,\cdot)$ and every coefficient c_β by $c^{n_\beta}c_\beta$. Then $D_{\lambda,c}S(\theta_0\bar P\cdot P')=S(\theta_0\overline{P^{\lambda,c}}\cdot P'^{\lambda,c})$, and $\Psi'(P)\mapsto\Psi_0(P^{\lambda,c})$ extends to a unitary map V of the Hilbert space $\mathscr{H}'$ of $D_{\lambda,c}S$ onto $\mathscr{H}_0$ with $V\Omega'=\Omega_0$. With $H'=\lambda^{-1}V^{-1}H_0V$, $\Psi'(\tau_{te_0}P)=e^{-tH'}\Psi'(P)$ for $t\ge0$, and $\sigma(H')=\lambda^{-1}\sigma(H_0)$; when $D_{\lambda,c}S$ is itself a limit of Theorem 274, its Hamiltonian in the time direction is H', and for nontrivial S, $m(D_{\lambda,c}S)=m(S)/\lambda$. If S is nontrivial and $D_{\lambda,c}S=S$, then $\lambda=c=1$.*

Proof. Since θ_0 is linear, $(f\circ\theta_0)(\lambda\,\cdot)=f(\lambda\,\cdot)\circ\theta_0$, which gives the identity; it shows that the form of $D_{\lambda,c}S$ is positive and that V is isometric, and V has dense range because $P=(P^{1/\lambda,1/c})^{\lambda,c}$. Further $(f(\cdot-te_0))(\lambda\,\cdot)=(f(\lambda\,\cdot))(\cdot-\frac{t}{\lambda}e_0)$, so $V\Psi'(\tau_{te_0}P)=\Psi_0(\tau_{(t/\lambda)e_0}P^{\lambda,c})=e^{-(t/\lambda)H_0}V\Psi'(P)$. A semigroup with this property is determined by its values on the dense set of the $\Psi'(P)$, which gives the claim on the Hamiltonian of a limit. If $D_{\lambda,c}S=S$, the two constructions coincide, $\sigma(H_0)=\lambda^{-1}\sigma(H_0)$, and the least point $m(S)$ of $\sigma(H_0)\setminus\{0\}$ satisfies $m(S)=m(S)/\lambda$; so $\lambda=1$, and $c^nS_n=S_n$ with $S_n\ne0$ for some $n\ge2$ by Theorem 274(vi) gives $c=1$. □

The normalisation of Proposition 276(ii) fixes the size of the field, not the scale: assigning the states to the spacings with a shift produces a dilated limit with the same normalisation.

Proposition 291. *Assume the hypothesis of Theorem 66, and suppose that the moments $\omega_k(\prod_j Z_kA_k(f_j))$ converge for every separated tuple to a nontrivial family S. Let $j\ge1$, and let $\omega_k''=\omega_k$ for even k and $\omega_k''=\omega_{k+j}$ for odd k, with the packets A_k'' formed from ω_k'' at the spacing a_k.*

(i) *(ω_k'') satisfies the hypothesis of Theorem 66 with $2^j\ell$ in place of ℓ.*

(ii) *For every choice of $Z_k''>0$ for which ($\mathrm{M_s}$) holds along (ω_k''), the moments $\omega_k''(\prod_jZ_k''A_k''(f_j))$ either fail to converge for some separated tuple or converge to the trivial family.*

(iii) *If $C_k''(0)>0$ for large k and ($\mathrm{M_s}$) holds along (ω_k'') for $Z_k''=C_k''(\sigma_0)^{-1/2}$, then ($\mathrm{U_s}$) fails for it.*

Proof. (i) For odd k, ω_{k+j} satisfies $\mathrm{BP}(n_{k+j},\varpi)$ and $n_{k+j}a_k=2^jn_{k+j}a_{k+j}\le2^j\ell$.

(ii) For odd k, $a_k=2^ja_{k+j}$ gives $A_k''(f)=2^{4j}A_{k+j}(f(2^j\,\cdot))$. Write $Z_k''=\zeta_kZ_k$ for even k and $Z_k''=\zeta_k2^{-4j}Z_{k+j}$ for odd k. The even moments are ζ_k^n times moments that converge to $S_n(f_1,\dots,f_n)$, and the odd ones ζ_k^n times moments that converge to $(D_{2^j,1}S)_n(f_1,\dots,f_n)$. Both families are nontrivial, so ($\mathrm{M_s}$) bounds ζ_k. Along a subsequence $\zeta_k\to\zeta_e$ on the even and $\zeta_k\to\zeta_o$ on the odd k, and the two limits are $D_{1,\zeta_e}S$ and $D_{2^j,\zeta_o}S$. If they coincide and $\zeta_e\zeta_o\ne0$, then $D_{2^j,\zeta_o/\zeta_e}S=S$, contrary to Lemma 290; if exactly one of ζ_e,ζ_o vanishes, one limit is trivial and the other not. So a common limit has $\zeta_e=\zeta_o=0$ and is trivial.

(iii) By Proposition 276(ii) every limit is nontrivial, so by (ii) some separated moment does not converge, and by Corollary 275 ($\mathrm{U_s}$) fails. □

The scale is fixed by a mass. With f_0, σ_0, C_k as in Proposition 276 and γ_k the norm of $\mathbb{T}_k$ on $\Omega_k^\perp$, consider for $m_0>0$:

(MC_{m_0}) $\liminf_k a_k^{-1}\log(1/\gamma_k)\ge m_0$, and there are $c_0>0$ and k_1 with $C_k(t)\ge c_0e^{-m_0t}C_k(\sigma_0)$ for $k\ge k_1$ and $t\in a_k\mathbb{Z}$, $t\ge\sigma_0$;

and the statement that the limits differ at most by the scale and the size of the field:

($\mathrm{O_s}$) for any two limits S, S' of Theorem 274 along subsequences there are $\lambda, c > 0$ with $S' = D_{\lambda,c} S$.

Theorem 292. *Assume the hypothesis of Theorem 66, $C_k(0) > 0$ for large k, $Z_k = C_k(\sigma_0)^{-1/2}$ and* ($\mathrm{M_s}$).

(i) *Every limit S of Theorem 274 satisfies*

$$\langle \Psi_0(f_0), e^{-\sigma_0 H_0} \Psi_0(f_0) \rangle = 1,$$

and under (MC_{m_0}) *it has $m(S) = m_0$.*

(ii) *Under* (MC_{m_0}) *and* ($\mathrm{O_s}$), ($\mathrm{U_s}$) *holds and the moments converge for every separated tuple.*

(iii) *For the Gaussian field of Proposition 207(a) at mass m,* (MC_m) *holds; for the same field at two masses $m \neq m'$, interleaved,* ($\mathrm{O_s}$) *holds and* (MC_{m_0}) *fails for every m_0.*

Proof. (i) The first claim and nontriviality are Proposition 276(ii). The proof of Theorem 274(iv) with γ_k^{1/a_k} in place of $e^{-m_* t}$ gives $\langle v, e^{-tH_0} v \rangle \le \limsup_k \gamma_k^{t/a_k} \|v\|^2 \le e^{-m_0 t} \|v\|^2$ on a dense set of $v \in \Omega_0^\perp$, so $m(S) \ge m_0$. By the approximation of $\theta_{0,k}$ by θ_0, as in the proof of Proposition 276(ii), $\langle \Psi_0(f_0), e^{-tH_0} \Psi_0(f_0) \rangle = \lim_k Z_k^2 C_k(t) \ge c_0 e^{-m_0 t}$ for dyadic $t \ge \sigma_0$. Since $\Psi_0(f_0) \perp \Omega_0$, its spectral measure for H_0 is carried by $[m(S), \infty)$, and the left side is at most $e^{-m(S)t} \|\Psi_0(f_0)\|^2$; letting $t \to \infty$ gives $m(S) \le m_0$.

(ii) Let S, S' be two limits and $S' = D_{\lambda,c} S$. By (i) and Lemma 290, $m_0 = m(S') = m(S)/\lambda = m_0/\lambda$, so $\lambda = 1$; then $S_2' = c^2 S_2$, and the normalisation in (i) gives $c = 1$. Convergence follows as in the proof of Corollary 275.

(iii) The transfer matrix has $\gamma_k = q_k = e^{-m a_k}$. The packet $\vartheta(\overline{A_k(f_0)})$ involves sites at negative times and $\tau_{1/a_k} A_k(f_0)$ sites at positive times, so the time factor of the covariance is $q_k^{y_0 - x_0}$ and $C_k(t) = e^{-mt} C_k(0)$; this is (MC_m). At two masses the limits are the two Gaussian families with kernels $\kappa e^{-m\|z\|_1}$ and $\kappa' e^{-m'\|z\|_1}$, and $D_{\lambda,c}$ with $\lambda = m/m'$ and $c^2 = \kappa' \lambda^8 / \kappa$ maps the first to the second, Gaussian families being determined by S_2. The first part of (MC_{m_0}) needs $m_0 \le \min(m, m')$, and the second, since $C_k(t)/C_k(\sigma_0) = e^{-m_k(t-\sigma_0)}$ with $m_k \in \{m, m'\}$, needs $m_0 \ge \max(m, m')$. □

($\mathrm{O_s}$) is the form, for the separated moments, of the statement that the continuum theory has no free dimensionless parameter; it is not proved here, nor is (MC_{m_0}) for the Wilson measure along any tuning. The second part of (MC_{m_0}) asks that the reflected correlation of f_0 decay no faster than at the rate m_0, which holds when f_0 couples to the lowest state; together with the first part it fixes the mass of every limit. A condition fixing the ratio $C_k(\sigma_0)/C_k(0)$, the analogue of (RC), fixes the scale within a set of limits related by dilations only where the corresponding ratio of the limits is injective along that set, which is not proved.

The pointwise bound on an axis. Reflection positivity in the coordinate directions bounds the separated two-point moments at every point by their values on a coordinate axis, where they form a completely monotone function of the distance. For $n = 2$ write $g_k(s) = M_k(0, s e_0)$ for $s \in \mathbb{Z}$, and $\lfloor t \rceil = \lfloor t + \frac{1}{2} \rfloor$. Let $h \in \{1, -1\}$ be such that $\vartheta x = (h - x_0, \mathbf{x})$ for $x = (x_0, \mathbf{x}) \in \mathbb{Z}^4$; the plane fixed by ϑ is $\{x_0 = h/2\}$.

Proposition 293. *Assume the hypothesis of Theorem 66, and let γ_k be the norm of $\mathbb{T}_k$ on $\Omega_k^\perp$.*

(i) *For integers $s \ge 2r+3$ and $j \ge 0$, $0 \le g_k(s+j) \le \gamma_k^j g_k(s)$, and for $s \ge 4r+6$*

$$0 \le g_k(s) - g_k(s+j) \le \frac{4j}{s} g_k\big(\lceil s/2 \rceil\big).$$

(ii) *For $y \in \mathbb{Z}^4$ with $|y|_\infty \ge 2r+4$, $|M_k(0,y)| \le g_k(|y|_\infty - 1)$.*

(iii) (SB_2) *holds if and only if $\limsup_k Z_k^2 g_k(\lceil \delta / a_k \rceil) < \infty$ for every $\delta > 0$. In that case, for every $\delta > 0$ and with $b(\delta) = \limsup_k Z_k^2 g_k(\lceil \delta/(4a_k) \rceil)$,*

$$Z_k^2 |M_k(0,y)| \le \big(b(\delta) + 1\big) e^{-m_*(a_k |y|_\infty - \delta/2)}$$

for all large k and all $y \in \mathbb{Z}^4$ with $|y| \ge \delta/a_k$; in particular $b_2(\delta, \Lambda) \le b(\delta) + 1$ for every $\Lambda > \delta$.

(iv) *Under* (SB_2), (RI_2) *holds if and only if for $0 < \delta < \Lambda$*

$$\limsup_k \sup_y Z_k^2 \big| M_k(0,y) - g_k(\lfloor |y| \rceil) \big| = 0,$$

the supremum over the $y \in \mathbb{Z}^4$ with $\delta/a_k \le |y| \le \Lambda/a_k$.

(v) *Under* (SB_2), *every limit of Theorem 274 has $S_2(f,g) = \iint f(x) g(y) K(x-y)\, dx\, dy$ for the separated pairs, with K measurable on $\mathbb{R}^4 \setminus \{0\}$, $K(-z) = K(z)$, $K \circ g = K$ almost everywhere for $g \in B_4$, and $|K(z)| \le (b(\delta)+1) e^{-m_*(|z|_\infty - \delta/2)}$ for almost every z with $|z| \ge 2\delta$, for every $\delta > 0$.*

Proof. For $j \ge r+1$ let $F_j = \Phi_{j e_0} - c_k$, a real function of the links with both endpoints in $\{x_0 \ge 1\}$. Since $\vartheta \Phi_x = \Phi_{\vartheta x}$, $\vartheta F_j = \Phi_{(h-j)e_0} - c_k$, and by translation invariance

$$\omega_k\big(\vartheta(F_j) \tau_t F_{j'}\big) = M_k\big((h-j)e_0, (j'+t)e_0\big) = g_k(j + j' + t - h)$$

for integers $t \ge 0$. In particular $N_0(F_j)^2 = g_k(2j - h)$.

(i) By the proof of Lemma 273(iii), with $s_0 = 2r + 2 - h$ and $\Psi = \Psi_{F_{r+1}}$, $g_k(s_0 + t) = \langle \Psi, \mathbb{T}_k^t \Psi \rangle$ for integers $t \ge 0$, and $\langle \Omega_k, \Psi \rangle = \omega_k(F_{r+1}) = 0$. Since $\mathbb{T}_k$ is self-adjoint, fixes Ω_k and has spectrum on $\Omega_k^\perp$ in $[0, \gamma_k]$, the spectral measure ν of $\mathbb{T}_k$ at Ψ is a positive measure on $[0, \gamma_k]$, and $g_k(s_0 + t) = \int \lambda^t \, d\nu(\lambda)$, $0^0 = 1$. As $s_0 \le 2r+3$, this gives $g_k(s) \ge 0$ and $g_k(s+j) \le \gamma_k^j g_k(s)$ for $s \ge 2r+3$. Let $s \ge 4r+6$ and $u = s - s_0$. From $1 - \lambda^j \le j(1-\lambda)$ and $\max_{0 \le x \le 1} x^p (1-x) = p^p (p+1)^{-p-1} \le (p+1)^{-1}$, applied with $p = \lfloor u/2 \rfloor$ to $\lambda^u (1-\lambda) = \lambda^{\lceil u/2 \rceil} \lambda^p (1-\lambda)$,

$$\begin{aligned} g_k(s) - g_k(s+j) &= \int \lambda^u (1 - \lambda^j)\, d\nu \le \frac{j}{p+1} \int \lambda^{\lceil u/2 \rceil} d\nu \\ &= \frac{j\, g_k(s_0 + \lceil u/2 \rceil)}{\lfloor u/2 \rfloor + 1}. \end{aligned}$$

Here $s_0 + \lceil u/2 \rceil = \lceil (s + s_0)/2 \rceil \ge \lceil s/2 \rceil \ge 2r+3$, so the numerator is at most $j\, g_k(\lceil s/2 \rceil)$, and $\lfloor u/2 \rfloor + 1 \ge (s - s_0 + 1)/2 \ge (s - 2r - 2)/2 \ge s/4$.

(ii) Let $s = |y|_\infty$. Choose $g \in B_4$, a permutation of the axes followed by a sign change, with $(gy)_0 = s$; then $M_k(0,y) = M_k(0, gy)$, since ω_k is invariant under B_4 and $g \Phi_x = \Phi_{gx}$. Write $gy = s e_0 + v$ with $v_0 = 0$, and let $j = \lfloor (s+h)/2 \rfloor$ and $j' =$

$s+h-j$, both at least $\lfloor (s-1)/2 \rfloor \geq r+1$. By translation invariance

$$M_k(0,gy) = M_k\big((h-j)e_0, j'e_0+v\big) = \omega_k\big(\vartheta(F_j)\tau_v F_{j'}\big),$$

and $\tau_v F_{j'}$ is again a function of the links in $\{x_0 \geq 1\}$. The Cauchy–Schwarz inequality for the positive form of Lemma 273(iii) and $N_0(\tau_v F_{j'}) = N_0(F_{j'})$ give $|M_k(0,y)| \leq (g_k(2j-h)g_k(2j'-h))^{1/2}$. If $s+h$ is even, $2j-h = 2j'-h = s$; otherwise the two arguments are $s-1$ and $s+1$. By (i) both products are at most $g_k(s-1)^2$.

(iii) Let $a_k \leq \delta$. The pair $(0, \lceil \delta/a_k \rceil e_0)$ lies in $\mathscr{W}_k(\delta, 2\delta)$, and (SB_2) gives

$$\limsup_k Z_k^2 g_k(\lceil \delta/a_k \rceil) \leq b_2(\delta, 2\delta).$$

Conversely let $\delta > 0$, $s_1 = \lceil \delta/(4a_k) \rceil$ and $|y| \geq \delta/a_k$. Then $|y|_\infty \geq |y|/2 \geq \delta/(2a_k)$, which for large k is at least $2r+4$ and at least s_1+1. By (ii) and (i), and $\gamma_k \leq e^{-m_* a_k}$,

$$|M_k(0,y)| \leq g_k(|y|_\infty - 1) \leq e^{-m_* a_k (|y|_\infty - 1 - s_1)} g_k(s_1),$$

and $a_k(1+s_1) \leq 2a_k + \delta/4 \leq \delta/2$ for $a_k \leq \delta/8$. Since $Z_k^2 g_k(s_1) \leq b(\delta)+1$ for large k, this is the bound, and $a_k|y|_\infty \geq \delta/2$ gives the last claim.

(iv) Suppose (RI_2), and let $\delta/a_k \leq |y| \leq \Lambda/a_k$. The point $y' = \lfloor |y| \rceil e_0$ has $||y| - |y'|| \leq \frac{1}{2}$, both pairs $(0,y)$, $(0,y')$ lie in $\mathscr{W}_k(\delta/2, 2\Lambda)$ for large k, and $M_k(0,y') = g_k(\lfloor |y| \rceil)$; so (RI_2) with $(\delta/2, 2\Lambda)$ gives the condition. Conversely, let $x, x' \in \mathscr{W}_k(\delta, \Lambda)$ with $||x_1 - x_2| - |x_1' - x_2'|| \leq 4$, $y = x_2 - x_1$, $y' = x_2' - x_1'$, $s = \lfloor |y| \rceil$ and $s' = \lfloor |y'| \rceil$. By translation invariance $M_k(x) = M_k(0,y)$ and $M_k(x') = M_k(0,y')$, and

$$\begin{aligned} Z_k^2 |M_k(x) - M_k(x')| \leq{}& Z_k^2 |M_k(0,y) - g_k(s)| \\ &+ Z_k^2 |g_k(s) - g_k(s')| \\ &+ Z_k^2 |M_k(0,y') - g_k(s')|. \end{aligned}$$

The first and last terms tend to 0 uniformly by the condition. Since $|s-s'| \leq 5$ and $\min(s,s') \geq \delta/(2a_k)$ for large k, (i) bounds the middle term by $40 a_k \delta^{-1} Z_k^2 g_k(\lceil \delta/(4a_k) \rceil)$, which tends to 0 by (iii).

(v) Let S be a limit along a subsequence, f, g separated, supported in a cube Q of side L at ℓ^∞ distance $\eta > 0$, and $\Theta(z) = \int f(u) g(u+z)\, du$, continuous and supported in $\{|z|_\infty \geq \eta\} \cap [-L,L]^4$. With ρ_k as in the proof of Theorem 288 put $K_k(z) = Z_k^2 M_k(0, \rho_k(z))$. By translation invariance,

$$\begin{aligned} Z_k^2 \omega_k\big(A_k(f)A_k(g)\big) - \int K_k \Theta\, dz = \sum_w Z_k^2 M_k(0,w) \\ \times \Big(a_k^8 \sum_x f(a_k x) g(a_k x + a_k w) - \int_{\rho_k^{-1}(w)} \Theta\, dz \Big). \end{aligned}$$

The bracket vanishes unless $|w|_\infty \geq \eta/a_k - 2$ and $|w|_\infty \leq L/a_k + 2$, and it is at most $a_k^4 \varepsilon_k$, where $\varepsilon_k \to 0$ by the uniform continuity of f, g and Θ, the inner sum being a Riemann sum for $\Theta(a_k w)$ with error uniform in w. At most $(2L/a_k + 5)^4$ vectors w contribute, and by (iii) with $\delta = \eta/2$ each has $Z_k^2 |M_k(0,w)| \leq b(\eta/2) + 1$ for large k. So the difference tends to 0. By (iii), K_k is bounded on every set $\{|z| \geq \delta\}$, $\delta > 0$, for large k, and a further subsequence converges weakly-$*$ in $L^\infty(\{|z| \geq \delta\})$ for every $\delta = 1/i$ to some K. Since Θ is integrable and supported in $\{|z| \geq \eta\}$, $\int K_k \Theta\, dz \to \int K \Theta\, dz$, which is the representation. For $|z| \geq 2\delta$ and large k, $|\rho_k(z)| \geq |z|/a_k - 1 \geq \delta/a_k$ and $a_k |\rho_k(z)|_\infty \geq |z|_\infty - a_k$, so (iii) gives $|K_k(z)| \leq (b(\delta)+1) e^{-m_*(|z|_\infty - a_k - \delta/2)}$, a bound that passes to the weak-$*$ limit. Finally $M_k(0,-w) = M_k(0,w)$ and $M_k(0,gw) = M_k(0,w)$ for $g \in B_4$, while $\rho_k(-z) = -\rho_k(z)$ and $\rho_k(gz) = g\rho_k(z)$ off a null set; so K_k and K are even and B_4-invariant almost everywhere. □

So (SB_2) is a bound on one function of one variable, the renormalised two-point moment on a coordinate axis, and (RI_2) asks that the renormalised moment at a point be close to its value on the axis at the same Euclidean length. Under (SB_2) the kernel of (v) is square integrable on every compact subset of $\mathbb{R}^4 \setminus \{0\}$, so Proposition 287 applies to it, and $(\mathrm{W_s})$ for $n = 2$ is the radial symmetry of K. Neither part follows from the other properties. For (RI_2) this is Proposition 289, whose fields satisfy (SB_n) for every n. For (SB_2) it is part (b) of the following proposition, whose fields satisfy the separated moment bound with constants independent of the separation, the mass condition and $(\mathrm{U_s})$, and concentrate the two-point function on the coordinate axes, where the bound of $(\mathrm{M_s})$ on averages over regions of physical size does not control the pointwise values.

Proposition 294. *Let $m > 0$ and $q_k = e^{-m a_k}$.*

(a) *Let J be a finite set, $q_{j,\mu} \in [0,1)$ for $j \in J$ and $\mu = 0, \dots, 3$, and let $\psi = (\psi_j(x))_{j \in J, x \in \mathbb{Z}^4}$ be the centred Gaussian field with $\mathbb{E}\psi_j(x)\psi_{j'}(y) = \delta_{jj'} \prod_\mu q_{j,\mu}^{|x_\mu - y_\mu|}$, $0^0 = 1$. Its law is invariant under translations and reflection positive for the reflections in coordinate hyperplanes through sites and bisecting links, and in the direction μ its transfer matrix $\mathbb{T}$ has $0 \leq \mathbb{T} \leq 1$, $\mathbb{T}\Omega = \Omega$ and norm at most $\max_j q_{j,\mu}$ on $\Omega^\perp$.*

(b) *Let $J = \{0,1,2,3\}$, $q_{j,\mu} = q_k$ for $j = \mu$ and 0 otherwise, $\Phi_x = \sum_j \psi_j(x)$, $A_k(f) = \sum_x a_k^4 f(a_k x)\Phi_x$, $f_0 \geq 0$ not zero, C_k formed from f_0, and $Z_k = C_k(\sigma_0)^{-1/2}$. The law is invariant under B_4, acting by $\psi_j(x) \mapsto \psi_{\bar g j}(gx)$ with $\bar g$ the permutation of the axes underlying g, so that $g\Phi_x = \Phi_{gx}$. Moreover $C_k(t) = e^{-mt} C_k(0)$ for $t \in a_k \mathbb{Z}_{\geq 0}$ and $a_k^{-3} C_k(0) \to \mathscr{J} = \int_{\mathbb{R}^3} (\int_{\mathbb{R}} f_0(s, \mathbf{x}) e^{-ms} ds)^2 d\mathbf{x} > 0$. There is $B < \infty$ such that for large k, every n and all $f_j \in C_c(\mathbb{R}^4)$ supported in a cube of side $L \geq a_k$,*

$$\Big| \omega_k\Big(\prod_{j=1}^n Z_k A_k(f_j) \Big) \Big| \leq (n-1)!! \, \big(B(2L)^4\big)^{n/2} \prod_{j=1}^n \|f_j\|_\infty$$

for even n, the moments vanishing for odd n; so $(\mathrm{M_s})$ holds with constants independent of δ, and the growth condition $(\mathrm{G_s})$ posed below holds with $p = 0$. The moments converge along the whole sequence to the Gaussian family with

$$S_2(f,g) = \frac{e^{m\sigma_0}}{\mathscr{J}} \sum_{\mu=0}^3 \int_{\mathbb{R}^4} \int_{\mathbb{R}} f(x) g(x + u e_\mu) e^{-m|u|}\, du\, dx,$$

so ($\mathrm{U_s}$) *holds, and* (MC_m) *holds. Further* (SB_2) *and* (RI_2) *fail,* $G_S = B_4$*, and there is no locally integrable* K *on* $\mathbb{R}^4 \setminus \{0\}$ *with* $S_2(f,g) = \iint f(x)g(y)K(x-y)\,dx\,dy$ *for all separated pairs.*

(c) *Let* $m' > m$*,* $q'_k = e^{-m'a_k}$*, and let* ω_k *be the law of the field* $\Phi = \psi_1$ *of* (a) *with* $J = \{1\}$*,* $q_{1,\mu} = q_k$*, for even* k*, and of* $\Phi = \psi_1 + \psi_2$ *with* $J = \{1,2\}$*,* $q_{1,\mu} = q_k$*,* $q_{2,\mu} = q'_k$*, for odd* k*; let* $f_0 \geq 0$ *be not zero and* $Z_k = C_k(\sigma_0)^{-1/2}$*. The laws are invariant under translations and* B_4*, reflection positive as in* (a)*, with transfer matrices of norm at most* q_k *on* $\Omega_k^{\perp}$*.* ($\mathrm{M_s}$) *and* (MC_m) *hold along the whole sequence. The moments converge along the even and along the odd* k *to Gaussian families* S *and* S' *with continuous kernels* $\kappa e^{-m\|z\|_1}$ *and* $\kappa'(e^{-m\|z\|_1} + e^{-m'\|z\|_1})$*,* $\kappa, \kappa' > 0$*, and* $S' \neq D_{\lambda,c}S$ *for all* $\lambda, c > 0$*. So* ($\mathrm{O_s}$) *and* ($\mathrm{U_s}$) *fail.*

Proof. (a) Fix $\mu = 0$ as time; the other directions are the same. The field at time 0 has a Gaussian law γ with covariance $C = \delta_{jj'} \prod_{i \geq 1} q_{j,i}^{|x_i - y_i|}$, positive definite as a product of the one-dimensional covariances $q^{|n|}$, $0 \leq q < 1$, whose Fourier transforms $(1-q^2)/(1-2q\cos p + q^2)$ are positive. Let $(Qh)_j(\mathbf{x}) = q_{j,0}h_j(\mathbf{x})$ on the finitely supported h; Q commutes with C, so the stationary chain $\psi_{t+1} = Q\psi_t + (1-Q^2)^{1/2}\xi_{t+1}$, with ξ_t independent of law γ, has covariance $Q^{|t-s|}C$, which is that of the field. As in the proof of Proposition 207(a), its transition operator P satisfies $P\mathscr{E}_h = \mathscr{E}_{Qh}$, so on the N-th Wiener chaos, the closure of the parts of the $\mathscr{E}_h$ homogeneous of degree N in h, it acts as the N-th tensor power of Q on the symmetric tensors over the space of the h with the inner product $\langle h, Ch' \rangle$. Q is self-adjoint there with $0 \leq Q \leq q_{\max} = \max_j q_{j,0}$, so $0 \leq P \leq 1$, P fixes the constants and has norm at most $q_{\max}^N \leq q_{\max}$ on the chaos of order $N \geq 1$. The rest of that proof gives reflection positivity for both kinds of planes and identifies the transfer matrix with P.

(b) Here $\mathbb{E}\Phi_x\Phi_y = w_k(y-x)$ with $w_k(z) = q_k^{|z_\mu|}$ if $z \neq 0$ is a multiple of e_μ, $w_k(0) = 4$, and $w_k(z) = 0$ otherwise. Since $(gz)_{\bar{g}j} = \pm z_j$, the covariance of the transformed field is that of ψ, which gives the B_4 invariance; $\omega_k(\Phi_0) = 0$, so $c_k = 0$. For $f_0(a_k x)f_0(a_k y) \neq 0$ the time coordinate of $y + (t/a_k)e_0 - \vartheta x$ is $x_0 + y_0 - h + t/a_k \geq 1$, so only the term $\mu = 0$ contributes, with $\mathbf{x} = \mathbf{y}$, and

$$C_k(t) = e^{-mt}e^{mha_k}a_k^3 \cdot a_k^3 \sum_{\mathbf{x}} \Big(a_k \sum_{x_0} f_0(a_k x) e^{-ma_k x_0} \Big)^2 .$$

The last factor is a Riemann sum converging to $\mathscr{J}$, which is positive because $f_0 \geq 0$ is not zero. For the pairs,

$$\omega_k\big(A_k(f)A_k(g)\big) = a_k^3 \sum_{\mu} a_k^5 \sum_{x \in \mathbb{Z}^4, s \in \mathbb{Z}} f(a_k x) g(a_k x + a_k s e_\mu) e^{-ma_k|s|}.$$

The inner sums are Riemann sums converging to $\iint f(x)g(x + ue_\mu)e^{-m|u|}du\,dx$, and since at most $(L+a_k)^4/a_k^4 \leq (2L)^4/a_k^4$ sites have $a_k x$ in the cube and $a_k \sum_s e^{-ma_k|s|} = a_k(1+q_k)/(1-q_k) \leq 4/m$ for $ma_k \leq 1$, they are at most $(4/m)(2L)^4\|f\|_\infty\|g\|_\infty$. With $Z_k^2 a_k^3 = e^{m\sigma_0}a_k^3/C_k(0) \to e^{m\sigma_0}/\mathscr{J}$ this gives the bound for $n = 2$ with $B = 32e^{m\sigma_0}/(m\mathscr{J})$ and the limit of S_2, and Wick's formula, a sum over the $(n-1)!!$ pairings, gives the bound for all n and the convergence of all moments. Since $(n-1)!! \leq (n!)^{1/2}$ and $(B(2L)^4)^{n/2} \leq (4B^{1/2})^n(1+L)^{2n}$, the bound has the form of ($\mathrm{G_s}$) with $\beta = \frac{1}{2}$, $q = 2$ and $p = 0$. By (a) the transfer matrix has norm at most q_k on $\Omega_k^{\perp}$, so $\liminf_k a_k^{-1}\log(1/\gamma_k) \geq m$, and $C_k(t) = e^{-m(t-\sigma_0)}C_k(\sigma_0)$; this is ($\mathrm{MC}_m$) with $c_0 = e^{m\sigma_0}$.

For $s \geq 1$, $g_k(s) = w_k(se_0) = q_k^s$, so $Z_k^2 g_k(\lceil \delta/a_k \rceil) \geq (e^{m\sigma_0}/(2\mathscr{J}))a_k^{-3}e^{-m(\delta + a_k)} \to \infty$; as the pair $(0, \lceil \delta/a_k \rceil e_0)$ lies in $\mathscr{W}_k(\delta, 2\delta)$ for large k, (SB_2) fails. With $N = \lceil \delta/a_k \rceil + 1$, the points $y = Ne_0$ and $y' = (N-1)e_0 + e_1$ have $N - 1 \leq |y'| \leq |y| = N$, both pairs $(0,y)$, $(0,y')$ lie in $\mathscr{W}_k(\delta, 2\delta)$ for large k, $M_k(0,y') = w_k(y') = 0$ and $Z_k^2 M_k(0,y) = Z_k^2 q_k^N \to \infty$; so ($\mathrm{RI}_2$) fails.

Let $T(\varphi) = \frac{e^{m\sigma_0}}{\mathscr{J}} \sum_\mu \int \varphi(ue_\mu)e^{-m|u|}du$ for $\varphi \in C_c(\mathbb{R}^4 \setminus \{0\})$; then $S_2(f,g) = T(\Theta)$ with $\Theta(z) = \int f(u)g(u+z)\,du$. If S_2 had a locally integrable kernel K, the argument of Proposition 287(i) would give $T(\varphi) = \int K\varphi\,dz$ for all φ, T being even. Take φ_ε with values in $[0,1]$, equal to 1 on the segment $\{ue_0 : 1 \leq u \leq 2\}$ and supported in its ε-neighbourhood N_ε; then $T(\varphi_\varepsilon) \geq \frac{e^{m\sigma_0}}{\mathscr{J}} \int_1^2 e^{-mu}du > 0$, while $\int |K|\varphi_\varepsilon \leq \int_{N_\varepsilon} |K| \to 0$ as $\varepsilon \to 0$, since N_ε decreases to a null set. Let $R \in G_S$ and suppose $R^{-1}e_0$ is not one of the $\pm e_\nu$; then it has positive distance 3ε from the union A of the coordinate axes. Take $f, g \geq 0$ supported in the balls of radius ε about 0 and $R^{-1}e_0$, with $f(0) > 0$ and $g(R^{-1}e_0) > 0$. Then Θ is supported in the ball of radius 2ε about $R^{-1}e_0$, which misses A, so $S_2(f,g) = 0$, while $S_2(f \circ R^{-1}, g \circ R^{-1}) = T(\Theta \circ R^{-1}) \geq \frac{e^{m\sigma_0}}{\mathscr{J}} \int \Theta(R^{-1}ue_0)e^{-m|u|}du > 0$, because $\Theta(R^{-1}e_0) > 0$. So $R^{-1}e_0 \in \{\pm e_\nu\}$, and applying this to $\pi_\mu R \in G_S$ gives $R^{-1}e_\mu \in \{\pm e_\nu\}$ for every μ, that is, $R \in B_4$.

(c) The symmetries, positivity and the norm of the transfer matrices follow from (a), the covariances $\prod_\mu q^{|z_\mu|}$ being invariant under B_4. Let $C_k^{(i)}$ be formed with the covariance $\prod_\mu q_i^{|z_\mu|}$, $q_1 = q_k$, $q_2 = q'_k$. As in (b), the time factor of the covariance between the two packets of $C_k^{(i)}(t)$ is q_i^{t/a_k} times that at $t = 0$, so $C_k^{(1)}(t) = e^{-mt}C_k^{(1)}(0)$, $C_k^{(2)}(t) = e^{-m't}C_k^{(2)}(0)$, and $C_k^{(i)}(0)$ converges to $\mathscr{J}_1 = \iint f_0(x)f_0(y)e^{-m\|\theta_0 x - y\|_1}dx\,dy > 0$ and $\mathscr{J}_2$, defined with m' in place of m, by Riemann sums. For even k, $C_k = C_k^{(1)}$; for odd k, $C_k = C_k^{(1)} + C_k^{(2)}$, and for $t \geq \sigma_0$

$$\frac{C_k(t)}{C_k(\sigma_0)} \geq \frac{e^{-mt}C_k^{(1)}(0)}{e^{-m\sigma_0}(C_k^{(1)}(0) + C_k^{(2)}(0))} \geq \frac{\mathscr{J}_1}{2(\mathscr{J}_1 + \mathscr{J}_2)}e^{-m(t-\sigma_0)}$$

for large k. With the norm bound q_k this is (MC_m) with $c_0 = e^{m\sigma_0}\mathscr{J}_1/(2(\mathscr{J}_1 + \mathscr{J}_2))$. As in the proof of Proposition 207(a), the moments converge along the even and along the odd k to the Gaussian families with the stated kernels, $\kappa = e^{m\sigma_0}/\mathscr{J}_1$, $\kappa' = (e^{-m\sigma_0}\mathscr{J}_1 + e^{-m'\sigma_0}\mathscr{J}_2)^{-1}$, and the moments of packets supported in a cube of side L are bounded by $(n-1)!!$ times $((2L)^4 \sup_k Z_k^2 a_k^4 \sum_z (q_k^{|z|_1} + q_k'^{|z|_1}))^{n/2} \prod_j \|f_j\|_\infty$, where $a_k^4 \sum_z q_k^{|z|_1} = (a_k(1+q_k)/(1-q_k))^4$ is bounded; this gives ($\mathrm{M_s}$). The kernel of $D_{\lambda,c}S$ is $c^2\lambda^{-8}\kappa e^{-(m/\lambda)\|z\|_1}$, by the

substitution $u=\lambda x$, $v=\lambda y$. If $S'=D_{\lambda,c}S$, then the two continuous kernels coincide, since by Proposition 287(i) a continuous kernel is determined by S_2; at $z=te_0$, after multiplication by e^{mt},

$$\kappa'\big(1+e^{-(m'-m)t}\big)=c^2\lambda^{-8}\kappa\, e^{(m-m/\lambda)t}$$

for all $t>0$. The left side tends to $\kappa'>0$ as $t\to\infty$, which forces $\lambda=1$, and then the right side is constant while the left side is strictly decreasing. □

The mass. The gap of Theorem 66 gives the first half of (MC_{m_0}) with $m_0=m_*$, and reflection positivity with ($\mathrm{M_s}$) gives the second half, but with a rate that is in general larger.

Proposition 295. *Assume the hypothesis of Theorem 66, $C_k(0)>0$ for large k, $Z_k=C_k(\sigma_0)^{-1/2}$ and* ($\mathrm{M_s}$). *Let Q be a cube that contains* $\operatorname{supp} f_0$ *and $\theta_{0,k}(\operatorname{supp} f_0)$ for all k, and $m_1=\sigma_0^{-1}\log(M_2(Q,t_0)\|f_0\|_\infty^2)$.*

(i) *$m_1\ge m_*$; $\liminf_k a_k^{-1}\log(1/\gamma_k)\ge m_*$; and for large k and $t\in a_k\mathbb{Z}$ with $t\ge\sigma_0$,*

$$C_k(t)\ge e^{-m_1(t-\sigma_0)}C_k(\sigma_0).$$

(ii) *Every limit S of Theorem 274 has $m_*\le m(S)\le m_1$. If* (MC_{m_0}) *holds, then $m_*\le m_0\le m_1$.*

(iii) *If S and S' are limits of Theorem 274 with $S'=D_{\lambda,c}S$, then $\lambda=m(S)/m(S')$ lies in $[m_*/m_1,m_1/m_*]$.*

Proof. (i) By Proposition 276(i),(ii), $(M_2(Q,t_0)\|f_0\|_\infty^2)^{-1}\le C_k(\sigma_0)/C_k(0)\le e^{-m_*\sigma_0}$ for large k, which gives $m_1\ge m_*$ and $C_k(\sigma_0)/C_k(0)\ge e^{-m_1\sigma_0}$. The second claim is $\gamma_k\le e^{-m_*a_k}$. By the proof of Proposition 276(i), $C_k(t)=\int\lambda^{t/a_k}d\nu(\lambda)$ with a positive measure ν, so by Hölder's inequality, with $i=t/a_k$ and $p=\sigma_0/a_k$, $C_k(\sigma_0)^i\le C_k(0)^{i-p}C_k(t)^p$. Hence

$$C_k(t)\ge C_k(\sigma_0)\Big(\frac{C_k(\sigma_0)}{C_k(0)}\Big)^{(t-\sigma_0)/\sigma_0}\ge e^{-m_1(t-\sigma_0)}C_k(\sigma_0).$$

(ii) $m(S)\ge m_*$ is Theorem 274(iv). As in the proof of Theorem 292(i), (i) gives $\langle\Psi_0(f_0),e^{-tH_0}\Psi_0(f_0)\rangle=\lim_k Z_k^2C_k(t)\ge e^{-m_1(t-\sigma_0)}$ for dyadic $t\ge\sigma_0$, while the left side is at most $e^{-m(S)t}\|\Psi_0(f_0)\|^2$; so $m(S)\le m_1$. Under (MC_{m_0}), $m(S)=m_0$ by Theorem 292(i).

(iii) By Lemma 290, $m(S')=m(S)/\lambda$, and (ii) applies to both. □

So the two halves of (MC_{m_0}) hold separately, with the rates m_* and m_1, and (MC_{m_0}) asks that they coincide. Since $C_k(t)\le\gamma_k^{(t-\sigma_0)/a_k}C_k(\sigma_0)$ for $t\ge\sigma_0$, by the spectral representation, its second half gives $a_k^{-1}\log(1/\gamma_k)\le m_0$ for $k\ge k_1$, so (MC_{m_0}) says that the lattice gaps in physical units converge to m_0 and that the reflected correlation of f_0 decays at this rate uniformly in k. This is not implied by the other properties: the interleaved two-mass field of Theorem 292(iii) has them and violates (MC_{m_0}) for every m_0. Nor is it necessary for ($\mathrm{U_s}$): for the products of Proposition 289 with $p\ge2$, the transfer matrix has norm q_k on $\Omega_k^\perp$, attained on the first chaos of one factor, while the covariance of Φ is $e^{-pma_k\|z\|_1}$ and $C_k(t)=e^{-pmt}C_k(0)$ for $f_0\ge0$; so (MC_{m_0}) would need $m_0\le m$ and $m_0\ge pm$, while ($\mathrm{U_s}$) holds, also with $Z_k=C_k(\sigma_0)^{-1/2}$, which converges to a positive limit.

The dilations. ($\mathrm{O_s}$) is a statement about the set of limits. It is equivalent to a statement about the lattice moments alone: convergence after a change of the spacing by a bounded factor. Write $B_k(f_1,\dots,f_n)=\omega_k(\prod_j Z_kA_k(f_j))$ and $(D_{\lambda,c}B_k)(f_1,\dots,f_n)=c^nB_k(f_1(\lambda\cdot),\dots,f_n(\lambda\cdot))$. Since $A_k(f(\lambda\cdot))=\lambda^{-4}\sum_x(\lambda a_k)^4f(\lambda a_kx)(\Phi_x-c_k)$, $D_{\lambda,c}B_k$ is the moment family of the packets of the same states formed at the spacing λa_k, with the constants $c\lambda^{-4}Z_k$. Consider

(DS) there are $\Lambda\ge1$, $\lambda_k\in[\Lambda^{-1},\Lambda]$, $c_k>0$ and a nontrivial family T with $D_{\lambda_k,c_k}B_k\to T$ on every separated tuple.

A family is trivial if it vanishes on all separated tuples of length $n\ge1$.

Proposition 296. *Assume the hypothesis of Theorem 66, $C_k(0)>0$ for large k, $Z_k=C_k(\sigma_0)^{-1/2}$ and* ($\mathrm{M_s}$), *and let m_1 be as in Proposition 295.*

(i) (DS) *implies* ($\mathrm{O_s}$)*: every limit S of Theorem 274 has $T=D_{\lambda,c}S$ for some $\lambda,c>0$.*

(ii) ($\mathrm{O_s}$) *implies* (DS) *with $\Lambda=m_1/m_*$, and T can be taken to be any limit of Theorem 274.*

(iii) *Under* (DS), ($\mathrm{U_s}$) *holds if and only if (λ_k,c_k) converges in $(0,\infty)^2$.*

Proof. For $\lambda>0$ the map $f\mapsto f(\lambda\cdot)$ is a bijection of the separated tuples. Let $(f_j)\in\mathscr{C}_n(Q,\delta)$, $\lambda_k\to\lambda\in[\Lambda^{-1},\Lambda]$. For large k the tuples $(f_j(\lambda_k\cdot))$, $(f_j(\lambda\cdot))$ and the telescoping tuples with one factor $f_i(\lambda_k\cdot)-f_i(\lambda\cdot)$ lie in one $\mathscr{C}_n(Q',\delta/(2\Lambda))$, and $f_i(\lambda_k\cdot)\to f_i(\lambda\cdot)$ uniformly; so ($\mathrm{M_s}$) and multilinearity give $B_k(f_1(\lambda_k\cdot),\dots)-B_k(f_1(\lambda\cdot),\dots)\to0$. Hence, along a subsequence with $B_k\to S$ on every separated tuple,

$$B_k\big(f_1(\lambda_k\cdot),\dots,f_n(\lambda_k\cdot)\big)\to S_n\big(f_1(\lambda\cdot),\dots,f_n(\lambda\cdot)\big).\qquad(*)$$

By Proposition 276(ii) every limit is nontrivial.

(i) Let S be the limit along a subsequence, and pass to a further one with $\lambda_k\to\lambda$ and $c_k\to c\in[0,\infty]$. By $(*)$, $T_n(f)=\lim_k c_k^nB_k(f(\lambda_k\cdot))$ with $B_k(f(\lambda_k\cdot))\to S_n(f(\lambda\cdot))$. If $c=\infty$, then $S_n(f(\lambda\cdot))=0$ for every separated tuple of length $n\ge1$, and S is trivial; if $c=0$, T is trivial. So $0<c<\infty$ and $T=D_{\lambda,c}S$. For two limits, $D_{\lambda,c}S=D_{\lambda',c'}S'$ gives $S'=D_{\lambda/\lambda',c/c'}S$.

(ii) Let $\mathscr{G}$ be the countable set of the proof of Theorem 274(i), $F_1,F_2,\dots$ the separated tuples of its elements, and $d(B,B')=\sum_i2^{-i}\min(1,|B(F_i)-B'(F_i)|)$ for families B,B'. By the approximation in that proof, a family bounded on every $\mathscr{C}_n(Q,\delta)$ by a multiple of $\prod_j\|f_j\|_\infty$ that vanishes at every F_i vanishes on all separated tuples; such bounds hold for every $D_{\lambda,c}S$ with S a limit. Fix a limit S, let $I=[m_*/m_1,m_1/m_*]$, $e_k(\lambda,c)=d(D_{\lambda,c}B_k,S)$, and choose $(\lambda_k,c_k)\in I\times(0,\infty)$ with $e_k(\lambda_k,c_k)\le\inf_{I\times(0,\infty)}e_k+1/k$. Suppose $e_k(\lambda_k,c_k)\ge\varepsilon>0$ along a subsequence. A further subsequence has $B_k\to S'$ on every separated tuple, and ($\mathrm{O_s}$) gives $S=D_{\lambda,c}S'$ with $\lambda\in I$ by Proposition 295(iii). Then $D_{\lambda,c}B_k\to S$ on every separated tuple by $(*)$ with constant $\lambda_k=\lambda$, so $e_k(\lambda,c)\to0$ and $e_k(\lambda_k,c_k)\to0$, a contradiction. Hence $e_k(\lambda_k,c_k)\to0$. Now take any subsequence and a further one with $\lambda_k\to\lambda\in I$

and $B_k \to S'$. Some F_i, of length $n_i \geq 1$, has $S(F_i) \neq 0$, and $c_k^{n_i} B_k(F_i(\lambda_k \cdot)) \to S(F_i)$. If $c_k \to \infty$ along a further subsequence, then $B_k(F_l(\lambda_k \cdot)) \to 0$ for every l with $n_l \geq 1$, so by $(*)$ $D_{\lambda,1} S'$ vanishes at every F_l, and S' is trivial; if $c_k \to 0$, then $S(F_i) = 0$. So a further subsequence has $c_k \to c \in (0,\infty)$, and $D_{\lambda_k,c_k} B_k \to D_{\lambda,c} S'$ on every separated tuple by $(*)$. This family agrees with S at every F_l, hence everywhere. Every subsequence thus has a further subsequence along which $D_{\lambda_k,c_k} B_k \to S$, which is (DS) with $T = S$.

(iii) If $(\lambda_k, c_k) \to (\lambda, c)$, the proof of (i) gives $S = D_{1/\lambda,1/c} T$ for every limit S, which is ($\mathrm{U_s}$). Conversely, under ($\mathrm{U_s}$) let S be the limit. By the proof of (i), (c_k) has no subsequence tending to 0 or ∞, so (λ_k, c_k) lies in a compact subset of $(0,\infty)^2$, and every limit point (λ, c) has $D_{\lambda,c} S = T$. Two limit points (λ, c), (λ', c') give $D_{\lambda/\lambda', c/c'} S = S$, so $\lambda = \lambda'$ and $c = c'$ by Lemma 290. □

So ($\mathrm{O_s}$) says that the spacings a_k can be corrected by bounded factors λ_k so that the moments of the same states converge, and under it ($\mathrm{U_s}$) is the convergence of the corrections and of the constants, which (MC_{m_0}) gives by Theorem 292(ii). Neither ($\mathrm{O_s}$) nor ($\mathrm{U_s}$) follows from the properties proved here together with (MC_{m_0}): the fields of Proposition 294(c) are reflection positive, invariant under translations and B_4, have the gap $m a_k$ in every coordinate direction and satisfy ($\mathrm{M_s}$) and (MC_m), and their two limits are not related by any $D_{\lambda,c}$. Conversely ($\mathrm{O_s}$) does not give (MC_{m_0}) (Theorem 292(iii)). The proofs of Propositions 293, 295 and 296 use the averaged condition only through Lemma 273(iii) and the bound $\gamma_k \leq e^{-m_* a_k}$, which the fields of Proposition 294 have with m in place of m_*.

Temperedness. The Schwinger functions of [95] are linear functionals on the space $\mathscr{S}_0(\mathbb{R}^{4n})$ of the $F \in \mathscr{S}(\mathbb{R}^{4n})$ that vanish, with all their derivatives, at every $x = (x_1, \ldots, x_n)$ with $x_i = x_j$ for some $i \neq j$, and [96] adds the linear growth condition: $|S_n(F)| \leq \sigma_n |F|_{sn}$ on $\mathscr{S}_0(\mathbb{R}^{4n})$ for a fixed integer s and $\sigma_n \leq \alpha (n!)^{\beta'}$, with the norms $|F|_w = \sup_{x, |\gamma| \leq w} (1 + |x|^2)^{w/2} |\partial^\gamma F(x)|$ of Lemma 203. Theorem 274 gives the S_n only on $C_c^\infty(\mathfrak{D}_n)$. The test functions of $\mathscr{S}_0(\mathbb{R}^{4n})$ approach the coinciding points, where ($\mathrm{M_s}$) gives no control, since its constants $M_n(Q,\delta)$ may grow arbitrarily as $\delta \to 0$. The following rate closes this gap.

($\mathrm{G_s}$) *Growth at coinciding points.* There are $K, \beta, p, q \geq 0$ such that for every $n \geq 1$, every cube Q of side L and every $\delta \in (0,1]$ there is k_0 with

$$\left| \omega_k \Big(\prod_{j=1}^n Z_k A_k(f_j) \Big) \right| \leq K^n (n!)^\beta (1+L)^{qn} \delta^{-pn} \prod_{j=1}^n \|f_j\|_\infty$$

for all $k \geq k_0$ and $(f_1, \ldots, f_n) \in \mathscr{C}_n(Q, \delta)$.

Since $\mathscr{C}_n(Q,\delta) \subset \mathscr{C}_n(Q,1)$ for $\delta \geq 1$, ($\mathrm{G_s}$) implies ($\mathrm{M_s}$), and the limits of Theorem 274(i) obey the same bound.

For $x \in \mathbb{R}^{4n}$ write $x_{j\nu}$ for the coordinates of x_j, $|x|$ for the Euclidean norm, $|x|_\infty = \max_{j,\nu} |x_{j\nu}|$, and $d(x) = \min_{i \neq j} |x_i - x_j|_\infty$. For a coordinate direction μ let $\mathscr{S}_0^{\mu+}(\mathbb{R}^{4n})$ be the set of $F \in \mathscr{S}_0(\mathbb{R}^{4n})$ with $F(x) = 0$ whenever $x_{j\mu} \leq 0$ for some j, and $V_\mu = \{x \in \mathfrak{D}_n : x_{j\mu} > 0 \text{ for all } j\}$. Such an F vanishes with all its derivatives on $\{x_{j\mu} \leq 0\}$, by continuity from the interior of this set. Fix $\eta_0 \in C_c^\infty((-\frac{1}{2}, \frac{1}{2}))$ with $\eta_0 \geq 0$ and $\int \eta_0 = 1$, let $\psi = \mathbf{1}_{[-1/2,1/2)} * \eta_0$, so that $0 \leq \psi \leq 1$, $\operatorname{supp} \psi \subset (-1,1)$ and $\sum_{m \in \mathbb{Z}} \psi(t-m) = 1$, and put $C_\psi = \max(1, \|\psi'\|_\infty, \|\psi''\|_\infty)$. For $h > 0$ and $m = (m_1, \ldots, m_n) \in (\mathbb{Z}^4)^n$ let $\psi_m^h(x) = \prod_{j,\nu} \psi(x_{j\nu}/h - m_{j\nu})$; call m *separated* if $|m_i - m_j|_\infty \geq 4$ for $i \neq j$, and let E_h be the sum of ψ_m^h over the separated m.

Lemma 297. (i) *E_h is smooth with values in $[0,1]$, $E_h = 0$ on $\{d < 2h\}$ and $E_h = 1$ on $\{d \geq 6h\}$. For every multi-index $\gamma \neq 0$, $|\partial^\gamma E_h| \leq 2^{4n} h^{-|\gamma|} \prod_{j,\nu} \|\psi^{(\gamma_{j\nu})}\|_\infty$; in particular $|\partial^\gamma (1 - E_h)| \leq (2C_\psi)^{4n} h^{-|\gamma|}$ for all $\gamma \in \{0,1,2\}^{4n}$.*

(ii) *Let $F \in \mathscr{S}_0(\mathbb{R}^{4n})$, α a multi-index and w, N, w' integers with $w' \geq \max(w, |\alpha| + N)$. If $|x_i - x_j|_\infty = t \leq 1$ for some $i \neq j$, then*

$$(1 + |x|^2)^{w/2} |\partial^\alpha F(x)| \leq 3^w \frac{(4t)^N}{N!} |F|_{w'}.$$

If $F \in \mathscr{S}_0^{\mu+}(\mathbb{R}^{4n})$ and $0 < x_{j\mu} = t \leq 1$ for some j, the same holds with $2^w t^N / N!$ in place of $3^w (4t)^N / N!$.

(iii) *$C_c^\infty(\mathfrak{D}_n)$ is dense in $\mathscr{S}_0(\mathbb{R}^{4n})$ for every norm $|\cdot|_w$. Every $F \in \mathscr{S}_0^{\mu+}(\mathbb{R}^{4n})$ is, for every w, a $|\cdot|_w$-limit of finite sums of products $f_1 \otimes \cdots \otimes f_n$ with $f_j \in C_c^\infty(\mathbb{R}^4)$ supported in $\{x_\mu > 0\}$ and with pairwise disjoint supports.*

Proof. (i) The sum of all ψ_m^h is 1, and at each x at most two integers $m_{j\nu}$ have $\psi(x_{j\nu}/h - m_{j\nu}) \neq 0$, so at most 2^{4n} terms are nonzero near x; this gives smoothness and the derivative bounds, the same ones holding for $1 - E_h$, the sum over the other m. If $\psi_m^h(x) \neq 0$, then $|x_j - h m_j|_\infty < h$ for all j. For separated m this gives $|x_i - x_j|_\infty > h|m_i - m_j|_\infty - 2h \geq 2h$, so $E_h = 0$ where $d < 2h$. If $d(x) \geq 6h$, it gives $h|m_i - m_j|_\infty > 4h$ for every m with $\psi_m^h(x) \neq 0$, so all these m are separated and $E_h(x) = 1$.

(ii) Let $u = x_i - x_j$, let $x(\tau)$ be x with x_i replaced by $x_j + \tau u$, and $g(\tau) = \partial^\alpha F(x(\tau))$. All derivatives of g vanish at $\tau = 0$, where two arguments coincide, so Taylor's formula gives $|g(1)| \leq \sup_\tau |g^{(N)}(\tau)| / N!$, and $g^{(N)} = (u \cdot \nabla_{x_i})^N \partial^\alpha F$ is at most $|u|_1^N \leq (4t)^N$ times $\max_{|\gamma| = N} |\partial^{\gamma+\alpha} F(x(\tau))| \leq |F|_{w'} (1 + |x(\tau)|^2)^{-w/2}$. Since $|x - x(\tau)| \leq |u| \leq 2t \leq 2$, $(1 + |x|^2)^{1/2} \leq 3(1 + |x(\tau)|^2)^{1/2}$. In the second case move $x_{j\mu}$ from 0 to t instead; the path has length $t \leq 1$, which gives the factor 2^w.

(iii) Let $F \in \mathscr{S}_0(\mathbb{R}^{4n})$, fix w, let $\chi \in C_c^\infty(\mathbb{R}^{4n})$ with values in $[0,1]$ equal to 1 on $\{|x| \leq 1\}$, $\chi_R = \chi(\cdot/R)$ with $R \geq 1$, and $0 < h \leq \frac{1}{6}$. Then $\chi_R E_h F \in C_c^\infty(\mathfrak{D}_n)$ by (i), and $F - \chi_R E_h F = (1 - E_h) F + (1 - \chi_R) E_h F$. By the Leibniz rule and (i), $(1 + |x|^2)^{w/2} |\partial^\gamma ((1 - E_h) F)(x)|$ for $|\gamma| \leq w$ is at most $c_w h^{-w} \max_{\alpha \leq \gamma} (1 + |x|^2)^{w/2} |\partial^\alpha F(x)|$ with c_w depending only on w, n and ψ, and it vanishes unless $d(x) < 6h$; there (ii) with $N = w + 1$ bounds it by $c_w 3^w 24^{w+1} h |F|_{2w+1}$. The second term is supported in $\{|x| \geq R\}$ and its $|\cdot|_w$ norm is at most $c_{w,h} (1 + R^2)^{-1/2} |F|_{w+1}$. Taking h small and then R large proves the first claim. For the second, let $\eta \in C^\infty(\mathbb{R})$ with values in $[0,1]$, $\eta = 0$ on $(-\infty, 1]$ and $\eta = 1$ on $[2,\infty)$, and $G = \chi_R E_h \prod_j \eta(x_{j\mu}/h) F$, which is smooth with compact support in V_μ. The additional term $E_h (1 - \prod_j \eta(x_{j\mu}/h)) F$ vanishes unless $0 < x_{j\mu} < 2h$ for some j, and the second case of

(ii) bounds it in the same way. Finally, every point of V_μ has a neighbourhood $U_1\times\cdots\times U_n$ of open cubes in $\{x_\mu>0\}$ with pairwise disjoint closures; finitely many of them cover $\operatorname{supp}G$, a partition of unity subordinate to them [71, Sect. 1.4] writes G as a finite sum of smooth functions supported in such products, and for each the partial sums of the Fourier series in the proof of Theorem 274(i) are finite sums of products of the required form converging in C^∞ with supports in a fixed compact set, hence in every $|\cdot|_w$. □

Lemma 298. *Let $n\ge 2$, $C,p',q'\ge 0$, and let S be a multilinear form on the separated n-tuples with*

$$|S(f_1,\dots,f_n)|\le C(1+L)^{q'}\delta^{-p'}\prod_{j=1}^n\|f_j\|_\infty$$

for every cube Q of side L, every $\delta\in(0,1]$ and $(f_1,\dots,f_n)\in\mathscr{C}_n(Q,\delta)$. Let S also denote the distribution on $\mathfrak{D}_n$ with $S(f_1\otimes\cdots\otimes f_n)=S(f_1,\dots,f_n)$ for the separated tuples of smooth functions, which exists and is unique by the proof of Theorem 274(i). There is a number $c_\star$, independent of n, C, p', q' and S, such that, with $w=\lceil q'\rceil+\lceil p'\rceil+12n+1$,

$$|S(F)|\le c_\star^n C 6^{q'}16^{p'}\,|F|_w\qquad\text{for } F\in C_c^\infty(\mathfrak{D}_n).$$

Hence S extends uniquely to a linear functional on $\mathscr{S}_0(\mathbb{R}^{4n})$ that is continuous for $|\cdot|_w$, with the same bound.

Proof. One piece. Let $h\in(0,\frac{1}{16}]$, m separated, $z_j=hm_j$, and $G\in C^\infty(\mathbb{R}^{4n})$ supported in $\prod_j(z_j+(-h,h)^4)$. The open cubes $U_j=z_j+(-\frac32h,\frac32h)^4$ are at mutual ℓ^∞ distance at least h, and lie in a cube Q of side $L\le 2\max_j|z_j|_\infty+3h$. Let $B=\prod_j(z_j+[-2h,2h]^4)$, $e_\kappa(x)=\prod_j\exp(i\pi\kappa_j\cdot(x_j-z_j)/(2h))$ for $\kappa\in(\mathbb{Z}^4)^n$, $c_\kappa=(4h)^{-4n}\int_B G\bar e_\kappa$, and $\varphi_j\in C_c^\infty(U_j)$ with values in $[0,1]$, equal to 1 on $z_j+[-h,h]^4$. With $\mathsf{D}=\prod_{i=1}^{4n}(1-(2h/\pi)^2\partial_i^2)$ over the $4n$ coordinates, $\mathsf{D}\bar e_\kappa=\prod_i(1+\kappa_i^2)\bar e_\kappa$, so integration by parts gives $|c_\kappa|\le\prod_i(1+\kappa_i^2)^{-1}\|\mathsf{D}G\|_\infty$, and the same with powers of D shows that the partial sums of $\sum_\kappa c_\kappa\prod_j\varphi_j(x_j)e_{\kappa_j}(x_j)$ converge to G in C^∞ with supports in $\prod_j\operatorname{supp}\varphi_j$. Hence $S(G)$ is the sum of the values of S on the terms. Each term is c_κ times a product in $\mathscr{C}_n(Q,h)$ with factors of norm at most 1, and $\sum_{k\in\mathbb{Z}}(1+k^2)^{-1}\le 1+2\int_0^\infty\frac{dt}{1+t^2}<5$, so

$$|S(G)|\le C(1+L)^{q'}h^{-p'}\,5^{4n}\,\|\mathsf{D}G\|_\infty.$$

Decomposition. Let $F\in C_c^\infty(\mathfrak{D}_n)$, $h_l=2^{-l-4}$ and $E_l=E_{h_l}$. Since d is bounded below on $\operatorname{supp}F$, Lemma 297(i) gives $E_lF=F$ for large l, and with $E_{l+1}-E_l=E_{l+1}(1-E_l)-E_l(1-E_{l+1})$,

$$F=E_0F+\sum_{l\ge0}\big(E_{l+1}(1-E_l)F-E_l(1-E_{l+1})F\big),$$

a finite sum. Each term has the form $\sum_m\psi_m^hH$ over separated m, with (h,H) equal to (h_0,F), $(h_{l+1},(1-E_l)F)$ or $(h_l,(1-E_{l+1})F)$; in the last two cases $H=(1-E_{h'})F$ with $h/h'\in\{\frac12,2\}$, and H vanishes where $d\ge12h$.

Derivatives. Write $\mathsf{D}=\prod_i\mathsf{D}_i$ and $\psi_m^h=\prod_i\psi_i$. Since $\mathsf{D}_i(\psi_iG)=\psi_iG-(2h/\pi)^2(\psi_i''G+2\psi_i'\partial_iG+\psi_i\partial_i^2G)$ and $|\psi_i^{(k)}|\le C_\psi h^{-k}$, induction over the coordinates gives

$$|\mathsf{D}(\psi_m^hH)|\le b_\psi^{4n}\max_{\gamma\in\{0,1,2\}^{4n}}h^{|\gamma|}|\partial^\gamma H|,\qquad b_\psi=1+2C_\psi,$$

on the box $X_m=\prod_j(hm_j+[-h,h]^4)$, outside which $\psi_m^h=0$, since $1+16C_\psi/\pi^2\le b_\psi$. For $H=(1-E_{h'})F$, the Leibniz rule, Lemma 297(i), $h/h'\le2$ and $h\le1$ give $h^{|\gamma|}|\partial^\gamma H|\le 3^{|\gamma|}(2C_\psi)^{4n}\max_{\alpha\le\gamma}|\partial^\alpha F|$. Put $w_0=\lceil q'\rceil+8n$ and $N=\lceil p'\rceil+4n+1$, so that $w\ge\max(w_0,8n+N)$. At a point of the support of H, $d\le12h\le\frac34$, and Lemma 297(ii) gives $(1+|x|^2)^{w_0/2}|\partial^\alpha F(x)|\le 3^{w_0}(48h)^N|F|_w/N!$ for $|\alpha|\le8n$; for $H=F$ the same quantity is at most $|F|_w$.

Summation. For $x\in X_m$, $|z_j|_\infty\le|x|_\infty+h$, so the cube of the first step has $1+L\le2(1+|x|_\infty)\le2(1+|x|^2)^{1/2}$. Evaluating at the point of the compact box X_m where $(1+|x|^2)^{-w_0/2}$ is largest, and using $q'-w_0\le-8n$ and $(1+|x|^2)^{4n}\ge\prod_{j,\nu}(1+x_{j\nu}^2)$, the piece ψ_m^hH of a term of the second or third kind contributes at most

$$C2^{q'}h^{-p'}\big(5b_\psi\cdot2C_\psi\cdot9\big)^{4n}3^{w_0}\frac{(48h)^N}{N!}|F|_w\sup_{x\in X_m}\prod_{j,\nu}(1+x_{j\nu}^2)^{-1}.$$

The sum over $m\in(\mathbb{Z}^4)^n$ of the supremum is the $4n$-th power of $\sum_{k\in\mathbb{Z}}\sup_{|t-hk|\le h}(1+t^2)^{-1}\le3+2\int_0^\infty\frac{du}{1+h^2u^2}<\frac{7}{h}$. With $48^N/N!\le e^{48}$, $3^{w_0}\le3^{q'+1}3^{8n}$ and $h^{N-p'-4n}\le h$, the terms of the second and third kinds at the scale h contribute at most $C6^{q'}c_1^nh|F|_w$ with $c_1=3^{10}e^{48}(5\cdot7\cdot9\cdot2C_\psi b_\psi)^4$, and each h_l occurs in at most two of them, with $\sum_l2h_l=\frac14$. The term E_0F contributes at most $C2^{q'}16^{p'}(5\cdot7\cdot16\,b_\psi)^{4n}|F|_w$ by the same count without the Taylor factor. This gives the bound with $c_\star=c_1+(5\cdot7\cdot16\,b_\psi)^4$.

The extension follows from the density of $C_c^\infty(\mathfrak{D}_n)$ in $\mathscr{S}_0(\mathbb{R}^{4n})$ for $|\cdot|_w$ (Lemma 297(iii)), and it is unique by the same density. □

For a function F of n points put

$$(\Theta_\mu F)(x)=F(\theta_\mu x_1,\dots,\theta_\mu x_n),$$
$$(\tau_vF)(x)=F(x_1-v,\dots,x_n-v),$$
$$(F\otimes F')(x,x')=F(x)F'(x').$$

For $F_n\in\mathscr{S}_0^{\mu+}(\mathbb{R}^{4n})$ and $F'_{n'}\in\mathscr{S}_0^{\mu+}(\mathbb{R}^{4n'})$, $\Theta_\mu\bar F_n\otimes F'_{n'}\in\mathscr{S}_0(\mathbb{R}^{4(n+n')})$, because at a coincidence of a reflected point with an unreflected one some argument of F_n or of $F'_{n'}$ has μ-coordinate at most 0.

Theorem 299. *Assume the hypothesis of Theorem 66 and* (M$_\mathrm{s}$)*, and let S_n be limits as in Theorem 274.*

(i) *Let $n\ge2$, and suppose that there are $C,p',q'\ge0$ such that*

$$|S_n(f_1,\dots,f_n)|\le C(1+L)^{q'}\delta^{-p'}\prod_j\|f_j\|_\infty$$

on $\mathscr{C}_n(Q,\delta)$ for every cube Q of side L and $\delta \in (0,1]$. Then S_n extends uniquely to a linear functional on $\mathscr{S}_0(\mathbb{R}^{4n})$ continuous for the topology of $\mathscr{S}(\mathbb{R}^{4n})$, and $|S_n(F)| \le c_\star^n C 6^{q'} 16^{p'} |F|_w$ with w and $c_\star$ as in Lemma 298.

(ii) *Under* ($\mathrm{G_s}$)*, for every $n \ge 1$ and $F \in \mathscr{S}_0(\mathbb{R}^{4n})$,*

$$|S_n(F)| \le \sigma_n |F|_{sn}, \qquad \sigma_n = (6^q 16^p c_\star K)^n (n!)^\beta,$$

with $s = \lceil p \rceil + \lceil q \rceil + 13$, and $\sigma_n \le e^{6^q 16^p c_\star K} (n!)^{\beta+1}$, since $y^n \le e^y n!$ for $y \ge 0$; this is the linear growth condition of [96].

(iii) *Suppose that the hypothesis of* (i) *holds for every $n \ge 2$, with constants depending on n, and put $S_0 = 1$. Then the extended S_n are symmetric, invariant under the translations, B_4 and G_S, and satisfy $S_n(\bar F) = \overline{S_n(F)}$. For every μ and every finite family $F_n \in \mathscr{S}_0^{\mu+}(\mathbb{R}^{4n})$, $0 \le n \le n_0$ ($F_0 \in \mathbb{C}$),*

$$\sum_{n,n'=0}^{n_0} S_{n+n'}\big(\Theta_\mu \bar F_n \otimes F_{n'}\big) \ge 0,$$

and there are vectors $\Psi_\mu(F)$ of the space $\mathscr{H}_\mu$ of Theorem 274(iii) *whose inner products are these sums, which agree with $\Psi_\mu(P)$ for separated polynomials P of smooth functions, are dense in $\mathscr{H}_\mu$, and satisfy $e^{-tH_\mu}\Psi_\mu(F) = \Psi_\mu(\tau_{te_\mu}F)$ for $t \ge 0$. For $y \perp e_\mu$, $y \ne 0$, and $F_n \in \mathscr{S}_0^{\mu+}(\mathbb{R}^{4n})$, $F'_{n'} \in \mathscr{S}_0^{\mu+}(\mathbb{R}^{4n'})$,*

$$\lim_{\lambda\to\infty} S_{n+n'}\big(\Theta_\mu \bar F_n \otimes \tau_{\lambda y} F'_{n'}\big) = S_n(\Theta_\mu \bar F_n)\, S_{n'}(F'_{n'}).$$

Proof. (i) This is Lemma 298; continuity for $|\cdot|_w$ is continuity in $\mathscr{S}(\mathbb{R}^{4n})$.

(ii) For $n = 1$, $S_1 = 0$. For $n \ge 2$ the limits satisfy the hypothesis of (i) with $C = K^n (n!)^{\bar\beta}$, $q' = qn$ and $p' = pn$, and $\lceil qn \rceil + \lceil pn \rceil + 12n + 1 \le sn$, while $|F|_w \le |F|_{sn}$ for $w \le sn$.

(iii) Let g be a permutation of the arguments, or the diagonal action of a translation, of an element of B_4 or of an element of G_S. Then $F \mapsto F \circ g^{-1}$ maps $C_c^\infty(\mathfrak{D}_n)$ onto itself and is continuous for the topology of $\mathscr{S}(\mathbb{R}^{4n})$. The distributions $F \mapsto S_n(F \circ g^{-1})$ and S_n agree on the products of separated smooth functions by the symmetry of S_n, by Theorem 274(ii) and by the definition of G_S, hence on $C_c^\infty(\mathfrak{D}_n)$ by the uniqueness in Theorem 274(i), hence on $\mathscr{S}_0(\mathbb{R}^{4n})$ by density; the conjugation identity follows in the same way. Choose an integer s' such that the bounds of (i) for $n \le 2n_0$ hold with $|\cdot|_{s'n}$, and by Lemma 297(iii) finite sums $P_n^{(i)}$ of products of smooth functions with disjoint supports in $\{x_\mu > 0\}$ with $|P_n^{(i)} - F_n|_{2s'n_0} \to 0$, and $P_0^{(i)} = F_0$. Then $P^{(i)} = \sum_n P_n^{(i)}$ is a separated polynomial at positive μ-times, and $\sum_{n,n'} S_{n+n'}(\Theta_\mu \bar P_n^{(i)} \otimes P_{n'}^{(i)}) = S(\theta_\mu \bar P^{(i)} \cdot P^{(i)}) \ge 0$ by Theorem 274(iii), each $\Theta_\mu \bar P_n^{(i)} \otimes P_{n'}^{(i)}$ being a finite sum of products of separated smooth functions. Since $|G \otimes G'|_w \le |G|_w |G'|_w$ and $|\Theta_\mu G|_w = |G|_w$, $\Theta_\mu \bar P_n^{(i)} \otimes P_{n'}^{(i)} \to \Theta_\mu \bar F_n \otimes F_{n'}$ in $|\cdot|_{s'(n+n')}$, and the positivity passes to the limit. The same convergence, applied to $P^{(i)} - P^{(i')}$, shows that $\Psi_\mu(P^{(i)})$ is a Cauchy sequence in $\mathscr{H}_\mu$; its limit $\Psi_\mu(F)$ does not depend on the approximants and has the stated inner products. Separated polynomials of smooth functions are among the families F, and their vectors are dense in $\mathscr{H}_\mu$: those of separated polynomials of continuous functions are dense by construction, and each of them is a limit of vectors of smooth ones, by uniform approximation with supports in slightly larger sets and the bound of Theorem 274(i). The relation with e^{-tH_μ} holds for the $P^{(i)}$ by Theorem 274(iv) and passes to the limit because τ_{te_μ} preserves $\mathscr{S}_0^{\mu+}$ and is continuous. For the last claim, by Theorem 274(v) applied to every pair of monomials, $\langle \Psi_\mu(P), U_\mu(\lambda y)\Psi_\mu(P')\rangle = S(\theta_\mu \bar P \cdot \tau_{\lambda y} P')$ tends to $S(\theta_\mu \bar P) S(P') = \langle \Psi_\mu(P), \Omega_\mu\rangle\langle \Omega_\mu, \Psi_\mu(P')\rangle$ for separated polynomials P, P' at positive μ-times; the $U_\mu(\lambda y)$ are unitary and these vectors are dense, so $U_\mu(\lambda y)$ converges weakly to the projection on $\mathbb{C}\Omega_\mu$. With $U_\mu(\lambda y)\Psi_\mu(F') = \Psi_\mu(\tau_{\lambda y}F')$, obtained as for e^{-tH_μ}, and $\langle \Psi_\mu(F_n), \Omega_\mu\rangle = S_n(\Theta_\mu \bar F_n)$, $\langle \Omega_\mu, \Psi_\mu(F'_{n'})\rangle = S_{n'}(F'_{n'})$, this is the claim. □

With ($\mathrm{W_s}$) and ($\mathrm{U_s}$), Corollary 275 and Theorem 299 give the axioms of [95, 96]. The positivity of Theorem 299(iii) contains that of [95], whose test functions are supported where $0 < x_{10} < \cdots < x_{n0}$, and the symmetry of the S_n makes the order of the arguments irrelevant.

Corollary 300. *Assume the hypothesis of Theorem 66,* ($\mathrm{G_s}$)*,* ($\mathrm{W_s}$) *and* ($\mathrm{U_s}$)*. Then the moments converge for every separated tuple, and their limits, extended to $\mathscr{S}_0(\mathbb{R}^{4n})$ by Theorem 299 and with $S_0 = 1$, satisfy the linear growth condition of [96] and the axioms of [95] of Euclidean invariance, reflection positivity, symmetry and clustering. The Hilbert space and the Hamiltonian of their reconstruction are $\mathscr{H}_0$ and H_0, with $\sigma(H_0) \subset \{0\} \cup [m_*, \infty)$ and $\ker H_0 = \mathbb{C}\Omega_0$. If moreover $C_k(0) > 0$ for large k and $Z_k = C_k(\sigma_0)^{-1/2}$, they are not the Schwinger functions of the vacuum alone.*

Proof. ($\mathrm{G_s}$) gives ($\mathrm{M_s}$), and Corollary 275 gives convergence, invariance under the translations and $\mathrm{O}(4)$, so $G_S = \mathrm{O}(4)$, and properties (i)–(vi) of Theorem 274. Theorem 299(ii) gives the growth condition, and part (iii) for $\mu = 0$ gives the invariance on $\mathscr{S}_0$, positivity and clustering, and identifies the Hilbert space built from $\mathscr{S}_0^{0+}$ with $\mathscr{H}_0$ and its time translations with e^{-tH_0}; Theorem 274(iv) gives the spectrum. Proposition 276(ii) gives $\mathscr{H}_0 \ne \mathbb{C}\Omega_0$, so some S_n with $n \ge 2$ is not zero, by Theorem 274(vi). □

By the reconstruction theorem of [96], Schwinger functions with these properties are those of a Wightman theory on $\mathscr{H}_0$ with Hamiltonian H_0. So the averaged condition along the tuning, ($\mathrm{G_s}$) with the constants of Proposition 276(ii), ($\mathrm{W_s}$) and ($\mathrm{U_s}$) together give (E) for the field Φ at separated points, with mass gap at least m_*. None of these four statements is proved. For $n = 2$ the rate in ($\mathrm{G_s}$) is a rate in (SD):

($\mathrm{SD_p}$) there are $C_0, p_0 \ge 0$ such that for every $\varepsilon' \in (0, \varepsilon]$ there is k_0 with $\mathscr{N}_k(\varepsilon') \le C_0(\varepsilon/\varepsilon')^{p_0} \mathscr{N}_k(\varepsilon)$ for $k \ge k_0$.

Proposition 301. *Assume the hypothesis of Theorem 280.*

(a) ($\mathrm{SD_p}$) *holds if and only if there are $C, p', q' \ge 0$ such that for every cube Q of side L and $\delta \in (0,1]$ there is k_0 with*

$$Z_k^2 \big|\omega_k\big(A_k(f)A_k(g)\big)\big| \le C(1+L)^{q'} \delta^{-p'} \|f\|_\infty \|g\|_\infty$$

for $k \ge k_0$ and $(f,g) \in \mathscr{C}_2(Q,\delta)$. One can take $q' = 8$ and $p' = 8 + p_0$ in one direction and $p_0 = p'$ in the other.

(b) *Under* ($\mathrm{SD_p}$) *every limit S_2 of Corollary 281*(b) *extends uniquely to a linear functional on $\mathscr{S}_0(\mathbb{R}^8)$ with $|S_2(F)| \le C'|F|_{\lceil p_0\rceil+41}$, invariant under the translations and B_4, with $S_2(\Theta_\mu \bar f \otimes f) \ge 0$ for $f \in \mathscr{S}_0^{\mu+}(\mathbb{R}^4)$.*

Proof. (a) Let ($\mathrm{SD_p}$) hold, $(f,g) \in \mathscr{C}_2(Q,\delta)$ with $\delta \le 1$, and $\varepsilon' = \min(\varepsilon, \delta/8)$. The proof of Corollary 281(a) gives, for large k depending only on δ, $Z_k^2|\omega_k(A_k(f)A_k(g))| \le \mathsf{K}^2_{4L/\delta}\mathsf{K}^2_{\delta/4}\mathscr{N}_k(\varepsilon')\mathscr{N}_k(\varepsilon)^{-1}\|f\|_\infty\|g\|_\infty$, and $\mathscr{N}_k(\varepsilon')/\mathscr{N}_k(\varepsilon) \le \max(1,C_0)\max(1,8\varepsilon)^{p_0}\delta^{-p_0}$. Since $\mathsf{K}^2_{4L/\delta} = (8L/\delta+3)^8 \le 11^8(1+L)^8\delta^{-8}$ and $\mathsf{K}^2_{\delta/4} \le (\frac{7}{2})^8$, the bound holds with $q' = 8$, $p' = 8+p_0$. Conversely, the proof of Corollary 281(a) gives $\mathscr{N}_k(\varepsilon') \le M\mathscr{N}_k(\varepsilon)$ for large k, where M is the constant of the bound for a cube Q_0 of side $L_0 = 2(\varepsilon+2+2a_0)$, which contains $B_{\varepsilon'}$ and $\theta_{0,k}(B_{\varepsilon'})$ for all k and $\varepsilon' \le \varepsilon$, and for the separation $\min(\varepsilon',1)$. So $M \le C(1+L_0)^{q'}\max(1,\varepsilon^{-p'})(\varepsilon/\varepsilon')^{p'}$, which is ($\mathrm{SD_p}$) with $p_0 = p'$.

(b) By (a) the limits satisfy the hypothesis of Lemma 298 with $n=2$, $q'=8$, $p'=8+p_0$, and $\lceil q'\rceil + \lceil p'\rceil + 25 = \lceil p_0\rceil + 41$. Invariance and positivity follow as in the proof of Theorem 299(iii), from those of Corollary 281(b) and Lemma 297(iii) with $n=1$. □

The rate cannot be dropped: the structure that Theorem 274 and Corollary 281(b) give at separated points does not imply temperedness.

Proposition 302. (i) *Let $K \ge 0$ be integrable on the compact subsets of $\mathbb{R}^4 \setminus \{0\}$ and $S_2(F) = \iint F(x,y)K(x-y)\,dx\,dy$ for $F \in C_c^\infty(\mathfrak{D}_2)$. If $|S_2(F)| \le C|F|_w$ on $C_c^\infty(\mathfrak{D}_2)$, then $\int_{\delta\le|z|_\infty\le 1} K \le c_w C\delta^{-w}$ for $\delta \in (0,1]$, with c_w depending only on w. Conversely, if $\int_{\delta\le|z|_\infty\le\rho} K \le C(1+\rho)^{q'}\delta^{-p'}$ for $0<\delta\le 1\le\rho$, then S_2 extends to $\mathscr{S}_0(\mathbb{R}^8)$ as in Lemma 298.*

(ii) *Let $K(z) = \int_1^\infty e^{\sqrt{\lambda}-\lambda\|z\|_1}d\lambda$ for $z \ne 0$, $\|z\|_1 = \sum_\nu |z_\nu|$, and $S_2(f,g) = \iint f(x)g(y)K(x-y)\,dx\,dy$ for separated pairs. Then S_2 is given by a distribution on $\mathfrak{D}_2$, is symmetric, invariant under the translations and B_4, real on real functions, bounded on $\mathscr{C}_2(Q,\delta)$ by $L^8\mathsf{k}(\delta)\|f\|_\infty\|g\|_\infty$ with $\mathsf{k}(\delta) = \int_1^\infty e^{\sqrt{\lambda}-\lambda\delta}d\lambda$, and satisfies $|S_2(f,g(\cdot-\lambda y))| \le e\mathsf{k}(1)L^8\|f\|_\infty\|g\|_\infty e^{-(\lambda|y|_\infty - L)}$ for f,g supported in a cube of side L and $\lambda|y|_\infty \ge L+1$. For g supported in $\{x_\mu > 0\}$, $S_2(\bar g\circ\theta_\mu, g) \ge 0$; the time translations define on the completion $\mathscr{H}^{(1)} \ne 0$ of the functions supported in $\{x_0>0\}$ for the form $S_2(\bar g\circ\theta_0, g')$ a semigroup $e^{-tH^{(1)}}$ with $\sigma(H^{(1)}) \subset [1,\infty)$. These are the properties of Corollary 281*(b)*, with $m_* = 1$. But $\int_{\delta\le|z|_\infty\le 2\delta} K \ge 240\,\delta^4 e^{1/(64\delta)}$ for $\delta \le \frac{1}{32}$, so S_2 is bounded by no norm $|\cdot|_w$ on $C_c^\infty(\mathfrak{D}_2)$ and does not extend to a continuous functional on $\mathscr{S}_0(\mathbb{R}^8)$.*

Proof. (i) Let $\chi_0 \in C^\infty(\mathbb{R})$ be even, nonincreasing on $[0,\infty)$, equal to 1 on $[0,1]$ and to 0 on $[2,\infty)$, $\chi_1(z) = \prod_\nu \chi_0(z_\nu)$, and $\psi_\delta(z) = \chi_1(z) - \chi_1(2z/\delta)$. Then $0 \le \psi_\delta \le 1$, $\psi_\delta = 1$ on $\{\delta \le |z|_\infty \le 1\}$, $\psi_\delta = 0$ on $\{|z|_\infty \le \delta/2\} \cup \{|z|_\infty \ge 2\}$, and $|\partial^\gamma \psi_\delta| \le c_\gamma \delta^{-|\gamma|}$. With $\phi \in C_c^\infty((-1,1)^4)$, $\phi \ge 0$, $\int\phi = 1$, the function $F_\delta(x,y) = \phi(x)\psi_\delta(x-y)$ lies in $C_c^\infty(\mathfrak{D}_2)$, has $|F_\delta|_w \le c_w\delta^{-w}$, and $S_2(F_\delta) = \int \psi_\delta K \ge \int_{\delta\le|z|_\infty\le 1} K$. For the converse, $|S_2(f,g)| \le \|f\|_\infty\|g\|_\infty \int_Q dx \int_{\delta\le|z|_\infty\le\max(L,1)} K \le 2^{q'}C(1+L)^{q'+4}\delta^{-p'}\|f\|_\infty\|g\|_\infty$ on $\mathscr{C}_2(Q,\delta)$, and S_2 is the distribution of Lemma 298 by uniqueness.

(ii) For $|z|_\infty \ge \delta$, $\|z\|_1 \ge \delta$ and $K(z) \le \mathsf{k}(\delta)$; for $\|z\|_1 = u \ge 1$ and $\lambda \ge 1$, $\lambda u \ge \lambda + u - 1$, so $K(z) \le e\mathsf{k}(1)e^{-u}$. Hence K is continuous on $\mathbb{R}^4\setminus\{0\}$, which gives the distribution, and the two bounds follow, since $\|x-y\|_1 \ge |x-y|_\infty$. Symmetry, reality and invariance hold because K is real, even and a function of $\|z\|_1$, which B_4 preserves. Let g, g' be supported in $\{x_0 \ge t_0\}$, $t_0 > 0$. Substituting $x \mapsto \theta_0 x$ and $y \mapsto y + te_0$,

$$S_2\big(\bar g\circ\theta_0, g'(\cdot - te_0)\big) = \int_1^\infty e^{\sqrt{\lambda}-\lambda t}\,\mathsf{B}_\lambda(G_\lambda, G'_\lambda)\,d\lambda,$$

where G'_λ is formed from g' as G_λ from g, with

$$G_\lambda(\mathbf{x}) = \int g(x_0,\mathbf{x})e^{-\lambda x_0}dx_0,$$

$$\mathsf{B}_\lambda(u,v) = \iint \bar u(\mathbf{x})v(\mathbf{y})e^{-\lambda\|\mathbf{x}-\mathbf{y}\|_1}d\mathbf{x}\,d\mathbf{y} \qquad (u,v\colon \mathbb{R}^3 \to \mathbb{C}).$$

The integrand is bounded by $e^{\sqrt{\lambda}-2\lambda t_0}\|g\|_1\|g'\|_1$. From $e^{-\lambda|\varsigma|} = \frac{1}{\pi}\int_{\mathbb{R}} \frac{\lambda}{\lambda^2+k^2}e^{ik\varsigma}dk$ one obtains

$$\mathsf{B}_\lambda(u,u) = \pi^{-3}\int |\hat u(k)|^2 \prod_{i=1}^3 \frac{\lambda}{\lambda^2+k_i^2}\,dk \ge 0.$$

So the form is positive, and

$$\|g(\cdot - te_0)\|^2 = \int_1^\infty e^{\sqrt{\lambda}-2\lambda t}\mathsf{B}_\lambda(G_\lambda,G_\lambda)\,d\lambda \le e^{-2t}\|g\|^2.$$

The time translations therefore induce symmetric contractions $\mathsf{T}(t)$ of $\mathscr{H}^{(1)}$ with $\mathsf{T}(s)\mathsf{T}(t) = \mathsf{T}(s+t)$ and $\|\mathsf{T}(t)\| \le e^{-t}$, strongly continuous by dominated convergence, so $\mathsf{T}(t) = e^{-tH^{(1)}}$ with $H^{(1)} \ge 1$, as in the proof of Theorem 201(iv). For $g \ge 0$ not zero, $G_\lambda \ge 0$ is not zero and $\mathsf{B}_\lambda(G_\lambda,G_\lambda) > 0$, so $\mathscr{H}^{(1)} \ne 0$. Positivity for $\mu \ne 0$ follows from $\theta_\mu\pi_\mu = \pi_\mu\theta_0$ and the invariance under π_μ. Finally, let $0 < |z|_\infty \le 2\delta \le \frac{1}{16}$, so that $u = \|z\|_1 \le 8\delta \le \frac{1}{4}$, and $\lambda_0 = (4u)^{-2} \ge 1$. For $\lambda \in [\lambda_0, 2\lambda_0]$, $\sqrt{\lambda} - \lambda u \ge \frac{1}{4u} - \frac{1}{8u} = \frac{1}{8u}$, so $K(z) \ge \lambda_0 e^{1/(8u)} \ge e^{1/(64\delta)}$, and $\{\delta \le |z|_\infty \le 2\delta\}$ has volume $240\delta^4$. By (i), a bound $|S_2(F)| \le C|F|_w$ would give $240\delta^4 e^{1/(64\delta)} \le c_w C\delta^{-w}$ for all small δ, which is false; and a continuous functional on $\mathscr{S}_0(\mathbb{R}^8)$ is bounded by some $C|\cdot|_w$. □

The kernel of (ii) is the superposition, with weight $e^{\sqrt{\lambda}}d\lambda$, of the kernels $e^{-\lambda\|z\|_1}$ of the limits of Proposition 207(a). By (i), for a two-point function with a nonnegative kernel, a power rate as in Lemma 298 is also necessary for temperedness.

Remark 303. *Coinciding points.* The axioms of [95, 96] involve only $\mathscr{S}_0(\mathbb{R}^{4n})$, and positivity involves only test functions whose reflected and unreflected arguments are separated. Under ($\mathrm{G_s}$) the bound of Theorem 299(ii) extends by the Hahn–Banach theorem [101] to a functional on all

of $\mathscr{S}(\mathbb{R}^{4n})$ with the same bound; averaging over the finite group generated by the permutations of the arguments and the diagonal action of B_4, which preserve every $|\cdot|_w$, makes it symmetric and B_4-invariant. Every such extension agrees with S_n on $\mathscr{S}_0(\mathbb{R}^{4n})$ and so satisfies the positivity of Theorem 299(iii); two of them differ by a distribution supported on the coinciding points. Translation invariance of the extension is not claimed here. For the action density the extension cannot be an integral: under the hypothesis of Proposition 278, the functions F_δ of the proof of Proposition 302(i) are supported in one compact set and bounded by $\|\phi\|_\infty$, while, since $|z| \le 2|z|_\infty$, $S_2(F_\delta) \ge \int_{\delta\le|z|_\infty\le 2\delta} K \ge 240\cdot 4^{-8} c\,\delta^{-4}(\log\frac{1}{\delta})^{-p}$ for $2\delta \le \min(1, r_0/2)$; so no extension of S_2 to $\mathbb{R}^8$ is a distribution of order zero, and its values at coinciding points are fixed only by a renormalisation, which the axioms do not see. The same law satisfies the hypothesis of Theorem 299(i) for $n=2$: if in addition $K(z) \le C|z|^{-8}$ for $z \ne 0$, then $|S_2(f,g)| \le \|f\|_\infty\|g\|_\infty L^4 C \int_{|z|\ge\delta} |z|^{-8} dz = \frac{\pi^2}{2} C L^4 \delta^{-4}\|f\|_\infty\|g\|_\infty$ on $\mathscr{C}_2(Q,\delta)$, which is that hypothesis with $q' = p' = 4$, while (M) fails by Proposition 278. At strong coupling (Corollary 277) every limit vanishes at separated points, and its extension is 0.

Asymptotic freedom at short distance. Renormalised perturbation theory predicts how the two-point function of Φ behaves at short distance, and this behaviour can be posed as a condition on the lattice states. Let $G_k(z) = \omega_k(\Phi_0\Phi_z) - c_k^2$ for $z \in \mathbb{Z}^4$, as in the proof of Theorem 288(iv), so that $\omega_k((\Phi_x - c_k)(\Phi_y - c_k)) = G_k(y-x)$; let $\log^+ t = \max(\log t, 0)$ and, for $\gamma \ge 0$ and $\rho > 0$,

$$\ell(\rho) = 1 + \log^+\frac{1}{\rho}, \qquad w_\gamma(\rho) = \rho^{-8}\ell(\rho)^{-\gamma}.$$

(AF) There are $\gamma \ge 0$, $0 < c_1 \le C_1$, $r_0 \in (0,1]$ and k_0 such that for all $k \ge k_0$ and $z \in \mathbb{Z}^4$ with $|z|_\infty \ge 2r+1$

$$Z_k^2 |G_k(z)| \le C_1 w_\gamma(a_k|z|),$$

and $Z_k^2 G_k(z) \ge c_1 w_\gamma(a_k|z|)$ if moreover $a_k|z| \le r_0$.

(AF_∞) There are $\gamma \ge 0$ and $C_{\mathrm{AF}} > 0$ such that every limit S_2 of $Z_k^2\omega_k(A_k(f)A_k(g))$ on the separated pairs along a subsequence is $S_2(f,g) = \iint f(x)g(y)K(x-y)\,dx\,dy$ with K locally integrable on $\mathbb{R}^4\setminus\{0\}$ and

$$\lim_{\rho\to 0} \operatorname*{ess\,sup}_{0<|z|\le\rho} \Big| |z|^8 \Big(\log\frac{1}{|z|}\Big)^\gamma K(z) - C_{\mathrm{AF}}\Big| = 0.$$

(AF_β) For some $\Lambda > 0$, with $L_k = \log(1/(a_k\Lambda))$, the couplings are $\beta_{W,k} = 2N/g_k^2$ with $g_k^{-2} = 2b_0 L_k + (b_1/b_0)\log L_k + O(1)$.

(AF) is the two-sided form, at every lattice distance from $2r+1$ to r_0/a_k, of the law $|x-y|^{-8}(\log\frac{1}{|x-y|})^{-\gamma}$, with only a power bound at physical distances; (AF_∞) is the exact law of the limit at short distance; (AF_β) is the two-loop tuning of Remark 247. None of the three is proved here for the Wilson measure.

Remark 304. *The perturbative values.* Renormalised perturbation theory gives the following values; they are predictions and no proof below uses them. Let $F^a_{\mu\nu}$ be the field strength normalised by the action $\frac14\int F^a_{\mu\nu}F^a_{\mu\nu}$, with generators $\mathrm{Tr}(T^aT^b) = \frac12\delta^{ab}$, and let $\beta(g) = -b_0 g^3 + O(g^5)$, $b_0 = 11N/(48\pi^2)$ [63, 99]. The trace of the energy–momentum tensor, $\Theta = (\beta(g)/2g)F^a_{\mu\nu}F^a_{\mu\nu}$, is finite and invariant under the renormalisation group [35], so $\langle\Theta(x),\Theta(0)\rangle$, the connected two-point function, is $|x|^{-8}F(\bar g(|x|))$ with $\bar g(|x|)$ the running coupling at the scale $1/|x|$, and $F(g) = \frac14 b_0^2 g^4 T_0(1+O(g^2))$, where $T_0|x|^{-8}$ is the connected two-point function of $F^a_{\mu\nu}F^a_{\mu\nu}$ in the free theory. There, in the Feynman gauge, the propagator of the gauge field is $\delta^{ab}\delta_{\mu\nu}D(x)$ with $D(x) = 1/(4\pi^2|x|^2)$, and that of the field strength, which is gauge independent, is $\langle F^a_{\mu\nu}(x), F^b_{\rho\sigma}(0)\rangle = -\delta^{ab}P_{\mu\nu\rho\sigma}$ with

$$P_{\mu\nu\rho\sigma} = \delta_{\nu\sigma}D_{\mu\rho} - \delta_{\nu\rho}D_{\mu\sigma} - \delta_{\mu\sigma}D_{\nu\rho} + \delta_{\mu\rho}D_{\nu\sigma},$$

$D_{\mu\rho} = \partial_\mu\partial_\rho D = M_{\mu\rho}/(2\pi^2|x|^4)$, where $M = 4uu^{\mathsf T} - 1$, $u = x/|x|$. Since $\mathrm{tr}M = 0$ and $\mathrm{tr}M^2 = 12$, each of the four squared terms of the bracket, summed over μ,ν,ρ,σ, gives $4\cdot 12$, the two cross terms with a trace vanish and the other four give -12 each, so the sum of the squares is $(192-96)(2\pi^2|x|^4)^{-2} = 24\pi^{-4}|x|^{-8}$, and Wick's theorem gives $T_0 = 2(N^2-1)\cdot 24\pi^{-4} = 48(N^2-1)/\pi^4$. With $\bar g(|x|)^{-2} = 2b_0\log\frac{1}{|x|\Lambda}(1+o(1))$ this gives, as $|x|\to 0$,

$$\langle\Theta(x),\Theta(0)\rangle = \frac{3(N^2-1)}{\pi^4}\,|x|^{-8}\Big(\log\frac{1}{|x|}\Big)^{-2}(1+o(1)),$$

which is (AF_∞) with $\gamma = 2$ for every N and with $C_{\mathrm{AF}} = 3(N^2-1)/\pi^4$, a constant free of b_0 and Λ.[33] For the sum Φ of $N - \mathrm{Re}\,\mathrm{Tr}\,U_p$ over the 24 plaquettes with a corner at 0, $U_p = \exp(ia_k^2 g_k F^a_{\mu\nu}T^a + \cdots)$ gives $N - \mathrm{Re}\,\mathrm{Tr}\,U_p = \frac14 a_k^4 g_k^2 F^a_{\mu\nu}F^a_{\mu\nu} + \cdots$ (no sum) for p in the plane $\{\mu,\nu\}$, and each plane carries four of the 24 plaquettes, so $\Phi_x = \frac12 a_k^4 g_k^2 F^a_{\mu\nu}F^a_{\mu\nu} + \cdots = -(a_k^4/b_0)\Theta(1+O(g_k^2))$, up to a constant and fields of higher dimension. Thus $Z_k a_k^4 \to b_0$ makes $Z_k A_k(f)$ tend to $-\Theta(f)$; then (AF) holds with $\gamma = 2$ and Z_k of order a_k^{-4}, and $G_k(z)$ at a fixed lattice distance is of order g_k^4, that is of order $\beta_{W,k}^{-2}$ under (AF_β).

Theorem 305. *Assume the hypothesis of Theorem 66 and* (AF) *for the constants Z_k.*

(i) *Let Q be a cube of side L and $\delta \in (0,1]$. For $k \ge k_0$ with $a_k \le \min(1, \delta/(2r+1))$ and for $(f,g) \in \mathscr{C}_2(Q,\delta)$,*

$$Z_k^2\big|\omega_k\big(A_k(f)A_k(g)\big)\big| \le 100\,C_1(1+L)^4\delta^{-4}\|f\|_\infty\|g\|_\infty.$$

So ($\mathrm{M}_{\mathrm{s},2}$) *holds; every limit S_2 along a subsequence satisfies*

$$|S_2(f,g)| \le C_1 \iint |f(x)||g(y)| w_\gamma(|x-y|)\,dx\,dy$$

[33] For SU(2), $C_{\mathrm{AF}} = 9/\pi^4 \approx 0.092$; for SU(3), $C_{\mathrm{AF}} = 24/\pi^4 \approx 0.246$.

on the separated pairs, and extends uniquely to a continuous linear functional on $\mathscr{S}_0(\mathbb{R}^8)$ with $|S_2(F)| \le C|F|_{33}$.

(ii) *If $f,g \ge 0$ form a separated pair and $|x-y| \le r_0$ for $x \in \operatorname{supp} f$, $y \in \operatorname{supp} g$, then $S_2(f,g) \ge c_1 \iint f(x)g(y)w_\gamma(|x-y|)\,dx\,dy$. In particular no limit S_2 vanishes identically, and the moment bound* (M) *of Sec. XI D, posed for the packets $Z_k A_k(f)$, fails for $n=2$ and $Q=[0,1]^4$.*

(iii) *For every $\varepsilon > 0$, $\mathscr{N}_k(\varepsilon) > 0$ for large k and*

$$0 < \liminf_k Z_k^2 \mathscr{N}_k(\varepsilon) \le \limsup_k Z_k^2 \mathscr{N}_k(\varepsilon) < \infty.$$

(iv) (SD$_\mathrm{p}$) *holds with $p_0 = 4$, hence* (SD) *and* (UV), *and* (NE) *holds. Every limit S_2 of Corollary 281*(b) *is not identically zero, extends to $\mathscr{S}_0(\mathbb{R}^8)$ as in Proposition 301*(b), *and its Hamiltonian $H^{(1)}$ has a nonempty spectrum in $[m_*,\infty)$.*

(v) *If the transfer matrices $\mathbb{T}_k$ have spectrum on $\Omega_k^\perp$ in $[0,e^{-m_{\mathrm{lat}}}]$ for a fixed $m_{\mathrm{lat}} > 0$ and all k, then* (AF) *fails for every choice of Z_k; this is the case under the hypotheses of Corollary 277.*

Proof. (i) By Lemma 273(i) and translation invariance,

$$\omega_k(A_k(f)A_k(g)) = \sum_{x,y} a_k^8 f(a_k x) g(a_k y) G_k(y-x).$$

A nonzero term has $|a_k x - a_k y|_\infty \ge \delta$, so $|y-x|_\infty \ge \delta/a_k \ge 2r+1$, and (AF) with $\ell \ge 1$ bounds $Z_k^2|G_k(y-x)|$ by $C_1(a_k|y-x|)^{-8} \le C_1(a_k|y-x|_\infty)^{-8}$. There are $(2j+1)^4-(2j-1)^4 = 64j^3+16j \le 80j^3$ points $v \in \mathbb{Z}^4$ with $|v|_\infty = j \ge 1$, so with $J = \lceil \delta/a_k \rceil$

$$\sum_{|v|_\infty \ge J} a_k^4 (a_k|v|_\infty)^{-8} \le 80 a_k^{-4} \sum_{j \ge J} j^{-5} \le 80 a_k^{-4} \Big(J^{-5} + \frac{J^{-4}}{4} \Big) \le 100\,\delta^{-4},$$

while at most $(L/a_k+1)^4$ sites x have $a_k x \in Q$, and $a_k^4(L/a_k+1)^4 \le (1+L)^4$. This gives the bound, and (M$_{\mathrm{s},2}$) follows since $\mathscr{C}_2(Q,\delta) \subset \mathscr{C}_2(Q,1)$ for $\delta \ge 1$. In the same sum $Z_k^2|G_k(y-x)| \le C_1 w_\gamma(a_k|y-x|)$, and $(x,y) \mapsto |f(x)||g(y)|w_\gamma(|x-y|)$ is continuous with compact support in $\mathbb{R}^8$, since w_γ is continuous on $(0,\infty)$; its Riemann sums converge to its integral, which gives the bound on S_2. The extension is Lemma 298 with $n=2$, $C = 100C_1$, $q' = p' = 4$ and $w = 4+4+24+1 = 33$.

(ii) For large k every nonzero term of the sum has $|y-x|_\infty \ge 2r+1$ and $a_k|y-x| \le r_0$, so the lower bound of (AF) gives $Z_k^2\omega_k(A_k(f)A_k(g)) \ge c_1 \sum_{x,y} a_k^8 f(a_k x)g(a_k y) w_\gamma(a_k|y-x|)$, and the Riemann sums converge as in (i). The integral is positive when f and g are not zero. For (M), let $s_0 = r_0/2$ and $0 < \delta \le r_0/16$; let f be continuous with values in $[0,1]$, equal to 1 on $[\frac12-2\delta,\frac12-\delta]\times[\frac{s_0}{4},\frac{3s_0}{4}]^3$ and supported in $[\frac12-3\delta,\frac12-\frac{\delta}{2}]\times[0,s_0]^3$, and g continuous with values in $[0,1]$, equal to 1 on $[\frac12+\delta,\frac12+2\delta]\times[\frac{s_0}{8},\frac{7s_0}{8}]^3$ and supported in $[\frac12+\frac{\delta}{2},\frac12+3\delta]\times[0,s_0]^3$. The supports lie in $[0,1]^4$, at distance δ, and all their points are within $(36\delta^2+3s_0^2)^{1/2} \le r_0$ of each other. The pairs with x in the first box, $x_0+\delta \le y_0 \le x_0+4\delta$ with $y_0 \in [\frac12+\delta,\frac12+2\delta]$, and $|y_i-x_i| \le \delta$ for $i=1,2,3$ have measure $\frac{\delta s_0^3}{8}\cdot\delta\cdot 8\delta^3 = s_0^3\delta^5$, $f(x)g(y)=1$ and $2\delta \le |x-y| \le \sqrt{19}\,\delta < 5\delta$, where $w_\gamma \ge (5\delta)^{-8}\ell(2\delta)^{-\gamma}$. Hence $S_2(f,g) \ge c_1 5^{-8} s_0^3 \delta^{-3}(1+\log\frac{1}{2\delta})^{-\gamma}$, which is unbounded as $\delta \to 0$, while $\|f\|_\infty\|g\|_\infty = 1$; under (M) the limit would be bounded by $M_2(Q)$, as in the proof of Proposition 278.

(iii) Let Q_0 be the cube of side $L_0 = 2(\varepsilon+2+2a_0)$ of the proof of Proposition 301(a), which contains $B_{\varepsilon'}$ and $\theta_{0,k}(B_{\varepsilon'})$ for all k and $\varepsilon' \le \varepsilon$. For h supported in $B_{\varepsilon'}$ with $\|h\|_\infty \le 1$ and $a_k \le \varepsilon'$, $\omega_k(\vartheta(\overline{A_k(h)})A_k(h)) = \omega_k(A_k(\bar h \circ \theta_{0,k})A_k(h))$, and the pair lies in $\mathscr{C}_2(Q_0,\varepsilon')$; so by (i)

$$Z_k^2 \mathscr{N}_k(\varepsilon') \le 100 C_1 (1+L_0)^4 \max(1, \varepsilon'^{-4})$$

for large k, which with $\varepsilon' = \varepsilon$ is the upper bound. Suppose that $Z_k^2\mathscr{N}_k(\varepsilon) \to 0$ along a subsequence. By (M$_{\mathrm{s},2}$) and the proof of Theorem 274(i) a further subsequence has a limit S_2 on all separated pairs. Let g be supported in a cube of side L in $\{x_0 \ge \eta\}$, $\eta > 0$, and $t \ge 4\varepsilon+1$ dyadic. For large k, $\bar g \circ \theta_{0,k}$ is supported in $\{x_0 \le -\eta + a_k\}$ and $g(\cdot - te_0)$ in $\{x_0 \ge \eta + t\}$, so with $c = 2\varepsilon - \eta + a_k$ and $d = 0$ the inequality of Theorem 280(i), multiplied by $\mathscr{N}_k(\varepsilon)$, whose proof does not use $\mathscr{N}_k(\varepsilon) > 0$, gives

$$Z_k^2 \big|\omega_k\big(A_k(\bar g \circ \theta_{0,k})A_k(g(\cdot - te_0))\big)\big| \le \mathsf{K}_L^2 Z_k^2 \mathscr{N}_k(\varepsilon)\|g\|_\infty^2 \to 0.$$

By the approximation of $\theta_{0,k}$ by θ_0 in the proof of Theorem 274(iii), $S_2(\bar g \circ \theta_0, g(\cdot - te_0)) = 0$. The construction in the proof of Corollary 281(b), which uses only (M$_{\mathrm{s},2}$), gives a Hilbert space $\mathscr{H}^{(1)}$ and a self-adjoint $H^{(1)} \ge 0$ with $S_2(\bar g \circ \theta_0, g(\cdot - te_0)) = \|e^{-tH^{(1)}/2}w\|^2$, w the vector of g. Since $e^{-tH^{(1)}/2}$ is injective, $w = 0$, that is $S_2(\bar g \circ \theta_0, g) = 0$, for every g supported in $\{x_0 > 0\}$, and by the proof of Corollary 281(b) S_2 vanishes identically. This contradicts (ii). So $\liminf_k Z_k^2 \mathscr{N}_k(\varepsilon) > 0$, and in particular $\mathscr{N}_k(\varepsilon) > 0$ for large k.

(iv) Let $c_\varepsilon = \frac12 \liminf_k Z_k^2 \mathscr{N}_k(\varepsilon)$ and $C_2 = 100 C_1 (1+L_0)^4$. By (iii), for $\varepsilon' \in (0,\varepsilon]$ and large k,

$$\mathscr{N}_k(\varepsilon') \le \frac{C_2 \max(1,\varepsilon'^{-4})}{c_\varepsilon} \mathscr{N}_k(\varepsilon) \le C_0 \Big(\frac{\varepsilon}{\varepsilon'}\Big)^4 \mathscr{N}_k(\varepsilon),$$

where $C_0 = C_2 \max(1,\varepsilon^{-4})/c_\varepsilon$; the second inequality holds because $\max(1,\varepsilon'^{-4})$ is at most $\max(1,\varepsilon^{-4})(\varepsilon/\varepsilon')^4$. This is (SD$_\mathrm{p}$) with $p_0 = 4$; it gives (SD), and (UV) by Proposition 282(ii). Let $S_2^\sharp$ be a limit of Corollary 281(b), formed with $\mathscr{N}_k(\varepsilon)^{-1}$ in place of Z_k^2. By (i) and (iii) a further subsequence has $Z_k^2\mathscr{N}_k(\varepsilon) \to \lambda \in (0,\infty)$ and a limit S_2 with the constants Z_k, and then $S_2^\sharp = S_2/\lambda$, which is not identically zero by (ii). By Proposition 284(i) this is (NE), and Corollary 281(b) and Proposition 301(b) give the rest.

(v) Under (AF), (iii) and (iv) give $\mathscr{N}_k(\varepsilon) > 0$ for large k and (SD), which Proposition 282(iii) excludes under a lattice gap; Corollary 277 provides the gap. □

So the short-distance law, in the two-sided form (AF), gives the two inputs (SD) and (NE) of Sec. XI H for $n=2$, with

the power rate ($\mathrm{SD_p}$) that temperedness needs, and it excludes the trivial limits of the strong-coupling range. It says nothing about $n \geq 3$: the bounds (RN_m) with $m \geq 2$ of Proposition 285 are not consequences of it. It also fixes the constants Z_k up to bounded factors.

Proposition 306. *Assume* (AF) *for the constants Z_k, and let $z_0 \in \mathbb{Z}^4$ with $|z_0|_\infty \geq 2r+1$.*

(i) *For large k, $G_k(z_0) > 0$ and*

$$c_1 |z_0|^{-8} \leq Z_k^2 a_k^8 \, \ell(a_k|z_0|)^\gamma G_k(z_0) \leq C_1 |z_0|^{-8}.$$

(ii) *$Z_k \geq c_1^{1/2} |z_0|^{-4} \|\Phi\|_\infty^{-1} a_k^{-4} \ell(a_k|z_0|)^{-\gamma/2}$ for large k; in particular $Z_k \to \infty$.*

(iii) *For large k and every $z \in \mathbb{Z}^4$ with $|z|_\infty \geq 2r+1$ and $a_k|z| \leq r_0$,*

$$\frac{c_1}{C_1} \frac{w_\gamma(a_k|z|)}{w_\gamma(a_k|z_0|)} \leq \frac{G_k(z)}{G_k(z_0)} \leq \frac{C_1}{c_1} \frac{w_\gamma(a_k|z|)}{w_\gamma(a_k|z_0|)}.$$

(iv) *Under the hypothesis of Theorem 66, if* (AF) *also holds for constants Z_k' with an exponent γ', then $\gamma' = \gamma$ and Z_k'/Z_k is bounded above and below by positive constants; both are comparable to $\mathscr{N}_k(\varepsilon)^{-1/2}$ for every $\varepsilon > 0$.*

(v) *Under* (AF_β), *$Z_k a_k^4$ is bounded above and below by positive constants if and only if $\beta_{W,k}^\gamma G_k(z_0)$ is.*

Proof. (i) is (AF) at $z = z_0$, valid for large k because $a_k|z_0| \to 0$. (ii) follows from (i), since $|G_k(z_0)| = |\mathrm{Cov}(\Phi_0, \Phi_{z_0})| \leq \mathrm{Var}(\Phi_0) \leq \|\Phi\|_\infty^2$ by the Cauchy–Schwarz inequality and translation invariance, and $\ell(a_k|z_0|)^\gamma a_k^8 \to 0$. (iii) is the quotient of the two bounds of (AF) at z and z_0. For (iv), Theorem 305(iii) applied to Z_k and to Z_k' makes $Z_k^2 \mathscr{N}_k(\varepsilon)$ and $Z_k'^2 \mathscr{N}_k(\varepsilon)$ bounded above and below, hence Z_k'/Z_k; and (i) for both gives $\ell(a_k|z_0|)^{\gamma-\gamma'}$ bounded above and below, which forces $\gamma' = \gamma$ since $\ell(a_k|z_0|) \to \infty$. For (v), ($\mathrm{AF}_\beta$) gives $\beta_{W,k}/\log(1/a_k) \to 4Nb_0 > 0$, and $\ell(a_k|z_0|)/\log(1/a_k) \to 1$, so by (i) $Z_k^2 a_k^8$ is bounded above and below exactly when $\beta_{W,k}^\gamma G_k(z_0)$ is. □

By Remark 304, perturbation theory predicts $\gamma = 2$ and $G_k(z_0)$ of order $\beta_{W,k}^{-2}$, hence Z_k of order a_k^{-4}, and for the plaquette sum $Z_k a_k^4 \to b_0$ in the normalisation of Θ. Part (iii) is the content of (AF) without reference to Z_k: the connected correlation of Φ decays from the lattice distance $|z_0|$ to every distance up to r_0/a_k like the eighth power of the distance, with the logarithmic correction, within fixed factors. Its lower half is a lower bound on correlations at physical separations, the kind of input that upper bounds do not supply (compare Proposition 209).

The two-sided law is consistent with the other properties of Sec. XI H and implies neither ($\mathrm{W_s}$) nor the exact law (AF_∞).

Proposition 307. *Let $\gamma \geq 0$, $\mu_j = 2^{4(j-1)}(1+j)^{-\gamma/2}$ for $j \geq 0$, let $a_k \leq 1$ and $J_k = \lfloor \log_2(2/a_k) \rfloor$, let $\phi_k^{(j)}$, $0 \leq j \leq J_k$, be independent copies of the field of Proposition 207*(a) *with q_k replaced by $q_{k,j} = e^{-2^{j-1} a_k}$, and put $\phi_k = \sum_{j \leq J_k} \mu_j \phi_k^{(j)}$, $A_k(f) = \sum_x a_k^4 f(a_k x) \phi_k(x)$, $r = 0$ and $Z_k = 1$.*

(a) *The law of ϕ_k is invariant under translations and B_4 and reflection positive for the reflections in coordinate hyperplanes through sites and bisecting links, and the spectrum of its transfer matrix in each coordinate direction on $\Omega_k^\perp$ lies in $[0, e^{-a_k/2}]$ and contains $e^{-a_k/2}$: the gap is $\frac{1}{2}$ in physical units.*

(b) (AF) *holds with this γ, $r_0 = \frac{1}{2}$, $c_1 = e^{-1} 2^{-16-\gamma}$ and $C_1 = \max(2^\gamma (A_\gamma + B), B')$, where $A_\gamma = \sum_{i \geq 0} 2^{-8i} (1+i)^\gamma$, $B = \sum_{i \geq 1} 2^{8i} e^{-2^{i-1}}$ and $B' = \sum_{j \geq 0} \sup_{t \geq 2^j} t^8 e^{-t}$.*

(c) *The moments converge along the whole sequence to the Gaussian family with $S_2(f,g) = \iint f(x) g(y) K(x-y) \, dx \, dy$, $K(v) = h(\|v\|_1)$, $h(u) = \sum_{j \geq 0} \mu_j^2 e^{-2^{j-1} u}$.* ($\mathrm{G_s}$) *holds with $\beta = \frac{1}{2}$ and $p = q = 2$, and* ($\mathrm{U_s}$) *holds; the conclusions of Theorem 305(i)–(iv) hold with $m_* = \frac{1}{2}$. The group G_S is B_4, so* ($\mathrm{W_s}$) *fails, and* (AF_∞) *fails.*

Proof. (a) As in the proof of Proposition 283(a), with $J_k + 1$ independent fields in place of two: the tuple is a reversible Markov chain in each coordinate direction with transfer operator $\bigotimes_j P_j$, of spectrum $\{\prod_j q_{k,j}^{N_j}\}$ with 1 simple, the functions of ϕ_k at positive times are functions of the tuple there, and the vector of $\phi_k(x)$ at positive time has a nonzero component of eigenvalue $q_{k,0} = e^{-a_k/2}$, the largest $q_{k,j}$.

(b) The covariance is $G_k(z) = \sum_{j \leq J_k} \mu_j^2 e^{-2^{j-1} u}$ with $u = a_k \|z\|_1$; for $z \neq 0$ let $\rho = a_k|z|$, so that $\rho \leq u \leq 2\rho$ and $u \geq a_k$. If $u \leq 2$, let $j_* = \lfloor \log_2(2/u) \rfloor \geq 0$, so that $\frac{1}{2} < 2^{j_*-1} u \leq 1$. The terms with $j = j_* - i$, $i \geq 0$, are at most $\mu_{j_*}^2 2^{-8i} (1+i)^\gamma$, because $(1+j_*)/(1+j) \leq 1+i$; those with $j = j_* + i$, $i \geq 1$, at most $\mu_{j_*}^2 2^{8i} e^{-2^{i-1}}$, because $2^{j-1} u > 2^{i-1}$. So $G_k(z) \leq (A_\gamma + B)(2^{j_*-1})^8 (1+j_*)^{-\gamma} \leq (A_\gamma + B) \rho^{-8} (1+j_*)^{-\gamma}$. Here $1 + j_* \geq \max(1, \log_2(2/u)) \geq \max(1, \log_2(1/\rho))$, which is at least $\frac{1}{2}\ell(\rho)$, so $G_k(z) \leq 2^\gamma (A_\gamma + B) w_\gamma(\rho)$. If $u > 2$, then $\rho > 1$, $2^{j-1} u \geq 2^j$ and $\mu_j^2 \leq 2^{8(j-1)}$, so $G_k(z) \leq u^{-8} \sum_j (2^{j-1} u)^8 e^{-2^{j-1} u} \leq B' u^{-8} \leq B' w_\gamma(\rho)$. For the lower bound let $\rho \leq \frac{1}{2}$, so $u \leq 1$; then $j_* \leq J_k$ because $u \geq a_k$, and the term j_* alone is at least $(2u)^{-8} (1+j_*)^{-\gamma} e^{-1}$. Since $(2u)^{-8} \geq (4\rho)^{-8}$ and $1 + j_* \leq 1 + \log_2(2/\rho) = 2 + \log(1/\rho)/\log 2 \leq 2\ell(\rho)$, because $1/\log 2 < 1.5$, this is at least $e^{-1} 2^{-16-\gamma} w_\gamma(\rho)$. The finiteness of A_γ, B, B' follows from the ratio test, since $t^8 e^{-t}$ decreases for $t \geq 8$.

(c) $\omega_k(A_k(f) A_k(g)) = \sum_{x,y} a_k^8 f(a_k x) g(a_k y) h_k(a_k \|y - x\|_1)$ with h_k the partial sum of h up to J_k. On a separated pair at distance δ only $u \geq \delta$ occurs, where $0 \leq h - h_k \leq \sum_{j > J_k} \mu_j^2 e^{-2^{j-1} \delta} \to 0$, and h is continuous on $(0, \infty)$; so the Riemann sums converge to $\iint f(x) g(y) K(x-y) \, dx \, dy$. The fields are centred Gaussian, so the moments of separated tuples are sums over the $(n-1)!!$ pairings of products of these two-point functions, and they converge as well; this gives ($\mathrm{U_s}$). Theorem 305(i) and $(n-1)!! \leq (n!)^{1/2}$ bound the moments of tuples in $\mathscr{C}_n(Q, \delta)$, $\delta \leq 1$, by $(100 C_1)^{n/2} (n!)^{1/2} (1+L)^{2n} \delta^{-2n} \prod_j \|f_j\|_\infty$ for large k, which is ($\mathrm{G_s}$). The proofs of Theorem 305(i)–(iv) use the states only through their invariance, reflection positivity and gap, as in Remark 319. K is continuous on $\mathbb{R}^4 \setminus \{0\}$ and h is strictly decreasing, so by Proposition 287(i) an $R \in \mathrm{O}(4)$ lies in G_S only if $\|Rv\|_1 = \|v\|_1$ for all v; then R permutes the vertices $\pm e_\mu$ of the unit ball of

$\|\cdot\|_1$, and $R \in B_4$. Finally, choose i_1 with $((1+i)/(2+i))^\gamma \geq \frac{1}{2}$ for $i \geq i_1$. Since $\mu_{i+1}^2 = 2^8((1+i)/(2+i))^\gamma \mu_i^2$,

$$h(\rho) \geq \sum_{i \geq i_1} \mu_{i+1}^2 e^{-2^i \rho} \geq 2^7 \Big(h(2\rho) - \sum_{i<i_1} \mu_i^2 \Big),$$

and $h(2\rho) \to \infty$ as $\rho \to 0$, so $\liminf_{\rho\to 0} h(\rho)/h(2\rho) \geq 2^7$. With $v = \rho e_0$ and $v' = \frac{\rho}{2}(1,1,1,1)$, $|v| = |v'| = \rho$, $\|v\|_1 = \rho$ and $\|v'\|_1 = 2\rho$, while (AF_∞), K being continuous, would give $K(v)/K(v') \to 1$. □

The example is not a lattice gauge theory. With (AF_∞) in place of (AF), the leading term at short distance is rotation invariant: if K is as in (AF_∞) and square integrable on compact subsets of $\mathbb{R}^4 \setminus \{0\}$, the components $K_{2j}(\rho,\cdot)$, $j \geq 1$, of Proposition 287(ii) satisfy $\rho^8 (\log \frac{1}{\rho})^\gamma \|K_{2j}(\rho,\cdot)\|_{L^2(S^3)} \to 0$ as $\rho \to 0$ along almost every ρ, because the projection onto $\mathscr{Y}_{2j}$, $j \geq 1$, has norm one and annihilates the constant C_{AF}. This is not ($\mathrm{W_s}$), which asks that these components vanish.

Remark 308. *Status of (AF).* No part of (AF), (AF_∞) or (AF_β) is proved for the Wilson measure, in any regime of couplings that tends to the continuum, and (AF) fails in the strong-coupling range (Theorem 305(v)). Bałaban's renormalisation group [6, 7, 10–12] constructs effective actions in a small-field approximation with a coupling constant renormalisation by recursive renormalisation-group equations [10], and proves the ultraviolet stability of four-dimensional pure lattice gauge theories, that is, bounds on the partition function in a box of fixed physical size uniform in the spacing [12]; these bounds concern the partition function, not the two-point function of a local field, and the box is fixed. Magnen, Rivasseau and Sénéor [86] give the basis for a construction of the Schwinger functions of the SU(2) gauge field with a fixed infrared cut-off and no ultraviolet cut-off, in a regularised axial gauge; these are correlations of the gauge field in a gauge, with an infrared cut-off, not of a gauge-invariant composite field such as Φ, and they are not states of the Wilson measure along a lattice tuning. An asymptotically free ultraviolet limit has been constructed by convergent expansions in the running coupling for the two-dimensional Gross–Neveu model [61], not for a gauge theory in four dimensions. What is missing is exactly the following inequality, for Φ the plaquette sum, the torus limits ω_k along the tuning (AF_β), $\gamma = 2$ and some $Z_k > 0$, $0 < c_1 \leq C_1$, $r_0 > 0$: for large k and all $z \in \mathbb{Z}^4$ with $|z|_\infty \geq 2r+1$, where $G_k(z) = \omega_k(\Phi_0 \Phi_z) - \omega_k(\Phi_0)^2$,

$$Z_k^2 |G_k(z)| \leq C_1 w_2(a_k|z|),$$

and $Z_k^2 G_k(z) \geq c_1 w_2(a_k|z|)$ if moreover $a_k|z| \leq r_0$; equivalently, by Proposition 306(iii), the Z_k-free two-sided comparison of $G_k(z)$ with $G_k(z_0) w_2(a_k|z|)/w_2(a_k|z_0|)$ for $2r+1 \leq |z|_\infty$, $a_k|z| \leq r_0$, together with the upper bound for $a_k|z| > r_0$ with $Z_k^2 = w_2(a_k|z_0|)/G_k(z_0)$. It is open, both halves, already in a box of fixed physical size. Given it, Theorem 305 supplies ($\mathrm{SD_p}$) and (NE) for $n = 2$; (RN_m) for $m \geq 2$, ($\mathrm{W_s}$), ($\mathrm{U_s}$) and the averaged condition along the tuning stay open.

Products and the two-point properties. After Proposition 285 the separated moments of order $n \geq 3$ rest on (RN_m), $m \geq 2$, and the rate ($\mathrm{G_s}$) on a rate in these bounds. Functions of two independent copies of the field of Proposition 207(a) show what the two-point properties do not give. Write γ_2 for the standard Gaussian measure on $\mathbb{R}^2$, $\|\cdot\|_p$ for the norm of $L^p(\gamma_2)$, and $T_M(u) = \max(-M, \min(M,u))$ for $M > 0$.

Lemma 309. *Let $m > 0$, $q_k = e^{-m a_k}$, let $\phi_k^{(1)}, \phi_k^{(2)}$ be independent copies of the field of Proposition 207(a), $\xi_x = (\phi_k^{(1)}(x), \phi_k^{(2)}(x))$, let $U : \mathbb{R}^2 \to \mathbb{R}$ be continuous, in $L^p(\gamma_2)$ for every $p < \infty$, with $\int U \, d\gamma_2 = 0$, and put $\Phi_x = U(\xi_x)$ and $A_k(f) = \sum_x a_k^4 f(a_k x) \Phi_x$, with $r = 0$.*

(i) *The law of Φ is invariant under translations and B_4 and reflection positive for the reflections in coordinate hyperplanes through sites and bisecting links, and in each coordinate direction the spectrum of its transfer matrix on $\Omega_k^\perp$ lies in $[0, q_k]$. It contains q_k if $\int U(u) u_i \, d\gamma_2(u) \neq 0$ for $i = 1$ or $i = 2$.*

(ii) *Let $U_M = T_M \circ U - \int T_M \circ U \, d\gamma_2$, so that $|U_M| \leq 2M$. For $n \geq 1$ and random vectors $\zeta_1, \dots, \zeta_n$ in $\mathbb{R}^2$, each of law γ_2,*

$$\Big| \mathbb{E} \prod_i U(\zeta_i) \Big| \leq \|U\|_n^n,$$

$$\Big| \mathbb{E} \prod_i U_M(\zeta_i) - \mathbb{E} \prod_i U(\zeta_i) \Big| \leq e_n(M),$$

with $e_n(M) = 2n(2\|U\|_n)^{n-1} \|U \mathbf{1}_{\{|U|>M\}}\|_n \to 0$ as $M \to \infty$. Hence, if f_j is supported in a cube of side L_j, the moments $\omega_k(\prod_j A_k(f_j))$ formed with U_M and with U differ by at most $e_n(M) \prod_j (L_j + a_k)^4 \|f_j\|_\infty$.

Proof. (i) As in the proof of Proposition 283(a), the pair $(\phi_k^{(1)}, \phi_k^{(2)})$ is invariant, reflection positive, and in each coordinate direction a reversible Markov chain with transfer operator $P \otimes P$, of spectrum $\{q_k^{N+N'}\}$ with 1 simple; the functions of Φ at positive times are functions of the pair there, so Φ has the stated invariance and positivity, and its transfer matrix is the restriction of $P \otimes P$ to an invariant subspace. For a site x at time $s \geq 1$, g_{Φ_x} is $(P \otimes P)^s$ applied to $U(\xi_x)$, viewed as a function of the field at one time, and $P \otimes P$ acts as q_k on the linear functions of that field. By Gaussian integration by parts the orthogonal projection of $U(\xi_x)$ on these linear functions has inner product $\int U(u) u_i \, d\gamma_2(u)$ with $\phi_k^{(i)}(x)$, so it is not zero, and g_{Φ_x} has a nonzero component of eigenvalue q_k.

(ii) The first bound is Hölder's inequality with n exponents n. Since $|T_M u| \leq |u|$, $|T_M u - u| \leq |u| \mathbf{1}_{\{|u|>M\}}$ and $\int U \, d\gamma_2 = 0$,

$$\|U_M\|_n \leq 2\|U\|_n, \qquad \|U_M - U\|_n \leq 2\|U \mathbf{1}_{\{|U|>M\}}\|_n.$$

The difference of the two products is the sum over j of the products $\prod_{i<j} U_M(\zeta_i) \, (U_M - U)(\zeta_j) \prod_{i>j} U(\zeta_i)$, and Hölder's inequality for each of them gives the second bound. By dominated convergence $e_n(M) \to 0$, as $|U|^n$ is integrable. The moments are $\sum_{x_1,\dots,x_n} a_k^{4n} \prod_j f_j(a_k x_j) \mathbb{E} \prod_j U(\xi_{x_j})$, each ξ_{x_j} has law γ_2, and $\sum_x a_k^4 |f_j(a_k x)| \leq (L_j + a_k)^4 \|f_j\|_\infty$. □

The moments of exponentials are explicit. For a centred Gaussian vector $(X_1, \dots, X_n)$ with unit variances and covariances $\rho_{ii'}$, $t > 0$, and $u_{ii'} = e^{t^2 \rho_{ii'}} - 1$,

$$\mathbb{E} \prod_{i=1}^n \big(e^{t X_i - t^2/2} - 1 \big) = \sum_G \prod_{\{i,i'\} \in G} u_{ii'}, \tag{30}$$

the sum over the sets G of pairs whose union is $\{1,\dots,n\}$. Indeed $\mathbb{E}\prod_{i\in S} e^{tX_i - t^2/2} = \prod_{\{i,i'\}\subset S}(1+u_{ii'})$ for $S\subset\{1,\dots,n\}$; expanding the product of the factors $e^{tX_i-t^2/2}-1$ and these products, the coefficient of $\prod_{\{i,i'\}\in G} u_{ii'}$ is the sum of $(-1)^{n-|S|}$ over the S that contain every pair of G, which is 1 if G covers $\{1,\dots,n\}$ and 0 otherwise. When all $\rho_{ii'}\ge 0$, every term is nonnegative and nondecreasing in each $\rho_{ii'}$.

Proposition 310. *In the setting of Lemma 309 let $t_k>0$ with $t_k\to\infty$, $\lambda_k = e^{-t_k^2/2}$ and*

$$U_k(u) = u_1 + \lambda_k\big(e^{t_k u_2 - t_k^2/2} - 1\big),$$

let $\varepsilon>0$, form $\mathscr{N}_k(\varepsilon)$ with $\Phi_x = U_k(\xi_x)$, and put $Z_k = \mathscr{N}_k(\varepsilon)^{-1/2}$ and $\ell_0 = (\frac{3}{8})^8 e^{-m(2\varepsilon+6)}$.

(a) *The spectrum of the transfer matrix on $\Omega_k^{\perp}$ lies in $[0,q_k]$ and contains q_k: the gap is m in physical units. For large k, $\ell_0 \le \mathscr{N}_k(\varepsilon) \le 2(1+2a_0)^8$. ($\mathrm{SD_p}$) holds with $p_0=0$; hence* (SD), (UV), (RN_1) *and* ($\mathrm{M_{s,2}}$) *hold.* (NE) *holds, and every limit S_2 of Corollary 281*(b) *is $S_2(f,g) = z\iint f(x)g(y)e^{-m\|x-y\|_1}\,dx\,dy$ with $z\in[\frac12(1+2a_0)^{-8},\ell_0^{-1}]$.*

(b) *Let $n\ge 3$, $\eta = 1/(4m(3n+5))$, $c_j = 3j\eta e_1$, and let $f_1,\dots,f_n$ be continuous with values in $[0,1]$, f_j equal to 1 on $c_j+[-\frac{\eta}{2},\frac{\eta}{2}]^4$ and supported in $c_j+[-\eta,\eta]^4$. Then $(f_j)\in\mathscr{C}_n(Q,\eta)$ for a cube Q of side $3n\eta$, and for large k*

$$Z_k^n\,\omega_k\Big(\prod_{j=1}^n A_k(f_j)\Big) \ge \frac{(\eta/2)^{4n} e^{nt_k^2/4}}{2^{n(n-1)/2}(2(1+2a_0)^8)^{n/2}} \to\infty.$$

So ($\mathrm{M_{s,}}n$) fails for every $n\ge 3$.

(c) *Let $j\ge 2$, $\eta' = 1/(4m(3j+10))$, and let $f_1,\dots,f_j$ be as in* (b)*, with η' in place of η and centres $2\eta' e_0 + 3i\eta' e_1$, $i=1,\dots,j$. They are supported in $\{x_0\ge\eta'\}$, $(f_i)\in\mathscr{C}_j(Q',\eta')$ for a cube Q' of side $3j\eta'$, and for large k, with $P_k=\prod_i A_k(f_i)$,*

$$Z_k^{2j}\,\omega_k\big(\vartheta(\bar P_k)P_k\big) \ge \frac{(\eta'/2)^{8j} e^{jt_k^2/2}}{2^{j(2j-1)}(2(1+2a_0)^8)^j} \to\infty.$$

So (RN_j) fails for every $j\ge 2$.

(d) *There are $M_k\to\infty$ such that* (a)–(c) *hold for the fields $\Phi_x = U_{k,M_k}(\xi_x)/(2M_k)$, with $U_{k,M}$ formed from U_k as in Lemma 309*(ii)*, which take values in $[-1,1]$, with $\frac12\ell_0$ and $4(1+2a_0)^8$ in place of ℓ_0 and $2(1+2a_0)^8$ and the right sides of* (b), (c) *halved. For these fields $Z_k\to\infty$.*

Proof. (a) The spectrum is given by Lemma 309(i), since $\int U_k u_1\,d\gamma_2 = 1$. Let $\rho(z) = \prod_\mu q_k^{|z_\mu|} = e^{-ma_k\|z\|_1}$ be the covariance of $\phi_k^{(i)}$ and $E_x = e^{t_k\phi_k^{(2)}(x) - t_k^2/2}$. By independence and centring, for h supported in $B_{\varepsilon'}$ with $\|h\|_\infty\le 1$ and $(r+1)a_k\le\varepsilon'$,

$$\omega_k\big(\vartheta(\overline{A_k(h)})A_k(h)\big) = \sum_{x,y} a_k^8 \overline{h(a_k x)}h(a_k y)\big(\rho + \lambda_k^2(e^{t_k^2\rho}-1)\big),$$

with $\rho = \rho(y-\vartheta x)$, since $\mathbb{E}[(E_x-1)(E_y-1)] = e^{t_k^2\rho(x-y)}-1$ by (30). For $a_k x_0, a_k y_0 \ge \varepsilon'$ the time component of $y-\vartheta x$ is at least $2\varepsilon'/a_k - 1$, so $\rho\le e^{-m(2\varepsilon'-a_k)}$, and $\lambda_k^2 e^{t_k^2\rho} \le e^{-t_k^2(1-e^{-m(2\varepsilon'-a_k)})}\to 0$. As $\sum_x a_k^4|h(a_k x)| \le (1+2a_0)^4$ and $\rho\le 1$, $\mathscr{N}_k(\varepsilon')\le 2(1+2a_0)^8$ for large k. For the function $h_*\ge 0$ of the proof of Proposition 283(b), formed for ε with $\tau=\frac14$, both parts are nonnegative, and that proof gives $\sum_{x,y} a_k^8 h_*(a_k x)h_*(a_k y)\rho(y-\vartheta x)\ge\ell_0$; so $\mathscr{N}_k(\varepsilon)\ge\ell_0$, and the ratio in (NE) with $g_0=h_*$ is at least $\ell_0/(2(1+2a_0)^8)$. Hence $\mathscr{N}_k(\varepsilon')\le C_0\mathscr{N}_k(\varepsilon)$ with $C_0 = 2(1+2a_0)^8/\ell_0$ for every $\varepsilon'\in(0,\varepsilon]$ and large k, which is ($\mathrm{SD_p}$) with $p_0=0$. (UV) follows by Proposition 282(ii), (RN_1) by Proposition 285(iii) and ($\mathrm{M_{s,}}2$) by Corollary 281(a), whose proofs use the state only through the properties of Lemma 309(i). For $(f,g)\in\mathscr{C}_2(Q,\delta)$, Q of side L, the sites carrying f and g have $\|a_k(x-y)\|_1\ge\delta$, so the part of $Z_k^2\omega_k(A_k(f)A_k(g))$ with λ_k^2 is at most $Z_k^2(L+1)^8 e^{-t_k^2(1-e^{-m\delta})}\|f\|_\infty\|g\|_\infty\to 0$, and the other part is Z_k^2 times a Riemann sum of the continuous function $f(x)g(y)e^{-m\|x-y\|_1}$; along a subsequence with $Z_k^2\to z$ the limit is the stated one.

(b) Distinct c_j differ by 3η along e_1, so the supports are at ℓ^∞ distance at least η; they lie in $[2\eta,(3n+1)\eta]\times[-\eta,\eta]^3$, of ℓ^1 diameter at most $(3n-1)\eta+6\eta = 1/(4m)$. Expanding $\prod_j U_k(\xi_{x_j})$ over the sets I of indices that take the first summand, independence gives

$$\mathbb{E}\prod_j U_k(\xi_{x_j}) = \sum_I \mathbb{E}\Big[\prod_{j\in I}\phi_k^{(1)}(x_j)\Big]\lambda_k^{n-|I|}\mathbb{E}\Big[\prod_{j\notin I}(E_{x_j}-1)\Big].$$

The first factor is a sum of products of the covariances $\rho>0$ by Wick's formula, and the second is nonnegative by (30); the term $I=\emptyset$ and the term G of all pairs in (30) give

$$\mathbb{E}\prod_j U_k(\xi_{x_j}) \ge \lambda_k^n \prod_{i<i'}\big(e^{t_k^2\rho(x_i-x_{i'})}-1\big).$$

For sites in the supports $\rho\ge e^{-1/4}\ge\frac34$; for $\frac34 t_k^2\ge\log 2$, each factor is at least $\frac12 e^{t_k^2\rho}$, and the right side is at least $2^{-n(n-1)/2}\exp(t_k^2 n(\frac38(n-1)-\frac12))\ge 2^{-n(n-1)/2}e^{nt_k^2/4}$, since $n\ge 3$. All terms of the moment are nonnegative; keeping the sites with $a_k x_j\in c_j+[-\frac{\eta}{2},\frac{\eta}{2}]^4$, at least $(\eta/(2a_k))^4$ for each j when $a_k\le\eta/2$, and using $Z_k^2\ge\frac12(1+2a_0)^{-8}$ from (a) gives the bound.

(c) The supports lie in $[\eta',3\eta']\times[2\eta',(3j+1)\eta']\times[-\eta',\eta']^2$, at ℓ^∞ distance at least η' from each other. By Lemma 273(i), $\vartheta(\bar P_k) = \prod_i A_k(f_i\circ\theta_{0,k})$, and the $2j$ functions $f_i\circ\theta_{0,k}$, f_i are nonnegative with supports in $\{x_0\le -\eta'+a_k\}$ and $\{x_0\ge\eta'\}$, whose union has ℓ^1 diameter at most $(6\eta'+a_k)+(3j-1)\eta'+4\eta' \le (3j+10)\eta' = 1/(4m)$ for $a_k\le\eta'$. The proof of (b) with $n=2j$ applies, the map $x\mapsto\vartheta x$ being a bijection of $\mathbb{Z}^4$.

(d) Choose $M_k\ge k$ with $e_n(M_k)\le 1/k$ for $n\le k$, e_n formed with U_k, and $\|U_{k,M_k}-U_k\|_2\le\frac12$. Then $\int U_{k,M_k}(u)u_1\,d\gamma_2\ge\frac12$, and Lemma 309 gives the spectrum and shows that, for each fixed n and fixed supports, the moments and the reflected norms formed with U_{k,M_k} differ from those of U_k by $O(1/k)$, uniformly in the functions of norm at most 1 with supports in the given cubes; in particular $\mathscr{N}_k(\varepsilon')$ changes by at most $(1+2a_0)^8/k$. The field $U_{k,M_k}(\xi_x)/(2M_k)$ has packets $A_k/(2M_k)$, reflected norms $\mathscr{N}_k/(4M_k^2)$ and Z_k multiplied by $2M_k$, so the moments of $Z_k A_k(f_j)$ are those formed with U_{k,M_k} and its own $\mathscr{N}_k(\varepsilon)$, and the claims follow from (a)–(c). □

The fields of Proposition 310 have every property that this subsection uses at separated points, the gap included, and (SD), (SD$_\mathrm{p}$), (UV), (NE) and nontrivial two-point limits. The exponential part has variance $\lambda_k^2(e^{t_k^2}-1)=1-e^{-t_k^2}$ at each site, correlations that vanish at every fixed distance, and moments of every order $n\ge 3$ that grow like $e^{nt_k^2/4}$ at points closer than $1/(4m)$; it does not enter the two-point functions. So (RN$_m$), $m\ge 2$, does not follow from the invariance, the reflection positivity in the coordinate hyperplanes, the transfer-matrix gap and the two-point properties; any argument that uses the state only through these, such as a chessboard or multiple-reflection estimate or a bound by powers of the transfer matrix, gives neither (RN$_2$) nor (M$_{\mathrm{s},3}$). The choice of Z_k does not matter: the unnormalised separated two-point moments are bounded in k, so every normalisation for which some separated two-point limit is nonzero has $\liminf_k Z_k>0$, and for it the moments in (b) and (c) still tend to infinity; this applies to $Z_k=C_k(\sigma_0)^{-1/2}$ of Proposition 276(ii) for $f_0\ge 0$, $f_0\ne 0$. Part (d) takes into account that Φ is bounded for the Wilson measure: for bounded fields (M$_\mathrm{s}$) is automatic when Z_k is bounded, and in (d) $Z_k\to\infty$.

The rate (G$_\mathrm{s}$) is a rate in (RN). Consider

(RN$_\mathrm{p}$) there are $K_1,\beta_1,p_1,q_1\ge 0$ such that for every $m\ge 1$, every cube Q of side L and every $\delta\in(0,1]$ there is k_0 with

$$Z_k^{2m}\omega_k\big(\vartheta(\bar P_k)P_k\big)\le K_1^m(m!)^{\beta_1}(1+L)^{q_1 m}\delta^{-p_1 m}\prod_{j=1}^m\|f_j\|_\infty^2$$

for $P_k=\prod_{j=1}^m A_k(f_j)$, all $k\ge k_0$ and all $(f_1,\dots,f_m)\in\mathscr{C}_m(Q,\delta)$ with supports in $\{x_0\ge\delta\}$.

Proposition 311. *Assume the hypothesis of Theorem 66.*

(i) (G$_\mathrm{s}$) *implies* (RN$_\mathrm{p}$) *with* $K_1=4^\beta 7^{2q}K^2$, $\beta_1=2\beta$, $q_1=2q$, $p_1=2p$.

(ii) (RN$_\mathrm{p}$) *implies* (G$_\mathrm{s}$) *with* $K=2^8e^4 5^{q_1/2}(16e^2)^{p_1/2}K_1^{1/2}$, $\beta=4+\frac12(\beta_1+p_1)$, $q=4+\frac12 q_1$, $p=4+\frac12 p_1$.

(iii) *With* $Z_k=\mathscr{N}_k(\varepsilon)^{-1/2}$ *as in Theorem 280, the case* $m=1$ *of the bound in* (RN$_\mathrm{p}$)*, with its own constants, holds if and only if* (SD$_\mathrm{p}$) *holds. So under* (SD$_\mathrm{p}$), (G$_\mathrm{s}$) *holds if and only if the cases* $m\ge 2$ *hold with constants of the form in* (RN$_\mathrm{p}$).

Proof. (i) Let $(f_j)\in\mathscr{C}_m(Q,\delta)$ have supports in $\{x_0\ge\delta\}$, $Q=\prod_\nu[q_\nu,q_\nu+L]$, let $s\ge\delta$ be the least time in the supports, $n_0=\lfloor(s-\delta)/a_k\rfloor$, $v_\nu=\lfloor q_\nu/a_k\rfloor$ for $\nu\ne 0$, $v_\perp=(0,v_1,v_2,v_3)$, and $f_j'=f_j(\cdot+a_k(n_0e_0+v_\perp))$, supported in $[\delta,L+2]\times[0,L+1]^3$ for $a_k\le 1$. With $P_k'=\prod_j A_k(f_j')$, $P_k=\tau_{n_0e_0}\tau_{v_\perp}P_k'$ by Lemma 273(i), and, as in the proof of Lemma 279, $\omega_k(\vartheta(\bar P_k)P_k)=\|\mathbb{T}_k^{n_0}\Psi_{\tau_{v_\perp}P_k'}\|^2\le\omega_k(\vartheta(\bar P_k')P_k')$. The last is the moment of the $2m$ functions $\bar f_j'\circ\theta_{0,k}$, f_j', which lie in $Q''=[-L-3,L+3]^4$ and are pairwise at ℓ^∞ distance at least δ for $a_k\le\delta$, the reflected ones being supported in $\{x_0\le-\delta+a_k\}$. (G$_\mathrm{s}$) with $n=2m$ and Q'', of side $2L+6$, gives the bound with $((2m)!)^\beta\le 4^{\beta m}(m!)^{2\beta}$ and $7+2L\le 7(1+L)$.

(ii) The moments of order one vanish. For $n\ge 2$, part (ii) of Proposition 285 gives (M$_{\mathrm{s},n}$) with $M_n(Q,\delta)=\mathsf{K}_{2nL/\delta}^n\max_i(R_iR_{n-i})^{1/2}$, R_i taken at Q^*, of side $2L+4$, and $\delta_n=\delta/(16(n+1))\le 1$. By (RN$_\mathrm{p}$) and $i!(n-i)!\le n!$,

$$(R_iR_{n-i})^{1/2}\le K_1^{n/2}(n!)^{\beta_1/2}\big(5(1+L)\big)^{q_1n/2}\times\big(16(n+1)\big)^{p_1n/2}\delta^{-p_1n/2}.$$

From $e^n\ge n^n/n!$, $(n+1)^n\le e\,n^n\le e^{n+1}n!\le e^{2n}n!$. For $\delta\le 1$, $4nL/\delta+3\le 4n(1+L)/\delta$, so $\mathsf{K}_{2nL/\delta}^n\le 4^{4n}n^{4n}(1+L)^{4n}\delta^{-4n}\le 4^{4n}e^{4n}(n!)^4(1+L)^{4n}\delta^{-4n}$. Multiplying gives the constants stated.

(iii) If (SD$_\mathrm{p}$) holds, Lemma 279 gives, for f supported in a cube of side L in $\{x_0\ge\delta\}$ and large k,

$$Z_k^2\omega_k\big(\vartheta(\overline{A_k(f)})A_k(f)\big)\le\mathsf{K}_L^2\|f\|_\infty^2\frac{\mathscr{N}_k(\delta)}{\mathscr{N}_k(\varepsilon)},$$

with $\mathsf{K}_L^2\le 3^8(1+L)^8$. The ratio is at most $C_0\max(1,\varepsilon)^{p_0}\delta^{-p_0}$ for $\delta\le\varepsilon$ and at most K^2 for $\delta>\varepsilon$, by Proposition 282(i). Conversely, if the case $m=1$ holds, let $\varepsilon'\in(0,\varepsilon]$ and h be supported in $B_{\varepsilon'}\subset[0,L_0]^4$, $L_0=\varepsilon+1+a_0$; the bound with $\delta=\min(\varepsilon',1)$ gives $\mathscr{N}_k(\varepsilon')\le C_0(\varepsilon/\varepsilon')^{p_1}\mathscr{N}_k(\varepsilon)$ with $C_0=K_1(1+L_0)^{q_1}\max(1,\varepsilon^{-p_1})$. The last claim follows from (i), (ii). □

Bounds on each moment do not give the factor $K^n(n!)^\beta$: in the next fields every moment is bounded uniformly in the separation, and (G$_\mathrm{s}$) fails.

Proposition 312. *In the setting of Lemma 309 let* $t>0$, $U(u)=e^{tu_2-t^2/2}-1$, $\varepsilon>0$, *form* $\mathscr{N}_k(\varepsilon)$ *with* $\Phi_x=U(\xi_x)$, *and put* $Z_k=\mathscr{N}_k(\varepsilon)^{-1/2}$, *with* ℓ_0 *as in Proposition 310.*

(a) *The gap is* m *in physical units. For large* k, $t^2\ell_0\le\mathscr{N}_k(\varepsilon)\le(1+2a_0)^8(e^{t^2}-1)$, (SD$_\mathrm{p}$) *holds with* $p_0=0$, (NE) *holds, and for every* n *and all* f_j *supported in a cube of side* L, *separated or not,*

$$\Big|\omega_k\Big(\prod_{j=1}^n Z_kA_k(f_j)\Big)\Big|\le\frac{2^n e^{t^2n(n-1)/2}(L+1)^{4n}}{(t^2\ell_0)^{n/2}}\prod_{j=1}^n\|f_j\|_\infty;$$

in particular (M$_\mathrm{s}$) *and* (RN) *hold.*

(b) *Let* n *be even with* $n\ge e\log 2/t^2+\frac12$, $s=\lceil n^{1/4}\rceil$, $\eta=1/(12(1+m)s)$, *let* $v_1,\dots,v_n$ *be distinct points of* $\{0,\dots,s-1\}^4$, *and let* f_j *be as in Proposition 310*(b) *with centres* $c_j=3\eta v_j$. *Then* $(f_j)\in\mathscr{C}_n(Q,\eta)$ *for a cube* Q *of side at most* 1, *and for large* k

$$Z_k^n\,\omega_k\Big(\prod_{j=1}^n A_k(f_j)\Big)\ge\frac{(\eta/2)^{4n}e^{t^2n(n-1)/(2e)}}{2^{n+1}\big((1+2a_0)^8(e^{t^2}-1)\big)^{n/2}}.$$

(G$_\mathrm{s}$) *fails for all* K,β,p,q, *and no limit of Theorem 274 satisfies the bound of* (G$_\mathrm{s}$).

(c) *For every sequence* $M_k\to\infty$, (a) *and* (b) *hold for the fields* $U_{M_k}(\xi_x)/(2M_k)$, *with values in* $[-1,1]$, *with* 4^n *in place of* 2^n, $\frac12 t^2\ell_0$ *in place of* $t^2\ell_0$, *and the right side of* (b) *halved, for* k *large depending on* n; *for these fields* $Z_k\to\infty$.

Proof. (a) The gap is Lemma 309(i), with $\int U(u)u_2\,d\gamma_2=t$. As in the proof of Proposition 310(a), the reflected norm of

$A_k(h)$ is $\sum_{x,y} a_k^8 \overline{h(a_k x)} h(a_k y)(e^{t^2\rho}-1)$, with $\rho = \rho(y-\vartheta x) \in (0,1]$; the upper bound follows, and the lower one from $e^{t^2\rho} - 1 \ge t^2\rho$ and the function h_*, which also bounds the ratio in (NE) with $g_0 = h_*$ below by $t^2\ell_0/((1+2a_0)^8(e^{t^2}-1))$, and ($\mathrm{SD_p}$) with $p_0 = 0$. By Lemma 309(ii) the moments are at most $\|U\|_n^n \prod_j (L+1)^4 \|f_j\|_\infty$, and $\|U\|_n \le \|e^{tu_2 - t^2/2}\|_n + 1 = e^{t^2(n-1)/2} + 1 \le 2e^{t^2(n-1)/2}$. This gives ($\mathrm{M_s}$), and (RN) by Proposition 285(i).

(b) Distinct v_j differ by at least 1 in some coordinate, so the supports are at ℓ^∞ distance at least η; they lie in a cube of side $3\eta s \le 1$, of ℓ^1 diameter at most $12\eta s \le 1/m$, so $\rho \ge e^{-1}$ for all sites in them. By (30) and its monotonicity, $\mathbb{E}\prod_j U(\xi_{x_j})$ is at least its value for the Gaussian vector $X_i = e^{-1/2}Y + (1-e^{-1})^{1/2} W_i$ with $Y, W_1, \dots, W_n$ independent standard normal, whose covariances are all e^{-1}. Conditionally on Y the factors are independent, and with $\sigma = te^{-1/2}$ this value is $\mathbb{E}[(e^{\sigma Y - \sigma^2/2} - 1)^n]$. For n even the integrand is nonnegative, and it is at least $2^{-n} e^{n\sigma Y - n\sigma^2/2}$ where $y_0 = (\log 2 + \sigma^2/2)/\sigma \le Y$. Since $\mathbb{E}[e^{n\sigma Y} \mathbf{1}_{\{Y \ge y_0\}}] = e^{n^2\sigma^2/2} \mathbb{P}(Y \ge y_0 - n\sigma) \ge \frac{1}{2} e^{n^2\sigma^2/2}$ for $n\sigma \ge y_0$, that is $n \ge \log 2/\sigma^2 + \frac{1}{2}$, the value is at least $2^{-n-1} e^{\sigma^2 n(n-1)/2}$. Keeping the sites with $a_k x_j \in c_j + [-\frac{\eta}{2}, \frac{\eta}{2}]^4$, as in the proof of Proposition 310(b), gives the bound. If ($\mathrm{G_s}$) held, these moments would be at most $K^n (n!)^\beta 2^{qn} \eta^{-pn}$, and $\eta^{-1} \le 24(1+m) n^{1/4}$; the logarithm of this is $O(n \log n)$, while the logarithm of the lower bound is $t^2 n(n-1)/(2e) - O(n \log n)$, a contradiction for large even n. The limits of Theorem 274(i) satisfy the same lower bound.

(c) By Lemma 309(ii), $|\mathbb{E}\prod_j U_{M_k}(\zeta_j)| \le (2\|U\|_n)^n$, the moments and reflected norms change by at most $e_n(M_k) \prod_j (L_j+1)^4 \|f_j\|_\infty \to 0$ for each fixed n, and $\int U_{M_k}(u) u_2 \, d\gamma_2 \to t$; the rescaling by $2M_k$ is that of the proof of Proposition 310(d). □

One inequality between the moments and the separated two-point functions gives ($\mathrm{M_s}$) and ($\mathrm{G_s}$) from (SD) and ($\mathrm{SD_p}$). For a cube Q and $\delta > 0$ put

$$\mu_k(Q,\delta) = \sup Z_k^2 \big| \omega_k\big(A_k(h) A_k(h')\big)\big|,$$

the supremum over the $(h,h') \in \mathscr{C}_2(Q,\delta)$ with $\|h\|_\infty, \|h'\|_\infty \le 1$.

(GD) There are $K_0, \beta_0 \ge 0$ such that for every $n \ge 3$, every cube Q and every $\delta > 0$ there is k_0 with

$$\Big|\omega_k\Big(\prod_{j=1}^n Z_k A_k(f_j)\Big)\Big| \le K_0^n (n!)^{\beta_0} \mu_k(Q,\delta)^{n/2} \prod_{j=1}^n \|f_j\|_\infty$$

for all $k \ge k_0$ and $(f_1, \dots, f_n) \in \mathscr{C}_n(Q,\delta)$.

Proposition 313. *Assume the hypothesis of Theorem 66.*

(i) *Under* (GD), ($\mathrm{M_{s,2}}$) *implies* ($\mathrm{M_s}$). *With* $Z_k = \mathscr{N}_k(\varepsilon)^{-1/2}$ *as in Theorem 280,* (GD) *and* (SD) *imply* ($\mathrm{M_s}$)*, and* (GD) *and* ($\mathrm{SD_p}$) *imply* ($\mathrm{G_s}$) *with* $K = \max(1, K_0) C_2^{1/2}$, $\beta = \beta_0$, $q = 4$, $p = 4 + \frac{1}{2} p_0$, *where* $C_2 = (\frac{77}{2})^8 \max(1, C_0) \max(1, 8\varepsilon)^{p_0}$.

(ii) *If for every k the packets* $A_k(f)$ *form a centred Gaussian family,* (GD) *holds with* $K_0 = 1$, $\beta_0 = \frac{1}{2}$.

(iii) (GD) *fails for* $n = 3$ *in the fields of Proposition 310.*

Proof. (i) ($\mathrm{M_{s,2}}$) gives $\mu_k(Q,\delta) \le M_2(Q,\delta)$ for large k, and Corollary 281(a) gives ($\mathrm{M_{s,2}}$) from (SD). Under ($\mathrm{SD_p}$) the proof of Proposition 301(a) gives $\mu_k(Q,\delta) \le C_2(1+L)^8 \delta^{-8-p_0}$ for Q of side L, $\delta \le 1$ and large k, which is the case $n = 2$; the moments of order one vanish.

(ii) By Wick's formula the moment is 0 for odd n and, for even n, the sum over the $(n-1)!!$ perfect matchings of the products of the $\omega_k(A_k(f_a)A_k(f_b))$; each pair lies in $\mathscr{C}_2(Q,\delta)$, and $((n-1)!!)^2 \le (n-1)!!\, n!! = n!$.

(iii) There ($\mathrm{M_{s,2}}$) holds, so $\mu_k(Q,\eta)$ is bounded, while the left side of (GD) for the functions of Proposition 310(b) with $n = 3$ tends to infinity. □

So the case $m = 1$ of (RN) is (SD), and its rate is ($\mathrm{SD_p}$) (Propositions 285(iii) and 311(iii)); the cases $m \ge 2$ are a further input on the Wilson measure, which the two-point properties, reflection positivity and the gap do not supply (Proposition 310); ($\mathrm{G_s}$) is (RN) with constants $K_1^m (m!)^{\beta_1} (1+L)^{q_1 m} \delta^{-p_1 m}$ (Proposition 311), and bounds on each moment do not give the dependence on n (Proposition 312). The single inequality (GD), of Gaussian domination type, gives ($\mathrm{M_s}$) from (SD) and ($\mathrm{G_s}$) from ($\mathrm{SD_p}$); it holds for Gaussian packets, fails in the fields of Proposition 310, and is not proved here for the Wilson measure.

The effective energy. Along each lattice of distances from the reflection plane the largest reflected norm is logarithmically convex, and (SD) becomes a bound, in physical units, on its rate of decay. For $(r+2)a_k \le s$ with $\mathscr{N}_k(s) > 0$ put

$$\mathfrak{E}_k(s) = \frac{1}{2a_k} \log \frac{\mathscr{N}_k(s)}{\mathscr{N}_k(s+a_k)}.$$

[34]

Lemma 314. *Assume the hypothesis of Theorem 66, let* $(r+2)a_k \le s$ *and* $\mathscr{N}(n) = \mathscr{N}_k(s + n a_k)$ *for integers* $n \ge 0$.

(i) $\mathscr{N}(n)$ *is the supremum of* $\|\mathbb{T}_k^n \Psi_{A_k(h)}\|^2$ *over h supported in* B_s *with* $\|h\|_\infty \le 1$; $\mathscr{N}(n+1) \le \gamma_k^2 \mathscr{N}(n)$, *and* $\mathscr{N}(n)^2 \le \mathscr{N}(n-1)\mathscr{N}(n+1)$ *for* $n \ge 1$.

(ii) *If* $\mathscr{N}(0) > 0$, *then* $\mathscr{N}(n) > 0$ *for every n, and* $n \mapsto \mathfrak{E}_k(s + n a_k)$ *is nonincreasing, with values in* $[a_k^{-1} \log(1/\gamma_k), \infty) \subset [m_*, \infty)$.

Proof. (i) Since $B_{s+na_k} = B_s + n a_k e_0$, h is supported in B_{s+na_k} if and only if $h' = h(\cdot + n a_k e_0)$ is supported in B_s, and then $A_k(h) = \tau_n A_k(h')$ by Lemma 273(i), so $\Psi_{A_k(h)} = \mathbb{T}_k^n \Psi_{A_k(h')}$ as in the proof of Lemma 273(iii). The packets are centred, so $\Psi_{A_k(h')} \perp \Omega_k$ and $\|\mathbb{T}_k^{n+1} \Psi_{A_k(h')}\| \le \gamma_k \|\mathbb{T}_k^n \Psi_{A_k(h')}\|$. For a vector Ψ and $f(n) = \|\mathbb{T}_k^n \Psi\|^2 = \langle \mathbb{T}_k^{n-1}\Psi, \mathbb{T}_k^{n+1}\Psi\rangle$, the Cauchy–Schwarz inequality gives $f(n)^2 \le f(n-1) f(n+1) \le \mathscr{N}(n-1)\mathscr{N}(n+1)$ when $\Psi = \Psi_{A_k(h')}$; the supremum over h' gives the last claim.

(ii) For h' supported in B_s, $h'' = h'(\cdot + a_k e_0)$ is supported in $\{x_0 \ge s - a_k\}$ and $(r+1)a_k \le s - a_k$, so $\Psi_{A_k(h')} = \mathbb{T}_k \Psi_{A_k(h'')}$

[34] For one correlator this is the effective mass of lattice spectroscopy; here it is taken of the largest reflected norm, a supremum over packets.

lies in the range of $\mathbb{T}_k$, which is orthogonal to $\ker\mathbb{T}_k = \ker\mathbb{T}_k^n$. So $\mathbb{T}_k^n\Psi_{A_k(h')} \neq 0$ when $\Psi_{A_k(h')} \neq 0$, and $\mathscr{N}(n) > 0$ by (i). The convexity in (i) says that $\log\mathscr{N}(n-1) - \log\mathscr{N}(n) \geq \log\mathscr{N}(n) - \log\mathscr{N}(n+1)$, and the bound $\mathscr{N}(n+1) \leq \gamma_k^2\mathscr{N}(n)$ gives the lower bound, with $\gamma_k \leq e^{-m_* a_k}$. □

For a single packet, $(2a_k)^{-1}\log(\|\mathbb{T}_k^n\Psi\|^2/\|\mathbb{T}_k^{n+1}\Psi\|^2)$ decreases, as $n\to\infty$, to the least point of the support of its spectral measure, written in physical energy; so $\mathfrak{E}_k(s)$ is the energy, in physical units, at which the largest reflected norm decays at the distance s from the plane.

Proposition 315. *Assume the hypothesis of Theorem 280, and let $C_{\varepsilon'}$ be as in Proposition 282*(iii).

(i) (SD) *holds if and only if* $\limsup_k \mathfrak{E}_k(s) < \infty$ *for every* $s \in (0,\varepsilon)$. *Under* (SD), $\limsup_k \mathfrak{E}_k(s) \leq \log(\mathsf{K}^4 C_{s/2})/s$; *conversely, for* $\varepsilon' \in (0,\varepsilon)$,

$$\limsup_k \frac{\mathscr{N}_k(\varepsilon')}{\mathscr{N}_k(\varepsilon)} \leq \mathsf{K}^2 \exp\Big(2(\varepsilon-\varepsilon')\limsup_k \mathfrak{E}_k(\varepsilon')\Big).$$

(ii) *Under* ($\mathrm{SD_p}$), *for every* $s \in (0,\varepsilon]$ *and all large* k,

$$\mathfrak{E}_k(s) \leq \frac{\log(\mathsf{K}^4 C_0) + p_0\log(2\varepsilon/s)}{s-2a_k}.$$

Conversely, if there are $p, E_0 \geq 0$ *and, for every* $\varepsilon' \in (0,\varepsilon]$, *a* k_0 *with* $\mathfrak{E}_k(s) \leq p/(2s) + E_0$ *for all* $s \in [\varepsilon',\varepsilon]$ *and* $k \geq k_0$, *then* ($\mathrm{SD_p}$) *holds with* $p_0 = p$.

Proof. For $s \in (0,\varepsilon]$ and large k, $\mathscr{N}_k(s) \geq \mathsf{K}^{-2}\mathscr{N}_k(\varepsilon) > 0$ by Proposition 282(i), so $\mathfrak{E}_k(s)$ is defined. Let $n = \lfloor s/(2a_k)\rfloor$. By Lemma 314(ii), applied at $s - na_k$, $\mathfrak{E}_k(s) \leq \mathfrak{E}_k(s-ia_k)$ for $1 \leq i \leq n$, so

$$\mathfrak{E}_k(s) \leq \frac{1}{n}\sum_{i=1}^n \mathfrak{E}_k(s-ia_k) = \frac{1}{2a_k n}\log\frac{\mathscr{N}_k(s-na_k)}{\mathscr{N}_k(s)}.$$

Since $s - na_k \geq s/2$ and $s \leq \varepsilon$, Proposition 282(i) gives $\mathscr{N}_k(s-na_k) \leq \mathsf{K}^2\mathscr{N}_k(s/2)$ and $\mathscr{N}_k(s) \geq \mathsf{K}^{-2}\mathscr{N}_k(\varepsilon)$, and $2a_k n \geq s - 2a_k$. Hence

$$\mathfrak{E}_k(s) \leq \frac{1}{s-2a_k}\log\Big(\mathsf{K}^4\frac{\mathscr{N}_k(s/2)}{\mathscr{N}_k(\varepsilon)}\Big),$$

which gives the bound in (i) under (SD), and the bound in (ii) under ($\mathrm{SD_p}$) at $\varepsilon' = s/2$.

Conversely, let $\varepsilon' \in (0,\varepsilon)$, $E' > \limsup_k \mathfrak{E}_k(\varepsilon')$ and $n = \lceil(\varepsilon-\varepsilon')/a_k\rceil$. For large k, Lemma 314(ii) gives

$$\log\frac{\mathscr{N}_k(\varepsilon')}{\mathscr{N}_k(\varepsilon'+na_k)} = 2a_k\sum_{j=0}^{n-1}\mathfrak{E}_k(\varepsilon'+ja_k) \leq 2a_k n E',$$

and $2a_k n \leq 2(\varepsilon-\varepsilon'+a_k)$. Since $\varepsilon'+na_k \geq \varepsilon$, Proposition 282(i) gives $\mathscr{N}_k(\varepsilon'+na_k) \leq \mathsf{K}^2\mathscr{N}_k(\varepsilon)$. Letting $k\to\infty$ and $E' \downarrow \limsup_k \mathfrak{E}_k(\varepsilon')$ gives the inequality of (i), hence (SD). Under the hypothesis of the converse in (ii) the same sum is at most $\sum_j (pa_k/(\varepsilon'+ja_k) + 2a_k E_0) \leq pa_k/\varepsilon' + p\log((\varepsilon+a_k)/\varepsilon') + 2(\varepsilon+a_k)E_0$, so for $a_k \leq \min(\varepsilon',\varepsilon,1)$

$$\frac{\mathscr{N}_k(\varepsilon')}{\mathscr{N}_k(\varepsilon)} \leq \mathsf{K}^2 e^{p+2(\varepsilon+1)E_0} 2^p\Big(\frac{\varepsilon}{\varepsilon'}\Big)^p,$$

which is ($\mathrm{SD_p}$) for $\varepsilon' < \varepsilon$; for $\varepsilon' = \varepsilon$ it holds with $C_0 \geq 1$. □

So (SD) says that at every distance from the reflection plane the largest reflected norm decays at a rate bounded in physical units, and ($\mathrm{SD_p}$) that this rate grows at most like $p_0/(2s)$ as $s\to 0$, up to the logarithm in (ii). Under a gap m_{lat} in lattice units the rate is at least m_{lat}/a_k at every distance, which is the failure of (SD) in Proposition 282(iii). By Proposition 282(ii) the bound on the rate is equivalent to the spectral condition (UV).

The abelian theory. For $G = \mathrm{U}(1)$ the positivity behind Proposition 284(ii) is a theorem.

Corollary 316. *Let $G = \mathrm{U}(1)$ with the Wilson weight and $\beta_k \geq 0$, assume the hypothesis of Theorem 280, and let $\Phi = b_0 - \sum_j b_j\cos\theta_j$ with $b_j \geq 0$ for $j \geq 1$ and each θ_j an integer combination of the angles of the links with both endpoints in $[-r,r]^4$; the sum of $1-\cos\theta_p$ over the 24 plaquettes with a corner at 0 is an example. Then $K_k(z) \geq 0$ for all $z \in \mathbb{Z}^4$. If* (SD) *holds, then* (NE) *holds, every limit S_2 of Corollary 281*(b) *is not identically zero, and $\mathscr{N}_k(\varepsilon) \leq \omega_k(\vartheta(A_k(g_0))A_k(g_0))$ for large k and every g_0 as in Proposition 284*(ii).

Proof. Let μ_L be the Wilson measure on Λ_L. Its density with respect to the product of the normalised Lebesgue measures on the link angles is $Z^{-1}\exp(\beta_k\sum_p\cos\theta_p)$, where each θ_p is a signed sum of four link angles. By Ginibre's inequality [59], $\int\cos\theta\cos\theta' d\mu_L \geq \int\cos\theta\, d\mu_L\int\cos\theta' d\mu_L$ for any two integer combinations θ,θ' of link angles. So $\mathrm{Cov}_{\mu_L}(\Phi_0,\Phi_z) = \sum_{j,l\geq 1} b_j b_l \mathrm{Cov}_{\mu_L}(\cos\theta_j,\cos\tau_z\theta_l) \geq 0$ once L is large, and the torus limit preserves this inequality for the local functions Φ_0, Φ_z. This is the hypothesis of Proposition 284(ii), whose proof gives the last claim, and Proposition 284(i) gives the others. □

For $\mathrm{U}(1)$ the averaged condition fails in the Coulomb phase of the Villain theory (Corollary 68), and (SD) fails under a gap in lattice units (Proposition 282(iii)); whether the hypotheses of the corollary hold together for $\mathrm{U}(1)$ along any tuning is not decided here. What the corollary excludes is the mechanism of the example below, whose two-point function changes sign.

Spatial momentum. The spatial translations act on the reconstructed space. For $v \in \mathbb{Z}^4$ with $v_0 = 0$, $\omega_k(\vartheta(\tau_v\bar F)\tau_v F') = \omega_k(\tau_v(\vartheta(\bar F)F')) = \omega_k(\vartheta(\bar F)F')$ for F, F' at positive times, so $\Psi_F \mapsto \Psi_{\tau_v F}$ extends to a unitary $U_k(v)$; U_k is a representation of $\mathbb{Z}^3$ that commutes with $\mathbb{T}_k$, since $\tau_v\tau_1 = \tau_1\tau_v$. By the spectral theorem for the commuting unitaries $U_k(e_i)$ there is a projection-valued measure P_k on $(-\pi,\pi]^3$ with $U_k(v) = \int e^{ip\cdot v}d\mathsf{P}_k(p)$. For $K > 0$ let $\mathbb{P}_k(K) = \mathsf{P}_k(\{|p|_\infty \leq Ka_k\})$, the states of spatial momentum at most K in physical units; it commutes with $\mathbb{T}_k$ and $\mathbb{E}_k(\Lambda)$. For $(r+1)a_k \leq s$ let $\mathscr{N}_k^{\mathrm{sp}}(s;K)$ be the supremum of $\|\mathbb{P}_k(K)\Psi_{A_k(h)}\|^2$ over h supported in B_s with $\|h\|_\infty \leq 1$, and consider

(LM) for some $K < \infty$, $c > 0$ and $\varepsilon_1 \in (0,\varepsilon]$, $\mathscr{N}_k^{\mathrm{sp}}(\varepsilon_1;K) \geq c\,\mathscr{N}_k(\varepsilon)$ for all large k.

It says that, up to a fixed fraction, the largest reflected norm is carried by spatial momenta bounded in physical units, as (UV) says for the energy. The proofs below use the Fejér weights:

for an integer $N \ge 1$,

$$F_N(q) = \sum_{|n|<N} \frac{1}{N}\Big(1-\frac{|n|}{N}\Big)e^{iqn} = \Big(\frac{\sin(Nq/2)}{N\sin(q/2)}\Big)^2$$

for $0 < |q| \le \pi$, $F_N(0) = 1$. Then $0 \le F_N \le 1$, $F_N(q) \ge (2/\pi)^2$ for $N|q| \le \pi$, because $\sin x \ge 2x/\pi$ on $[0, \frac{\pi}{2}]$ and $N\sin(|q|/2) \le N|q|/2$, and $F_N(q) \le \pi^2/(Nq)^2$ for $0 < |q| \le \pi$, because $\sin(|q|/2) \ge |q|/\pi$. For $v \in \mathbb{Z}^4$ with $v_0 = 0$ put $f_v^N = \prod_{i=1}^3 N^{-1}(1-|v_i|/N)_+$, so that $\sum_v f_v^N = 1$ and $\sum_v f_v^N e^{ip\cdot v} = F_N^{(3)}(p) = \prod_i F_N(p_i)$.

Proposition 317. *Assume the hypothesis of Theorem 280 and* (SD). *Then* (NE) *holds if and only if* (LM) *holds.*

Proof. Two facts are used. First, for g supported in a cube of side L in $\{x_0 \ge s\}$, $(r+1)a_k \le s$, the proof of Lemma 279 writes $\Psi_{A_k(g)}$ as a sum of at most K_L vectors $\mathbb{T}_k^n U_k(v)\Psi_{A_k(h')}$ with h' supported in B_s and $\|h'\|_\infty \le \|g\|_\infty$; since $\mathbb{P}_k(K)$ commutes with $\mathbb{T}_k$ and U_k,

$$\|\mathbb{P}_k(K)\Psi_{A_k(g)}\| \le \mathsf{K}_L \mathscr{N}_k^{\rm sp}(s;K)^{1/2}\|g\|_\infty.$$

Second, for g supported in $\{x_0 \ge s\}$ and $g^N = \sum_v f_v^N g(\cdot - a_k v)$, Lemma 273(i) gives $A_k(g^N) = \sum_v f_v^N \tau_v A_k(g)$, so $\Psi_{A_k(g^N)} = F_N^{(3)}(\mathsf{P}_k)\Psi_{A_k(g)}$, where $F_N^{(3)}(\mathsf{P}_k) = \int F_N^{(3)}\, d\mathsf{P}_k$. By (SD) and Proposition 282(i), for every $s > 0$ there is $C(s)$ with $\mathscr{N}_k(s) \le C(s)\mathscr{N}_k(\varepsilon)$ for large k.

Let (NE) hold with g_0 supported in $\{x_0 \ge t_0\}$, $t_0 > 0$, and in a cube of side L, and $c_0 > 0$ with $4c_0 < \liminf_k R_k(g_0)$. Let $\delta \in (0,1]$, $a_k \le \delta$ and $N = \lfloor \delta/a_k \rfloor \ge \delta/(2a_k)$. The function $g_0^N - g_0$ is supported in a cube of side $L+2$ in $\{x_0 \ge t_0\}$ and bounded by $\rho(\delta) = \sup\{|g_0(y) - g_0(y')| : |y-y'|_\infty \le \delta\}$, because $f_v^N \ne 0$ only for $|a_k v|_\infty < \delta$. For $|p|_\infty > Ka_k$ some $|p_i| > Ka_k$, so $F_N^{(3)}(p) \le (\pi/(NKa_k))^2 \le (2\pi/(\delta K))^2$. With Lemma 279,

$$\begin{aligned}\|(1-\mathbb{P}_k(K))\Psi_{A_k(g_0)}\| &\le \|\Psi_{A_k(g_0-g_0^N)}\| + \|(1-\mathbb{P}_k(K))F_N^{(3)}(\mathsf{P}_k)\Psi_{A_k(g_0)}\| \\ &\le \Big(\mathsf{K}_{L+2}\rho(\delta) + \frac{4\pi^2\mathsf{K}_L\|g_0\|_\infty}{\delta^2K^2}\Big)\mathscr{N}_k(t_0)^{1/2}.\end{aligned}$$

Fix δ, and then K, so that the bracket is at most $(c_0/C(t_0))^{1/2}$. For large k this gives

$$\|\mathbb{P}_k(K)\Psi_{A_k(g_0)}\| \ge (2c_0^{1/2} - c_0^{1/2})\mathscr{N}_k(\varepsilon)^{1/2},$$

and the first fact gives $\mathscr{N}_k^{\rm sp}(t_0;K) \ge c_0\mathsf{K}_L^{-2}\|g_0\|_\infty^{-2}\mathscr{N}_k(\varepsilon)$. If $t_0 \le \varepsilon$ this is (LM) with $\varepsilon_1 = t_0$. If $t_0 > \varepsilon$, every h supported in B_{t_0} is $h'(\cdot - na_k e_0)$ with $n \ge 0$ and h' supported in a cube of side $1+a_0$ in $\{x_0 \ge \varepsilon\}$, so $\|\mathbb{P}_k(K)\Psi_{A_k(h)}\| \le \|\mathbb{P}_k(K)\Psi_{A_k(h')}\|$ and $\mathscr{N}_k^{\rm sp}(t_0;K) \le \mathsf{K}^2\mathscr{N}_k^{\rm sp}(\varepsilon;K)$, which is (LM) with $\varepsilon_1 = \varepsilon$.

Conversely, let (LM) hold and, for large k, h_k be supported in B_{ε_1} with $\|h_k\|_\infty \le 1$ and $\|\mathbb{P}_k(K)\Psi_k\|^2 \ge \frac{c}{2}\mathscr{N}_k(\varepsilon)$, $\Psi_k = \Psi_{A_k(h_k)}$.

Energy. Let $n = \lfloor \varepsilon_1/(2a_k) \rfloor$ and $h_k' = h_k(\cdot + na_k e_0)$, supported in a cube of side $1+a_0$ in $\{x_0 \ge \varepsilon_1/2\}$. Then $\Psi_k = \mathbb{T}_k^n \Psi_{A_k(h_k')}$ and $\|\mathbb{T}_k^n(1-\mathbb{E}_k(\Lambda))\| \le e^{-\Lambda a_k n}$, so for $a_k \le \varepsilon_1/4$, by Lemma 279,

$$\|(1-\mathbb{E}_k(\Lambda))\Psi_k\| \le e^{-\Lambda\varepsilon_1/4}\mathsf{K}C(\tfrac{\varepsilon_1}{2})^{1/2}\mathscr{N}_k(\varepsilon)^{1/2}.$$

Fix Λ with

$$e^{-\Lambda\varepsilon_1/4}\mathsf{K}C(\tfrac{\varepsilon_1}{2})^{1/2} \le (c/8)^{1/2}.$$

Since $\mathbb{E}_k(\Lambda)$ and $\mathbb{P}_k(K)$ commute, $\|\mathbb{E}_k(\Lambda)\mathbb{P}_k(K)\Psi_k\| \ge (c/8)^{1/2}\mathscr{N}_k(\varepsilon)^{1/2}$.

Smoothing. Let $\sigma > 0$, $M = \lfloor \sigma/a_k \rfloor$, $N = \lfloor \pi/(Ka_k) \rfloor$, both at least 1, $w_n = M^{-1}(1-|n-M|/M)_+$ for $n \in \mathbb{Z}$, which vanish unless $1 \le n \le 2M-1$ and add up to 1, and

$$g_k = \sum_{n,v} w_n f_v^N h_k(\cdot - a_k(ne_0 + v)).$$

By Lemma 273(i), $\Psi_{A_k(g_k)} = W(\mathbb{T}_k)F_N^{(3)}(\mathsf{P}_k)\Psi_k$ with $W(\lambda) = \sum_n w_n\lambda^n \ge \lambda^{2M}$ on $[0,1]$. On the range of $\mathbb{E}_k(\Lambda)\mathbb{P}_k(K)$ the joint spectrum of $\mathbb{T}_k$ and U_k lies where $\lambda \ge e^{-\Lambda a_k}$, so $W \ge e^{-2\Lambda\sigma}$, and $|p|_\infty \le Ka_k$, so $N|p_i| \le \pi$ and $F_N^{(3)} \ge (2/\pi)^6$. Hence

$$\omega_k\big(\vartheta(\overline{A_k(g_k)})A_k(g_k)\big) \ge 4c'\mathscr{N}_k(\varepsilon),$$

with $4c' = \frac{c}{8}e^{-4\Lambda\sigma}(2/\pi)^{12}$.

Compactness. $A_k(g)$ depends only on the values of g on $a_k\mathbb{Z}^4$. Let $u(t) = M^{-1}(1-|t/a_k - M|/M)_+$, $\varsigma(t) = N^{-1}(1-|t/a_k|/N)_+$, $\kappa(y) = u(y_0)\prod_{i=1}^3 \varsigma(y_i)$ and

$$G_k(y) = \sum_{z\in\mathbb{Z}^4} \kappa(y - a_k z)h_k(a_k z).$$

Then $G_k(a_k x) = g_k(a_k x)$, so $A_k(G_k) = A_k(g_k)$. The function G_k is supported in $S = B_{\varepsilon_1} + [0,2\sigma]\times[-\pi/K, \pi/K]^3$, a compact subset of $\{x_0 \ge \varepsilon_1\}$ independent of k. For $t, t' \in \mathbb{R}$, at most $2M$ terms of $\sum_{n\in\mathbb{Z}} u(t-a_k n)$ are nonzero, so this sum is at most 2, and $\sum_n |u(t-a_k n) - u(t'-a_k n)| \le 4|t-t'|/(Ma_k) \le 8|t-t'|/\sigma$; likewise $\sum_n \varsigma(t-a_k n) \le 2$ and $\sum_n |\varsigma(t-a_k n) - \varsigma(t'-a_k n)| \le 8K|t-t'|/\pi$, for $a_k \le \min(\sigma, \pi/K)$. Writing the difference of two products of four factors as a telescoping sum, $|G_k| \le 16$ and G_k is Lipschitz with constant $8(8/\sigma + 24K/\pi)$ for the ℓ^∞ norm. By the Arzelà–Ascoli theorem the G_k lie in a compact subset of $C(S)$.

One packet. Let L_S be the side of a cube containing S and $\eta > 0$ with $\mathsf{K}_{L_S}C(\varepsilon_1)^{1/2}\eta \le c'^{1/2}$. A finite set $G^1, \dots, G^J$ of the G_k is an η-net, so for each large k there is j with $\|G_k - G^j\|_\infty \le \eta$, and Lemma 279 gives $\|\Psi_{A_k(G_k)} - \Psi_{A_k(G^j)}\| \le c'^{1/2}\mathscr{N}_k(\varepsilon)^{1/2}$, hence $R_k(G^j) \ge c'$. Let $g_0 = \sum_j G^j(\cdot - jDe_1)$ with D dyadic. As in the proof of Proposition 284(i), for large k

$$R_k(g_0) \ge \sum_j R_k(G^j) - 16^2J^2\mathsf{K}_{L_S}^2 e^{-m_*(D-L_S-4\varepsilon-1)},$$

and $R_k \ge 0$. For D large the right side is at least $c'/2$ for all large k, and g_0 is continuous and supported in $\{x_0 \ge \varepsilon_1\}$: this is (NE). □

An example. (NE) is not implied by (SD) together with the properties used so far: in the following Gaussian field the largest reflected norms are carried by spatial momenta of order π/a_k at energies bounded in physical units, so that (UV) holds and (LM) fails.[35]

Proposition 318. *Let $m>0$, $A_k=(128(\cosh(ma_k)-1))^{-1}$,*

$$D_k(p)=1+A_k\prod_{\mu=0}^{3}(2-2\cos p_\mu),\qquad p\in(-\pi,\pi]^4,$$

let ϕ_k be the centred Gaussian field on $\mathbb{Z}^4$ with covariance $\int e^{ip\cdot(x-y)}D_k(p)^{-1}dp/(2\pi)^4$, $A_k(f)=\sum_x a_k^4 f(a_kx)\phi_k(x)$, and form $\mathscr{N}_k(\varepsilon)$ and R_k with these packets and $r=0$.

(a) *The law of ϕ_k is invariant under translations and B_4 and reflection positive for the reflections in coordinate hyperplanes through sites and bisecting links. Its transfer matrix in each coordinate direction has spectrum on $\Omega_k^\perp$ in $[0,e^{-ma_k}]$ containing e^{-ma_k}: the gap is m in physical units.*

(b) *For every $\varepsilon>0$, $\varepsilon'\in(0,\varepsilon]$ and all large k,*

$$2^{-15}me^{-4m(\varepsilon+2)}a_k^4\le\mathscr{N}_k(\varepsilon),\qquad \mathscr{N}_k(\varepsilon')\le(2+a_0)^4a_k^4;$$

so (SD$_\mathrm{p}$) *holds with $p_0=0$, and* (UV).

(c) *$R_k(g)\to0$ for every $g\in C_c(\mathbb{R}^4)$ supported in $\{x_0>0\}$:* (NE) *fails, and every limit S_2 of Corollary 281*(b) *vanishes identically.*

(d) *With $Z_k=\mathscr{N}_k(\varepsilon)^{-1/2}$,* (M$_\mathrm{s}$) *holds and every limit of Theorem 274 is trivial. With $Z_k=C_k(\sigma_0)^{-1/2}$, formed from any f_0 as before Proposition 276 with $C_k(0)>0$,* (M$_{\mathrm{s},2}$) *fails.*

Proof. (a) D_k is positive and invariant under permutations and sign changes of the coordinates, so the covariance is positive definite, real, and invariant under translations and B_4. Write $u_\mu=2-2\cos p_\mu$, and for $p\in(-\pi,\pi]^3$ let $b(p)=A_ku_1u_2u_3$, $\lambda(p)\in[0,1)$ the root of $\lambda+\lambda^{-1}=2+1/b(p)$ ($\lambda=0$ if $b(p)=0$), $E(p)=-\log\lambda(p)$ and $w(p)=(1-\lambda(p))/(1+\lambda(p))=\tanh(E(p)/2)$. Then $1+bu_0=b\lambda^{-1}(1-\lambda e^{ip_0})(1-\lambda e^{-ip_0})$, and expanding $(1-\lambda^2)/((1-\lambda e^{ip_0})(1-\lambda e^{-ip_0}))=\sum_t\lambda^{|t|}e^{-ip_0t}$ and using $b(\lambda^{-1}-\lambda)=1/w$,

$$\frac{1}{1+bu_0}=w\sum_{t\in\mathbb{Z}}\lambda^{|t|}e^{-ip_0t},$$

also for $b=0$. So the covariance of $\phi_k(t,\mathbf{x})$ and $\phi_k(s,\mathbf{y})$ is $\int e^{ip\cdot(\mathbf{x}-\mathbf{y})}w(p)\lambda(p)^{|t-s|}dp/(2\pi)^3$. Let γ be the centred Gaussian law on $\mathbb{R}^{\mathbb{Z}^3}$ with covariance the multiplier w, Λ the convolution with multiplier λ, and ξ_t independent of law the centred Gaussian with multiplier $w(1-\lambda^2)\ge0$. The stationary chain $\phi_{t+1}=\Lambda\phi_t+\xi_{t+1}$ has covariance with multiplier $w\lambda^{|t-s|}$, so it is the field; it is reversible because Λ and the covariance of γ commute and are symmetric. The proof of Proposition 207(a) applies with the operator Λ in place of q: the transition operator satisfies $P\mathscr{E}_h=\mathscr{E}_{\Lambda h}$, so it acts on the N-th Wiener chaos as the N-th tensor power of the restriction of Λ to the one-particle space, the completion of the finitely supported h for $\int|\hat h|^2w\,dp/(2\pi)^3$, on which Λ is multiplication by λ. Since $b\le64A_k$ with equality only at $p=(\pi,\pi,\pi)$, $\cosh E=1+1/(2b)\ge1+1/(128A_k)=\cosh(ma_k)$, so $0\le\lambda\le e^{-ma_k}$, with equality at that point. Hence $0\le P\le e^{-ma_kN}$ on the N-th chaos for $N\ge1$, and e^{-ma_k} lies in the spectrum on the first, λ being continuous and w positive. Reflection positivity and the identification of the transfer matrix with P follow as there, and B_4 gives the other directions.

For h supported in $\{x_0\ge1\}$ and ϑ fixing $\{x_0=c\}$, $|c|\le\frac12$, the time component of $y-\vartheta x$ is $x_0+y_0-2c\ge0$, and

$$\omega_k\big(\vartheta(\overline{A_k(h)})A_k(h)\big)=\int w|\hat H|^2\frac{dp}{(2\pi)^3},$$

where

$$\hat H(p)=\sum_{x_0}a_k\lambda^{x_0-c}\hat h_{x_0}(p),\qquad \hat h_{x_0}(p)=a_k^3\sum_{\mathbf{x}}h(a_kx)e^{ip\cdot\mathbf{x}}.$$

Let h be supported in $\{x_0\ge t\}$ with $t\ge a_k$. Since $\sum_{x_0\ge t/a_k}\lambda^{x_0-c}\le\lambda^{t/a_k-1/2}/(1-\lambda)$ and $w/(1-\lambda)\le1$, the Cauchy–Schwarz inequality with the weights λ^{x_0-c} gives

$$w|\hat H|^2\le a_k^2\lambda^{2t/a_k-1}\sum_{x_0}|\hat h_{x_0}|^2.\qquad(*)$$

(b) For h supported in $B_{\varepsilon'}$ with $\|h\|_\infty\le1$ and $a_k\le\min(\varepsilon',1)$, $(*)$ with $\lambda\le1$ and Parseval's identity give at most a_k^8 times the number of sites x with $a_kx\in B_{\varepsilon'}$, which is at most $(2+a_0)^4a_k^{-4}$. For the lower bound let $a_k\le\frac{1}{20}$, $ma_k\le2$, let b_0 be continuous with values in $[0,1]$, supported in $[\varepsilon,\varepsilon+1]$ and equal to 1 on $[\varepsilon+\frac14,\varepsilon+\frac34]$, let b_1 be the function equal to 1 on $[\frac14,\frac34]$, to 0 outside $[0,1]$ and linear in between, and $h_k(y)=b_0(y_0)\prod_{i=1}^3\cos(\pi y_i/a_k)b_1(y_i)$, supported in B_ε with $\|h_k\|_\infty\le1$. Since $\cos(\pi x_i)=(-1)^{x_i}$, $\hat H(p)=T(\lambda)\prod_i\beta(p_i+\pi)$ with $T(\lambda)=\sum_{x_0}a_kb_0(a_kx_0)\lambda^{x_0-c}$ and $\beta(q)=\sum_n a_kb_1(a_kn)e^{iqn}$. On $Q=\{p:|p_i-\pi|\le\frac{\pi}{3}\bmod2\pi\}$, $u_i\ge3$, so $b\ge27A_k$ and $\cosh E-1\le1/(54A_k)\le4(\cosh(ma_k)-1)\le\cosh(2ma_k)-1$, since $\cosh2x-1=2(\cosh x-1)(\cosh x+1)$; so $ma_k\le E\le2ma_k$ on Q. Then $w\ge\tanh(ma_k/2)\ge ma_k/4$, and $T(\lambda)\ge e^{-2ma_k((\varepsilon+1)/a_k+1/2)}\sum_{x_0}a_kb_0(a_kx_0)\ge e^{-2m(\varepsilon+2)}/4$. By Parseval's identity $\int_{-\pi}^{\pi}|\beta|^2dq/2\pi=a_k^2\sum_n b_1(a_kn)^2\ge a_k/4$, and since $|1-e^{iq}|\ge1$ for $|q|\ge\frac{\pi}{3}$ and b_1 is 4-Lipschitz,

$$\int_{\frac{\pi}{3}\le|q|\le\pi}|\beta|^2\frac{dq}{2\pi}\le a_k^2\sum_n|b_1(a_kn)-b_1(a_kn-a_k)|^2\le48a_k^3,$$

so $\int_{|q|\le\pi/3}|\beta|^2dq/2\pi\ge a_k/8$. Restricting the integral to Q gives

$$\mathscr{N}_k(\varepsilon)\ge\frac{ma_k}{4}\cdot\frac{e^{-4m(\varepsilon+2)}}{16}\Big(\frac{a_k}{8}\Big)^3.$$

(SD$_\mathrm{p}$) follows with $C_0=2^{15}(2+a_0)^4e^{4m(\varepsilon+2)}/m$, and (UV) by Proposition 282(ii), whose proof uses only the invariance, the reflection positivity and the gap.

[35] D_k depends on p only through the product of the four factors $2-2\cos p_\mu$, so the energy of a spatial momentum is small only where all three spatial factors are large, near the corner (π,π,π) of the Brillouin zone; the packets that carry the largest reflected norm live there, and smooth test functions do not see them.

(c) Let g be supported in $\{x_0 \geq t_0\}$, $t_0 > 0$, and in a cube of side L, and $\delta \in (0,1]$. Let V be the set of p with $|p_i| < \delta$ for some i. There $u_i < \delta^2$ for that i, so $b \leq 16 A_k \delta^2$ and $\cosh E - 1 \geq 4(\cosh(ma_k) - 1)/\delta^2 \geq 2(ma_k/\delta)^2$; since $\cosh x - 1 \leq x^2$ for $0 \leq x \leq 1$, $E \geq ma_k/\delta$ for $ma_k \leq \delta$. By $(*)$ and Parseval's identity the part of the integral over V is at most $e^{-(m/\delta)(2t_0 - a_k)} a_k^8$ times $\sum_x |g(a_k x)|^2 \leq (L+1)^4 a_k^{-4} \|g\|_\infty^2$. On the complement, $|p_1| \geq \delta$ and $|1 - e^{ip_1}| \geq 2\delta/\pi$, and $(*)$ with $\lambda \leq 1$ bounds that part by

$$\Big(\frac{\pi}{2\delta}\Big)^2 a_k^8 \sum_x |g(a_k x) - g(a_k x - a_k e_1)|^2 \leq \Big(\frac{\pi}{2\delta}\Big)^2 (L+2)^4 a_k^4 \rho(a_k)^2,$$

with ρ the modulus of continuity of g for $|\cdot|_\infty$. With (b),

$$\limsup_k R_k(g) \leq \frac{2^{15}(L+2)^4 \|g\|_\infty^2}{m e^{-4m(\varepsilon+2)}} e^{-2mt_0/\delta},$$

and $\delta \to 0$ gives $R_k(g) \to 0$. The last claim is Corollary 281(b).

(d) By (b) and Corollary 281(a), whose proof uses only the invariance, the reflection positivity and the gap, $(\mathrm{M}_{s,2})$ holds for $Z_k = \mathscr{N}_k(\varepsilon)^{-1/2}$. The moments of a centred Gaussian field are sums over pairings of products of covariances, and every pair of a separated tuple is separated; so (M_s) holds with $M_n \leq (n-1)!! M_2^{n/2}$, and by (c) every limit vanishes for $n \geq 1$. For $Z_k = C_k(\sigma_0)^{-1/2}$, $(\mathrm{M}_{s,2})$ would give, as in the proof of Corollary 281(a), $\mathscr{N}_k(\varepsilon') \leq M_2(Q, \varepsilon') C_k(\sigma_0)$ for $\varepsilon' < \varepsilon$, while $C_k(\sigma_0) \leq C_k(0) = R_k(f_0)\mathscr{N}_k(\varepsilon)$ by Proposition 276(i), $R_k(f_0) \to 0$ by (c), and $\mathscr{N}_k(\varepsilon') \geq \mathsf{K}^{-2}\mathscr{N}_k(\varepsilon)$. □

So (NE) is an input of its own: reflection positivity, the invariance, a gap bounded above and below in physical units and (SD_p) do not give it, and with the normalisation $Z_k = \mathscr{N}_k(\varepsilon)^{-1/2}$ the separated moment bound holds for every n while every limit is trivial. By Proposition 317, under (SD) it is exactly the statement that bounded spatial momenta carry a fixed fraction of the largest reflected norm, and by Corollary 316 it follows from (SD) for $\mathrm{U}(1)$. The example is not a lattice gauge theory. For the Wilson measure along the tuning the regimes are these: under a gap in lattice units, in particular in the strong-coupling range of Corollary 277, (SD) fails and the rate $\mathfrak{E}_k$ diverges like a_k^{-1}; under the short-distance law (AF) of Theorem 305, which asymptotic freedom predicts, (SD_p) and (NE) hold; which of the two occurs, if either, is not decided here.

Remark 319. *What is proved.* Given the averaged condition along the tuning, Theorem 274 turns (M_s) into Schwinger functions at separated points that are symmetric, invariant under the translations and B_4, reflection positive in every coordinate direction, with a Hamiltonian of gap m_* in every coordinate direction and exponential clustering in every direction; with the constants of Proposition 276(ii) they are nontrivial, and with (W_s) and (U_s) they converge and are $\mathrm{O}(4)$-invariant (Corollary 275). The averaged condition enters only through Lemma 273(iii), that is, through Corollary 65. For the two-point functions, Theorem 280 and Corollary 281 reduce (M_s) for $n = 2$ and nontriviality to (SD) and (NE), statements about reflected norms of single packets under the Wilson measure. For the rotations, Proposition 287 shows that the anisotropy of a separated two-point kernel has no components of degree below four and the angular profile Y_4 in degree four, and Theorem 288 derives (W_s) from (SB_n) and (RI_n); for the limit, Theorem 292 derives (U_s) from (MC_{m_0}) and (O_s). For the test functions of [95], Theorem 299 extends every S_n whose separated moments grow at most like a power of the inverse separation, and under (G_s) gives the linear growth condition of [96], reflection positivity and clustering on $\mathscr{S}_0$ (Corollary 300); for $n = 2$, (G_s) is the rate (SD_p) in (SD) (Proposition 301). For every n, (G_s) is (RN) with constants $K_1^m (m!)^{\beta_1} (1+L)^{q_1 m} \delta^{-p_1 m}$ (Proposition 311), and the inequality (GD) gives (M_s) from (SD) and (G_s) from (SD_p) (Proposition 313).

Which inputs are needed. (M_s) with constants that keep a reflected correlation fixed is not implied by the averaged condition, by Corollary 277. (W_s) is not implied by the other properties: the Gaussian field of Proposition 207(a) is invariant under translations and B_4, reflection positive in the coordinate hyperplanes, has gap ma_k in every coordinate direction, satisfies (M) and hence (M_s) by Theorem 204(i), and the proofs above use the averaged condition only through these properties, with m in place of m_*. Its limit has the continuous kernel $\kappa e^{-m\|x-y\|_1}$, so invariance under R on the separated pairs forces $\|Rz\|_1 = \|z\|_1$ for $z \neq 0$, and $G_S = B_4$. The same example at two interleaved masses violates (U_s), by the argument of Proposition 206(ii); it satisfies (O_s) and violates (MC_{m_0}) for every m_0 (Theorem 292(iii)). Products of independent copies of the field are not Gaussian, have the same properties and $G_S = B_4$, and violate (RI_2) (Proposition 289). The normalisation by $C_k(\sigma_0)$ does not fix the scale (Proposition 291). The bounds (RN_m), $m \geq 2$, are not implied by the two-point properties: the fields of Proposition 310, bounded ones included, have the invariance, the reflection positivity, gap m, (SD), (SD_p), (UV), (NE) and nonzero two-point limits, and violate $(\mathrm{M}_{s,n})$ for every $n \geq 3$ and (RN_m) for every $m \geq 2$. Bounds at each order do not give (G_s): the fields of Proposition 312 satisfy (M_s), (RN) and (SD_p), with moment bounds uniform in the separation, and their moments grow faster than $K^n (n!)^\beta$ for all K, β.

Short-distance law. The two-sided condition (AF), the lattice form of the law $|x-y|^{-8}(\log \frac{1}{|x-y|})^{-\gamma}$ that asymptotic freedom predicts with $\gamma = 2$ (Remark 304), gives $(\mathrm{M}_{s,2})$, (SD_p) with $p_0 = 4$, (SD), (UV) and (NE), excludes trivial limits and fails in the strong-coupling range (Theorem 305). It fixes Z_k up to bounded factors, with $Z_k \to \infty$ and Z_k comparable to $\mathscr{N}_k(\varepsilon)^{-1/2}$ (Proposition 306), it is consistent with a gap in physical units and with (U_s), and it implies neither (W_s) nor the exact law (AF_∞) (Proposition 307). It is not proved for the Wilson measure; the missing inequality is stated in Remark 308.

Relation to Theorem 233. Under the hypothesis of Theorem 45, Corollary 234 shows that, when $C_k(0) > 0$, (SR_0) fails at every spacing for the slice field of Φ or for that of $\Delta\Phi$. Nothing here uses (SR_0): the positivity used is that of the Wilson measure for products of packets with supports at distance at least ε from the reflection plane, that is, for functions of links at lattice distance at least $\varepsilon/a_k - r$ from it, and $r \geq 1$ for every nonconstant Φ. The nontriviality of Proposition 276(ii) comes

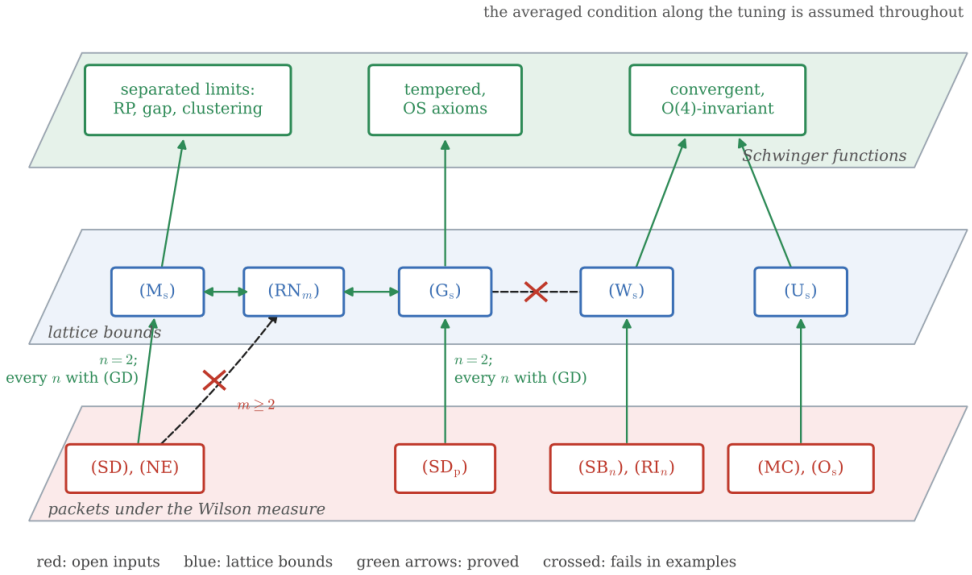


FIG. 25. The conditions of Sec. XI H, given the averaged condition along the tuning. Bottom: inputs on reflected norms of packets under the Wilson measure, none of them proved. Middle: lattice bounds; (M_s) is (RN) (Proposition 285), and (G_s) is (RN) with factorial constants (Proposition 311). Top: what the bounds give in the limit (Theorems 274 and 299, Corollaries 300 and 275). From the bottom row: Theorem 280 and Proposition 301 for $n = 2$, Proposition 313 for every n, Theorems 288 and 292. Crossed arrows fail in examples: the two-point properties do not give (RN_m), $m \geq 2$ (Proposition 310), and the bounds of the middle row do not give (W_s) (Proposition 207(a)).

from (M_s) and the choice of Z_k, not from positivity at short distance.

What is not proved. (M_s) for $n \geq 3$ with the constants of Proposition 276(ii), equivalently the bounds (RN_m), $m \geq 2$, on reflected norms of products of separated packets (Proposition 285), which do not follow from the two-point properties, reflection positivity and the gap (Proposition 310), and the inequality (GD) that gives them from (SD) (Proposition 313); (SD), equivalently the spectral condition (UV) (Proposition 282), which fails under a gap in lattice units, is not implied by reflection positivity and a gap bounded in physical units (Proposition 283), is a bound in physical units on the rate $\mathfrak{E}_k$ at which the largest reflected norm decays (Proposition 315), and follows from the short-distance law (AF) (Theorem 305); (NE), which under (SD) is the nontriviality of every limit (Proposition 284) and the condition (LM) that spatial momenta bounded in physical units carry a fixed fraction of the largest reflected norm (Proposition 317), which follows from (SD) for $\mathrm{U}(1)$ (Corollary 316) and is not implied by (SD_p), reflection positivity, the invariance and a gap bounded above and below in physical units (Proposition 318); (W_s), and (SB_n), (RI_n), of which (SB_2) is a bound on the axis and is not implied by the separated moment bounds (Propositions 293 and 294(b)); (U_s), and (MC_{m_0}), of which the two halves hold separately with different rates (Proposition 295), and (O_s), equivalently (DS) (Proposition 296); whether $\mathrm{O}(4)$ invariance of S_2 together with the other properties gives (W_s); the rate (G_s) at coinciding points, equivalently (RN_p) (Proposition 311), and its form (SD_p) for $n = 2$, which with (M_s) does not give (G_s) (Proposition 312); and the averaged condition along the tuning. None of these is proved here, and the mass gap of the continuum theory is not proved. The extension of the S_n to the test functions of [95] with the linear growth condition of [96] is therefore proved only under (G_s), and it does not follow from the properties at separated points: the two-point function of Proposition 302(ii) has those that Corollary 281(b) gives and is not tempered.

I. The conditional chain for every compact simple group

The lattice gap of Theorem 5 holds for every compact simple group in the range $N|\beta| < 2\kappa_G/29$. The continuum chain from Sec. X F to Corollary 300 is posed in the setting of Sec. X F, for any compact nontrivial $G \subset \mathrm{U}(N)$ and any positive continuous class weight, but several of its statements are proved for $\mathrm{SU}(2)$ or $\mathrm{SU}(N)$ only. This subsection lists where a property of these groups enters, states the chain for every compact simple group with each hypothesis named, and treats the statements that use $\mathrm{SU}(2)$ or $\mathrm{SU}(N)$: the centre sectors, the strong-coupling range of the averaged condition, the response of one link that the sharp quarter bounds for $\mathrm{SU}(2)$, the obstruction at fixed cube

size, and the holonomy flow.

Throughout, $G, N, \langle X,Y\rangle = -\operatorname{Tr}(XY)$, κ_G, $d = \dim G$, $K = \kappa_G/(d-1)$ and $C_0(G)$ are as in Sec. XIII Q, the inclusion $G \subset \mathrm{U}(N)$ is irreducible, as for every row of Table I, and the Wilson measure of Λ_L has the weight $\varphi = e^{N\beta \operatorname{Re}\operatorname{Tr}(\cdot)}$, that is, $\beta_W = N^2\beta$. The centre $Z(G)$ is finite, since $\mathfrak{g}$ is simple, and $\widehat{Z(G)}$ is its group of characters. Lemma 589 gives, for $N\beta \geq K$ and every plaquette,

$$\varepsilon_G(\beta) = 2\mu(N - \operatorname{Re}\operatorname{Tr} U_p) \leq \frac{2}{N\beta}\Big(\frac{d}{3}\log(N^2\beta) + C_0(G)\Big), \tag{31}$$

since $N - \operatorname{Re}\operatorname{Tr} U_p = N s_p$ and the plaquettes are equivalent under the symmetries of Λ_L. This replaces Lemma 78: for $\mathrm{SU}(2)$ it reads $\mu(2 - \operatorname{Tr} U_p) \leq (\log\beta + 5.71)/(2\beta)$.

Where the group enters. Table III lists the statements of Secs. X F–X K and XI H, the property of G that each proof uses, and its status for a general G. The forest gauge, the plaquette blocks and the chart bounds do not enter the chain: the averaged condition is posed without gauge fixing.

Centre sectors. For a direction ν, $t \in \mathbb{Z}/L\mathbb{Z}$ and $z \in Z(G)$ let $Z^\nu_{t,z}$ multiply every link (y,ν) with $y_\nu = t$ by z and fix the other links; for $\mathrm{SU}(N)$, $Z^\nu_t = Z^\nu_{t,\omega I}$. For $r = (r_0,\dots,r_3) \in \widehat{Z(G)}^4$ let $\mathscr{A}_r$ be the set of $f \in \mathscr{A}_\mathbb{C}$ with $f \circ Z^\nu_{t,z} = r_\nu(z) f$ for all ν, t and z, and let $\mathrm{BP}_0(n,\varpi)$ be the inequality of $\mathrm{BP}(n,\varpi)$ for the real $f \in \mathscr{A}_0$, 0 the trivial character. For $\mathrm{SU}(N)$ these are the objects of Sec. X H.

Proposition 320 (centre sectors for every group)**.** *Statements* (i)–(v) *of Proposition 82 hold for G, with $Z^\nu_{t,z}$ in place of Z^ν_t, $r \in \widehat{Z(G)}^4$, and the condition $f \circ Z^\nu_{t,z} = r_\nu(z) f$ in place of $f \circ Z^\nu = \omega^{r_\nu} f$. Theorem 84 holds with the clauses for* $\mathrm{SU}(N)$ *read for G:* $\mathrm{BP}(n,\varpi)$ *implies* $\mathrm{BP}_0(n,\varpi)$*, which implies* $\mathrm{BP}_{\mathrm{loc}}(n,\varpi)$*, and in* (iv)*, $h \in \mathscr{A}_0$. If $Z(G) = \{I\}$, then $\mathscr{A}_0 = \mathscr{A}_\mathbb{C}$ and* $\mathrm{BP}_0(n,\varpi)$ *is* $\mathrm{BP}(n,\varpi)$.

Proof. The proof of Proposition 82(i) uses of z only that it commutes with every element of G: then every plaquette variable and the density of $\gamma^0_{V_x}(\cdot \mid U)$ are unchanged, Haar measure is invariant under multiplication by z, $Z^\nu_{t,z}$ commutes with the gauge group, and $Z^\nu_{t,z}$ is invertible, with inverse $Z^\nu_{t,z^{-1}}$. In (ii) the gauge transformation equal to z^{-1} at the sites with $y_\nu = t+1$ and to I elsewhere shows that $f \circ Z^\nu_{t,z}$ does not depend on t for $f \in \mathscr{A}_\mathbb{C}$; write Z^ν_z for the induced map. The maps $f \mapsto f \circ Z^\nu_z$ are unitary on $\mathscr{A}_\mathbb{C}$, commute with each other, and $Z^\nu_z Z^\nu_{z'} = Z^\nu_{zz'}$, so they form a unitary representation of the finite abelian group $Z(G)^4$. For (iii) put

$$\Pi_r f = \frac{1}{|Z(G)|^4} \sum_{w \in Z(G)^4} \overline{r(w)}\, f \circ Z^0_{w_0} \cdots Z^3_{w_3},$$

with $r(w) = \prod_\nu r_\nu(w_\nu)$; the orthogonality of the characters of $Z(G)^4$ makes the Π_r orthogonal projections onto the $\mathscr{A}_r$ with $\sum_r \Pi_r = I$ and $\Pi_r \Pi_{r'} = 0$ for $r \neq r'$, and by (i) they commute with every $\mathscr{P}_x$. The proofs of (iv) and (v) and of the clauses of Theorem 84 use only (i)–(iii) and the fact that spectral projections of $\mathscr{K}$ commute with the Π_r. If $Z(G) = \{I\}$, the only character is trivial. □

For the groups of Table I the centre is cyclic of order N for $\mathrm{SU}(N)$, of order two for $\mathrm{SO}(2m)$, $\mathrm{Sp}(n)$ and E_7, of order three for E_6, and trivial for $\mathrm{SO}(2m+1)$, G_2, F_4 and E_8; for the last four groups every gauge-invariant function is neutral.

The chain for every group. Let $a_k \to 0$ be the lattice spacings of Definition 44, $\varpi > 0$, $\ell > 0$ and $n_k \geq 2$ integers with $n_k a_k \leq \ell$, $\beta_k \geq 0$, and $\mathscr{L}_k$ infinite sets of even integers $L \geq 2n_k + 4$.

($\mathrm{BP}^G_{\mathrm{loc}}$) For every k and every $L \in \mathscr{L}_k$, $\mathrm{BP}_{\mathrm{loc}}(n_k,\varpi)$ holds for the Wilson measure of G on Λ_L at β_k.

Theorem 321 (the conditional chain for every compact simple group)**.** (a) *Each of the following implies* ($\mathrm{BP}^G_{\mathrm{loc}}$)*:* (a1) $\mathrm{BP}(n_k,\varpi)$ *on the same tori;* (a2) $\mathrm{BP}_0(n_k,\varpi)$ *on the same tori;* (a3) $a_k = 2^{-k}a_0$, $n_k = 2^k n_0$, *the sets $\mathscr{L}_k$ of Corollary 102, and its hypotheses* (H1) *and* (H2) *for block-spin maps of G, with* $\varpi = \min(\varpi_0, 1/C_A)/\Gamma$.

(b) *Assume* ($\mathrm{BP}^G_{\mathrm{loc}}$). *Every torus limit ω_k along $\mathscr{L}_k$ is invariant under the translations and B_4, reflection positive, and its transfer matrix has gap at least $16\varpi/(625e^2 n_k)$ on $\Omega_k^\perp$. Every continuum limit of (ω_k, O_k), and of $(\omega_k, O_k \circ \Phi_{s_k})$ for the spatial flow of Proposition 47, satisfies $\sigma(H) \subset \{0\} \cup [m_*,\infty)$ with $m_* = 16\varpi/(625e^2\ell)$, and 0 is a simple eigenvalue.*

(c) *Assume* ($\mathrm{BP}^G_{\mathrm{loc}}$). *For every field Φ as in Sec. XI H and constants Z_k, every lemma, proposition, theorem and corollary of Sec. XI H other than Corollary 277 holds for G under the hypotheses stated in it; in particular Theorem 274 under* ($\mathrm{M_s}$)*, Proposition 276, and Theorem 299.*

(d) *Assume* ($\mathrm{BP}^G_{\mathrm{loc}}$), ($\mathrm{G_s}$), ($\mathrm{W_s}$) *and* ($\mathrm{U_s}$). *Then the conclusions of Corollary 300 hold for G: the limits extend to $\mathscr{S}_0(\mathbb{R}^{4n})$, satisfy the linear growth condition of [96] and the axioms of [95] of Euclidean invariance, reflection positivity, symmetry and clustering, the reconstructed Hamiltonian has $\sigma(H_0) \subset \{0\} \cup [m_*,\infty)$ and $\ker H_0 = \mathbb{C}\Omega_0$, and for $C_k(0) > 0$ and $Z_k = C_k(\sigma_0)^{-1/2}$ the limit is nontrivial.*

Proof. The proofs cited below use of G only the following: (G1) $G \subset \mathrm{U}(N)$ is a compact group with Haar measure, and the weight $e^{N\beta \operatorname{Re}\operatorname{Tr}(\cdot)}$ is a positive continuous class function; (G2) for $\beta \geq 0$ and even L the Wilson measure is reflection positive for reflections in planes through sites and bisecting links [97, 105], which the proof of Corollary 65 uses in the setting of Sec. X F; (G3) $\operatorname{Re}\operatorname{Tr}$ is not constant on G, since $\operatorname{Re}\operatorname{Tr} g = N$ only for $g = I$; (G4) gauge invariance of the measures, the conditions (B1) and (B2), and the invariance of Haar measure under translations. The straight-path map of Sec. X J is a block-spin map of range one for every G.

(a) Part (a1) is Theorem 84(i), whose first clause holds in the setting of Sec. X F, and (a2) is Proposition 320. For (a3), the proofs of Lemma 98, Theorems 100 and 101 and Corollary 102 use (G4) only, and give $\mathrm{BP}(n_k,\varpi)$; (a1) applies.

(b) This is Theorem 84(ii), (iii). Its proof uses Lemma 63, which uses the locality of the cube moves only, the bound $\varpi \leq 4$, which uses (G3), and the proofs of Corollary 65, Theorem 66 and Proposition 47(iii), which use (G1), (G2) and Appendix F; the spatial flow is defined for a bi-invariant metric on any compact G.

(c) By Remark 319, Sec. XI H uses the averaged condition only through Lemma 273(iii), that is, through the reflection

TABLE III. The statements of the conditional chain, the property of the gauge group used in their proofs, and their status for every compact simple group G.

Statement	Property of G used	Every compact simple G
Lemma 63, Theorem 64	locality of the cube moves; $\mathrm{Re}\,\mathrm{Tr}\,U_p$ is not constant	as stated
Corollary 65, Theorem 66, Proposition 47(iii)	reflection positivity for $\beta \geq 0$ [97]; Appendix F	as stated
Propositions 67, 75, 87, 88, Theorem 72, Lemma 76	a bi-invariant metric; Haar measure	as stated
Corollary 74(i), (ii)	the sharp quarter on S^3; the SU(3) bounds	part (iii), with κ_G; Proposition 323
Lemma 78	SU(N); Haar density of SU(2)	(31)
Proposition 82, Definition 83(b), Theorem 84(i), (iv)	the centre $\{\omega^j I\}$ of SU(N)	Proposition 320
Theorem 84, other clauses; Lemma 95	none beyond Sec. X F	as stated
Lemma 77, Theorem 79, Corollaries 80, 85, 86	centre of SU(N); commuting pairs in SU(N)	under (CP), Proposition 324; (CP) fails for Sp(n), n even
Lemma 91, Theorem 92	polar coordinates of SU(2)	Proposition 325
Theorem 94, Corollary 96	(B1), (B2)	as stated
Lemma 98, Theorems 100, 101, Corollaries 102, 104, Proposition 103	gauge invariance; translation invariance of Haar measure	as stated
Sec. XI H except Corollary 277	reflection positivity and the transfer gap of Corollary 65	as stated
Corollary 277	Corollary 74(i)	Corollary 322

positivity and the transfer gap of Corollary 65, which (b) supplies. The remaining arguments concern ω_k, the field Φ, its covariance under B_4 and the translations, and Gaussian comparison fields, and contain no property of G; the one statement that does, Corollary 277, is excluded.

(d) The proof of Corollary 300 uses Corollary 275, Theorems 274 and 299 and Proposition 276(ii), which hold for G by (c). □

None of the hypotheses of Theorem 321(b)–(d) is proved for any group along the continuum tuning: neither $(\mathrm{BP}^G_{\mathrm{loc}})$ with $n_k a_k \leq \ell$, nor (H1) and (H2), nor $(\mathrm{M_s})$, $(\mathrm{G_s})$, $(\mathrm{W_s})$, $(\mathrm{U_s})$. The theorem places the open inputs of the continuum step in the same statements for every compact simple group. At strong coupling the averaged condition holds for every group, and the limits are trivial.

Corollary 322. *Let $0 \leq N\beta_k \leq \kappa_G/48$ for every k. Then $\mathrm{BP}(2, \frac{21}{32})$ holds for the Wilson measure of G on every Λ_L with $L \geq 8$ even, so $(\mathrm{BP}^G_{\mathrm{loc}})$ holds with $\varpi = \frac{21}{32}$, $n_k = 2$ and $\ell = 2a_0$; every torus limit has transfer-matrix gap at least $m_{\mathrm{lat}} = 21/(2500e^2) > 1.13 \cdot 10^{-3}$ in lattice units, and for every Φ and every choice of Z_k satisfying $(\mathrm{M_s})$ every limit of Theorem 274 is trivial.*

Proof. At $n = 2$ Corollary 74(iii) requires $N|\beta| < 4\kappa_G/129$, which holds since $129 < 192$, and gives

$$\alpha_2 = \frac{33N|\beta|}{4(\kappa_G - 24N|\beta|)},$$

increasing in $N|\beta|$ and equal to $\frac{33}{96} = \frac{11}{32}$ at $N|\beta| = \kappa_G/48$. So $\mathrm{BP}(2, \frac{21}{32})$ holds on Λ_L, $L \geq 2n+4 = 8$, and $n_k a_k = 2a_k \leq 2a_0$. By Corollary 65 with $q_2 = \frac{625}{16}$ the gap is at least $\varpi/(2e^2 q_2) = 8\varpi/(625e^2) = 21/(2500e^2)$, and $e < 2.7183$ gives $2500e^2 < 18473$, so the gap exceeds $21/18473 > 1.13 \cdot 10^{-3}$. Proposition 276(iii) holds for G by Theorem 321(c), and applies. □

For SU(2) the range of Corollary 322 is $\beta_W \leq \frac{1}{24}$, a third of the range $\beta_W \leq \frac{1}{8}$ of Corollary 277; the difference comes from the sharp quarter.

The response of one link. For $S \in M_N(\mathbb{C})$ let ν_S be the probability measure on G with density proportional to $e^{\mathrm{Re}\,\mathrm{Tr}(SU)}$ with respect to Haar measure; given the other links, the law of a link under the Wilson measure is ν_S with $S = N\beta$ times the sum of the six staples. For $A \in M_N(\mathbb{C})$ put

$$\Lambda_S(A) = \sup_f \mathrm{Cov}_{\nu_S}\big(f, \mathrm{Re}\,\mathrm{Tr}(AU)\big),$$

the supremum over the real f on G with $|f(U) - f(V)| \leq \|U - V\|_{\mathrm{HS}}$. This is the derivative of $\nu_{S+tA}(f)$ at $t = 0$, maximised over these f, so by the Kantorovich–Rubinstein duality $W_1(\nu_S, \nu_{S'}) \leq \sup_{0 \leq t \leq 1} \Lambda_{S+t(S'-S)}(S' - S)$ for the Hilbert–Schmidt distance. Moving one partner link by δ in this distance moves one staple by δ, so $C^0_{ee'} \leq N|\beta|\Lambda_G(6N|\beta|)$ for the Hilbert–Schmidt metric, where

$$\Lambda_G(r) = \sup\big\{\Lambda_S(A) : \|S\|_{\mathrm{op}} \leq r,\ \|A\|_{\mathrm{HS}} = 1\big\}.$$

The sharp quarter (Theorem 682) computes Λ_G for SU(2); for a general group the following holds.

Proposition 323. (i) *For $\|A\|_{\mathrm{HS}} = 1$, $\Lambda_0(A) \leq 1/\sqrt{d}$, and $\Lambda_0(e^{i\theta}A) \geq 1/(2N)$ for some $\theta \in \mathbb{R}$; so $1/(2N) \leq \Lambda_G(0) \leq 1/\sqrt{d}$.*

(ii) *For* $0 \le r < \kappa_G$, $\Lambda_G(r) \le 1/(\kappa_G - r)$.

(iii) *For* $G = \mathrm{SU}(2)$, $\Lambda_S(A) \le \frac{1}{2}\|A\|_{\mathrm{HS}}$ *for all* S *and* A, *with equality for some* $A \neq 0$ *at* $S = 0$; *so* $\Lambda_G(r) = \frac{1}{2}$ *for every* $r \ge 0$.

Proof. (i) Let $g(U) = \operatorname{Re}\operatorname{Tr}(AU)$ and $X_1, \dots, X_d$ an orthonormal basis of $\mathfrak{g}$. Since the inclusion is irreducible, $\sum_a X_a^2 = -cI$ by Schur's lemma, and taking traces, $cN = d$. With the left-invariant fields $\tilde{X}_a f(U) = \frac{d}{ds} f(U e^{sX_a})|_{s=0}$, the Laplacian is $\Delta = \sum_a \tilde{X}_a^2$ and $\Delta g = \operatorname{Re}\operatorname{Tr}(AU \sum_a X_a^2) = -\frac{d}{N} g$. The a-th component of ∇g is $\operatorname{Re}\operatorname{Tr}(AUX_a)$, the real Hilbert–Schmidt product of $(AU)^*$ with X_a, so $|\nabla g| \le \|A\|_{\mathrm{HS}} = 1$ and g is 1-Lipschitz for the Hilbert–Schmidt distance. By Schur orthogonality, $\int |\operatorname{Tr}(AU)|^2 dU = \|A\|_{\mathrm{HS}}^2/N$ and $\int \operatorname{Tr}(AU) dU = 0$, and

$$\int g^2 \, dU = \frac{1}{2} \int |\operatorname{Tr}(AU)|^2 dU + \frac{1}{2} \operatorname{Re} \int \operatorname{Tr}(AU)^2 dU$$

lies between 0 and $1/N$. A function that is 1-Lipschitz for the Hilbert–Schmidt distance is 1-Lipschitz for the geodesic distance, which is larger, so $|\nabla f| \le 1$ almost everywhere, and

$$\operatorname{Cov}(f,g) = -\frac{N}{d} \int f \Delta g = \frac{N}{d} \int \langle \nabla f, \nabla g \rangle \le \frac{N}{d} \Big(\int |\nabla g|^2 \Big)^{1/2},$$

with $\int |\nabla g|^2 = -\int g \Delta g = \frac{d}{N} \int g^2 \le \frac{d}{N} \cdot \frac{1}{N}$; so $\operatorname{Cov}(f,g) \le 1/\sqrt{d}$. Choose θ with $e^{2i\theta} \int \operatorname{Tr}(AU)^2 dU \ge 0$; for $A' = e^{i\theta} A$ and $g' = \operatorname{Re}\operatorname{Tr}(A'U)$, $\int g'^2 \ge 1/(2N)$, and $f = g'$ is admissible, so $\Lambda_0(A') \ge 1/(2N)$.

(ii) Let $\|S\|_{\mathrm{op}} \le r$ and $V(U) = -\operatorname{Re}\operatorname{Tr}(SU)$. Along a geodesic $e^{tX}U$ the second derivative of V is $-\operatorname{Re}\operatorname{Tr}(SX^2U)$, whose absolute value is at most

$$\|S\|_{\mathrm{op}} \|X^2\|_1 = \|S\|_{\mathrm{op}} |X|^2,$$

so $\mathrm{Ric}_G + \mathrm{Hess}\, V \ge \kappa_G - r > 0$. As in the proof of Lemma 73, $|\operatorname{Cov}_{\nu_S}(f,g)|$ is at most $\|\nabla f\|_\infty \|\nabla g\|_\infty / (\kappa_G - r)$, and both gradients are at most one for $\|A\|_{\mathrm{HS}} = 1$.

(iii) Write $U = x_0 I + x_1 \mathbf{i} + x_2 \mathbf{j} + x_3 \mathbf{k}$ with $x \in S^3 \subset \mathbb{R}^4$, and U' likewise with x', so that $\|U - U'\|_{\mathrm{HS}} = \sqrt{2}|x - x'|$ and $\operatorname{Re}\operatorname{Tr}(AU) = \langle w, x \rangle$, where w_a is the real Hilbert–Schmidt product of A^* with the a-th basis matrix, of norm $\sqrt{2}$; hence $|w| \le \sqrt{2}\|A\|_{\mathrm{HS}}$, with equality when A^* is a real multiple of an element of $\mathrm{SU}(2)$. Then $\nu_S = \nu_b$ in the notation of Theorem 682, with $b \in \mathbb{R}^4$ defined by $\operatorname{Re}\operatorname{Tr}(SU) = \langle b, x \rangle$, and f is $\sqrt{2}$-Lipschitz for the chordal distance of the unit sphere. So, for $w \neq 0$,

$$\begin{aligned} \operatorname{Cov}_{\nu_S}(f, \langle w, x \rangle) &= \sqrt{2}|w| \operatorname{Cov}_{\nu_b}\big(f/\sqrt{2}, \langle w/|w|, x \rangle\big) \\ &\le \sqrt{2}|w| \, L(b, w/|w|) \le \tfrac{1}{2}\|A\|_{\mathrm{HS}}, \end{aligned}$$

with equality at $b = 0$ for $A^* \in \mathbb{R}\,\mathrm{SU}(2)$ and $f = \sqrt{2}\langle w/|w|, x \rangle$. □

Lemma 73 is the bound of Proposition 323(ii) with $r = 6N|\beta|$, in the geodesic metric, and Corollary 74(i) rests on part (iii): $C^0_{ee'} \le N|\beta| \cdot \frac{1}{2} = |\beta|$, uniformly in β. Through Theorem 72, a bound $\Lambda_G(6N|\beta|) \le \lambda$ gives $\mathrm{BP}(n, 1 - \alpha_n)$ with $\alpha_n < 1$ for large n whenever $18N|\beta|\lambda < 1$. With $\lambda = 1/(\kappa_G - 6N|\beta|)$ this is the range $N|\beta| < \kappa_G/24$ that Corollary 74(iii) reaches as $n \to \infty$; for $\mathrm{SU}(2)$, $\lambda = \frac{1}{2}$ gives $|\beta| < \frac{1}{18}$, that is $\beta_W < \frac{2}{9}$, the range of Corollary 74(i). The analogue of the sharp quarter for G is a bound λ of the order of $\Lambda_G(0) \le 1/\sqrt{d}$, uniform in $r \le 6N|\beta|$, which would give a range of order $\sqrt{d}$ in place of κ_G. At $r = 0$ the ratio of the bound of part (ii) to that of part (i) is $\sqrt{d}/\kappa_G$, which is $2\sqrt{N^2 - 1}/N < 2$ for $\mathrm{SU}(N)$ and $4\sqrt{248} > 62$ for E_8. The value of $\Lambda_G(r)$ for $r > 0$ and $G \neq \mathrm{SU}(2)$ is not determined here. The proof of Theorem 682 uses that $\mathrm{SU}(2)$ is the unit sphere of $\mathbb{R}^4$, on which ν_S is a von Mises–Fisher law, and the harmonic extension of a Lipschitz function to the ball; it is not extended here to other groups.

The obstruction at fixed cube size. Theorem 79 uses the centre of $\mathrm{SU}(N)$ twice: the charged traces of Polyakov loops have mean zero, and on commuting pairs some charged trace is bounded below. For a general group the first part carries over whenever $Z(G) \neq \{I\}$, and the second becomes a condition on G. Let $A = P_0$ and B be the Polyakov loops of directions 0 and 1 based at the site 0, as in Sec. X G.

(CP) There are continuous $F_1, \dots, F_m : G \times G \to \mathbb{C}$ and $\ell > 0$ such that for every k: $F_k(gA'g^{-1}, gB'g^{-1}) = F_k(A', B')$; there are $z, z' \in Z(G)$ and $\lambda_k \neq 1$ with $F_k(zA', z'B') = \lambda_k F_k(A', B')$; $|F_k| \le 1$ and $|F_k(HA', H'B') - F_k(A', B')| \le \ell(\|H - I\|_{\mathrm{op}} + \|H' - I\|_{\mathrm{op}})$; for all $g, A', B', H, H' \in G$. Moreover $\sum_k |F_k(A', B')|^2 > 0$ for every commuting pair (A', B').

Under (CP), by compactness there are $c_F > 0$ and $\eta_F > 0$ with $\sum_k |F_k(A', B')|^2 \ge c_F$ whenever $\|A'B'A'^{-1}B'^{-1} - I\|_{\mathrm{HS}} \le \eta_F$, as in the proof of Lemma 77.

Proposition 324. *Let* $\beta \ge 0$ *and* $n \ge 2$.

(i) *If* $Z(G) \neq \{I\}$, *there is an irreducible unitary representation* σ *of* G, *of dimension* N_σ, *whose central character is not trivial, and* $\mathrm{BP}(n, \varpi)$ *on* Λ_L *implies*

$$\frac{1}{L^3} \sum_{y : y_0 = 0} \mu\big(\chi_\sigma(P_0) \overline{\chi_\sigma(P_y)}\big) \le \frac{2N_\sigma^2 n^2}{\varpi L^2}, \quad \chi_\sigma = \operatorname{Tr}\sigma.$$

(ii) *Under* (CP), *for* $N\beta \ge K$, $\mathrm{BP}(n, \varpi)$ *on* Λ_{2n+4} *implies*

$$\varpi \le 2048\, n^4 \varepsilon_G(\beta) \max\Big(\frac{m\ell^2}{c_F}, \frac{1}{\eta_F^2} \Big),$$

with $\varepsilon_G(\beta)$ *bounded by* (31); *so along any sequence with* $n_k^4 \log(N^2 \beta_k)/\beta_k \to 0$ *no* $\varpi > 0$ *satisfies* $\mathrm{BP}(n_k, \varpi)$ *uniformly in the torus.*

(iii) $\mathrm{SU}(N)$ *satisfies* (CP) *with* $F_{ab} = \operatorname{Tr}(A^a B^b)/N$, $(a,b) \in I_N$, $m = N^2 - 1$ *and* $\ell = N - 1$. $\mathrm{Sp}(n') \subset \mathrm{U}(2n')$ *with* n' *even does not satisfy* (CP): *there is a commuting pair at which every function with the first two properties of* (CP) *vanishes. If* $Z(G) = \{I\}$, *no function satisfies the second property.*

Proof. (i) Let $I \neq z \in Z(G)$. The inclusion $G \subset \mathrm{U}(N)$ is faithful and irreducible, so z acts as a scalar $\zeta \neq 1$ and σ may be the inclusion; in general any irreducible σ with $\sigma(z) \neq I$ serves. The twist $Z^0_{0,z}$ preserves μ (Proposition 320) and maps $\chi_\sigma(P_y)$ to $\zeta_\sigma \chi_\sigma(P_y)$, $\zeta_\sigma \neq 1$, so $\mu(\chi_\sigma(P_y)) = 0$. The proof of

Theorem 79(i) applies to $g_y = \operatorname{Re}\chi_\sigma(P_y)$ and to the imaginary parts, with $|\chi_\sigma| \le N_\sigma$ in place of $|\operatorname{Tr}| \le N$.

(ii) Let $L = 2n+4 \le 4n$, $\varepsilon = \varepsilon_G(\beta) = \mu(\|U_p - I\|_{\mathrm{HS}}^2)$, and f the real or the imaginary part of some $F_k(A,B)$. The twist that multiplies the links $(y,0)$ with $y_0 = 0$ by z and the links $(y,1)$ with $y_1 = 0$ by z' preserves μ and maps $F_k(A,B)$ to $\lambda_k F_k(A,B)$, so $\mu(F_k(A,B)) = 0$.

If V_x meets neither A nor B, $\mathscr{P}_x f = f$. Otherwise the proof of Theorem 79(ii) gives loops A', B' that avoid V_x, a holonomy Γ, and $A = H_0 \Gamma A' \Gamma^{-1}$, $B = H_1 \Gamma B' \Gamma^{-1}$ with $H_0, H_1 \in G$ and $\|H_0 - I\|_{\mathrm{op}} + \|H_1 - I\|_{\mathrm{op}} \le S_0 + S_1$, the sum of $\|U_p - I\|_{\mathrm{op}}$ over at most $2Ln$ plaquettes. By the invariance under simultaneous conjugation, $F_k(\Gamma A'\Gamma^{-1}, \Gamma B'\Gamma^{-1}) = F_k(A', B')$, a function of the links outside V_x, so $\|f - \mathscr{P}_x f\|_2 \le \ell \|S_0 + S_1\|_2 \le 2\ell Ln\sqrt{\varepsilon}$. At most $2Ln^3$ points x have V_x meeting $A \cup B$, so $\mathscr{E}_n(f) \le 8\ell^2 L^3 n \varepsilon$.

By Lemma 76 the commutator of A and B is within $\sum_p \|U_p - I\|_{\mathrm{HS}}$ of I, a sum over L^2 plaquettes, so it exceeds η_F with probability at most $L^2\sqrt{\varepsilon}/\eta_F$, and $\sum_k \mu(|F_k(A,B)|^2) \ge c_F(1 - L^2\sqrt{\varepsilon}/\eta_F)$. As $\mu(F_k(A,B)) = 0$, one of the $2m$ functions f has $\operatorname{Var}_\mu(f) \ge c_F(1 - L^2\sqrt{\varepsilon}/\eta_F)/(2m)$. If $L^2\sqrt{\varepsilon} \le \eta_F/2$, $\mathrm{BP}(n, \varpi)$ gives $\varpi \le 32 m \ell^2 L^3 n \varepsilon / c_F \le 2048 m \ell^2 n^4 \varepsilon / c_F$. Otherwise $\eta_F^2 < 4L^4\varepsilon \le 1024 n^4 \varepsilon$, and $\varpi \le 2$ by Proposition 75(ii) gives $\varpi \le 2048 n^4 \varepsilon / \eta_F^2$.

(iii) For $\mathrm{SU}(N)$, F_{ab} is invariant under simultaneous conjugation; for (c,d) with $ac + bd \not\equiv 0$ modulo N, $z = \omega^c I$ and $z' = \omega^d I$ give $\lambda = \omega^{ac+bd} \ne 1$; $|F_{ab}| \le 1$; telescoping, $\|(HA')^a - A'^a\|_{\mathrm{op}} \le a\|H - I\|_{\mathrm{op}}$, so $\ell = N - 1$; and Lemma 77 gives the last property.

For $\mathrm{Sp}(n') = \{g \in \mathrm{U}(2n') : g^T \Omega g = \Omega\}$, $\Omega = \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$, n' even, put $A' = \operatorname{diag}(iI, -iI)$ and $B' = \operatorname{diag}(D, D)$ with $D = \operatorname{diag}(I_{n'/2}, -I_{n'/2})$. Both lie in $\mathrm{Sp}(n')$ and commute, and $Z(\mathrm{Sp}(n')) = \{\pm I\}$. The element Ω lies in $\mathrm{Sp}(n')$ and exchanges the two diagonal blocks, so $\Omega A' \Omega^{-1} = -A'$ and $\Omega B' \Omega^{-1} = B'$. With Q the permutation matrix that exchanges the two halves of $\{1, \dots, n'\}$, $q = \operatorname{diag}(Q, Q) \in \mathrm{Sp}(n')$, $qA'q^{-1} = A'$ and $qB'q^{-1} = -B'$. So (A', B') is simultaneously conjugate to $(zA', z'B')$ for all $z, z' \in \{\pm I\}$, and a function invariant under simultaneous conjugation with $F(zA', z'B') = \lambda F(A', B')$, $\lambda \ne 1$, has $F(A', B') = \lambda F(A', B') = 0$. If $Z(G) = \{I\}$, $F(zA', z'B') = F(A', B')$. □

For $\mathrm{Sp}(1) = \mathrm{SU}(2)$ the pair of Proposition 324(iii) does not exist, and (CP) holds. For $\mathrm{Sp}(n')$ with n' even the proof of Theorem 79(ii), (iii) fails at the variance bound when the law of (A,B) is near the orbit of (A', B'), and for $\mathrm{SO}(2m+1)$, G_2, F_4 and E_8 there is no charged function at all. Whether $\mathrm{BP}(n, \varpi)$ fails on Λ_{2n+4} at weak coupling for these groups, and whether (CP) holds for $\mathrm{Sp}(n')$ with $n' \ge 3$ odd, $\mathrm{SO}(2m)$, E_6 and E_7, is not determined here. Part (i) holds for every G with nontrivial centre and has no content for the four groups with trivial centre.

The holonomy flow. Let χ be a smooth real class function on G with $\chi(zW) = \chi(W)$ for $z \in Z(G)$, and define $J(W) \in \mathfrak{g}$ by $\langle J(W), X\rangle = \frac{d}{ds}\chi(e^{sX}W)|_{s=0}$. Put $\ell_0 = \sup|\chi(W) - \chi(W')|/\|W - W'\|_{\mathrm{op}}$, $\ell_1 = \sup\|J(W) - J(W')\|_{\mathrm{HS}}/\|W - W'\|_{\mathrm{HS}}$ and $D_\chi = \sup|\Delta\chi|$, all finite since G is a compact submanifold of $M_N(\mathbb{C})$. On G^E let $\mathscr{V}$ have the component $L^{-1}J(P_b)U_{(b,0)}$ at the link $(b,0)$, with P_b the Polyakov loop of direction 0 based at b, and vanish at the links of the other directions; put $S = -N\beta\sum_p \operatorname{Re}\operatorname{Tr} U_p$, $H = \sum_p (N - \operatorname{Re}\operatorname{Tr} U_p)$ and $\Theta = \mathscr{V} S - \operatorname{div}\mathscr{V}$. For $\mathrm{SU}(2)$ and $\chi(W) = -\frac{1}{2}(\operatorname{Tr} W)^2$, $J(W) = \operatorname{Tr} W \cdot \frac{1}{2}(W - W^*)$ is the field of Sec. XI, with $\ell_1 \le 4$ and $\Delta\chi = 2(\operatorname{Tr} W)^2 - 2$.

Proposition 325 (the holonomy flow for every group)**.** (i) $J(gWg^{-1}) = gJ(W)g^{-1}$, $J(W)$ *commutes with* W, $J(zW) = J(W)$ *for* $z \in Z(G)$, *and* $J(z) = 0$.

(ii) $\mathscr{V} P_y = J(P_y)P_y$ *and* $\mathscr{V}\chi(P_y) = |J(P_y)|^2$ *for every site* y.

(iii) $\operatorname{div}\mathscr{V} = \sum_{y:\, y_0 = 0} \Delta\chi(P_y)$.

(iv) $|\mathscr{V} S| \le 2N\beta\ell_1 H$, *and* $\|\Theta\|_2 \le 4\ell_1(M+2) + L^3 D_\chi$ *with* $M = \log\mu(e^{N\beta H/2})$; *for* $N\beta \ge 2K$, $M \le 6L^4(\frac{d}{3}\log\frac{N^2\beta}{2} + C_0(G))$.

(v) $\mu(\mathscr{V} h) = \mu(h\Theta)$ *and* $\mu(\mathscr{V} h)^2 \le \operatorname{Var}_\mu(h)\|\Theta\|_2^2$ *for every real* h *of class* C^1.

(vi) *If* $N\beta \ge 2K$, $n \ge 2$, $L \ge 2n+4$ *is even and* $\mathrm{BP}_0(n, \varpi)$ *holds on* Λ_L, *then*

$$\mu\big(|J(P_0)|^2\big)^2 \le \frac{\ell_0^2 L^3 n\, \varepsilon_G(\beta)}{\varpi}\, \|\Theta\|_2^2.$$

Proof. (i) Since χ is a class function and the metric is invariant under Ad, $\langle J(gWg^{-1}), X\rangle = \langle J(W), g^{-1}Xg\rangle$, which is the first identity. Conjugating by e^{sX}, $\chi(We^{sX}) = \chi(e^{sX}W)$, and $\chi(We^{sX}) = \chi(e^{s\,\mathrm{Ad}_W X}W)$; so $\langle J(W), X\rangle = \langle J(W), \mathrm{Ad}_W X\rangle$ for all X, and $J(W)$ is fixed by $\mathrm{Ad}_{W^{-1}}$. The third identity follows from $\chi(e^{sX}zW) = \chi(e^{sX}W)$. For z central, $J(z) = gJ(z)g^{-1}$ for all g, so $J(z)$ lies in the centre of $\mathfrak{g}$, which is zero.

(ii) The computation in the proof of Lemma 91(i) uses only the first identity of (i). Then $\mathscr{V}\chi(P_y) = \frac{d}{ds}\chi(e^{sJ(P_y)}P_y)|_{s=0} = \langle J(P_y), J(P_y)\rangle$.

(iii) As in the proof of Lemma 91(ii), the right translation $U_e \mapsto W = U_e R$ carries the component at e to $L^{-1}J(W)W$, which is L^{-1} times the gradient of χ for the bi-invariant metric; its divergence for Haar measure is $L^{-1}\Delta\chi(W)$, and W is a conjugate of the loop through the initial site of e, so $\Delta\chi(W) = \Delta\chi(P_y)$. Each of the L^3 loops has L links.

(iv) The computation of Lemma 91(iii) uses only the first identity of (i) and gives $\mathscr{V}\operatorname{Re}\operatorname{Tr} U_p = L^{-1}\operatorname{Re}\operatorname{Tr}(U_p\Delta_p)$ with $\Delta_p = J(P_b) - J(VP_{b+e_\nu}V^{-1}) \in \mathfrak{g}$ and $\|\Delta_p\|_{\mathrm{HS}} \le \ell_1 Q_\sigma$. Since $\operatorname{Re}\operatorname{Tr}\Delta_p = 0$, $|\operatorname{Re}\operatorname{Tr}(U_p\Delta_p)| \le \|U_p - I\|_{\mathrm{HS}}\ell_1 Q_\sigma$, and the Cauchy–Schwarz inequality over each strip and $\|U - I\|_{\mathrm{HS}}^2 = 2(N - \operatorname{Re}\operatorname{Tr} U)$ give $|\mathscr{V} S| \le 2N\beta\ell_1 H$. Put $X = N\beta H/2$, so $\mu(e^X) = e^M$; the inequality $X^2 \le M^2 + (2M+2)e^{X-M}$ of that proof gives $\mu(X^2) \le (M+2)^2$, so $\|\mathscr{V} S\|_2 \le 4\ell_1\|X\|_2 \le 4\ell_1(M+2)$, and $|\operatorname{div}\mathscr{V}| \le L^3 D_\chi$ by (iii). With $\phi(t) = \log\int e^{-NtH}dU$, convexity gives $M = \phi(\beta/2) - \phi(\beta) \le \frac{N\beta}{2}\mu_{\beta/2}(H)$, where $\mu_{\beta/2}$ is the Wilson measure at $\beta/2$, and Lemma 589 at $\beta_W = N^2\beta/2 \ge KN$ bounds $\mu_{\beta/2}(H)$ by $6L^4N(\frac{d}{3}\log\frac{N^2\beta}{2} + C_0(G))/(N^2\beta/2)$.

(v) This is the proof of Lemma 91(iv).

(vi) Let $h = \chi(P_0)$, real and gauge invariant. The twist $Z^0_{0,z}$ maps P_0 to zP_0, and $Z^\nu_{t,z}$, $\nu \ne 0$, fixes it, so $h \in \mathscr{A}_0$. As in the proof of Theorem 92(i), for each of the Ln^3 points x with V_x meeting P_0 there is a loop A' avoiding V_x with $P_0 = H_0\Gamma A'\Gamma^{-1}$ and $\|H_0 - I\|_{\mathrm{op}} \le S_0$, a sum over at most Ln plaquettes; then

$|h-\chi(A')| = |\chi(H_0W)-\chi(W)| \le \ell_0 S_0$ with $W = \Gamma A' \Gamma^{-1}$, and $\mu((h-\mathscr{P}_x h)^2) \le \ell_0^2 L^2 n^2 \varepsilon_G(\beta)$. So $\mathscr{E}_n(h) \le \ell_0^2 L^3 n \varepsilon_G(\beta)$, $\mathrm{BP}_0(n,\varpi)$ bounds $\mathrm{Var}_\mu(h)$ by this over ϖ, and (v) with $\mathscr{V} h = |J(P_0)|^2$ from (ii) gives the claim. □

On Λ_{2n+4}, part (iv) gives $\|\Theta\|_2 \le c_1 n^4(\log(N^2\beta)+c_2)$ with c_1, c_2 depending on G and χ only, so $\mathrm{BP}_0(n,\varpi)$ with a fixed ϖ at $\beta \gg n^{12}\log^3\beta$ forces $\mu(|J(P_0)|^2)$ to zero: the law of the Polyakov loop concentrates near the critical set of χ, which contains $Z(G)$ by (i). For SU(2) and $\chi = -\frac{1}{2}(\mathrm{Tr}W)^2$, $|J(W)|^2 = \frac{1}{2}(\mathrm{Tr}W)^2(4-(\mathrm{Tr}W)^2)$ and $\ell_0 \le 4$, and (vi) reproduces Theorem 92(i) with (31) in place of Lemma 78. For SU(2) every admissible field vanishes on $\{\pm I\} \cup \{\mathrm{Tr}W = 0\}$ (the discussion after Theorem 92); for other groups the common critical set of the admissible χ is not determined here. As for SU(2), a lower bound on $\mu(|J(P_0)|^2)$ at weak coupling on Λ_{2n+4} is not proved for any χ, so the failure of BP_0 there is not proved for any group.

Remark 326 (what is proved for every group and what remains). Proved for every compact simple G: the conditional chain of Theorem 321, with the hypothesis ($\mathrm{BP}^G_{\mathrm{loc}}$) and, for the Schwinger functions, ($\mathrm{M_s}$), ($\mathrm{G_s}$), ($\mathrm{W_s}$) and ($\mathrm{U_s}$); the centre sectors for $Z(G)$ (Proposition 320); the averaged condition $\mathrm{BP}(2,\frac{21}{32})$ for $N\beta \le \kappa_G/48$ and the triviality of the limits there (Corollary 322); two-sided bounds on the response of one link at $S=0$ and the Bakry–Émery bound for $\|S\|_{\mathrm{op}} < \kappa_G$ (Proposition 323); the obstruction at fixed cube size under (CP), which holds for SU(N) and fails for Sp(n') with n' even (Proposition 324); the holonomy flow for every centre-invariant class function (Proposition 325). Not proved: the hypotheses of Theorem 321 along the tuning for any group; an analogue of the sharp quarter, a bound on $\Lambda_G(r)$ of the order of $\Lambda_G(0)$, for any $G \ne \mathrm{SU}(2)$; (CP) for the groups other than SU(N) and Sp(n'), n' even; any obstruction at fixed cube size for the groups with trivial centre; and the mass gap.

XII. BLOCK AVERAGES

Proposition 187 locates the failure of (D) in the quadratic approximation: gauge transformations constant on the lines of F move the whole cube and change few links outside it. By Proposition 188 every gauge fixed on an invariant forest fails in the same way, through the constant fields. This section takes up the programme of the report [82] of the third author, which keeps the cube condition but changes the gauge and conditions the cube on block averages of the field, as in Bałaban's renormalisation group [6, 7]. The natural block averages of the line field leave one class of lines invisible and fail at every size (Proposition 327). In a gauge whose fixed links are trees inside blocks of side two, the averages over whole blocks remove every zero mode, the constrained quadratic form has a gap that does not depend on the volume (Theorem 330), and the constrained cube condition holds at every coupling once the cube is large (Theorem 334). Integrating out one scale bounds the cube coefficients by the constrained ones plus those of the averages (Corollary 342), and the averages carry the coarse quadratic form at every scale up to a factor between 1 and 256, through forms that are local uniformly in the scale (Proposition 335 and Theorems 338 and 340), and the constrained cube condition holds at every scale with one cube side (Theorem 344). For the group-valued links themselves, the straight paths between block roots satisfy a coercivity bound for all fields but do not compose, a continuous covariant map cannot treat the transverse positions alike, and a cut-off of Bałaban's averaging operation is coercive for all fields and agrees with the block averages to first order (Sec. XII B). It composes across all scales on fields of small curvature for every compact group (Sec. XII C), and under the SU(2) Wilson measure the fields that are not small form clusters of vanishing physical size (Sec. XII E).

A. Block-constrained cubes

In this subsection everything concerns the quadratic approximation: real fields φ on links, $(d\varphi)_p$ the signed sum around the plaquette p, and $(dh)_{(x,\mu)} = h(x) - h(x+e_\mu)$ for a function h on sites.

Proposition 327 ([82]). *In the setting of Proposition 187, condition the cube in addition on any family of the face averages*

$$V^{\mathrm{face}}_{y,\mu}(\varphi) = \frac{1}{8}\sum_x \big(\varphi_{(x,\mu)} + \varphi_{(x+e_\mu,\mu)}\big),$$

where x runs over the eight sites of the block $2y+\{0,1\}^4$ with $x_\mu = 2y_\mu$. Then $\sum_{c\in C_I} K_c \ge 12 N_n$ for the twelve classes

$$C_I = \{(0111,\mu),(1111,\mu),(0111+e_\mu,\mu),(1111+e_\mu,\mu) : \mu = 1,2,3\},$$

written as residue of the base and direction. Hence $K_c \ge N_n$ for some class, and the constrained condition fails for every $\alpha < 1$, at every even n and every b.

Proof. By Table VI, $\sigma(0111) = \sigma(1111) = 0$, so the lines of F through the sites with x_1, x_2, x_3 odd have direction 0; call them invisible. Let ℓ be one of them and h_ℓ its indicator. The sum of dh_ℓ along a path of length two from x is $h_\ell(x) - h_\ell(x+2e_\mu)$. For $\mu \ne 0$ both end points have even μ-coordinate and are off ℓ; for $\mu = 0$ they have the same transverse coordinates and are both on ℓ or both off it. So dh_ℓ changes no face average, and the translation argument of Proposition 187 applies to the constrained laws: $\sum_{i\in V}|(dh_\ell)_i| \le \sum_{j\notin V} C_{V,j}|(dh_\ell)_j|$. An active link $e = (x,\mu)$ has $|(dh_\ell)_e| = 1$ for at most one invisible ℓ, since its two ends lie on different lines and both cannot be invisible (for $\mu \ne 0$ they differ in the coordinate μ, and for $\mu = 0$ the link would lie in F). It has one such ℓ exactly when its base or its end has residue 0111 or 1111, which by Table VI are the classes of C_I: the six classes with base residue 0111 or 1111 and direction $\mu \ne 0$, and the six with base residue $r+e_\mu$, $r \in \{0111, 1111\}$, whose directions μ differ from $\sigma(r+e_\mu)$ (for instance $\sigma(0011) = 3 \ne 1$). Summing over the invisible lines gives $12N_n \le \sum_{c\in C_I} K_c$. □

Definition 328. The blocks are $B(y) = 2y+\{0,1\}^4$, $y \in \mathbb{Z}^4$, and $o = x \bmod 2$ is the offset of a site. A link (x,μ) is a *tree*

link if $o_\mu = 0$ and $o_\nu = 0$ for every $\nu > \mu$, and *active* otherwise; fields vanish on tree links. For a coarse link (y,μ) the *block average* is

$$V_{y,\mu}(\varphi) = \frac{1}{16} \sum_{x \in B(y)} \big(\varphi_{(x,\mu)} + \varphi_{(x+e_\mu,\mu)}\big),$$

the mean of φ summed along the sixteen paths of length two in the direction μ that start in $B(y)$.

Lemma 329. (i) *The fifteen tree links of a block form a spanning tree of its sixteen sites. Each block carries* 49 *active links: the* 17 *active links inside it, with $o_\mu = 0$, and the* 32 *links with $o_\mu = 1$, which join $B(y)$ to $B(y+e_\mu)$.*

(ii) *If dh vanishes on the tree links, h is constant on every block, and $V_{y,\mu}(\varphi + dh) = V_{y,\mu}(\varphi) + h_{B(y)} - h_{B(y+e_\mu)}$.*

(iii) *With $S^\mu_y = \sum_{x \in B(y),\, o_\mu = 0} \varphi_{(x,\mu)}$,*

$$V_{y,\mu}(\varphi) = \frac{1}{8} \sum_{x \in B(y),\, o_\mu = 1} \varphi_{(x,\mu)} + \frac{1}{16}\big(S^\mu_y + S^\mu_{y+e_\mu}\big).$$

Proof. (i) A tree link of direction μ has $o_\mu = 0$ and stays in its block; there are 2^μ of them for each μ, fifteen in all. From any site, the tree links of direction 3 lead to $o_3 = 0$, then those of direction 2 to $o_2 = 0$, and so on to the corner, so the tree links connect the sixteen sites, and fifteen edges connecting sixteen vertices form a tree. Of the 64 links with base in the block, the 32 with $o_\mu = 0$ stay inside it. (ii) h is constant along the edges of a spanning tree. The path of length two from $x \in B(y)$ ends at $x + 2e_\mu \in B(y+e_\mu)$, and the sum of dh along it is $h(x) - h(x+2e_\mu)$. (iii) The path from x consists of (x,μ) and $(x+e_\mu,\mu)$; a link with $o_\mu = 1$ is the second link of one path and the first of another, and the links of $S^\mu_{y+e_\mu}$ are the second links of the paths that start with $o_\mu = 1$. □

By (ii) the residual gauge freedom is the coarse gauge group, which acts on the block averages as on the links of the coarse lattice. Let W be a set of blocks and let Φ_W be the space of finitely supported real φ on the active links with base in $\bigcup W$, with $V_{y,\mu}(\varphi) = 0$ for every $y \in W$ and every μ. Write $Q(\varphi) = \sum_p (d\varphi)^2_p$ over all plaquettes, φ being zero off its support.

Theorem 330. *For every set of blocks W and every $\varphi \in \Phi_W$,*

$$\sum_e \varphi_e^2 \le 52\, Q(\varphi).$$

Proof. Inside a block. Let $e = (x,\mu)$ be an active link with $o_\mu = 0$, and let $\nu_1 > \cdots > \nu_k$ be the directions above μ in which o equals one; $1 \le k \le 3$. Put $x_0 = x$ and $x_i = x_{i-1} - e_{\nu_i}$. The plaquette p_i at x_i in the plane $\{\mu,\nu_i\}$ has the links (x_i,μ), (x_{i-1},μ) and the tree links (x_i,ν_i), (x_i+e_μ,ν_i), whose offsets vanish in ν_i and above it; and (x_k,μ) is a tree link. Hence $\varphi_e = \pm(d\varphi)_{p_1} \pm \cdots \pm (d\varphi)_{p_k}$ and $\varphi_e^2 \le k \sum_i (d\varphi)^2_{p_i}$. The plaquette at x' in the plane $\{\mu,\nu\}$ is used only by the links $(x' + e_\nu + \sum_{\rho \in R} e_\rho, \mu)$ with $R \subset \{\nu+1,\dots,3\}$, each with $k = c + 1 + |R|$, where c is the number of ones of the offset of x' strictly between μ and ν. Its total weight $2^{3-\nu}(c+1) + (3-\nu)2^{2-\nu}$ is at most 8, the value at $\mu = 0$, $\nu = 1$. So the sum I of φ_e^2 over the active links inside the blocks is at most $8Q_{\text{in}}$, where Q_{in} is the sum over the plaquettes inside a block.

Between blocks. Fix $y \in W$ and μ, and label the eight links (x,μ), $x \in B(y)$, $o_\mu = 1$, by the transverse offset $a \in \{0,1\}^3$. If a and a' differ in the coordinate ν, with $o_\nu = 0$ at a, the plaquette at x in the plane $\{\mu,\nu\}$ contains these two links, (x,ν) in $B(y)$ and $(x+e_\mu,\nu)$ in $B(y+e_\mu)$, the last two inside their blocks. So $\varphi_a - \varphi_{a'} = \pm(d\varphi)_p \pm \psi \pm \psi'$ with ψ, ψ' inside blocks, and these twelve face plaquettes correspond to the twelve edges of the cube $\{0,1\}^3$. By Lemma 329(iii) and $V_{y,\mu}(\varphi) = 0$, the mean of the φ_a is $m = -(S^\mu_y + S^\mu_{y+e_\mu})/16$, so $8m^2 \le \frac{1}{2}$ times the sum of the squares of the sixteen links of S^μ_y and $S^\mu_{y+e_\mu}$. The Laplacian of the graph $\{0,1\}^3$ has smallest nonzero eigenvalue 2, so

$$\begin{aligned}\sum_a \varphi_a^2 &= 8m^2 + \sum_a (\varphi_a - m)^2 \\ &\le 8m^2 + \frac{3}{2} \sum_{\text{edges}} \big((d\varphi)_p^2 + \psi^2 + \psi'^2\big).\end{aligned}$$

Summed over $y \in W$ and μ, a link (z,ν) inside a block occurs as ψ or ψ' once for each $\mu \ne \nu$ (as ψ if $o_\mu = 1$, as ψ' of the face of the block below if $o_\mu = 0$) and in two of the sums S^ν, so it receives weight at most $3 \cdot \frac{3}{2} + 2 \cdot \frac{1}{2} = \frac{11}{2}$. The face plaquettes are distinct and lie in no block, so the links between blocks contribute at most $\frac{3}{2} Q_{\text{face}} + \frac{11}{2} I$, and

$$\sum_e \varphi_e^2 \le \frac{13}{2} I + \frac{3}{2} Q_{\text{face}} \le 52\, Q_{\text{in}} + \frac{3}{2} Q_{\text{face}} \le 52\, Q(\varphi). \quad \square$$

The constant 52 is not sharp. A certificate of one of us (T. Romanov) lowers it to within one part in a thousand of the bulk value; it rests on a computation, and no result of the paper uses it.

Remark 331 (a computation, used by no theorem). Let $c = \frac{537}{5000}$, and let $S(y) = B(y) \cup B(y+e_0) \cup \cdots \cup B(y+e_3)$, whose active links number $5 \cdot 49 = 245$. The certificate of T. Romanov consists of 196 rational numbers $l_{\mu,b}$, $\mu = 0,\dots,3$, indexed by the 49 active links b of a block, and a rational symmetric matrix G of size 245 on the active links of $S(0)$. With $L_{y,\mu}(\varphi) = \sum_b l_{\mu,b} \varphi_{(y,b)}$, where (y,b) is the link b of $B(y)$, and G_y the translate of G to $S(y)$, these are stated to satisfy the identity of quadratic forms in finitely supported φ

$$Q(\varphi) - c \sum_e \varphi_e^2 - \sum_{y,\mu} V_{y,\mu}(\varphi) L_{y,\mu}(\varphi) = \sum_y G_y(\varphi|_{S(y)}), \quad (32)$$

and G is positive definite. Both sides of (32) are invariant under translations by $2\mathbb{Z}^4$, so the identity amounts to finitely many equalities of coefficients, one for each pair of active links of $B(0)$ and of a block near it. Granted these, take $\varphi \in \Phi_W$. If $y \in W$ then $V_{y,\mu}(\varphi) = 0$; if $y \notin W$ then φ vanishes on the links with base in $B(y)$, so $L_{y,\mu}(\varphi) = 0$. Every term of the middle sum vanishes, and

$$\sum_e \varphi_e^2 \le \frac{5000}{537}\, Q(\varphi) = 9.31098\ldots\, Q(\varphi)$$

for every set of blocks W and every $\varphi \in \Phi_W$. On a periodic lattice of even side both sides of (32) are the periodisations of the same forms, so the same holds there.

The certificate was found by semidefinite programming and rounded to rationals: the $l_{\mu,b}$ have denominators dividing 2^{24} and the entries of G the denominator $2^{30}\cdot 5^5$. The coefficient identity of (32) and the positivity of all 245 leading principal minors of the integer matrix $2^{30}5^5 G$ are checked by computer in exact arithmetic, not by hand: by the program of T. Romanov and independently by our program in the archive, which rebuilds the four forms from Definition 328 alone. With W the set of all blocks the best constant is the inverse of the Bloch value 0.1074878 quoted below, that is 9.3034, so $\frac{5000}{537}$ is within 0.09 per cent of it; the semidefinite optimum for this window and this class of multipliers is $c^* = 0.107488$. These floating-point values are not used. The results of the paper are stated with 52, which is proved by hand (Theorem 330); with the computation, those that use only Theorem 330 would hold with 52 replaced by $\frac{5000}{537}$. No result of the paper uses this remark.

With W all blocks the bound extends to ℓ^2, so the block-constrained quadratic form on $\mathbb{Z}^4$ has a spectral gap at least $\frac{1}{52}$; it removes in particular the four flat connections that are zero modes of the unconstrained form. In the Bloch decomposition the exact gap is $0.10748776948684\ldots$, attained at the momentum $(0,\pi,\pi,\pi)$, a floating-point value computed in [82] and reproduced in the archive; the theorem does not use it. The cube conditions now take the following form. Let n be even, V the active links with base in $[0,n)^4$, W its blocks, and for an exterior η and values $v = (v_{y,\mu})_{y\in W}$ let $\gamma_V^{\mathrm{c}}(\cdot \mid \eta, v)$ be the law on $\mathbb{R}^V$ with density proportional to $\exp(-\frac{b}{2}\sum_{p\cap V\neq\emptyset}(d\varphi)_p^2)$ on the affine set $\{V_{y,\mu}(\varphi) = v_{y,\mu}\}$, with $\varphi = \eta$ off V. Define $C^{\mathrm{c}}_{V,j}$, K_c^{c} and $D_{\mathrm{c}}(n,\alpha)$ as $C_{V,j}$, K_c and $D(n,\alpha)$, with v held fixed and the 49 classes of active links modulo two.

Lemma 332. *Let $H \geq 0$ be symmetric on $\mathbb{R}^m$ with $\|H\| \leq h$, let $B : \mathbb{R}^m \to \mathbb{R}^k$ with $BB^{\top} \geq \beta > 0$ and $\langle x, Hx\rangle \geq a|x|^2$ on $\ker B$, and $s > 0$. Then $K = \left(\begin{smallmatrix} H & sB^{\top} \\ sB & 0 \end{smallmatrix}\right)$ is invertible, and the solution of $K(x,y) = (f,g)$ has $|x| \leq (|f| + h|x_1|)/a + |x_1|$ with $|x_1| \leq |g|/(s\sqrt{\beta})$, and $|y| \leq (|f| + h|x|)/(s\sqrt{\beta})$.*

Proof. Put $x_1 = B^{\top}(BB^{\top})^{-1} g/s$, so $sBx_1 = g$ and $|x_1| \leq |g|/(s\sqrt{\beta})$. On $\ker B$ the compression of H is at least a, so there is a unique $x_0 \in \ker B$ with $Hx_0 - f + Hx_1 \perp \ker B$, and $a|x_0| \leq |f| + h|x_1|$. Then $f - H(x_0 + x_1)$ lies in $(\ker B)^{\perp} = \operatorname{ran} B^{\top}$, and $y = (BB^{\top})^{-1}B(f - Hx)/s$ solves $sB^{\top}y = f - Hx$. For $f = g = 0$ the construction gives zero, so K is injective. □

Lemma 333 (Combes–Thomas [39])**.** *Let K be an invertible symmetric matrix indexed by a finite set X, with $\|K^{-1}\| \leq M_0$ and $\sum_b |K_{ab}| \leq S$ for every a, and let $d : X \to \mathbb{R}$ satisfy $|d(a) - d(b)| \leq 1$ whenever $K_{ab} \neq 0$. If $(e^{\lambda} - 1)SM_0 \leq \frac{1}{2}$, then $\|e^{\lambda d} K^{-1} e^{-\lambda d}\| \leq 2M_0$.*

Proof. $e^{\lambda d} K e^{-\lambda d} = K + E$ with $|E_{ab}| \leq (e^{\lambda} - 1)|K_{ab}|$, so $\|E\| \leq (e^{\lambda} - 1)S$ by the Schur test, $\|K^{-1}E\| \leq \frac{1}{2}$, and $(K + E)^{-1} = (1 + K^{-1}E)^{-1}K^{-1}$ has norm at most $2M_0$. □

Theorem 334. *For every $b > 0$, every even n and every exterior link j:*

(i) *$\gamma_V^{\mathrm{c}}(\cdot \mid \eta, v)$ is a Gaussian law whose covariance depends on neither η nor v and whose mean is affine in (η, v), and $C^{\mathrm{c}}_{V,j} = \|u_j\|_1$, where $u_j \in \mathbb{R}^V$ is the change of the mean when η_j increases by one; u_j does not depend on b;*

(ii) *$\sum_{e\in V} e^{2\lambda_0 d(e)} u_j(e)^2 \leq 3500^2$ and $\|u_j\|_1 \leq 10^{14}$, where $\lambda_0 = 2.4\cdot 10^{-5}$ and $d(e)$ is the ℓ^{∞} distance between the midpoints of e and j;*

(iii) *$D_{\mathrm{c}}(n,\alpha)$ holds for every $\alpha \in (0,1)$ and every even $n \geq 2\cdot 10^{15}/\alpha$, at every $b > 0$.*

Proof. (i) Let H be the matrix of $\sum_{p\cap V\neq\emptyset}(d\varphi)_p^2$ on $\mathbb{R}^V$ and B that of the averages $V_{y,\mu}$, $y \in W$, restricted to V. By Theorem 330 with this W, $H \geq \frac{1}{52}$ on $\ker B$, so the density is that of a non-degenerate Gaussian on the affine set, its covariance is b^{-1} times the inverse of H on $\ker B$, and its mean m minimises the quadratic form on the affine set. The mean is affine in (η, v), and two exteriors that differ at j give translates by u_j, which minimises $Q(u + \delta_j)$ subject to $Bu = g$, g being minus the coefficients of j in the averages. The translation argument of Proposition 187 gives $C^{\mathrm{c}}_{V,j} = \|u_j\|_1$.

(ii) The pair (u_j, y), with y the Lagrange multipliers, solves $K(u_j, y) = (f, sg)$ with K as in Lemma 332, $s = 96$ and $f = -\sum_{p\ni j}(d\delta_j)_p\, d_p|_V$, where $d_p|_V$ is the row of the plaquette p restricted to V. Each link lies in six plaquettes of four links, so the row sums of $|H|$ are at most 24 and $h \leq 24$; $|f| \leq 6\sqrt{3}$; and $|g| \leq \frac{1}{16}$, since an exterior link enters at most one average of W, with weight $\frac{1}{16}$. By Lemma 329(iii) each average has squared norm at least $8\cdot\frac{1}{64} = \frac{1}{8}$, two averages share links only if they are (y,μ) and $(y \pm e_{\mu}, \mu)$, and then their inner product is at most $8\cdot\frac{1}{256} = \frac{1}{32}$; so $BB^{\top} \geq \frac{1}{16}$ by Gershgorin's theorem. With $a = \frac{1}{52}$, Lemma 332 gives $\|K^{-1}\| \leq M_0 = 105$. The rows of K have absolute sums at most $S = \max(24 + \frac{96}{8}, 96\cdot 2) = 192$, because an average has eight coefficients $\frac{1}{8}$ and sixteen of $\frac{1}{16}$ and a link enters averages with total weight at most $\frac{1}{8}$. Index the rows of B by the centre $2y + \frac{1}{2}(1,1,1,1) + e_{\mu}$ of the two blocks of the average and put d equal to the ℓ^{∞} distance to the midpoint of j. Links of one plaquette, and an average and its links, are within distance one, so Lemma 333 applies with $\lambda = \log(1 + 1/40320) > \lambda_0$, and

$$\sum_e e^{2\lambda d(e)} u_j(e)^2 \leq \big(2M_0 e^{\lambda}(|f| + s|g|)\big)^2 \leq 3500^2,$$

since f and g live at distance at most one from j. At most $4(64k^3 + 16k)$ links have their base at ℓ^{∞} distance $k \geq 1$ from the base of j, and their midpoints are at distance at least $k - 1$, so Cauchy–Schwarz gives

$$\|u_j\|_1^2 \leq 3500^2 \cdot 4e^{2\lambda}\Big(1 + \sum_{k\geq 1}(64k^3 + 16k)e^{-2\lambda k}\Big) \leq (10^{14})^2.$$

(iii) $C^{\mathrm{c}}_{V_x,j} = 0$ unless j shares a plaquette with V_x or enters an average of V_x, which forces the base z of j into $x + [-1, n+1]^4$, while $z \notin x + [0,n)^4$ when $j \notin V_x$. For each coordinate at most

$\frac{n}{2}+2$ even x_i satisfy the first condition and exactly $\frac{n}{2}$ the second, so

$$K_c^{\mathrm{c}} \le \Big(\Big(\frac{n}{2}+2\Big)^4 - \Big(\frac{n}{2}\Big)^4\Big)10^{14} \le \frac{17}{n}\,10^{14}N_n$$

for $n \ge 10^4$, which is at most αN_n when $n \ge 2\cdot 10^{15}/\alpha$. □

The constants are crude. The computations of [82] give $C^{\mathrm{c}}_{V,j} \le 12.48$ at $n=4$, and an exact face expansion $K_c^{\mathrm{c}}/N_n = a_c/n + O(n^{-2})$ with $\max_c a_c = 61.18$, so that $D_{\mathrm{c}}(n, 0.95)$ first holds at $n = 64$; they are floating-point values, reproduced in the archive for $n \le 4$, and the theorem does not use them. The coefficients are large because the constrained responses obey no maximum principle: an exterior link changes the curvature of up to six plaquettes, and keeping the curvature small moves a dozen links inside the cube with alternating signs. Re-weighting the classes does not help much: if $\sum_{j\in c\setminus V}\sum_{c'} w_{c'}C^{c'}_{V,j} \le \alpha N_n w_c$ for positive weights w, where $C^{c'}_{V,j}$ is the part of $\|u_j\|_1$ on the links of the class c', then αN_n is at least the spectral radius of the non-negative matrix of these sums, hence at least $\min_c K_c^{\mathrm{c}}$, by the Collatz–Wielandt inequality [82].

The block averages can be integrated out one scale at a time. For a field v on the coarse links put $Q_1(v) = \sum_P (dv)_P^2$ over the plaquettes P of the coarse lattice, and write $V(\varphi) = (V_{y,\mu}(\varphi))_{y,\mu}$. The averages of averages are formed in the same way on the coarse lattice, and $V^{(k)}$, Q_k denote the k-fold averages and the plaquette form of the lattice of spacing 2^k.

Proposition 335. *Let φ be a finitely supported field on $\mathbb{Z}^4$ that vanishes on the tree links.*

(i) *For the coarse plaquette P at y in the plane $\{\mu,\nu\}$, $16\,(dV(\varphi))_P$ is the sum over $x \in B(y)$ of the curvatures of the squares of side two spanned by x, $x+2e_\mu$, $x+2e_\nu$, and every plaquette of $\mathbb{Z}^4$ lies in exactly four of these squares. Hence $Q_1(V(\varphi)) \le Q(\varphi)$, and $Q_k(V^{(k)}(\varphi)) \le Q(\varphi)$ for every $k \ge 1$.*

(ii) *For every finitely supported v on the coarse links, the field $\varphi^*(v)$ equal to $v_{y,\mu}$ on the eight links (x,μ), $x \in B(y)$, $o_\mu = 1$, and to zero on the other links, vanishes on the tree links, has $V(\varphi^*(v)) = v$ and $Q(\varphi^*(v)) = 4\,Q_1(v)$.*

(iii) *The effective form $S_1(v) = \inf\{Q(\varphi) : V(\varphi) = v\}$ satisfies $Q_1 \le S_1 \le 4\,Q_1$, and $S_k(w) = \inf\{Q(\varphi) : V^{(k)}(\varphi) = w\} \ge Q_k(w)$ for every k, the infima over finitely supported fields that vanish on the tree links.*

(iv) *For every k, every set W of blocks of the lattice of spacing 2^k and every w in the space Φ_W of that lattice, $\sum_e w_e^2 \le 52\,S_k(w)$.*

Proof. (i) The path of length two from $x \in B(y)$ in the direction μ ends at $x+2e_\mu \in B(y+e_\mu)$, at the same offset, and so on around the square; so the four averages of $(dV)_P$ pair the sixteen paths into sixteen closed squares, and the sum of φ around a square is the sum of the curvatures of its four plaquettes. A plaquette at z in the plane $\{\mu,\nu\}$ lies in the square from x exactly when $z - x \in \{0,1\}e_\mu + \{0,1\}e_\nu$, which gives four points x, each in one block. By Cauchy–Schwarz, $(dV)_P^2 \le \frac{1}{16}\sum_x \ell_x^2 \le \frac{1}{4}\sum_x \sum_{p\subset\square_x} (d\varphi)_p^2$, where ℓ_x is the curvature of the square from x; summing over P gives $Q_1(V(\varphi)) \le Q(\varphi)$. The argument uses neither the tree links nor the spacing, so it applies on every lattice, and induction gives the bound for $V^{(k)}$.

(ii) The tree links have $o_\mu = 0$. By Lemma 329(iii) the average of φ^* is $\frac{1}{8}\cdot 8 v_{y,\mu}$. A plaquette inside a block has only links with $o_\mu = 0$ in its direction μ, and so vanishes. A plaquette at x in the plane $\{\mu,\nu\}$ with $o_\mu = 1$, $o_\nu = 0$ has the two links (x,μ), $(x+e_\nu,\mu)$ of the same block, both equal to $v_{y,\mu}$, and two links with offset zero in the direction ν; its curvature vanishes, and so does that of the plaquettes with $o_\mu = 0$, $o_\nu = 1$. If $o_\mu = o_\nu = 1$, its four links carry $v_{y,\mu}$, $v_{y+e_\mu,\nu}$, $v_{y+e_\nu,\mu}$, $v_{y,\nu}$, so its curvature is $(dv)_P$, and there are four such plaquettes for each P.

(iii) The lower bound is (i) and the upper bound is (ii); for S_k apply (i) to the k-fold averages.

(iv) Theorem 330 holds on every lattice, so $\sum_e w_e^2 \le 52\,Q_k(w) \le 52\,S_k(w)$ by (iii). □

In the Bloch decomposition the eigenvalues of S_1 relative to Q_1 lie in $[1,4]$ by (iii). The value four is attained at the momentum $(\pi,\pi,0,0)$, and on the torus of side L the least eigenvalue is at most t_L^{-2}, which tends to one, by a long-wavelength field (Theorem 644(i)). So at this order the averages carry the coarse quadratic form with no loss at long wavelengths; the coupling does not run here, and its running is the order-g^2 effect of Remark 345(c). Part (iv) is a gap for the averages at every scale, with the constant of Theorem 330; it does not give the localisation of Theorem 334(ii) for S_k, since S_k is not a nearest-neighbour form.

The effective forms are equivalent to the coarse plaquette forms uniformly in the scale. Write $L = 2^k$ and $B_k(y) = Ly + \{0,\dots,L-1\}^4$.

Lemma 336. (i) *For every $k \ge 1$, every coarse link (y,μ) and every field φ,*

$$V^{(k)}_{y,\mu}(\varphi) = \frac{1}{L^4}\sum_{x\in B_k(y)}\sum_{t=0}^{L-1}\varphi_{(x+te_\mu,\mu)}.$$

(ii) *For a function h on sites, $V^{(k)}_{y,\mu}(dh) = \bar h(y) - \bar h(y+e_\mu)$, where $\bar h(y)$ is the mean of h over $B_k(y)$.*

(iii) *$S_k(w)$ equals the infimum of $Q(\varphi)$ over all finitely supported fields with $V^{(k)}(\varphi) = w$, whether or not they vanish on the tree links.*

Proof. (i) For $k=1$ this is the definition. If it holds for k, then $V^{(k)}_{y',\mu} + V^{(k)}_{y'+e_\mu,\mu}$ is L^{-4} times the sum over $x \in B_k(y')$ of the sums of φ along the straight paths of length $2L$ from x, because $B_k(y'+e_\mu) = B_k(y') + Le_\mu$. The blocks $B_k(y')$, $y' \in 2y + \{0,1\}^4$, tile $B_{k+1}(y)$, and averaging over y' with weight $\frac{1}{16}$ gives (i) for $k+1$. (ii) The sum of dh along the path of length L from x is $h(x) - h(x+Le_\mu)$; average over $x \in B_k(y)$. (iii) Let φ be finitely supported. On each block $B(y')$ of side two the

tree links form a spanning tree (Lemma 329(i)), so there is h_0 on $B(y')$ with $(dh_0)_e = -\varphi_e$ on its tree links, unique up to a constant, which we take to be zero where φ vanishes on the tree links of $B(y')$. Each $B_k(y)$ is a union of blocks of side two, since $k \ge 1$; adding a constant to h_0 on one of them makes the mean over $B_k(y)$ vanish, and this changes h_0 on finitely many blocks. The resulting h is finitely supported, $\varphi + dh$ vanishes on the tree links, $Q(\varphi+dh) = Q(\varphi)$, and $V^{(k)}(dh) = 0$ by (ii). □

The extension uses the lowest-order Whitney forms of the coarse cubical complex whose vertices are the centres

$$c_z = Lz + \tfrac{L-1}{2}(1,1,1,1), \qquad z \in \mathbb{Z}^4,$$

of the blocks $B_k(z)$, rather than the coarse sites. Let $\lambda(t) = \max(0, 1-|t|)$, let χ^μ_z be the indicator of $\{\xi \in \mathbb{R}^4 : c_{z,\mu} \le \xi_\mu < c_{z,\mu} + L\}$, and $\lambda^\nu_z(\xi) = \lambda((\xi_\nu - c_{z,\nu})/L)$. For a square-summable field u on the coarse links put

$$\omega_\mu(\xi) = \frac{1}{L}\sum_z u_{(z,\mu)}\, \chi^\mu_z(\xi) \prod_{\nu \ne \mu} \lambda^\nu_z(\xi),$$

$$(\Pi u)_{(x,\mu)} = \int_0^1 \omega_\mu(x + te_\mu)\, dt.$$

Fig. 26 shows the construction along one direction.

Lemma 337. *Let $k \ge 1$, u a square-summable field on the coarse links, and let T act on each component of u by $T = \tau_1\tau_2\tau_3\tau_4$, where $(\tau_\nu f)(z) = \frac{3}{4}f(z) + \frac{1}{8}\big(f(z-e_\nu) + f(z+e_\nu)\big)$.*

(i) *For the plaquette p at x in the plane $\{\mu,\nu\}$,*

$$(d\Pi u)_p = \int_0^1\int_0^1 \Omega_{\mu\nu}(x + se_\mu + te_\nu)\, ds\, dt,$$

$$\Omega_{\mu\nu} = \frac{1}{L^2}\sum_z (du)_{z,\mu\nu}\, \chi^\mu_z \chi^\nu_z \prod_{\rho \ne \mu,\nu} \lambda^\rho_z,$$

where $(du)_{z,\mu\nu}$ is the curvature of u at the coarse plaquette at z in the plane $\{\mu,\nu\}$. In particular $\Pi(db)$ is a gradient.

(ii) $Q(\Pi u) \le Q_k(u)$.

(iii) $V^{(k)}(\Pi u) = Tu$.

Proof. (i) The function ω_μ is continuous and piecewise linear in every coordinate ξ_ν, $\nu \ne \mu$, and ω_ν in ξ_μ. Hence

$$(d\Pi u)_p = \int_0^1 \big(\omega_\mu(x+te_\mu) - \omega_\mu(x+e_\nu+te_\mu)\big)dt + \int_0^1 \big(\omega_\nu(x+e_\mu+te_\nu) - \omega_\nu(x+te_\nu)\big)dt$$

is the integral over the unit square of $\partial_\mu\omega_\nu - \partial_\nu\omega_\mu$, which exists almost everywhere. Since $\partial_\mu \lambda^\mu_z = \frac{1}{L}(\chi^\mu_{z-e_\mu} - \chi^\mu_z)$ and χ^ν_z and λ^ρ_z, $\rho \ne \mu$, do not change when z is shifted by e_μ, a shift of the summation index gives

$$\partial_\mu\omega_\nu = \frac{1}{L^2}\sum_z \big(u_{(z+e_\mu,\nu)} - u_{(z,\nu)}\big)\chi^\mu_z\chi^\nu_z \prod_{\rho\ne\mu,\nu}\lambda^\rho_z,$$

and the same with μ,ν exchanged. The difference is $\Omega_{\mu\nu}$, with $(du)_{z,\mu\nu} = u_{(z,\mu)} + u_{(z+e_\mu,\nu)} - u_{(z+e_\nu,\mu)} - u_{(z,\nu)}$, the sign convention of the fine plaquettes. If $u = db$ then $\Omega = 0$, every curvature of Πu vanishes, and Πu is a gradient.

(ii) By Cauchy–Schwarz, $(d\Pi u)^2_p$ is at most the integral of $\Omega^2_{\mu\nu}$ over the unit square of p. At a point ξ exactly one pair (z_μ, z_ν) has $\chi^\mu_z\chi^\nu_z = 1$, and the hat functions form a partition of unity, $\sum_{z_\rho,z_\sigma}\lambda^\rho_z\lambda^\sigma_z = 1$ for $\{\rho,\sigma\}$ the other two directions; so $\Omega^2_{\mu\nu} \le L^{-4}\sum_{z_\rho,z_\sigma}\lambda^\rho_z\lambda^\sigma_z (du)^2_{z,\mu\nu}$. Summing over the plaquettes of the plane $\{\mu,\nu\}$ integrates over ξ_μ,ξ_ν, which gives the factor L^2 for each pair (z_μ,z_ν), and sums over the integers x_ρ, x_σ. Since $c_{z,\rho} - x_\rho$ lies in $\mathbb{Z} + \frac12$, $\sum_{x_\rho\in\mathbb{Z}}\lambda^\rho_z = 2\sum_{j=0}^{L-1}\big(1 - \frac{j+1/2}{L}\big) = L$, and the same for σ. So the plaquettes of the plane $\{\mu,\nu\}$ contribute at most $L^{-4}\cdot L^2\cdot L^2 \sum_z (du)^2_{z,\mu\nu}$.

(iii) By Lemma 336(i), $V^{(k)}_{y,\mu}(\Pi u)$ is the mean over $x = Ly + s$, $s \in \{0,\dots,L-1\}^4$, of $\int_0^L \omega_\mu(x+te_\mu)\,dt$. The term of $u_{(z,\mu)}$ in it is a product of a factor of the coordinate μ and one of each $\nu \ne \mu$, and the mean over s is the product of the means over the single coordinates. Put $\delta = s_\mu - \frac{L-1}{2}$. The interval $[x_\mu, x_\mu + L)$ meets $[c_{z,\mu}, c_{z,\mu}+L)$ in length $L - |\delta|$ if $z_\mu = y_\mu$, in $\max(0,-\delta)$ if $z_\mu = y_\mu - 1$, in $\max(0,\delta)$ if $z_\mu = y_\mu + 1$, and not otherwise. As L is even, $\sum_s |\delta| = 2\sum_{j=0}^{L/2-1}(j+\frac12) = \frac{L^2}{4}$, so the means of these lengths divided by L are $\frac34$, $\frac18$, $\frac18$. For $\nu \ne \mu$ put $\delta = s_\nu - \frac{L-1}{2}$; then $\lambda^\nu_z(x)$ is $1 - |\delta|/L$ if $z_\nu = y_\nu$, $\max(0, \mp\delta)/L$ if $z_\nu = y_\nu \mp 1$, and zero otherwise, with the same means. The product of the four means is the coefficient of $u_{(z,\mu)}$ in $(Tu)_{(y,\mu)}$. □

Theorem 338. *For every $k \ge 1$ and every finitely supported field w on the links of the lattice of spacing 2^k,*

$$Q_k(w) \le S_k(w) \le 256\, Q_k(w).$$

Proof. The lower bound is Proposition 335(iii). The operator T of Lemma 337 is a convolution, the same on every component, so it commutes with d; its symbol is $\prod_\nu(\frac34 + \frac14\cos\theta_\nu) \ge \frac{1}{16}$, so it is invertible on square-summable fields with $\|T^{-1}\| \le 16$. Let $u = T^{-1}w$. By Lemma 337, $V^{(k)}(\Pi u) = w$ and

$$Q(\Pi u) \le Q_k(u) = \|T^{-1}dw\|^2 \le 256\, Q_k(w).$$

It remains to replace Πu by a finitely supported field. Let u_R be u on the coarse links with base in $[-R,R]^4$ and zero elsewhere, and $e_R = w - Tu_R = T(u - u_R)$, which is finitely supported and tends to zero. Let ψ_R equal $e_R(y,\mu)$ on the links (x,μ) with $x \in B_k(y)$ and $x_\mu \equiv L-1$ modulo L, and zero elsewhere. Each straight path of length L from a site of $B_k(y)$ in the direction μ contains exactly one such link of $B_k(y)$, so $V^{(k)}(\psi_R) = e_R$ by Lemma 336(i), and $Q(\psi_R) \le 24\|\psi_R\|^2 = 24L^3\|e_R\|^2$, since a link lies in six plaquettes of four links. The field $\Pi u_R + \psi_R$ is finitely supported, its k-fold averages are w, and $Q(\Pi u_R + \psi_R)^{1/2} \le Q_k(u_R)^{1/2} + (24L^3)^{1/2}\|e_R\|$, which tends to $Q_k(u)^{1/2}$. Lemma 336(iii) gives $S_k(w) \le Q_k(u) \le 256\, Q_k(w)$. □

So at every scale the averages carry the coarse plaquette form up to a factor at most 256, independent of k. The construction

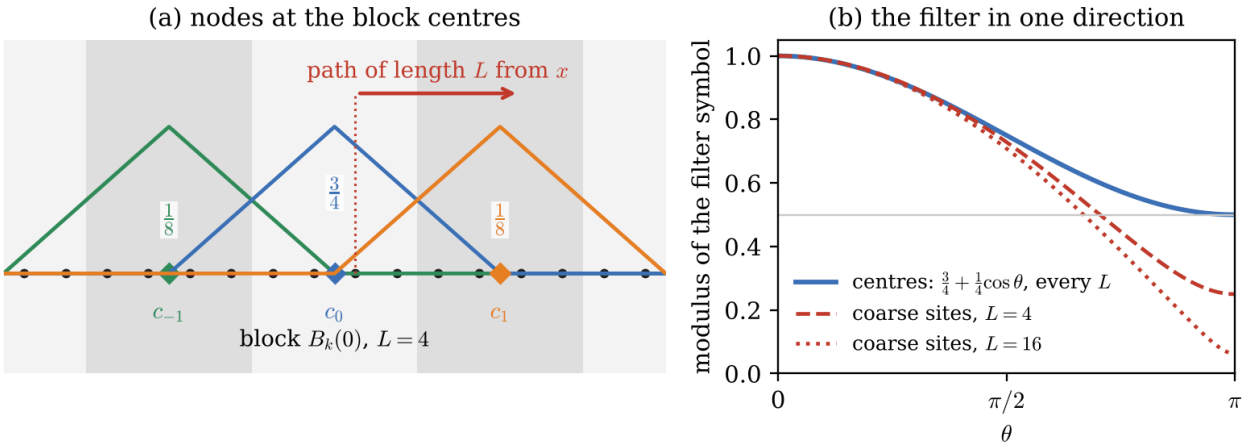


FIG. 26. The extension of Theorem 338 along one direction. (a) A block of side $L=4$: the nodes are the block centres c_z, the hat functions λ_z interpolate between them, and the straight path of length L from a site x meets the segments of three centres; averaged over the block, their weights are $\frac{1}{8}$, $\frac{3}{4}$, $\frac{1}{8}$ at every scale. (b) The symbol of this filter stays at least $\frac{1}{2}$, while the filter obtained with the coarse sites as nodes has symbol of modulus $1/L$ at $\theta=\pi$.

inverts the averages by a fixed convolution: the filter $(\frac{1}{8},\frac{3}{4},\frac{1}{8})$ does not depend on the scale because the nodes are the block centres, and its symbol is at least $\frac{1}{2}$. With the coarse sites as nodes the corresponding filter is $\frac{L+1}{2L}\delta_0+\frac{L-1}{2L}\delta_1$, whose symbol has modulus $1/L$ at $\theta=\pi$ (Fig. 26(b)). In floating point, on a grid of 12^4 momenta, the largest ratio of S_k to Q_k on curvatures is 3.9, 7.8 and 10.2 for $k=1,2,3$, and the kernel of S_k decreases by the factors 0.17, 0.23 and 0.25 per coarse spacing; these values are reproduced in the archive. The theorem gives no decay. The decay comes from an exact formula for S_k in Fourier variables: in a Feynman gauge for the fine field the effective form is diagonal in the six planes, and the pole of the fine propagator at zero momentum cancels between numerator and denominator.

For $t\in\mathbb{C}$ put $D_L(t)=\frac{1}{L}\sum_{j=0}^{L-1}e^{ijt}$, and for $\vartheta\in\mathbb{C}^4$ put $\Psi(\vartheta)=\prod_\nu D_L(\vartheta_\nu)$, $E_\mu(\vartheta)=e^{i\vartheta_\mu}-1$ and $\varepsilon(\vartheta)=\sum_\nu(2-2\cos\vartheta_\nu)$. For an entire function f let $f^\sharp(\vartheta)=\overline{f(\bar\vartheta)}$, so that $ff^\sharp=|f|^2$ on real arguments. For $\theta\in\mathbb{C}^4$ and $n\in\{0,\dots,L-1\}^4$ let $\theta^{(n)}=(\theta+2\pi n)/L$; every function below is 2π-periodic in each coordinate of $\theta^{(n)}$, and we take the real parts of these coordinates in $[-\pi,\pi]$. Write $\Psi_n=(\Psi\Psi^\sharp)(\theta^{(n)})$, $D_{n,\sigma}=(D_LD_L^\sharp)(\theta_\sigma^{(n)})$, and put

$$A_\sigma=\Psi_0D_{0,\sigma},\qquad \hat\varepsilon=L^2\varepsilon(\theta^{(0)}),\qquad \tilde a_\sigma=A_\sigma+\hat\varepsilon r_\sigma,$$
$$r_\sigma=\sum_{n\neq0}\frac{\Psi_nD_{n,\sigma}}{L^2\varepsilon(\theta^{(n)})},$$

$$Y=\Psi_0^3\prod_\tau D_{0,\tau}+\sum_\sigma E_\sigma E_\sigma^\sharp\sum_{\emptyset\neq J\subset\{1,2,3,4\}\setminus\{\sigma\}}\hat\varepsilon^{|J|-1}\prod_{\tau\in J}r_\tau\prod_{\tau\notin J\cup\{\sigma\}}A_\tau,\tag{33}$$

all functions of θ, and for a plane $P=\{\lambda,\rho\}$

$$K_P=\frac{\prod_{\sigma\notin P}\tilde a_\sigma}{Y}.$$

For $L=1$ these are $\tilde a_\sigma=Y=K_P=1$. For a coarse field w let $\hat w(\theta)=\sum_y e^{-i\theta\cdot y}w_y$, and let $F_P=(dw)_P$ be its curvature in the plane P, a function of the coarse sites.

Lemma 339. *Let $\gamma_1,\dots,\gamma_4>0$, $\Gamma=\mathrm{diag}(\gamma_\mu)$ and $0\neq E\in\mathbb{C}^4$, and for $P=\{\lambda,\rho\}$, $\lambda<\rho$, let c_P be the row vector with $c_Pv=E_\lambda v_\rho-E_\rho v_\lambda$. Then*

$$\Gamma-\frac{\Gamma EE^*\Gamma}{E^*\Gamma E}=\frac{1}{E^*\Gamma E}\sum_P\gamma_\lambda\gamma_\rho\,c_P^*c_P.$$

Proof. Both sides are Hermitian. The diagonal entry (μ,μ) of the right side is $\sum_{\lambda\neq\mu}\gamma_\mu\gamma_\lambda|E_\lambda|^2/E^*\Gamma E$, which is $\gamma_\mu-\gamma_\mu^2|E_\mu|^2/E^*\Gamma E$; the entry (μ,ν), $\mu\neq\nu$, comes only from $P=\{\mu,\nu\}$, where c_P has $-E_\nu$ in place μ and E_μ in place ν, and equals $-\gamma_\mu\gamma_\nu\bar E_\mu E_\nu/E^*\Gamma E$. □

Theorem 340. *Let $k\geq1$ and $L=2^k$.*

(i) *The functions K_P are real-analytic, even, 2π-periodic and positive on $\mathbb{R}^4$, $K_P(0)=1$, and for every finitely supported field w on the coarse links*

$$S_k(w)=\sum_P\int_{[-\pi,\pi]^4}K_P(\theta)\,|\hat F_P(\theta)|^2\,\frac{d\theta}{(2\pi)^4}.$$

(ii) *There are $\eta>0$ and $C<\infty$, independent of k, such that every K_P extends analytically to $\{|\mathrm{Im}\,\theta_\nu|\leq\eta\}$ and is bounded there by C.*

(iii) *Hence $S_k(w)=\sum_P\sum_{y,y'}\mathscr{K}_P(y-y')F_P(y)F_P(y')$ with real kernels $\mathscr{K}_P(y)=(2\pi)^{-4}\int K_P(\theta)e^{i\theta\cdot y}\,d\theta$ that satisfy $|\mathscr{K}_P(y)|\leq Ce^{-\eta|y|_1}$ for every k.*

Proof. Step 1: the form in Fourier variables. Let $c_\mu(q) = L\Psi(q)D_L(q_\mu)$. Lemma 336(i) and the Poisson summation formula give, for every square-summable field φ,

$$\widehat{V^{(k)}(\varphi)}_\mu(\theta) = L^{-4}\sum_n c_\mu(\theta^{(n)})\,\hat\varphi_\mu(\theta^{(n)}),$$

with $\hat\varphi(q) = \sum_x e^{-iq\cdot x}\varphi_x$. Write $x_n = \hat\varphi(\theta^{(n)}) \in \mathbb{C}^4$ and $e_n = (e^{i\theta_\mu^{(n)}} - 1)_\mu$, so that $|e_n|^2 = \varepsilon(\theta^{(n)}) =: \varepsilon_n$. By the Lagrange identity the plaquette form at the momentum $\theta^{(n)}$ is $\varepsilon_n|x_n|^2 - |e_n^* x_n|^2$, and the Plancherel formula gives

$$Q(\varphi) = \int_{[-\pi,\pi]^4} L^{-4}\sum_n \big(\varepsilon_n|x_n|^2 - |e_n^* x_n|^2\big)\,\frac{d\theta}{(2\pi)^4}.$$

Since $LD_L(t)(e^{it}-1) = e^{iLt}-1$ and $e^{iL\theta_\mu^{(n)}} = e^{i\theta_\mu}$, the averages map the gauge direction at every alias to the coarse gauge direction: $c_\mu(\theta^{(n)})(e_n)_\mu = \Psi(\theta^{(n)})E_\mu(\theta)$. For $|t| \le \pi/L$ we have $|D_L(t)| = |\sin(Lt/2)|/(L|\sin(t/2)|) \ge 2/\pi$, so $\Psi(\theta^{(0)}) \ne 0$ for real θ.

Step 2: the minimum at one momentum. Fix real $\theta \notin 2\pi\mathbb{Z}^4$; then $E = E(\theta) \ne 0$ and every $\varepsilon_n > 0$. Let $z \in \mathbb{C}^4$. Minimising $L^{-4}\sum_n \varepsilon_n|y_n|^2$ under $L^{-4}\sum_n c_\mu(\theta^{(n)})(y_n)_\mu = z_\mu$ separates in μ, and the Cauchy–Schwarz inequality gives the minimum $\sum_\mu |z_\mu|^2/a_\mu$, attained at $(y_n)_\mu = z_\mu\overline{c_\mu(\theta^{(n)})}/(\varepsilon_n a_\mu)$, where

$$a_\mu = L^{-4}\sum_n \frac{|c_\mu(\theta^{(n)})|^2}{\varepsilon_n} = \frac{\tilde a_\mu}{\hat\varepsilon}.$$

If x satisfies the constraint with right side $\hat w(\theta)$, write $x_n = x_n^T + \beta_n e_n$ with $e_n^* x_n^T = 0$. Then $L^{-4}\sum_n(\varepsilon_n|x_n|^2 - |e_n^*x_n|^2) = L^{-4}\sum_n \varepsilon_n|x_n^T|^2$, and by Step 1 the vectors x_n^T satisfy the constraint with right side $\hat w(\theta) + tE$ for some $t \in \mathbb{C}$. Hence the plaquette form at θ is at least

$$\min_{t\in\mathbb{C}}\sum_\mu \frac{|\hat w_\mu + tE_\mu|^2}{a_\mu} = \hat w^*\Big(\Gamma - \frac{\Gamma E E^*\Gamma}{E^*\Gamma E}\Big)\hat w$$

with $\Gamma = \operatorname{diag}(a_\mu^{-1})$. Conversely, take t optimal, y optimal for $z = \hat w + tE$, and replace y_0 by $y_0 - \beta e_0$ with $\beta = L^4 t/\Psi(\theta^{(0)})$: the constraint becomes $\hat w(\theta)$ and the plaquette form does not increase, since e_0 is a gauge direction. By Lemma 339 the minimum is

$$\sum_P \frac{\gamma_\lambda\gamma_\rho}{E^*\Gamma E}|c_P\hat w|^2, \qquad \frac{\gamma_\lambda\gamma_\rho}{E^*\Gamma E} = \frac{\prod_{\sigma\notin P} a_\sigma}{\sum_\sigma |E_\sigma|^2\prod_{\tau\ne\sigma} a_\tau}.$$

We multiply numerator and denominator by $\hat\varepsilon^3$ and expand $\tilde a_\tau = A_\tau + \hat\varepsilon r_\tau$. The terms without r add up to $\Psi_0^3\prod_\tau D_{0,\tau}\sum_\sigma |E_\sigma|^2/D_{0,\sigma}$, and $|E_\sigma|^2 = L^2 D_{0,\sigma}|e^{i\theta_\sigma^{(0)}} - 1|^2$ by the identity of Step 1, so they equal $\hat\varepsilon\,\Psi_0^3\prod_\tau D_{0,\tau}$; every other term carries a factor $\hat\varepsilon$. So the denominator is $\hat\varepsilon Y$, the numerator $\hat\varepsilon\prod_{\sigma\notin P}\tilde a_\sigma$, and the quotient is $K_P(\theta)$. Since $|c_P\hat w| = |\hat F_P|$, the minimum at θ is $\sum_P K_P(\theta)|\hat F_P(\theta)|^2$.

Step 3: integration. Integrating the lower bound over θ gives $Q(\varphi) \ge \sum_P \int K_P|\hat F_P|^2$ for every square-summable φ with $V^{(k)}(\varphi) = w$. The minimisers of Step 2 are bounded by a constant times $|\hat w(\theta)|$: for $n \ne 0$ because $\varepsilon_n \ge 4/L^2$ and $a_\mu \ge A_\mu/\hat\varepsilon$; for $n = 0$ because $|c_\mu(\theta^{(0)})|/(\varepsilon_0 a_\mu) \le L^3/A_\mu$; and $|t||E| \le |\hat w|(\max\gamma/\min\gamma)^{1/2}$ by the Cauchy–Schwarz inequality for the inner product Γ, while $|e_0| \le |\theta|/L \le \frac{\pi}{2}|E|/L$. Here $A_\mu \ge (2/\pi)^{10}$, and $\gamma_\mu/\gamma_\nu = \tilde a_\nu/\tilde a_\mu$ is bounded because the $\tilde a$ are continuous and at least A. So the minimisers define a square-summable field φ^* with $V^{(k)}(\varphi^*) = w$ and $Q(\varphi^*) = \sum_P\int K_P|\hat F_P|^2$. Since $V^{(k)}$ and Q are bounded on square-summable fields, truncating φ^* and correcting with ψ_R as in the proof of Theorem 338 gives finitely supported fields with the same averages and plaquette forms tending to $Q(\varphi^*)$, and Lemma 336(iii) gives (i) for S_k. The functions are even since $|D_L|^2$ and ε are, and positive since $\tilde a_\sigma \ge A_\sigma > 0$ and $Y \ge \Psi_0^3\prod_\tau D_{0,\tau} \ge (2/\pi)^{32}$ on $\mathbb{R}^4$, all other terms of (33) being non-negative there. At $\theta = 0$, $\Psi_0 = D_{0,\tau} = 1$ and $\hat\varepsilon = 0$, so $K_P(0) = 1$.

Step 4: the strip. Let $|\operatorname{Im}\theta_\nu| \le \frac12$, and for n let $s = L\theta^{(n)}$, so $|\operatorname{Re}s_\nu| \le \pi L$ and $|\operatorname{Im}s_\nu| \le \frac12$. For $q = x + iy$ with $|x| \le \pi$, $|e^{iq}-1|^2 = (e^{-y}-1)^2 + 2e^{-y}(1-\cos x) \ge e^{-|y|}\frac{4}{\pi^2}x^2$, so

$$|D_L(q)| \le \min\Big(e^{1/2}, \frac{c_1}{|\operatorname{Re}Lq|}\Big)$$

for $|\operatorname{Im}q| \le 1/(2L)$, where $c_1 = (e^{1/2}+1)e^{1/4}\frac{\pi}{2} < 5.35$, and the same holds for $D_L^\sharp$. Also $\operatorname{Re}(2-2\cos q) \ge (2-2\cos x) - 2(\cosh y - 1)$, so $\operatorname{Re}L^2\varepsilon(\theta^{(n)}) \ge \frac{4}{\pi^2}|\operatorname{Re}s|^2 - 1.2$. For $n \ne 0$ some $|\operatorname{Re}s_\nu| \ge \pi$, and then $|L^2\varepsilon(\theta^{(n)})| \ge 0.28\,|\operatorname{Re}s|^2$. Writing $s = \theta + 2\pi m$ with $m \in \mathbb{Z}^4\setminus\{0\}$, $|\operatorname{Re}s_\nu| \ge \pi|m_\nu|$, and

$$|r_\sigma| \le \frac{c_1^{10}}{0.28\,\pi^2}\sum_m\prod_\nu f(m_\nu) = \frac{c_1^{10}}{0.28\,\pi^2}\Big(\frac43\Big)^4 < 2.2\cdot 10^7,$$

with $f(0) = 1$ and $f(j) = 1/(\pi j)^2$. The r_σ are finite sums of functions analytic on the strip, since no $\varepsilon(\theta^{(n)})$, $n \ne 0$, vanishes there. Further $|A_\sigma| \le e^5$, $|\hat\varepsilon| \le 4(\pi^2 + \frac14)e^{1/2} < 70$ and $|E_\sigma E_\sigma^\sharp| \le (e^{1/2}+1)^2 < 7.1$, so on the strip $|Y| \le M = 10^{28}$ and $|\tilde a_\sigma| \le 1.6\cdot 10^9$, uniformly in k. By the Cauchy estimate on discs of radius $\frac14$ in each variable, $|\partial_\nu Y| \le 4M$ where $|\operatorname{Im}\theta| \le \frac14$, so $|Y(\theta) - Y(\operatorname{Re}\theta)| \le 16\eta M \le \frac12(2/\pi)^{32}$ for $|\operatorname{Im}\theta_\nu| \le \eta = 10^{-36}$. There $|Y| \ge \frac12(2/\pi)^{32}$, $K_P = \prod_{\sigma\notin P}\tilde a_\sigma/Y$ is analytic, and $|K_P| \le C = 10^{26}$, which is (ii). Moving the contour of each θ_ν to $\operatorname{Im}\theta_\nu = \pm\eta$, according to the sign of y_ν, gives (iii); the kernels are real because K_P is real and even. □

The proof makes η explicit but very small. In floating point the kernels decrease much faster: $\max|\mathscr{K}_P(y)|$ over $|y|_1 = d$ falls by factors 0.17, 0.24 and 0.26 per unit of d for $k = 1,2,3$, the largest value of K_P on $\mathbb{R}^4$ is 4.4, 8.7 and 11.3, and its smallest value is 1, attained at $\theta = 0$ (Fig. 27); these computations are reproduced in the archive. Each K_P is a Riemann sum in n of a function of $\theta + 2\pi n$, so the kernels have a limit as $k \to \infty$; the theorem does not use it. Uniform decay of the propagators of the averaging operations of lattice gauge theories is the subject of Bałaban's work [6]; here the hard constraint and the Feynman gauge make the effective form itself explicit. Together with Theorem 338, the effective forms of the quadratic

approximation are equivalent to the coarse plaquette forms and local, with constants independent of the scale.

Lemma 341. *Let (X,d_X) and (Y,d_Y) be Polish spaces, ρ,ρ' Borel probability measures on Y, and κ,κ' probability kernels from Y to X, all with finite first moments. If $W_1(\kappa(\cdot\mid v),\kappa'(\cdot\mid v))\le\varepsilon$ and $W_1(\kappa'(\cdot\mid v),\kappa'(\cdot\mid v'))\le d_Y(v,v')$ for all v,v', then the mixtures $\pi=\int\kappa(\cdot\mid v)\,\rho(dv)$ and $\pi'=\int\kappa'(\cdot\mid v)\,\rho'(dv)$ satisfy*

$$W_1(\pi,\pi')\le\varepsilon+W_1^{d_Y}(\rho,\rho').$$

Proof. Let F be 1-Lipschitz on X and $\kappa F(v)=\int F\,d\kappa(\cdot\mid v)$. Then $\int F\,d\pi-\int F\,d\pi'=\int(\kappa F-\kappa' F)\,d\rho+\big(\int\kappa' F\,d\rho-\int\kappa' F\,d\rho'\big)$. The first term is at most ε. The function $\kappa' F$ is 1-Lipschitz for d_Y, so the second is at most $W_1^{d_Y}(\rho,\rho')$. The Kantorovich–Rubinstein duality gives the claim. □

A law γ on $\mathbb{R}^V$ *decomposes along the averages* if $\gamma(\cdot\mid\eta)=\int\gamma_V^{\mathrm c}(\cdot\mid\eta,v)\,\rho_V(dv\mid\eta)$, with $\rho_V(\cdot\mid\eta)$ the law of $(V_{y,\mu})_{y\in W}$. Every law with density proportional to $\exp(-\frac{b}{2}\sum_{p\cap V\neq\emptyset}(d\varphi)_p^2)\,g(V(\varphi))$, on a set cut out by conditions on the averages, decomposes in this way, because Lebesgue measure disintegrates along the fibres of a surjective linear map into multiples of Lebesgue measure on the fibres. Here g can carry any action of the averages, in particular a coarse gauge fixing and a coarse effective action, which is the form a law takes after the fine scale has been integrated out.

Corollary 342. *Let $r_w\in\mathbb{R}^V$ minimise $\sum_{p\cap V\neq\emptyset}(d\varphi)_p^2$, with zero exterior, subject to $V_{y',\mu'}(r_w)=\delta_{w,(y',\mu')}$ for all (y',μ'), $y'\in W$; put $L_w=\|r_w\|_1$ and $d_L(v,v')=\sum_w L_w|v_w-v'_w|$. For every law that decomposes along the averages,*

$$C_{V,j}\le C_{V,j}^{\mathrm c}+C_{V,j}^{\rho},\qquad C_{V,j}^{\rho}=\sup\frac{W_1^{d_L}(\rho_V(\cdot\mid\eta),\rho_V(\cdot\mid\eta'))}{|\eta_j-\eta'_j|},$$

the supremum over exteriors that differ only at j, and $L_w\le 4\cdot10^{14}$. Hence $K_c\le K_c^{\mathrm c}+K_c^{\rho}$ with $K_c^{\rho}=\sum_{j\in c\setminus V}C_{V,j}^{\rho}$, and if $K_c^{\rho}\le\frac{\alpha}{2}N_n$ for every class and $n\ge4\cdot10^{15}/\alpha$, then $D(n,\alpha)$ holds for the law, with the 49 classes of active links.

Proof. Apply Lemma 341 with $\kappa=\gamma_V^{\mathrm c}(\cdot\mid\eta,\cdot)$, $\kappa'=\gamma_V^{\mathrm c}(\cdot\mid\eta',\cdot)$, $\rho=\rho_V(\cdot\mid\eta)$, $\rho'=\rho_V(\cdot\mid\eta')$. The first condition holds with $\varepsilon=C_{V,j}^{\mathrm c}|\eta_j-\eta'_j|$. By Theorem 334(i) the laws $\gamma_V^{\mathrm c}(\cdot\mid\eta',v)$ and $\gamma_V^{\mathrm c}(\cdot\mid\eta',v')$ are translates by $\sum_w(v'_w-v_w)r_w$, so their distance is at most $d_L(v,v')$. For the bound on L_w, the pair (r_w,y) solves $K(r_w,y)=(0,se_w)$ with K as in the proof of Theorem 334. Measure d from the centre of the row w; Lemma 333 gives $\sum_e e^{2\lambda d(e)}r_w(e)^2\le(2M_0s)^2=20160^2$. With $z_0=2y+e_\mu$ for $w=(y,\mu)$, a link whose base is at distance k from z_0 has its midpoint at distance at least $k-1$ from the centre, and the count in that proof gives $L_w^2\le20160^2\cdot4e^{2\lambda}\big(1+\sum_{k\ge1}(64k^3+16k)e^{-2\lambda k}\big)\le(4\cdot10^{14})^2$. Summing over the translates as in Definition 42 gives $K_c\le K_c^{\mathrm c}+K_c^{\rho}$, and Theorem 334(iii) with $\alpha/2$ in place of α gives $K_c^{\mathrm c}\le\frac{\alpha}{2}N_n$. □

The responses are much smaller than the bound: in floating point, $\max_w L_w$ is 23.8, 44.1 and 50.9 for the block (m,m,m,m), $m=\lfloor n/4\rfloor$, of the cubes $n=4$, 6, 8. Since Theorem 334 controls the first term at every coupling, the corollary reduces $D(n,\alpha)$ for a law that decomposes along the averages to the cube condition K_c^{ρ} for its averages, a problem of the same form one scale up, in the metric d_L. Iterating it needs Theorem 334 and the corollary with S_k in place of the plaquette form, which Theorem 344 below gives with constants independent of k, and ends in a cube condition for the coarsest averages, which is where a gap has to come from: the quadratic approximation describes a massless field and has none. The corollary is one step of the composition of Remark 345(a), not its conclusion.

On the lattice of spacing 2^k the blocks, the tree links, the active links and the averages are those of Definition 328, and after k steps the averages of the quadratic approximation carry the form S_k. Let (S_{ab}) be its matrix, $S_k(w)=\sum_{a,b}S_{ab}w_aw_b$ for finitely supported w, and $d(a,b)$ the ℓ^∞ distance between the midpoints of the links a and b. Each link lies in six plaquettes of four links, and the midpoints of the links of the plaquettes at y and y' are at distance at most $|y-y'|_1+1$, so Theorem 340(iii) gives, for every non-decreasing $\phi\ge0$,

$$\sum_b|S_{ab}|\,\phi(d(a,b))\le24C\sum_{z\in\mathbb{Z}^4}e^{-\eta|z|_1}\phi(|z|_1+1),\qquad(34)$$

with C and η of that theorem. In particular $\sum_b S_{ab}\eta_b$ converges for every bounded field η. For n, V, W as before, a bounded exterior η and values v, let $\gamma_V^{\mathrm c,k}(\cdot\mid\eta,v)$ be the law on $\mathbb{R}^V$ with density proportional to

$$\exp\Big(-\frac{b}{2}\Big(\sum_{a,b\in V}S_{ab}w_aw_b+2\sum_{a\in V,b\notin V}S_{ab}w_a\eta_b\Big)\Big)$$

on the affine set $\{V_{y,\mu}(w)=v_{y,\mu},\ y\in W\}$, the conditional law of the Gaussian with the form bS_k given the exterior and the averages. Define $C_{V,j}^{\mathrm c,k}$, $K_c^{\mathrm c,k}$, $D_{\mathrm c}^k(n,\alpha)$, $r_w^{(k)}$ and $L_w^{(k)}$ as in Theorem 334 and Corollary 342, with S_k in place of the plaquette form; for $k=0$ they are the former ones.

Lemma 343. *Let K be an invertible symmetric matrix indexed by a finite set X, with $\|K^{-1}\|\le M_0$, and let $d:X\to\mathbb{R}$ and $\lambda>0$. If $\sum_b|K_{ab}|(e^{\lambda|d(a)-d(b)|}-1)\le\frac{1}{2M_0}$ for every a, then $\|e^{\lambda d}K^{-1}e^{-\lambda d}\|\le2M_0$.*

Proof. $e^{\lambda d}Ke^{-\lambda d}=K+E$ with $|E_{ab}|\le|K_{ab}|(e^{\lambda|d(a)-d(b)|}-1)$, a symmetric bound, so $\|E\|\le\frac{1}{2M_0}$ by the Schur test, and the proof of Lemma 333 applies. □

Theorem 344. *Let $\lambda_1=10^{-215}$. For every $k\ge1$, every $b>0$, every even n and every exterior link j:*

(i) *$\gamma_V^{\mathrm c,k}(\cdot\mid\eta,v)$ is a Gaussian law whose covariance depends on neither η nor v and whose mean is affine in (η,v), and $C_{V,j}^{\mathrm c,k}=\|u_j\|_1$, where $u_j\in\mathbb{R}^V$ minimises $S_k(u+\delta_j)$ subject to $Bu=g$, with B and g as in Theorem 334;*

(ii) *$\sum_{e\in V}e^{2\lambda_1 d(e)}u_j(e)^2\le10^{350}$ and $\|u_j\|_1\le10^{607}e^{-\lambda_1 r_j/2}$, where $d(e)=d(e,j)$ and $r_j=\min_{e\in V}d(e)$; and $L_w^{(k)}\le10^{438}$ for every w;*

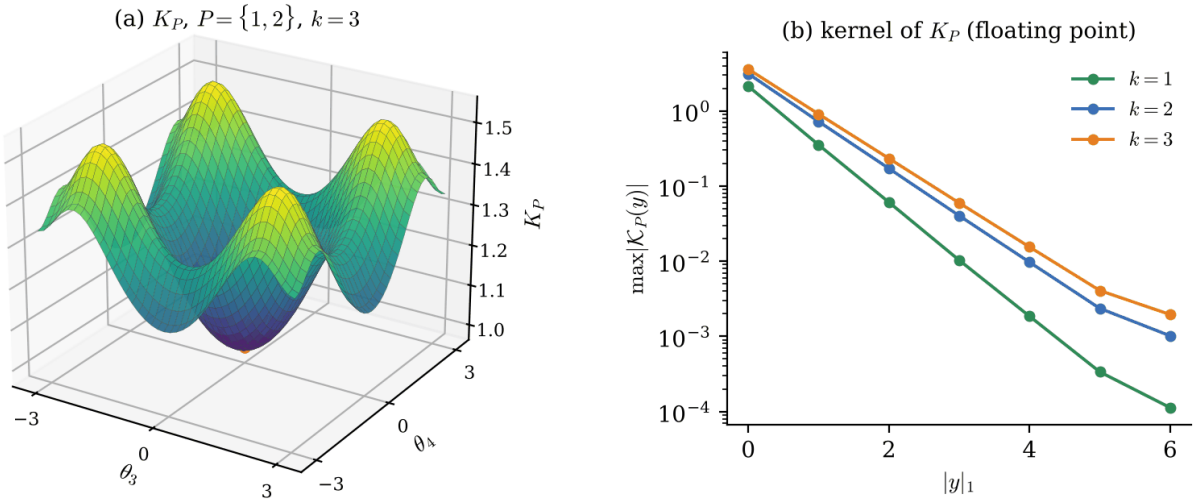


FIG. 27. The effective forms of Theorem 340. (a) The multiplier K_P for the plane $P=\{1,2\}$ at $k=3$, as a function of (θ_3,θ_4) with $\theta_1=\theta_2=0$; it equals 1 at $\theta=0$ and stays bounded. (b) The largest value of the kernel $|\mathcal{K}_P(y)|$ on $|y|_1=d$ for $k=1,2,3$, in logarithmic scale: the decay per coarse spacing hardly changes with k.

(iii) *$D_c^k(n,\alpha)$ holds for every $\alpha\in(0,1)$ and every even $n\geq 10^{827}/\alpha$;*

(iv) *Corollary 342 holds on the lattice of spacing 2^k, with S_k in place of the plaquette form, $L_w^{(k)}\leq 10^{438}$ and $n\geq 2\cdot 10^{827}/\alpha$.*

None of these constants depends on k or on b.

Proof. (i) Let $H=(S_{ab})_{a,b\in V}$. A vector $w\in\ker B$, extended by zero, lies in the space Φ_W of the lattice of spacing 2^k, so Proposition 335(iv) gives $\langle w,Hw\rangle=S_k(w)\geq\frac{1}{52}|w|^2$ on $\ker B$. The proof of Theorem 334(i) applies, the exterior entering through the linear term $\sum_{b\notin V}S_{ab}\eta_b$; changing η_j by one translates the mean by the minimiser of $\langle u,Hu\rangle+2\sum_{a\in V}u_aS_{aj}=S_k(u+\delta_j)-S_k(\delta_j)$ subject to $Bu=g$.

(ii) The pair (u_j,y) solves $K(u_j,y)=(f,sg)$ with K built from H and B as in Lemma 332 and $f=-(S_{aj})_{a\in V}$. Since $Q_k(w)\leq 24|w|^2$, Theorem 338 gives $\|H\|\leq h=6144$; take $s=4h=24576$. The bounds of Lemma 332 for (h,s) are at most those for $(24,96)$, because $|x_1|$ and $|f|/s$ decrease while $h|x_1|$ and h/s do not change, so $\|K^{-1}\|\leq M_0=105$ as in the proof of Theorem 334. Index the rows of B by centres as there. For $\phi(r)=e^{\lambda r}-1\leq\lambda re^{\lambda r}$ and $\lambda\leq\eta/2$, (34) bounds the row sums of Lemma 343 over H by

$$24C\lambda e^{\lambda}\sum_z(|z|_1+1)e^{-\eta|z|_1/2}\leq 24C\lambda e^{\lambda}\Big(1+16\cdot 24\,e^{\eta/2}\Big(\frac{2}{\eta}\Big)^5\Big),$$

since at most $8r^3$ points have $|z|_1=r\geq 1$ and $\sum_{r\geq1}r^pe^{-xr}\leq e^xp!/x^{p+1}$. The entries of $sB^\top$ add at most $\frac{s}{8}(e^\lambda-1)$ and a row of sB gives $2s(e^\lambda-1)$. At $\lambda=\lambda_1$, with $C=10^{26}$, $\eta=10^{-36}$ and $(2/\eta)^5=3.2\cdot10^{181}$, the first sum is at most

$$2.4\cdot10^{-188}e^{\lambda_1}(1+384e^{\eta/2}\cdot3.2\cdot10^{181})<2.95\cdot10^{-4},$$

while the other two add at most $\frac{17}{8}s(e^{\lambda_1}-1)<10^{-209}$. So all the sums are below $3\cdot10^{-4}<\frac{1}{2M_0}$, and Lemma 343 gives

$$\|e^{\lambda_1 d}K^{-1}e^{-\lambda_1 d}\|\leq 210.$$

By (34),

$$\|e^{\lambda_1 d}f\|\leq 24Ce^{\lambda_1}\sum_z e^{-(\eta-\lambda_1)|z|_1}=24Ce^{\lambda_1}\coth^4\frac{\eta-\lambda_1}{2}<4\cdot10^{172},$$

since $\coth x\leq\frac{1}{x}+\frac{x}{3}$ and $2/(\eta-\lambda_1)\leq 2.0001\cdot10^{36}$, and $\|e^{\lambda_1 d}sg\|\leq e^{\lambda_1}s/16$, which gives the first bound, as $(210(4\cdot10^{172}+2\cdot10^3))^2<10^{350}$. The links with base at ℓ^∞ distance m from the base of j number at most $4(64m^3+16m)$, with midpoints at distance at least $m-1$, so

$$Z(x)=\sum_e e^{-xd(e)}\leq 4e^x\Big(1+\frac{384e^x}{x^4}+\frac{16e^x}{x^2}\Big),$$

and $Z(\lambda_1)\leq 4.004(1+384.4\cdot10^{860}+16.1\cdot10^{430})<10^{864}$, by $e^{2\lambda_1}\leq 1.001$. As $d\geq r_j$ on V, the Cauchy–Schwarz inequality gives $\|u_j\|_1^2\leq 10^{350}e^{-\lambda_1 r_j}Z(\lambda_1)$. Finally $(r_w^{(k)},y)$ solves $K(r_w^{(k)},y)=(0,se_w)$; measuring d from the centre of the row w, we get $\sum_e e^{2\lambda_1 d(e)}r_w^{(k)}(e)^2\leq(210s)^2$ and, since $Z(2\lambda_1)\leq 4.004(1+24.1\cdot10^{860}+4.1\cdot10^{430})<9.7\cdot10^{861}<(9.85\cdot10^{430})^2$ and $210s<5.17\cdot10^6$, $L_w^{(k)}\leq 210sZ(2\lambda_1)^{1/2}<10^{438}$.

(iii) The exterior links of a class have their bases at ℓ^∞ distance $m\geq1$ from $[0,n-1]^4$; at most $4((n+2m)^4-(n+2m-2)^4)\leq 128(n^3+8m^3)$ of them have distance m, and for them $r_j\geq m-1$. With $q=\lambda_1/2$, (ii) gives

$$K_c^{c,k}\leq 10^{607}\cdot128\,e^{2q}\Big(\frac{n^3}{q}+\frac{48}{q^4}\Big),$$

which is at most $\alpha N_n = \alpha(n/2)^4$ when $n \geq 10^{827}/\alpha$: with $1/q = 2\cdot 10^{215}$ and $e^{2q} \leq 1.001$ the first term is at most $2.57\cdot 10^{824} n^3 \leq \alpha n^4/32$ for $n \geq 8.3\cdot 10^{825}/\alpha$, and the second at most $9.9\cdot 10^{1471} \leq \alpha n^4/32$ for $n^4 \geq 3.2\cdot 10^{1473}/\alpha$, which holds since $n^4 \geq 10^{3308}/\alpha^4$.

(iv) The proof of Corollary 342 uses the plaquette form only through Theorem 334, the translation property of (i) and the bound on L_w, which (i)–(iii) provide. □

The constants are poor because the strip of Theorem 340 has width 10^{-36}. What the theorem adds is that none of them depends on the scale: the constrained cube condition holds with one cube side at every scale, and the weights $L_w^{(k)}$ of the metrics d_L are bounded uniformly. It does not compose the scales. The cube condition K_c^{ρ} that Corollary 342 leaves at scale k concerns the law of the averages given the fine exterior, in the metric d_L, and it has to be turned into the cube condition of the law one scale up; that step and the cube condition for the coarsest averages are open, and in the quadratic approximation the coarsest averages carry a massless Gaussian, so a gap has to come from beyond it.

Remark 345. Theorem 334 is the weak-coupling half of the constrained route at leading order, and Theorem 45 cannot use it as it stands. Four steps remain, each an open problem.

(a) *Composition.* D_{c} holds the block averages fixed. A gap in physical units needs a theorem that combines the constrained conditions at all scales with a cube condition for the coarsest averages, in the place of Theorem 45. Corollary 342 is one step of it, Proposition 335(iv) gives the gap at every scale, and Theorem 338 the equivalence of S_k and Q_k uniformly in k, Theorem 340 the locality of S_k uniformly in k, and Theorem 344 the constrained cube condition and the bounds on the weights L_w at every scale with constants independent of k. Turning the cube condition for the averages into that of the law one scale up, and a cube condition for the coarsest averages, are open.

(b) *Beyond the quadratic approximation.* For the Wilson measure the block averages are products of group elements along paths, their level sets are curved, and the Bakry–Émery tensor of Sec. VI is negative at large fields; Theorem 330 controls only the small fields, and the large fields have to be split off as in Bałaban's analysis [12]. For the straight paths between block roots the group-valued estimate holds for all fields (Theorem 347), but that map does not compose across scales (Proposition 350), and for every compact connected group no continuous covariant map treats the transverse positions of a block alike (Proposition 351). The cut-off average of Theorem 353 is coercive for all fields and agrees with the averages to first order, with constants for one step; on torus-valued fields of small curvature it composes (Theorem 355), and for every compact group it composes on fields whose curvature is small at every scale with a small sum (Theorem 359); under the Wilson measure the remaining large fields form microscopic clusters (Corollary 384), but the conditional estimate needed near them is open. Away from them, on the fibres of the cut-off average, the Hessian of the action along the fibre is at least $\lambda(\frac{1}{26} - c_G\kappa)$ at every background with plaquettes at most κ (Theorem 361), and on a convex chart of the fibre the constrained law satisfies a Brascamp–Lieb inequality for large λ (Theorem 363), and on a product chart its covariances decay exponentially (Theorem 367), and the mass off that chart is small on most fibres of the SU(2) Wilson measure (Theorem 370) and on every fibre through a field with small plaquettes (Corollary 375), and on every fibre the law concentrates near the least action of its regular fields (Proposition 377); the remaining small-field steps are listed in Remark 381.

(c) *Second order.* Let $b_0 = 11N/(48\pi^2)$ for SU(N), $t = 2b_0\log 2$, and let $g_m^{-2} = g_*^{-2} + tm$ be the one-loop coupling m steps above a matching scale. Comparison with integrals gives $\sum_{m\leq M} g_m^2 = t^{-1}\log M + O(1)$ and $\sum_m g_m^4 \leq \pi^2/(6t^2)$ [82]. A loss $1 + cg_m^2$ per step therefore compounds to a power of M, and a step from one scale to the next has to be exact at order g^2, with that term absorbed into the running coupling and the normalisation of the field, and remainders of order g^4.

(d) *Crossover and existence.* The passage from weak coupling to the strong-coupling range of Sec. IV–VIII and the existence of the limit, (E), are untouched.

The renormalisation group needed for (a) and (b) is the ultraviolet half of (E), so the constrained route places both residues on one construction.

B. Path blocks for the group-valued links

Remark 345(b) asks for estimates on the actual group-valued links. They can be proved for a coarser block map, the straight path between block roots, and this map exhibits the second half of the difficulty: it does not compose across scales. A topological argument shows why no continuous map that treats the transverse positions alike can take its place.

Let $G \subset \mathrm{U}(r)$ be a compact matrix group, $\delta(A,B) = \|A - B\|_{\mathrm{HS}}$, which is invariant under left and right multiplication, and work on the periodic lattice of side $2L$ with the block tree of Definition 328, all fifteen tree links of each block equal to I. Write $\mathscr{B}$ for the number of blocks. The *path map* is

$$V^{\mathrm{p}}_{y,\mu}(U) = U_{2y+e_\mu,\mu},$$

the product along the straight path from $2y$ to $2y+2e_\mu$, whose first link is a tree link; it transforms as a link under the gauge transformations at the block roots. Fixing $V^{\mathrm{p}} = v$ fixes four of the 49 active links of each block, so the fibre is $G^{45\mathscr{B}}$, and the Wilson law in the tree gauge restricts to it with product Haar measure as reference, since the tree gauge has unit Jacobian. The *reconstruction* $R(v)$ puts $R_{2y+o,\mu} = v_{y,\mu}$ if $o_\mu = 1$ and $R_{2y+o,\mu} = I$ otherwise. For U in the fibre of v put

$$D_v(U)^2 = \sum_e \delta(U_e, R_e(v))^2, \qquad Q(U) = \sum_p \delta(U_p, I)^2,$$

the first sum having zero terms on the tree and path links, and let $Q_1(v) = \sum_P \delta(v_P, I)^2$ on the coarse lattice.

Lemma 346. *For vectors z_a of a real Hilbert space indexed by $a \in \{0,1\}^3$, with $z_0 = 0$,*

$$\sum_a \|z_a\|^2 \le \frac{3+\sqrt{7}}{2} \sum_{\{a,a'\}} \|z_a - z_{a'}\|^2 \le 3 \sum_{\{a,a'\}} \|z_a - z_{a'}\|^2,$$

the sums on the right over the twelve edges of the cube.

Proof. It suffices to treat real numbers. The right side is $\langle z, L_0 z\rangle$, where L_0 is the Laplacian of the cube with the vertex 0 deleted, and the claim is that its least eigenvalue is $3-\sqrt{7}$. The permutations of the coordinates commute with L_0. On the functions f_k of the level $k = |a|$, L_0 acts by $f_1 \mapsto 3f_1 - 2f_2$, $f_2 \mapsto 3f_2 - 2f_1 - f_3$, $f_3 \mapsto 3f_3 - 3f_2$, whose characteristic polynomial in $t = 3-\lambda$ is $t^3 - 7t$; the eigenvalues are 3 and $3 \pm \sqrt{7}$. On the functions of levels one and two whose sum over each level vanishes, indexed by the coordinate that is one at level one and zero at level two, L_0 acts by the matrix $\left(\begin{smallmatrix} 3 & 1 \\ 1 & 3 \end{smallmatrix}\right)$, with eigenvalues 2 and 4; at level three these functions vanish. The two spaces have dimensions 3 and 4 and span $\mathbb{R}^7$. Finally $1/(3-\sqrt{7}) = (3+\sqrt{7})/2 < 3$.[36] □

Theorem 347. *For every $G \subset \mathrm{U}(r)$, every $L \ge 1$, every v and every U in the fibre of v,*

$$D_v(U)^2 \le 224\, Q(U).$$

Proof. Let I_0 be the sum of $\delta(U_e, I)^2$ over the active links inside the blocks and J_0 the sum of $\delta(U_e, v_{y,\mu})^2$ over the links that leave $B(y)$ in the direction μ, so $D_v^2 = I_0 + J_0$. For an active link inside a block, the chain of plaquettes in the proof of Theorem 330 applies to group elements: each plaquette of the chain has its two transverse links on the tree, so the holonomy is the product of two consecutive links of the chain, and invariance of δ with the triangle inequality gives $\delta(U_e, I)^2 \le k \sum_i \delta(U_{p_i}, I)^2$ with $k \le 3$. The loads are those of that proof, so $I_0 \le 8 Q_{\mathrm{in}}$. Fix y, μ and label the eight links leaving $B(y)$ by the transverse offset $a \in \{0,1\}^3$; the link $a = 0$ is the path link and equals $v_{y,\mu}$. For a transverse edge $\{a, a'\}$ the face plaquette has holonomy $U_a \psi' U_{a'}^{-1} \psi^{-1}$ with ψ, ψ' links inside the two blocks, so $\delta(U_a, U_{a'}) \le \delta(U_p, I) + \delta(\psi, I) + \delta(\psi', I)$. Lemma 346 with $z_a = U_a - v_{y,\mu}$ in the real Hilbert–Schmidt space and Cauchy–Schwarz bound the eight terms by 9 times the sum of the three squares over the twelve edges. Each face plaquette occurs once, and each link inside a block occurs once for each of the three directions other than its own, as in the proof of Theorem 330. Hence $J_0 \le 9 Q_{\mathrm{face}} + 27 I_0$ and $D_v^2 \le 28 I_0 + 9 Q_{\mathrm{face}} \le 224\, Q$. □

The bound holds for all fields, with no small-field condition and no logarithmic chart; Theorem 330 is its linearisation for the averages, with the better constant 52.

Proposition 348. *$Q_1(V^{\mathrm{p}}(U)) \le 4 Q(U)$, $Q(R(v)) = 4 Q_1(v)$ and $Q(U) \le 8 Q_1(v) + 48 D_v(U)^2$ for U in the fibre of v; if $Q_1(v) = 0$, then $Q(U) \le 24 D_v(U)^2$.*

Proof. The holonomy of a coarse plaquette is that of the square of side two at $2y$, a product of four conjugated fine plaquette holonomies, so $\delta(v_P, I) \le \sum_{p \subset P} \delta(U_p, I)$; these squares tile each coordinate plane, which gives the first bound. In $R(v)$ only the plaquettes with offset one in both directions carry holonomy, equal to that of the coarse plaquette, four for each P. For any plaquette, telescoping gives $\delta(U_p, R_p) \le \sum_{e \in p} \delta(U_e, R_e)$, so $\delta(U_p, I)^2 \le 2\delta(R_p, I)^2 + 8 \sum_{e \in p} \delta(U_e, R_e)^2$, and each link lies in six plaquettes. If $Q_1(v) = 0$, every R_p is I and the factor 2 is not needed. □

Let $m = 45\mathscr{B}$, let the action be λQ with $\lambda \ge 0$ (for $\mathrm{SU}(N)$, $\lambda = \beta/(2N)$ up to a constant), and put

$$z_G(t) = \int_G e^{-t\delta(g,I)^2} dg, \qquad Z_\lambda(v) = \int_{G^m} e^{-\lambda Q(U)} dU,$$

with normalised Haar measures, the integral over the fibre of v.

Theorem 349. *For every v,*

$$e^{-8\lambda Q_1(v)} z_G(48\lambda)^m \le Z_\lambda(v) \le e^{-\lambda Q_1(v)/8} z_G(\lambda/448)^m,$$

and if $Q_1(v) = 0$, then $z_G(24\lambda)^m \le Z_\lambda(v) \le z_G(\lambda/224)^m$. Let $A_\lambda = z_G(\lambda/448)/z_G(24\lambda)$. Under the conditional law at such a v,

$$\mathbb{E}\, e^{\lambda D_v^2/448} \le A_\lambda^m, \qquad \mathbb{E}\, \delta(U_e, I)^2 \le \frac{20160}{\lambda} \log A_\lambda$$

for each link e when $v \equiv I$, and $A_\lambda \to 10752^{q/2}$ as $\lambda \to \infty$ if G has dimension $q \ge 1$.

Proof. By Theorem 347 and Proposition 348, $Q \ge \frac{1}{2}(D_v^2/224 + Q_1/4)$ and $Q \le 8Q_1 + 48 D_v^2$. Each term of D_v^2 involves one free link, and right multiplication by $R_e(v)^{-1}$ preserves Haar measure, so $\int e^{-tD_v^2} dU = z_G(t)^m$; this gives the two bounds, and the flat case is the same with $Q \le 24 D_v^2$. Next, $-\lambda Q + \lambda D_v^2/448 \le -\lambda D_v^2/448$, and integration with the lower bound gives the exponential moment; Jensen's inequality gives $\mathbb{E} D_v^2 \le (448m/\lambda)\log A_\lambda$. When $v \equiv I$ the law is invariant under translation by blocks, so the $\mathscr{B}$ links of each of the 45 classes have equal expectations, and $448 \cdot 45 = 20160$. Finally, in exponential coordinates $\delta(e^X, I)^2 = \|X\|^2 + O(\|X\|^4)$ and the Haar density is smooth and positive at 0, while $\delta(\cdot, I)$ is bounded below away from I; so $z_G(t) \sim c_G t^{-q/2}$, and the ratio at $\lambda/448$ and 24λ tends to $(24 \cdot 448)^{q/2}$. □

So the fluctuations of the free links around a flat path field are of order $\lambda^{-1/2}$ uniformly in the volume, for the group variables themselves. The path map cannot be iterated, however, and this is where it differs from the averages. Write $V^{\mathrm{p},k}$ for the k-fold path map, the straight path of length $s = 2^k$, and Q_k for the plaquette energy of the lattice of spacing s.

Proposition 350. *On the periodic lattice of side $3s$, $s = 2^k$, there are $\mathrm{SU}(2)$ fields in the tree gauge with*

$$\frac{Q(U)}{Q_k(V^{\mathrm{p},k}(U))} = s \sin^2 \frac{\pi}{2s} \le \frac{\pi^2}{4s}.$$

For the averages, by contrast, $Q_k(V^{(k)}(\varphi)) \le Q(\varphi)$ for every k in the quadratic approximation (Proposition 335(i)).

[36] The constant $(3+\sqrt{7})/2$ is attained by a function of the level $|a|$ alone, so it is the best constant in the lemma.

Proof. Let $T = \mathrm{diag}(i\pi/2, -i\pi/2)$. Put the $s/2$ links in the direction 0 at $(j,0,0,0)$, j odd, $j < s$, equal to $e^{2T/s}$ and all other links equal to I. These links have offset one in the direction 0, so the tree gauge holds, and the links at even j are tree links. The path from 0 to se_0 carries e^T and every other path carries I, so one coarse link is e^T, and it lies in six coarse plaquettes, each with $\delta^2 = \|e^T - I\|_{\mathrm{HS}}^2 = 4$: $Q_k = 24$. No fine plaquette contains two of the changed links, since the two links of a plaquette in the direction 0 differ in a transverse coordinate, so there are $3s$ nonzero fine plaquettes, each with $\delta^2 = 8\sin^2(\pi/2s)$, and $Q = 24s\sin^2(\pi/2s)$. Gauge covariance of the path map makes the ratio gauge invariant. □

A field concentrated on one thin path carries a coarse curvature of order one at a fine cost of order $1/s$. In the averages it is one transverse position among s^3 and enters with the weight s^{-3}, which is why the averages compose and the paths do not. The two properties needed for Remark 345(a) and (b) are therefore held by two different maps: the paths give a global estimate for the group-valued links on fibres that do not compose, and the averages compose on linear fibres that exist only in the quadratic approximation.

A block map with both properties cannot treat the transverse positions alike and be continuous, for any compact connected gauge group. Let $s = 2^k$ and let V be a map of the links with both ends in $B(y) \cup B(y+e_\mu)$, blocks of side s, to G. Call it *covariant* if $V(U^h) = h_1 V(U) h_2^{-1}$ for every gauge transformation h equal to h_1 on $B(y)$ and to h_2 on $B(y+e_\mu)$, and *normalised* if $V(I) = I$. The path map and its iterates are covariant, normalised and continuous. Let T be the set of the s^3 links in the direction μ from $B(y)$ to $B(y+e_\mu)$, and for $u \in G^T$ let $U(u)$ carry u_t on the link $t \in T$ and I on every other link (Fig. 28). Let $S \subset G$ be a maximal torus, of dimension ℓ; a basis of the kernel of its exponential map identifies S with $(\mathbb{R}/2\pi\mathbb{Z})^\ell$ and its first homology with $\mathbb{Z}^\ell$.

Proposition 351. *Let $G \subset \mathrm{U}(r)$ be a compact connected Lie group of positive dimension, let V be covariant and normalised, and let $F(u) = V(U(u))$ be continuous on G^T. Then F maps S^T into S, and the integer matrices W_t by which the maps $u_t \mapsto F(u)$, $u_t \in S$, $u_{t'} = I$ for $t' \neq t$, act on $H_1(S) = \mathbb{Z}^\ell$ satisfy*

$$\sum_{t \in T} W_t = I_\ell .$$

Hence some $W_t \neq 0$, so that the map $u_t \mapsto V(U(u))$ on fields equal to I off the link t is not homotopic to a constant on S, and, if $|T| \geq 2$, no transitive group of permutations of T leaves F invariant. For $G = \mathrm{SU}(2)$, $\ell = 1$, the W_t are integers adding up to one, and some field equal to I off one link of T has $V = -I$.

Proof. The gauge transformation equal to g on $B(y) \cup B(y+e_\mu)$ maps $U(u)$ to $U(gug^{-1})$, since the links equal to I stay I, and the one equal to g on $B(y)$ and to I on $B(y+e_\mu)$ maps it to $U(gu)$. So $F(gug^{-1}) = gF(u)g^{-1}$ and $F(gu) = gF(u)$, and with $F(I) = I$ the second gives $F(g,\dots,g) = g$. If every u_t lies in S, every $g \in S$ fixes u under conjugation, so $F(u)$ commutes with S; in a compact connected Lie group a maximal torus is its own centraliser [66], so $F(u) \in S$. Therefore F restricts to a continuous map $f : S^T \to S$. The first homology of S^T is $\bigoplus_{t \in T} \mathbb{Z}^\ell$, the t-th summand coming from the inclusion of the t-th factor with the identity elsewhere, so f acts on it by $(x_t)_t \mapsto \sum_t W_t x_t$. The diagonal map $S \to S^T$ acts by $x \mapsto (x,\dots,x)$, and f composed with it is the identity of S because $F(g,\dots,g) = g$. Hence $\sum_t W_t = I_\ell$. If all W_t vanished the sum would vanish, and a map of tori that is homotopic to a constant acts by zero on homology. A transitive permutation group preserving F would make all W_t equal to one matrix W, and $|T|\,W = I_\ell$ has no integer solution for $|T| \geq 2$. For $\mathrm{SU}(2)$, S is a circle through $-I$, and a loop of nonzero degree in a circle is onto. □

For the path map, W_t is the identity at the position of the path and zero elsewhere. Averages of the path holonomies weigh the positions alike, so by the proposition they have no continuous covariant extension to all of G^T, for $\mathrm{SU}(N)$ and for every compact simple group. Bałaban's averages are defined only on fields whose plaquettes are small, and the large fields are split off [6, 12]. A block map with both properties for all fields is therefore discontinuous or, like the path map, not invariant under the transverse symmetries of the block. Theorem 353 below gives a continuous one whose departure from the symmetry is confined to large fields.[37]

The proposition leaves room for a map that treats the positions alike to first order and breaks the symmetry only where the field is large. For a coarse link (y,μ) and $x \in B(y)$ put

$$P_x = U_{x,\mu} U_{x+e_\mu,\mu},$$

the product along the straight path of length two from x to $x+2e_\mu$. In the tree gauge it is the parallel transport from $2y$ to $2y+2e_\mu$ through x and $x+2e_\mu$, it transforms as a link under the gauge transformations at the block roots, and $P_{2y} = V^{\mathrm{p}}_{y,\mu}$, since the first link of that path is a tree link. Let $\mathfrak{g}$ be the Lie algebra of G, $\|X\|$ the Hilbert–Schmidt norm and log the principal logarithm of unitary matrices without the eigenvalue -1.

Lemma 352. *If g is unitary and $\delta(g,I) \leq 1$, then $\|\log g\| \leq \frac{\pi}{3}\,\delta(g,I)$. If X is skew-Hermitian, then $\delta(e^X, I) \leq \|X\|$.*

Proof. Let $e^{i\theta_j}$, $|\theta_j| \leq \pi$, be the eigenvalues of g. Then $\delta(g,I)^2 = \sum_j 4\sin^2(\theta_j/2)$ and $\|\log g\|^2 = \sum_j \theta_j^2$. Each term of the first sum is at most 1, so $|\theta_j| \leq \pi/3$, and $\theta/(2\sin(\theta/2))$ increases on $(0,\pi)$ and equals $\pi/3$ at $\theta = \pi/3$. The second statement is $|e^{i\theta} - 1| \leq |\theta|$ for the eigenvalues of X. □

Let $\rho_G \in (0,1]$ be such that $\log g \in \mathfrak{g}$ for every $g \in G$ with $\delta(g,I) \leq \rho_G$.[38] Let $\phi(t) = \min\{1, \max\{0, 2-2t\}\}$, which

[37] That spheres and tori carry no continuous symmetric mean goes back to Aumann [2] and Eckmann [49]; the proof above is their homology argument in gauge-covariant form, with the links of the slice T as the arguments of the mean and a maximal torus in place of the circle.

[38] The exponential map of G is a diffeomorphism from a ball of $\mathfrak{g}$ of operator-norm radius less than π onto a neighbourhood N of I in G, on which it inverts the principal logarithm, and N contains $\{g \in G : \delta(g,I) \leq \rho\}$ for small $\rho > 0$. For $\mathrm{U}(r)$ one may take $\rho_G = 1$. For $\mathrm{SU}(N)$ one may take $\rho_G = \min\{1, 5/\sqrt{N}\}$: by Lemma 352 the trace of $\log g$ has modulus at most $\sqrt{N}\,\pi\rho_G/3 < 2\pi$, and it lies in $2\pi i\mathbb{Z}$. Some restriction is needed: on the circle $\{\mathrm{diag}(z,z^7)\} \subset \mathrm{U}(2)$ the point $z = e^{2\pi i/7}$ has $\delta(g,I) < 1$ but $\log g \notin \mathfrak{g}$.

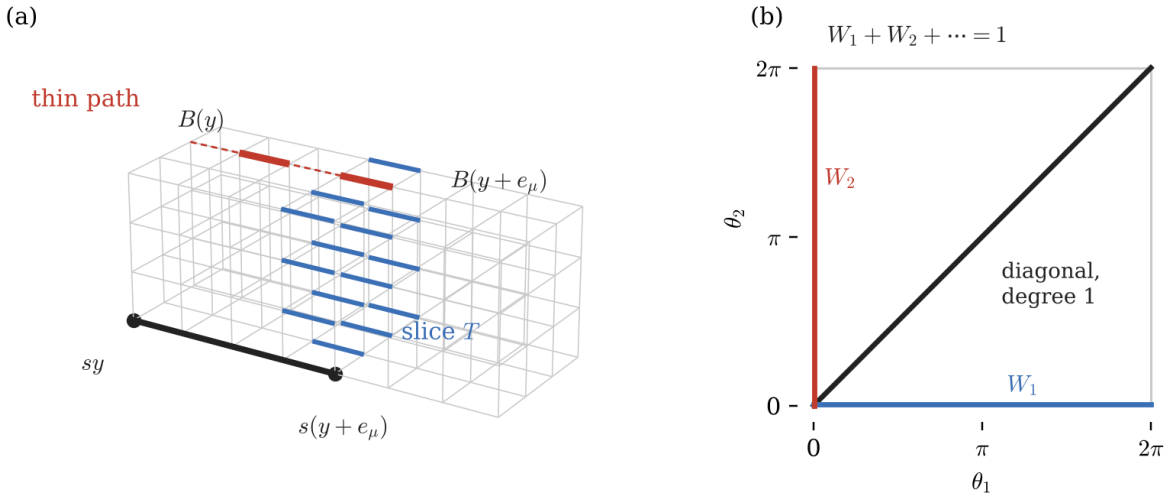


FIG. 28. (a) Two adjacent blocks $B(y)$, $B(y+e_\mu)$ of side $s=4$, drawn in three of the four dimensions. The black segment is the straight path between the roots, the coarse link of the path map. The blue links form the slice T of Proposition 351. The red links carry the field of Proposition 350: a path whose holonomy is that of a coarse link of order one, at a fine energy of order $1/s$. (b) For SU(2), two coordinates of the torus $(\mathbb{R}/2\pi\mathbb{Z})^T$ of slice fields with values in a maximal torus. The diagonal is pure gauge and is sent to a loop of degree one, so the degrees W_t of the coordinate loops add up to one.

equals 1 on $[0,\frac{1}{2}]$ and 0 on $[1,\infty)$, put

$$\chi_{y,\mu}(U)=\phi\Big(\rho_G^{-1}\max_{x\in B(y)}\delta(P_x,P_{2y})\Big),$$

and define the *cut-off average*

$$V^\sharp_{y,\mu}(U)=P_{2y}\exp\Big(\frac{\chi_{y,\mu}(U)}{16}\sum_{x\in B(y)}\log\big(P_{2y}^{-1}P_x\big)\Big),\qquad(35)$$

the exponential factor being I when $\chi_{y,\mu}(U)=0$ (Fig. 29). When every path is close to the straight one it is the average of the sixteen paths taken in the chart at P_{2y}; when some path is far from it, it is the path map.

Theorem 353. *For every compact $G\subset \mathrm{U}(r)$ the cut-off average is continuous, covariant and normalised, and the following hold, with $c=\pi/3$.*

(i) *If $\delta(P_x,P_{2y})\le\rho_G/2$ for every $x\in B(y)$, then $V^\sharp_{y,\mu}(U)=P_{2y}\exp\big(\frac{1}{16}\sum_x\log(P_{2y}^{-1}P_x)\big)$. For φ with values in $\mathfrak{g}$, vanishing on the tree links,*

$$V^\sharp_{y,\mu}(e^{t\varphi})=I+t\,V_{y,\mu}(\varphi)+O(t^2),$$

with $V_{y,\mu}$ the block average of Definition 328.

(ii) *For every field U in the tree gauge,*

$$\sum_{y,\mu}\sum_{x\in B(y)}\delta(P_x,P_{2y})^2\le 896\,Q(U),$$

$$\sum_{y,\mu}\delta\big(V^\sharp_{y,\mu}(U),V^{\mathrm{p}}_{y,\mu}(U)\big)^2\le 56c^2\,Q(U),$$

$$\sum_e\delta\big(U_e,R_e(V^\sharp(U))\big)^2\le(448+896c^2)\,Q(U),$$

$$Q_1\big(V^\sharp(U)\big)\le(8+2688c^2)\,Q(U),$$

and $448+896c^2<1431$, $8+2688c^2<2956$.

(iii) *In the setting of Proposition 351 with $s=2$, W_t is the identity at the position of the straight path and 0 at the seven others.*

(iv) *If U equals I except on one link from $B(y)$ to $B(y+e_\mu)$, where it equals g with $\delta(g,I)\le\rho_G/2$, then $V^\sharp_{y,\mu}(U)=\exp(\frac{1}{8}\log g)$, at each of the eight transverse positions of the link; the path map gives g at one position and I at the seven others.*

Proof. Write $P=P_{2y}$, $v_x=P^{-1}P_x$ and $m=\max_x\delta(v_x,I)=\max_x\delta(P_x,P)$. Where $m<\rho_G$, every $\log v_x$ lies in $\mathfrak{g}$, depends continuously on U by Lemma 352, which keeps the eigenvalues of v_x away from -1, and the exponent of (35) lies in $\mathfrak{g}$, so $V^\sharp\in G$. Where $m\ge\rho_G$, $V^\sharp=P$. As m increases to ρ_G the exponent has norm at most $\chi c\rho_G\to 0$, so $V^\sharp$ is continuous. Let h be a gauge transformation equal to h_1 on $B(y)$ and to h_2 on $B(y+e_\mu)$. The middle point $x+e_\mu$ of each path lies in one of the two blocks, so $P_x\mapsto h_1P_xh_2^{-1}$ and $v_x\mapsto h_2v_xh_2^{-1}$; δ is invariant and log is equivariant under unitary conjugation, so χ is unchanged and $V^\sharp\mapsto h_1V^\sharp h_2^{-1}$. At $U=I$ every v_x is I, so $V^\sharp(I)=I$.

(i) The first statement is $\chi=1$. If $U=e^{t\varphi}$ with $\varphi=0$ on the tree, then $P=I+t\varphi_{2y+e_\mu,\mu}+O(t^2)$ and $\log v_x=t(\varphi_{x,\mu}+\varphi_{x+e_\mu,\mu}-\varphi_{2y+e_\mu,\mu})+O(t^2)$, and for small t the condition of the first statement holds. The mean over the sixteen x is $t(V_{y,\mu}(\varphi)-\varphi_{2y+e_\mu,\mu})+O(t^2)$, since $\varphi_{2y,\mu}=0$, and multiplying by P gives the claim.

(ii) In the tree gauge $P=v_{y,\mu}:=V^{\mathrm{p}}_{y,\mu}(U)$. If $o_\mu=0$, then $U_{x+e_\mu,\mu}$ is the link leaving $B(y)$ at the transverse offset of x, and invariance of δ with the triangle inequality gives $\delta(P_x,P)\le\delta(U_{x,\mu},I)+\delta(U_{x+e_\mu,\mu},v_{y,\mu})$; if $o_\mu=1$, the roles of the two links are exchanged. Each link leaving a block in the direction μ occurs for two sites x of that block, and each link in the

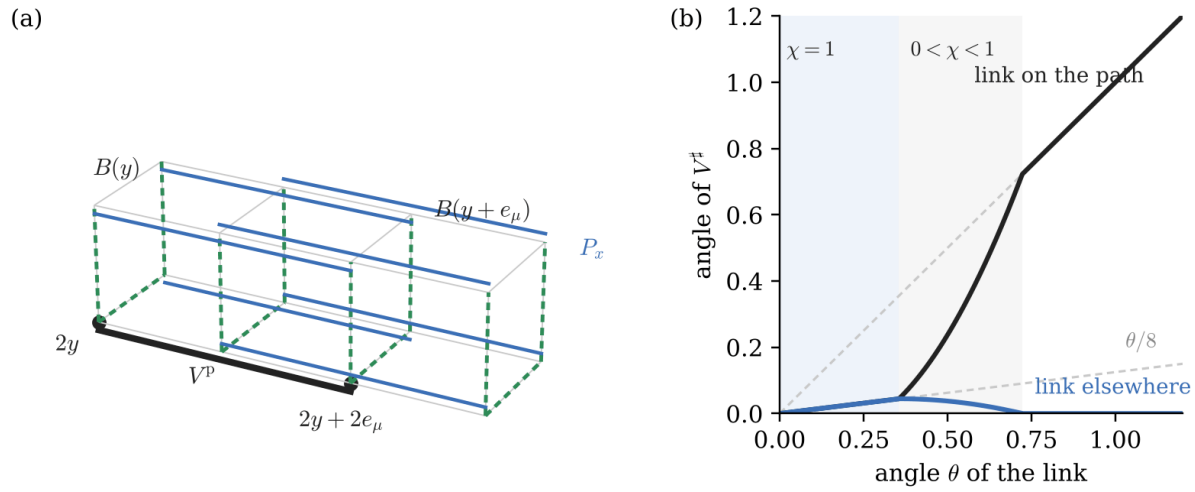


FIG. 29. (a) A block $B(y)$ of side two and the next block in the direction μ, drawn in three of the four dimensions. Green dashed: the tree links, equal to I. Blue: the paths P_x of length two from the sites of $B(y)$, each joining $2y$ to $2y+2e_\mu$ through the trees; black: the straight path P_{2y}, the coarse link of the path map. The cut-off average is the mean of the sixteen paths in the chart at P_{2y} while every path is close to it. (b) For SU(2) and one link $e^{\theta X}$, $X = \mathrm{diag}(i,-i)$, the angle of $V^\sharp$ when the link lies on the straight path (black) or at another transverse position (blue), with $\rho_G = 1$. Both equal $\theta/8$, the block average, while $\chi = 1$; past the cut-off they are the path map, θ and 0, so the slice windings are 1 and 0.

direction μ with $o_\mu = 0$ occurs once for the block containing it and once for the block before it. With $(a+b)^2 \le 2a^2+2b^2$ and the notation of the proof of Theorem 347, the sum is at most $2(2I_0+2J_0) = 4D_v^2 \le 896Q$. Next, if $\chi > 0$, Lemma 352 and the Cauchy–Schwarz inequality give

$$\delta(V^\sharp, P) \le \Big\| \frac{\chi}{16} \sum_x \log v_x \Big\| \le \frac{c}{16} \sum_x \delta(v_x, I) \le \frac{c}{4} \Big(\sum_x \delta(P_x, P)^2 \Big)^{1/2},$$

and the second bound follows from the first, $896c^2/16 = 56c^2$. The reconstruction $R(V^\sharp)$ agrees with $R(v)$ on the links with $o_\mu = 0$, and on the eight links with $o_\mu = 1$ of each block and direction $\delta(U_e, V^\sharp)^2 \le 2\delta(U_e, v)^2 + 2\delta(V^\sharp, v)^2$; so the third sum is at most $2D_v^2 + 16 \cdot 56c^2 Q \le (448 + 896c^2)Q$. Finally, write $V^\sharp = vM$. Removing the four factors M from the holonomy of a coarse plaquette one at a time changes $\delta(\cdot, I)$ by at most $\delta(M, I)$ each time, so $\delta(V_P^\sharp, I)^2 \le 2\delta(v_P, I)^2 + 8\sum_{\ell \in P} \delta(M_\ell, I)^2$. Each coarse link lies in six coarse plaquettes, so $Q_1(V^\sharp) \le 2Q_1(v) + 48 \cdot 56c^2 Q$, and $Q_1(v) \le 4Q$ by Proposition 348. The numerical bounds use $c^2 < 1.0967$.

(iii) For $s = 2$ the slice T consists of the eight links leaving $B(y)$, and the links inside the blocks are I, so $P_x = u_a$ for the transverse offset a of x, two sites for each a. Let t be a position other than the straight path and $u_{t'} = I$ for $t' \ne t$. Then $P = I$, two of the v_x equal u_t and the others I, so $F(u) = \exp(\frac{1}{8}\chi \log u_t)$ with $\chi = \phi(\delta(u_t, I)/\rho_G)$. On S the maps $u_t \mapsto \exp(\frac{\lambda}{8}\chi \log u_t)$, $0 \le \lambda \le 1$, are continuous by the argument above and take values in S, since $\log u_t$ commutes with S and the centraliser of a maximal torus in $\mathfrak{g}$ is its Lie algebra; they join F to the constant I, so $W_t = 0$. By Proposition 351 the W_t add up to the identity, which is therefore the matrix at the straight path.

(iv) At the straight path, $P_x = g$ for the two sites with transverse offset 0 and $P_x = I$ for the others, so $V^\sharp = g\exp(-\frac{14}{16}\log g) = \exp(\frac{1}{8}\log g)$; at another position, $P = I$ and two of the P_x equal g, so $V^\sharp = \exp(\frac{2}{16}\log g)$. □

The small-field part of (35) is Bałaban's averaging operation [7]; the cut-off extends it to all fields with the bounds of (ii). The theorem shows how far Proposition 351 reaches. The cut-off average is continuous, covariant and coercive for every field, and it agrees with the block averages to first order. Its slice windings are those of the path map, as the proposition demands of any such map, but the departure from the transverse symmetry happens only where some path differs from the straight one by more than $\rho_G/2$. Below that the field of a single link enters with the weight $\frac{1}{8}$ of the averages, whichever position it occupies.

What the theorem does not give is composition. The constants of (ii) are those of one step, and iterating them gives a factor that grows exponentially in the number of steps; on its large-field branch the map is the path map, for which Proposition 350 rules out any bound uniform in that number. Whether the k-fold cut-off average obeys $Q_k(V^{\sharp,k}(U)) \le C\,Q(U)$ and the corresponding coercivity bound with C independent of k is open for general fields; it is the form in which Remark 345(b) remains. Section XII C proves it, with $C = 1.001$, for fields with values in a maximal torus whose curvature stays small at every scale, and with $C = e^{12060\sigma}$ for fields of any compact group whose curvatures at the successive scales are small with sum σ.

C. Composition across scales

This subsection proves that the cut-off average composes across all scales on fields with values in a maximal torus whose curvature stays small, which includes the thin paths that defeat the path map, and then that it composes for every compact group on fields whose curvature is small, with a summable loss over the scales.

Let S be a maximal torus of $G \subset \mathrm{U}(r)$ and $\mathfrak{s}$ its Lie algebra. The exponential maps $\mathfrak{s}$ onto S, and its kernel Λ consists of diagonalisable matrices with eigenvalues in $2\pi i\mathbb{Z}$, so every nonzero element of Λ has norm at least 2π. For an S-valued field with $\delta(U_p, I) \le \rho_G$ on every plaquette put $\Theta_p = \log U_p$; it lies in $\mathfrak{g}$, commutes with S, and so lies in $\mathfrak{s}$. For a fine site x and a coordinate plane $\{\mu,\nu\}$ let $\square_x$ be the square of side two at x in that plane and $F_x = \sum_{p \subset \square_x} \Theta_p$, the sum of its four plaquettes, oriented as the square. Put

$$\eta_G = \rho_G / 12.$$

The k-fold cut-off average is defined by putting the coarse field back into the tree gauge, by an S-valued gauge transformation, before each step. The tree gauge fixes such a transformation up to one that is constant on blocks, under which $V^\sharp$ is covariant, so the coarse plaquette holonomies $U_P^{(j)}$ of the j-th iterate $U^{(j)}$ are well defined; we write $\Theta_P^{(j)} = \log U_P^{(j)}$ when $\delta(U_P^{(j)}, I) \le \rho_G$.

Lemma 354. *Let U be an S-valued field in the tree gauge with $\|\Theta_p\| \le \eta_G$ for every plaquette. Then $\chi_{y,\mu}(U) = 1$ for every coarse link, and the coarse plaquette P at y in the plane $\{\mu,\nu\}$ has*

$$V^\sharp(U)_P = \exp \bar F_P, \qquad \bar F_P = \frac{1}{16} \sum_{o \in \{0,1\}^4} F_{2y+o},$$

with $\|\bar F_P\| \le 4 \max_p \|\Theta_p\|$, so that $\log V^\sharp(U)_P = \bar F_P$.

Proof. Let $\eta = \max_p \|\Theta_p\|$. Fix (y,μ) and $x = 2y + o$. The tree path τ from $2y$ to x sets the coordinates of o one at a time in increasing order, so it has at most four edges, at most three of them transverse to μ. In the tree gauge its links are I, so $P_{2y}^{-1} P_x$ is the holonomy of the loop τ, then the straight path from x to $x + 2e_\mu$, then $\tau + 2e_\mu$ backwards, then the straight path from $2y + 2e_\mu$ back to $2y$. This loop is the boundary of the surface swept by the transverse edges of τ along two steps in the direction μ: at most six distinct plaquettes, each with coefficient ± 1. The group is abelian, so $P_{2y}^{-1} P_x = \exp d_x$ with d_x the signed sum of their Θ_p, and $\|d_x\| \le 6\eta \le \rho_G/2$. By Lemma 352, $\delta(P_x, P_{2y}) \le \rho_G/2$, so $\chi = 1$, and since the eigenvalues of d_x have modulus at most $\frac{1}{2}$, $\log(P_{2y}^{-1} P_x) = d_x$. Hence $V^\sharp_{y,\mu} = \exp(X_{y,\mu} + \bar d_{y,\mu})$, where $X_{y,\mu} \in \mathfrak{s}$ is any logarithm of P_{2y} and $\bar d_{y,\mu}$ is the mean of the d_x.

Let the sides of P be the coarse links ℓ with signs ε_ℓ, and write $d_{o,\ell}$ for the d_x of the side ℓ at the offset o. For each o the four straight paths of length two at the offset o of the four sides bound the square $\square_{2y+o}$, so

$$Y_o = \sum_\ell \varepsilon_\ell \big(X_\ell + d_{o,\ell} \big) - F_{2y+o} \in \Lambda.$$

Since $d_{0,\ell} = 0$, $\|Y_o - Y_0\| \le 4 \cdot 6\eta + 8\eta = 32\eta \le \frac{8}{3} < 2\pi$, so $Y_o = Y_0$ for every o. Averaging over o gives $\sum_\ell \varepsilon_\ell (X_\ell + \bar d_\ell) = \bar F_P + Y_0$, and $\exp Y_0 = I$. Finally $\|\bar F_P\| \le \max_o \|F_{2y+o}\| \le 4\eta < \pi/3$, so $\bar F_P$ is the principal logarithm. □

Theorem 355. *Let U be an S-valued field in the tree gauge on the periodic lattice of side divisible by 2^k, with $\delta(U_p, I) \le \rho_G$ on every plaquette, and put $\eta_0 = \max_p \|\Theta_p\|$, $A = \sum_p \|\Theta_p\|$. Suppose that*

$$\max_P \|\Theta_P^{(j)}\| \le \eta_G \quad \text{for } 0 \le j < k, \tag{36}$$

which holds in particular if $\eta_0 A \le \eta_G^2$. Then the cut-off is inactive at every step, $\Theta^{(j+1)} = \bar F(\Theta^{(j)})$, and for $j < k$

$$\sum_P \|\Theta_P^{(j+1)}\|^2 \le \sum_P \|\Theta_P^{(j)}\|^2, \qquad \sum_P \|\Theta_P^{(j+1)}\| \le \frac{1}{4} \sum_P \|\Theta_P^{(j)}\|,$$
$$\max_P \|\Theta_P^{(j+1)}\| \le 4 \max_P \|\Theta_P^{(j)}\|.$$

Consequently

$$Q_k\big(V^{\sharp,k}(U)\big) \le \sum_p \|\Theta_p\|^2 \le 1.001\, Q(U),$$
$$Q_k\big(V^{\sharp,k}(U)\big) \le 16^{-k} A^2.$$

Proof. The first assertions follow from Lemma 354 by induction, the S-valued gauge fixing between steps leaving the plaquette holonomies unchanged. The coarse plaquettes P in a plane together with the offsets o are in bijection with the fine squares $\square_x$ in that plane, $x = 2y + o$, and a fine plaquette lies in four such squares. With $\|\bar F_P\|^2 \le \frac{1}{16} \sum_o \|F_{2y+o}\|^2$ and $\|F_x\|^2 \le 4 \sum_{p \subset \square_x} \|\Theta_p\|^2$ this gives $\sum_P \|\bar F_P\|^2 \le \frac{1}{16} \cdot 4 \cdot 4 \sum_p \|\Theta_p\|^2$; the same count without Cauchy–Schwarz gives the factor $\frac{1}{4}$ in the second inequality, and the third is $\|F_x\| \le 4\eta$.

For the sufficient condition, let η_j and A_j be the maximum and the sum of $\|\Theta_P^{(j)}\|$. While (36) holds up to $j-1$, the inequalities give $\eta_j \le 4^j \eta_0$ and $\eta_j \le A_j \le 4^{-j} A$. The product of the two bounds is $\eta_0 A \le \eta_G^2$, so the smaller one is at most η_G, and (36) holds at j.

Finally $\delta(\exp\Theta, I) \le \|\Theta\|$ (Lemma 352), so $Q_k(V^{\sharp,k}(U)) \le \sum_P \|\Theta_P^{(k)}\|^2$, which is at most both $\sum_p \|\Theta_p\|^2$ and $\eta_k A_k \le A_k^2 \le 16^{-k} A^2$. The eigenvalues of each Θ_p are $i\theta$ with $|\theta| \le \eta_0 \le \eta_G \le \frac{1}{12}$, since $\eta_0^2 \le \eta_0 A$ in the sufficient condition and $\eta_0 \le \eta_G$ is the case $j = 0$ of (36); and $\theta^2 \le 1.001 \cdot 4 \sin^2(\theta/2)$ there, which gives $\sum_p \|\Theta_p\|^2 \le 1.001\, Q(U)$. □

Corollary 356. *For $G = \mathrm{SU}(2)$ and the fields of Proposition 350 with $s = 2^k \ge 2^{14}$,*

$$Q_k\big(V^{\sharp,k}(U)\big) \le \frac{18\pi^2}{s^2} \le \frac{4}{s}\, Q(U),$$
$$Q_k\big(V^{\mathrm{p},k}(U)\big) = 24 \ge \frac{4s}{\pi^2}\, Q(U).$$

Proof. Here $\rho_G = 1$ and S is the diagonal torus. The $3s$ nonzero plaquettes have $\Theta_p = \pm 2T/s$, of norm $\sqrt{2}\pi/s$, so $\eta_0 A = 6\pi^2/s \le \frac{1}{144}$ when $s \ge 864\pi^2$, which holds for $s \ge 2^{14}$, and $A^2 = 18\pi^2$. Proposition 350 gives $Q(U) = 24 s \sin^2(\pi/2s)$, which lies between $(6\pi^2/s)(1 - \pi^2/(12s^2))$ and $6\pi^2/s$. □

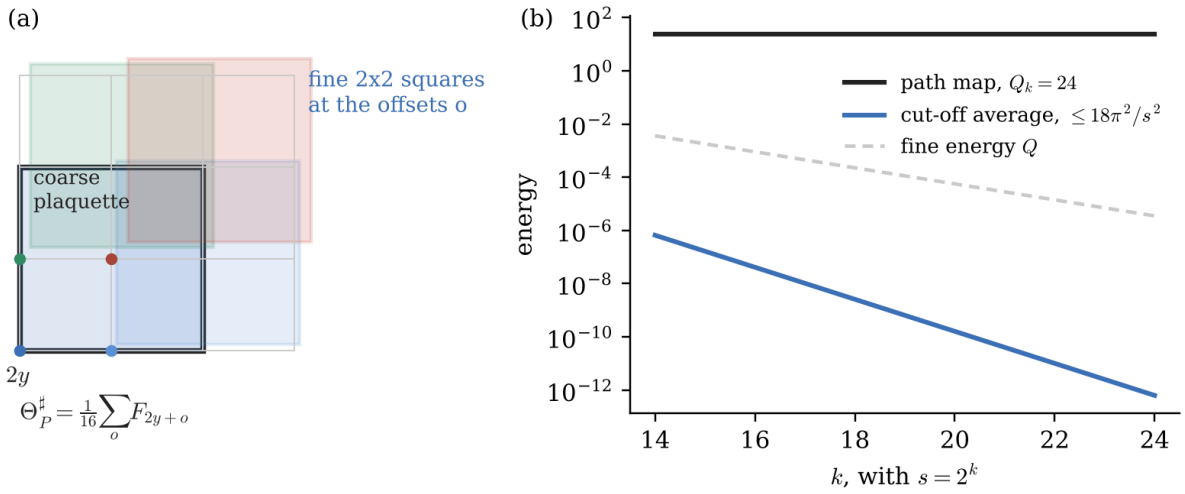


FIG. 30. (a) In the torus, the coarse plaquette of the cut-off average is the mean of the fluxes F_{2y+o} of the sixteen fine squares of side two at the offsets o (four of them drawn), whatever the gauge of the coarse links (Lemma 354). (b) For the thin-path fields of Proposition 350, the coarse energy after k steps: 24 for the path map, at most $18\pi^2/s^2$ for the cut-off average (Corollary 356); the fine energy is about $6\pi^2/s$.

So on torus-valued fields of small curvature the cut-off average behaves like the linear averages: the curvature is averaged over the sixteen offsets at every step, the energy never increases, and a thin path is spread out and contracted instead of being carried to the coarse lattice (Fig. 30).

For general fields the paths do not commute, and the logarithms carry commutators from the second order on. The next two lemmas bound these commutators by the square of the local curvature. The same composition then holds for every compact group, with a loss of at most $1+O(\eta_j)$ at scale j.

Lemma 357. *Let $Z_1,\dots,Z_n \in \mathfrak{g}$ with $r = \sum_i \|Z_i\| \le \min\{\frac14, \rho_G\}$, and let $g = e^{Z_1}\cdots e^{Z_n}$. Then $\delta(g,I) \le r$, $\log g \in \mathfrak{g}$, and*

$$\Big\| \log g - \sum_i Z_i \Big\| \le 1.6\, r^2.$$

Proof. The distance δ is invariant under multiplication by unitary matrices on either side, so $\delta(g,I) \le \sum_i \delta(e^{Z_i},I) \le r$ by Lemma 352, and $\log g \in \mathfrak{g}$ by the choice of ρ_G. Expanding the exponentials, $g - I - \sum_i Z_i$ is a sum of products of at least two factors. Since the Hilbert–Schmidt norm is submultiplicative, its norm is at most $e^r - 1 - r \le \frac12 r^2 e^r \le 0.643\, r^2$, and $M = g - I$ has $\|M\| \le e^r - 1 \le 1.137\, r \le 0.285$. The operator norm of M is smaller than 1. The principal logarithm is holomorphic on the disc $|z-1|<1$, which contains the spectrum of g, so it is given there by its power series, $\log g = \sum_{k\ge1} (-1)^{k+1} M^k/k$. Hence

$$\|\log g - M\| \le \frac{\|M\|^2}{2(1-\|M\|)} \le 0.905\, r^2,$$

and the two bounds add up to less than $1.6\, r^2$. □

Let U be a G-valued field in the tree gauge with $\delta(U_p,I) \le \rho_G$, and put $\Theta_p = \log U_p$. A change of base point conjugates Θ_p and does not change its norm. Let P be a coarse plaquette at y in the plane $\{\mu,\nu\}$. Its sides are $\ell_1 = (y,\mu)$, $\ell_2 = (y+e_\mu,\nu)$, $\ell_3 = (y+e_\nu,\mu)$ and $\ell_4 = (y,\nu)$, with initial points y_i. For an offset $o \in \{0,1\}^4$ let $P_{i,o}$ be the path product of ℓ_i through $2y_i + o$, and put

$$K_{P,o} = P_{1,o} P_{2,o} P_{3,o}^{-1} P_{4,o}^{-1}, \qquad \bar F_P = \frac{1}{16} \sum_o \log K_{P,o}.$$

The tree segments between consecutive sides cancel. Hence $K_{P,o}$ is the holonomy of the square $\square_{2y+o}$, carried to $2y$ along the tree. In a torus, $\log K_{P,o} = F_{2y+o}$, and $\bar F_P$ is the mean of Lemma 354. Let $\sigma_{P,o}$ be the sum of $\|\Theta_p\|$ over the four plaquettes of $\square_{2y+o}$.

Lemma 358. *Let U be a G-valued field in the tree gauge with $\delta(U_p,I) \le \rho_G$ and $\|\Theta_p\| \le \eta \le \rho_G/1000$ for every plaquette. Then $\chi_{y,\mu}(U) = 1$ for every coarse link. Every coarse plaquette has $\delta(V^\sharp(U)_P, I) \le 0.03\, \rho_G$ and*

$$\begin{aligned} \big\| \log V^\sharp(U)_P - \bar F_P \big\| &\le 3.2\, r_P^2, \\ \|\bar F_P\| &\le \frac{1+6.4\eta}{16} \sum_o \sigma_{P,o}, \end{aligned} \tag{37}$$

with $r_P \le 28.26\,\eta$ defined in the proof. Let E, A be the sum of $\|\Theta_p\|^2$ and of $\|\Theta_p\|$, and let E', A', η' be the sum of squares, the sum and the maximum of $\|\log V^\sharp(U)_P\|$. Then

$$\sqrt{E'} \le (1+6030\eta)\sqrt{E}, \qquad A' \le (1+9147\eta)\frac{A}{4},$$
$$\eta' \le 4(1+647\eta)\,\eta.$$

Proof. Paths. Fix a coarse link (y,μ) and $x = 2y+o$. As in the proof of Lemma 354, $P_{2y}^{-1} P_x$ is the holonomy of a loop bounding a surface of at most six plaquettes, swept by the transverse edges of the tree path. Build the surface one plaquette at a time. This writes the holonomy as a product of at most six

factors $hU_p^{\pm1}h^{-1} = \exp(\pm h\Theta_p h^{-1})$. Let $\sigma_{(y,\mu),o} \le 6\eta$ be the sum of their $\|\Theta_p\|$. By Lemma 357, $\delta(P_x, P_{2y}) \le 6\eta \le \rho_G/2$, so $\chi = 1$. The logarithm $D_{(y,\mu),o} = \log(P_{2y}^{-1}P_x)$ lies in $\mathfrak{g}$, with

$$\|D_{(y,\mu),o}\| \le (1+9.6\eta)\,\sigma_{(y,\mu),o}.$$

In the same way $K_{P,o}$ is a product of four such factors, so $\|\log K_{P,o}\| \le (1+6.4\eta)\sigma_{P,o}$. Averaging over o gives the second inequality of (37).

The plaquette. Write $D_{i,o}$ for $D_{\ell_i,o}$. Then $P_{i,o} = P_{i,0}e^{D_{i,o}}$ and $D_{i,0} = 0$, and $V_{\ell_i}^\sharp = P_{i,0}e^{m_i}$ with $m_i = \frac{1}{16}\sum_o D_{i,o}$. The paths and $V^\sharp$ transform as links under gauge transformations constant on blocks. These transformations preserve the tree gauge and conjugate $D_{i,o}$, m_i, $K_{P,o}$ and $V_P^\sharp$ by fixed elements. Choose one with $P_{1,0} = P_{2,0} = P_{3,0} = I$. Then $P_{4,0} = K_{P,0}^{-1}$, and with $c = \log K_{P,0}$,

$$V_P^\sharp = e^{m_1}e^{m_2}e^{-m_3}e^{-m_4}e^{c},$$
$$K_{P,o} = e^{D_{1,o}}e^{D_{2,o}}e^{-D_{3,o}}e^{-D_{4,o}}e^{c}.$$

Put $r_P = \sum_i \max_o\|D_{i,o}\| + \|c\|$. Since $\|m_i\| \le \max_o\|D_{i,o}\|$, the five exponents in each product have norms adding up to at most r_P, and

$$r_P \le 24(1+9.6\eta)\eta + 4(1+6.4\eta)\eta \le 28.26\,\eta.$$

Lemma 357 gives $\delta(V_P^\sharp, I) \le r_P \le 0.03\,\rho_G$ and

$$\log V_P^\sharp = m_1 + m_2 - m_3 - m_4 + c + R,$$
$$\log K_{P,o} = D_{1,o} + D_{2,o} - D_{3,o} - D_{4,o} + c + R_o,$$

with $\|R\|, \|R_o\| \le 1.6\,r_P^2$. Average the second line over o and subtract it from the first; since m_i is the mean of the $D_{i,o}$, this is the first inequality of (37).

Sums. Each fine plaquette lies in four squares $\square_x$, and the pairs (P,o) are in bijection with these squares. The second inequality of (37) and Cauchy–Schwarz therefore give $\sum_P\|\bar F_P\|^2 \le (1+6.4\eta)^2 E$; without Cauchy–Schwarz they give $\sum_P\|\bar F_P\| \le (1+6.4\eta)A/4$. For the remainders, let $\mathscr{U}_\ell$ be the set of plaquettes of the surfaces of the side $\ell = (y,\mu)$ for all sixteen offsets. These lie in planes $\{\mu,\alpha\}$, $\alpha \ne \mu$, at μ-positions $2y_\mu$, $2y_\mu+1$, $2y_\mu+2$, and there are at most 36 of them. A plaquette lies in $\mathscr{U}_\ell$ for at most four coarse links. With $m'_\ell = \sum_{p\in\mathscr{U}_\ell}\|\Theta_p\|$,

$$r_P \le 1.0096\Big(\sum_{\ell\subset P} m'_\ell + \sigma_{P,0}\Big).$$

A coarse link is a side of six coarse plaquettes, and a plaquette lies in $\square_{2y}$ for at most one coarse plaquette. Therefore

$$\sum_P r_P \le 1.0096\,(6\cdot 4A + A) \le 25.24A,$$
$$\sum_P r_P^2 \le 1.0096^2\cdot 5\,(6\cdot 36\cdot 4E + 4E) \le 4424E.$$

Hence, with $r_P \le 28.26\,\eta$,

$$\sum_P 3.2\,r_P^2 \le 3.2\cdot 28.26\cdot 25.24\,\eta A \le 2283\,\eta A,$$
$$\Big(\sum_P \big(3.2\,r_P^2\big)^2\Big)^{1/2} \le 3.2\cdot 28.26\,\eta\,(4424E)^{1/2} \le 6015\,\eta\sqrt{E}.$$

The triangle inequality, for the Euclidean norm over P in the first case, gives $\sqrt{E'} \le (1+6.4\eta+6015\eta)\sqrt{E}$ and $A' \le (1+6.4\eta)A/4 + 2283\,\eta A$. For the maximum, $\eta' \le (1+6.4\eta)4\eta + 3.2\,(28.26\,\eta)^2 \le 4(1+647\eta)\eta$. □

For a G-valued field, the k-fold cut-off average is defined as before, with G-valued gauge transformations. The tree gauge fixes them up to transformations constant on blocks, under which $V^\sharp$ is covariant. These transformations conjugate the plaquette holonomies and leave the norms $\|\Theta_P^{(j)}\|$ unchanged.

Theorem 359. *Let $G \subset \mathrm{U}(r)$ be compact and let U be a G-valued field on the periodic lattice of side divisible by 2^k, with $\delta(U_p, I) \le \rho_G$ on every plaquette. Let η_j, A_j, E_j be the maximum, the sum and the sum of squares of $\|\Theta_P^{(j)}\|$ over the plaquettes of the j-th iterate.*

(i) *If $\eta_j \le \rho_G/1000$ for $0 \le j < k$, the cut-off is inactive at every step. With $\sigma = \sum_{j<k}\eta_j$,*

$$E_k \le e^{12060\sigma}E_0, \qquad A_k \le e^{9147\sigma}4^{-k}A_0,$$
$$\eta_k \le e^{647\sigma}4^k\eta_0.$$

(ii) *If $\eta_0 A_0 \le 2\cdot 10^{-10}\rho_G^2$, then (i) applies for every k with $\sigma \le 7.25\sqrt{\eta_0 A_0}$, and*

$$Q_k\big(V^{\sharp,k}(U)\big) \le e^{8.8\cdot 10^4\sqrt{\eta_0 A_0}}E_0 \le 1.0001\,e^{8.8\cdot 10^4\sqrt{\eta_0 A_0}}\,Q(U),$$
$$Q_k\big(V^{\sharp,k}(U)\big) \le e^2\,16^{-k}A_0^2.$$

Proof. (i) Apply Lemma 358 at each step to the field in the tree gauge; the gauge fixing does not change the norms. The lemma gives $\delta(U_P^{(j+1)}, I) \le \rho_G$, so $\Theta^{(j+1)}$ is defined, and with $1+t \le e^t$ the three inequalities of the lemma multiply over the steps.

(ii) Suppose that for $i < j$ we have $\eta_i \le \rho_G/1000$ and $\sum_{i<j}\eta_i \le 7.25\sqrt{\eta_0 A_0} \le 1/9747$. Then (i) gives $A_j \le e\,4^{-j}A_0$ and $\eta_j \le e\,4^j\eta_0$, and $\eta_j \le A_j$. So $\eta_j \le e\min\{4^j\eta_0, 4^{-j}A_0\} \le e\sqrt{\eta_0 A_0} \le \rho_G/1000$. Let i_* be the last index with $4^{i_*}\eta_0 \le 4^{-i_*}A_0$. The terms of $\min\{4^i\eta_0, 4^{-i}A_0\}$ increase by the factor 4 up to i_* and decrease by the factor 4 after it. The terms at i_* and i_*+1 are at most $\sqrt{\eta_0 A_0}$, so the sum over all i is at most $\frac{8}{3}\sqrt{\eta_0 A_0}$, and $\sum_{i\le j}\eta_i \le \frac{8e}{3}\sqrt{\eta_0 A_0} \le 7.25\sqrt{\eta_0 A_0}$. This completes the induction. Then $E_k \le e^{12060\cdot 7.25\sqrt{\eta_0 A_0}}E_0$. By Lemma 352, $Q_k(V^{\sharp,k}(U)) \le E_k$, and $E_k \le \eta_k A_k \le A_k^2 \le e^2\,16^{-k}A_0^2$. Finally the eigenvalues $i\theta$ of each Θ_p have $|\theta| \le \eta_0 \le \sqrt{\eta_0 A_0} < 10^{-4}$, and $\theta^2 \le 1.0001\cdot 4\sin^2(\theta/2)$ there. □

For a torus Theorem 355 is sharper, and its hypothesis is weaker; the content of Theorem 359 is the non-abelian case. Neither theorem gives the supremum over boundary conditions in (D). Theorem 353 has the constants of one step. Theorems 355 and 359 need the curvature to be small at every scale: in Theorem 359 the losses $e^{6030\eta_j}$ must have a bounded product, and a field whose curvature is smooth on a scale ℓ grows by a factor four per step until the coarse spacing reaches ℓ. The large fields, where the cut-off is active, and the Wilson measure itself are outside both; Sec. XII E shows where the large fields lie under that measure.

D. Small fields on the fibres of the cut-off average

Theorem 330 is a gap on the linear fibres of the block averages. For group-valued links the fibres of the cut-off average are curved, and Remark 345(b) asks whether the gap survives on them. It does at every background whose plaquettes are small, however far its links are from I, with a loss linear in the curvature.

Let $G \subset \mathrm{U}(r)$ be compact, work on the periodic lattice of side $2L$ in the tree gauge of Definition 328, and let W be a set of blocks. Write A_W for the active links with base in $\bigcup W$ and W^+ for the blocks at ℓ^∞ distance at most one from W. Fix U and let F be the set of fields that agree with U off A_W and have the same cut-off averages $V^\sharp_{y,\mu}$ as U for every $y \in W$ and every μ. The group G^{A_W} carries the bi-invariant metric with $|\xi|^2 = T(\xi) = \sum_e \|\xi_e\|^2$, $\xi_e \in \mathfrak{g}$, and its geodesics through U are the curves $U_e \exp(t\xi_e)$; write $\mathrm{Ad}(g)X = gXg^{-1}$. For a plaquette p at x in the plane $\{\mu,\nu\}$ with links $e_1 = (x,\mu)$, $e_2 = (x+e_\mu,\nu)$, $e_3 = (x+e_\nu,\mu)$, $e_4 = (x,\nu)$ and $U_p = U_1U_2U_3^{-1}U_4^{-1}$, the *covariant curvature*

$$D_p(\xi) = \mathrm{Ad}(U_1)\xi_1 + \mathrm{Ad}(U_1U_2)(\xi_2 - \xi_3) - \mathrm{Ad}(U_1U_2U_3^{-1})\xi_4$$

is the derivative of $U_p(t)U_p^{-1}$ along these curves. Reversing the orientation replaces it by $-\mathrm{Ad}(U_p^{-1})D_p(\xi)$, of the same norm, so $Q_U(\xi) = \sum_p \|D_p(\xi)\|^2$ does not depend on the orientations. At $U \equiv I$, $D_p(\xi) = (d\xi)_p$ and Q_I is the form Q of Theorem 330. For $y \in W$ and a direction μ let $v = U_{2y+e_\mu,\mu}$ be the path link, let $\tilde\xi$ equal $\mathrm{Ad}(v^{-1})\xi_e$ on the links of $B(y)$ and ξ_e on the other links, and let $N_{y,\mu}(\xi)^2$ be the sum of $\|\xi_e\|^2$ over the 24 links that enter $V_{y,\mu}$.

Lemma 360. *Suppose that $\delta(U_p,I) \le \kappa \le 10^{-8}\rho_G$ for every plaquette with a vertex in a block of W^+, and let $y \in W$.*

(i) *Every active link e inside a block of W^+ has $\delta(U_e,I) \le 3\kappa$, and every link e that leaves $B(y)$ in the direction μ has $\delta(U_e,v) \le 21\kappa$.*

(ii) *For a plaquette inside a block of W^+, $\|D_p(\xi) - (d\xi)_p\| \le 18\kappa \sum_{e\in p}\|\xi_e\|$.*

(iii) *Let $x \in B(y)$ with x_μ odd and x_ν even, and let p be the plaquette at x in the plane $\{\mu,\nu\}$, oriented as above, so that e_1, e_3 leave $B(y)$, e_2 lies in $B(y+e_\mu)$ and e_4 in $B(y)$. Then*

$$\big\|\mathrm{Ad}(v^{-1})D_p(\xi) - \big(\xi_1 + \xi_2 - \xi_3 - \mathrm{Ad}(v^{-1})\xi_4\big)\big\| \le 48\kappa \sum_{e\in p}\|\xi_e\|.$$

(iv) *Near U the cut-off average is*

$$V^\sharp_{y,\mu} = v\exp\Big(\frac{1}{16}\sum_x \log(v^{-1}P_x)\Big),$$

and its derivative satisfies

$$\Big\|V^\sharp_{y,\mu}(U)^{-1}\frac{d}{dt}V^\sharp_{y,\mu}\big(Ue^{t\xi}\big)\Big|_{t=0} - V_{y,\mu}(\tilde\xi)\Big\| \le 260\,\kappa N_{y,\mu}(\xi).$$

Proof. We use $\|\mathrm{Ad}(g)X - X\| \le 2\delta(g,I)\|X\|$, which holds since $gXg^{-1} - X = ((g-I)X - X(g-I))g^{-1}$, and $\delta(gh,I) \le \delta(g,I) + \delta(h,I)$.

(i) The chains of plaquettes in the proof of Theorem 330 have their transverse links on the tree, so $\delta(U_e,I) \le \sum_{i\le k}\delta(U_{p_i},I) \le 3\kappa$. For the links leaving $B(y)$, the face plaquettes give $\delta(U_a,U_{a'}) \le \kappa + 3\kappa + 3\kappa$ for adjacent transverse offsets, as in the proof of Theorem 347, and every offset is joined to the straight path, which carries v, by at most three such steps.

(ii) The links of p and the products U_1U_2, $U_1U_2U_3^{-1}$ are within 3κ, 6κ and 9κ of I by (i).

(iii) By (i), $\delta(v^{-1}U_1,I) \le 21\kappa$ and $\delta(v^{-1}U_1U_2,I) \le 24\kappa$, and $v^{-1}U_1U_2U_3^{-1} = v^{-1}U_pU_4$ with $\delta(U_pU_4,I) \le 4\kappa$.

(iv) Write $P_x = U_{x,\mu}U_{x+e_\mu,\mu}$, $w_x = v^{-1}P_x$, $X_x = \log w_x$ and $M = \frac{1}{16}\sum_x X_x$; the path P_{2y} is v, its first link being on the tree. One of the two links of P_x leaves $B(y)$ and the other lies in $B(y)$ or $B(y+e_\mu)$, so $\delta(P_x,v) \le 24\kappa < \rho_G/2$ by (i); hence $\chi_{y,\mu} = 1$ near U, and $\|X_x\|, \|M\| \le 8\pi\kappa$ by Lemma 352. Since $\|[X,Y]\| \le 2\|X\|\,\|Y\|$, the operators ad_{X_x} and ad_M have norm at most $a = 16\pi\kappa \le 50.3\kappa$. Along $Ue^{t\xi}$ put $\alpha = \xi_{2y+e_\mu,\mu}$ and $\beta_x = \mathrm{Ad}(U^{-1}_{x+e_\mu,\mu})\xi_{x,\mu} + \xi_{x+e_\mu,\mu}$, the right logarithmic derivatives of v and P_x. Then $w_x^{-1}\dot w_x = \beta_x - \mathrm{Ad}(w_x^{-1})\alpha$, and the formula for the derivative of the exponential map gives $\dot X_x = \Phi(\mathrm{ad}_{X_x})(\beta_x - \mathrm{Ad}(w_x^{-1})\alpha)$ and

$$(V^\sharp)^{-1}\dot V^\sharp = \mathrm{Ad}(e^{-M})\alpha + \Psi(\mathrm{ad}_M)\dot M,$$

with $\Phi(z) = z/(1-e^{-z})$ and $\Psi(z) = (1-e^{-z})/z$. The operators ad_X are skew-adjoint, and from the power series, $|\Phi(z)-1| \le |z|/2 + 0.09|z|^2 \le |z|$ and $|\Psi(z)-1| \le |z|$ for $|z| \le 1$, the Bernoulli numbers giving $\sum_{k\ge1}|B_{2k}|/(2k)! < 0.09$. By Lemma 329(iii), $V_{y,\mu}(\tilde\xi) = \frac{1}{16}\sum_x(\tilde\xi_{x,\mu} + \tilde\xi_{x+e_\mu,\mu}) = \alpha + \frac{1}{16}\sum_x(\beta_x - \alpha) - \frac{1}{16}\sum_x \varepsilon_x$, where $\varepsilon_x = \mathrm{Ad}(U^{-1}_{x+e_\mu,\mu})\xi_{x,\mu} - \tilde\xi_{x,\mu}$. If x_μ is even, $U_{x+e_\mu,\mu}$ leaves $B(y)$ and $\tilde\xi_{x,\mu} = \mathrm{Ad}(v^{-1})\xi_{x,\mu}$; otherwise $U_{x+e_\mu,\mu}$ lies in $B(y+e_\mu)$ and $\tilde\xi_{x,\mu} = \xi_{x,\mu}$; so $\|\varepsilon_x\| \le 42\kappa\|\xi_{x,\mu}\|$ by (i). With m_β and m_ξ the means of $\|\beta_x\|$ and $\|\xi_{x,\mu}\|$, the four differences $\mathrm{Ad}(e^{-M})\alpha - \alpha$, $(\Psi(\mathrm{ad}_M)-1)\dot M$, $\dot M - \frac{1}{16}\sum_x(\beta_x - \alpha)$ and $\frac{1}{16}\sum_x\varepsilon_x$ have norms at most $51\kappa\|\alpha\|$, $51.6\kappa(m_\beta + \|\alpha\|)$, $51\kappa m_\beta + 99\kappa\|\alpha\|$ and $42\kappa m_\xi$, using $\|\mathrm{Ad}(w_x^{-1}) - 1\| \le 48\kappa$. The links leaving $B(y)$ occur twice among the $\xi_{x,\mu}$, $\xi_{x+e_\mu,\mu}$ and the other sixteen links once, so the Cauchy–Schwarz inequality gives $m_\beta \le \frac{\sqrt{48}}{16}N_{y,\mu} \le 0.434N_{y,\mu}$ and $m_\xi \le \frac14 N_{y,\mu}$, and the sum is at most $(202 + 103\cdot0.434 + 42/4)\kappa N_{y,\mu} \le 260\kappa N_{y,\mu}$. □

Theorem 361. *Under the hypothesis of Lemma 360:*

(i) *for every $\xi \in \mathfrak{g}^{A_W}$ tangent to F at U, that is with $\frac{d}{dt}V^\sharp_{y,\mu}(Ue^{t\xi})|_{t=0} = 0$ for all $y \in W$ and all μ,*

$$T(\xi) \le 52\big(1 + 1.4\cdot10^6\kappa\big)\,Q_U(\xi);$$

(ii) *for every $\xi \in \mathfrak{g}^{A_W}$,*

$$\frac{d^2}{dt^2}Q\big(Ue^{t\xi}\big)\Big|_{t=0} \ge 2Q_U(\xi) - 84\kappa T(\xi),$$

so on the tangent space of F the Hessian of Q is at least $(\frac{1}{26} - 5.4\cdot 10^4\kappa)T(\xi)$; for $\mathrm{SU}(2)$ the Hessian of the Wilson action $\beta\sum_p(1-\frac12\operatorname{Re}\operatorname{Tr}U_p) = \frac{\beta}{4}Q$ on the linearised constrained directions is at least $\frac{\beta}{2}(\frac{1}{52} - 2.7\cdot 10^4\kappa)T(\xi)$;

(iii) *near U, F is a smooth submanifold of G^{A_W} of codimension $4|W|\dim G$, and the Hessian of $Q|_F$ for the induced metric satisfies $\operatorname{Hess}_F Q(\xi,\xi) \geq (\frac{1}{26} - c_G\kappa)T(\xi)$, where c_G depends only on G.*

Proof. (i) We follow the proof of Theorem 330, using $(a+b)^2 \leq (1+\kappa)a^2 + (1+\kappa^{-1})b^2$ for each error term. *Inside a block.* For the chain $p_1,\dots,p_k$ of an active link $e=(x_0,\mu)$, $\xi_e = \sum_i \pm(d\xi)_{p_i}$ with tree links on the transverse sides, and Lemma 360(ii) gives

$$\|\xi_e\| \leq \sum_i \|D_{p_i}\| + 36\kappa \sum_{j<k} \|\xi_{x_j,\mu}\|.$$

Hence $\|\xi_e\|^2$ is at most $(1+\kappa)k\sum_i\|D_{p_i}\|^2$ plus $3888\kappa(1+\kappa)\sum_{j<k}\|\xi_{x_j,\mu}\|^2$. A link (x',μ) is a point x_j of the chains of at most eight links, which differ from x' by a set of directions above μ, so

$$I \leq 8(1+\kappa)Q_{\mathrm{in}} + 31104\kappa(1+\kappa)I.$$

Between blocks. Fix $y\in W$ and μ, and work in the frame of Lemma 360(iii),(iv); the norms do not depend on it. For a transverse edge $\{a,a'\}$, part (iii) gives $\|\xi_a - \xi_{a'}\| \leq \|D_p\| + \|\xi_\psi\| + \|\xi_{\psi'}\| + 48\kappa(\|\xi_a\| + \|\xi_{a'}\| + \|\xi_\psi\| + \|\xi_{\psi'}\|)$, so the square is at most $3(1+\kappa)(\|D_p\|^2 + \|\xi_\psi\|^2 + \|\xi_{\psi'}\|^2)$ plus $9216\kappa(1+\kappa)$ times the sum of the four squares. Since ξ is tangent, part (iv) gives $\|V_{y,\mu}(\tilde\xi)\| \leq 260\kappa N_{y,\mu}$, and by Lemma 329(iii) the mean m of the ξ_a is $V_{y,\mu}(\tilde\xi)$ minus a sixteenth of the sum of the sixteen framed links of S^μ_y and $S^\mu_{y+e_\mu}$; so $8\|m\|^2 \leq (1+\kappa)\frac12\sum_{16}\|\xi_e\|^2 + 540800\kappa(1+\kappa)N^2_{y,\mu}$. With the Laplacian of the cube as before,

$$\begin{aligned}\sum_a\|\xi_a\|^2 \leq{}& (1+\kappa)\Big(\frac12\sum_{16}\|\xi_e\|^2 + \frac32\sum_{\text{edges}}\Sigma_p\Big)\\ &+\kappa(1+\kappa)\Big(540800N^2_{y,\mu}\\ &+4608\big(3\sum_a\|\xi_a\|^2 + \sum_\Psi\|\xi_e\|^2\big)\Big),\end{aligned}$$

where $\Sigma_p = \|D_p\|^2 + \|\xi_\psi\|^2 + \|\xi_{\psi'}\|^2$ and Ψ is the set of the 24 links ψ,ψ'. A link lies in at most two of the sets of $N_{y,\mu}$ and in at most three of the sets Ψ, so summing over $y\in W$ and μ as in the proof of Theorem 330, with $J = T - I$,

$$\begin{aligned}J \leq{}& (1+\kappa)\Big(\frac{11}{2}I + \frac32 Q_{\text{face}}\Big)\\ &+1.1\cdot 10^6\kappa(1+\kappa)T.\end{aligned}$$

Adding I and inserting the bound for I gives $T \leq 52(1+\kappa)^2Q_U(\xi) + 1.31\cdot 10^6\kappa(1+\kappa)^2T$, and the claim follows for $\kappa \leq 10^{-8}$.

(ii) Write $D_p(\xi) = B_1+B_2+B_3+B_4$ with

$$\begin{aligned}B_1 &= \operatorname{Ad}(U_1)\xi_1, \quad B_2 = \operatorname{Ad}(U_1U_2)\xi_2,\\ B_3 &= -\operatorname{Ad}(U_1U_2)\xi_3, \quad B_4 = -\operatorname{Ad}(U_1U_2U_3^{-1})\xi_4.\end{aligned}$$

Then $U_p(t) = e^{tB_1}e^{tB_2}e^{tB_3}e^{tB_4}U_p$, so $\ddot U_p = (D_p^2 + C_p)U_p$ with $D_p = \sum_i B_i$ and $C_p = \sum_{i<j}[B_i,B_j]$. Since $\delta(U_p,I)^2 = 2r - 2\operatorname{Re}\operatorname{Tr}U_p$, $\operatorname{Tr}C_p = 0$ and $-\operatorname{Tr}D_p^2 = \|D_p\|^2$,

$$\begin{aligned}\frac{d^2}{dt^2}\delta(U_p,I)^2 = 2\|D_p\|^2 &- 2\operatorname{Re}\operatorname{Tr}\big(D_p^2(U_p-I)\big)\\ &- 2\operatorname{Re}\operatorname{Tr}\big(C_p(U_p-I)\big),\end{aligned}$$

and $\|C_p\| \leq \sum_{i<j}2\|B_i\|\|B_j\| \leq 3\sum_{e\in p}\|\xi_e\|^2$. So the second derivative is at least $2(1-\kappa)\|D_p\|^2 - 6\kappa\sum_{e\in p}\|\xi_e\|^2$ when $\delta(U_p,I)\leq\kappa$. Every plaquette that contains a link of A_W has a vertex in a block of W^+, each link lies in six plaquettes, and $Q_U(\xi) \leq 24T(\xi)$; this gives the bound, and with (i) $\frac{2}{52(1+1.4\cdot10^6\kappa)} - 84\kappa \geq \frac{1}{26} - 5.4\cdot10^4\kappa$.

(iii) Let $C_{y,\mu}(U') = \log(V^\sharp_{y,\mu}(U)^{-1}V^\sharp_{y,\mu}(U'))$ near U, so that F is $C^{-1}(0)$ there, and let $\mathscr{G} = DCDC^*$ at U, a block matrix indexed by the pairs $k=(y,\mu)$. By Lemma 360(iv), $DC_k = \tilde B_k + R_k$, where $\tilde B_k\xi = V_{y,\mu}(\tilde\xi)$ and $\|R_k\| \leq 260\kappa$ as maps from the 24 links. Each $\operatorname{Ad}(v^{-1})$ is an isometry, so $\tilde B_k\tilde B_k^*$ is the sum of the squared coefficients of $V_{y,\mu}$ on A_W, at least $\frac18$, $\|\tilde B_k\| \leq \sqrt{3/16} \leq 0.434$, and $\|\tilde B_k\tilde B_l^*\| \leq \frac{1}{32}$ for the two neighbours $l = (y\pm e_\mu,\mu)$, the other blocks being zero. So the diagonal blocks of $\mathscr{G}$ are at least $\frac18 - 227\kappa$, the two off-diagonal blocks of a row have norm at most $\frac{1}{32} + 227\kappa$, and $\mathscr{G} \geq \frac{1}{16} - 700\kappa$. Hence DC is onto and F is a submanifold near U; and the Neumann series gives $\|\mathscr{G}^{-1}\| \leq 17$ in the norm $\max_k\|z_k\|$. Let γ be a geodesic of F with $\gamma(0)=U$, $\dot\gamma(0)=\xi$, written as $Ue^{X(t)}$ link by link. The curves Ue^{tX} are the geodesics of G^{A_W}, so $\eta = \ddot X(0)$ is the second fundamental form at (ξ,ξ); it is normal to F, and differentiating $C(\gamma(t)) = 0$ twice gives $\eta = -DC^*\mathscr{G}^{-1}h$, where h_k is the second derivative of $C_k(Ue^{t\xi})$ at $t=0$. Then

$$\operatorname{Hess}_F Q(\xi,\xi) = \frac{d^2}{dt^2}Q\big(Ue^{t\xi}\big)\Big|_{t=0} + dQ(\eta),$$

and $dQ(\eta) = -\langle\mathscr{G}^{-1}DC\nabla Q, h\rangle$. As in (ii), $dQ(\xi) = -2\sum_p\operatorname{Re}\operatorname{Tr}(D_p(U_p-I))$, so $\|\nabla_e Q\| \leq 12\kappa$, $\|(DC\nabla Q)_k\| \leq (0.434+260\kappa)\,12\kappa\sqrt{24} \leq 26\kappa$ and $\|(\mathscr{G}^{-1}DC\nabla Q)_k\| \leq 442\kappa$. The map $C_k(Ue^X)$ depends only on the 24 links of $V_{y,\mu}$ and is real-analytic in (U,X) on a neighbourhood of the compact set where $\delta(P_x,P_{2y}) \leq \rho_G/4$ for every x, so $\|h_k\| \leq c_2N_{y,\mu}(\xi)^2$ with c_2 depending only on G. Summing, $|dQ(\eta)| \leq 884c_2\kappa T(\xi)$, and (ii) gives the claim with $c_G = 5.4\cdot10^4 + 884c_2$. □

So on the fibres of the cut-off average the action is uniformly convex along every geodesic of the fibre on which all plaquettes stay below κ, with modulus $\lambda(\frac{1}{26} - c_G\kappa)$, and the background enters only through its curvature. The constants are crude: on the periodic lattice of side four, for $\mathrm{SU}(2)$ fields equal to block

gauge transforms of fields with κ up to 0.08 and W all sixteen blocks, the least value of Q_U/T on the tangent space stays within 0.4% of the gap 0.1075 of the quadratic approximation, and with κ up to 0.24 the frame errors of Lemma 360 stay below 1.3κ, a floating-point computation reproduced in the archive.

Remark 362. Theorem 361 is the curvature input of the small-field half of Remark 345(b); three steps remain open.

(a) With respect to the Riemannian volume of F, the constrained law has the density $e^{-\lambda Q}$ times the coarea factor $\det(DC\,DC^{*})^{-1/2}$, and its Bakry–Émery tensor also contains the Ricci curvature of F. These terms do not depend on λ, but they have not been estimated here.

(b) The set of points of F at which every plaquette is at most κ is not known to be geodesically convex in F, so the theorem gives neither a Brascamp–Lieb inequality on it nor a Helffer–Sjöstrand representation of the constrained influences, and the decay through Lemma 333 is not reached.

(c) The mass of the constrained law outside that set is bounded by Theorem 349 only for the fibres of the path map.

The hypothesis $\kappa \le 10^{-8}\rho_G$ concerns every plaquette near W; under the SU(2) Wilson measure the plaquettes that are not small form the clusters of Sec. XII E, and near them the theorem does not apply.

The first two steps can be taken on a chart of F whose domain is convex. Write $T_U \subset \mathfrak{g}^{A_W}$ for the tangent space of F at U, N_U for its orthogonal complement, $\|X\|_\infty = \max_e \|X_e\|$ and $K_s = \{\eta \in T_U : \|\eta\|_\infty \le s\}$, a convex set. The *constrained law* is the conditional law of $e^{-\lambda Q}\prod_{e\in A_W} dU_e$, the other links being those of U, given the averages $V^{\sharp}_{y,\mu}$, $y \in W$, in the version given by the coarea formula.

Theorem 363. *Assume the hypothesis of Lemma 360. There are constants $c_3 \ge 1$ and c_6 depending only on G such that, if $0 < s \le 1/(6000c_3)$ and $\kappa + 6s \le 10^{-8}\rho_G$:*

(i) *there is a smooth map $h : K_s \to N_U$ with $\|h(\eta)\|_\infty \le s/2$ and $\Phi(\eta) = Ue^{\eta+h(\eta)} \in F$; Φ is an embedding, its image contains every $Ue^X \in F$ with $\|X\|_\infty \le s/40$, and every point of $\Phi(K_s)$ satisfies the hypothesis of Lemma 360 with $\kappa + 6s$ in place of κ;*

(ii) *for η in the interior of K_s and $\zeta \in T_U$,*

$$\frac{d^2}{dt^2} Q\big(\Phi(\eta + t\zeta)\big)\Big|_{t=0} \ge \Big(\frac{1}{26} - 4\cdot 10^5 c_3(\kappa + s)\Big) T(\zeta);$$

(iii) *on $\Phi(K_s)$ the constrained law has, in the coordinate η, a density proportional to $e^{-\lambda Q(\Phi(\eta)) - \psi(\eta)}$ with respect to Lebesgue measure on T_U, where ψ is smooth, does not depend on λ, and $|\frac{d^2}{dt^2}\psi(\eta + t\zeta)| \le c_6 T(\zeta)$ on the interior of K_s;*

(iv) *if moreover $4\cdot 10^5 c_3(\kappa + s) \le \frac{1}{195}$ and $\lambda \ge 60c_6$, the constrained law conditioned on $\Phi(K_{s/2})$, written as a law μ on $K_{s/2}$, satisfies*

$$\mathrm{Var}_\mu(f) \le \frac{60}{\lambda}\int |\nabla f|^2\, d\mu$$

for every smooth f on T_U.

The constants do not depend on L, W, U or λ.

For SU(2) and the Wilson action, $\lambda = \beta/4$, so (iv) is a Brascamp–Lieb inequality [25] with constant $240/\beta$ on a chart of the fibre around every background with small plaquettes, for $\beta \ge 240c_6$.

Proof. Local constants. For $k = (y,\mu)$ let $c_k(X) = C_{y,\mu}(Ue^X)$, which depends only on the 24 links of $V_{y,\mu}$, and let $j(X_e) = \det\Psi(\mathrm{ad}_{X_e})$, the density of Haar measure in the coordinate $U_e e^{X_e}$, with Ψ as in Lemma 360. Both are real-analytic on a neighbourhood of the compact set of backgrounds with $\delta(P_x, P_{2y}) \le 10^{-6}\rho_G$ for every x and of X with $\|X\|_\infty \le 10^{-3}\rho_G$, and $j > 0$ there. So there is $c_3 \ge 1$ depending only on G, computed in Proposition 364, such that there the m-th derivative of c_k in X is at most $c_3\prod_{i\le m} N_{y,\mu}(Y^i)$ on $Y^1,\dots,Y^m$ for $m = 1,2,3$, and the first two derivatives of $\log j$ are at most c_3 in norm. By Lemma 360(i) the background U lies in this set.

Let $B = Dc(0)$, a map from $\mathfrak{g}^{A_W}$ to $\mathfrak{g}^K$, K the set of pairs k, so that $T_U = \ker B$ and $N_U = B^*\mathfrak{g}^K$, and $\mathscr{G} = BB^*$. By the proof of Theorem 361(iii), $\mathscr{G} \ge \frac{1}{17}$ and $\|\mathscr{G}^{-1}\| \le 17$ in the norms $\max_k\|z_k\|$ and, $\mathscr{G}$ being symmetric, $\sum_k\|z_k\|$; each B_k has norm at most 0.44 on the 24 links of $V_{y,\mu}$, and every link lies in at most two of these sets. Hence $\|B\| \le 0.63$, $\|BX\|_\infty \le 2.2\|X\|_\infty$, $\|B^*z\|_\infty \le 0.88\max_k\|z_k\|$ and $\|B^*z\|_1 \le 10.6\sum_k\|z_k\|$, where $\|X\|_1 = \sum_e\|X_e\|$.

(i) Write $c(X) = BX + R(X)$. For $\|X\|_\infty, \|\hat X\|_\infty \le 2s$, Taylor's formula gives $\|R_k(X)\| \le \frac{c_3}{2}N_{y,\mu}(X)^2 \le 12c_3\|X\|_\infty^2$ and $\|R_k(X) - R_k(\hat X)\| \le 48c_3 s\|X - \hat X\|_\infty$. For $\eta \in K_s$ the equation $c(\eta + B^*z) = 0$ reads $z = \Theta(z) = -\mathscr{G}^{-1}R(\eta + B^*z)$, and on $\max_k\|z_k\| \le s$ the map Θ has values of norm at most $17\cdot 12c_3\cdot 4s^2 = 816c_3s^2 \le s$ and Lipschitz constant at most $17\cdot 48\cdot 0.88\,c_3 s \le 720c_3 s \le \frac12$. So it has a unique fixed point $z(\eta)$, smooth in η by the implicit function theorem, and $h = B^*z$ has $\|h\|_\infty \le 0.88\cdot 816c_3s^2 \le s/2$. Since $h(\eta) \perp T_U$, η is the orthogonal projection of $\eta + h(\eta)$, so Φ is injective; its differential $1 + Dh$ is injective, and $\dim T_U = \dim F$. If $Ue^X \in F$ with $\|X\|_\infty \le s/40$, then $X = \eta + B^*z$ with $z = \mathscr{G}^{-1}BX$, $\max_k\|z_k\| \le 17\cdot 2.2\,s/40 < s$ and $\|\eta\|_\infty \le \|X\|_\infty + 0.88\max_k\|z_k\| < s$, and z is a fixed point of Θ, so $Ue^X = \Phi(\eta)$. Finally $\|\eta + h\|_\infty \le \frac32 s$, and by bi-invariance $\delta(U'_p, U_p) \le \sum_{e\in p}\delta(e^{X_e}, I) \le 6s$ for $U' = Ue^X$; the links off A_W are unchanged, so U' is in the tree gauge and F is its fibre.

(ii) Let $X(t) = \eta + t\zeta + h(\eta + t\zeta)$, $\gamma(t) = Ue^{X(t)}$, and let $A = Dc(X)$. Differentiating $c(X(t)) = 0$ gives $X' = \zeta - B^*(AB^*)^{-1}A\zeta$ and $X'' = -B^*(AB^*)^{-1}D^2c(X)[X',X']$. As above $\|(A-B)Y\| \le c_3N_{y,\mu}(X)N_{y,\mu}(Y) \le 7.4c_3sN_{y,\mu}(Y)$ in each component, so $\|A - B\| \le 10.5c_3s$ and $\sum_k\|((A-B)Y)_k\| \le 14.8c_3s\|Y\|_1$. Since $A\zeta = (A-B)\zeta$ and $AB^* = \mathscr{G} + (A-B)B^*$, the Neumann series gives $\|(AB^*)^{-1}\| \le 34$ in the Euclidean norm and in the norm $\sum_k\|z_k\|$, the latter because

$17\cdot 14.8\cdot 10.6\,c_3 s\le\frac12$; hence $\|Dh\|\le 0.63\cdot 34\cdot 10.5 c_3 s\le\frac16$, $T(X')\le 1.03T(\zeta)$, the two parts of X' being orthogonal, and $\|X''\|_1\le 10.6\cdot 34\cdot 2c_3\cdot 1.03T(\zeta)\le 744c_3T(\zeta)$.

Let $w=\gamma^{-1}\gamma'=\Psi(\mathrm{ad}_X)X'$, link by link. Then $\gamma(t+\tau)=\gamma(t)\exp(\tau w+\frac{\tau^2}{2}\dot w+O(\tau^3))$, so

$$\frac{d^2}{dt^2}Q(\gamma)=\frac{d^2}{d\sigma^2}Q\big(\gamma e^{\sigma w}\big)\Big|_{\sigma=0}+dQ(\dot w).$$

The vector w is tangent to F at $\gamma(t)$, which satisfies the hypothesis with $\kappa'=\kappa+6s$, so by Theorem 361(i), (ii) at $\gamma(t)$ the first term is at least $(\frac{1}{26}-5.4\cdot 10^4\kappa')T(w)$. From the series of Ψ and $\|\mathrm{ad}_X\|\le 2\|X\|$, $\|w_e-X_e'\|\le 2\|X_e\|\,\|X_e'\|\le 3s\|X_e'\|$, so $T(w)\ge(1-6s)T(\zeta)$; and $\dot w=D\Psi(\mathrm{ad}_X)[\mathrm{ad}_{X'}]X'+\Psi(\mathrm{ad}_X)X''$, where the term of first order in ad vanishes because $\mathrm{ad}_{X'}X'=0$, so $\|\dot w_e\|\le 3s\|X_e'\|^2+(1+3s)\|X_e''\|$ and $\|\dot w\|_1\le 750c_3T(\zeta)$. With $\|\nabla_e Q\|\le 12\kappa'$,

$$\frac{d^2}{dt^2}Q(\gamma)\ge\Big(\frac{1}{26}-\frac{s}{4}-(5.4\cdot 10^4+9000c_3)\kappa'\Big)T(\zeta),$$

and $6(5.4\cdot 10^4+9000c_3)+1\le 4\cdot 10^5 c_3$.

(iii) In the coordinate X, Haar measure is $\prod_e j(X_e)\,dX$, and the coarea formula for c writes dX as the integral over the values of c of the area measure of its level sets divided by $\det(AA^*)^{1/2}$. On the level set F the graph $X=\eta+h(\eta)$ has area element $\det(1+Dh^*Dh)^{1/2}\,d\eta$. So the density is $e^{-\lambda Q(\Phi)-\psi}$ with

$$\psi(\eta)=-\sum_e\log j(X_e)+\tfrac12\log\det(AA^*)\\ -\tfrac12\log\det(1+Dh^*Dh).$$

Along $\eta+t\zeta$, the first sum has second derivative at most $c_3(T(X')+\|X''\|_1)\le 745c_3^2T(\zeta)$. Next, $A'=D^2c(X)[X',\cdot]$ and $A''=D^3c(X)[X',X',\cdot]+D^2c(X)[X'',\cdot]$; their components are maps of rank at most $\dim G$ on the 24 links of $V_{y,\mu}$, of norms at most $c_3N_{y,\mu}(X')$ and $c_3(N_{y,\mu}(X')^2+N_{y,\mu}(X''))$. Hence the Hilbert–Schmidt norm satisfies $\|A'\|_2^2\le 2.1\dim G\,c_3^2T(\zeta)$, and the trace norm $\|A''\|_1\le\dim G\,c_3(2.1T(\zeta)+2\|X''\|_1)$. Since $AA^*\ge(17^{-1/2}-10.5c_3s)^2\ge\frac{1}{25}$ and $\|A\|\le 1$,

$$\Big|\frac{d^2}{dt^2}\log\det(AA^*)\Big|\le 25\big(2\|A''\|_1+2\|A'\|_2^2\big)+2500\|A'\|_2^2,$$

by the identity

$$\frac{d^2}{dt^2}\log\det M=\mathrm{Tr}(M^{-1}M'')-\mathrm{Tr}\big((M^{-1}M')^2\big)$$

and the bounds $|\mathrm{Tr}(XY)|\le\|X\|\,\|Y\|_1$ and $\|XY\|_1\le\|X\|_2\|Y\|_2$. Finally $Dh=-\Pi$ on T_U with $\Pi=B^*(AB^*)^{-1}A$; Π' is a sum of two terms that contain A' once, and Π'' of terms that contain A'' once or A' twice, so $\|\Pi'\|_2\le 481\|A'\|_2$ and $\|\Pi''\|_1\le c(\|A''\|_1+\|A'\|_2^2)$ with c absolute, and the same identity with $M=1+Dh^*Dh\ge 1$ and $\|Dh\|\le\frac16$ bounds the last term. Each bound is a multiple of $T(\zeta)$ by a constant that depends only on c_3 and $\dim G$; this is c_6, computed in Proposition 364.

(iv) Let $H=\lambda Q\circ\Phi+\psi$ on the interior of K_s. By (ii) and (iii), $\mathrm{Hess}\,H\ge\lambda(\frac{1}{26}-\frac{1}{195})-c_6=\frac{\lambda}{30}-c_6\ge m=\frac{\lambda}{60}$, so for $a,\eta\in K_{s/2}$, $H(\eta)\ge H(a)+\nabla H(a)\cdot(\eta-a)+\frac{m}{2}|\eta-a|^2$, the segment lying in the convex set $K_{s/2}$. The supremum $\tilde H$ over $a\in K_{s/2}$ of the right-hand sides is finite, equals H on $K_{s/2}$, and $\tilde H-\frac{m}{2}|\eta|^2$ is convex. With d the distance to $K_{s/2}$, a convex function, the functions $\tilde H+nd$ are m-convex on T_U, and so are their mollifications, which are smooth; for these the Bakry–Émery criterion [5] gives the Poincaré inequality with constant $1/m$. Letting the mollification and then $n\to\infty$, by dominated convergence ($\tilde H$ grows quadratically and $d=0$ exactly on $K_{s/2}$), the inequality passes to $e^{-H}\mathbf{1}_{K_{s/2}}$, which is μ up to normalisation by (iii). □

The constants of Theorem 363 can be computed. Cauchy's estimates on a complex neighbourhood of the real chart bound the derivatives of the averages, and the rest of the proof turns these bounds into c_6.

Proposition 364. *Let $G\subset\mathrm{U}(r)$ be compact. Theorem 363 holds with*

$$c_3=\max\{1.5\cdot 10^4\sqrt{r},\dim G\},\qquad c_6=6.81\cdot 10^5\dim G\,c_3^2.$$

For SU(2) *these are $c_3\le 2.13\cdot 10^4$ and $c_6\le 9.3\cdot 10^{14}$, so part* (iv) *holds for $\lambda\ge 5.6\cdot 10^{16}$, that is, for the Wilson action at $\beta\ge 2.3\cdot 10^{17}$, once $\kappa+s\le 6.0\cdot 10^{-13}$.*

Proof. Write $\|\cdot\|_{\mathrm{op}}$ for the operator norm, which is at most the Hilbert–Schmidt norm, while $\|X\|\le\sqrt{r}\|X\|_{\mathrm{op}}$ on $\mathfrak{g}$. Since $\rho_G\le 1$, every background of the set of the proof has its sites $w_x=v^{-1}P_x$ within $a=10^{-6}$ of I, and every X of that set has $\|X_e\|\le x_0=10^{-3}$.

Three estimates. Let X be skew-Hermitian and Z complex with $\|Z\|_{\mathrm{op}}\le t$. Duhamel's formula $e^{X+Z}-e^X=\int_0^1 e^{\sigma(X+Z)}Ze^{(1-\sigma)X}\,d\sigma$, with $\|e^{\sigma(X+Z)}\|_{\mathrm{op}}\le e^{\sigma t}$ by the Lie–Trotter product formula, gives $e^{X+Z}=e^X(I+F)$ and likewise $e^{-X-Z}=(I+F')e^{-X}$, with $\|F\|_{\mathrm{op}},\|F'\|_{\mathrm{op}}\le\varphi=e^t-1$. The same formula gives $\|e^A-e^B\|\le\|A-B\|$ for skew-Hermitian A,B, in both norms. Finally, on $\|w-I\|_{\mathrm{op}}<1$ the series $\log w=\sum_{k\ge1}(-1)^{k+1}(w-I)^k/k$ is holomorphic, equals the principal logarithm at unitary w, and its derivative satisfies $\|D\log(w)[H]\|\le\|H\|/(1-\|w-I\|_{\mathrm{op}})$ in both norms.

The complex chart. Fix U and a real X in the set, put $U'=Ue^X$, and let Z be complex with $\|Z_e\|_{\mathrm{op}}\le t$ on the 24 links of $V_{y,\mu}$. For a path $P_x=U_1U_2$ with path link v, the site at the complex field $U_ee^{X_e+Z_e}$ is

$$w_x=e^{-X_v-Z_v}v^{-1}U_1e^{X_1+Z_1}U_2e^{X_2+Z_2}\\ =(I+F_v')\,w_x'\,(I+\tilde F_1)(I+F_2),$$

where w_x' is the site at U', a unitary matrix with $\delta(w_x',I)\le a+3x_0=e_0$ since each link moves by $\delta(e^{X_e},I)\le x_0$, and $\tilde F_1$ is F_1 conjugated by the unitary $U_2e^{X_2}$. Hence $\|w_x-w_x'\|_{\mathrm{op}}\le\psi=(1+\varphi)^3-1$, and if $e_0+\psi<1$ then $\|\log w_x-\log w_x'\|_{\mathrm{op}}\le d=\psi/(1-e_0-\psi)$, so the mean M of the sixteen logarithms is within d of its value M' at U', which is skew-Hermitian.

The average is $V^\sharp = v e^{X_v+Z_v} e^M$, and with $\|e^M\|_{\mathrm{op}} \le e^d$ and $\|e^M - e^{M'}\|_{\mathrm{op}} \le e^d - 1$,

$$r = V^\sharp (U')^{-1} V^\sharp = e^{-M'}(I+F_v)e^M,$$
$$\|r - I\|_{\mathrm{op}} \le (1+\varphi)e^d - 1 = \eta.$$

At real points $\chi = 1$, as $e_0\rho_G < \rho_G/2$. The unitary $q' = V^\sharp(U)^{-1}V^\sharp(U') = e^{-M_0}e^{X_v}e^{M'}$ has $\delta(q',I) \le x_0 + \|M' - M_0\| \le x_0 + 3x_0/(1-e_0) = \gamma_0 < 0.0041$. So if $\gamma_0 + \eta < 1$, then $c(X+Z) = \log(q'r)$ is holomorphic in Z, extends c, and

$$\|c(X+Z) - c(X)\|_{\mathrm{op}} \le S(t) = \frac{\eta}{1-\gamma_0-\eta}.$$

Derivatives of c_k. For $N_{y,\mu}(Y) = 1$ the function $f(z) = c(X+zY)$ is holomorphic on $|z| < 0.1$, where $e_0 + \psi < 0.36$ and $\gamma_0 + \eta < 0.91$, and $\|zY_e\|_{\mathrm{op}} \le |z|$. Cauchy's estimate $\|f^{(m)}(0)\| \le m! R^{-m} \sup_{|z|=R}\|f(z) - f(0)\|$ gives

$$\|D^m c(X)[Y,\dots,Y]\| \le \sqrt{r}\, m!\, S(R)/R^m.$$

At $R = 10^{-3}$, 0.05 and 0.07 the values of $m!S(R)/R^m$ for $m = 1,2,3$ are below 4.2, 307 and 14811; at $R = 0.07$, for instance, $\varphi < 0.0726$, $\psi < 0.2337$, $d < 0.3062$, $\eta < 0.4567$ and $S < 0.847$. By Banach's theorem [13] a symmetric multilinear form on a real Hilbert space has the same norm as its restriction to the diagonal. Applied to $\langle u, D^m c(X)[\cdot]\rangle$ for unit vectors $u \in \mathfrak{g}$, with the Euclidean norm $N_{y,\mu}$ on the 24 links, this gives $\|D^m c(X)[Y^1,\dots,Y^m]\| \le 1.5\cdot 10^4 \sqrt{r}\prod_i N_{y,\mu}(Y^i)$ for $m = 1,2,3$.

Derivatives of $\log j$. Since $\mathfrak{g}$ is reductive, $\mathrm{Tr}\,\mathrm{ad}_X = 0$, and $\Psi(z) = e^{-z/2}\sinh(z/2)/(z/2)$, so $\log j(X) = \mathrm{Tr}\, g(\mathrm{ad}_X)$ with $g(w) = \log(\sinh(w/2)/(w/2)) = \sum_{k\ge1} g_{2k} w^{2k}$ for $|w| < 2\pi$. On $|w| = 3$, $|\sinh(w/2)/(w/2) - 1| \le \sinh(1.5)/1.5 - 1 < 0.4196$, so $|g| \le 0.5441$ and $|g_{2k}| \le 0.5441\cdot 3^{-2k}$. For complex X with $\|X\|_{\mathrm{op}} \le b$ the operator ad_X on $\mathfrak{g}\otimes\mathbb{C}$ has norm at most $2b$, and $|\mathrm{Tr}\, A^{2k}| \le \dim G\, \|A\|_{\mathrm{op}}^{2k}$, so $|\log j(X)| \le 0.5441 \dim G\, u^2/(1-u^2)$ with $u = 2b/3$. Along $X + zH$, $\|H\| = 1$, $|z| \le 0.8$, this is at most $0.218\dim G$, and Cauchy's estimate bounds the first two derivatives of $\log j$ by $0.28\dim G$ and $0.69\dim G$.

The constant c_6. In the proof of Theorem 363(iii), with c_3 as above, $a' = \|A'\|_2^2 \le 2.1\dim G\, c_3^2 T(\zeta)$ and $b' = \|A''\|_1 \le \dim G\, c_3(2.1 + 1488c_3)T(\zeta) \le 1490.1\dim G\, c_3^2 T(\zeta)$. Write $K = (AB^*)^{-1}$, so that

$$K' = -KA'B^*K, \quad K'' = 2KA'B^*KA'B^*K - KA''B^*K,$$
$$\Pi' = B^*K'A + B^*KA', \quad \Pi'' = B^*K''A + 2B^*K'A' + B^*KA''.$$

With $\|B\| \le 0.63$, $\|K\| \le 34$, $\|A\| \le 1$ and $k = 34\cdot 0.63 = 21.42$ these give $\|\Pi'\|_2 \le (k+k^2)\|A'\|_2 \le 481\|A'\|_2$ and $\|\Pi''\|_1 \le (2k^3+2k^2)\|A'\|_2^2 + (k^2+k)\|A''\|_1 \le 20576\,a' + 481\,b'$. For $M = 1 + Dh^*Dh \ge 1$ with $\|Dh\| \le \frac16$, the identity for $\log\det$ gives $|(\log\det M)''| \le \frac13\|\Pi''\|_1 + \frac{19}{9}\|\Pi'\|_2^2$. Adding the three terms of ψ,

$$\Big|\frac{d^2}{dt^2}\psi\Big| \le 745c_3^2T + 25b' + 1275a' + \tfrac12\Big(\tfrac13(20576a' + 481b') + \tfrac{19}{9}\,481^2 a'\Big),$$

which is at most $6.81\cdot 10^5 \dim G\, c_3^2 T(\zeta)$.

For SU(2), $r = 2$ and $\dim G = 3$. Then $60c_6 < 5.6\cdot 10^{16}$, and $4\cdot 10^5 c_3(\kappa + s) \le \frac{1}{195}$ holds when $\kappa + s \le 6.0\cdot 10^{-13}$; it implies $s \le 1/(6000c_3)$ and $\kappa + 6s \le 10^{-8}$. □

The bounds are far from the true sizes: on random SU(2) backgrounds the first three derivatives of c_k along unit directions stay below 0.14, 0.03 and 0.02, a floating-point computation reproduced in the archive. The loss comes from the radius 0.07 of the complex neighbourhood, which enters the third derivative as 0.07^{-3}.

Remark 365. Theorem 363 removes the coarea factor and the curvature of F for $\lambda \ge 60c_6$ and replaces the small-field set by the convex chart domain $K_{s/2}$. What it does not give:

(a) the mass of the constrained law outside $\Phi(K_{s/2})$, which is Remark 362(c);

(b) decay: (iv) is a variance bound for the law on the chart, not a decay of its covariances between distant links. That needs a Helffer–Sjöstrand representation on the convex set $K_{s/2}$, which has a boundary, and the locality of $\mathrm{Hess}\, H$, which through Dh contains $\mathscr{G}^{-1}$, exponentially decaying by Lemma 333 but not of finite range; Theorem 367 below reaches it on a chart whose domain is a product;

(c) numbers: the constants of Proposition 364 are far from optimal, so for SU(2) the threshold is $\lambda \ge 5.6\cdot 10^{16}$ and κ and s are at the scale 10^{-13}.

On the periodic lattice of side four, for SU(2) backgrounds as in the check of Theorem 361, two to sixteen blocks and s up to 0.03, the chart Hessian of Q is at least 0.21 and $|\psi''|/T$ stays below one, a floating-point computation reproduced in the archive.

The obstacle in Remark 365(b) is the shape of $K_{s/2}$, which is not a product in any coordinates adapted to the links. A slightly different chart domain is. For $k = (y,\mu)$ let $d_k = (2y + e_\mu, \mu)$ be the path link of $V_{y,\mu}$. It has $o_\mu = 1$, so it is active and lies in A_W; it enters $V_{y,\mu}$ with weight $\frac18$ by Lemma 329(iii); and it enters no other average, since the links with $o_\mu = 1$ that enter $V_{y',\mu}$ have their base in $B(y')$. Let $L' = A_W \setminus \{d_k\}$ and, for $r > 0$,

$$\Omega_r = \{\eta \in T_U : \|\eta_e\| \le r \text{ for every } e \in L'\}.$$

Give every link the position of its midpoint and every pair k that of d_k, and let $d(a,b)$ be the ℓ^∞ distance between the positions on the periodic lattice. The links that enter $V_{y,\mu}$ have midpoints within distance 1 of that of d_k. Let ∇' be the gradient in the coordinates $(\eta_e)_{e\in L'}$, and, for a block matrix $M = (M_{ab})$ whose indices are links or pairs and whose blocks are maps between copies of $\mathfrak{g}$, and for $\beta \ge 0$, let

$$\|M\|_\beta = \max\Big(\sup_a \sum_b e^{\beta d(a,b)}\|M_{ab}\|,\ \sup_b \sum_a e^{\beta d(a,b)}\|M_{ab}\|\Big).$$

At most two indices share a position, so the sum

$$N(\beta) = \sup_a \sum_b e^{-\beta d(a,b)}$$

is finite for $\beta > 0$ and depends only on β.

Lemma 366. (i) *The operator norm satisfies $\|M\| \le \|M\|_\beta$, and $\|M_1M_2\|_\beta \le \|M_1\|_\beta\|M_2\|_\beta$ and $\|M^*\|_\beta = \|M\|_\beta$.*

(ii) *If $|\varphi(a)-\varphi(b)| \le d(a,b)$ for all a,b and $0 \le \alpha \le \beta$, then $\|e^{\alpha\varphi}Me^{-\alpha\varphi} - M\| \le \frac{\alpha}{\beta}\|M\|_\beta$.*

(iii) *If M is invertible, $\|M^{-1}\| \le M_0$ and $\|M\|_\beta \le M_1$, then $\|M^{-1}\|_\gamma \le 2M_0N(\gamma)$ with $\gamma = \beta/(4M_0M_1)$.*

Proof. (i) is the Schur test and the triangle inequality for d. (ii) The blocks of the difference are $(e^{\alpha(\varphi(a)-\varphi(b))} - 1)M_{ab}$, and $|e^{\alpha(\varphi(a)-\varphi(b))} - 1| \le e^{\alpha d(a,b)} - 1 \le \frac{\alpha}{\beta}e^{\beta d(a,b)}$ by convexity of $\alpha \mapsto e^{\alpha d}$; apply (i) with $\beta = 0$. (iii) Let $\alpha = 2\gamma \le \beta$, since $M_0M_1 \ge \|M^{-1}\|\|M\| \ge 1$. By (ii), $e^{\alpha\varphi}Me^{-\alpha\varphi} = M+E$ with $\|M^{-1}E\| \le \frac12$, so $\|e^{\alpha\varphi}M^{-1}e^{-\alpha\varphi}\| \le 2M_0$ as in Lemma 333. With $\varphi = d(\cdot,b)$ the block (a,b) gives $e^{2\gamma d(a,b)}\|(M^{-1})_{ab}\| \le 2M_0$ for all a,b, and summing $e^{\gamma d(a,b)}\|(M^{-1})_{ab}\| \le 2M_0e^{-\gamma d(a,b)}$ over b or over a gives the claim. □

Theorem 367. *Assume the hypotheses of Theorem 363(iv), with $c_6 \ge 1$, as we may, and let $r = s/32$.*

(i) *The coordinates $(\eta_e)_{e\in L'}$ identify T_U with $\mathfrak{g}^{L'}$, Lebesgue measure on T_U being a constant multiple of that on $\mathfrak{g}^{L'}$; $\Omega_{r'}$ is a product of balls in them, and $K_{r'} \subset \Omega_{r'} \subset K_{15.1r'}$ for every $r' > 0$.*

(ii) *There are $\alpha_G > 0$ and c_9 depending only on G with the following property. Let μ be the constrained law conditioned on $\Phi(\Omega_r)$, written as a law on Ω_r. For every $\varphi : L' \to \mathbb{R}$ with $|\varphi(e)-\varphi(e')| \le d(e,e')$, every $0 \le \alpha \le \alpha_G$ and all smooth F, G on T_U,*

$$|\mathrm{Cov}_\mu(F,G)| \le \frac{120}{\lambda}\left\|e^{-\alpha\varphi}\nabla' F\right\|_{L^2(\mu)}\left\|e^{\alpha\varphi}\nabla' G\right\|_{L^2(\mu)}.$$

(iii) *If $X_1, X_2 \subset A_W$, $\tilde F, \tilde G$ are smooth functions of the links in X_1 and in X_2 whose left-invariant gradients have norm at most 1 in each link, $F = \tilde F \circ \Phi$ and $G = \tilde G \circ \Phi$, then*

$$|\mathrm{Cov}_\mu(F,G)| \le \frac{c_9}{\lambda}\,|X_1|\,|X_2|\,e^{-\alpha_G d(X_1,X_2)}.$$

The constants do not depend on L, W, U, λ, s or κ.

So the covariances of local observables of the field under the constrained law, conditioned on the chart, decay exponentially at a rate that depends only on G, with amplitude $O(1/\lambda)$; for SU(2) and the Wilson action the amplitude is $O(1/\beta)$.

Proof. All constants called C depend only on G.

The section. By Lemma 360(iv), $B_k = \tilde B_k + R_k$ with $\|R_k\| \le 260\kappa$, and the block of $\tilde B_k$ at d_k is $\frac18$ times an isometry of $\mathfrak{g}$, so $\|B_{k,d_k}^{-1}\| \le (\frac18 - 260\kappa)^{-1} \le 8.01$. For $\xi \in \mathfrak{g}^{L'}$ let $(S\xi)_e = \xi_e$ for $e \in L'$ and $(S\xi)_{d_k} = -B_{k,d_k}^{-1}\sum_{e\ne d_k} B_{k,e}\xi_e$. Since d_k enters only V_k, $BS = 0$; S is injective, and $\dim T_U = |L'|\dim G$ because F has codimension $4|W|\dim G$, so S is an isomorphism onto T_U whose inverse is the restriction to L'. The other coefficients of $\tilde B_k$ add up to $\frac78 + 1 = \frac{15}{8}$, so $\|(S\xi)_{d_k}\| \le 8.01(\frac{15}{8} + \sqrt{24}\cdot 260\kappa)\max_e\|\xi_e\| \le 15.1\max_e\|\xi_e\|$. This proves (i), and S has blocks of norm at most 1.1 that vanish beyond distance 1. Write $\mathrm{Box}_{r'}$ for the product of balls $S^{-1}\Omega_{r'}$. Since $15.1\cdot 2r < s$, the estimates of Theorem 363 hold on $S(\mathrm{Box}_{2r})$.

The density. Put $X(\xi) = S\xi + h(S\xi)$, $A = Dc(X)$, $P = 1 - B^*(AB^*)^{-1}A$ and $\bar P = PS$. In the proof of Theorem 363(ii) the derivative of X along $\zeta \in T_U$ is $P\zeta$, and $P\zeta = \zeta + Dh\,\zeta$ with $Dh\,\zeta \perp T_U$; so $\partial_\xi X = \bar P$, $\bar P^*\bar P = S^*(1+Dh^*Dh)S \ge S^*S \ge 1$, and $\det\bar P^*\bar P = \det(S^*S)\det(1+Dh^*Dh)$. By Theorem 363(iii) the law μ has, in the coordinate ξ, the density $e^{-\bar H}\mathbf{1}_{\mathrm{Box}_r}$ up to normalisation, with

$$\bar H(\xi) = \lambda Q\big(Ue^X\big) - \sum_e \log j(X_e) + \tfrac12\log\det(AA^*) - \tfrac12\log\det(\bar P^*\bar P),$$

and $\mathrm{Hess}\,\bar H \ge m = \frac{\lambda}{60}$ on Box_{2r}, because $|S\zeta| \ge |\zeta|$.

Locality of the Hessian. We show that $\|\mathrm{Hess}\,\bar H(\xi)\|_\beta \le c_8\lambda$ on Box_{2r} for some $\beta > 0$ and $c_8 \ge 1$ depending only on G, the Hessian being the block matrix of second derivatives in the coordinates ξ_e. The maps B, A, S, the second derivative of $Q(Ue^X)$ in X and that of $\sum_e \log j(X_e)$ have blocks bounded by C (the last two by compactness, as c_3) that vanish beyond distance 2, so their norms $\|\cdot\|_1$ are at most C. By the proof of Theorem 363, $\|(AB^*)^{-1}\| \le 34$ and $\|(AA^*)^{-1}\| \le 25$, and $\|(\bar P^*\bar P)^{-1}\| \le 1$. Applying Lemma 366(iii) to AB^* and AA^*, then (i) to P, $\bar P$ and $\bar P^*\bar P$, then (iii) to $\bar P^*\bar P$, gives $\beta_0 \in (0,1]$ such that $(AB^*)^{-1}$, $(AA^*)^{-1}$, P, $\bar P$, $\bar P^*\bar P$ and $(\bar P^*\bar P)^{-1}$ have norm $\|\cdot\|_{\beta_0}$ at most C on Box_{2r}. Let $\gamma = \beta_0/2$, and for links e,e' and $v,v' \in \mathfrak{g}$ write v,v' also for the vectors of $\mathfrak{g}^{L'}$ supported at e, e'. Call a block matrix M, depending linearly on v, *concentrated at e* if $\|M_{ab}\| \le C|v|e^{-\gamma(d(a,e)+d(b,e))}$, and, depending bilinearly on (v,v'), *concentrated at (e,e')* if $\|M_{ab}\| \le C|v||v'|e^{-\gamma(d(a,e)+d(a,e')+d(b,e)+d(b,e'))}$; a vector is concentrated if it is so as a column. If $\|Y\|_{\beta_0} \le C$, then YM and MY are concentrated where M is, because $e^{-\gamma(d(c,e)+d(c,e'))} \le e^{\beta_0 d(a,c)}e^{-\gamma(d(a,e)+d(a,e'))}$. For M concentrated at e and M' at e', using $d(a,e)+d(a,e') \ge d(e,e')$ in both indices,

$$|\mathrm{Tr}(MM')| \le C|v||v'|N(\gamma/2)^2e^{-\gamma d(e,e')},$$

and if M is concentrated at (e,e'), then

$$|\mathrm{Tr}M| \le C|v||v'|N(\gamma)e^{-\gamma d(e,e')}.$$

Now $x_v = \partial_v X = \bar P v$ has $\|(x_v)_a\| \le Ce^{-\beta_0 d(a,e)}|v|$, so it is concentrated at e. The vector $D^2c(X)[x_v,x_{v'}]$ has components at most $c_3N_k(x_v)N_k(x_{v'}) \le C|v||v'|e^{-\beta_0(d(k,e)+d(k,e'))}$, so $x_{vv'} = \partial_v\partial_{v'}X = -B^*(AB^*)^{-1}D^2c(X)[x_v,x_{v'}]$ is concentrated at (e,e'). In the same way $A_v = \partial_v A = D^2c(X)[x_v,\cdot]$ is concentrated at e, $A_{vv'} = D^3c(X)[x_v,x_{v'},\cdot] + D^2c(X)[x_{vv'},\cdot]$ is concentrated at (e,e'), $\partial_v P = -B^*(AB^*)^{-1}A_vP$ is concentrated at e, and $\partial_v\partial_{v'}P$ is a sum of products of operators bounded in $\|\cdot\|_{\beta_0}$ with either one factor concentrated at (e,e') or one factor concentrated at e and one at e'. The same holds for the derivatives of $\mathscr{G}_1 = AA^*$

and $\mathscr{G}_2 = \bar{P}^* \bar{P}$. Then

$$\partial_v \partial_{v'} \bar{H} = \lambda \big(D^2 Q[x_v, x_{v'}] + DQ[x_{vv'}] \big) - \sum_e \partial_v \partial_{v'} \log j(X_e) + \sum_{i=1,2} \pm \tfrac{1}{2} \big(\operatorname{Tr}(\mathscr{G}_i^{-1} \partial_v \partial_{v'} \mathscr{G}_i) - \operatorname{Tr}(\mathscr{G}_i^{-1} \partial_v \mathscr{G}_i \mathscr{G}_i^{-1} \partial_{v'} \mathscr{G}_i) \big),$$

where D^2Q and DQ are the derivatives of $Q(Ue^X)$ in X. By the bounds above every term is at most

$$C(\lambda+1)|v||v'| e^{-\gamma d(e,e')/2};$$

for instance

$$|D^2 Q[x_v, x_{v'}]| \le C \sum_a e^{-\beta_0 (d(a,e)+d(a,e')-2)} |v||v'|.$$

Hence, using $\lambda \ge 60 c_6 \ge 60$, every block of the Hessian of $\bar{H}$ satisfies

$$\| (\operatorname{Hess} \bar{H})_{ee'} \| \le C\lambda e^{-\beta_0 d(e,e')/4},$$

and $\| \operatorname{Hess} \bar{H} \|_\beta \le c_8 \lambda$ with $\beta = \beta_0/8$.

Approximation. Let ρ be smooth on $[0,\infty)$ with $0 \le \rho' \le 1$, $\rho(t) = t$ for $t \le \frac{3}{2} r$, $\rho \le \frac{7}{4} r$ and ρ constant on $[2r,\infty)$, and let $\tau(\xi)_e = \rho(\|\xi_e\|) \xi_e / \|\xi_e\|$, which maps $\mathfrak{g}^{L'}$ into $\mathrm{Box}_{7r/4}$. Let v be smooth and convex on $[0,\infty)$, zero on $[0,r]$, with $0 < v'' \le 1$ on (r,∞) and $v'' = 1$ on $[\frac{5}{4} r, \infty)$, and $V(\xi) = \sum_e v(\|\xi_e\|)$. For $t \ge \frac{3}{2} r$, $v''(t) = 1$ and $v'(t)/t \ge 1 - \frac{5}{4} r/t \ge \frac{1}{6}$, so $\operatorname{Hess} V \ge \frac{1}{6}$ in the links with $\|\xi_e\| \ge \frac{3}{2} r$. Let $H_n = \bar{H} \circ \tau + nV$. Then

$$\operatorname{Hess} H_n = D\tau^* \big(\operatorname{Hess} \bar{H} \circ \tau \big) D\tau + \Delta_n,$$

where $D\tau$ and Δ_n are block diagonal, $D\tau$ is the identity in the links with $\|\xi_e\| \le \frac{3}{2} r$, $\|D\tau\| \le 1$, and $\Delta_n \ge n \frac{1}{6} - C_1$ in the other links, C_1 bounding $\nabla \bar{H}$ times the second derivative of τ. So $\operatorname{Hess} H_n \ge m$ on $\mathfrak{g}^{L'}$ for $n \ge 6(m + C_1)$, and all derivatives of H_n of order at least two are bounded. For φ as in (ii) and $\alpha \le \alpha_G = \beta/(120 c_8)$, the block-scalar matrix $e^{\alpha\varphi}$ commutes with $D\tau$ and Δ_n, so by Lemma 366(ii)

$$\big\| e^{\alpha\varphi} \operatorname{Hess} H_n e^{-\alpha\varphi} - \operatorname{Hess} H_n \big\| \le \frac{\alpha}{\beta} c_8 \lambda \le \frac{m}{2}.$$

Weighted Helffer–Sjöstrand. Let μ_n have density proportional to e^{-H_n} on $\mathfrak{g}^{L'}$ and $\mathscr{L}_n = -\Delta + \nabla H_n \cdot \nabla$ on $L^2(\mu_n)$. For H_n smooth with $\operatorname{Hess} H_n \ge m > 0$ and bounded derivatives of order at least two, and F, G smooth with bounded derivatives, the Helffer–Sjöstrand representation [69, 92] reads $\operatorname{Cov}_{\mu_n}(F,G) = \langle \nabla F, \mathscr{A}^{-1} \nabla G \rangle_{L^2(\mu_n)}$ with $\mathscr{A} = \mathscr{L}_n \otimes 1 + \operatorname{Hess} H_n$ acting on $L^2(\mu_n; \mathfrak{g}^{L'})$. With $w = e^{\alpha\varphi}$, constant in ξ, $\mathscr{A}_w = w \mathscr{A} w^{-1} = \mathscr{L}_n \otimes 1 + w \operatorname{Hess} H_n w^{-1}$, and $\operatorname{Re} \langle u, \mathscr{A}_w u \rangle \ge \frac{m}{2} \|u\|^2$ because $\mathscr{L}_n \ge 0$; the adjoint satisfies the same bound, so $\|\mathscr{A}_w^{-1}\| \le \frac{2}{m} = \frac{120}{\lambda}$; this is the weighted form of the representation used in [26, 65]. Since $\mathscr{A}^{-1} = w^{-1} \mathscr{A}_w^{-1} w$,

$$|\operatorname{Cov}_{\mu_n}(F,G)| = \big| \langle w^{-1} \nabla F, \mathscr{A}_w^{-1} w \nabla G \rangle \big| \le \frac{120}{\lambda} \|w^{-1} \nabla F\|_{L^2(\mu_n)} \|w \nabla G\|_{L^2(\mu_n)}.$$

As $n \to \infty$, e^{-H_n} decreases to $e^{-\bar{H}} \mathbf{1}_{\mathrm{Box}_r}$ off the boundary of Box_r, since τ is the identity there and $V > 0$ off it, so $\mu_n \to \mu$ in total variation. For F, G smooth on T_U, apply the bound to $(F \circ S)\chi$ and $(G \circ S)\chi$ with χ smooth, compactly supported and equal to 1 near Box_r, and let $n \to \infty$; this proves (ii).

(iii) The left-invariant derivative of $U_a e^{X_a}$ along δX_a is $\Psi(\operatorname{ad}_{X_a}) \delta X_a$, of norm at most $2\|\delta X_a\|$, so $\|\partial_{\xi_f} F\| \le 2 \sum_{a \in X_1} \|\bar{P}_{af}\|$, and similarly for G. Take $\varphi = d(\cdot, X_2)$ and $\alpha = \alpha_G \le \beta_0$. Then $e^{-\alpha d(f,X_2)} \le e^{\beta_0 d(a,f)} e^{-\alpha d(a,X_2)}$, so $|e^{-\alpha\varphi} \nabla' F| \le 2|X_1| \|\bar{P}\|_{\beta_0} e^{-\alpha d(X_1, X_2)}$, and $e^{\alpha d(f,X_2)} \le e^{\beta_0 d(b,f)}$ for $b \in X_2$ gives $|e^{\alpha\varphi} \nabla' G| \le 2|X_2| \|\bar{P}\|_{\beta_0}$. Insert these in (ii); with C a bound for $\|\bar{P}\|_{\beta_0}$ on Box_{2r}, $c_9 = 480 C^2$. □

The rate α_G and the amplitude c_9 can be computed, and they turn out not to depend on G. The proof above passes through the norms $\|\cdot\|_\beta$, whose Schur sums bring in the large factors $N(\beta)$; the computation below replaces them by weighted operator norms, in which inverses and products cost nothing beyond their operator norms. For a block matrix M indexed by links and pairs and for $\vartheta \ge 0$ let

$$\|M\|_{(\vartheta)} = \sup_\varphi \big\| e^{\vartheta\varphi} M e^{-\vartheta\varphi} \big\|,$$

the supremum over all φ with $|\varphi(a) - \varphi(b)| \le d(a,b)$ for all indices a, b, the factors $e^{\pm\vartheta\varphi}$ acting as scalars on each block. For such φ write $w = e^{\vartheta\varphi}$ and $M^w = wMw^{-1}$.

Lemma 368. (i) *For $\vartheta \le \vartheta'$, $\|M\| \le \|M\|_{(\vartheta)} \le \|M\|_{(\vartheta')}$; moreover $\|M_1 M_2\|_{(\vartheta)} \le \|M_1\|_{(\vartheta)} \|M_2\|_{(\vartheta)}$, $\|M^*\|_{(\vartheta)} = \|M\|_{(\vartheta)}$, and $\|M_{ab}\| \le \|M\|_{(\vartheta)} e^{-\vartheta d(a,b)}$. If the blocks of M vanish beyond distance R, then $\|M\|_{(\vartheta)} \le e^{\vartheta R} (R_M C_M)^{1/2}$, where R_M and C_M are the largest row and column sums of the norms of the blocks.*

(ii) *If M is Hermitian and $0 \le \vartheta \le \vartheta'$, then $\|\operatorname{Re} M^w - M\| \le 2(\vartheta/\vartheta')^2 \|M\|_{(\vartheta')}$ for every φ, where $\operatorname{Re} T = \frac{1}{2}(T + T^*)$.*

(iii) *If $\operatorname{Re} M^w \ge c > 0$ for every φ, then $\|M^{-1}\|_{(\vartheta)} \le 1/c$. If $\|M^{-1}\|_{(\vartheta)} \|E\|_{(\vartheta)} < 1$, then $\|(M+E)^{-1}\|_{(\vartheta)} \le \|M^{-1}\|_{(\vartheta)} / (1 - \|M^{-1}\|_{(\vartheta)} \|E\|_{(\vartheta)})$.*

Proof. (i) Take $\varphi = 0$ for the first inequality and replace φ by $\vartheta\varphi/\vartheta'$ for the second. Products and adjoints follow from $(M_1 M_2)^w = M_1^w M_2^w$ and $(M^*)^w = (M^{w'})^*$ with $w' = e^{-\vartheta\varphi}$. With $\varphi = d(\cdot, b)$ the block (a,b) of M^w is $e^{\vartheta d(a,b)} M_{ab}$, of norm at most $\|M^w\|$. The last bound is the Schur test, since the blocks of M^w are $e^{\vartheta(\varphi(a) - \varphi(b))} M_{ab}$.

(ii) Let $f(z) = e^{z\varphi} M e^{-z\varphi}$, an entire function of $z \in \mathbb{C}$ with values in matrices. For $z = x + iy$, $f(z) = e^{iy\varphi} f(x) e^{-iy\varphi}$ with $e^{iy\varphi}$ unitary, so $\|f(z)\| \le K = \|M\|_{(\vartheta')}$ for $|x| \le \vartheta'$. Since M is Hermitian, $f(x)^* = f(-x)$ for real x, so $\operatorname{Re} M^w - M = g(\vartheta)$ with $g(z) = \frac{1}{2}(f(z) + f(-z)) - f(0)$. The function g is even, vanishes at 0 and is bounded by $2K$ on $|z| < \vartheta'$; so $g(z)/z^2$ is holomorphic there, and the maximum principle on the circles $|z| = \rho < \vartheta'$, applied to $\ell(g(z))/z^2$ for every linear functional ℓ of norm one, gives $\|g(z)\| \le 2K|z|^2/\vartheta'^2$.

(iii) If $\operatorname{Re} T \ge c$, then $\|Tu\| \|u\| \ge \operatorname{Re} \langle u, Tu \rangle \ge c\|u\|^2$, so $\|T^{-1}\| \le 1/c$; apply this to $T = M^w$, whose inverse is $(M^{-1})^w$.

The second statement is the Neumann series for $(M^w+E^w)^{-1}$. □

Proposition 369. *Let $G\subset \mathrm{U}(r)$ be compact. Theorem 367 holds with*

$$\alpha_G = 2.5\cdot 10^{-4}, \qquad c_9 = 1.2\cdot 10^4,$$

and in (iii) *the factor $|X_1||X_2|$ may be replaced by $(|X_1||X_2|)^{1/2}$.*

So the covariances of local observables under the constrained law on the chart decay at least like $e^{-d/4000}$, with amplitude $1.2\cdot 10^4/\lambda$, for every compact group; for $\mathrm{SU}(2)$ and the Wilson action the amplitude is $4.8\cdot 10^4/\beta$.

Proof. Throughout, $\xi\in \mathrm{Box}_{2r}$, $X=S\xi+h(S\xi)$, and the notation is that of the proof of Theorem 367. The hypothesis $4\cdot 10^5 c_3(\kappa+s)\le \frac{1}{195}$ of Theorem 363(iv) gives $c_3 s\le 1.29\cdot 10^{-8}$ and $c_3\kappa'\le 7.7\cdot 10^{-8}$ for $\kappa'=\kappa+6s$, and $\kappa\le 10^{-8}$. Every φ below is defined on all indices; a function on L' with the Lipschitz property extends to all indices with the same property, by $\varphi(a)=\min_{e\in L'}(\varphi(e)+d(a,e))$.

Local operators. The row k of B has blocks of norm at most $\frac18+260\kappa$, with sum at most $2+24\cdot 260\kappa$, by Lemma 360(iv) and the coefficients of $V_{y,\mu}$, eight of them $\frac18$ and sixteen $\frac{1}{16}$; every link lies in at most two rows, and the blocks vanish beyond distance 1. So $b=\|B\|_{(\vartheta)}\le e^{\vartheta}(\frac12+10^{-4})^{1/2}$ by Lemma 368(i). The row k of $A-B$ has norm at most $c_3N_{y,\mu}(X)\le 7.35c_3 s=\varepsilon_A$, since $\|X\|_\infty\le\frac32 s$, so $\|A-B\|_{(\vartheta)}\le\sqrt{48}e^{\vartheta}\varepsilon_A$. The row d_k of S is $s_k\xi=-B^{-1}_{k,d_k}\sum_{e\neq d_k}B_{k,e}\xi_e$, of norm $\sigma\le(\frac{\sqrt{11}}{8}+260\kappa)/(\frac18-260\kappa)\le 3.32$, the squares of the other coefficients adding up to $\frac{11}{64}$; the other rows of S are those of the identity. If D_k is the diagonal matrix with entries $e^{\vartheta(\varphi(d_k)-\varphi(e))}$ on the links e of $V_{y,\mu}$, then the row d_k of S^w is s_kD_k, and every link of L' lies in at most two of these sets, so

$$\bar s=\|S\|_{(\vartheta)}\le(1+2\sigma^2e^{2\vartheta})^{1/2}.$$

Finally $Q(Ue^X)=\sum_p\|U_p-I\|^2=\sum_p(2r-2\operatorname{Re}\operatorname{Tr}U_p)$. By Duhamel's formula the derivative of $U_ee^{X_e}$ in X_e is an average of $U_ee^{\theta X_e}Ye^{(1-\theta)X_e}$, and its second derivative an average, of total weight one, of products with Y and Z inserted between unitary factors. With $|\operatorname{Tr}(V_1YV_2ZV_3)|\le\|Y\|\|Z\|$ for unitary V_i this gives

$$\big|D^2Q_p[Y,Z]\big|\le 2\sum_{e\in p}\|Y_e\|\sum_{e\in p}\|Z_e\|,$$

and, since $\operatorname{Re}\operatorname{Tr}(VY)=\operatorname{Re}\operatorname{Tr}((V-I)Y)$ for skew-Hermitian Y and a conjugate V of U_p, $|DQ_p[Y]|\le 2\kappa'\sum_{e\in p}\|Y_e\|$. A link lies in six plaquettes, so $\|D^2Q\|_{(\vartheta)}\le 48e^{\vartheta}$ and $|DQ[Y]|\le 12\kappa'\|Y\|_1$.

Inverses. The diagonal blocks of $\mathscr{G}=BB^*$ have norm at most $\frac{3}{16}+227\kappa$, each row has two other nonzero blocks, of norm at most $\frac{1}{32}+227\kappa$, at distance 2, and $\mathscr{G}\ge\frac{1}{16}-700\kappa$, by the proof of Theorem 361(iii). The blocks of $\operatorname{Re}\mathscr{G}^w$ are $\cosh(\vartheta(\varphi(k)-\varphi(l)))\mathscr{G}_{kl}$, so $\operatorname{Re}\mathscr{G}^w\ge\frac{1}{16}-700\kappa-(\cosh 2\vartheta-1)(\frac{1}{16}+454\kappa)$, and Lemma 368(iii) bounds $g_0=\|\mathscr{G}^{-1}\|_{(\vartheta)}$ by the inverse of the right-hand side. With $E=(A-B)B^*$, the same lemma gives

$$\|(AB^*)^{-1}\|_{(\vartheta)}\le\frac{g_0}{1-g_0\|E\|_{(\vartheta)}}=h_0,$$
$$\|(AA^*)^{-1}\|_{(\vartheta)}\le\frac{g_0}{1-g_0(2\|E\|_{(\vartheta)}+\|A-B\|^2_{(\vartheta)})}=g_1,$$

since $AB^*=\mathscr{G}+E$ and $AA^*=\mathscr{G}+E+E^*+(A-B)(A-B)^*$. Let $Y=B^*(AB^*)^{-1}$, so $y=\|Y\|_{(\vartheta)}\le bh_0$. Since $BS=0$, $\bar P=PS=S-Y(A-B)S$, so $\bar p=\|\bar P\|_{(\vartheta)}\le\bar s(1+y\|A-B\|_{(\vartheta)})$; and $M=Y(A-B)S$ takes values in N_U, orthogonal to the values of S, so $\bar P^*\bar P=S^*S+M^*M$. Here $S^*S=1+\sum_k s_k^*s_k$, and $(s_k^*s_k)^w=v^*u$ with $u=s_kD_k$, $v=s_kD_k^{-1}$, so $\operatorname{Re}(s_k^*s_k)^w=\frac14((u+v)^*(u+v)-(u-v)^*(u-v))\ge-\sigma^2\sinh^2\vartheta\,J_k$, J_k the projection on the links of $V_{y,\mu}$; and $\operatorname{Re}(M^*M)^w\ge-2(\vartheta/\vartheta'')^2\|M\|^2_{(\vartheta'')}$ by Lemma 368(ii), for $\vartheta\le\vartheta''$. Hence

$$g_2=\|(\bar P^*\bar P)^{-1}\|_{(\vartheta)}\le\Big(1-2\sigma^2\sinh^2\vartheta-2\frac{\vartheta^2}{\vartheta''^2}\|M\|^2_{(\vartheta'')}\Big)^{-1}.$$

For the ℓ^1 norms, $\|\mathscr{G}^{-1}\|\le 17$ in the norm $\sum_k\|z_k\|$ and E has column sums at most $2(2+10^{-4})\varepsilon_A$, so $(AB^*)^{-1}$ has norm $h_1\le 17.001$ there, and $\|B^*z\|_1\le(2+10^{-4})\sum_k\|z_k\|$.

The Hessian. Let $u,u'\in\mathfrak{g}^{L'}$, $p=wu$ and $q=w^{-1}u'$. Since $\langle u,(\mathrm{Hess}\,\bar H)^wu'\rangle=\partial_p\partial_q\bar H$, it suffices to bound $|\partial_p\partial_q\bar H|$ by $K|u||u'|$ for every φ. Write $x_p=\bar Pp=w\tilde x$ and $x_q=w^{-1}\tilde y$, with $|\tilde x|\le\bar p|u|$ and $|\tilde y|\le\bar p|u'|$. The links of $V_{y,\mu}$ lie within distance 1 of the position of k, so the weights of x_p and x_q on them differ from $e^{\pm\vartheta\varphi(k)}$ by factors at most e^{ϑ}, and

$$\sum_k N_{y,\mu}(x_p)N_{y,\mu}(x_q)\le 2e^{2\vartheta}|\tilde x||\tilde y|,$$

every link lying in at most two of these sets. In the same way $A_p=\partial_pA=D^2c(X)[x_p,\cdot]$ equals $w\check A_p$ and $A_q=w^{-1}\check A_q$, where $\check A_p$ and $\check A_q$ have rows of rank at most $\dim G$, so their Hilbert–Schmidt norms satisfy

$$\|\check A_p\|_2\|\check A_q\|_2\le 2\dim G\,c_3^2e^{2\vartheta}\bar p^2|u||u'|,$$

and conjugating them by w multiplies these norms by at most e^{ϑ}. The second derivative $x_{pq}=-YD^2c(X)[x_p,x_q]$ has $\|x_{pq}\|_1\le c_3x_1|u||u'|$ with $x_1=2(2+10^{-4})h_1e^{2\vartheta}\bar p^2$, and $A_{pq}=D^3c(X)[x_p,x_q,\cdot]+D^2c(X)[x_{pq},\cdot]$ has trace norm $\bar a\le 2\dim G\,c_3^2(e^{2\vartheta}\bar p^2+x_1)|u||u'|$, the weights cancelling in both.

The four parts of $\bar H$ are bounded as follows. First, $\partial_p\partial_q\lambda Q=\lambda(\langle\tilde x,(D^2Q)^w\tilde y\rangle+DQ[x_{pq}])$, at most $\lambda(48e^{\vartheta}\bar p^2+12c_3\kappa'x_1)|u||u'|$. Second, the Hessian of $\sum_e\log j(X_e)$ in X is block diagonal and commutes with w, so its part is at most $c_3(\bar p^2+c_3x_1)|u||u'|$. Third, for $\mathscr{G}_1=AA^*$,

$$\partial_p\partial_q\log\det\mathscr{G}_1=\operatorname{Tr}(\mathscr{G}_1^{-1}\partial_p\partial_q\mathscr{G}_1)-\operatorname{Tr}(\mathscr{G}_1^{-1}\partial_p\mathscr{G}_1\mathscr{G}_1^{-1}\partial_q\mathscr{G}_1),$$

with $\partial_p\mathscr{G}_1=A_pA^*+AA_p^*$ and $\partial_p\partial_q\mathscr{G}_1=A_{pq}A^*+A_pA_q^*+A_qA_p^*+AA_{pq}^*$. Each term is the trace of a product in which

w and w^{-1} occur once each, next to $\check{A}_p$ and $\check{A}_q$, or not at all, next to A_{pq}. A cyclic permutation writes it as $\mathrm{Tr}(wFw^{-1}F')$, or $\mathrm{Tr}(w^{-1}FwF')$, and $|\mathrm{Tr}(wFw^{-1}F')| \le \|wFw^{-1}\|_2\|F'\|_2$ or $\|wFw^{-1}\|\|F'\|_1$. With $a = \|A\|_{(\vartheta)} \le b + \|A-B\|_{(\vartheta)}$ this gives

$$\tfrac{1}{2}|\partial_p\partial_q \log\det \mathscr{G}_1| \le g_1 a\bar{a} + (g_1 + 2g_1^2a^2)e^{2\vartheta}\|\check{A}_p\|_2\|\check{A}_q\|_2.$$

Fourth, $\partial_p\bar{P} = -YA_p\bar{P}$ and $\partial_p\partial_q\bar{P} = -YA_{pq}\bar{P} + YA_qYA_p\bar{P} + YA_pYA_q\bar{P}$, and the same counting for $\mathscr{G}_2 = \bar{P}^*\bar{P}$ gives

$$\tfrac{1}{2}|\partial_p\partial_q \log\det \mathscr{G}_2| \le g_2\bar{p}^2 y\bar{a} + (3g_2\bar{p}^2y^2 + 2g_2^2\bar{p}^4y^2)e^{2\vartheta}\|\check{A}_p\|_2\|\check{A}_q\|_2.$$

So

$$\|\mathrm{Hess}\,\bar{H}\|_{(\vartheta)} \le K = \lambda k_1 + \dim G\, c_3^2 k_0,$$

where $k_1 = 48e^{\vartheta}\bar{p}^2 + 12c_3\kappa' x_1$ and k_0 collects the other three parts, divided by $\dim G\, c_3^2|u||u'|$, which is possible as $c_3 \ge 1$. At $\vartheta = \vartheta' = 0.197$, with $\vartheta'' = 0.6$, these give $b \le 0.862$, $g_0, h_0, g_1 \le 17.37$, $y \le 14.96$, $\bar{s}, \bar{p} \le 5.80$, $g_2 \le 7.41$, $k_1 \le 1966$ and $k_0 \le 4.16\cdot 10^9$; and since $\lambda \ge 60c_6 = 4.086\cdot 10^7 \dim G\, c_3^2$ by Proposition 364, $K \le 2068\lambda$.

Decay. In the proof of Theorem 367, replace the bound on $e^{\alpha\varphi}\,\mathrm{Hess}\,H_n e^{-\alpha\varphi} - \mathrm{Hess}\,H_n$ by the following. Since $D\tau$ and Δ_n are block diagonal and symmetric and commute with $w = e^{\alpha\varphi}$, Lemma 368(ii) with $\vartheta = \alpha$ gives

$$\left\|\mathrm{Re}(\mathrm{Hess}\,H_n)^w - \mathrm{Hess}\,H_n\right\| \le 2\frac{\alpha^2}{\vartheta'^2}K \le \frac{\lambda}{120} = \frac{m}{2}$$

for $\alpha \le \vartheta'(240\cdot 2068)^{-1/2}$, which holds for $\alpha \le 2.5\cdot 10^{-4}$. Then $\mathrm{Re}\langle u, \mathscr{A}_w u\rangle \ge \frac{m}{2}\|u\|^2$ as before, and (ii) follows. For (iii), $\nabla' F = \bar{P}^* g$ with g supported on X_1 and $|g_a| \le 2$, so with $\varphi = d(\cdot, X_2)$,

$$|e^{-\alpha\varphi}\nabla' F| = |(\bar{P}^*)^{w^{-1}}e^{-\alpha\varphi}g| \le 2\bar{p}_\alpha |X_1|^{1/2} e^{-\alpha d(X_1,X_2)},$$

where $\bar{p}_\alpha = \|\bar{P}\|_{(\alpha)} \le 4.8$ at $\alpha = 2.5\cdot 10^{-4}$, and $|e^{\alpha\varphi}\nabla' G| \le 2\bar{p}_\alpha |X_2|^{1/2}$. Insert these in (ii): $c_9 = 480\bar{p}_\alpha^2 \le 1.2\cdot 10^4$. □

The chart misses only points of the fibre at which some plaquette near W is not small, and under the Wilson measure such points carry little mass on most fibres.

Theorem 370. (i) *Assume the hypotheses of Theorem 367 and $s \ge 1.24\cdot 10^5\kappa$. Every $U' \in F$ with $\delta(U'_p, I) \le \kappa$ for every plaquette p with a vertex in a block of W^+ lies in $\Phi(\Omega_r)$. Hence the constrained law gives the complement of $\Phi(\Omega_r)$ at most the mass of the points of F at which such a plaquette exceeds κ.*

(ii) *Let $G = \mathrm{SU}(2)$, let the side $2L$ be a power of two, let μ be the Wilson measure in the tree gauge at $\beta_W \ge 1$, so that $\lambda = \beta_W/4$, and let E be the set of fields with $\delta(U_p, I) \le \kappa$ for every plaquette with a vertex in a block of W^+. For μ-almost every U let μ_U be the constrained law on the fibre $F(U)$ through U. Then*

$$\mathbb{E}_\mu\left[\mu_U(E^c)\right] = \mu(E^c) \le 10^4|W|\, q_\kappa(\beta_W),$$

with q_κ as in Theorem 383. If κ, s and λ satisfy the hypotheses of (i) *and $0 < \varepsilon < 1$, the fields $U \in E$ with $\mu_U(F(U) \setminus \Phi_U(\Omega_r)) \le \varepsilon$, where Φ_U is the chart at U, have probability at least $1 - (1+\varepsilon^{-1})10^4|W|q_\kappa(\beta_W)$.*

(iii) *At every such U, for $X_1, X_2 \subset A_W$ and $\tilde{F}, \tilde{G}$ as in Theorem 367*(iii)*,*

$$|\mathrm{Cov}_{\mu_U}(\tilde{F}, \tilde{G})| \le \Big(\frac{4c_9}{\beta_W}e^{-\alpha_G d(X_1,X_2)} + 30\varepsilon\Big)|X_1|\,|X_2|.$$

Proof. (i) The fields U and U' satisfy the hypothesis of Lemma 360 and agree off A_W, in particular on the tree links. An active link e with base in a block $B(y)$, $y \in W$, either lies inside $B(y)$ or leaves it in a direction μ. In the first case part (i) of the lemma, at U and at U', gives $\delta(U_e, U'_e) \le 6\kappa$. In the second, let v and v' be the path links of $V_{y,\mu}$ at U and at U'. Part (iv), at both fields, writes the common cut-off average as $ve^M = v'e^{M'}$ with $\|M\|, \|M'\| \le 8\pi\kappa$, so by bi-invariance and Lemma 352

$$\delta(v, v') = \delta\left(e^M e^{-M'}, I\right) \le \|M\| + \|M'\| \le 16\pi\kappa < 50.3\kappa,$$

and part (i) gives $\delta(U_e, U'_e) \le \delta(U_e, v) + \delta(v, v') + \delta(v', U'_e) \le 92.3\kappa$. Let $X_e = \log(U_e^{-1}U'_e)$. By Lemma 352, $\|X_e\| \le \frac{\pi}{3}92.3\kappa < 96.7\kappa \le s/1280$ for every $e \in A_W$. Theorem 363(i) with $s/32$ in place of s, whose chart is the restriction of Φ by the uniqueness of the fixed point in its proof, puts $U' = Ue^X$ in $\Phi(K_{s/32})$, and $K_{s/32} \subset \Omega_r$ by Theorem 367(i).

(ii) The tree links span every block, so for every field there is exactly one gauge transformation, equal to I at the corner of every block, that makes the tree links equal to I. Given the tree links, it multiplies each active link on both sides by elements that do not depend on that link, which preserves Haar measure, and S is gauge invariant; so in the tree gauge the active links have density proportional to $e^{-\beta_W S}$ with respect to product Haar measure. Since $\beta_W S = \frac{\beta_W}{4}\sum_p \delta(U_p, I)^2$, conditionally on the links off A_W the links of A_W have density proportional to $e^{-\lambda Q}$, and conditioning further on the averages gives the constrained law. So μ_U is a version of the conditional law of μ given the links off A_W and the averages, and the identity is the tower property of conditional expectation. If $U \notin E$, some plaquette with a vertex in a block $B(y')$, $y' \in W^+$, is bad for $\rho = \kappa$. Its corner z differs from that vertex by an element of $\{0,1\}^4$, so z lies in a block at ℓ^∞ distance at most two from W, and the plaquette lies in the unit cube $\mathscr{C}_z$. There are at most $5^4|W|$ such blocks, with 16 sites each, and E is gauge invariant, so Theorem 383 with $|A| = 1$ gives $\mu(E^c) \le 10^4|W|q_\kappa(\beta_W)$. For $U \in E$, part (i) gives $\mu_U(F(U) \setminus \Phi_U(\Omega_r)) \le \mu_U(E^c)$, so the exceptional fields lie in $E^c \cup \{\mu_U(E^c) > \varepsilon\}$, which by Markov's inequality has probability at most $\mu(E^c)(1+\varepsilon^{-1})$.

(iii) Let $A = \Phi_U(\Omega_r)$, $m = \mu_U$, $m_A = m(\cdot \mid A)$ and $\varepsilon' = m(A^c) \le \varepsilon$. By Theorem 367(iii) with $\lambda = \beta_W/4$, $|\mathrm{Cov}_{m_A}(\tilde{F}, \tilde{G})| \le \frac{4c_9}{\beta_W}|X_1||X_2|e^{-\alpha_G d(X_1,X_2)}$. The group SU(2)

has diameter $\sqrt{2}\pi$, and $\tilde F$ has gradient at most 1 in each link, so after subtracting a constant, which changes no covariance, $\|\tilde F\|_\infty \le \frac{\sqrt{2}\pi}{2}|X_1| \le 2.23|X_1|$, and likewise for $\tilde G$. Writing $m = (1-\varepsilon')m_A + \varepsilon' m_{A^c}$ gives $|m(fg) - m_A(fg)| \le 2\varepsilon'\|f\|_\infty\|g\|_\infty$ and $|m(f) - m_A(f)| \le 2\varepsilon'\|f\|_\infty$ for bounded f, g, so

$$|\mathrm{Cov}_m(f,g) - \mathrm{Cov}_{m_A}(f,g)| \le 6\varepsilon'\|f\|_\infty\|g\|_\infty,$$

and $6 \cdot 2.23^2 < 30$. □

So on a fixed set of blocks the constrained law of the SU(2) Wilson measure decorrelates, coarea factor and curvature of the fibre included, on every fibre outside a set of fibres whose probability tends to zero as $\beta_W \to \infty$. The bound grows linearly in $|W|$, which is harmless for the fixed cubes of the cube condition, but the thresholds are far from the one-loop range: with the constants of Proposition 364 the hypotheses force $\kappa + s \le 6.0 \cdot 10^{-13}$ and $s \ge 1.24 \cdot 10^5 \kappa$, so $\kappa < 4.9 \cdot 10^{-18}$, and $q_\kappa(\beta_W) < 1$ needs $\beta_W \kappa^2 > 96(\log \beta_W + C_0)$, so β_W exceeds $3.8 \cdot 10^{38}$ (Fig. 31).

The mass off the chart can also be bounded on a single fibre, by a Peierls argument inside the fibre. Two properties of the cut-off average make this possible: along a fixed set of links it is a submersion with a constant that does not depend on the field, and its law under Haar measure on those links has a bounded density.

Fix $y \in W$ and a direction μ. Write the offsets of the sites of $B(y)$ in the three directions other than μ as $a \in \{0,1\}^3$, and let x_a and x'_a be the sites of offset a with $o_\mu = 0$ and $o_\mu = 1$. Let $\ell_a = (x'_a, \mu)$, which joins $B(y)$ to $B(y + e_\mu)$, and let $u_a = (x_a, \mu)$ and $w_a = (x'_a + e_\mu, \mu)$, which lie inside $B(y)$ and $B(y + e_\mu)$. The links u_0 and w_0 are tree links and $\ell_0 = d_k$ is the path link, so in the tree gauge

$$P_{2y} = \ell_0, \qquad P_{x_a} = u_a \ell_a, \qquad P_{x'_a} = \ell_a w_a,$$

and $v_x = P_{2y}^{-1} P_x$ equals I at the two sites of offset 0. The path of $V^\sharp_{y',\nu}$ from a site x consists of the link (x,ν) and the next link $(x + e_\nu, \nu)$, so ℓ_a, whose base has $o_\mu = 1$, enters only the paths from x_a and x'_a of the pair (y,μ). We call $\ell_0, \dots, \ell_7$ the *private links* of (y,μ). The 32 links with $o_\mu = 1$ of $B(y)$ are the private links of the four pairs (y,μ); the other 17 active links of $B(y)$ lie inside it.

Let

$$\alpha = 2 \arcsin \frac{1}{2\sqrt{2}} = 0.7227\ldots, \qquad \Lambda = \frac{\alpha}{\sin \alpha} = 1.0926\ldots,$$

so that $1 - \frac{7}{8}\Lambda > 0.0439$. Two facts about $\mathrm{ad}X$ for $X \in \mathfrak{g}$, which is skew-adjoint for the Hilbert–Schmidt product, will be used throughout. If the eigenvalues $i\psi$ of $\mathrm{ad}X$ satisfy $|\psi| < 2\alpha$, then

$$\log\left(e^{tZ} e^X\right) = X + t L_X Z + O(t^2),$$
$$L_X = \frac{\mathrm{ad}X}{e^{\mathrm{ad}X} - 1}, \qquad \|L_X\| \le \Lambda,$$

since the eigenvalues of L_X have modulus $\frac{|\psi|/2}{\sin(|\psi|/2)}$, which increases with $|\psi|$. For every $Y \in \mathfrak{g}$,

$$\frac{d}{dt} e^{Y+tZ}\Big|_{t=0} e^{-Y} = E_Y Z,$$
$$E_Y = \frac{e^{\mathrm{ad}Y} - 1}{\mathrm{ad}Y}, \qquad \|E_Y\| \le 1,$$

because $|e^{i\psi} - 1| \le |\psi|$; in particular $d(e^A, e^B) \le \|A - B\|$ for the Riemannian distance d of the bi-invariant metric, along the curve $e^{A + t(B-A)}$. If $\delta(v, I) < 1$ and $e^{i\theta_j}$, $|\theta_j| \le \pi$, are the eigenvalues of v, then $\sum_j 4\sin^2(\theta_j/2) < 1$; each $|\theta_j|$ is less than $\frac{\pi}{3}$, where $\sin^2(t/2)$ is convex, so $|\theta_j| + |\theta_k| < 2\alpha$ for all j, k, and $X = \log v$ satisfies the hypothesis on $\mathrm{ad}X$.

Lemma 371. *Let $\sigma_0 = (1 - \frac{7}{8}\Lambda)/\sqrt{2} > 0.031$. At every field at which $V^\sharp_{y,\mu}$ is differentiable, its derivative $J : \mathfrak{g}^8 \to \mathfrak{g}$ with respect to its eight private links, right trivialised, satisfies $JJ^* \ge \sigma_0^2$. The fields at which it is not differentiable form a null set for Haar measure.*

Proof. Write $m = \max_x \delta(v_x, I)$ and $\chi = \phi(m/\rho_G)$, so that $V^\sharp = P_{2y} e^{\chi N}$ with $N = \frac{1}{16}\sum_x \log v_x$. Where $m > \rho_G$, the cut-off vanishes near the field, $V^\sharp = \ell_0$, and the derivative along ℓ_0 is the identity. Suppose $m < \rho_G$, and that the maximum is attained at one path x_*, after identifying identical path functions, and $m \ne \rho_G/2$. Copies that coincide because u_a, w_a are tree links are counted together. Other ties lie in finitely many proper real-analytic subsets, a null set. The variation below fixes both copies of an identical pair, since then $u_b = w_b = I$ and $A - 1 = 0$. Then every $\delta(v_x, I)$ is less than 1, the logarithms are defined, χ is differentiable, and every variation that fixes v_{x_*} to first order fixes χ to first order. The site x_* has offset $b \ne 0$.

Fix $\zeta \in \mathfrak{g}$ and put $\eta = \mathrm{Ad}(\ell_0)\zeta$. Move $\ell_0 \mapsto e^{t\eta}\ell_0$, and, if $x_* = x'_b$, $\ell_b \mapsto e^{t\eta}\ell_b$, or, if $x_* = x_b$, $\ell_b \mapsto e^{t\xi}\ell_b$ with $\xi = \mathrm{Ad}(u_b^{-1})\eta$, so that $u_b\ell_b \mapsto e^{t\eta}u_b\ell_b$. In both cases P_{2y} and P_{x_*} are multiplied on the left by $e^{t\eta}$, so v_{x_*} and χ are fixed to first order. The other site $\bar x$ of offset b has $P_{\bar x} \mapsto e^{t\,\mathrm{Ad}(u_b^{\pm 1})\eta} P_{\bar x}$, and the paths from the remaining twelve sites with $v_x \ne I$ do not move. With the isometry $A = \mathrm{Ad}(\ell_0^{-1} u_b^{\pm 1} \ell_0)$,

$$v_{\bar x} \mapsto e^{t(A-1)\zeta} v_{\bar x} + O(t^2), \qquad v_x \mapsto e^{-t\zeta} v_x + O(t^2)$$

for the twelve others, so N moves by $t\,\delta N$ with $16\,\delta N = L_{X_{\bar x}}(A-1)\zeta - \sum_x L_{X_x}\zeta$, twelve terms, and $\|\delta N\| \le \frac{14}{16}\Lambda\|\zeta\|$. Since χ is fixed,

$$\frac{d}{dt} V^\sharp (V^\sharp)^{-1} = \eta + \mathrm{Ad}(\ell_0) E_{\chi N}(\chi\,\delta N) = \mathrm{Ad}(\ell_0)\, T\zeta,$$

with $\|T\zeta - \zeta\| \le \frac{7}{8}\Lambda\|\zeta\|$, as $\chi \le 1$. The variation is $K\zeta$ for a linear map $K : \mathfrak{g} \to \mathfrak{g}^8$ with two nonzero components of norm $\|\zeta\|$, so $\|K\| = \sqrt{2}$ and $JK = \mathrm{Ad}(\ell_0)T$. For a unit vector $e \in \mathfrak{g}$,

$$\|J^* e\| \ge \frac{\|K^* J^* e\|}{\sqrt{2}} = \frac{\|T^* \mathrm{Ad}(\ell_0)^* e\|}{\sqrt{2}} \ge \frac{1 - \frac{7}{8}\Lambda}{\sqrt{2}} = \sigma_0,$$

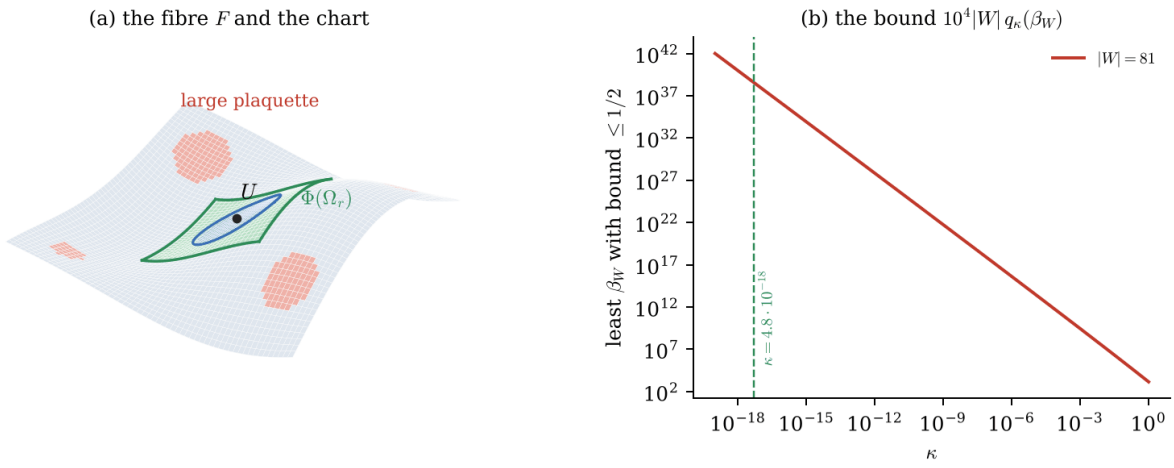


FIG. 31. Theorem 370. (a) A fibre F of the cut-off average, drawn as a surface: the product chart $\Phi(\Omega_r)$ around the background U (green) contains every point of F whose plaquettes near W are at most κ (inside the blue curve), and the rest of F meets only points with a large plaquette (red). (b) The least β_W at which the bound $10^4|W|q_\kappa(\beta_W)$ is at most $\frac{1}{2}$, for the 81 blocks of a cube of side three; it grows like $\kappa^{-2}\log(1/\kappa)$, and at the largest κ the hypotheses allow it is about $4.0\cdot 10^{38}$.

since the least singular value of T is at least $1-\frac{7}{8}\Lambda$. At the other points of differentiability the same bound follows by continuity: the cut-off has finitely many smooth branch formulas, at such a point the formulas of the adjacent full-dimensional branches agree with its differential, and the lower bound on the singular value is preserved under limits. □

Corollary 372. *Let $C=(V^\sharp_{y,\mu})_{y\in W,\mu}$, as a function of the links of A_W, the other links being fixed. Wherever C is differentiable,*

$$\det\bigl(DC\,DC^*\bigr)^{-1/2}\le \sigma_0^{-4|W|\dim G}<33^{4|W|\dim G}.$$

Proof. The private links of distinct pairs are distinct, and each enters only the average of its pair, so the columns of DC that belong to private links form a block diagonal matrix with the blocks of Lemma 371. Hence $DC\,DC^*\ge\bigoplus_k J_kJ_k^*\ge\sigma_0^2$. □

The coarea factor of the constrained law is therefore bounded on the whole fibre by a constant per block, for every value of the cut-off and however far the links inside the blocks are from I. The volume of the fibre is bounded through the law of the averages. From here on $G=\mathrm{SU}(2)$, which with the metric $\|X\|^2=\mathrm{Tr}(X^*X)$ is the round sphere of radius $\sqrt{2}$, $\rho_G=1$, and h is normalised Haar measure.

Lemma 373. *Let $G=\mathrm{SU}(2)$, fix every link other than the private links of (y,μ), and let these eight links be independent and Haar distributed. Then the law of $V^\sharp_{y,\mu}$ has a density with respect to h that is at most*

$$M=1+2\Bigl(1-\frac{7}{8}\Lambda\Bigr)^{-3}<2.4\cdot 10^4$$

at every point, in the version given by the formula (38) of the proof.

Proof. Let $g=\ell_0$, and let m_1 and m_2 be the largest values of $\delta(v_{x_a},I)$ and of $\delta(v_{x'_a},I)$. Up to the null set $m=1$, split G^8 into the three regions

$$R_0=\{m>1\},$$
$$R_1=\{m_1\ge m_2,\ m_1<1\},\qquad R_2=\{m_2>m_1,\ m_2<1\};$$

ties between the two families, including two path functions that coincide identically, belong to R_1. On R_0, $\chi=0$ and $V^\sharp=g$.

On R_1 use the coordinates $(g,h_1,\dots,h_7)$ with $h_a=v_{x_a}=g^{-1}u_a\ell_a$. For fixed g the map $\ell_a\mapsto h_a$ is a left translation, so these coordinates carry product Haar measure, and

$$v_{x'_a}=g^{-1}\ell_a w_a=g^{-1}u_a^{-1}g\,h_a w_a.$$

Fix h and put $m_h=\max_a\delta(h_a,I)<1$. The set S_h of g with $(g,h)\in R_1$ is the measurable set where $\delta(v_{x'_a},I)\le m_h$ for every a; on it $m=m_h$, so $\chi=\phi(m_h)$ does not depend on g, and $V^\sharp=g\,e^{\chi N(g)}$ with $N(g)=\frac{1}{16}\sum_a(\log h_a+\log v_{x'_a})$. The set $\{\delta(v,I)<1\}$ is the ball of angular radius α about I on the sphere, which is geodesically convex, and on it $\|\log v-\log v'\|\le\Lambda\,d(v,v')$. By bi-invariance $d(g^{-1}u^{-1}g,g'^{-1}u^{-1}g')\le 2d(g,g')$, so for $g,g'\in S_h$

$$\|N(g)-N(g')\|\le\frac{7}{16}\,2\Lambda\,d(g,g')=\frac{7}{8}\Lambda\,d(g,g').$$

If $V^\sharp(g)=V^\sharp(g')=c$, then $g=c\,e^{-\chi N(g)}$ and $g'=c\,e^{-\chi N(g')}$, so $d(g,g')\le\frac{7}{8}\Lambda\,d(g,g')$ and $g=g'$: the map $g\mapsto V^\sharp(g)$ is injective on S_h. Its derivative is computed as in the proof of Lemma 371: with $\eta=\mathrm{Ad}(g)\zeta$ the sites x'_a move by $e^{t(A_a-1)\zeta}$, $A_a=\mathrm{Ad}(g^{-1}u_a^{-1}g)$, the sites x_a do not move, and the right trivialised derivative is $\mathrm{Ad}(g)T$ with $\|T-1\|\le\frac{7}{16}2\Lambda=\frac{7}{8}\Lambda$, so its determinant is at least $(1-\frac{7}{8}\Lambda)^3$ in modulus. On R_2 the coordinates $(g,h'_1,\dots,h'_7)$ with $h'_a=v_{x'_a}=g^{-1}\ell_a w_a$ and $v_{x_a}=g^{-1}u_a g\,h'_a w_a^{-1}$ give the same conclusions.

For fixed h, the smooth formula with $\chi = \phi(m_h)$ fixed extends to a neighbourhood of each point of S_h, where no logarithm meets $-I$; its restriction to the measurable set S_h is injective and has the Jacobian bounds above. The change of variables formula for measurable restrictions therefore applies, even if part of the boundary of S_h has positive measure, and likewise on R_2. The law of $V^\sharp$ has the density

$$\rho(c) = h^{\otimes 7}\big(\{\ell : (c,\ell) \in R_0\}\big) + \sum_{i=1,2} \int \sum_{g \in S_h^{(i)},\ V^\sharp = c} \frac{dh^{\otimes 7}}{|\det DV^\sharp(g)|}, \quad (38)$$

the inner sum having at most one term, and $\rho(c) \le 1 + 2(1 - \frac{7}{8}\Lambda)^{-3}$. Finally $1 - \frac{7}{8}\Lambda = 0.04391\ldots$ and $M = 23621\ldots$. □

The four averages of a block depend on disjoint sets of private links, and on the links inside the blocks, which are private to none. So, given the links inside the blocks of W, the averages $V_k^\sharp$ are independent, and the formula (38) for each pair defines, for every value c of C, a measure ν_c on the fibre $F_c = C^{-1}(c)$: in the coordinates of the proof of Lemma 373, chosen for each pair according to its region, it is the product Haar measure of the links inside the blocks and of the seven coordinates other than g, divided by the product of the determinants $|\det DV_k^\sharp(g)|$. Then $h = \int \nu_c\, dh(c)$ on G^{A_W}, so ν_c is a version of the conditional law, and

$$\nu_c(F_c) \le M^{4|W|} \qquad \text{for every } c. \quad (39)$$

Where C is differentiable,

$$\nu_c = \mathrm{vol}(G)^{-45|W|} \det\big(DC\,DC^*\big)^{-1/2}\, d\mathscr{H}.$$

Indeed, if $C(z,g) = c$ is solved for g by $g = \psi(z)$, then $DC = \partial_g C\,(-D\psi, 1)$, so

$$\det\big(DC\,DC^*\big)^{1/2} = |\det \partial_g C|\, \det\big(1 + D\psi^* D\psi\big)^{1/2},$$

and the second factor is the Jacobian of the graph of ψ. So ν_c is the coarea version on the charts of Theorems 363 and 367, where C is smooth, and we use it on the whole fibre.

Proposition 374. *Let $G = \mathrm{SU}(2)$, $\lambda > 0$ and $0 < K \le 10^{-8}$, and let $n_P \le 384|W|$ be the number of plaquettes that contain a link of A_W. Suppose the fibre F contains a field U with $\delta(U_p, I) \le \kappa$ for every plaquette p with a vertex in a block of W^+, where $\kappa \le K/(4\sqrt{n_P})$. Then the constrained law μ_U on F gives the set of fields with a plaquette p with a vertex in a block of W^+ and $\delta(U'_p, I) > K$ mass at most*

$$\varepsilon_W(K,\lambda) = \Big(\frac{2\cdot 10^9\, n_P^{3/2}}{K^3}\Big)^{45|W|} e^{-3\lambda K^2/4}.$$

Proof. Let $c = C(U)$, let B be the set in question, and let Q_W be the sum of $\delta(U'_p, I)^2$ over the n_P plaquettes that contain a link of A_W; on F it differs from Q by a constant, so $\mu_U = e^{-\lambda Q_W}\nu_c / \nu_c(e^{-\lambda Q_W})$. A plaquette that contains no link of A_W has $U'_p = U_p$ for $U' \in F$, so on B the large plaquette is one of the n_P and $Q_W > K^2$. By (39),

$$\int_B e^{-\lambda Q_W}\, d\nu_c \le M^{4|W|} e^{-\lambda K^2}.$$

For the denominator we solve the constraints for the path links. Let $L' = A_W \setminus \{d_k\}$, $\rho = K/(360\sqrt{n_P})$ and $R = 44\rho$, and let D be the set of $z \in G^{L'}$ with $d(z_e, U_e) \le \rho$ for every $e \in L'$. The field U satisfies the hypothesis of Lemma 360, as $\kappa \le K/4$, and part (i) of the lemma gives $\delta(U_e, I) \le 3\kappa$ for the links inside the blocks and $\delta(U_e, U_{d_k}) \le 21\kappa$ for the private links of k, so $\delta(v_x, I) \le 24\kappa$ and, by Lemma 352, $d(v_x, I) \le 8\pi\kappa$ at every site of every pair. Fix $z \in D$ and a pair k, and let ℓ range over the ball B_k of radius R about U_{d_k}. Each v_x is ℓ^{-1} times a product of at most two links of L' and fixed links, so at the field with links z and ℓ it lies within $R + 2\rho$ of its value at U, and within $8\pi\kappa + 46\rho < 0.01$ of I. Hence the cut-off equals 1 near this field and $V_k^\sharp = \ell\, e^{N_k(\ell)}$, with N_k depending on z and ℓ. Each v_x is moved by $d(\ell, \ell')$ when ℓ is replaced by ℓ', so the map $\Psi(\ell) = c_k\, e^{-N_k(\ell)}$ satisfies $d(\Psi(\ell), \Psi(\ell')) \le \frac{7}{8}\Lambda\, d(\ell, \ell')$. At U the path link solves $U_{d_k} = c_k e^{-N_k}$, and the sites move by at most $R + 2\rho$ between U and the field with links z and ℓ, so

$$d\big(\Psi(\ell), U_{d_k}\big) \le \frac{7}{8}\Lambda\,(R + 2\rho) < 0.9561 \cdot 46\rho < R.$$

By the contraction principle Ψ has a unique fixed point $\psi_k(z) \in B_k$. The path link d_k is private to k, so the field $\Gamma(z)$ with links z on L' and $\psi_k(z)$ on d_k lies in F, and Γ is injective.

Near $\Gamma(D)$ every cut-off equals 1, so C is smooth, and $\partial_g C$ is block diagonal with the blocks $\partial_{d_k} V_k^\sharp$. Moving $d_k \mapsto e^{t\eta} d_k$ multiplies every v_x of k by $e^{-t\zeta}$, $\zeta = \mathrm{Ad}(d_k^{-1})\eta$, so the block is $\mathrm{Ad}(d_k)(1 - E_{N_k}\frac{1}{16}\sum_x L_{X_x})$, with determinant at most $(1 + \frac{7}{8}\Lambda)^3 < 1.957^3$ in modulus. By the identity above, ν_c restricted to $\Gamma(D)$ is the image of $h^{\otimes L'}$ on D divided by $|\det \partial_g C|$, so

$$\nu_c\big(\Gamma(D)\big) \ge 1.957^{-12|W|} h(B_\rho)^{45|W|},$$
$$h(B_\rho) = \frac{2\theta - \sin 2\theta}{2\pi} \ge \frac{\rho^3}{14},$$

with $\theta = \rho/\sqrt{2}$ the angular radius, as $2\theta - \sin 2\theta \ge \frac{4}{3}\theta^3(1 - \frac{\theta^2}{5})$. A plaquette has at most two path links: two path links $(2y + e_\mu, \mu)$ and $(2y' + e_\mu, \mu)$ of one direction are never opposite sides of a plaquette, since their bases have exactly one odd coordinate, μ, and opposite sides differ by e_ρ, $\rho \ne \mu$; two of different directions can be adjacent sides. So on $\Gamma(D)$ each plaquette moves by at most $2R + 2\rho = 90\rho$ and $\delta(U'_p, I) \le \kappa + 90\rho \le K/(2\sqrt{n_P})$. Hence $Q_W \le K^2/4$ there and

$$\int_F e^{-\lambda Q_W}\, d\nu_c \ge e^{-\lambda K^2/4}\, 1.957^{-12|W|} \Big(\frac{\rho^3}{14}\Big)^{45|W|}.$$

Dividing,

$$\mu_U(B) \le \big(M^4 1.957^{12}\big)^{|W|} \Big(\frac{14\cdot 360^3 n_P^{3/2}}{K^3}\Big)^{45|W|} e^{-3\lambda K^2/4},$$

and $(M^4 1.957^{12})^{1/45} < 2.93$, $2.93 \cdot 14 \cdot 360^3 < 2 \cdot 10^9$. Finally $n_P \le 384|W|$, since each plaquette that contains a link of A_W has the base of that link, a site of a block of W, as a vertex, and each site is a vertex of 24 plaquettes. □

Corollary 375. *In addition to the hypotheses of Proposition 374, assume those of Theorem 367 with K in place of κ, and $s \ge 1.24 \cdot 10^5 K$. Then $\mu_U(F \setminus \Phi_U(\Omega_r)) \le \varepsilon_W(K,\lambda)$, and for $X_1, X_2 \subset A_W$ and $\tilde F, \tilde G$ as in Theorem 367*(iii)*,*

$$|\mathrm{Cov}_{\mu_U}(\tilde F, \tilde G)| \le \Big(\frac{c_9}{\lambda} e^{-\alpha_G d(X_1,X_2)} + 30\,\varepsilon_W(K,\lambda) \Big) |X_1|\,|X_2|.$$

Proof. The field U satisfies the hypothesis of Lemma 360 with K, so Theorem 370(i) with K in place of κ puts every point of F outside $\Phi_U(\Omega_r)$ in the set of Proposition 374. The covariance bound follows as in the proof of Theorem 370(iii). □

So the constrained law decorrelates on every fibre through a field whose plaquettes near W are at most $K/(4\sqrt{n_P})$, with no average over fibres, once λK^2 is a large multiple of $|W|\log(n_P/K)$; with $K < 4.9 \cdot 10^{-18}$ this needs β_W above $1.4 \cdot 10^{39}|W|$ under the Wilson measure. The restriction is on the fibre: the links off A_W are fixed on it, so a fibre whose boundary links carry a large plaquette near W contains no such field, and neither may a fibre with small boundary links, and the proposition does not cover these fibres. It is a lower bound for the fibre measure near one good field against an upper bound for the whole fibre; it does not compare different fibres.

The hypothesis on U enters the proof twice: through Lemma 360, to place the sixteen sites v_x of every pair near I, and through the bound on Q_W near U. Neither needs the links off A_W or the links inside the blocks to be small, and every fibre contains a field at which the sites are exactly I.

Lemma 376. *For every value c of C and every choice of the links off A_W, the fibre F_c contains a field at which $v_x = I$ for every site x of every pair (y,μ), $y \in W$.*

Proof. Fix μ. The links of A_W in the direction μ are, for each $y \in W$, the links u_a and ℓ_a of the pair (y,μ), and the links w_a of that pair are the links u_a of the pair $(y+e_\mu,\mu)$ when $y+e_\mu \in W$, and fixed links otherwise. Whether u_a is a tree link depends only on the offset, so u_a and w_a are both tree links or neither. For a pair with path link g and given links w_a put $\ell_0 = g$ and, for $a \ne 0$,

$$\ell_a = g, \quad \text{or} \quad \ell_a = g\,w_a^{-1},\ u_a = g\,w_a\,g^{-1},$$

the first when u_a is a tree link and the second otherwise. Then $v_{x_a} = g^{-1}u_a\ell_a = I$ and $v_{x'_a} = g^{-1}\ell_a w_a = I$, as $u_a = w_a = I$ in the first case. The blocks of W along a line in the direction μ form runs $y_1,\dots,y_n$ of consecutive blocks. If $y_n + e_\mu \notin W$, choose the links of the pairs $(y_n,\mu),\dots,(y_1,\mu)$ in this order, with $g = c_{y_j,\mu}$, the links w_a of (y_j,μ) being the fixed links for $j = n$ and the links u_a just chosen for (y_{j+1},μ) otherwise. If the run closes up around the torus, put $u_a = I$ and $\ell_a = g$ for every pair of the run. At the resulting field every v_x equals I, so the cut-off equals 1, $N = 0$ and $V^\sharp_{y,\mu} = \ell_0 = c_{y,\mu}$. The directions do not interact, since the pairs (y,μ) use only links in the direction μ. □

Call a field $U \in F$ *regular* if $\delta(v_x, I) \le \frac14$ at every site x of every pair, and let $q(F)$ be the infimum of Q_W over the regular fields of F, finite by Lemma 376. Write $\mu_F = e^{-\lambda Q_W}\nu_c / \nu_c(e^{-\lambda Q_W})$ for the constrained law on $F = F_c$.

Proposition 377. *Let $G = \mathrm{SU}(2)$, $\lambda > 0$ and $0 < t \le 1$. For every fibre F, whatever the links off A_W, and every regular $U \in F$,*

$$\mu_F\big(\{U' \in F : Q_W(U') > Q_W(U) + t\}\big) \le \varepsilon'_W(t,\lambda),$$
$$\varepsilon'_W(t,\lambda) = \Big(\frac{3.5 \cdot 10^{11}\, n_P^3}{t^3} \Big)^{45|W|} e^{-3\lambda t/4}.$$

In particular $\mu_F(\{Q_W > q(F) + t\}) \le \varepsilon'_W(t,\lambda)$ on every fibre.

Proof. The numerator is bounded by (39) as in the proof of Proposition 374, by $M^{4|W|}e^{-\lambda(Q_W(U)+t)}$. For the denominator take $\rho = t/(2040\,n_P) \le 10^{-3}$, $R = 44\rho$, and D and B_k as in that proof. At the field with links $z \in D$ on L' and $\ell \in B_k$ on d_k, each v_x of the pair k lies within $R + 2\rho = 46\rho < 0.043$ of its value at U, and $\delta \le d$, so $\delta(v_x, I) < \frac14 + 0.043 < \frac12$. Hence the cut-off equals 1 near this field, the logarithms are Λ-Lipschitz along the moves, and the contraction argument of that proof gives the injective map $\Gamma : D \to F$, near whose image C is smooth with $|\det \partial_g C| \le 1.957^{3\cdot4|W|}$. So

$$\nu_c\big(\Gamma(D)\big) \ge 1.957^{-12|W|} \Big(\frac{\rho^3}{14} \Big)^{45|W|}.$$

On $\Gamma(D)$ each of the n_P plaquettes moves by at most 90ρ, as in that proof, and $\delta(U_p, I) \le 2\sqrt{2}$, so

$$\delta(U'_p, I)^2 \le \delta(U_p, I)^2 + 180 \cdot 2\sqrt{2}\rho + 90^2\rho^2 < \delta(U_p, I)^2 + 510\rho,$$

and $Q_W \le Q_W(U) + 510 n_P \rho = Q_W(U) + \frac{t}{4}$ there. Dividing as before,

$$\mu_F(\{Q_W > Q_W(U) + t\}) \le \big(M^4 1.957^{12}\big)^{|W|} \Big(\frac{14 \cdot 2040^3 n_P^3}{t^3} \Big)^{45|W|} e^{-3\lambda t/4},$$

and $2.93 \cdot 14 \cdot 2040^3 < 3.5 \cdot 10^{11}$. For the last statement apply this to regular fields with $Q_W(U)$ decreasing to $q(F)$ and use the continuity of ε'_W in t. □

Corollary 378. *Let $G = \mathrm{SU}(2)$, $\lambda > 0$ and $0 < K \le 1$. Suppose the fibre F contains a regular field U with $\delta(U_p, I) \le K/(4\sqrt{n_P})$ on every plaquette p that contains a link of A_W, whatever the plaquettes made of fixed links. Then μ_F gives the fields with such a plaquette above K mass at most $\varepsilon'_W(\frac{15}{16}K^2, \lambda)$.*

Proof. At U, $Q_W \le K^2/16$, and a field with one of these plaquettes above K has $Q_W > K^2 \ge Q_W(U) + \frac{15}{16}K^2$. Apply Proposition 377 with $t = \frac{15}{16}K^2 \le 1$. □

The field of Proposition 374 is regular, since its sites lie within 24κ of I, so Corollary 378 contains that proposition up to the constants and drops its hypothesis on the plaquettes made

of fixed links and on the links inside the blocks. It reaches every fibre whose boundary links carry large plaquettes, provided the sites of the averages can be made small there together with the plaquettes that move. Proposition 377 holds on every fibre with no hypothesis at all, but its reference value $q(F)$ need not be small. Where it is of order one, as it can be when the averages are far from what the boundary links suggest, the constrained law sits within t of $q(F)$ but away from the charts of Theorem 367, which are built at fields with all plaquettes near W small. The proposition also does not compare regular with irregular fields, at which some site is far from I.

Different fibres with the same links off A_W are compared through the law of the averages. Call the links off A_W the *exterior* E, write h for product Haar measure on G^{A_W}, and let

$$\pi_E = \frac{e^{-\lambda Q_W} h}{h(e^{-\lambda Q_W})}$$

on G^{A_W}. Under the SU(2) Wilson measure in the tree gauge, π_E with $\lambda = \beta_W/4$ is the conditional law of the links of A_W given the exterior. Since $h = \int \nu_c \, dh(c)$,

$$\pi_E = \int \mu_{F_c} \frac{\nu_c(e^{-\lambda Q_W})}{h(e^{-\lambda Q_W})} \, dh(c),$$

so μ_{F_c} is the conditional law of π_E given $C = c$, and the law of the fibre through a field drawn from π_E is the law of C under π_E. Let $Q_*(E)$ be the least value of Q_W on G^{A_W}, which is attained since G^{A_W} is compact.

Proposition 379. *Let $G = \mathrm{SU}(2)$, $\lambda > 0$ and $0 < t \le 1$. For every exterior E,*

$$\pi_E\big(\{Q_W > Q_*(E) + t\}\big) \le \varepsilon''_W(t,\lambda),$$
$$\varepsilon''_W(t,\lambda) = \Big(\frac{1.05 \cdot 10^7 \, n_P^3}{t^3}\Big)^{49|W|} e^{-3\lambda t/4}.$$

In particular the law of the averages under π_E gives the fibres F with $\inf_F Q_W > Q_*(E) + t$ mass at most $\varepsilon''_W(t,\lambda)$.

Proof. Since h is a probability measure,

$$\int_{\{Q_W > Q_* + t\}} e^{-\lambda Q_W} \, dh \le e^{-\lambda(Q_* + t)}.$$

For the normalisation let U_* be a field with $Q_W(U_*) = Q_*$, let $\rho = t/(90.6 n_P)$, and let D be the set of fields with $d(U_e, U_{*,e}) \le \rho$ for each of the $49|W|$ links of A_W, the $45|W|$ links of L' and the $4|W|$ path links. Then $h(D) = h(B_\rho)^{49|W|} \ge (\rho^3/14)^{49|W|}$. Each plaquette contains at most four links of A_W, so on D it moves by at most 4ρ, and as $\delta \le d$ and $\delta(U_p, I) \le 2\sqrt{2}$,

$$\delta(U_p,I)^2 \le \delta(U_{*,p},I)^2 + 16\sqrt{2}\rho + 16\rho^2 < \delta(U_{*,p},I)^2 + 22.63\rho,$$

because each link of A_W lies in six plaquettes, so $n_P \ge 74$ and $\rho < 1.5 \cdot 10^{-4}$. Hence $Q_W \le Q_* + 22.63 n_P \rho < Q_* + \frac{t}{4}$ on D, and

$$\pi_E\big(\{Q_W > Q_* + t\}\big) \le \Big(\frac{14 \cdot 90.6^3 n_P^3}{t^3}\Big)^{49|W|} e^{-3\lambda t/4},$$

with $14 \cdot 90.6^3 < 1.05 \cdot 10^7$. A field U with $Q_W(U) \le Q_* + t$ lies in its own fibre, so $\inf_{F(U)} Q_W \le Q_* + t$, which gives the last statement. □

Call a field that agrees with E off A_W a *completion* of E.

Corollary 380. *Let $G = \mathrm{SU}(2)$, $\lambda > 0$, $0 < K \le 10^{-8}$ and $\kappa = K/(4\sqrt{n_P})$. Suppose the exterior E has a completion U^0 with $\delta(U^0_p, I) \le \kappa/(2\sqrt{n_P})$ for every plaquette p with a vertex in a block of W^+. Let $\mathscr{E}$ be the set of completions U with $\delta(U_p, I) \le \kappa$ for every such plaquette. Then:*

(i) $\pi_E(\mathscr{E}^c) \le \varepsilon''_W(\kappa^2/2, \lambda)$;

(ii) *every $U \in \mathscr{E}$ is regular and its fibre has $q(F(U)) < \kappa^2$, so the law of the averages under π_E gives the fibres with $q(F) \ge \kappa^2$, among them every fibre where $q(F)$ is of order one, mass at most $\varepsilon''_W(\kappa^2/2, \lambda)$;*

(iii) *if λ, s and K satisfy the hypotheses of Corollary 375, then for every $U \in \mathscr{E}$ the constrained law on $F(U)$ gives the complement of the chart $\Phi_U(\Omega_r)$ at U mass at most $\varepsilon_W(K, \lambda)$, and its covariances obey the bound of that corollary.*

Proof. (i) $Q_*(E) \le Q_W(U^0) \le n_P \kappa^2/(4 n_P) = \kappa^2/4$. If $Q_W(U) \le Q_* + \kappa^2/2$, then $Q_W(U) \le \frac{3}{4}\kappa^2$, so every plaquette that contains a link of A_W has $\delta(U_p, I) < \kappa$, and a plaquette with a vertex in a block of W^+ that contains no link of A_W has the value it has at U^0, at most $\kappa/(2\sqrt{n_P})$. So $\mathscr{E}^c \subset \{Q_W > Q_* + \kappa^2/2\}$, and Proposition 379 with $t = \kappa^2/2 \le 1$ applies. (ii) A field $U \in \mathscr{E}$ satisfies the hypothesis of Lemma 360, which places every site of every pair within $24\kappa < \frac{1}{4}$ of I, as in the proof of Proposition 374; so U is regular and $q(F(U)) \le Q_W(U) < \kappa^2$. Hence a fibre with $q(F) \ge \kappa^2$ contains no field of $\mathscr{E}$, and the law of the averages gives these fibres at most the mass $\pi_E(\mathscr{E}^c)$. (iii) The fibre $F(U)$ contains the field U, whose plaquettes with a vertex in a block of W^+ are at most $\kappa = K/(4\sqrt{n_P})$, which is the hypothesis of Proposition 374; Corollary 375 applies at U. □

So on every exterior whose plaquettes near W can all be made small by the links of A_W, the fibres where $q(F)$ is of order one, and more generally the fibres that contain no field with small plaquettes near W, carry total mass at most $\varepsilon''_W(\kappa^2/2, \lambda)$ in the law of the averages, and every other fibre has the chart decay of Corollary 375. This is a bound for every such exterior, not a mean over exteriors as in Theorem 370(ii). If $\lambda\kappa^2 \ge 196|W|\ln(8.4 \cdot 10^7 n_P^3/\kappa^6)$, then $\varepsilon''_W(\kappa^2/2, \lambda) \le e^{-\lambda\kappa^2/8}$; with $n_P \le 384|W|$ and $K = 4.8 \cdot 10^{-18}$ this holds under the Wilson measure for $\beta_W \ge 6.4 \cdot 10^{43}|W|^2(1 + \ln|W|)$. The fibres where $q(F)$ is of order one remain possible in two places: on an exterior with no completion whose plaquettes near W are small, and as values of c of small mass on an exterior with one.

Remark 381. Theorem 367 is the decay that Remark 365(b) asks for, on the chart $\Phi(\Omega_r)$ with $r = s/32$, which contains $\Phi(K_{s/32})$ and so every $Ue^X \in F$ with $\|X\|_\infty \le s/1280$. Its constants are explicit by Propositions 364 and 369: the rate $\alpha_G = 2.5 \cdot 10^{-4}$ is the same for every group and lies far below the decay seen in the computation described at the end of this remark. What remains open for the small fields:

(a) the mass of the constrained law outside $\Phi(\Omega_r)$ on the fibres that contain no field whose plaquettes near W are small: Corollary 375 bounds it on every fibre that contains one, Corollary 380 shows that on every exterior with a completion whose plaquettes near W are small these fibres, among them those where $q(F)$ is of order one, have mass at most ε''_W in the law of the averages, and Theorem 370 bounds them in mean over all exteriors; the constrained cube condition of Remark 345(a) takes a supremum over boundary conditions, which include these fibres of small mass and the exteriors with no such completion, and on them no chart of Theorem 367 is available; by Proposition 397 these are the typical case on cubes of fixed physical size along the tuning;

(b) the passage from the decay of the constrained law on one fibre, at a fixed background, to the constrained cube condition of Remark 345(a), whose supremum runs over boundary conditions that include large fields.

On the periodic lattice of side twelve, for $\mathrm{SU}(2)$ backgrounds as in the check of Theorem 361 and W a row of five blocks, the section S has row sums at most 14, the blocks of $\bar{P}$ beyond the range of S decay by a factor about 0.4 per unit of distance, and the blocks of the chart Hessian of Q fall from 15 at distance zero to 10^{-8} at distance nine, a floating-point computation reproduced in the archive.

E. Large fields under the Wilson measure

Theorem 359 needs every plaquette to be small. This subsection shows that, under the $\mathrm{SU}(2)$ Wilson measure, the plaquettes that are not small form clusters of microscopic size, and that the cut-off average departs from the average only next to them. It also shows that such plaquettes are present along the one-loop tuning, so the composition has to pass through them.

Let $\Lambda_L = (\mathbb{Z}/L\mathbb{Z})^4$ with $L = 2^m \geq 4$, and let μ_L be the $\mathrm{SU}(2)$ Wilson measure on Λ_L, with density proportional to $e^{-\beta_W S(U)}$ with respect to product Haar measure, where

$$S(U) = \sum_p s_p, \qquad s_p = 1 - \tfrac{1}{2}\operatorname{Tr} U_p = \tfrac{1}{4}\,\delta(U_p, I)^2.$$

This is the Wilson measure of Sec. III on the torus, without gauge fixing; it has $4L^4$ links and $6L^4$ plaquettes. For $0 < \rho \leq 1$ call a plaquette *bad* if $\delta(U_p, I) > \rho$. The closed unit cube $\mathscr{C}_x = x + [0,1]^4$ contains 24 plaquettes. Let B_x be the event that one of them is bad. A plaquette lies in exactly four unit cubes, one for each choice of the two transverse coordinates of the cube's corner.

Lemma 382. *Let Z_L be the normalising constant of μ_L and $C_0 = \frac{1}{2} + \frac{2}{3}\log(24\pi^3)$. For $\beta_W \geq 1$,*

$$-\log Z_L \leq 6L^4(\log\beta_W + C_0), \qquad \mathbb{E}_{\mu_L} S \leq 6L^4\,\frac{\log\beta_W + C_0}{\beta_W}.$$

Proof. Write $U = \cos\theta\, I + \sin\theta\, J$ with $0 \leq \theta \leq \pi$ and $J^2 = -I$. The angle θ of $g^{-1}h$ is a bi-invariant distance, and under Haar measure θ has density $\frac{2}{\pi}\sin^2\theta$. Put $r = (4\sqrt{\beta_W})^{-1}$. If every link has angle at most r, every plaquette has angle at most $4r \leq 1$ and $s_p \leq (4r)^2/2 = 1/(2\beta_W)$, so $S \leq 3L^4$. Each link angle is at most r with probability

$$\frac{2}{\pi}\int_0^r \sin^2 t\, dt \geq \frac{8r^3}{3\pi^3} = \frac{1}{24\pi^3\beta_W^{3/2}},$$

by $\sin t \geq 2t/\pi$. Hence $Z_L \geq e^{-3L^4}(24\pi^3\beta_W^{3/2})^{-4L^4}$, which is the first bound. The function $F(b) = -\log Z_L$ at the coupling b is concave with $F(0) = 0$ and $F'(b) = \mathbb{E}S$, so $bF'(b) \leq F(b)$, which is the second. □

Theorem 383. *For $\beta_W \geq 1$, $0 < \rho \leq 1$ and every set A of unit cubes,*

$$\mu_L\Big(\bigcap_{x\in A} B_x\Big) \leq q^{|A|},$$
$$q = q_\rho(\beta_W) = \min\big\{1, \exp\big(6(\log\beta_W + C_0) - \beta_W\rho^2/16\big)\big\}.$$

Proof. The measure μ_L is reflection positive in every hyperplane through sites [97]: the plaquettes are either in the hyperplane or on one side of it, so conditionally on the links in the hyperplane the weight factors into two mirror images, and reflection preserves Haar measure and $\operatorname{Tr} U_p$. On a torus of side 2^m this gives the chessboard estimate for functions of the links of closed unit cubes [54], [28, Section 5.3]:

$$\mu_L\Big(\bigcap_{x\in A} B_x\Big) \leq \mu_L\Big(\bigcap_{x\in\Lambda_L} B_x\Big)^{|A|/L^4},$$

since the reflection of B_x into another cube is the event B_y of that cube. If every cube is bad, there are at least $L^4/4$ bad plaquettes, so $S \geq \rho^2 L^4/16$ and the event has unnormalised weight at most $e^{-\beta_W\rho^2 L^4/16}$. Dividing by Z_L and using Lemma 382 gives the bound. □

Corollary 384. *Join two unit cubes when their corners are at ℓ^∞ distance one on the torus. For an integer $r \geq 1$,*

$$\mu_L\{\text{some connected set of } r \text{ bad cubes}\} \leq L^4(6400\,q)^r.$$

Hence, on a torus of fixed physical side La, if $a \to 0$ along $L = 2^m$ and $\beta_W(a) \to \infty$, the physical diameters of all connected clusters of bad cubes tend to 0 in probability, for every fixed ρ.

Proof. A connected set of r cubes containing a given cube is recovered from a walk of length $2(r-1)$ around a spanning tree, and each cube has 80 neighbours, so there are at most $80^{2(r-1)} \leq 6400^r$ such sets. Apply Theorem 383 and sum over the L^4 roots. A cluster of physical diameter at least R contains a connected set of $\lceil R/a \rceil$ cubes, and once $6400\,q \leq \frac{1}{2}$ the bound is at most $(La/a)^4 2^{-R/a} \to 0$. □

The bad cubes control where the cut-off average differs from the average. For a coarse link (y,μ) let $\mathscr{Z}_{y,\mu}$ be the 24 unit cubes with corners $2y + te_\mu + w$, $t \in \{0,1,2\}$, $w \in \{0,1\}^4$, $w_\mu = 0$ (Fig. 32).

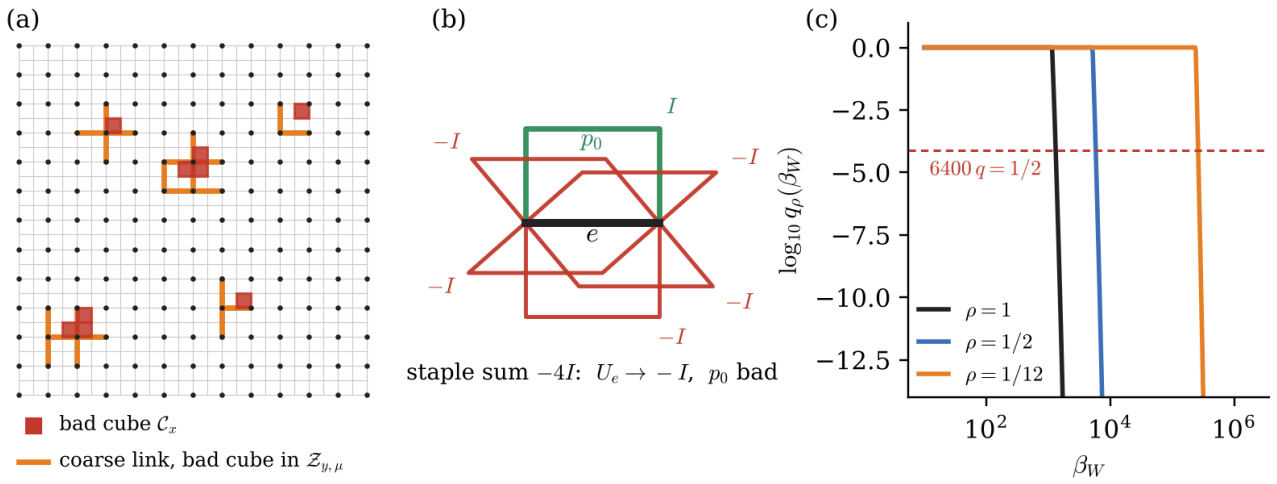


FIG. 32. Large fields under the SU(2) Wilson measure. (a) A plane section of the lattice: bad unit cubes (red) form clusters, and the cut-off average can differ from the path average only on coarse links (orange) whose 24 cubes $\mathcal{Z}_{y,\mu}$ meet a bad cube (Corollary 385). (b) The frustrated exterior of a link e: the staple of p_0 is I and the other five staples are $-I$, so U_e is driven to $-I$ and p_0 is bad. (c) The weight $q_\rho(\beta_W)$ of Theorem 383 for three values of ρ; below the dashed line the cluster bound of Corollary 384 is summable.

Corollary 385. *Let $\rho = \frac{1}{12}$. If no cube of $\mathcal{Z}_{y,\mu}$ is bad, then $\chi_{y,\mu}(U) = 1$ in the tree gauge, and $V^\sharp_{y,\mu}$ is the average of the sixteen paths in the chart at P_{2y}. A bad cube lies in $\mathcal{Z}_{y,\mu}$ for at most eight coarse links, so the expected fraction of coarse links with $\chi < 1$ is at most $4.5 \cdot 10^5 (\log\beta_W + C_0)/\beta_W$. Along the limits of Corollary 384, with probability tending to one these links lie within physical distance $2a$ of clusters of bad cubes of physical diameter less than any fixed $R > 0$.*

Proof. As in the proof of Lemma 358, $P_{2y}^{-1} P_x$ is a product of at most six conjugates of plaquette holonomies or their inverses. These plaquettes lie in planes $\{\mu,\alpha\}$ with corners $2y + te_\mu + w$ as above, and a plaquette lies in the unit cube at its corner. So $\delta(P_x, P_{2y}) \le 6\rho = \frac{1}{2}$, and $\chi = 1$. A cube with corner z lies in $\mathcal{Z}_{y,\mu}$ only if $z_\mu - 2y_\mu \in \{0,1,2\}$ and $z_\alpha - 2y_\alpha \in \{0,1\}$ for $\alpha \ne \mu$, which leaves at most two coarse links in each direction. There are $L^4/4$ coarse links, the number of bad cubes is at most four times the number of bad plaquettes, and a bad plaquette has $s_p > \rho^2/4$, so by Lemma 382 the expected fraction is at most $4 \cdot 8 \cdot 4 \cdot 24 \cdot 144 (\log\beta_W + C_0)/\beta_W$. The last assertion is Corollary 384. □

Small plaquettes everywhere, which Theorem 359 assumes, are nevertheless improbable along the one-loop tuning. For SU(2), $\beta_W = 4g^{-2}$, and the one-loop coupling of Remark 345(c) at spacing $a = 1/L$ gives $\beta_W = 8b_0 \log L + O(1) = \frac{11}{3\pi^2} \log L + O(1)$.

Proposition 386. *Let $0 < \rho \le 1$, $v_* = \frac{2}{\pi} \int_{\arccos(3/4)}^{\pi/3} \sin^2 t\, dt$ and $p = v_* e^{-3\beta_W}$. Conditionally on all other links, a plaquette through a link e is bad with probability at least p. Hence*

$$\mu_L\{\text{no plaquette is bad}\} \le e^{-\lfloor L/3 \rfloor^4 p},$$

which tends to 0 when $\beta_W \le B \log L + O(1)$ with $B < 4/3$, in particular along the one-loop tuning.

Proof. Identify SU(2) with the unit quaternions. Conditionally on the other links, U_e has density proportional to $e^{\kappa h \cdot u}$ with $0 \le \kappa \le 6\beta_W$ and $|h| = 1$, the six staples contributing at most β_W each. The plaquette holonomy is U_e or its inverse multiplied by a fixed element, so it has a density of the same form. Writing the density as a superposition of indicators of caps centred at h, the probability of a cap about I is largest when $h = I$, and for $h = I$ it increases with κ, because its derivative is the covariance of an increasing function of $\mathrm{Tr}\, U$ with $\frac{1}{2}\mathrm{Tr}\, U$. So the probability of being bad is at least that of the angle exceeding $\arccos(1 - \rho^2/4) \le \arccos(3/4)$ under the density proportional to $e^{6\beta_W \cos t} \sin^2 t$, which is at least p: bound the numerator from below on $[\arccos(3/4), \pi/3]$, where $\cos t \ge \frac{1}{2}$, and the denominator by $e^{6\beta_W} \pi/2$.

Let $M = \lfloor L/3 \rfloor$. Choose the links in direction 0 at the sites $3z$, $z \in \{0,\dots,M-1\}^4$, and their plaquettes in the plane $\{0,1\}$. No chosen plaquette contains another chosen link. Removing one chosen link at a time therefore bounds the probability that all chosen plaquettes are good by $(1-p)^{M^4}$. With $\beta_W = B \log L + O(1)$ the exponent is of order L^{4-3B}. □

The composition of Theorem 359 is local, so it continues across the scales away from the defects, and the defects spread by at most three spacings of the current scale at each step. For a coarse plaquette P at y let $\mathcal{B}_P$ be the fine box $2y + [0,4]^4$.

Lemma 387. *Let U be a G-valued field in the tree gauge and P a coarse plaquette. If every plaquette p in $\mathcal{B}_P$ has $\delta(U_p, I) \le \rho_G$ and $\|\Theta_p\| \le \eta \le \rho_G/1000$, then $\chi = 1$ on the four sides of P, $\delta(V^\sharp(U)_P, I) \le 0.03\,\rho_G$, the inequalities (37) hold for P, and $\|\log V^\sharp(U)_P\| \le 4(1 + 647\eta)\eta$.*

Proof. For the plaquette P the proof of Lemma 358 uses only the plaquettes of the surfaces $\mathcal{U}_\ell$ of the four sides ℓ of P and those of the sixteen squares $\square_{2y+o}$. A plaquette of $\mathcal{U}_\ell$, $\ell = (y_i, \mu_i)$, has its corner x at $x_{\mu_i} - 2(y_i)_{\mu_i} \in \{0,1,2\}$ and

$x_\alpha - 2(y_i)_\alpha \in \{0,1\}$ for $\alpha \neq \mu_i$, with $y_i \in \{y, y+e_\mu, y+e_\nu\}$; a plaquette of $\square_{2y+o}$ has $x-2y \in \{0,1,2\}^4$. So $x-2y \in \{0,1,2,3\}^4$ in every case, and the plaquette lies in $\mathscr{B}_P$. The bounds on χ, on $\delta(V_P^\sharp, I)$, the inequalities (37) and the bound $\|\bar F_P\| + 3.2\, r_P^2 \le 4(1+647\eta)\eta$ in that proof use only the norms of these plaquettes. □

Let a be the fine spacing and $a_j = 2^j a$. Given $\eta_0 > 0$ put $\eta_{j+1} = 4(1+647\eta_j)\eta_j$. Let $\mathfrak{B}_0$ be the set of fine plaquettes with $\delta(U_p, I) > \rho_G$ or $\|\Theta_p\| > \eta_0$, and let $\mathfrak{B}_{j+1}$ be the set of plaquettes P of the lattice of spacing a_{j+1} whose box $\mathscr{B}_P$ contains a plaquette of $\mathfrak{B}_j$. Let E_j° and A_j° be the sum of squares and the sum of $\|\Theta_P^{(j)}\|$ over the plaquettes $P \notin \mathfrak{B}_j$ of the j-th iterate.

Theorem 388. *Let $G \subset \mathrm{U}(r)$ be compact and U a G-valued field on the periodic lattice of side divisible by 2^k, and suppose that $\eta_j \le \rho_G/1000$ for $j < k$.*

(i) *For $j \le k$, every plaquette $P \notin \mathfrak{B}_j$ of the j-th iterate has $\delta(U_P^{(j)}, I) \le \rho_G$ and $\|\Theta_P^{(j)}\| \le \eta_j$. For $j < k$ the cut-off is inactive on the sides of every plaquette outside $\mathfrak{B}_{j+1}$, and (37) holds there.*

(ii) *Every point of a plaquette of $\mathfrak{B}_j$ lies within ℓ^∞ distance $3(a_j - a)$ of a plaquette of $\mathfrak{B}_0$.*

(iii) *With $\sigma = \sum_{j<k} \eta_j$,*

$$E_k^\circ \le e^{12060\sigma} E_0^\circ, \qquad A_k^\circ \le e^{9147\sigma} 4^{-k} A_0^\circ.$$

(iv) *If $4^k\eta_0 \le 10^{-4}\rho_G$, the hypothesis holds, $\eta_j \le 1.023 \cdot 4^j\eta_0$ for $j \le k$, $\sigma \le 0.35 \cdot 4^k\eta_0$, and $E_k^\circ \le 1.53\, E_0^\circ$.*

Proof. (i) By induction on j. If $P \notin \mathfrak{B}_{j+1}$, no plaquette of $\mathscr{B}_P$ lies in $\mathfrak{B}_j$, so by the induction hypothesis all of them have $\delta \le \rho_G$ and norm at most $\eta_j \le \rho_G/1000$. Lemma 387, applied to the j-th iterate in the tree gauge, gives $\delta(U_P^{(j+1)}, I) \le 0.03\,\rho_G$ and $\|\Theta_P^{(j+1)}\| \le \eta_{j+1}$; the gauge fixing between steps does not change these norms.

(ii) Let P at y lie in $\mathfrak{B}_{j+1}$, and let $p \in \mathfrak{B}_j$ lie in $\mathscr{B}_P$, with corner x, $x - 2y \in \{0,1,2,3\}^4$. In units of a_j every coordinate of a point of P lies in $[2y_c, 2y_c+2]$, and p contains a point whose coordinates are x_c or lie in $[x_c, x_c+1]$; so the two are at distance at most 3 in each coordinate. Hence every point of a plaquette of $\mathfrak{B}_{j+1}$ is within $3a_j$ of a plaquette of $\mathfrak{B}_j$, and $\sum_{i<j} 3a_i = 3(a_j - a)$.

(iii) Repeat the part *Sums* of the proof of Lemma 358 with the plaquettes $P \notin \mathfrak{B}_{j+1}$ only. Their squares $\square_{2y+o}$ and surfaces $\mathscr{U}_\ell$ consist of plaquettes outside $\mathfrak{B}_j$, so the same counts give $\sqrt{E_{j+1}^\circ} \le (1+6030\eta_j)\sqrt{E_j^\circ}$ and $A_{j+1}^\circ \le (1+9147\eta_j)A_j^\circ/4$, and $1 + t \le e^t$.

(iv) The recursion gives $\eta_j \le e^{647\sigma_j} 4^j \eta_0$ with $\sigma_j = \sum_{i<j}\eta_i$. Suppose $\sigma_j \le 0.35 \cdot 4^j\eta_0$, which holds at $j = 0$. Then $\sigma_j \le 3.5 \cdot 10^{-5}$, $e^{647\sigma_j} \le 1.023$, $\eta_j \le 1.023 \cdot 4^j\eta_0 \le 1.023 \cdot 10^{-4}\rho_G$, and $\sigma_{j+1} \le 1.373 \cdot 4^j\eta_0 \le 0.35 \cdot 4^{j+1}\eta_0$. Finally $e^{12060 \cdot 3.5 \cdot 10^{-5}} \le 1.53$. □

Under the Wilson measure the defects of Theorem 388 are the bad plaquettes of Theorem 383. For SU(2), $\rho_G = 1$; a plaquette with $\delta(U_p, I) = 2\sqrt{2}\sin(\theta/2) \le \rho \le \frac{1}{10}$ has $\|\Theta_p\| = \sqrt{2}\,\theta \le 1.01\,\delta(U_p, I)$, so with $\eta_0 = 1.01\rho$ every plaquette of $\mathfrak{B}_0$ is bad, and lies in a bad cube.

Corollary 389. *Let $G = \mathrm{SU}(2)$, $C_1 = C_0 + \frac{1}{6}\log 12800$, and for $\beta_W \ge 1$ let $\rho(\beta_W)$ be given by $\rho^2 = 96(\log\beta_W + C_1)/\beta_W$, so that $6400\, q_\rho(\beta_W) = \frac{1}{2}$. Let k_* be the largest integer with $4^{k_*} \cdot 1.01\rho \le 10^{-4}$. If $1.01\rho \le 10^{-4}$, then $k_* \ge 0$,*

$$a_{k_*} \ge 1.5 \cdot 10^{-3} \Big(\frac{\beta_W}{\log\beta_W + C_1} \Big)^{1/4} a,$$

and with μ_L-probability at least $1 - L^4 2^{-r}$ every connected cluster of bad cubes has fewer than r cubes, while for every $j \le k_*$ the defects of the j-th iterate lie within ℓ^∞ distance $3a_j$ of these clusters and the iterate satisfies Theorem 388(i), (iii) with $E_{k_*}^\circ \le 1.53\, E_0^\circ$. Along the limits of Corollary 384 with $a\,\rho(\beta_W)^{-1/2} \to 0$, which holds along the one-loop tuning, the defects of all iterates up to the spacing a_{k_*} lie, with probability tending to one, within clusters of physical size tending to 0.

Proof. With this ρ, $q_\rho(\beta_W) = \exp(-\log 12800)$, and Corollary 384 gives the probability of a connected set of r bad cubes at most $L^4(6400q)^r = L^4 2^{-r}$. By maximality, $4^{k_*+1} \cdot 1.01\rho > 10^{-4}$, so $2^{k_*} > \frac{1}{2} \cdot 10^{-2}(1.01\rho)^{-1/2}$, and $\rho^{-1/2} = 96^{-1/4}(\beta_W/(\log\beta_W + C_1))^{1/4}$ with $96^{1/4} < 3.14$. The rest is Theorem 388 with $\eta_0 = 1.01\rho$, whose hypothesis holds by part (iv), and $\mathfrak{B}_0$ contained in the bad cubes; the bound $\|\Theta_p\| \le 1.01\,\delta(U_p, I)$ applies since $\rho \le \frac{1}{10}$. For the limits take $r = \lceil 8\log_2 L \rceil$, so that $L^4 2^{-r} \le L^{-4}$ and $ra \to 0$ for a torus of fixed physical side La. The defects then lie in sets of physical diameter at most $ra + 6a_{k_*}$, and $a_{k_*} \le 10^{-2}(1.01\rho)^{-1/2} a \to 0$. □

The scale a_{k_*} is of the order $(\beta_W/\log\beta_W)^{1/4}a$, the order of the cube side that the condition (D) needs by Proposition 267. So the cut-off average carries the field through the defects of the Wilson measure, without spreading them, up to the scale at which (D) can first hold, but not beyond: past a_{k_*} the bound $\eta_j \le 1.023 \cdot 4^j\eta_0$ no longer keeps the curvature small. Along the one-loop tuning β_W grows like $\log L$, so k_* grows like $\frac{1}{4}\log_2\log L$, a vanishing fraction of the $\log_2 L$ scales.

The joint bound of Theorem 383 is not a conditional one, and (D) needs a supremum over boundary conditions. The two differ genuinely. Fix a link e and a plaquette p_0 through it. Put all links on the six staples of e at I, except the middle links of the five staples other than that of p_0, which are $-I$. Then the conditional density of U_e is proportional to $e^{-2\beta_W \mathrm{Tr}\, U_e}$, and p_0, whose holonomy is $U_e^{\pm 1}$, is bad with conditional probability tending to 1 as $\beta_W \to \infty$, for every $\rho < 2\sqrt{2}$. Nearby exteriors behave the same way. So the clusters of Corollary 384 are small in probability, but the boundary response of a cube to a frustrated exterior is not small, and the composition beyond the scale a_{k_*} of Corollary 389 and the conditional estimate in (D) remain open.

For exteriors that are calm on the boundary of a box and lie in a maximal torus, the conditional estimate does hold. Let

$R=[0,l]^4$, $l\geq 2$, be a box of Λ_L with $l<L$. Its boundary ∂R is a cubical 3-sphere. Call a link or plaquette of R a *boundary* link or plaquette if it lies in ∂R, and *inner* otherwise. A plaquette that contains an inner link lies in R, so conditionally on all other links the inner links have the law

$$\nu_\eta(dU)=Z_\eta^{-1}\exp\Big(-\beta_W\sum_{p\in P^\circ}s_p\Big)\prod_e dU_e,$$

the product over the inner links, where P° is the set of inner plaquettes. It depends on the exterior η only through the boundary links. Let $T\subset \mathrm{SU}(2)$ be the diagonal torus of the elements $\exp(tJ)$, $J=\mathrm{diag}(i,-i)$, and for $u\in\mathrm{SU}(2)$ let $\angle u\in[0,\pi]$ be its angle, so that $\delta(u,I)=2\sqrt{2}\sin(\angle u/2)$.

Lemma 390. *Let $0\leq\rho_s<\pi/3$. Suppose the boundary links of R lie in T and $\angle U_p\leq\rho_s$ for every boundary plaquette. Then the inner links can be chosen in T so that $\angle U_p\leq C\rho_s$ for every plaquette of R, where*

$$C=\sqrt{3}\,(24\cdot 13118+9)<5.46\cdot 10^5.$$

Proof. Write a T-valued field as $U_e=\exp(a_eJ)$ with $a_e\in\mathbb{R}/2\pi\mathbb{Z}$. For a plaquette p let $(da)_p$ be the signed sum of a over its four links, so that $U_p=\exp((da)_pJ)$. For a boundary plaquette let $\theta_p\in(-\pi,\pi]$ be the representative of $(da)_p$; then $|\theta_p|=\angle U_p\leq\rho_s$.

Closedness. The signed angles of the six plaquettes of a unit 3-cube of ∂R add up to 0 modulo 2π, since each link occurs twice with opposite signs. Their representatives add up to a number of modulus at most $6\rho_s<2\pi$, hence to 0. So θ is a closed real 2-cochain on ∂R.

Lift. We construct a real 1-cochain α on the boundary links with $d\alpha=\theta$ and $|\alpha_e|\leq 13118\,l\rho_s$. The boundary is the union of the eight facets $F_i^0=\{x\in R: x_i=0\}$ and $F_i^l=\{x\in R: x_i=l\}$, each a cube $[0,l]^3$. Take them in the order F_1^l, F_2^l, F_2^0, F_3^l, F_3^0, F_4^l, F_4^0, F_1^0, and let Γ_k be the union of the intersections of the k-th facet with the earlier ones. In the cube of the k-th facet, Γ_k is empty for $k=1$, one face for $k=2,3$, three faces, two of them opposite, for $k=4,5$, five faces for $k=6,7$ and all six for $k=8$. So Γ_k is contractible or a 2-sphere, and its first cellular cohomology vanishes.

On a facet with coordinates t_1,t_2,t_3, the axial gauge gives α^0 with $d\alpha^0=\theta$ on the facet. Put $\alpha^0=0$ on the links in direction t_3. Determine α^0 on the links in direction t_2 from the plaquettes in the planes $\{t_2,t_3\}$, starting from 0 at $t_3=0$. On the links in direction t_1 at $t_3=0$, determine it from the plaquettes in the plane $\{t_1,t_2\}$, starting from 0 at $t_2=0$, and then from the plaquettes in the planes $\{t_1,t_3\}$. The plaquettes in the planes $\{t_1,t_2\}$ with $t_3>0$ then satisfy $d\alpha^0=\theta$ by closedness, by induction on t_3. Each value is a signed sum of at most l, respectively $2l$, angles, so $|\alpha^0|\leq 2l\rho_s$.

For $k=1$ put $\alpha=\alpha^0$, so $M_1=\max|\alpha|\leq 2l\rho_s$. For $k\geq 2$, α is already defined on the links of Γ_k, with $|\alpha|\leq M_{k-1}$, and $\omega=\alpha-\alpha^0$ is closed there. Hence $\omega=d\lambda$ for a function λ on the vertices of Γ_k, and $|\lambda(x)-\lambda(y)|\leq\|\omega\|_\infty d_\Gamma(x,y)$, where d_Γ is the path metric of Γ_k.

We claim $d_\Gamma\leq 3d_1$, where d_1 is the ℓ^1 metric of the facet. If x and y lie in one face of Γ_k, that face contains an ℓ^1 geodesic between them. If they lie in adjacent faces $\{t_i=s\}$ and $\{t_j=s'\}$, the path through the point with $t_i=s$, $t_j=s'$ and third coordinate that of x has length $d_1(x,y)$. If they lie in opposite faces $\{t_i=0\}$ and $\{t_i=l\}$, then $d_1(x,y)\geq l$. In that case Γ_k contains a face adjacent to both, and the path from x to that face, across it and on to y has length at most $d_1(x,y)+2l\leq 3d_1(x,y)$.

So λ is $3\|\omega\|_\infty$-Lipschitz for d_1 on Γ_k. The function

$$\tilde\lambda(x)=\min_y\big(\lambda(y)+3\|\omega\|_\infty d_1(x,y)\big)$$

extends it to the vertices of the facet with the same constant [89]. Put $\alpha=\alpha^0+d\tilde\lambda$ on the links of the facet. It agrees with the earlier values on Γ_k, satisfies $d\alpha=\theta$ on the facet, and has

$$|\alpha|\leq 2l\rho_s+3(M_{k-1}+2l\rho_s).$$

So $M_k\leq 3M_{k-1}+8l\rho_s$, which gives $M_8\leq 13118\,l\rho_s$.

Extension. Let c be the centre of R, $R_0=l/2$, $r(x)=|x-c|_\infty$ and $\varphi(x)=c+R_0(x-c)/r(x)$, the radial projection of $R\setminus\{c\}$ onto ∂R. Let $\hat\alpha$ and $\hat\theta=d\hat\alpha$ be the lowest-order Whitney forms of α and θ on the unit cubes of ∂R [115]. On a unit cube with coordinates t_1,t_2,t_3, $\hat\alpha=\sum_i a_i\,dt_i$, where a_i is the multilinear interpolation of the values of α on the four links in direction t_i; similarly $\hat\theta=\sum_{i<j}\theta_{ij}\,dt_i\wedge dt_j$. These forms have continuous tangential components across the faces of the unit cubes, satisfy $\int_e\hat\alpha=\alpha_e$ for every boundary link, and for tangent vectors

$$|\hat\alpha(u)|\leq\sqrt{3}\,M_8|u|,\qquad |\hat\theta(u,w)|\leq\sqrt{3}\,\rho_s|u||w|.$$

Put $\tilde B=(r/R_0)^2\varphi^*\hat\alpha$ on $R\setminus\{c\}$. Since $|\nabla r|=1$ almost everywhere and $|x-c|\leq 2r$, we have $|d\varphi(v)|\leq 3R_0|v|/r$. Then

$$d\tilde B=d\big(r^2/R_0^2\big)\wedge\varphi^*\hat\alpha+(r/R_0)^2\varphi^*\hat\theta$$

gives

$$|d\tilde B(v,w)|\leq\big(12\sqrt{3}\,M_8/R_0+9\sqrt{3}\,\rho_s\big)|v||w|\leq C\rho_s|v||w|,$$

and $|\tilde B|\leq 3\sqrt{3}\,M_8 r/R_0$ is bounded.

The set $R\setminus\{c\}$ is the union of the cones from c over the unit cubes of ∂R. These cones are convex polytopes cut out by hyperplanes through c. On each of them $\tilde B$ is smooth, and its tangential components are continuous across them since φ is continuous. Put $B_e=\int_e\tilde B$ for every link e of R. Apply Stokes' theorem on each piece of a plaquette: the boundary terms on the interior cuts cancel, and the point c contributes nothing since $\tilde B$ is bounded. This gives $(dB)_p=\int_p d\tilde B$, so $|(dB)_p|\leq C\rho_s$. On ∂R, $r=R_0$ and φ is the identity on tangent vectors, so $B=\alpha$ on the boundary links.

Gauge. The cochain $a-\alpha$ on the boundary links has $(d(a-\alpha))_p\equiv\theta_p-\theta_p=0$ modulo 2π on every boundary plaquette. So it is a flat U(1)-field on the 2-skeleton of ∂R, which is simply connected, and $a-\alpha\equiv d\mu$ modulo 2π for a function μ on the boundary vertices. Extend μ by 0 to the inner vertices and put $U_e=\exp((B_e+(d\mu)_e)J)$ on the inner links. On the boundary links $B+d\mu\equiv a$, and every plaquette of R has angle at most $|(dB)_p|\leq C\rho_s$. □

For a general SU(2)-valued boundary field the curvature is not a closed cochain, and the lift in the proof of Lemma 390 is not available. A lift by gluing local gauges still works when $l^2\rho_s$ is small. Identify $\mathfrak{su}(2)$ with the imaginary quaternions and write $|X|$ for the operator norm, which is the Euclidean norm, so that $|[X,Y]| \le 2|X||Y|$, $\|e^X - I\| \le |X|$ and $\angle e^X = |X|$ for $|X| \le \pi$. With $d(g,h) = \angle(g^{-1}h)$, the map $X \mapsto e^X$ is 1-Lipschitz, since its differential has singular values 1 and $\sin|X|/|X|$, and the principal logarithm is 2-Lipschitz on $\{\angle h \le \frac{1}{2}\}$, a geodesically convex ball on which the inverse differential has singular values at most $\frac{1}{2}/\sin\frac{1}{2} < 2$. Put

$$C_1 = 2, \qquad C_k = 11\,C_{k-1} + 24 \quad (2 \le k \le 8),$$

so that $C_8 = 85\,743\,550$, and

$$K = 42\,C_8 + 20, \qquad \varepsilon_0 = \frac{1}{100\,C_8^2}.$$

Lemma 391. *Let the boundary links η of R take values in* SU(2)*, with $\angle\eta_p \le \rho_s$ for every boundary plaquette and $l^2\rho_s \le \varepsilon_0$. Then the inner links can be chosen so that $\angle U_p \le K\rho_s$ for every plaquette of R.*

Proof. For a vertex gauge g write $\eta^g_{xy} = g_x \eta_{xy} g_y^{-1}$. A product of conjugates of m elements of angle at most ρ_s has angle at most $m\rho_s$, since d is bi-invariant.

One facet. On a facet with coordinates t_1, t_2, t_3 let P_x be the path from the corner to x that moves first in direction t_1, then t_2, then t_3, and put $g^0_x = \eta(P_x)$, the ordered product of the links along P_x. Then $\eta^{g^0}_{xy}$ is the holonomy of the loop $P_x\,(xy)\,P_y^{-1}$. For a link in direction t_3 this loop retraces itself. For a link in direction t_2 at height $t_3 = c$ it bounds c plaquettes of the planes $\{t_2, t_3\}$, and for a link in direction t_1 at $(t_2,t_3) = (b,c)$ it bounds b plaquettes of a plane $\{t_1,t_2\}$ and c of a plane $\{t_1,t_3\}$. So the holonomy is a product of conjugates of at most $2l$ plaquette holonomies, and every link of the facet has $\angle\eta^{g^0}_e \le 2l\rho_s$.

Gluing. Take the facets in the order of the proof of Lemma 390, with the same sets Γ_k; there $d_\Gamma \le 3d_1$. Two vertices of Γ_k joined by a link of the facet lie in a common face of Γ_k, since $l \ge 2$, so that link lies in Γ_k. We construct a gauge g on the boundary vertices with $\angle\eta^g_e \le M_k = C_k l\rho_s$ on the links of the first k facets. For $k = 1$ take $g = g^0$. For $k \ge 2$, let g^0 be the gauge of the k-th facet and $t_x = g_x (g^0_x)^{-1}$ for $x \in \Gamma_k$. For a link xy of Γ_k, $\eta^g_{xy} = t_x \eta^{g^0}_{xy} t_y^{-1}$, so

$$d(t_x,t_y) = \angle\big(t_x^{-1}t_y\big) \le \angle\eta^g_{xy} + \angle\eta^{g^0}_{xy} \le M_{k-1} + 2l\rho_s,$$

and t is $3(M_{k-1}+2l\rho_s)$-Lipschitz for d_1 on Γ_k. Fix $x_0 \in \Gamma_k$ and put $h_x = t_{x_0}^{-1} t_x$. The facet has d_1-diameter $3l$, so

$$\angle h_x \le 9l(M_{k-1}+2l\rho_s) = 9(C_{k-1}+2)\,l^2\rho_s \le 9(C_8+2)\varepsilon_0 < \tfrac{1}{2},$$

and $a_x = \log h_x$ is L-Lipschitz for d_1, with $L = 6(M_{k-1}+2l\rho_s)$. Extend each of the three real coordinates of a to the vertices of the facet by the formula of McShane [89], as in the proof of Lemma 390; the extension $\tilde a$ is $\sqrt{3}L$-Lipschitz. Put $g_x = t_{x_0} e^{\tilde a_x} g^0_x$ on the facet; on Γ_k this is the old value. For a link xy of the facet, $\eta^g_{xy} = \tilde t_x \eta^{g^0}_{xy} \tilde t_y^{-1}$ with $\tilde t_x = t_{x_0} e^{\tilde a_x}$, so

$$\begin{aligned}\angle\eta^g_{xy} &\le d(\tilde t_x, \tilde t_y) + 2l\rho_s \le \sqrt{3}L + 2l\rho_s \\ &= 6\sqrt{3}\,M_{k-1} + (12\sqrt{3}+2)\,l\rho_s \le M_k.\end{aligned}$$

After the eighth facet, $\angle\eta^g_e \le M = C_8 l\rho_s$ on every boundary link.

Potential. Since $M \le C_8\varepsilon_0/l < \frac{1}{2}$, the logarithms $A_e = \log\eta^g_e$ are defined, with $|A_e| \le M$ and $A_{-e} = -A_e$. Let $(dA)_p$ be the signed sum of A over the four links of a boundary plaquette. For $X_1, \dots, X_q \in \mathfrak{su}(2)$,

$$\Big\| e^{X_1}\cdots e^{X_q} - I - \sum_i X_i \Big\| \le \tfrac{1}{2}\Big(\sum_i |X_i|\Big)^2,$$

by induction on q from $e^X e^Y - I - X - Y = (e^X - I - X) + (e^Y - I - Y) + (e^X - I)(e^Y - I)$ and $\|e^X - I - X\| \le |X|^2/2$. Applied to the four links of p, and since $\|\eta^g_p - I\| = 2\sin(\angle\eta_p/2) \le \rho_s$, this gives

$$|(dA)_p| \le \sigma = \rho_s + 8M^2.$$

Let $\hat A$ be the lowest-order Whitney form of A on the unit cubes of ∂R, as in the proof of Lemma 390, now with values in $\mathfrak{su}(2)$. Its coefficients are interpolations of values of A, and those of $d\hat A$ are interpolations of values of dA, so

$$|\hat A(u)| \le \sqrt{3}\,M|u|, \qquad |d\hat A(u,w)| \le \sqrt{3}\,\sigma|u||w|.$$

With r, R_0 and φ as there, put $\tilde B = (r/R_0)^2 \varphi^* \hat A$ on $R \setminus \{c\}$ and $\tilde B(c) = 0$. The same computation gives

$$|\tilde B(v)| \le 3\sqrt{3}\,M\frac{r}{R_0}|v|, \qquad |d\tilde B(v,w)| \le \kappa_1 |v||w|,$$

where $\kappa_1 = 12\sqrt{3}M/R_0 + 9\sqrt{3}\,\sigma$. On each cone from c over a unit cube of ∂R, $\tilde B$ is smooth away from c, and its tangential components are continuous across the cones.

Transport. For a link e of R, parametrised by $s \in [0,1]$, let $V_e = T(1)$, where $T' = T\,\tilde B(\dot e)$ and $T(0) = I$; the coefficient is bounded and piecewise smooth, and V composes along paths in the order of the links. For a closed path γ of length ℓ,

$$T(1) - I - \int_\gamma \tilde B = \int_0^1 (T - I)\,\tilde B(\dot\gamma)\,ds,$$

and $\|T(s) - I\| \le \int_0^s |\tilde B(\dot\gamma)|$, so the holonomy V_γ satisfies $\|V_\gamma - I\| \le |\int_\gamma \tilde B| + \frac{1}{2}(\int_\gamma |\tilde B|)^2$. Let p be a plaquette of R and divide it into N^2 squares q of side $h = 1/N$. Its holonomy is a product of conjugates of the holonomies of the boundaries of the squares, since every inner edge of the subdivision is traversed once in each direction. For each square, Stokes' theorem on the finitely many convex polygons in which the cones cut q gives $\int_{\partial q}\tilde B = \int_q d\tilde B$: the integrals over the interior cuts cancel because the tangential components agree, and if $c \in q$ a disc of radius δ around c contributes $O(\delta^2)$, since $|\tilde B| \le 3\sqrt{3}Mr/R_0$. Hence

$$\|V_{\partial q} - I\| \le \kappa_1 h^2 + \tfrac{1}{2}\big(4h \cdot 3\sqrt{3}M\big)^2 = \kappa h^2,$$

with $\kappa = \kappa_1 + 216M^2$. Since $2\arcsin(x/2) \le x(1+x^2)$ for $0 \le x \le 1$, summing over the squares gives $\angle V_p \le \kappa(1+\kappa^2 N^{-4})$, and letting $N \to \infty$, $\angle V_p \le \kappa$. Now $M/R_0 = 2C_8\rho_s$ and $M^2 = C_8^2 l^2 \rho_s^2 \le \rho_s/100$, so

$$\begin{aligned}\kappa &\le \big(24\sqrt{3}C_8 + 9\sqrt{3}\big)\rho_s + \big(72\sqrt{3}+216\big)M^2 \\ &< (42C_8+20)\,\rho_s = K\rho_s.\end{aligned}$$

Boundary values. On ∂R, $r = R_0$ and φ is the identity on tangent vectors, and along a boundary link e the tangential coefficient of $\hat{A}$ is the constant A_e. So $V_e = e^{A_e} = \eta_e^g$. Extend g by I to the inner vertices and put $U_{xy} = g_x^{-1} V_{xy} g_y$. Then $U = \eta$ on the boundary links, and U_p is conjugate to V_p, so $\angle U_p \le K\rho_s$ for every plaquette of R. □

The boundary fields of Lemma 391 with strict inequalities form an open set of positive Haar measure that contains the identity field; unlike those of Lemma 390, they are not a null set. The constants are far from optimal.

For $0 < \varepsilon \le 1$ let E_ε be the event that at least εl^4 of the l^4 unit cubes $x + [0,1]^4$, $x \in \{0,\dots,l-1\}^4$, of R contain a bad plaquette.

Proposition 392. *Let $\beta_W \ge 1$ and $0 < \rho \le 1$. Suppose that no boundary plaquette of R is bad under η, and that the inner links can be chosen so that every inner plaquette has angle at most κ. Then*

$$\begin{aligned}-\log \nu_\eta(E_\varepsilon) \ge{}& \frac{\beta_W \rho^2 \varepsilon}{16}\, l^4 \\ &- 6(l+1)^4\big(\log\beta_W + C_0 + \tfrac{1}{2} + \beta_W\kappa^2\big),\end{aligned}$$

with C_0 as in Lemma 382. In particular, if the boundary links lie in T with angle at most $\rho_s < \pi/3$ on every boundary plaquette and $\rho \ge \sqrt{2}\rho_s$, the bound holds with $\kappa = C\rho_s$ and the constant C of Lemma 390. If the boundary links lie in $\mathrm{SU}(2)$ with angle at most ρ_s on every boundary plaquette, $l^2\rho_s \le \varepsilon_0$ and $\rho \ge \sqrt{2}\rho_s$, it holds with $\kappa = K\rho_s$.

Proof. On E_ε the bad plaquettes are inner, and each lies in at most four of the unit cubes. So there are at least $\varepsilon l^4/4$ of them, each with $s_p > \rho^2/4$, and $\sum_{P^\circ} s_p \ge \rho^2\varepsilon l^4/16$. Hence the unnormalised weight of E_ε is at most $\exp(-\beta_W\rho^2\varepsilon l^4/16)$.

For the normalisation let $\bar{U}$ be the inner links of the hypothesis, $t = (4\sqrt{\beta_W})^{-1}$, and consider the inner links $U_e = \bar{U}_e h_e$ with $\angle h_e \le t$. The angle of $g^{-1}h$ is a bi-invariant distance, so every inner plaquette has $\angle U_p \le \kappa + 4t$ and

$$s_p \le \tfrac{1}{2}(\kappa+4t)^2 \le \kappa^2 + 16t^2 = \kappa^2 + \beta_W^{-1}.$$

Each h_e has probability at least $8t^3/(3\pi^3) = (24\pi^3\beta_W^{3/2})^{-1}$, as in the proof of Lemma 382. There are $n_P \le 6l^2(l+1)^2$ inner plaquettes and $n_L \le 4l(l+1)^3$ inner links, so

$$Z_\eta \ge \exp\big(-n_P(\beta_W\kappa^2+1)\big)\big(24\pi^3\beta_W^{3/2}\big)^{-n_L}.$$

Since $C_0 = \frac{1}{2} + \frac{2}{3}\log(24\pi^3)$, $n_P + n_L\log(24\pi^3\beta_W^{3/2}) \le 6(l+1)^4(\log\beta_W + C_0 + \frac{1}{2})$, which gives the bound. For the last assertion, a boundary plaquette has $\delta(U_p, I) \le \sqrt{2}\angle U_p \le \sqrt{2}\rho_s \le \rho$, so it is not bad, and Lemma 390 or Lemma 391 gives the inner links. □

So in a box whose boundary is calm at a level ρ_s small compared with $\rho\sqrt{\varepsilon}/C$ and T-valued, or small compared with $\rho\sqrt{\varepsilon}/K$ and $\mathrm{SU}(2)$-valued with $l^2\rho_s \le \varepsilon_0$, the fraction of bad cubes exceeds ε only with probability exponentially small in the volume once $\beta_W\rho^2\varepsilon$ is large compared with $\log\beta_W$, uniformly over such exteriors and over l. The estimate is conditional, unlike Theorem 383.

Remark 393. The exteriors with boundary links in T form a null set for μ_L. Lemma 391 removes the torus at the price of the smallness $l^2\rho_s \le \varepsilon_0$. In the continuum the corresponding condition is that of Uhlenbeck's theorem [111], which gives a Coulomb gauge on a ball when the curvature is small in $L^{n/2}$; on 3-dimensional pieces of ∂R of side comparable to l and for curvature of size ρ_s this norm is of order $l^2\rho_s$. The lemma replaces the Coulomb gauge by the gluing of the facet gauges. With ρ_s of the order of the level $\rho(\beta_W)$ of Corollary 389, $\rho_s = c_0(\log\beta_W/\beta_W)^{1/2}$, it applies for

$$l \le (\varepsilon_0/c_0)^{1/2}\,(\beta_W/\log\beta_W)^{1/4},$$

the scale a_{k_*} of Corollary 389 up to a constant factor. Along the tuning $\beta_W \sim b\log(1/a)$ the physical side al then tends to 0, so the lemma does not reach cubes of a fixed physical size; this is a limitation of the argument, not a proof that such fillings fail. Moreover Proposition 392 bounds the fraction of bad cubes under one exterior and does not compare the laws under two exteriors, so it gives no contraction coefficient.

The calm boundary is needed only for the small-curvature filling. If the boundary layer is paid for instead of filled, every condition on the exterior disappears, at the price of a surface term that is small against the volume on large boxes.

Proposition 394. *Let $\beta_W \ge 1$, $0 < \rho \le 1$ and $0 < \varepsilon \le 1$ with $\varepsilon l \ge 16$. For every boundary field η,*

$$-\log\nu_\eta(E_\varepsilon) \ge l^4\Big(\frac{\beta_W\rho^2\varepsilon}{32} - 6\log\beta_W - C_1 - \frac{48\beta_W}{l}\Big),$$

with $C_1 = 3 + 4\log(24\pi^3) < 31$.

Proof. A unit cube $x + [0,1]^4$ with $x \in \{1,\dots,l-2\}^4$ does not meet ∂R, so all its plaquettes are inner; the other $l^4 - (l-2)^4 \le 8l^3 \le \varepsilon l^4/2$ unit cubes meet it. On E_ε at least $\varepsilon l^4/2$ cubes of the first kind contain a bad plaquette, which is inner, and a plaquette lies in at most four unit cubes. So at least $\varepsilon l^4/8$ inner plaquettes are bad, $\sum_{P^\circ} s_p \ge \rho^2\varepsilon l^4/32$, and the unnormalised weight of E_ε is at most $\exp(-\beta_W\rho^2\varepsilon l^4/32)$.

For the normalisation take inner links $U_e = h_e$ with $\angle h_e \le t = (4\sqrt{\beta_W})^{-1}$. An inner plaquette whose four links are inner has $\angle U_p \le 4t$ and $s_p \le (2\beta_W)^{-1}$. An inner plaquette spans two directions and has its other two coordinates in $\{1,\dots,l-1\}$; it contains a boundary link exactly when its cell in the plane of the two spanned directions meets the boundary of $[0,l]^2$, which happens for $4(l-1)$ of the l^2 cells. So $24(l-1)^3 \le 24l^3$ inner plaquettes contain a boundary link, and on them $s_p \le 2$. There are $n_P = 6l^2(l-1)^2 \le 6l^4$ inner plaquettes and $n_L = 4l(l-1)^3 \le 4l^4$ inner links, so, as in the proof of Proposition 392,

$$Z_\eta \ge \exp\big(-48\beta_W l^3 - 3l^4\big)\big(24\pi^3\beta_W^{3/2}\big)^{-4l^4}.$$

Dividing gives the bound, and $24\pi^3 < e^7$ gives $C_1 < 31$. □

Corollary 395. *Let $\beta_W \geq e^{44}$, $l \geq 6144\beta_W^{3/4}$ and $\rho = \varepsilon = \beta_W^{-1/4}$. Then $\nu_\eta(E_\varepsilon) \leq \exp(-\beta_W^{1/4} l^4/64)$ for every boundary field η. In particular, along $\beta_W \sim b\log(1/a)$ with $l = \lceil r/a \rceil$ and fixed $r, b > 0$, on cubes of fixed physical side r the fraction of unit cubes that contain a plaquette with $\delta(U_p, I) > \beta_W^{-1/4}$ is at most $\beta_W^{-1/4}$ with conditional probability tending to one, uniformly over all exteriors.*

Proof. Here $\varepsilon l \geq 6144\beta_W^{1/2} \geq 16$ and $\rho \leq 1$. The first term of Proposition 394 is $\beta_W^{1/4}/32$, and $48\beta_W/l \leq \beta_W^{1/4}/128$. For $x = \log\beta_W \geq 44$ we have $6x+31 \leq e^{x/4}/128$: at $x = 44$ the left side is 295 and the right side exceeds 467, and $x/4 - \log(6x+31)$ increases for $x \geq 44$. So $6\log\beta_W + C_1 \leq \beta_W^{1/4}/128$ and the bracket is at least $\beta_W^{1/4}(\frac{1}{32} - \frac{1}{128} - \frac{1}{128}) = \beta_W^{1/4}/64$. Along the tuning, $\beta_W \to \infty$ and $l/\beta_W^{3/4} \to \infty$, so the hypotheses hold for small a. □

Like Proposition 392, Corollary 395 bounds the bad cubes under one exterior and does not compare the laws under two, so it gives no contraction coefficient. The bad cubes are rare on cubes of fixed physical size for every exterior, and the conditional estimate in (D) remains open.

The same count locates the windows on which the fibre estimates of Sec. XII D apply. Let the side of the torus be even, let $B(y) = 2y + \{0,1\}^4$ be the blocks of Definition 328, and let l be even, so that R is a union of blocks.

Corollary 396. *Let $G = \mathrm{SU}(2)$, $\beta_W \geq 1$, let W_0 be a set of m blocks, $0 < K \leq 10^{-8}$, $\rho = K/(3072m)$, and $0 < \varepsilon \leq 1$ with $\varepsilon l \geq 16$. For every boundary field η of R, with ν_η-probability at least*

$$1 - \exp\Big(-l^4\Big(\frac{\beta_W\rho^2\varepsilon}{32} - 6\log\beta_W - 31 - \frac{48\beta_W}{l}\Big)\Big),$$

all but fewer than $256m\varepsilon l^4$ of the translates $W = y_0 + W_0$ whose blocks of W^+ lie in $[1, l-1]^4$ have the following property: in the tree gauge the exterior of W has a completion that satisfies the hypothesis of Corollary 380, namely the field itself.

Proof. Outside E_ε, which has the stated probability by Proposition 394, fewer than εl^4 unit cubes of R contain a bad plaquette. A plaquette with a vertex in a block of W^+ has all its vertices in R and lies in a unit cube $x + [0,1]^4$ of R; if it is bad, that cube contains a bad plaquette. The vertex lies in a block $B(y')$ with $y'_i \in \{\lfloor x_i/2 \rfloor, \lfloor (x_i+1)/2 \rfloor\}$, and $y' \in \{y\}^+$ for some $y \in W$, so y_i takes at most four values and y lies in a set of at most 256 blocks determined by x; then $y_0 \in y - W_0$ takes at most $256m$ values. Hence fewer than $256m\varepsilon l^4$ translates have a bad plaquette with a vertex in a block of W^+. At every other translate these plaquettes have $\delta(U_p, I) \leq \rho$. The number n_P of plaquettes that contain a link of A_W is at most $384m$, so $\rho \leq K/(8n_P) = \kappa/(2\sqrt{n_P})$ with $\kappa = K/(4\sqrt{n_P})$. A gauge transformation conjugates every plaquette holonomy and leaves $\delta(U_p, I)$ unchanged, so the field in the tree gauge is a completion of its own exterior with this property. □

So under every exterior of a box, the windows of a fixed number of blocks on which Corollary 380 applies fill all but at most $256m\varepsilon l^4$ of the positions, a fraction of about $4096m\varepsilon$. The bound is useful only when $\beta_W\rho^2\varepsilon$ is large, that is for windows of bounded size, and the next proposition shows that this restriction cannot be removed.

Proposition 397. *Let $G = \mathrm{SU}(2)$, $\beta_W \geq 1$ and $0 < \rho' \leq 1$.*

(i) *Conditionally on all links but one link of a plaquette p_0, the event $\delta(U_{p_0}, I) \leq \rho'$ has probability at most $14(\beta_W\rho'^2)^{3/2}$ under the Wilson measure.*

(ii) *Let W be a set of blocks with $W^+ \neq W$, $0 < K \leq 10^{-8}$ and $\kappa = K/(4\sqrt{n_P})$. Under the Wilson measure in the tree gauge, the exterior of W has a completion as in Corollary 380 with probability at most $7 \cdot 10^{-32}\beta_W^{3/2}|W|^{-3}$, and $\varepsilon''_W(\kappa^2/2, \beta_W/4) < 1$ only if $\beta_W > 10^{24}|W|^2$.*

Proof. (i) As in the proof of Proposition 386, the holonomy of p_0 has a conditional density proportional to $e^{\kappa' h\cdot u}$ with $0 \leq \kappa' \leq 6\beta_W$ and $|h| = 1$, and the probability of a cap about I is largest at $h = I$, $\kappa' = 6\beta_W$. Since $\delta(u, I) = 2\sqrt{2}\sin(\angle u/2)$ and $\sin x \geq 2x/\pi$ on $[0, \frac{\pi}{2}]$, the event forces $\angle U_{p_0} \leq \theta = \pi\rho'/(2\sqrt{2})$. With $\cos t - 1 \geq -t^2/2$ and $\sin t \geq 2t/\pi$ on $[0, \frac{\pi}{2}]$,

$$\int_0^\pi e^{6\beta_W(\cos t - 1)} \sin^2 t\, dt \geq \frac{4}{\pi^2}\int_0^{\pi/2} e^{-3\beta_W t^2} t^2\, dt \geq \frac{4 \cdot 0.44}{\pi^2(3\beta_W)^{3/2}},$$

because the integral of $e^{-s^2}s^2$ over $[0, \pi\sqrt{3}/2]$ exceeds 0.44. The integral of $e^{6\beta_W(\cos t-1)}\sin^2 t$ over $[0,\theta]$ is at most $\theta^3/3$, so the ratio is at most $9.72(\beta_W\theta^2)^{3/2}$, and $(\pi/(2\sqrt{2}))^3 < 1.371$.

(ii) Take $y' \in W^+ \setminus W$ and a plaquette p_0 inside $B(y')$. Its links have their base in $B(y')$, so none lies in A_W, and every completion of the exterior agrees with the field on p_0. A completion as in Corollary 380 therefore needs $\delta(U_{p_0}, I) \leq \kappa/(2\sqrt{n_P}) = K/(8n_P)$. This event is gauge invariant, so its probability is the same without gauge fixing, and by (i) it is at most $14(\beta_W K^2/(64n_P^2))^{3/2}$. Each link of A_W lies in six plaquettes and a plaquette contains at most four of them, so $n_P \geq 73.5|W|$, and $14 \cdot 10^{-24}/(512 \cdot 73.5^3) < 7 \cdot 10^{-32}$. Finally, $\varepsilon''_W(\kappa^2/2, \lambda) < 1$ means $\frac{3}{8}\lambda\kappa^2 > 49|W|\ln(3.45 \cdot 10^{11} n_P^6/K^6)$, and with $\lambda = \beta_W/4$, $\kappa^2 = K^2/(16n_P)$, $n_P \geq 73.5|W|$ and $K \leq 10^{-8}$ the logarithm exceeds 162.8, so $\beta_W > 512 \cdot 49 \cdot 73.5 \cdot 162.8 \cdot 10^{16}|W|^2/3 > 10^{24}|W|^2$. □

Along the tuning $\beta_W \sim b\log(1/a)$, a cube of fixed physical side r has $|W| = (r/2a)^4$ blocks, so both bounds of (ii) fail for small a: on such a cube the exterior has no calm completion with probability tending to one, and the fibre estimates of Sec. XII D are vacuous there. Their entropy factors are powers of $|W|$, so they are estimates for windows of bounded size, which Corollary 396 places at almost every position under every exterior. The exteriors with no calm completion and the rare fibres of Corollary 380 are thus the typical case at the physical scale and not an exceptional one. The cube condition

(D) needs estimates on the whole cube that are uniform in $|W|$, with the windows that are not calm split off as in Bałaban's analysis [12], and that remains open.

F. Conditioning on a fixed large-field set

The estimates of Sec. XII E are statements about the Wilson measure on the torus. The cube condition (D) takes a supremum over exteriors, and three facts about conditional laws hold under every exterior; the setting is that of a Gibbs field whose potential is itself random, the uniqueness theory of which goes back to Bassalygo and Dobrushin [3]. The first bounds the excess of the action over its minimum in a box, with no reflection positivity and no torus. The second says that restricting a single-link law to a fixed set, however irregular, leaves its Kantorovich response to the neighbouring links bounded by a constant times β_W. The third turns the one-point rarity of a bad set into closeness of any two Gibbs measures, at order of that rarity, and shows at the same time why one-point rarity gives closeness and not decay.

Lemma 398 (excess action)**.** *Let $G=\mathrm{SU}(2)$, let Λ be a set of N links of $\mathbb{Z}^4$ and fix the links outside Λ. Let P_Λ be the plaquettes that contain a link of Λ, $S_\Lambda=\sum_{p\in P_\Lambda}s_p$, and $S_{\min}$ the minimum of S_Λ over the links of Λ. For every $E>0$, under the Wilson measure conditioned on the links outside Λ,*

$$\Pr\big(S_\Lambda \ge S_{\min}+E\big) \le \Big(\frac{3\pi^3}{8r^3}\Big)^N e^{-\beta_W E/2},$$

where $r=\min\{E/(6\pi N),\pi/2\}$.

Proof. The plaquettes that contain no link of Λ do not depend on the links of Λ, so the conditional law has density $Z^{-1}e^{-\beta_W S_\Lambda}$ with respect to the product of Haar measures on Λ. The unnormalised mass of the event is at most $e^{-\beta_W(S_{\min}+E)}$. For the lower bound on Z, let U^* be a minimiser, which exists by compactness. If a link of a plaquette p moves by geodesic distance t, the angle $\delta(U_p,I)\in[0,\pi]$ moves by at most t, by bi-invariance, so $s_p=\frac14\delta(U_p,I)^2$ moves by at most $\frac14 t\cdot 2\pi=\frac{\pi}{2}t$. Each link lies in six plaquettes. If every link of Λ lies within distance r of its value in U^*, then $S_\Lambda\le S_{\min}+6\cdot\frac{\pi}{2}rN\le S_{\min}+E/2$. The Haar measure of a geodesic ball of radius $r\le\pi/2$ is $\frac{2}{\pi}\int_0^r\sin^2 t\,dt\ge 8r^3/(3\pi^3)$, by $\sin t\ge 2t/\pi$. Hence $Z\ge(8r^3/3\pi^3)^N e^{-\beta_W(S_{\min}+E/2)}$, and the quotient of the two bounds is the claim. □

The constant depends on N and E alone, so excess action is rare under every exterior, at a cost that is exponential in N. The minimum itself is a function of the exterior, and the lemma says nothing about the size of its plaquettes: it controls the fluctuation of the field about the configuration the exterior forces, not that configuration.

Lemma 399 (restriction to a fixed set)**.** *Let $G=\mathrm{SU}(2)$, let e be a link and let $h\ge 0$ be an integrable function on G with $\int h\,dU>0$ that does not depend on the other links. For a configuration σ of the other links let $\gamma_e^h(\cdot\mid\sigma)$ be the law on G with density proportional to $e^{-\beta_W S}h$. If σ and σ' differ only at a link $e'\neq e$, then*

$$W_1\big(\gamma_e^h(\cdot\mid\sigma),\gamma_e^h(\cdot\mid\sigma')\big)\le\frac{\pi^2}{8}\beta_W\,\delta(\sigma_{e'},\sigma'_{e'}),$$

and the two laws coincide unless e and e' lie in a common plaquette.

Proof. Only the plaquettes containing e enter the law, and two distinct links of $\mathbb{Z}^4$ lie in at most one common plaquette p_0; if there is none the laws are equal. Otherwise move $\sigma_{e'}$ along a unit-speed geodesic $U(t)$, $0\le t\le T=\delta(\sigma_{e'},\sigma'_{e'})$, and let γ_t be the corresponding law. The function $t\mapsto s_{p_0}$ has $|\frac{d}{dt}s_{p_0}|\le\frac{\pi}{2}$ as in the proof of Lemma 398, and the derivative $\dot s_{p_0}$ is a function of U_e with oscillation at most π. For a 1-Lipschitz f on G, which has oscillation at most the diameter π, $\frac{d}{dt}\gamma_t(f)=-\beta_W\operatorname{Cov}_{\gamma_t}(f,\dot s_{p_0})$ because h does not depend on t, and $|\operatorname{Cov}(f,\dot s_{p_0})|\le\sqrt{\operatorname{Var}f\,\operatorname{Var}\dot s_{p_0}}\le\frac{\pi}{2}\cdot\frac{\pi}{4}$. Integrating over t and taking the supremum over f gives the bound. □

When h is the indicator of a set C the lemma says that conditioning a link to lie in C costs nothing in its response, however small or irregular C is, as long as C is fixed. The responses that are not bounded are those to a set that moves with the neighbours, such as the intersection of balls centred at them; this is the difference between conditioning on a fixed set of bad links and conditioning on a plaquette being large.

Proposition 400 (one-point rarity)**.** *Let $\mathscr{A}$ be a set of links, D the diameter of G, and γ_e, $e\in\mathscr{A}$, single-link kernels with*

$$W_1\big(\gamma_e(\cdot\mid\sigma),\gamma_e(\cdot\mid\sigma')\big)\le\sum_{e'}c_{ee'}(\sigma,\sigma')\,\delta(\sigma_{e'},\sigma'_{e'})$$

for all σ,σ'. Let $\sigma\mapsto B(\sigma)\subset\mathscr{A}$ be measurable, and suppose there are constants $a_{ee'},A_{ee'}\ge0$ with $c_{ee'}(\sigma,\sigma')\le a_{ee'}$ whenever $e\notin B(\sigma)\cup B(\sigma')$, $c_{ee'}\le A_{ee'}$ always, $\sup_e\sum_{e'}a_{ee'}\le\theta<1$ and $\sup_e\sum_{e'}A_{ee'}\le\bar A$. If μ_1,μ_2 are invariant under every γ_e and $\mu_i(e\in B)\le q$ for every e and $i=1,2$, there is a coupling of μ_1 and μ_2 with

$$\sup_{e\in\mathscr{A}}\mathbb{E}\,\delta(\sigma_e,\sigma'_e)\le\frac{2\bar A D q}{1-\theta}.$$

In particular $|\mu_1(F)-\mu_2(F)|\le L_F|S_F|\,2\bar A D q/(1-\theta)$ for every F with support S_F that is L_F-Lipschitz in each link.

Proof. Fix $b_0\in\mathscr{A}$, $\lambda>0$, and let $w_b=e^{-\lambda|b-b_0|}$, which is summable. As in the proof of Lemma 665 the functional $\Psi(\pi)=\sum_b w_b\mathbb{E}_\pi\delta(\sigma_b,\sigma'_b)$ has a minimiser π^* among the couplings, and replacing the pair at one link e by a sample of an optimal coupling of $\gamma_e(\cdot\mid\sigma)$ and $\gamma_e(\cdot\mid\sigma')$ gives a coupling again. Only the term at e changes, so minimality gives, with $\rho(e)=\mathbb{E}_{\pi^*}\delta(\sigma_e,\sigma'_e)$,

$$\rho(e)\le\mathbb{E}_{\pi^*}\sum_{e'}c_{ee'}(\sigma,\sigma')\,\delta(\sigma_{e'},\sigma'_{e'})$$

for every e, with no weight left. On the event that e is good in both configurations the coefficients are at most $a_{ee'}$; on its

complement, whose probability is at most $\mu_1(e\in B)+\mu_2(e\in B)\le 2q$, they are at most $A_{ee'}$ and every distance is at most D. Hence $\rho(e)\le \theta\sup\rho+2\bar A D q$, and $\sup\rho\le D$ is finite, which gives the bound. The last statement follows by writing $F(\sigma)-F(\sigma')$ as a telescopic sum over the links of S_F. □

The proposition is approximate uniqueness: all Gibbs measures of the specification lie within $2\bar A D q/(1-\theta)$ of each other, link by link. It is not decay. Its proof sees the bad set only through the one-point probabilities q, and the term $2\bar A D q$ does not shrink with the distance from the region where μ_1 and μ_2 are allowed to differ. The bound D on the distance at a bad link is not what causes this: replacing it by Hölder's inequality gives $\rho\le\theta\rho+\bar A(qD)^{1/p}\rho^{1/p'}$, a recursion with the positive fixed point ρ^* of order $\bar A^p qD/(1-\theta)^p$. To obtain decay the bad set has to be held fixed while the comparison runs, as in the disagreement percolation of van den Berg and Maes [112], where the two configurations are coupled link by link and the disagreement has to percolate through the bad set; the price is that the laws conditioned on that set, not the Wilson measure, then have to satisfy the fibre estimates of Sec. XII D.

G. Rarity of coarse defects

Theorem 383 makes a large plaquette rare at the lattice scale. The cube condition (D) concerns a cube of fixed physical side, which along the tuning spans $m\sim\beta_W/b$ scales, and it needs the analogue of that rarity for the curvature of the block averages at every one of them. This subsection isolates the two sides of that question. On the energy side, the coarse curvature of a field is bounded by its action uniformly in the scale, for linear fields and for group-valued fields in a torus. On the entropy side, the chessboard estimate applies to coarse events, and it is the normalisation of Lemma 382, not the energy, that makes it useless beyond $O(\log\beta_W)$ scales. A right translation of the Haar measures gives rarity at every scale and under every exterior, with a rate governed by the second variation of the action along a fixed direction; for the abelian theory that second variation is the energy of the direction, while for SU(2) it grows like 4^k at scale k, and the growth is produced by the gauge orbit of the vacuum.

Throughout, Λ_L is the torus of side $L=2^m$, the scale k satisfies $2^{k+2}\le L$, and the lattice of spacing 2^k is the torus $\Lambda_{L'}$ with $L'=L/2^k\ge 4$. For a real field φ on the links, $d\varphi$ is its curvature, $d^{\mathsf T}$ the adjoint of d, and $\Delta=d^{\mathsf T}d+dd^{\mathsf T}$, which acts on each component of a field as the lattice Laplacian $(\Delta f)(x)=8f(x)-\sum_{\pm\nu}f(x\pm e_\nu)$. The Green function $G_{L'}$ of $\Lambda_{L'}$ is the function of mean zero with $\Delta G_{L'}=\delta_0-L'^{-4}$; it is invariant under the permutations and sign changes of the coordinates, and applied to each component it inverts Δ on fields whose components have mean zero. It commutes with d, $d^{\mathsf T}$ and the translations.

Lemma 401 (energy of a coarse defect)**.** *Let P be a plaquette of the lattice of spacing 2^k and $J_{k,P}$ the field with $(d_kV^{(k)}(\varphi))_P=\langle\varphi,J_{k,P}\rangle$ for every field φ on Λ_L, given by Lemma 336(i). Then $d^{\mathsf T}J_{k,P}=0$, every field φ satisfies*

$$(d_kV^{(k)}(\varphi))_P^2\le\gamma_k\,Q(\varphi),$$

where

$$\gamma_k=\langle J_{k,P},G_LJ_{k,P}\rangle=\sup_{\varphi}\frac{(d_kV^{(k)}(\varphi))_P^2}{Q(\varphi)},$$

and $\frac{1}{432}\le\gamma_k\le\frac12$ for every k. In particular $Q(\varphi)\ge 2r^2$ whenever $|(d_kV^{(k)}(\varphi))_P|\ge r$, at every scale.

Proof. Lemma 336(ii) shows that $V^{(k)}(dh)$ is the gradient of $\bar h$ on the coarse lattice for every function h on sites, so $\langle dh,J\rangle=0$, the curvature of a gradient vanishing, and $d^{\mathsf T}J=0$; J is supported on the links of directions μ, ν of the plane of P, and its component in each direction has mean zero, the two sides of P entering with opposite signs. Put $u=G_LJ$. Then $\Delta u=J$, $d^{\mathsf T}u=G_Ld^{\mathsf T}J=0$, and for every field φ

$$\begin{aligned}\langle\varphi,J\rangle=\langle\varphi,\Delta u\rangle&=\langle d\varphi,du\rangle+\langle d^{\mathsf T}\varphi,d^{\mathsf T}u\rangle\\&=\langle d\varphi,du\rangle\le\sqrt{Q(\varphi)Q(u)},\end{aligned}$$

with $Q(u)=\langle u,d^{\mathsf T}du\rangle=\langle u,\Delta u\rangle=\langle u,J\rangle=\gamma_k$, and equality at $\varphi=u$. This proves the inequality and the characterisation of γ_k as a supremum.

For the upper bound, by Proposition 335(i), whose proof is a local identity, $Q_k(V^{(k)}(\varphi))\le Q(\varphi)$ for every field, so γ_k is at most the supremum of $(d_kw)_P^2/Q_k(w)$ over the fields w of $\Lambda_{L'}$. The argument of the first paragraph on $\Lambda_{L'}$, with $j=d_k^{\mathsf T}e_P$ the field equal to ±1 on the four sides of P, identifies that supremum with $\langle j,G_{L'}j\rangle$. The two sides of P in the direction μ are the links (y,μ) and $(y+e_\nu,\mu)$ with opposite signs, so

$$\begin{aligned}\langle j,G_{L'}j\rangle&=4G_{L'}(0)-2G_{L'}(e_\nu)-2G_{L'}(e_\mu)\\&=4\big(G_{L'}(0)-G_{L'}(e_1)\big)=\tfrac12\big(1-L'^{-4}\big),\end{aligned}$$

by $(\Delta G_{L'})(0)=8G_{L'}(0)-8G_{L'}(e_1)=1-L'^{-4}$ and the symmetry of $G_{L'}$.

For the lower bound take P at $y=0$ in the plane $\{1,2\}$, write $L_k=2^k$, and let f be the function on $\mathbb Z/L\mathbb Z$ with $f(s)=s/L_k$ for $0\le s\le L_k$, $f(s)=1$ for $L_k\le s\le 2L_k$, $f(s)=1-(s-2L_k)/L_k$ for $2L_k\le s\le 3L_k$ and $f=0$ elsewhere; let g be the indicator of $[0,2L_k)$; and let $h(s)=(1-\operatorname{dist}(s,[0,L_k))/L_k)_+$. Put $\varphi_{(x,2)}=L_k^{-1}f(x_1)g(x_2)h(x_3)h(x_4)$ and $\varphi=0$ on the other links; all supports fit in the torus since $4L_k\le L$. By Lemma 336(i), $V^{(k)}_{y,2}(\varphi)=L_k^{-4}\sum_{x\in B_k(y)}\sum_{t<L_k}\varphi_{(x+te_2,2)}$, and for $y=0$ and $y=e_1$ the factors g and h equal 1 on the ranges that occur, so $V^{(k)}_{0,2}(\varphi)=\bar f(0)$ and $V^{(k)}_{e_1,2}(\varphi)=\bar f(1)$, the means of f over $[0,L_k)$ and $[L_k,2L_k)$, which are $(L_k-1)/(2L_k)$ and 1. Hence $(d_kV^{(k)}(\varphi))_P=\bar f(1)-\bar f(0)=(L_k+1)/(2L_k)\ge\frac12$. The curvature of φ lives on the plaquettes of the planes $\{1,2\}$, $\{2,3\}$, $\{2,4\}$, and g is never differenced, so

$$\begin{aligned}Q(\varphi)&=L_k^{-2}\big(\|\nabla f\|^2\|g\|^2\|h\|^4+2\|f\|^2\|g\|^2\|\nabla h\|^2\|h\|^2\big)\\&\le L_k^{-2}\Big(\frac{2}{L_k}2L_k(3L_k)^2+2\cdot3L_k\cdot2L_k\frac{2}{L_k}3L_k\Big),\end{aligned}$$

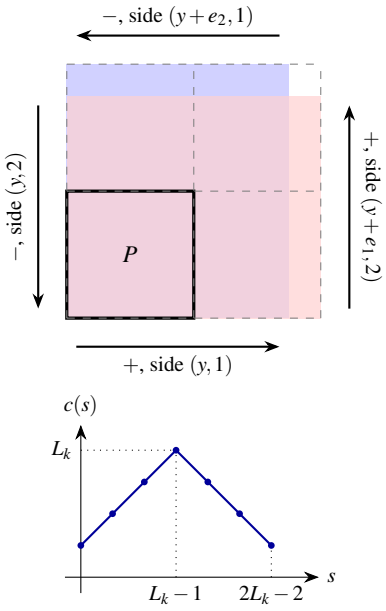


FIG. 33. The smeared current loop $J_{k,P}$ of Lemma 401 in the plane of P, drawn for $L_k = 4$. Left: each side of the coarse plaquette P (thick square) contributes a current of its own direction on the links of a slab of transverse size L_k^3 and length $2L_k - 1$ (blue for the direction 1, red for the direction 2), with the signs of the four sides; the slabs of opposite sides are disjoint and the two directions are carried by different links. Right: the longitudinal profile $L_k^4 J = \pm c(s)$ on a slab, the tent $c(s) = \min\{s+1, L_k, 2L_k - 1 - s\}$, which counts the straight paths of Lemma 336(i) through the link at position s. The current is divergence free and its energy $\gamma_k = \langle J_{k,P}, G_L J_{k,P}\rangle$ lies in $[\frac{1}{432}, \frac{1}{2}]$ at every scale.

where ∇ is the forward difference: f and h change by $1/L_k$ on $2L_k$ steps and by nothing elsewhere, and f^2, g^2, h^2 have sums at most $3L_k$, $2L_k$, $3L_k$. So $Q(\varphi) \le 108$, and $\gamma_k \ge \frac{1}{4}/108$. □

The fields $J_{k,P}$ are smeared current loops: by Lemma 336(i) each side of P contributes $\pm L_k^{-4} c(s)$ on the $L_k^3(2L_k - 1)$ links of its direction whose transverse coordinates lie in the block and whose longitudinal coordinate s ranges over $[0, 2L_k - 2]$, with the tent $c(s) = \min\{s+1, L_k, 2L_k - 1 - s\}$. A bare holonomy around a loop of n links forces action only of order r^2/n, since the loop can be thickened to a tube of any radius; the block average smears the loop over a four-dimensional region, and this is why its energy does not decrease with the scale. The lemma is the quadratic part of Theorem 338, seen from one coarse plaquette. The same scale invariance holds for group-valued fields in a torus.

Proposition 402 (energy of a coarse defect, torus-valued fields)**.** *Let U be an S-valued field in the tree gauge on Λ_L, as in Theorem 355, and let P be a plaquette of the lattice of spacing 2^k with $\delta(V^{\sharp,k}(U)_P, I) \ge r$. Then*

$$Q(U) \ge \min\Big\{ \frac{r^2}{1.001}, \frac{1}{144.2} \Big\} \ge \frac{r^2}{1154}.$$

Proof. If some plaquette has $\delta(U_p, I) > 1 = \rho_G$ then $Q(U) > 1 \ge r^2/8$, since $r \le \operatorname{diam} \mathrm{SU}(2) = 2\sqrt{2}$. Otherwise every Θ_p is defined, and for $\Theta_p = \theta_p J_p$ with $J_p^2 = -I$ one has $\|\Theta_p\|^2 = 2\theta_p^2$ and $\delta(U_p, I)^2 = 8\sin^2(\theta_p/2)$, so $\|\Theta_p\|^2 \le 1.001\, \delta(U_p, I)^2$ when $\theta_p \le 1/(12\sqrt{2})$, that is when $\|\Theta_p\| \le \eta_G = \frac{1}{12}$. Let j_* be the least $j \le k$ with $\max_P \|\Theta_P^{(j)}\| > \eta_G$, if there is one. If there is none, (36) holds for every $j < k$, and Theorem 355 gives $r^2 \le \delta(V^{\sharp,k}(U)_P, I)^2 \le Q_k(V^{\sharp,k}(U)) \le 1.001\, Q(U)$. If $j_* = 0$, some plaquette has $\theta_p > 1/(12\sqrt{2})$ and $Q(U) \ge 8\sin^2(\theta_p/2) \ge 1/144.1$. If $1 \le j_* \le k$, (36) holds for $j < j_*$, so Theorem 355 at the scale j_* shows that $\Theta^{(j_*)}$ is defined and $\sum_P \|\Theta_P^{(j_*)}\|^2 \le \sum_p \|\Theta_p\|^2 \le 1.001\, Q(U)$, every fine plaquette having $\|\Theta_p\| \le \eta_G$; one term of the left side exceeds $1/144$, so $Q(U) \ge 1/144.2$. □

The S-valued fields form a null set for the Wilson measure, and for general $\mathrm{SU}(2)$ fields Theorem 359 controls the cut-off average only while $\sum_j \eta_j$ stays small; whether a bound of the form $Q(U) \ge c\, r^2$, uniform in k, holds for every $\mathrm{SU}(2)$ field is not known. The next proposition shows that, whatever the energy of a coarse defect, the chessboard estimate cannot use it beyond a bounded number of scales.

Proposition 403 (chessboard for coarse events)**.** *Let $\beta_W \ge 1$, let $b = 2^j$ with $2b \mid L$, and let E be an event determined by the links of the closed cube $[0,b]^4$ on which the action of the plaquettes of that cube is at least A. For $z \in (b\mathbb{Z}/L\mathbb{Z})^4$ let E_z be the event obtained from E by the reflections that carry $[0,b]^4$ to $z + [0,b]^4$. Then for every set Z of such z,*

$$\mu_L\Big(\bigcap_{z \in Z} E_z \Big) \le \exp\Big(|Z| \Big(6b^4 (\log \beta_W + C_0) - \frac{\beta_W A}{4} \Big) \Big).$$

Proof. As in the proof of Theorem 383, μ_L is reflection positive in the hyperplanes through sites, and the chessboard estimate for functions of the links of the closed blocks $z + [0,b]^4$, $z \in (b\mathbb{Z}/L\mathbb{Z})^4$, bounds the probability of $\bigcap_{z \in Z} E_z$ by that of $\bigcap_z E_z$ over all $(L/b)^4$ blocks, raised to the power $|Z|(b/L)^4$. Reflection preserves the Haar measures and $\operatorname{Tr} U_p$, so on every E_z the plaquettes of the block $z + [0,b]^4$ carry action at least A. A plaquette lies in at most four closed blocks, one for each choice of the two transverse coordinates of the corner, so on the intersection $S \ge (L/b)^4 A/4$, and the event has unnormalised weight at most $e^{-\beta_W (L/b)^4 A/4}$. Dividing by $Z_L \ge e^{-6L^4(\log \beta_W + C_0)}$ of Lemma 382 and raising to the power $|Z|(b/L)^4$ gives the bound. □

A plaquette of the lattice of spacing 2^k depends, by Lemma 387 applied at each step, on the fine links of a cube of side $2^{k+2} - 4$, so the event that it carries a large cut-off average is of the kind in the proposition with $b = 2^{k+2}$. Whatever action A that event forces, the bound is below one only when $6 \cdot 16^{k+2}(\log \beta_W + C_0) < \beta_W A/4$, that is $16^k \lesssim \beta_W A / \log \beta_W$. For the events of Lemma 401 and Proposition 402, A is of the order of r^2 at every scale, and the loss is entirely in the normalisation: the lower bound of Lemma 382 charges the full fluctuation volume of the block, $6b^4(\log \beta_W + C_0)$, against a defect whose energy does not grow with the block. In the Gaussian theory the two fluctuation volumes, around the defect and around the vacuum, are the same, and the following

translation argument transfers that cancellation to the group, under every exterior.

Theorem 404 (Haar tilt)**.** *Let G be* SU(2) *or* U(1)*, let Λ be a finite set of links, either all links of Λ_L or the inner links of a box $R \subset \mathbb{Z}^4$ whose other links are fixed, and let μ be the law with density $Z^{-1}e^{-\beta_W S}$ with respect to the product of the Haar measures on Λ, where $S=\sum_p s_p$ runs over the plaquettes meeting Λ and $s_p = 1-\frac{1}{N}\operatorname{Re}\operatorname{Tr}U_p$. For $r=(r_e)_{e\in\Lambda}$ with $r_e\in\mathfrak{g}$ and $\rho_e=\|r_e\|_{\mathrm{op}}$ put $U_e^t = U_e e^{tr_e}$,*

$$D_rS(U)=\partial_t S(U^t)\big|_{t=0}, \qquad K_r=\sup_U \partial_t^2 S(U^t)\big|_{t=0}.$$

(i) *$K_r \le \sum_p\big(\sum_{e\subset p,\,e\in\Lambda}\rho_e\big)^2 \le 24\sum_{e\in\Lambda}\rho_e^2$. For $G=\mathrm{U}(1)$ and $r_e=ia_e$ with $a_e\in\mathbb{R}$, $K_r\le Q(a)=\sum_p (da)_p^2$.*

(ii) *$\mathbb{E}_\mu e^{\beta_W t D_rS}\le e^{\beta_W t^2K_r/2}$ for every $t\in\mathbb{R}$.*

(iii) *For every $x>0$, uniformly in the fixed links,*

$$\mu(D_rS\ge x)\le e^{-\beta_W x^2/(2K_r)}, \qquad \mu(D_rS\le -x)\le e^{-\beta_W x^2/(2K_r)}.$$

Proof. (i) Order the links of a plaquette p as traversed, so that $U_p=g_1^{\varepsilon_1}\cdots g_4^{\varepsilon_4}$ with $\varepsilon_i=\pm1$. Since $ge^{tr}=e^{t\,\mathrm{Ad}(g)r}g$ and $(ge^{tr})^{-1}=e^{-tr}g^{-1}$, moving the exponentials to the left gives $U_p(t)=e^{t\tilde a_1}\cdots e^{t\tilde a_4}U_p$ with $\tilde a_i=\pm\mathrm{Ad}(h_i)r_{e_i}$ for unitary h_i, so $\|\tilde a_i\|_{\mathrm{op}}=\rho_{e_i}$, and $\tilde a_i=0$ if $e_i\notin\Lambda$. Then

$$\partial_t^2\operatorname{Tr}U_p(t)\big|_{t=0}=\operatorname{Tr}\Big[\Big(\sum_i\tilde a_i^2+2\sum_{i<j}\tilde a_i\tilde a_j\Big)U_p\Big],$$

and $|\operatorname{Tr}(XU)|\le N\|X\|_{\mathrm{op}}$ for unitary U, so $|\partial_t^2 s_p|\le(\sum_i\rho_{e_i})^2\le 4\sum_{e\subset p}\rho_e^2$; each link lies in six plaquettes. The second derivative at any other t is the one at 0 for the field U^t, so the bound holds for K_r. For U(1) all factors commute, $U_p(t)=e^{it(da)_p}U_p$ and $\partial_t^2 s_p=(da)_p^2\operatorname{Re}\operatorname{Tr}U_p\le(da)_p^2$.

(ii) Right multiplication by the fixed element e^{-tr_e} preserves the Haar measure of each link of Λ, so

$$\int e^{-\beta_W S(Ue^{-tr})}\prod_{e\in\Lambda}dU_e=Z,$$

that is, $\mathbb{E}_\mu e^{-\beta_W[S(Ue^{-tr})-S(U)]}=1$. Taylor's formula along $s\mapsto Ue^{-sr}$ with the Lagrange remainder gives

$$\begin{aligned}S(Ue^{-tr})-S(U)&=-tD_rS(U)+\tfrac{t^2}{2}\partial_s^2S(Ue^{-sr})\big|_{s=s_*}\\&\le -tD_rS(U)+\tfrac{t^2}{2}K_r,\end{aligned}$$

the second derivative being even in r. Hence

$$\mathbb{E}_\mu e^{\beta_W tD_rS-\beta_W t^2K_r/2}\le 1.$$

(iii) By Chebyshev's inequality and (ii), $\mu(D_rS\ge x)\le e^{-\beta_W tx+\beta_W t^2K_r/2}$ for $t>0$; take $t=x/K_r$. The lower tail is the upper tail for $-r$. Nothing depends on the fixed links. □

The map $U\mapsto Ue^{-tr}$ is the group version of a translation of the Gaussian field; its Jacobian is one, and no smallness of the field enters. The observable D_rS is the component of the force along r: at a plaquette, $\partial_t s_p=-\frac{1}{N}\operatorname{Re}\operatorname{Tr}\big((\sum_i\tilde a_i)U_p\big)$, the covariant derivative of r in the background U paired with the sine part of U_p. For the abelian theory the second variation is the energy of the direction, and the theorem gives the rarity of coarse defects at every scale.

Corollary 405 (abelian theory)**.** *Let $G=\mathrm{U}(1)$, let μ_L be its Wilson measure on Λ_L, $U_e=e^{i\theta_e}$, $\theta_p=(d\theta)_p$ the angle of U_p in $(-\pi,\pi]$, and let $a=G_LJ_{k,P}$ be the minimiser of Lemma 401. For every k, every plaquette P of the lattice of spacing 2^k and every $x>0$,*

$$\mu_L\Big(\sum_p\sin\theta_p\,(da)_p\ge x\Big)\le e^{-\beta_W x^2}.$$

Here $\sum_p\theta_p(da)_p=(d_kV^{(k)}(\theta))_P$ for every choice of link angles θ_e with $(d\theta)_p\equiv\theta_p$ modulo 2π, up to $2\pi\sum_p n_p(da)_p$ with the integers $n_p=((d\theta)_p-\theta_p)/2\pi$. Under a fixed exterior of a box, the same bound holds with a replaced by its restriction $\tilde a$ to the inner links and x^2 replaced by $x^2/(2Q(\tilde a))$.

Proof. By Theorem 404(i) with $r_e=ia_e$, $K_r\le Q(a)=\gamma_k\le\frac12$, and $D_rS=\sum_p\sin\theta_p(da)_p$ since $s_p(t)=1-\cos(\theta_p+t(da)_p)$. Part (iii) gives the bound. For the identity, $\sum_p(d\theta)_p(da)_p=\langle\theta,d^{\mathsf T}da\rangle=\langle\theta,\Delta a\rangle=\langle\theta,J_{k,P}\rangle$ because $d^{\mathsf T}a=0$. In a box the tilt acts on the inner links only, and $K_r\le Q(\tilde a)$. □

For the abelian theory this is the statement that a smeared coarse flux of size x costs $\beta_W x^2$ at every scale, uniformly in the exterior; the four-dimensional abelian theory has no running coupling, and the bound is consistent with its known behaviour at weak coupling [55, 64]. For SU(2) the same tilt is available, but its second variation grows with the scale.

Proposition 406 (the gauge orbit of the vacuum)**.** *Let $G=\mathrm{SU}(2)$ on Λ_L, $T\in\mathfrak{g}$ with $\|T\|_{\mathrm{op}}=1$, $a=G_LJ_{k,P}$ for a plaquette P of the lattice of spacing 2^k, and $r_e=a_eT$. Then*

$$\Big(\sqrt{2\|a\|^2-\gamma_k}-\sqrt{\gamma_k}\Big)_+^2\le K_r\le 24\|a\|^2$$

and

$$\|a\|^2\ge\frac{\gamma_k^2}{\|J_{k,P}\|^2}\ge\frac{4^k}{1492992},$$

with $\gamma_k=\langle J_{k,P},G_LJ_{k,P}\rangle$ as in Lemma 401; in particular $K_r\ge 4^k/(3\cdot10^6)$ for $k\ge11$. The lower bound is attained, up to the stated loss, at a pure gauge field, where $Q(U)=0$.

Proof. The upper bound is Theorem 404(i). For the lower bound let U be the pure gauge field $U_e=h_xh_{x+e}^{-1}$ of a function h on sites with values in SU(2). The action is invariant under gauge transformations and $U_ee^{tr_e}=h_x\exp(t\,\mathrm{Ad}(h_{x+e}^{-1})r_e)h_{x+e}^{-1}$, so $S(Ue^{tr})=S(e^{tr'})$ with $r'_e=\mathrm{Ad}(h_{x+e}^{-1})r_e$. At the identity field, $U_p(t)=\exp(tX_p+O(t^2))$ with $X_p=(dr')_p$ the signed sum of the r'_e around p, and $s_p=1-\cos\|X_p(t)\|_{\mathrm{op}}$, so $\partial_t^2s_p|_{t=0}=$

$\|(dr')_p\|_{\mathrm{op}}^2$ and $K_r \geq \sum_p \|(dr')_p\|_{\mathrm{op}}^2$. Take $h_x = R^{x_2}$ with $R = i\sigma_1$ and $T = i\sigma_3$, which is well defined on the torus since $R^4 = I$ and $4 \mid L$, and satisfies $\mathrm{Ad}(R)T = -T$. Then $r'_e = \varepsilon_e a_e T$ with $\varepsilon_{(x,\mu)} = (-1)^{x_2}$ for $\mu \neq 2$ and $\varepsilon_{(x,2)} = (-1)^{x_2+1}$, and $K_r \geq Q(\varepsilon a)$. Since $J_{k,P}$ has components only in the directions 1, 2 of the plane of P and G_L acts on each component, $a_3 = a_4 = 0$; the reflection exchanging the coordinates 1 and 2 carries the component a_1 to $-a_2$, so $\|a_1\|^2 = \|a_2\|^2 = \frac{1}{2}\|a\|^2$. On the plaquette at x in the plane $\{1,2\}$ the signs give

$$(d(\varepsilon a))_p = (-1)^{x_2}\big[(a_{(x,1)} + a_{(x+e_2,1)}) - (a_{(x+e_1,2)} - a_{(x,2)})\big] = (-1)^{x_2}\big[\sigma_x - v_x\big],$$

so that $Q(\varepsilon a) \geq \|\sigma - v\|^2 \geq (\|\sigma\| - \|v\|)_+^2$, with $\|\sigma\|^2 = 4\|a_1\|^2 - \|\nabla_2 a_1\|^2$ and $\|v\|^2 = \|\nabla_1 a_2\|^2$, where ∇_ν is the forward difference in the direction ν. All directional differences together give $\sum_{\mu,\nu} \|\nabla_\nu a_\mu\|^2 = \langle a, \Delta a\rangle = \langle G_L J, J\rangle = \gamma_k$, so $\|\sigma\|^2 \geq 2\|a\|^2 - \gamma_k$ and $\|v\|^2 \leq \gamma_k$, which is the lower bound. Finally $\gamma_k = \langle J, a\rangle \leq \|J\|\|a\|$, and by Lemma 336(i) each of the four sides of P contributes L_k^3 rows of $2L_k - 1$ links with coefficients $L_k^{-4} c(s)$, $c \leq L_k$, so $\|J\|^2 \leq 4L_k^{-8} L_k^3 (2L_k - 1) L_k^2 \leq 8 \cdot 4^{-k}$; with $\gamma_k \geq \frac{1}{432}$ this gives $\|a\|^2 \geq 4^k/(8 \cdot 432^2)$. For $k \geq 11$ that is at least 2, so $2\|a\|^2 - \gamma_k \geq 7\gamma_k$ and $K_r \geq (1 - 7^{-1/2})^2 \cdot \frac{7}{8} \cdot 2\|a\|^2 \geq 0.67\|a\|^2 \geq 4^k/(3 \cdot 10^6)$. □

So for SU(2) the tilt of Theorem 404 proves the rarity of the force component $D_r S$ at the rate $\beta_W x^2/(2K_r)$, which is of order $\beta_W x^2/4^k$ at scale k, and not more: the supremum in K_r is reached on the gauge orbit of the vacuum, where the field has no action and no curvature, because a fixed direction r is not gauge covariant and its covariant derivative in a pure gauge background is as large as r itself. The natural repair is to transport the direction with the field, $r_e \mapsto \mathrm{Ad}(h(U))r_e$. When the transport runs along a forest, the translation still preserves the Haar measures; but the residual gauge freedom of the forest then realises the field of the proof on the links between the trees, and the constant is reached at zero action in that gauge as well.

Proposition 407 (covariant tilt on a forest)**.** *Let $G = \mathrm{SU}(2)$, let F be a set of links of Λ_L without cycles, with one root chosen in every component, and for a site y let $h_y(U)$ be the ordered product of the link variables along the path in F from the root of its component to y, with $h_y = I$ at the roots. For $r = (r_e)$ with $r_e \in \mathfrak{g}$ and $r_e = 0$ on F put*

$$\Phi_t(U)_e = U_e \exp\big(t\,\mathrm{Ad}(h_y(U)^{-1})r_e\big), \quad y \text{ the end point of } e,$$

and $\Phi_t(U)_f = U_f$ for $f \in F$; let $D_r^F S(U) = \partial_t S(\Phi_t U)|_{t=0}$ and

$$K_r^F = \sup\big\{\partial_t^2 S(\tilde{U}e^{tr})\big|_{t=0} :\ \tilde{U}_f = I \text{ for all } f \in F\big\}.$$

(i) *Φ_t preserves the product of the Haar measures for every t.*

(ii) *$\mathbb{E}_\mu e^{\beta_W t D_r^F S} \leq e^{\beta_W t^2 K_r^F/2}$ for every $t \in \mathbb{R}$, and $\mu(\pm D_r^F S \geq x) \leq e^{-\beta_W x^2/(2K_r^F)}$ for every $x > 0$. $D_r^F S$ is invariant under the gauge transformations that are trivial at the roots.*

(iii) *Let F be one of the sixteen translates F_o of the forest of Definition 328, with blocks $2y + o + \{0,1\}^4$, let $a = G_L J_{k,P}$ and T be as in Proposition 406, and let $r_e = a_e T$ on the links outside F. Then*

$$K_r^F \geq 2\sum_{e \notin F} a_e^2,$$

and the bound is reached on pure gauge fields, where $Q(U) = 0$. For at least one o the right side is at least $\frac{29}{16}\|a\|^2 \geq \frac{29}{16} \cdot \frac{4^k}{1492992}$.

Proof. (i) Let $\Gamma(U) = (U_F, \tilde{U})$ with $\tilde{U}_e = h_x(U) U_e h_y(U)^{-1}$ for $e = (x,y) \notin F$. Given the variables on F, the map $U_e \mapsto \tilde{U}_e$ is a left and a right translation by elements that depend only on U_F, so Γ is a bijection of the configuration space that preserves the product of the Haar measures; this is the change of variables of Lemma 10. Let R_t multiply every coordinate $\tilde{U}_e$, $e \notin F$, on the right by e^{tr_e} and leave U_F alone. Then

$$(\Gamma^{-1} R_t \Gamma U)_e = h_x^{-1}\big(h_x U_e h_y^{-1} e^{tr_e}\big) h_y = U_e \exp\big(t\,\mathrm{Ad}(h_y^{-1}) r_e\big),$$

so $\Phi_t = \Gamma^{-1} R_t \Gamma$ is a composition of three maps that preserve the measure.

(ii) Since the action is gauge invariant, $S(\Phi_t U) = S(\tilde{U}e^{tr})$, and $\tilde{U}e^{tr}$ again has trivial links on F. Taylor's formula along $s \mapsto \Phi_{-s}U$ with the Lagrange remainder gives $S(\Phi_{-t}U) - S(U) \leq -tD_r^F S(U) + \frac{t^2}{2} K_r^F$, and $\int e^{-\beta_W S \circ \Phi_{-t}}\, dU = Z$ by (i); the rest is the proof of Theorem 404 (ii), (iii). Under a gauge transformation g the transports change to $g_o h_y g_y^{-1}$, with o the root of the component of y, so $\tilde{U}$ is conjugated by the root values and $D_r^F S$ is unchanged when they are trivial.

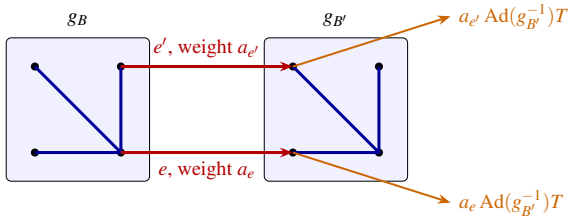
(iii) The tree links of F_o join sites of one block, so a pure gauge field $U = I^g$, $U_e = g_x g_{x+e}^{-1}$, with g constant on the blocks has trivial links on F_o, lies in the set of the supremum and has $Q(U) = 0$. By the proof of Proposition 406, $\partial_t^2 S(Ue^{tr})|_{t=0} = \sum_p \|(dr')_p\|_{\mathrm{op}}^2$ with $r'_e = \mathrm{Ad}(g_{x+e}^{-1}) r_e$, the end point of e taken. Since $a_3 = a_4 = 0$, a plaquette p at x in a plane $\{\mu,\nu\}$ with $\mu \in \{1,2\}$ and $\nu \in \{3,4\}$ has at most two tilted sides, $e = (x,\mu)$ with end point $x + e_\mu$ and $e' = (x + e_\nu, \mu)$ with end point $x + e_\mu + e_\nu$, and

$$\|(dr')_p\|_{\mathrm{op}}^2 = a_e^2 + a_{e'}^2 - 2a_e a_{e'} \langle \mathrm{Ad}(g_{x+e_\mu}^{-1})T, \mathrm{Ad}(g_{x+e_\mu+e_\nu}^{-1})T\rangle,$$

with the inner product of $\mathfrak{g}$ in which $\|T\| = 1$ and $a_e = 0$ on F_o. When $(x-o)_\nu$ is odd the two end points lie in different blocks. Take the block values g_B independent with Haar law. The adjoint representation of SU(2) is irreducible of dimension three, so $\mathbb{E}\,\mathrm{Ad}(g_B) = 0$, and the expectation of the inner product vanishes. Every link $e = (x,\mu) \notin F_o$ with $\mu \in \{1,2\}$ is, for each $\nu \in \{3,4\}$, a side of exactly one such plaquette, at x if $(x-o)_\nu$ is odd and at $x - e_\nu$ otherwise. Dropping the other plaquettes, whose terms are nonnegative,

$$\mathbb{E}_g\, \partial_t^2 S(I^g e^{tr})\big|_{t=0} \geq \sum_{\nu \in \{3,4\}} \sum_{e \notin F_o} a_e^2 = 2\sum_{e \notin F_o} a_e^2,$$

and the supremum over the section is at least the expectation. A link (x,μ) lies in F_o for one of the sixteen offsets o when $\mu = 1$



tree links (blue) are trivial in the section; the residual gauge group turns the tilt direction by one SU(2) element per block

FIG. 34. The covariant tilt on the block forest (Proposition 407). Transporting the direction along the trees keeps the Haar measures, but the section still contains every pure gauge field that is constant on the blocks. On a plaquette whose tilted sides e, e' end in different blocks the transported directions are turned by independent elements, and the average of $\|(dr')_p\|^2$ over them is $a_e^2+a_{e'}^2$ instead of the $(a_e-a_{e'})^2$ of the abelian energy.

and for two of them when $\mu=2$. Averaging over o and using $\|a_1\|^2=\|a_2\|^2=\frac{1}{2}\|a\|^2$ from the proof of Proposition 406,

$$\frac{1}{16}\sum_o\sum_{e\notin F_o}a_e^2=\frac{15}{16}\|a_1\|^2+\frac{14}{16}\|a_2\|^2=\frac{29}{32}\|a\|^2,$$

so one offset reaches $\frac{29}{32}\|a\|^2$; the last bound is the lower bound for $\|a\|^2$ of that proposition. □

Proposition 407 and Figure 34 show where the obstruction sits. The Haar translation survives gauge covariance along a forest, and for the block forest the constant it produces is at least $\frac{29}{16}\|a\|^2$, reached by fields with no action and no curvature at all; the chessboard estimate and the covariant tilt are therefore lost on the same fields, the gauge orbit of the vacuum, for different reasons. A bound of the form

$$\partial_t^2 S(\tilde U e^{tr})\le Q(r)+\mathscr{R}(\tilde U,r),$$

with $\mathscr{R}$ supported on the large-field region, is false for every forest section, since the left side exceeds $Q(r)$ by a fixed multiple of $\|a\|^2$ at $\tilde U=I^g$, where the large-field region is empty. A section that is smooth on the orbits, such as the Landau gauge, has no such pure gauge fields, but its Jacobian is a Faddeev–Popov determinant and not one, and part (ii) does not apply to it as it stands.[39]

Along the tuning $\beta_W\sim b\log(1/a)$ a cube of fixed physical side spans $m\sim\beta_W/b$ scales. Proposition 403 covers $O(\log\beta_W)$ of them, Propositions 406 and 407 cover $O(\log\beta_W)$ of them without the torus and uniformly in the exterior, and Corollary 405 covers all of them for the abelian theory. What the cube condition needs for SU(2) is a translation of the Haar measures whose second variation is the energy of the direction up to a constant independent of the scale, together with a non-perturbative treatment of the last $O(1)$ scales, where the running coupling is of order one and coarse defects are not rare at all. Proposition 407 shows that no translation transported along a forest does this: its constant is reached at zero action, with the growth 4^k of the energy of the smeared current under the residual gauge freedom, so the missing estimate is not a large-field remainder but a different section of the gauge orbits, with a Jacobian that enters the argument of part (ii). This is the content of Bałaban's large-field expansion [12], and it is not carried out here. The cube condition itself is a mixing condition of Dobrushin–Shlosman type, which for compact spins with finite range is equivalent to a logarithmic Sobolev inequality uniform in the volume [106]; the rarity of coarse defects is the part of that inequality that concerns the large fields.

[39] The Wilson measure satisfies no curvature-dimension condition at the values of β_W considered here, so the inequality cannot be obtained from a Bakry–Émery argument either: for one link equal to $-I$ with trivial neighbours the second variation of S along a unit direction on that link is -6, since each of its six plaquettes contributes $\partial_t^2(1+\cos t)=-1$, while the Ricci curvature of SU(2) with unit-speed geodesics e^{tr}, $\|r\|_{\mathrm{op}}=1$, is 2; the Hessian of $\beta_W S$ plus the curvature is negative as soon as $\beta_W>1/3$.

XIII. LABELS, ISLANDS AND THE ENVIRONMENT

The cube condition (D) is a statement about conditional laws under every exterior, and Sec. XII F showed that one-point rarity of a bad set gives closeness of Gibbs measures but not decay. The route that remains is the one of the large-field expansions: label each block of a window good or bad, prove decay on the good windows, bound the bad islands by their rarity, and control the covariance that the labels themselves carry. This section proves what each of these four steps gives for the cut-off average of Sec. XII B, and proves, by explicit examples, where the natural choices of labels fail. The statements are exact and finite; none of them closes (D), and Sec. XIII V lists what they leave.

Throughout, $G=\mathrm{SU}(2)$ with $\rho_G=1$, the lattice is in the tree gauge of Definition 328, $d(g,h)=\|\log(g^{-1}h)\|$ is the bi-invariant geodesic distance in the Hilbert–Schmidt norm, so that $d(g,I)=\|X\|$ for $g=e^X$ with $\|X\|\le\pi\sqrt{2}$, and δ is the chordal distance. A *pair* $k=(y,\mu)$ consists of a block and a direction; its *path link* is $\ell_k=U_{2y+e_\mu,\mu}$, its *private links* are the eight links leaving $B(y)$ in the direction μ, which enter no other cut-off average, and the seven private links other than ℓ_k are its *non-path* links. In the sixteen paths of $V_k^\sharp$ the path link occurs twice, as the two paths of transverse offset zero, and each non-path private link occurs twice, each time multiplied by one link inside $B(y)$ or $B(y+e_\mu)$. For a coarse field c the reconstruction $R(c)$ of Sec. XII B puts c_k on the private links of k and I on the links inside the blocks.

A. Backgrounds and the path links

The first question is which background the chart of Theorem 367 should be centred at when a window is conditioned on its labels. The natural label of a link is its distance to the reconstruction of the coarse field, and the natural background is the reconstruction itself. The first result shows that this is not enough: a coarse field all of whose plaquettes are large has

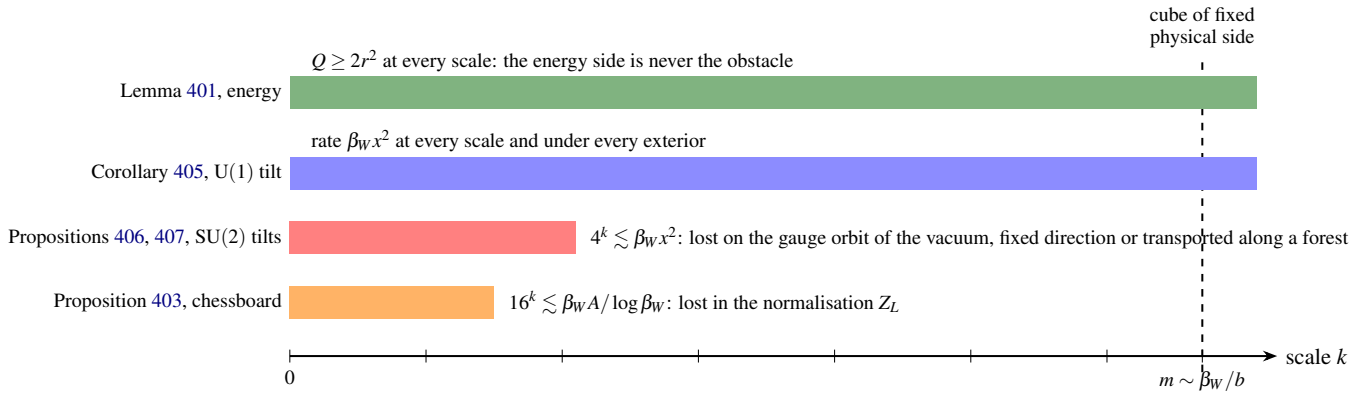


FIG. 35. Which scales the estimates of this subsection reach, along the tuning $\beta_W \sim b\log(1/a)$, where a cube of fixed physical side spans $m \sim \beta_W/b$ scales. The energy of a coarse defect is scale invariant (Lemma 401), and for the abelian theory the Haar tilt makes coarse defects rare at every scale (Corollary 405). For SU(2) the chessboard estimate stops after $O(\log\beta_W)$ scales because the normalisation charges the whole fluctuation volume of a block against the defect (Proposition 403), and the tilt stops after $O(\log\beta_W)$ scales because its second variation grows like 4^k on the gauge orbit of the vacuum, whether the direction is fixed or transported along a forest (Propositions 406, 407). The remaining scales are the open input of the cube condition.

a reconstruction at which every relative link label is good and the constrained action has a negative second variation along the fibre.

Proposition 408 (a rough background)**.** *On the periodic lattice of side four, with coarse torus $(\mathbb{Z}/2\mathbb{Z})^4$, let $c_{y,\mu} = (-I)^{\sum_{\nu<\mu} y_\nu}$. Then every coarse plaquette of c equals $-I$, the field $U = R(c)$ is in the tree gauge and has $V^\sharp(U) = c$, and $d(U_e, R_e(c)) = 0$ for every link. There is a curve $U(t) = U\exp(t\xi)$, ξ supported on four links, such that $V^\sharp(U(t)) = c$ for all small $|t|$, $d(U_e(t), R_e(c)) \leq |t|$, and the Wilson action $S = \sum_p (1 - \frac{1}{2}\operatorname{Re}\operatorname{Tr} U_p)$ satisfies*

$$\frac{d^2}{dt^2}\Big(\beta_W S(U(t))\Big)\Big|_{t=0} = -\beta_W, \qquad T(\xi) = \sum_e \|\xi_e\|^2 = 4.$$

Every first derivative of S at U vanishes, so the intrinsic Hessian of $\beta_W S$ on the fibre of c at U has an eigenvalue at most $-\beta_W/4$, in any smooth chart of the fibre.

Proof. The coarse plaquette at y in the plane $\{\mu,\nu\}$, $\mu<\nu$, is the product of $c_{y,\mu}$, $c_{y+e_\mu,\nu}$, $c^{-1}_{y+e_\nu,\mu}$ and $c^{-1}_{y,\nu}$; these are central, and the exponent of $-I$ is $\sum_{\rho<\mu} y_\rho + \sum_{\rho<\nu}(y+e_\mu)_\rho - \sum_{\rho<\mu}(y+e_\nu)_\rho - \sum_{\rho<\nu} y_\rho = 1$ modulo two, since $\mu<\nu$ puts e_μ inside the second sum and e_ν outside the third. So every coarse plaquette is $-I$, also across the periodic seam. In $U = R(c)$ the links inside the blocks are I, in particular the tree links, and all sixteen paths of a pair k equal c_k, so every $v_x = I$, the cut-off is one and $V^\sharp_k(U) = c_k$; the relative labels vanish by construction.

Let $T \in \mathfrak{su}(2)$ have $\|T\| = 1$, with eigenvalues $\pm i/\sqrt{2}$, and let $\xi_e = a_e T$ with $a_e = -1,+1,+1,-1$ on the links $((0,1,1,1),1)$, $((0,1,1,1),2)$, $((0,1,0,0),1)$, $((0,0,1,0),2)$ and $a_e = 0$ elsewhere. All four links are private links; the last two are path links. The pair $((0,0,0,0),1)$ has the non-path link $((0,1,1,1),1)$ with $a=-1$ and the path link $((0,1,0,0),1)$ with $a=+1$, the pair $((0,0,0,0),2)$ has $((0,1,1,1),2)$ with $a=+1$ and the path link $((0,0,1,0),2)$ with $a=-1$, and no other pair contains a link of the support. Since the background is central, $P_x(t) = c_k\exp(t a_x T)$ for the sixteen paths, with a_x the value on the private link of the path, so $v_x(t) = \exp(t(a_x - a_0)T)$ lies in the chart for small $|t|$, the cut-off is one, and

$$V^\sharp_k(U(t)) = c_k \exp\Big(\frac{t}{16}\sum_x a_x T\Big) = c_k,$$

because each private link occurs in exactly two paths and the two nonzero values of each pair cancel. The curve stays in the fibre, and $d(U_e(t), R_e(c)) = |t a_e| \leq |t|$.

Write $U_p = s_p I$ with $s_p = \pm 1$ and let $q_p = (d\,a)_p$ be the signed curl of the scalar field a around p. Since the perturbations commute with the background, $U_p(t) = s_p\exp(t q_p T)$ and $\operatorname{Re}\operatorname{Tr} U_p(t) = 2 s_p\cos(t q_p/\sqrt{2})$, so

$$S(U(t)) = \sum_p \big(1 - s_p\cos(t q_p/\sqrt{2})\big), \qquad S''(0) = \frac{1}{2}\sum_p s_p q_p^2.$$

The plaquettes with $q_p \neq 0$ are those containing a link of the support. A link (x,μ) lies in six plaquettes, two in each plane $\{\mu,\nu\}$, at x and at $x - e_\nu$. Listing the six plaquettes of each of the four links, the plaquette at $(0,1,1,1)$ in the plane $\{1,2\}$ contains two links of the support, $((0,1,1,1),1)$ and $((0,1,1,1),2)$, with $q_p = -1-1 = -2$; the other 22 plaquettes contain one link each, with $q_p = \pm 1$. The sign s_p of a plaquette is -1 exactly when it has an odd number of links on which U equals $-I$; counting through the four links, which is a finite check, gives twelve plaquettes with $s_p = 1$, $q_p^2 = 1$, ten with $s_p = -1$, $q_p^2 = 1$, and the one with $s_p = -1$, $q_p^2 = 4$. Hence $\sum_p s_p q_p^2 = 12 - 10 - 4 = -2$, $S''(0) = -1$, and $T(\xi) = 4$.

Every plaquette of U is central, so every first derivative of S at U vanishes and the second derivative along any curve through U with velocity ξ depends only on ξ: the intrinsic Hessian on the fibre has the value -1 at ξ, which has $T(\xi) = 4$.

The Haar density of the fibre, in a chart in which the path links are solved for, is the inverse Jacobian of the path-link equations, which at U is a fixed positive number independent of β_W; its logarithmic second derivatives are bounded in any fixed chart, so no choice of chart turns the Hessian of $\beta_W S$ plus the log-density positive for all large β_W. □

The proposition does not contradict Theorem 367, which assumes small plaquettes; it shows that the hypothesis of that theorem on the coarse field cannot be dropped in favour of relative link labels. A coarse-curvature label is therefore part of any labelling, and the next lemma shows that it has a witness at the fine scale, so that it is as rare as a fine defect.

Lemma 409 (a fine witness for coarse curvature)**.** *Let P be the coarse plaquette at y in the plane $\{\mu,\nu\}$ and suppose that $\delta(U_p,I)\le\kappa\le 10^{-8}$ for every fine plaquette p with a vertex in one of the blocks at ℓ^∞ distance at most one from $B(y)$, $B(y+e_\mu)$, $B(y+e_\nu)$ or $B(y+e_\mu+e_\nu)$. Then $c=V^\sharp(U)$ satisfies*

$$\delta(c_P,I)\le(4+28\pi)\kappa<92\kappa.$$

Proof. For a pair k among the four sides of P, Lemma 360(i) gives $\delta(U_e,I)\le 3\kappa$ for the link of each path inside a block and $\delta(U_e,\ell_k)\le 21\kappa$ for its private link, so $\delta(P_x,\ell_k)\le 24\kappa$ for every path, and by Lemma 352 $\|\log(\ell_k^{-1}P_x)\|\le 8\pi\kappa$. The two paths of offset zero contribute nothing, so $c_k=\ell_k M_k$ with $\|\log M_k\|\le\frac{14}{16}\,8\pi\kappa=7\pi\kappa$ and $\delta(c_k,\ell_k)\le 7\pi\kappa$. The four path links around P are the holonomy of the square of side two at $2y$: the other four links of its sides are the first links of the straight paths, which are tree links and equal I. That square is tiled by four fine plaquettes, and inserting them with their parallel transports gives $\delta(\ell_P,I)\le 4\kappa$, where ℓ_P is that holonomy. Replacing the four ℓ_k by c_k one at a time, bi-invariance of δ and the triangle inequality add $4\cdot 7\pi\kappa$. □

Remark 410 (flat fields far from the identity). Lemma 409 bounds coarse curvature, not the links, and no bound on the links in the tree gauge follows from flatness alone. On the periodic lattice of side $2L$ with blocks of side two, fix a direction μ, put $U_e=-I$ on every link in the direction μ that joins two different blocks, and $U_e=I$ on every other link. The tree links lie inside the blocks, so U is in the tree gauge. A plaquette in a plane containing μ has its two links in the direction μ at the same μ-coordinate, so both cross a block boundary or neither does, and every plaquette holonomy is I. Yet every crossing link is $-I$, the point of $\mathrm{SU}(2)$ farthest from I, every path of the cut-off average across a block boundary in the direction μ carries $-I$, and so does the coarse link. For odd L the Wilson line around the direction μ is $(-I)^L=-I$, which is central and hence gauge invariant, so no gauge transformation makes U trivial. A witness that forces a large plaquette near every link far from I in the tree gauge is therefore false; witnesses have to be taken relative to a reference such as the reconstruction of Lemma 411, as they are in this section. The example is due to Tristan Romanov.

With the coarse field calm, the chart has a centre. The next lemma shows that for a pair whose non-path links are near the reconstruction, the constraint $V_k^\sharp=c_k$ determines the path link, on every branch of the cut-off, and the fibre of the window over its non-path links is a single graph.

Lemma 411 (the path link is determined)**.** *Let k be a pair with cut-off average c_k, and suppose that each of its seven non-path private links and each link inside $B(y)$ and $B(y+e_\mu)$ is within geodesic distance r of its reconstruction value. Then every path other than the two copies of ℓ_k has $d(P_x,c_k)\le 2r$, and every ℓ_k with $V_k^\sharp=c_k$ satisfies*

$$d(\ell_k,c_k)\le 14r,\qquad d(v_x,I)\le 16r \text{ for every path.}$$

If moreover $r\le 10^{-4}$, the cut-off equals one at every such ℓ_k, and ℓ_k is unique: it is the only fixed point of

$$X\mapsto F(X)=-\frac{1}{16}\sum_x\log\bigl(e^{-X}c_k^{-1}P_x\bigr),\qquad \ell_k=c_k e^X,$$

in the ball $\|X\|\le 14r$, where the sum runs over the fourteen paths of nonzero offset, and F is a contraction of that ball with constant less than 0.9.

Proof. A path of nonzero offset is the product of a link inside a block, within r of I, and a non-path private link, within r of c_k, in one of the two orders; bi-invariance of d and the triangle inequality give $d(P_x,c_k)\le 2r$. If the cut-off is zero, $V_k^\sharp=\ell_k$, so $\ell_k=c_k$. Otherwise every $v_x=\ell_k^{-1}P_x$ has $\delta(v_x,I)<1$, its principal logarithm is defined and $\|\log v_x\|=d(v_x,I)=d(\ell_k,P_x)$. Put $D=d(\ell_k,c_k)$ and $N=\frac{1}{16}\sum_x\log v_x$, the two terms of offset zero being zero. Then $c_k=\ell_k\exp(\chi N)$ and

$$D=d(e^{\chi N},I)\le\chi\|N\|\le\frac{14}{16}\max_x d(\ell_k,P_x)\le\frac{7}{8}(D+2r),$$

so $D\le 14r$, and $d(v_x,I)\le D+2r\le 16r$. For $r\le 10^{-4}$ this is less than $\frac12$, so $\chi=1$, and $V_k^\sharp=c_k$ reads $X=F(X)$ for $\ell_k=c_k e^X$ with $\|X\|\le 14r$. On that ball each $A_x=c_k^{-1}P_x$ has $d(A_x,I)\le 2r$, so $d(e^{-X}A_x,I)\le 16r$ and $\|F(X)\|\le\frac{14}{16}\cdot 16r=14r$: F maps the ball into itself. Writing $e^{-X}A_x=I+M$ with $\|M\|\le 16r$, the series of the logarithm gives $\|D\log(I+M)\|\le 1/(1-16r)$, and the differential of $X\mapsto e^{-X}A_x$ has norm at most e^{14r}; hence $\|DF\|\le\frac78 e^{14r}/(1-16r)<0.9$, and the fixed point is unique by the contraction principle. □

Proposition 412 (good windows are graphs)**.** *Let W be a finite set of blocks, let c be a coarse field, let K_W be the set of pairs based in W or entering W from a block outside it, and let Z_W be the set of non-path active links with base in $\bigcup W$. Free the links of A_W and the path links of the pairs entering W, fix every other link, and condition on $V_k^\sharp=c_k$ for every $k\in K_W$ and on the event that every link of Z_W and every non-path link of a pair entering W is within $r\le 10^{-4}$ of its reconstruction value. Then the fibre of these constraints is the graph of a smooth map Γ from the product of the r-balls of the links of Z_W to the path links of the pairs of K_W, and the conditional law is the Wilson weight times the Haar disintegration of this graph, whose density with respect to the product Haar measure on Z_W is the product over $k\in K_W$ of the inverse Jacobians of $X\mapsto X-F(X)$ at the fixed point.*

Proof. The constraint of a pair $k \in K_W$ involves its own private links and the links inside two blocks, so it involves only its own path link, links of Z_W, and fixed links that are within r of their reconstruction values; no other constraint involves a freed link, since the other pairs have their supports off the free set. Lemma 411 applies to every $k \in K_W$ with the same r and gives one solution for each pair, a smooth function of the links of Z_W by the implicit function theorem since $I - DF$ is invertible. The fibre is therefore the graph of Γ, and the coarea formula for the submersion $U \mapsto (V_k^\sharp)_k$ restricted to the product of the balls gives the stated density, since the derivative of the constraint of k in its own path link is $I - DF$ in the chart at c_k and the cross derivatives between different pairs vanish. □

The proposition says that, once the coarse field is calm and the labels are placed on the non-path links, the good windows carry the product structure that the proof of Theorem 367 needs, with no branch of the cut-off to discard; the Hessian estimate of that theorem then applies on the graph. Two further examples show that the two obvious alternatives fail: labelling all physical links, including the path links, produces a nonconvex domain, and a bad pair can have several path links.

Proposition 413 (the all-link label domain is not convex)**.** *Let $c \equiv I$, freeze every link outside the eight private links of one pair at I, and write the seven non-path links as e^{z_j} and the path link as e^X. For every sufficiently small $r > 0$ the set of $(z_1, \dots, z_7)$ for which all eight links lie in the geodesic ball of radius r at I is not convex.*

Proof. With the cut-off equal to one, the constraint is $H(X,z) = X + \frac{1}{8}\sum_{j=1}^{7} \log(e^{-X}e^{z_j}) = 0$, since each non-path link occurs in two of the sixteen paths and the two paths of offset zero contribute nothing. Take $A, B \in \mathfrak{su}(2)$ orthonormal for the Hilbert–Schmidt norm, so that $[A,[A,B]] = -2B$ and $[B,[B,A]] = -2A$, and put $z_1 = -\frac{a}{7}A + hB$, $z_2 = -\frac{a}{7}A - hB$, $z_3 = \cdots = z_7 = -\frac{a}{7}A$ with $a > 0$ small. The inner automorphism of $\mathrm{SU}(2)$ that fixes A and negates B and $[A,B]$ exchanges the logarithms of $e^{-xA}e^{z_1}$ and $e^{-xA}e^{z_2}$, so their sum lies along A when $X = xA$, and the equation reduces to its A-component; by the implicit function theorem the solution is $X = x(a,h)A$ with x analytic, even in h by the same symmetry, and $x(a,0) = a$ since $\log(e^{-xA}e^{bA}) = (b-x)A$ gives $x + \frac{7}{8}(b-x) = 0$ at $h = 0$. Let $L(x,b,h)$ be the A-component of $\log(e^{-xA}e^{bA+hB})$. The Baker–Campbell–Hausdorff series through degree three has one term of degree two in h along A, namely $\frac{1}{12}[hB,[hB,-xA]] = \frac{xh^2}{6}A$, so $\partial_h^2 L(x,b,0) = \frac{x}{3} + O((|x|+|b|)^2)$. The A-component of H is $x + \frac{1}{8}\big(2L(x,-\frac{a}{7},h) + 5(-\frac{a}{7} - x)\big)$, and $\partial_x L(x,b,0) = -1$; differentiating $H = 0$ twice in h at $h = 0$ gives $\frac{1}{8}\partial_h^2 x + \frac{1}{4}\partial_h^2 L = 0$, that is

$$\partial_h^2 x(a,0) = -2\,\partial_h^2 L(a,-\tfrac{a}{7},0) = -\frac{2a}{3} + O(a^2) < 0.$$

Put $\eta = \frac{r}{2}$ and take a with $r < a < r(1 + \frac{\eta^2}{4})$. The points $z^\pm$ with $h = \pm\eta$ and their midpoint z^0 with $h = 0$ have all seven non-path links within $\sqrt{a^2/49 + \eta^2} < r$ of I; the path link of $z^\pm$ is at distance $x(a,\pm\eta) = a - \frac{a}{3}\eta^2 + O(r^4) \le r - \frac{r^3}{48} + O(r^4) < r$ from I for small r, while that of z^0 is at distance $x(a,0) = a > r$. So the endpoints lie in the domain and the midpoint does not. □

Proposition 414 (a bad pair with three path links)**.** *Let $c \equiv I$, freeze every link outside the eight private links of one pair at I, and take the private links in the one-parameter subgroup $\mathrm{diag}(e^{iq}, e^{-iq})$: one non-path link at the angle $-\frac{1}{2}$, the other six at the angle $\frac{49}{100}$. Then there are at least three values of the path-link angle q with the same cut-off average, and the derivative of the cut-off average with respect to q vanishes at some $q \in (0, \frac{1}{5})$.*

Proof. On the subgroup all logarithms are scalar and the cut-off average has the angle $f(q) = q + \frac{\chi(q)}{8}\big(-\frac{1}{2} + 6\cdot\frac{49}{100} - 7q\big)$, where $\chi(q) = \phi(m(q))$, $m(q) = \max_j 2\sqrt{2}\,|\sin((\theta_j - q)/2)|$ and $\theta_1 = -\frac{1}{2}$, $\theta_2 = \cdots = \theta_7 = \frac{49}{100}$, the factor $2\sqrt{2}|\sin(\theta/2)|$ being the chordal distance of $\mathrm{diag}(e^{i\theta}, e^{-i\theta})$ from I. At $q = 0$ the maximum is attained at θ_1 alone, with $m(q) = 2\sqrt{2}\sin((q + \frac{1}{2})/2)$ near 0 and $\frac{1}{2} < m(0) < 1$, so $\chi = 2 - 2m$ is smooth near 0 with $\chi'(0) = -2\sqrt{2}\cos\frac{1}{4}$, and

$$\begin{aligned} f'(0) &= 1 - \frac{7}{8}\chi(0) + \frac{\chi'(0)}{8}\Big(-\frac{1}{2} + \frac{294}{100}\Big) \\ &= -\frac{3}{4} + \sqrt{2}\Big(\frac{7}{2}\sin\frac{1}{4} - \frac{61}{100}\cos\frac{1}{4}\Big) \end{aligned}$$

after inserting $\chi(0) = 2 - 4\sqrt{2}\sin\frac{1}{4}$. With $\sin\frac{1}{4} < \frac{1}{4}$, $\cos\frac{1}{4} > \frac{31}{32}$ and $1 < \sqrt{2} < \frac{3}{2}$ on the positive and the negative term separately, $f'(0) < -\frac{91}{3200} < 0$. Put $\gamma = f(0)$, which lies in $(0,1)$. For $|q| = 1$ the maximal chordal distance exceeds one, so $\chi(\pm 1) = 0$ and $f(\pm 1) = \pm 1$. Since f decreases through γ at 0 and $f(-1) < \gamma < f(1)$, the intermediate value theorem gives one preimage of γ in $(-1,0)$, one in $(0,1)$, and $q = 0$. On $[0, \frac{1}{5}]$ the maximiser stays θ_1 and $\frac{9}{10} < m < 1$ at $q = \frac{1}{5}$, so $\chi(\frac{1}{5}) < \frac{1}{5}$ and $f'(\frac{1}{5}) > 1 - \frac{7}{40} - 3\cdot\frac{13}{100} > 0$; with $f'(0) < 0$, continuity gives a zero of f' in $(0, \frac{1}{5})$. □

Hence the Jacobian of the path-link constraint, which enters the density of Proposition 412, has no positive lower bound on the bad side, and a bad pair contributes a sum over branches to any estimate of the conditional law. On the good side none of this happens: the labels of Proposition 412 are placed on the non-path links only.

B. The path-link law on its two sides

The conditional law of the path link of a pair k, given its seven non-path private links $z = (\ell_a)_{a \neq 0}$, the links u, w inside its two blocks and the rest of the field σ, is

$$\pi(dg \mid z,u,w,\sigma) = \frac{1}{Z}\sum_{g\,:\,V(g;z,u,w) = c_k} \frac{e^{-\beta_W S(g;\cdot)}}{|\det D_g V(g;z,u,w)|}\,\delta_g, \tag{40}$$

where $V = V_k^\sharp$ as a function of the path link, the sum runs over all its preimages of c_k, and $|\det D_g V|$ is the Jacobian between tangent spaces of the bi-invariant metric. This is the

Haar disintegration along the Lipschitz map V on the compact three-manifold $\mathrm{SU}(2)$, by the coarea formula, times the Wilson weight; on the good side it is the one-point law of Proposition 412, and on the bad side it carries the branches of Proposition 414. We determine its response to each kind of exterior link. Throughout, W_1^{δ} and W_1^{d} are the Kantorovich distances for the chordal and the geodesic distance, and the facts $\|L_X\| \le \theta/\sin\theta$ for an eigenangle $\theta < \pi$ of $X/\sqrt{2}$, $\|E_Y\| \le 1$ and $d(e^A, e^B) \le \|A-B\|$ of Sec. XII D are used as stated there.

Lemma 415 (links that enter only the Wilson weight)**.** *Let e' be a link that is neither a private link of k nor inside $B(y)$ or $B(y+e_\mu)$, and let σ, σ' differ only at e'. Then for every (z,u,w) and c_k for which (40) is a finite nonzero measure,*

$$W_1^{\delta}\big(\pi(\cdot \mid z,u,w,\sigma), \pi(\cdot \mid z,u,w,\sigma')\big) \le \beta_W \, d(\sigma_{e'}, \sigma'_{e'}),$$

and the two laws coincide unless e' lies in a plaquette with a private link of k. For the law of all eight private links given u, w, σ and restricted to any fixed measurable set B of the non-path links, with the cost $\sum_a \delta(\ell_a, \ell'_a)$, the same holds with $8\beta_W$. Both bounds include every branch and the disintegration density.

Proof. Only S depends on $\sigma_{e'}$: V is a function of g, z, u, w, and so is the measure $\nu = \sum_g \delta_g / |\det D_g V|$, respectively $\mathbf{1}_B J_8^{-1} \mathrm{vol}_M$ with $M = \{V = c_k\}$ and J_8 the Jacobian of Lemma 371. An exterior link lies in at most one plaquette with a private link: the private links are the eight μ-links at the sites x'_a, whose μ-coordinates agree, so the two plaquettes in a plane $\{\mu, \nu\}$ through a ν-link, which sit at different μ-coordinates, cannot both contain one; and a μ-link (x, μ) with x not a site x'_a has $x + \varepsilon e_\nu = x'_a$ for at most one pair (ν, ε), since two such relations would put a coordinate of x outside $\{0,1\}$ in one direction and inside it in another. If there is no such plaquette p_0 the laws are equal. Otherwise move $\sigma_{e'}$ along a minimal geodesic $\sigma_{e'} e^{t\xi}$, $\|\xi\| = 1$, $0 \le t \le T = d(\sigma_{e'}, \sigma'_{e'})$, with laws π_t. For bounded f, $\frac{d}{dt}\pi_t(f) = -\beta_W \mathrm{Cov}_{\pi_t}(f, \dot{s}_{p_0})$, by differentiation under the integral against the fixed finite measure ν. Here $\frac{d}{dt} U_{p_0}(t) = U_{p_0}(t)\xi'$ with $\|\xi'\| = 1$, ξ' a conjugate of $\pm\xi$, so $\dot{s}_{p_0} = -\frac{1}{2}\mathrm{Re}\,\mathrm{Tr}(U_{p_0}\xi')$; in a basis in which $U_{p_0} = \mathrm{diag}(e^{i\varphi}, e^{-i\varphi})$ only the diagonal entries $\pm i\alpha$ of ξ' contribute, $2\alpha^2 \le 1$, and $\dot{s}_{p_0} = \alpha \sin\varphi$. Hence $|\dot{s}_{p_0}| \le \frac{1}{\sqrt{2}}$, and a link of a plaquette moving at geodesic speed v gives $|\dot{s}_p| \le v/\sqrt{2}$. A function that is 1-Lipschitz for δ has oscillation at most the chordal diameter $2\sqrt{2}$ of $\mathrm{SU}(2)$, and at most $8 \cdot 2\sqrt{2}$ for the sum cost on eight links. Since $|\mathrm{Cov}(f,h)| \le \frac{1}{4}\,\mathrm{osc}\, f\, \mathrm{osc}\, h$, $|\frac{d}{dt}\pi_t(f)| \le \beta_W$, respectively $8\beta_W$; integrate over $[0,T]$ and use the Kantorovich duality. □

This is Lemma 399 for the fibre law: the branches and the density are harmless for every link that enters only through the action. The links it leaves out are the seven non-path links, which are in the conditioning, and the inside links u_a, w_a, which enter V. On the good side both are harmless, and the radius of Lemma 411 and Proposition 412 can be taken as $1/32$ rather than 10^{-4}.

Proposition 416 (the good side to radius $1/32$)**.** *Let $r \le \frac{1}{32}$, and suppose $d(\ell_a, c_k) \le r$ for $a \ne 0$ and $d(u_a, I), d(w_a, I) \le r$. Put $\theta_0 = \frac{1}{2\sqrt{2}}$ and $\kappa_0 = \frac{7}{8}\theta_0/\sin\theta_0 < 0.8935$. Then every solution g of $V(g;z,u,w) = c_k$ has cut-off equal to one and $d(g, c_k) \le 14r$, and there is exactly one: writing $g = c_k e^X$, X is the unique fixed point of $F(X) = -\frac{1}{16}\sum_x \log(e^{-X} c_k^{-1} P_x)$ in $\{\|X\| \le 14r\}$, and F is a contraction of that ball with constant κ_0. At the solution the singular values of $D_g V$ lie in $[1-\kappa_0, 1+\kappa_0]$, so the density $1/|\det D_g V|$ lies in $[(1+\kappa_0)^{-3}, (1-\kappa_0)^{-3}] \subset [0.147, 828]$. The law (40) is the point mass at $g = g(z,u,w)$, and for two configurations satisfying the hypothesis*

$$d(g,g') \le 1.1986 \sum_{a \ne 0} d(\ell_a, \ell'_a) + 0.5993 \sum_{a \ne 0} \big(d(u_a, u'_a) + d(w_a, w'_a)\big),$$

with no dependence on β_W, on c_k or on any other link.

Proof. The first part is the proof of Lemma 411, which holds for every r up to the contraction: every solution has $D = d(g, c_k) \le 14r$ and $d(v_x, I) \le 16r \le \frac{1}{2}$, so $m \le \frac{1}{2}$ and $\chi = 1$, and $V = c_k$ reads $X = F(X)$, with F mapping the ball into itself. For the contraction constant, on the ball each $A_x = c_k^{-1} P_x$ has $d(A_x, I) \le 2r$, so $Y_x = \log(e^{-X} A_x)$ has $\|Y_x\| \le 16r$, an eigenangle at most $16r/\sqrt{2} \le \theta_0$. Since $e^{-(X+tZ)} A_x = (I - tE_{-X}Z + O(t^2)) e^{Y_x}$ and $\log(e^{tW} e^{Y}) = Y + tL_Y W + O(t^2)$, $\|DF\| \le \frac{14}{16} \cdot \|E_{-X}\| \max_x \|L_{Y_x}\| \le \kappa_0$, as $\theta/\sin\theta$ increases on $[0, \pi)$; the fixed point is unique by the contraction principle. For the Jacobian, vary $g \mapsto g e^{tZ}$: then $v_x \mapsto e^{-tZ} v_x$, $\log v_x \mapsto \log v_x - t L_{Y_x} Z + O(t^2)$, and with $N = \frac{1}{16}\sum_x \log v_x$ and $\bar{L} = \frac{1}{14}\sum_x L_{Y_x}$,

$$V^{-1} \frac{dV}{dt}\Big|_{t=0} = \mathrm{Ad}(e^{-N})\big(I - \tfrac{7}{8} E_N \bar{L}\big) Z,$$

by $\frac{d}{dt} e^{N+tW} e^{-N} = E_N W$. As $\mathrm{Ad}(e^{-N})$ is an isometry and $\|\frac{7}{8} E_N \bar{L}\| \le \kappa_0$, the singular values lie in $[1-\kappa_0, 1+\kappa_0]$. Finally, the law has one atom, so $W_1^d = d(g, g')$. Move one link at a time along a minimal geodesic; the geodesic ball of radius r is convex, so the hypothesis holds along the way, and $d(g,g') \le \|X - X'\|$. For $\ell_a \mapsto \ell_a e^{t\zeta}$, $\|\zeta\| = 1$, the two paths through ℓ_a move, $A_{x_a} \mapsto A_{x_a} e^{t\zeta}$ and $A_{x'_a} \mapsto A_{x'_a} e^{t\,\mathrm{Ad}(w_a^{-1})\zeta}$, and $\log(e^Y e^{t\zeta'}) = Y + tL_{-Y}\zeta' + O(t^2)$, so $\|\partial_t F\| \le \frac{2}{16}\theta_0/\sin\theta_0$ and, by the implicit function theorem, $\|dX/dt\| \le \frac{2}{16}\frac{\theta_0}{\sin\theta_0}\frac{1}{1-\kappa_0} < 1.1986$. For $u_a \mapsto u_a e^{t\zeta}$ one path moves, $A_{x_a} \mapsto A_{x_a} e^{t\,\mathrm{Ad}(\ell_a^{-1})\zeta}$, which gives half of it, and likewise for w_a. □

The only change from Lemma 411 is that the series bound $\|D\log(I+M)\| \le (1-16r)^{-1}$ is replaced by the exact norm $\theta/\sin\theta$ of L_Y. So on the good side the response of the path-link law to every exterior link is bounded, with a constant independent of β_W, and the bad side begins at $r = \frac{1}{32}$. On the bad side the response to the non-path links is not bounded at all. The configuration of Proposition 414 has a fold: a value c at which two branches are born, with infinite density, and the law jumps across it. When the paths of a pair lie on one torus,

the component of the cut-off average transverse to that torus has a closed form.

Lemma 417 (the transverse part of the cut-off average)**.** *Write* $\mathrm{SU}(2)$ *as the unit quaternions, with* $T=\{e^{iq}\}$*, and its Lie algebra as the pure quaternions, with the Euclidean norm* $|\cdot|$*. Let* $\theta_1,\dots,\theta_n$ *be real and* $\varpi_1,\dots,\varpi_n>0$*, and for* $g\in\mathrm{SU}(2)$ *put* $h_x=g^{-1}e^{i\theta_x}$*,* $\chi(g)=\phi(\max_x\delta(h_x,I))$ *and*

$$V(g)=g\exp Y(g),\qquad Y(g)=\chi(g)\sum_x\varpi_x\log h_x,$$

and $Y(g)=0$ *if* $\chi(g)=0$*. For real numbers* q *and* s *put* $g(q,s)=\cos s\,e^{iq}+\sin s\,k$*, and define* $\psi_x\in[0,\pi]$ *by* $\cos\psi_x=\cos s\cos(\theta_x-q)$*. Then the* j*-component of* $V(g(q,s))$ *is* 0 *and its* k*-component is* $\sin s\,\Xi(q,s)$*, where* $\Xi=1$ *if* $\chi=0$ *and otherwise*

$$\Xi=\cos|Y|-\frac{\sin|Y|}{|Y|}\,\chi\sum_x\varpi_x\,\psi_x\cot\psi_x,$$

with $\sin|Y|/|Y|=1$ *at* $Y=0$*. If* $\chi>0$*, then* $\psi_x<\frac{\pi}{4}$ *for every* x*. The function* Ξ *is continuous in* q*,* s *and the* θ_x*, and* $\Xi\le 1$*.*

Proof. Write a quaternion as $\alpha+\beta j$ with $\alpha,\beta\in\mathbb{C}=\mathbb{R}+\mathbb{R}i$. As $jz=\bar z j$,

$$(\alpha_1+\beta_1 j)(\alpha_2+\beta_2 j)=\alpha_1\alpha_2-\beta_1\bar\beta_2+(\alpha_1\beta_2+\beta_1\bar\alpha_2)j,$$

the inverse of a unit quaternion $\alpha+\beta j$ is $\bar\alpha-\beta j$, and βj has j-component $\Re\beta$ and k-component $\Im\beta$. Now $g=g(q,s)=\cos s\,e^{iq}+(i\sin s)j$, and with $\varphi_x=\theta_x-q$,

$$h_x=\cos s\,e^{i\varphi_x}-(i\sin s\,e^{-i\theta_x})j,$$

whose real part is $\cos\psi_x$ and whose pure part P_x has norm $\sin\psi_x$. If $\chi=0$, then $V=g$, whose j-part is $(i\sin s)j$. Let $\chi>0$. Then $\delta(h_x,I)=2(1-\cos\psi_x)^{1/2}<1$, so $\cos\psi_x>\frac34>\cos\frac{\pi}{4}$, and $\log h_x=w_xP_x$ with $w_x=\psi_x/\sin\psi_x$ ($w_x=1$ if $\psi_x=0$). Hence $Y=iy+\eta j$ with

$$y=\chi\cos s\sum_x\varpi_x w_x\sin\varphi_x,\qquad \eta=-i\chi\sin s\sum_x\varpi_x w_x e^{-i\theta_x},$$

and $\exp Y=(C+i\sigma y)+\sigma\eta j$ with $C=\cos|Y|$ and $\sigma=\sin|Y|/|Y|$. The j-part of $V=g\exp Y$ is βj with

$$\begin{aligned}\beta&=\cos s\,e^{iq}\sigma\eta+i\sin s\,(C-i\sigma y)\\&=i\sin s\Big(C-\sigma\chi\cos s\sum_x\varpi_x w_x e^{-i\varphi_x}-i\sigma y\Big).\end{aligned}$$

By the formula for y the imaginary part of the bracket vanishes, and $w_x\cos s\cos\varphi_x=w_x\cos\psi_x=\psi_x\cot\psi_x$, so the bracket is Ξ and $\beta=i\sin s\,\Xi$. Since $0<\psi\cot\psi\le 1$ on $[0,\frac{\pi}{4})$ and $\sigma\ge 0$, $\Xi\le C\le 1$. The cut-off χ is continuous, and so is Ξ where $\chi>0$; as $\chi\to 0$, $|Y|\le\frac{\pi}{4}\chi\sum_x\varpi_x\to 0$ and the subtracted term is at most $\chi\sum_x\varpi_x$, so $\Xi\to 1$. □

Proposition 418 (the fold of the path-link law)**.** *In the configuration of Proposition 414, with every link other than the eight private links equal to* I*, let* $c_\lambda=\mathrm{diag}(e^{i\lambda},e^{-i\lambda})$ *and let* π_λ *be the law (40) with* $c_k=c_\lambda$*. Let* f_0 *be the angle of the cut-off average on the torus, the function* f *of Proposition 414, let* $q^*\in(0.0741,0.0742)$ *be its minimiser on* $[-0.005,0.2227]$*,* $\gamma^*=f_0(q^*)\in[0.16978261,0.16978263]$*,* $g^*=e^{iq^*}$*, and* $\Lambda=[\gamma^*-10^{-3},\gamma^*+10^{-3}]$*. Then:*

(i) *For* $\lambda\in\Lambda$ *every preimage of* c_λ *lies on the torus. For* $\lambda<\gamma^*$ *there is one,* $g_L(\lambda)=e^{iq_L(\lambda)}$ *with* $q_L\in[-0.0172,-0.0156]$*; for* $\lambda>\gamma^*$ *there are three,* $g_L(\lambda)$ *and* $g_\pm(\lambda)=e^{iq_\pm}$ *with* $q_-<q^*<q_+$ *and* $|q_\pm-q^*|\le\sqrt{(\lambda-\gamma^*)/2.3}$*; for* $\lambda=\gamma^*$ *there are two,* g_L *and* g^**. Off* γ^* *the value* c_λ *is regular.*

(ii) *The weight* $e^{-\beta_W S}/|\det D_gV|$ *of* $g_L(\lambda)$ *is at most* 2.78*, and the weights of* $g_\pm(\lambda)$ *tend to infinity as* $\lambda\downarrow\gamma^*$*, for every* $\beta_W\ge 0$*.*

(iii) $\pi_\lambda\to\delta_{g^*}$ *as* $\lambda\downarrow\gamma^*$ *and* $\pi_\lambda\to\delta_{g_L^*}$ *as* $\lambda\uparrow\gamma^*$*,* $g_L^*=g_L(\gamma^*)$*, weakly, with* $\delta(g^*,g_L^*)\ge 0.1268$*. Hence for every* $\beta_W\ge 0$ *and* $\varepsilon>0$ *there are* $\lambda<\gamma^*<\lambda'$ *with* $\lambda'-\lambda<\varepsilon$ *and* $W_1^\delta(\pi_\lambda,\pi_{\lambda'})\ge 0.126$*.*

(iv) *Fix* $c=c_{\gamma^*}$ *and move the seven non-path links together,* $\ell_a\mapsto\ell_a e^{it}$*, with laws* $\pi^{(t)}$*. Then* $\pi^{(t)}\to\delta_{g^*}$ *as* $t\uparrow 0$ *and* $\pi^{(t)}\to\delta_{g_L^*}$ *as* $t\downarrow 0$*. So for every* $\beta_W\ge 0$ *and* $\varepsilon>0$ *there are two configurations of the non-path links at total distance* $7\sqrt2\,(t'-t)<10\varepsilon$ *whose path-link laws are at* W_1^δ *distance at least* 0.126*: the Kantorovich response of the path-link law to its non-path links is infinite, for every* β_W*.*

Proof. Write $g=g_0+g_1i+g_2j+g_3k$ as a unit quaternion, so that the torus T is $\{e^{iq}\}$, and θ_a for the angles of the non-path links, $-\frac12$ once and $\frac{49}{100}$ six times. Then $V(g)=g\exp(\chi(m)\frac18\sum_{a\ne0}\log(g^{-1}\ell_a))$ with $m=\max_a\delta(g^{-1}\ell_a,I)$, since each ℓ_a lies on two paths and $u=w=I$, and the Wilson weight of the path link is $e^{-\beta_W S(g)}$ with $S(g)=\frac34[\delta(g,I)^2+\delta(g,e^{0.49i})^2]\in[0,12]$, the three plaquettes through the path link and its partner and the three through the path link and I.

Step 1, the torus. For $q\in[q_{lo},q_{hi}]=[\frac{49}{100}-\alpha,-\frac12+\alpha]\approx[-0.2327,0.2227]$, where $\alpha=\arccos\frac34$, the maximiser of $\delta(g^{-1}\ell_a,I)$ at $g=e^{iq}$ is the far link when $q\ge-0.005$ and the six near links when $q\le-0.005$, $m=2\sqrt2\sin(d(q)/2)\in(\frac12,1)$ with $d(q)=\max\{q+\frac12,\frac{49}{100}-q\}$, and

$$f_0(q)=q+\frac{\chi(q)}{8}\Big(\frac{244}{100}-7q\Big),\qquad \chi(q)=2-4\sqrt2\sin\frac{d(q)}{2};$$

outside $[q_{lo},q_{hi}]$ some $\delta(g^{-1}\ell_a,I)\ge1$, $\chi=0$ and $f_0(q)=q$. On $[q_{lo},-0.005]$, $\chi'=2\sqrt2\cos\frac{0.49-q}{2}>0$ and $\frac{244}{100}-7q>0$, so, as $\chi(-0.005)=2-4\sqrt2\sin0.2475<0.6146$,

$$f_0'\ge1-\tfrac78\chi\ge1-\tfrac78\cdot0.6146>0.46:$$

f_0 increases from q_{lo} to $f_0(-0.005)=0.18501\dots$. On $[-0.005,q_{hi}]$,

$$\chi'=-2\sqrt2\cos\tfrac{q+1/2}{2}<0,\qquad \chi''=\sqrt2\sin\tfrac{q+1/2}{2}>0,$$

so $f_0'' = \frac{1}{8}\chi''(\frac{244}{100} - 7q) - \frac{7}{4}\chi' > 0$: f_0 is strictly convex, with $f_0'(-0.005^+) = -0.3858\ldots < 0$ and $f_0'(q_{hi}) = 0.7087\ldots > 0$, a unique minimiser q^*, and $f_0'(0.0741) < 0 < f_0'(0.0742)$ places it, while convexity gives $f_0(0.0741) + 10^{-4} f_0'(0.0741) \le \gamma^* \le \min\{f_0(0.0741), f_0(0.0742)\}$, the stated interval. For $\lambda \in \Lambda \subset (0.168, 0.171)$ a torus preimage e^{iq} with $q \notin [q_{lo}, q_{hi}]$ would need $q \equiv \lambda$ modulo 2π, impossible. So the torus preimages are: one root q_L in $[q_{lo}, -0.005]$, which lies in $[-0.0172, -0.0156]$ because f_0 is increasing there and $f_0(-0.0172) < \gamma^* - 10^{-3}$, $f_0(-0.0156) > \gamma^* + 10^{-3}$; and in $[-0.005, q_{hi}]$ none if $\lambda < \gamma^*$, the point q^* if $\lambda = \gamma^*$, and two if $\lambda > \gamma^*$, as $f_0(-0.005)$ and $f_0(q_{hi}) = q_{hi}$ exceed λ. These two lie in $[0.05, 0.1]$, where $f_0'' > 4.6$, since $f_0(0.05), f_0(0.1) > \gamma^* + 10^{-3}$, whence $\lambda - \gamma^* \ge \frac{4.6}{2}(q_\pm - q^*)^2$. The values of f_0, f_0' and f_0'' used in this step, and $\alpha \in (0.7227, 0.7228)$, are evaluations with sin and cos replaced by their Taylor polynomials of degree ten, whose remainders are below 10^{-9} for arguments in $[0, 0.73]$, and with $1.41421356 < \sqrt{2} < 1.41421357$; on $[0.05, 0.1]$ the factors χ'', $-\chi'$ and $\frac{244}{100} - 7q$ of f_0'' are monotone and positive, and their values at the ends give $f_0'' > 4.8$.

Step 2, off the torus. Conjugation by $h \in T$ fixes every ℓ_a and c_λ, so $V(hgh^{-1}) = hV(g)h^{-1}$ and the preimages of c_λ are invariant under it; every $g \notin T$ is conjugate to $g(q,s) = \cos s\, e^{iq} + \sin s\, k$ with $s \in (0, \frac{\pi}{2}]$. The map V is that of Lemma 417 with the seven angles θ_a and $\varpi_a = \frac{1}{8}$, so $V_j = 0$ and $V_k(g(q,s)) = \sin s\, \Xi(q,s)$. Where $\chi > 0$, $\cos s \cos(\theta_a - q) > \frac{3}{4}$ for every a, so $\cos(\theta_a - q) > 0$ and ψ_a is at least the distance of $\theta_a - q$ to $2\pi\mathbb{Z}$. These distances for the far link and for a near link add up to at least $\frac{99}{100}$, so $\max_a \psi_a \ge \frac{99}{200}$, $m \ge 2\sqrt{2}\sin\frac{99}{400}$ and, by $\sin x \ge x - \frac{x^3}{6}$ and $4\sqrt{2} > 5.6568$, $\chi \le 2 - 2m < 0.6143$. Then $|Y| \le \frac{\chi}{8}\sum_a \psi_a < 0.6143 \cdot \frac{7\pi}{32} < 0.4223$ and $\cos|Y| \ge 1 - \frac{1}{2}|Y|^2 > 0.9108$, while $\sin|Y|/|Y| \le 1$ and $0 < \psi_a \cot\psi_a \le 1$ bound the subtracted term of Ξ by $\frac{7}{8}\chi < 0.5376$. Hence $\Xi > \frac{1}{3}$ at every point, also where $\chi = 0$ and $\Xi = 1$, and $V_k(g(q,s)) > 0$ for $s \in (0, \frac{\pi}{2}]$: $V(g) \notin T$ for $g \notin T$. So $V^{-1}(T) = T$, which with Step 1 gives (i) except for regularity.

Step 3, the Jacobian at a torus point. $D_g V$ at $g = e^{iq}$ commutes with the differential of conjugation by $e^{i\varphi}$, which fixes the axial line $g \cdot i\mathbb{R}$ and rotates the transverse plane by 2φ; a linear map commuting with all these rotations preserves the line and the plane and is a complex scalar $\lambda_\perp e^{i\omega}$ on the plane. The axial eigenvalue is $f_0'(q)$, and $\lambda_\perp = \|\partial_s V(g(q,s))|_{s=0}\| = |\partial_s V_k(q,0)|$, since $\partial_s g = k$ is a unit transverse vector and $V_j = 0$. So $|\det D_g V| = |f_0'(q)|\lambda_\perp(q)^2$, and by Step 2, as Ξ is continuous, $\lambda_\perp(q) = \Xi(q,0) \in (\frac{1}{3}, 1]$. At $s = 0$ and $q \in [q_{lo}, q_{hi}]$, $\psi_a = |\theta_a - q|$ and $|Y| = \Delta(q) := \frac{\chi}{8}(\frac{244}{100} - 7q) = f_0(q) - q$, so

$$\lambda_\perp(q) = \cos\Delta - \frac{\sin\Delta}{\Delta}\,\frac{\chi}{8}\sum_a (\theta_a - q)\cot(\theta_a - q).$$

For $q \in [-0.0172, -0.0156]$ the near links are the maximiser, with $d = \frac{49}{100} - q \in [0.5056, 0.5072]$ and $\chi = 2 - 4\sqrt{2}\sin\frac{d}{2}$, so $0.07067 < \frac{\chi}{8} < 0.07315$ by $x - \frac{x^3}{6} \le \sin x \le x$ and $5.6568 < 4\sqrt{2} < 5.65686$. The partial fractions $x\cot x = 1 - 2\sum_{n\ge1} x^2/(n^2\pi^2 - x^2)$, with $\frac{1}{1-u} \ge 1 + u$, $\sum_n n^{-2} = \frac{\pi^2}{6}$ and $\sum_n n^{-4} = \frac{\pi^4}{90}$, give $x\cot x \le 1 - \frac{x^2}{3} - \frac{x^4}{45}$ for $0 < |x| < \pi$. The far link has $|\theta_a - q| \ge 0.4828$ and the near ones $|\theta_a - q| \ge 0.5056$, so $\sum_a(\theta_a - q)^2 > 1.7668$, $\sum_a(\theta_a - q)^4 > 0.4464$, and the subtracted term is at most $\frac{\sin\Delta}{\Delta}$ times $0.07315\,(7 - 0.5889 - 0.0099) < 0.4683$. Further $\Delta \in [0.1801, 0.1873]$, $\cos\Delta \ge 1 - \frac{1}{2}\Delta^2 > 0.9824$ and $\frac{\sin\Delta}{\Delta} \le 1 - \frac{1}{6}\Delta^2 + \frac{1}{120}\Delta^4 < 0.99461$, so $\lambda_\perp > 0.9824 - 0.99461 \cdot 0.4683 > 0.5166$ on this interval. At q_L the near links are the strict maximiser and at $q_\pm$ the far link is, with $m \in (\frac{1}{2}, 1)$, so V is real-analytic near these points; $f_0'(q_L) \in [1.36, 1.37]$ by the formula, $f_0'(q_\pm) \ne 0$ by strict convexity and $\lambda_\perp > 0$, so c_λ is regular for $\lambda \ne \gamma^*$, and (40) is the continuous disintegration there. The weight of g_L is at most $1/(1.36 \cdot 0.5166^2) < 2.78$, and those of $g_\pm$ are at least $e^{-12\beta_W}/|f_0'(q_\pm)|$, as $\lambda_\perp \le 1$, with $f_0'(q_\pm) \to f_0'(q^*) = 0$. This is (ii).

Step 4, the limits. Let $f(g) = \delta(g, g^*)$, 1-Lipschitz for δ and at most $2\sqrt{2}$. For $\lambda \downarrow \gamma^*$, $\pi_\lambda(f) \le 2.78 \cdot 2\sqrt{2}/(w_- + w_+) + \sqrt{2}\max_\pm |q_\pm - q^*| \to 0$, and as π_λ lives on three points converging to $\{g^*, g_L^*\}$, $\pi_\lambda \to \delta_{g^*}$. For $\lambda \uparrow \gamma^*$, $\pi_\lambda = \delta_{g_L(\lambda)} \to \delta_{g_L^*}$. For $\lambda \in \Lambda$, $q^* - q_L(\lambda) \in [0.0897, 0.0914]$, so $\pi_\lambda(f) = 2\sqrt{2}\sin\frac{q^* - q_L}{2} \ge 0.1268$, and the Kantorovich duality $W_1^\delta(\pi_\lambda, \pi_{\lambda'}) \ge |\pi_\lambda(f) - \pi_{\lambda'}(f)|$ gives (iii).

Step 5, the exterior form. Multiplying every path on the right by e^{it} gives $V(ge^{it}; z_t) = V(g; z_0)e^{it}$, the covariance of $V^\sharp$ with $h_1 = I$, $h_2 = e^{-it}$. So g solves $V(g; z_t) = c_{\gamma^*}$ iff $g' = ge^{-it}$ solves $V(g'; z_0) = c_{\gamma^* - t}$, with the same Jacobian, and the action at $g = g'e^{it}$ under z_t differs from $S(g')$ by at most $6|t|$. Hence $\pi^{(t)}$ is the image of $\pi_{\gamma^* - t}$ under the translation by e^{it}, which moves every point by $\sqrt{2}|t|$, with weights multiplied by factors in $[e^{-6\beta_W|t|}, e^{6\beta_W|t|}]$, and the limits of Step 4 persist as $t \to 0$; each non-path link moves by $\sqrt{2}|t' - t|$. This is (iv). □

The mechanism is the one Proposition 414 exhibits: at the fold the density $1/|\det D_g V|$ is infinite, two branches carry all the mass on one side and do not exist on the other, and the Wilson weight cannot repair it, since a vanishing Jacobian wins against any fixed $e^{-\beta_W S}$. The configuration is bad for every $r \le \frac{1}{32}$. So item (i) of Sec. XIII V in its original form, a Lipschitz bound for the law (40) in the non-path links, is false; what can be asked is the response of the law of all eight private links of a bad pair, in which the non-path links are integrated over the bad set and Lemma 371 removes both the branches and the density. Lemma 415 settles that response for every link that enters only the action; what remains is its response to the inside links u_a, w_a, which move the fibre $\{V = c_k\}$ while the bad set does not, and the question is the mass that the law gives to the set where the fibre is tangent to the boundary of the bad set.

The site of a bad pair. The response of the eight private links to the inside links u_a, w_a is avoided by putting those links into the variables of the law. Let $k = (y, \mu)$, $k' = (y - e_\mu, \mu)$ and $k'' = (y + e_\mu, \mu)$. The inside links of k' in $B(y)$ are the links u_a of k, and the inside links of k'' in $B(y + e_\mu)$ are the links w_a of k: consecutive pairs along μ share their inside μ-links. The *site* of k is the set $\mathscr{S}_k$ of the 24 links ℓ_a, u_a, w_a ($a \ne 0$ for the last two) and the path links ℓ_0', ℓ_0'' of k' and k'', with the cost $\sum_{e \in \mathscr{S}_k} \delta(\cdot, \cdot)$. Its links are based in $B(y - e_\mu) \cup B(y) \cup B(y + e_\mu)$, and the cut-

off averages that contain a link of $\mathscr{S}_k$ are $V_k^\sharp$, which contains site links only, $V_{k'}^\sharp$, whose other links are the non-path links z' of k' and its inside links u' in $B(y-e_\mu)$, and $V_{k''}^\sharp$, with z'' and w'' in $B(y+2e_\mu)$. Every other link enters the law of the site only through the Wilson weight.

Lemma 419 (the law of a site)**.** *Let $r \le \frac{1}{32}$, suppose $d(\ell'_a, c_{k'})$, $d(u'_a, I)$, $d(\ell''_a, c_{k''})$, $d(w''_a, I) \le r$ for $a \ne 0$, and let E be a measurable set of site variables contained in $\{d(u_a,I) \le r,\ d(w_a,I) \le r,\ a \ne 0\}$. Then the law of the site given the exterior σ, the three constraints $V_k^\sharp = c_k$, $V_{k'}^\sharp = c_{k'}$, $V_{k''}^\sharp = c_{k''}$ and E is*

$$\frac{1}{Z}\, e^{-\beta_W S}\mathbf{1}_E\, J_{22}^{-1}\mathrm{vol}_{M_k}(d\ell\, du\, dw)\, \frac{\delta_{\Gamma'}(d\ell'_0)}{|\det D_{\ell'_0} V_{k'}^\sharp|}\, \frac{\delta_{\Gamma''}(d\ell''_0)}{|\det D_{\ell''_0} V_{k''}^\sharp|},$$

where $M_k = \{V_k^\sharp = c_k\}$ in the 22 variables (ℓ, u, w), $J_{22} = \det(DV_k^\sharp DV_k^{\sharp *})^{1/2} \ge \sigma_0^3$ almost everywhere, $\Gamma' = \Gamma'(z', u', u)$ and $\Gamma'' = \Gamma''(z'', w'', w)$ are the unique solutions of Proposition 416 for k' and k'', and the two determinants lie in $[(1-\kappa_0)^3, (1+\kappa_0)^3]$.

Proof. Only the three named constraints involve site variables. For k' the hypothesis of Proposition 416 holds, its inside links in $B(y)$ being the u_a, which E keeps within r of I; so $\ell'_0 \mapsto V_{k'}^\sharp$ has the single regular preimage Γ', the Haar disintegration of ℓ'_0 along it is the atom divided by the determinant, and the determinant bounds are those of the proposition; likewise for k''. The Haar disintegration of (ℓ, u, w) along $V_k^\sharp$ is $J_{22}^{-1}\mathrm{vol}_{M_k}$ by the coarea formula. Since $DV_k^\sharp DV_k^{\sharp *}$ is the sum over the variables of the nonnegative terms $D_e V_k^\sharp (D_e V_k^\sharp)^*$, it dominates the sum over the eight private links, which is at least σ_0^2 by Lemma 371. The product of the three disintegrations is the disintegration of the product Haar measure along the three constraints, and the Wilson weight and $\mathbf{1}_E$ multiply it. □

The constraint of the site's own pair is thus a fixed measure on $M_k \cap E$: the exterior enters only through S, through the positions Γ', Γ'' of two atoms, and through two densities on the good side, whose variation the next lemma bounds.

Lemma 420 (the good-side density is log-Lipschitz)**.** *Under the hypothesis of Proposition 416, let one link of the pair move along a unit-speed geodesic. Then $\left|\frac{d}{dt}\log|\det D_g V|\right| \le 64.94$ when the link is a non-path link and ≤ 32.47 when it is an inside link.*

Proof. By the proof of Proposition 416, $|\det D_g V| = |\det T|$ with $T = I - \frac{7}{8}E_N\bar{L}$, so $\frac{d}{dt}\log|\det T| = \mathrm{tr}(T^{-1}\dot{T})$, which is at most $3\|T^{-1}\|\|\dot{T}\|$ on the three-dimensional $\mathfrak{g}$, with $\|T^{-1}\| \le (1-\kappa_0)^{-1}$ and $\dot{T} = -\frac{7}{8}(\dot{E}_N\bar{L} + E_N\dot{\bar{L}})$, $\|\bar{L}\| \le \Lambda_0 := \theta_0/\sin\theta_0 = 1.02115$, $\|E_N\| \le 1$.

For a power series φ whose radius of convergence exceeds the spectral radius of a normal operator A with spectral projections P_i and eigenvalues λ_i, the derivative is $d\varphi(A)[B] = \sum_{i,j}\varphi[\lambda_i,\lambda_j]P_iBP_j$, with the divided differences $\varphi[\lambda_i,\lambda_j]$ of Daleckii–Krein [41]; the blocks P_iBP_j are orthogonal for the Hilbert–Schmidt product, so $\|d\varphi(A)[B]\| \le \max_{i,j}|\varphi[\lambda_i,\lambda_j]|\|B\|_{HS}$, and by the Hermite–Genocchi formula $\varphi[\lambda_i,\lambda_j] = \int_0^1 \varphi'((1-u)\lambda_i + u\lambda_j)\,du$ this maximum is at most the maximum of $|\varphi'|$ on the convex hull of the spectrum. With $E_Y = e(\mathrm{ad}\,Y)$, $L_Y = f(\mathrm{ad}\,Y)$, $e(x) = (e^x-1)/x$, $f(x) = x/(e^x-1)$, and $\|\mathrm{ad}\,W\|_{HS} = 2\|W\|$, this gives $\|\dot{L}_Y\| \le 2M_L\|\dot{Y}\|$ and $\|\dot{E}_N\| \le 2M_E\|\dot{N}\|$, where M_L and M_E bound $|f'|$ and $|e'|$ on the segments $i[-a', a']$, $a' \le \sqrt{2}\cdot 16r \le \frac{\sqrt{2}}{2}$, and $i[-b', b']$, $b' \le \sqrt{2}\cdot 14r \le 0.6188$, which contain the spectra $\{0, \pm ia'\}$ of $\mathrm{ad}\,Y_x$ and $\{0, \pm ib'\}$ of $\mathrm{ad}\,N$. Since $e'(z) = \int_0^1 te^{tz}\,dt$, $|e'(iy)| \le \frac{1}{2}$, so $M_E = \frac{1}{2}$. Since $f'(iy) = -\frac{1}{2} - i\frac{\sin y - y}{4\sin^2(y/2)}$, and $y - \sin y \le \frac{y^3}{6}$, $\sin\frac{y}{2} \ge \frac{y}{2}(1-\frac{y^2}{24})$, $M_L \le \left(\frac{1}{4} + \frac{a^2}{36}(1-\frac{a^2}{24})^{-4}\right)^{1/2} \le 0.5149$ at $a = \frac{\sqrt{2}}{2}$.

Along the motion the fixed point X moves with $\|\dot{X}\| \le \dot{x}$, $\dot{x} = 1.1986$ for a non-path link and 0.5993 for an inside link, by the proof of Proposition 416. For a path that does not contain the moving link, $\dot{Y}_x = -L_{Y_x}E_{-X}\dot{X}$, so $\|\dot{Y}_x\| \le \Lambda_0\dot{x}$; for a path that contains it, $\log(e^Y e^{t\zeta'}) = Y + tL_{-Y}\zeta' + O(t^2)$ adds Λ_0. With $n=2$ moving paths for a non-path link and $n=1$ for an inside link, $\|\dot{N}\| \le \frac{\Lambda_0}{16}(14\dot{x}+n)$ and $\|\dot{\bar{L}}\| \le 2M_L\Lambda_0(\dot{x}+\frac{n}{14})$. Hence

$$\left|\frac{d}{dt}\log|\det T|\right| \le \frac{21}{8}\,\frac{2M_E\Lambda_0\|\dot{N}\| + \|\dot{\bar{L}}\|}{1-\kappa_0},$$

which is at most 64.94 and 32.47 in the two cases, with $(1-\kappa_0)^{-1} \le 9.390$. □

Theorem 421 (the site of a bad pair responds boundedly)**.** *Under the hypotheses of Lemma 419, let σ, σ' differ only at one link e' outside $\mathscr{S}_k$, and let $\pi(\cdot \mid \sigma)$ be the law of the site of that lemma. Then*

$$W_1\big(\pi(\cdot \mid \sigma), \pi(\cdot \mid \sigma')\big) \le C_{e'}\, d(\sigma_{e'}, \sigma'_{e'}),$$

for the cost $\sum_{e\in\mathscr{S}_k}\delta$, with $C_{e'} = 24\,n_p(e')\beta_W$ if e' enters only the Wilson weight, $n_p(e') \le 6$ the number of its plaquettes that contain a site link, $C_{e'} = 197\beta_W + 2206$ if e' is a non-path link of k' or k'', and $C_{e'} = 87\beta_W + 1103$ if e' is an inside link of k' in $B(y-e_\mu)$ or of k'' in $B(y+2e_\mu)$. The constants do not depend on c_k, $c_{k'}$, $c_{k''}$, E or the rest of σ, and every branch of $V_k^\sharp = c_k$ inside the site is integrated, not compared.

Proof. Move $\sigma_{e'}$ along a unit-speed geodesic, $0 \le t \le T = d(\sigma_{e'}, \sigma'_{e'})$; geodesic balls of radius $r \le \frac{1}{32}$ are convex, so the hypotheses of Lemma 419 hold along the way. By that lemma, $\pi_t = (\Phi_t)_*\mu_t$ with $\Phi_t(\ell, u, w) = (\ell, u, w, \Gamma'_t, \Gamma''_t)$ and μ_t the law on $M_k \cap E$ with density $h_t = e^{-\beta_W S_t} J_t'^{-1} J_t''^{-1}$ against the fixed measure $\mathbf{1}_E J_{22}^{-1}\mathrm{vol}_{M_k}$; plaquettes without a site link are constant on M_k and cancel. For f Lipschitz with constant one for the site cost, whose oscillation is at most the diameter $24\cdot 2\sqrt{2}$,

$$\pi_T(f) - \pi_0(f) = (\mu_T - \mu_0)(f\circ\Phi_T) + \mu_0(f\circ\Phi_T - f\circ\Phi_0).$$

The second term is at most $\dot{x}T$ by Proposition 416, since one atom moves, and it vanishes when e' enters only the Wilson weight. For the first, $\frac{d}{dt}\mu_t(\varphi) = \mathrm{Cov}_{\mu_t}(\varphi, \partial_t \log h_t)$ and $|\mathrm{Cov}(\varphi,\psi)| \le \frac{1}{4}\,\mathrm{osc}\,\varphi\,\mathrm{osc}\,\psi$, so the first term is at most

$24\sqrt{2}\,T\sup|\partial_t \log h_t|$, and $|\partial_t \log h_t| \le \beta_W|\partial_t S_t| + |\partial_t \log J'_t|$. Links of a plaquette moving at total geodesic speed v give $|\dot s_p| \le v/\sqrt{2}$, by the proof of Lemma 415. If e' enters only the Wilson weight, $\operatorname{osc}\partial_t S_t \le \sqrt{2}\,n_p(e')$ and J' does not move, which gives $24 n_p\beta_W$. If $e' = \ell'_a$, it shares at most one plaquette with a site link, and the atom Γ' moves at speed at most 1.1986 through its six plaquettes, so $|\partial_t S_t| \le (1+6\cdot 1.1986)/\sqrt{2} \le 5.793$, and Lemma 420 bounds the density term by 64.94; hence $C_{e'} \le 24\sqrt{2}(5.793\beta_W + 64.94) + 1.1986 < 197\beta_W + 2206$. If $e' = u'_a$, it shares no plaquette with a site link, the atom moves at speed at most 0.5993, $|\partial_t S_t| \le 6\cdot 0.5993/\sqrt{2} \le 2.543$, and $C_{e'} \le 24\sqrt{2}(2.543\beta_W + 32.47) + 0.5993 < 87\beta_W + 1103$. Kantorovich duality gives the bound on W_1. □

The eight private links alone. The fold of Proposition 418 is also reached by moving one inside link, and the law of the eight private links restricted to a fixed set of non-path links then has no response bound that holds for every such set. The direction matters: a link (x,μ) with $o_\mu = 0$ is a tree link when $o_\nu = 0$ for every $\nu > \mu$, so a block has $p_\mu = 8 - 2^\mu$ active inside μ-links, 7, 6, 4 and 0 for $\mu = 0,1,2,3$. For $\mu = 3$ every u_a and w_a is a tree link, the law of the eight private links does not depend on them, and Lemma 415 is the whole answer. For the other three directions the next two results show that the answer depends on the set over which the non-path links are integrated.

Proposition 422 (an inside link at the fold)**.** *Take the configuration of Proposition 418 for a direction $\mu \le 2$, with $c_k = c_{\gamma^*}$ and the offsets chosen so that the inside link u_b of one of the six non-path links at the angle $\frac{49}{100}$ is active. Replace u_b by $u_b(t) = \operatorname{diag}(e^{it}, e^{-it})$, keep every other link, and let $\pi^{(t)}$ be the law (40). There is $\tau > 0$ such that for $0 < |t| < \tau$ every preimage of c_k lies on the torus and is regular; for $t > 0$ there is one, $g_L(t)$, and for $t < 0$ there are three, $g_L(t)$ and $g_\pm(t) = e^{iq_\pm(t)}$ with $q_\pm(t) \to q^*$, where $g_L(t) \to g_L^*$. The weight of $g_L(t)$ tends to that of g_L^*, at most 2.78, and the weights of $g_\pm(t)$ tend to infinity as $t \uparrow 0$, for every $\beta_W \ge 0$. Hence $\pi^{(t)} \to \delta_{g^*}$ as $t \uparrow 0$ and $\pi^{(t)} \to \delta_{g_L^*}$ as $t \downarrow 0$, and for every $\beta_W \ge 0$ and $\varepsilon > 0$ there are $t < 0 < t'$ with $d(u_b(t), u_b(t')) = \sqrt{2}\,(t' - t) < \varepsilon$ and $W_1^\delta(\pi^{(t)}, \pi^{(t')}) \ge 0.126$.*

Proof. The action and the paths. The inside link u_b has μ-coordinate $2y_\mu$ and the private links $2y_\mu + 1$, and the two μ-links of a plaquette have the same μ-coordinate, so u_b shares no plaquette with the path link and $S(g)$ is that of Proposition 418. All links lie in T, so conjugation by T is still a symmetry, and the fourteen paths of nonzero offset have the angles $-\frac{1}{2}$ twice, $\frac{49}{100}$ eleven times, and $\frac{49}{100} + t$ once, on the path from x_b. For $g = g_0 + g_1 i + g_2 j + g_3 k$ with $g_0 + i g_1 = \rho e^{iq}$, $\rho > 0$, the real part of $g^{-1}e^{i\theta}$ is $\rho\cos(\theta - q)$, and $\delta(h, I)$ decreases in the real part of h; so the order of the distances $\delta(v_x, I)$ is the order of the numbers $|\theta - q|$, for every such g and not only on T.

Localisation. V is jointly continuous in (g,t) and SU(2) is compact, and by Proposition 418(i) the preimages of c_k at $t = 0$ are g_L^* and g^*. So for open neighbourhoods U_L of g_L^* and U^* of g^* there is $\tau > 0$ such that every preimage at $|t| < \tau$ lies in $U_L \cup U^*$; take $\tau \le 10^{-3}$.

Near g^*. For $q \in [0.05, 0.1]$ and $|t| \le 10^{-3}$, $(\frac{1}{2} + q) - (\frac{49}{100} + |t| - q) \ge 0.109$, so the far link is the strict maximiser, $m = 2\sqrt{2}\sin\frac{q+1/2}{2} \in (0.76, 0.84) \subset (\frac{1}{2}, 1)$, and V is real-analytic in (g,t) near $(g^*, 0)$. On T its angle is

$$f(q;t) = f_0(q) + \frac{\chi_0(q)}{16}\,t,$$
$$\chi_0(q) = 2 - 4\sqrt{2}\sin\tfrac{q+1/2}{2} \in (0.32, 0.48),$$

since only the angle sum changes, by t. Since $|\chi_0''| = \sqrt{2}\sin\frac{q+1/2}{2}$ is less than $\frac{1}{2}$, the second derivative of f exceeds $4.6 - 10^{-4}$ on $[0.05, 0.1]$, and by Step 1 of the proof of Proposition 418 both $f(0.05;t)$ and $f(0.1;t)$ exceed $\gamma^* + 10^{-3} - \frac{|t|}{32} > \gamma^*$. Hence for $t > 0$, $f(q;t) > f_0(q) \ge \gamma^*$ on $[0.05, 0.1]$ and there is no torus preimage in U^*, and for $t < 0$, $f(q^*;t) < \gamma^*$ and strict convexity give exactly two, $q_-(t) < q^* < q_+(t)$, with $f'(q_\pm(t);t) \ne 0$; since $f(\cdot;t) \to f_0$ in C^2 and q^* is the only zero of $f_0 - \gamma^*$ on $[0.05, 0.1]$, $q_\pm(t) \to q^*$ and $f'(q_\pm(t);t) \to f_0'(q^*) = 0$. Off the torus, every $g \in U^*$ is T-conjugate to $g(q,s) = \cos s\, e^{iq} + \sin s\, k$ with $s \ge 0$ small and q near q^*, and conjugation by T preserves $d(\cdot, g^*)$. The paths lie in T, so V is the map of Lemma 417 with the fourteen angles above and $\varpi_x = \frac{1}{16}$: $V_j = 0$ and $V_k(q,s;t) = \sin s\,\Xi_t(q,s)$, with Ξ_t continuous in (q,s,t), and $\Xi_0(q^*, 0) > \frac{1}{3}$ by Step 2 of the proof of Proposition 418. So $\Xi_t > 0$ near $(q^*, 0, 0)$ and $V_k(q,s;t) > 0$ for small $s > 0$: shrinking U^* and τ, there is no preimage off T in U^*. As in Step 3 of that proof, $|\det D_g V| = |f'|\lambda_\perp^2$ at torus points, with $\lambda_\perp = \Xi_t(q,0) \le 1$; so the weights of $g_\pm(t)$ are at least $e^{-12\beta_W}/|f'(q_\pm(t);t)| \to \infty$.

Near g_L^*. For $q \in [-0.0182, -0.0146]$ and $|t| \le 10^{-3}$, the six near paths are farther from g than the far link by at least $(\frac{49}{100} - q - 10^{-3}) - (\frac{1}{2} + q) \ge 0.0182$ in angle, and their chordal distance $2\sqrt{2}\sin\frac{\theta}{2}$, $\theta \in [0.503, 0.510]$, lies in $(0.70, 0.72) \subset (\frac{1}{2}, 1)$. Let Ψ_+ and Ψ_- be the cut-off averages computed with the maximum taken, respectively, at the path from x_b, of angle $\frac{49}{100} + t$, and at an unmoved near path, of angle $\frac{49}{100}$. Both are real-analytic in (g,t) near $(g_L^*, 0)$, and both equal $V(\cdot;0)$ near g_L^* at $t = 0$, when the two paths coincide. The difference of the real parts of the two path variables is

$$\rho\big(\cos(\tfrac{49}{100} - q) - \cos(\tfrac{49}{100} + t - q)\big) = 2\rho\sin(\tfrac{49}{100} + \tfrac{t}{2} - q)\sin\tfrac{t}{2},$$

and $\sin(\frac{49}{100} + \frac{t}{2} - q) > 0.48$, so near g_L^* the moved path is the maximiser for $t > 0$ and an unmoved one for $t < 0$: $V = \Psi_+$ for $t \ge 0$ and $V = \Psi_-$ for $t \le 0$. Since $D_g\Psi_\pm(g_L^*, 0) = D_g V(g_L^*; 0)$ is invertible, with determinant $f_0'(q_L^*)\lambda_\perp(q_L^*)^2 > 0$, the implicit function theorem gives a closed ball $\bar U_L$ about g_L^* and $\tau > 0$ such that each equation $\Psi_\pm(g;t) = c_k$, $|t| < \tau$, has exactly one solution in $\bar U_L$, at which $D_g\Psi_\pm$ is invertible. Conjugation by T fixes g_L^* and maps $\bar U_L$ onto itself, so the solution is fixed by it and lies in T. This is the one preimage $g_L(t)$ in U_L, for either sign of t; it is regular, it tends to g_L^*, and its weight tends to that of g_L^*, which is at most 2.78 by Proposition 418(ii).

The limits. For $t > 0$, $\pi^{(t)} = \delta_{g_L(t)} \to \delta_{g_L^*}$. For $t < 0$, with $f = \delta(\cdot, g^*)$ and $w_\pm$ the weights of $g_\pm(t)$, $\pi^{(t)}(f) \le 2.78(1 + o(1))\,2\sqrt{2}/(w_+ + w_-) + \max_\pm \delta(g_\pm(t), g^*) \to 0$, so $\pi^{(t)} \to \delta_{g^*}$. As $\delta(g^*, g_L^*) \ge 0.1268$ by Proposition 418(iii),

Kantorovich duality with f gives the last statement; $t \mapsto e^{it}$ has geodesic speed $\sqrt{2}$. □

So the path-link law responds to an inside link as it does to the non-path links in Proposition 418(iv). The law of all eight private links is the one that the bad side asks about, and the next theorem shows that its response to an inside link is not bounded independently of the set B of non-path links over which it is restricted, in contrast with Lemma 415.

Theorem 423 (no bound uniform in the restricting set)**.** *In the configuration of Proposition 422, let z_0 be its seven non-path links, let B_ε be the set of z with $d(\ell_a, \ell_a(z_0)) < \varepsilon$ for $a \neq 0$, and let $Q_{t,\varepsilon}$ be the law of the eight private links given $u_b = u_b(t)$, the other links, $V_k^\sharp = c_k$ and $z \in B_\varepsilon$, that is $\frac{1}{Z}\mathbf{1}_{B_\varepsilon} J_8^{-1} e^{-\beta_W S}\mathrm{vol}_M$. For every $\beta_W \geq 0$ and $L > 0$ there are $t < 0 < t'$ with $\sqrt{2}\,(t'-t) < 1/L$ and $\varepsilon > 0$ such that*

$$W_1\big(Q_{t,\varepsilon}, Q_{t',\varepsilon}\big) \geq 0.12$$

for the cost $\sum_a \delta(\ell_a, \ell'_a)$. For $\varepsilon < 0.4$ the set B_ε lies in the bad set $\{\max_{a\neq 0} d(\ell_a, c_k) > r\}$ for every $r \leq \frac{1}{32}$. So there is no constant C, for any β_W, with $W_1 \leq C\, d(u_b, u'_b)$ for the law of the eight private links restricted to every fixed measurable set of non-path links.

Proof. Step 1, the fibre over B_ε. Fix t with $0 < |t| < \tau$, and let $g_1, \dots, g_n$ be the preimages at z_0 of Proposition 422, all regular. By compactness and continuity, for small ε every solution of $V(g,z) = c_k$ with $z \in B_\varepsilon$ has g in one of disjoint balls U_i about the g_i. At $g_\pm(t)$ ($t < 0$) the far link is the strict maximiser, and at $g_L(t)$ for $t > 0$ the moved path is, by the margins of that proof; there V is real-analytic in (g,z) near (g_i, z_0), and the implicit function theorem gives a smooth G_i on B_ε with $M \cap (U_i \times B_\varepsilon) = \{(G_i(z), z)\}$. At $g_L(t)$ for $t < 0$ the maximum at (g_i, z_0) is attained by the six functions $m_a = \delta(g^{-1}\ell_a, I)$ of the near offsets a (for $a = b$ through the path from x'_b) and by no other path, since the moved path is shorter by a margin proportional to $|t|$ and the far link by 0.0182. Near (g_i, z_0), V equals at each point one of the six real-analytic functions $\Psi_a = g\exp(\phi(m_a)N)$, one with the largest m_a, and at $z = z_0$ the six coincide as functions of g. In the chart $F(X,z) = \log(c_k^{-1} V(g_i e^X, z))$, with F_a defined from Ψ_a in the same way, the derivatives $D_X F_a(0, z_0)$ are therefore one invertible map A_i, and the function $s \mapsto F(X + s(X' - X), z)$ is a continuous selection of finitely many C^1 functions; at a point of differentiability that is not an isolated point of any of the closed sets where it equals one of them, its derivative equals theirs, and the exceptional points are countable. Hence, for $\|X\|, \|X'\| \leq \rho$ and $z \in B_\varepsilon$,

$$\big\|F(X',z) - F(X,z) - A_i(X' - X)\big\| \leq \eta\, \|X' - X\|,$$

with $\eta = \eta(\rho, \varepsilon) \to 0$, as every $D_X F_a$ is within η of A_i there. With $\eta\, \|A_i^{-1}\| \leq \frac{1}{2}$ and ε small, $X \mapsto X - A_i^{-1} F(X,z)$ is a contraction of the ρ-ball into itself, and $V(\cdot, z) = c_k$ has exactly one solution $G_i(z)$ there, Lipschitz in z with constant at most $2\|A_i^{-1}\|\, \mathrm{Lip}_z V$. In all cases $M \cap (\mathrm{SU}(2) \times B_\varepsilon)$ is the disjoint union of the graphs Γ_i of the G_i, and $|\det D_g V|$, wherever V is differentiable on Γ_i, lies within a factor $1 \pm \eta'$ of $|\det A_i|$, $\eta' \to 0$ with ρ, ε.

Step 2, the density on a graph. At a point of Γ_i where V is differentiable and G_i is differentiable at z, write $DV = (A\ B)$ with $A = D_g V$, $B = D_z V$; then $DG_i = -A^{-1}B$, and by the determinant identity $\det(I + PQ) = \det(I + QP)$,

$$J_8 = \det(AA^* + BB^*)^{1/2} = |\det A|\, \det(I + DG_i^* DG_i)^{1/2}.$$

The second factor is the area element of the Lipschitz graph Γ_i over z, so $J_8^{-1}\mathrm{vol}_M = |\det D_g V|^{-1} dz$ on Γ_i, dz Haar measure on the seven links carried by G_i, provided the two differentiability conditions hold for almost every z. For G_i this is Rademacher's theorem. V is differentiable at $(G_i(z), z)$ except where two of the six m_a tie, and for $a \neq a'$ the tie set is null: write $\ell_a = he^{is}$ along the circles $\ell_a T$, along which Haar measure is uniform, fix the other coordinates, and let $\psi(s)$ be $m_a - m_{a'}$ at $(G_i(z_s), z_s)$. Its almost-everywhere derivative is at least $\partial_s m_a - \|D_g m_a - D_g m_{a'}\|\sqrt{2}\,\mathrm{Lip}\, G_i$; here $\partial_s m_a = \sqrt{2}\cos\frac{\theta}{2}$ on T, with θ the angle of $g^{-1}\ell_a$, which is at least 1.36 at (g_i, z_0) and exceeds 1.3 near it, while $D_g m_a - D_g m_{a'}$ vanishes at $z = z_0$, where $\ell_a = \ell_{a'}$. So ψ is strictly increasing for small ρ, ε, has one zero on each circle, and the tie set is null by Fubini, the circles meeting B_ε in arcs.

Step 3, the limit $\varepsilon \to 0$. By Steps 1 and 2,

$$Q_{t,\varepsilon}(\varphi) = \frac{\sum_i \int_{B_\varepsilon} \varphi(G_i(z), z)\, e^{-\beta_W S} |\det D_g V|^{-1}\, dz}{\sum_i \int_{B_\varepsilon} e^{-\beta_W S} |\det D_g V|^{-1}\, dz},$$

the integrands evaluated at $(G_i(z), z)$. As $\varepsilon \to 0$, $G_i(z) \to g_i$ uniformly on B_ε and the weights converge uniformly to those of g_i in $\pi^{(t)}$, so $Q_{t,\varepsilon}(\varphi) \to \pi^{(t)}(\varphi(\cdot, z_0))$ for every continuous φ.

Step 4, the gap. Let $\varphi(\ell) = \delta(\ell_0, g^*)$, which is 1-Lipschitz for the cost. By Proposition 422 choose $t < 0 < t'$ with $\sqrt{2}\,(t' - t) < 1/L$, $\pi^{(t)}(\varphi) \leq 0.0017$ and $\pi^{(t')}(\varphi) \geq 0.1268 - 0.0017$, then ε by Step 3 with $|Q_{t,\varepsilon}(\varphi) - \pi^{(t)}(\varphi)| \leq 0.0017$ and likewise for t'. Kantorovich duality gives $W_1 \geq 0.1268 - 4 \cdot 0.0017 = 0.12$. Finally $d(\ell_a(z_0), c_k) \geq \sqrt{2}\,(\frac{49}{100} - \gamma^*) > 0.45$ for the near links, so B_ε lies in the bad set for $\varepsilon < 0.4$. □

The proof uses that B_ε is chosen after t and t': the boundary of B_ε meets the fibre near a fold of its projection to the non-path links, which is the tangency named after Proposition 418. On a bad block the labels of Sec. XIII M restrict the non-path links to two sets only, and neither has its boundary near a fold.

Proposition 424 (the sets of the labels)**.** *Let $k = (y, \mu)$, $r \leq \frac{1}{32}$, and fix every link other than the eight private links of k. The event that the labels of a set of blocks containing y take given values, with y bad, restricts z to $\mathrm{SU}(2)^7$ if y is bad through a coarse plaquette or through a non-path active link that is not private to k, and to*

$$B_r^{\mathrm{bad}} = \Big\{z : \max_{a \neq 0} d(\ell_a, c_k) > r\Big\}$$

otherwise. In the second case $d(u_a, I), d(w_a, I) \leq r$ for $a \neq 0$, and over the shell $\{z : r \leq \max_{a\neq 0} d(\ell_a, c_k) \leq \frac{1}{32}\}$, which

contains the boundary of B_r^{bad}, the fibre M is the graph of one smooth map $z \mapsto g(z)$, with $|\det D_g V|^{-1} \in [0.147, 828]$ and $J_8^{-1}\mathrm{vol}_M = |\det D_g V|^{-1} dz$.

Proof. The private links of k are non-path active links of k only, so they enter the label of y and of no other block, and the label of y is the union of the events G_e^c over the non-path active links e of the four pairs based at y and of the coarse events at the corner y. Given the other links, this union is everything when one of its other members holds, and is B_r^{bad} otherwise; in the second case the fourteen inside μ-links of k, which are non-path active links of k or tree links, are within r of I. Over the shell the hypothesis of Proposition 416 holds, so $V = c_k$ has exactly one solution, regular, with the stated density, and it is smooth in z by the implicit function theorem, the cut-off being one; Step 2 of the proof of Theorem 423 gives the last identity. □

The relative chart. The folds of Proposition 422 and Theorem 423 are folds of the projection of the fibre $M = \{V_k^\sharp = c_k\}$ to the seven non-path links. In the coordinates $g = \ell_0$ and $\xi_a = g^{-1}\ell_a$, $a \neq 0$, the fibre is a single graph over ξ as soon as the inside links of $B(y)$ are within $\frac{1}{32}$ of I, and the response of the law of the eight private links to an inside link reduces to one condition on a Jacobian. In these coordinates the paths are

$$v_{x_a} = g^{-1} u_a g\, \xi_a, \qquad v_{x'_a} = \xi_a w_a, \qquad v_{2y} = I,$$

so $V_k^\sharp = g\,E(g,\xi)$ with $E = \exp(\chi N)$, $\chi = \phi(m)$, $m = \max_x \delta(v_x, I)$ and $N = \frac{1}{16}\sum_x \log v_x$, and E depends on g only through the conjugates $\tilde{u}_a = g^{-1}u_a g$. For fixed g the map $(\ell_a)_{a\neq 0} \mapsto \xi$ preserves Haar measure. With $\alpha = 2\arcsin\frac{1}{2\sqrt{2}}$ and $\Lambda = \alpha/\sin\alpha$ as in Sec. XII D, put

$$\lambda = \frac{1}{32}\Big(2\sqrt{2}\cdot\frac{7}{8}\sqrt{2}\,\alpha + \frac{7}{16}\sqrt{2}\,\Lambda\Big) < 0.1002,$$
$$L = \frac{2\cdot\frac{7}{8}\sqrt{2}\,\alpha + \frac{1}{16}\Lambda}{1-\lambda} < 2.064,$$

by $0.7227 < \alpha < 0.7228$, $\Lambda < 1.0927$ and $\sqrt{2} < 1.41422$.

Lemma 425 (the relative chart)**.** *Fix every link other than the eight private links of k, and suppose $d(u_a, I) \le \frac{1}{32}$ for $a \neq 0$; the links w_a are arbitrary.*

(i) *For fixed ξ, $d(E(g,\xi), E(g',\xi)) \le \lambda\, d(g,g')$, and $V_k^\sharp = c_k$ has exactly one solution $g = G(\xi)$, the fixed point of the contraction $g \mapsto c_k E(g,\xi)^{-1}$.*

(ii) *If one link u_b or w_b moves from e to e', the u_a staying within $\frac{1}{32}$ of I, then $d(G(\xi), G'(\xi)) \le L\,d(e,e')$ for every ξ, and the eight private links $G(\xi)$, $G(\xi)\xi_a$ move by at most $8L\,d(e,e')$ in the cost $\sum_a \delta$.*

(iii) *Wherever $g \mapsto F(g) = gE(g,\xi)$ is differentiable, its differential has singular values in $[1-\lambda, 1+\lambda]$, so $D(\xi) = |\det D_g F(G(\xi))| \in [(1-\lambda)^3, (1+\lambda)^3] \subset [0.728, 1.332]$.*

(iv) *For almost every c_k and every measurable set B of non-path links, the law $\frac{1}{Z}\mathbf{1}_B J_8^{-1} e^{-\beta_W S}\mathrm{vol}_M$ of the eight private links is the image under $\xi \mapsto (G(\xi), G(\xi)\xi)$ of*

$$\hat{Q}(d\xi) = \frac{1}{Z}\,\mathbf{1}_B(G(\xi)\xi)\, e^{-\beta_W S} D(\xi)^{-1}\, d\xi,$$
$$Z = \int \mathbf{1}_B(G(\xi)\xi)\, e^{-\beta_W S} D(\xi)^{-1} d\xi,$$

S evaluated at $(G(\xi), G(\xi)\xi)$ and $d\xi$ Haar measure on $\mathrm{SU}(2)^7$.

If every u_a equals I, then E does not depend on g, $D \equiv 1$, and (ii) holds with L replaced by 1.857. If every w_a equals I, the same holds with u arbitrary in the coordinates $\xi_a = \ell_a g^{-1}$, in which $V_k^\sharp = \exp(\chi N')\,g$ with $N' = \frac{1}{16}\sum_x \log(P_x g^{-1})$, $P_{x_a} g^{-1} = u_a \xi_a$ and $P_{x'_a} g^{-1} = \xi_a$.

More generally, put $\kappa = \frac{7}{4}\sqrt{2}\,\alpha + \frac{7}{16}\Lambda$, so that $\kappa < 2.2670$, and

$$\sigma_u = \max_{a\neq 0} \|\mathrm{Ad}(u_a) - 1\|, \qquad \sigma_w = \max_{a\neq 0} \|\mathrm{Ad}(w_a) - 1\|.$$

If $\sigma_u < 1/\kappa$, then (i)–(iv) hold for every w with λ replaced by $\lambda_u = \kappa\sigma_u$ and L by $L_u = (\frac{7}{4}\sqrt{2}\,\alpha + \frac{1}{16}\Lambda)/(1-\lambda_u)$, the range of D being $[(1-\lambda_u)^3, (1+\lambda_u)^3]$, provided in (ii) that $\sigma_u < 1/\kappa$ along the move. If $\sigma_w < 1/\kappa$, the same holds for every u in the coordinates $\xi_a = \ell_a g^{-1}$, with σ_w in place of σ_u; there $P_{x'_a} g^{-1} = \xi_a\, g w_a g^{-1}$. As $\|\mathrm{Ad}(u) - 1\| = 2\sin\theta$ for u with eigenangles $\pm\theta$, $\theta \in [0,\pi]$, the first condition holds when $\sin\theta \le 0.2205$ for every u_a, $a \neq 0$, and $d(u_a, I) \le \frac{1}{32}$ gives $\lambda_u \le \lambda$.

Proof. (i) Let $g_t = g e^{t\zeta}$, $\|\zeta\| = 1$. If $d(u,I) = \sqrt{2}\,\theta$, then $\mathrm{Ad}(u) - 1$ has the eigenvalues 0 and $e^{\pm 2i\theta} - 1$, so its norm is $2\sin\theta \le \sqrt{2}\,d(u,I)$, and by bi-invariance and $d(e^A, e^B) \le \|A - B\|$,

$$d(\tilde{u}_a(t), \tilde{u}_a(t')) = d\big(e^{(t'-t)\,\mathrm{Ad}(\tilde{u}_a^{-1})\zeta}, e^{(t'-t)\zeta}\big) \le \sqrt{2}\,d(u_a, I)\,|t'-t|.$$

So each v_{x_a} moves at geodesic speed at most $\frac{\sqrt{2}}{32}$, and the other paths are fixed. Since $\delta(\cdot, I)$ is 1-Lipschitz for d, m moves at speed at most $\frac{\sqrt{2}}{32}$ and χ at most $\frac{2\sqrt{2}}{32}$. On an interval where $\chi > 0$, every $\delta(v_x, I) < 1$, so every eigenangle θ of v_x has $2\sqrt{2}\sin\frac{\theta}{2} < 1$, $\theta < \alpha$, $\|\log v_x\| = \sqrt{2}\,\theta < \sqrt{2}\,\alpha$, and the differential of log along the path has norm at most Λ; the two paths of offset zero contribute nothing, so $\|N\| \le \frac{7}{8}\sqrt{2}\,\alpha$ and N moves at speed at most $\frac{7}{16}\Lambda\frac{\sqrt{2}}{32}$. Hence $Y = \chi N$ moves at speed at most λ there. The set where $\chi > 0$ is open and $Y = 0$ off it, so Y is λ-Lipschitz along the whole geodesic, and $d(E(g), E(g')) \le \|Y(g) - Y(g')\| \le \lambda\, d(g,g')$. Since $V_k^\sharp = gE(g,\xi)$, the equation $V_k^\sharp = c_k$ reads $g = c_k E(g,\xi)^{-1}$, and $d(c_k E^{-1}, c_k E'^{-1}) = d(E, E')$: the map is a contraction of the complete space $\mathrm{SU}(2)$, with a unique fixed point.

(ii) A unit-speed move of u_b moves $\tilde{u}_b$ and v_{x_b}, and a move of w_b moves $v_{x'_b}$, at speed one, and no other path; so m moves

at speed at most 1, χ at most 2, N at most $\frac{1}{16}\Lambda$, and Y at fixed (g,ξ) at most $2\cdot\frac{7}{8}\sqrt{2}\,\alpha+\frac{1}{16}\Lambda=(1-\lambda)L$. For the two fixed points, $d(G,G')\le d(E(G),E(G'))+(1-\lambda)Ld(e,e')$, with E,E' before and after the move at the same point, so $d(G,G')\le\lambda d(G,G')+(1-\lambda)Ld(e,e')$. The geodesic ball of radius $\frac{1}{32}$ is convex, so the hypothesis holds along the move. The eight private links are G and $G\xi_a$, each at distance $d(G,G')$ from its new value, and $\delta\le d$.

(iii) The derivative of F along $ge^{t\zeta}$, translated back to the identity, is $F^{-1}\dot F=\mathrm{Ad}(E^{-1})\zeta+E^{-1}\dot E$, and $E^{-1}\dot E=E_{-Y}\dot Y$ has norm at most $\|\dot Y\|\le\lambda$ by (i). So $D_gF=\mathrm{Ad}(E^{-1})(I+K)$ with $\|K\|\le\lambda$, and $\mathrm{Ad}(E^{-1})$ is an isometry.

(iv) For fixed ξ, $(1-\lambda)d(g,g')\le d(F(g),F(g'))\le(1+\lambda)d(g,g')$ by (i) and bi-invariance, and F is onto by (i), so F is a bi-Lipschitz homeomorphism of $\mathrm{SU}(2)$ with inverse $c\mapsto G_c(\xi)$. The area formula gives $\int h(F(g))|\det D_gF|\,dg=\int h(c)\,dc$, so for bounded measurable ψ and f,

$$\iint\psi(F(g,\xi))f(g,\xi)\,dg\,d\xi=\int\psi(c)\int\frac{f(G_c(\xi),\xi)}{|\det D_gF(G_c(\xi),\xi)|}\,d\xi\,dc,$$

the Jacobian being defined at $(G_c(\xi),\xi)$ for almost every ξ for almost every c, by Fubini and the area formula applied to the null set where F is not differentiable. The inner integral is a disintegration of Haar measure on the eight links along $V_k^\sharp$; so is $J_8^{-1}\mathrm{vol}_M$ by the coarea formula, and two disintegrations agree for almost every c. Multiplying by $\mathbf{1}_Be^{-\beta_WS}$ gives (iv). If every $u_a=I$, then $\tilde u_a=I$, $E=E(\xi)$, λ may be taken zero and the speed bound of (ii) is 1.857. If every $w_a=I$, multiplying the paths by g^{-1} on the right and conjugating the logarithms by g gives the stated form, in which g enters only as the right factor; the same proof applies with $\lambda=0$, and right translation preserves Haar measure.

For the last part, the hypothesis on u enters the proof only through the speed of $\tilde u_a$ in (i), which is at most $\|\mathrm{Ad}(\tilde u_a^{-1})-1\|=\|\mathrm{Ad}(u_a)-1\|\le\sigma_u$, since $\mathrm{Ad}(\tilde u_a^{-1})$ is conjugate to $\mathrm{Ad}(u_a)^{-1}$ and $\mathrm{Ad}(u_a)$ is orthogonal. With this bound, m moves at speed at most σ_u, χ at most $2\sigma_u$ and N at most $\frac{7}{16}\Lambda\sigma_u$, so Y is $\kappa\sigma_u$-Lipschitz in g, and (i)–(iv) follow as before. In the coordinates $\xi_a=\ell_ag^{-1}$, conjugating by g gives $V_k^\sharp=\exp(\chi N')g$ with $N'=\frac{1}{16}\sum_x\log(P_xg^{-1})$, $P_{x_a}g^{-1}=u_a\xi_a$ and $P_{x'_a}g^{-1}=\xi_a\,gw_ag^{-1}$, and $m=\max_x\delta(P_xg^{-1},I)$; so g enters N' and m only through gw_ag^{-1}, which moves at speed at most σ_w, and the fixed-point equation $g=E'(g,\xi)^{-1}c_k$, $E'=\exp(\chi N')$, is treated in the same way. Finally $\kappa<2.2670$ by $\alpha<0.7228$, $\Lambda<1.0927$ and $\sqrt{2}<1.41422$, and $2\cdot0.2205\cdot2.2670<1$. □

The normaliser has a lower bound that does not depend on the chart. Let S_{dep} be the sum of $1-\frac{1}{2}\mathrm{Re}\,\mathrm{Tr}\,U_p$ over the plaquettes that contain a private link of k; the other plaquettes do not depend on the eight links and cancel from the normalised law, so S may be replaced by S_{dep}, and Z below is defined with it.

Lemma 426 (the normaliser)**.** *There are* 36 *plaquettes that contain a private link of k, so $0\le S_{\mathrm{dep}}\le 72$. For every c_k, every inside and exterior link and every $\beta_W\ge0$,*

$$Z_{\mathrm{SU}(2)^7}\ge\tfrac{1}{2}e^{-72\beta_W},\qquad Z_{B_r^{\mathrm{bad}}}\ge\big(\tfrac{1}{2}-b_r\big)e^{-72\beta_W},$$

with $b_r=(2\vartheta-\sin2\vartheta)/(2\pi)$ and $\vartheta=r/\sqrt{2}$, where $Z_B=\int\mathbf{1}_Be^{-\beta_WS_{\mathrm{dep}}}J_8^{-1}\mathrm{vol}_M$, and $b_r<2.3\cdot10^{-6}$ for $r\le\frac{1}{32}$.

Proof. A private link (x'_a,μ) lies in two plaquettes of each plane $\{\mu,\nu\}$, at x'_a and at x'_a-e_ν, and the other μ-link of exactly one of them, $(x'_a\pm e_\nu,\mu)$, is private. So the 48 incidences cover 12 plaquettes twice and 24 once, 36 in all, each term lying in $[0,2]$. Let A be the set of configurations with $\ell_0=c_k$ and $\Re\,\mathrm{Tr}(c_k^{-1}u_1\ell_1)<0$. There $v_{x_1}=c_k^{-1}u_1\ell_1$ has $\delta(v_{x_1},I)^2=4-2\Re\,\mathrm{Tr}\,v_{x_1}>4$, so $m>1$ near A, the cut-off vanishes and $V_k^\sharp=\ell_0$ near A: the fibre contains A, and on it $D_\ell V_k^\sharp$ is the projection to the path link, so $J_8=1$ and vol_M is Haar measure in $\ell_1,\dots,\ell_7$. The map $\ell_1\mapsto c_k^{-1}u_1\ell_1$ preserves Haar measure and $X\mapsto-X$ exchanges the signs of $\Re\,\mathrm{Tr}X$, so A has Haar measure $\frac{1}{2}$ in $(\ell_a)_{a\ne0}$; with $S_{\mathrm{dep}}\le72$ this is the first bound. A point of A outside B_r^{bad} has $d(\ell_1,c_k)\le r$, a geodesic ball whose Haar measure is $\frac{2}{\pi}\int_0^\vartheta\sin^2t\,dt=b_r$, the Haar density of the eigenangle $t\in[0,\pi]$ being $\frac{2}{\pi}\sin^2t$ and $d=\sqrt{2}t$. Finally $2\vartheta-\sin2\vartheta\le\frac{4}{3}\vartheta^3$ gives $b_r\le\frac{2\vartheta^3}{3\pi}\le\frac{2}{3\pi(32\sqrt{2})^3}<2.3\cdot10^{-6}$. □

Theorem 427 (the response in the relative chart)**.** *Let k be as in Lemma 425, let B be $\mathrm{SU}(2)^7$ or B_r^{bad} with $r<\frac{1}{32}$, and in the second case suppose also $d(w_a,I)\le\frac{1}{32}$ for $a\ne0$. Let one of the links u_b, w_b, $b\ne0$, move from e to e', the hypotheses holding along the geodesic, and write Q,Q' for the two laws of the eight private links, $\hat Q,\hat Q'$, D,D' for the objects of Lemma 425(iv). Suppose that*

$$\text{(J)}\qquad\int\big|\log D-\log D'\big|\,d\hat Q\le K\,d(e,e')$$

for every such pair along the geodesic. Then, for the cost $\sum_a\delta$,

$$W_1(Q,Q')\le\big(16.52+2898\beta_W+41.37K+238.1\,\iota\,e^{72\beta_W}\big)\,d(e,e'),$$

with $\iota=0$ for $B=\mathrm{SU}(2)^7$ and $\iota=1$ for $B=B_r^{\mathrm{bad}}$. If every u_a equals I and e is a link w_b, or every w_a equals I and e is a link u_b, then (J) *holds with $K=0$, and for $B=\mathrm{SU}(2)^7$ the bound holds with u, respectively w, otherwise arbitrary. For $B=\mathrm{SU}(2)^7$ and $\sigma_u<1/\kappa$ along the move, or $\sigma_w<1/\kappa$ in the coordinates $\xi_a=\ell_ag^{-1}$, with λ_u,L_u of Lemma 425 (respectively λ_w,L_w) and (J) in that chart,*

$$W_1(Q,Q')\le\Big(8L_u+16\sqrt{2}R_u\big(24\sqrt{2}L_u\beta_W+K\big)\Big)\,d(e,e'),$$

where $R_u=(1+\lambda_u)^3/(1-\lambda_u)^3$.

Proof. Since W_1 is a metric, it suffices to prove the bound with an error $o(d(e,e'))$ for nearby pairs and to add along a subdivision of the geodesic. Couple $\hat Q$ and $\hat Q'$ so that $\xi=\xi'$ with probability $1-\mathrm{TV}(\hat Q,\hat Q')$, and use the maps of Lemma 425(iv): by

(ii) of that lemma the cost is at most $8Ld(e,e') < 16.52\,d(e,e')$ where $\xi = \xi'$, and at most the diameter $8\cdot 2\sqrt{2}$ otherwise. So $W_1 \le 16.52\,d(e,e') + 16\sqrt{2}\,\mathrm{TV}(\hat Q,\hat Q')$, and with f, f' the unnormalised densities, $\mathrm{TV}(\hat Q,\hat Q') \le \frac{1}{Z}\int |f-f'|\,d\xi$.

The action. The links u_b, w_b share no plaquette with a private link, the two μ-links of a plaquette having one μ-coordinate, so S_{dep} changes only through the eight links, each of which moves by at most $Ld(e,e')$. Each of the 48 incidences of a private link with a plaquette contributes at most $\frac{1}{\sqrt2}$ times its speed, by the proof of Lemma 415, so $|S_{\mathrm{dep}} - S'_{\mathrm{dep}}| \le \frac{48L}{\sqrt2} d(e,e') < 70.06\,d(e,e')$ at every ξ, and

$$\int_{B\cap B'} \frac{|e^{-\beta_W S} - e^{-\beta_W S'}|}{D'}\,d\xi \le \big(e^{70.06\beta_W d(e,e')} - 1\big)\Big(\frac{1+\lambda}{1-\lambda}\Big)^3 Z,$$

with $(\frac{1+\lambda}{1-\lambda})^3 < 1.828$ by Lemma 425(iii); here B, B' denote the sets of ξ with $G(\xi)\xi$, respectively $G'(\xi)\xi$, in B.

The Jacobian. With $\Delta = \log D - \log D'$, $|\Delta| \le 3\log\frac{1+\lambda}{1-\lambda}$ and $|e^\Delta - 1| \le 1.828|\Delta|$, so $\int_{B\cap B'} e^{-\beta_W S}|D^{-1} - D'^{-1}|\,d\xi \le 1.828\,Z\int|\Delta|\,d\hat Q \le 1.828\,KZ\,d(e,e')$.

The restricting set. For $B = \mathrm{SU}(2)^7$ there is nothing to do. For $B = B_r^{\mathrm{bad}}$ let $\mathscr{M}(\xi) = \max_a d(G(\xi)\xi_a, c_k)$ and $\varepsilon = Ld(e,e')$, with $r+\varepsilon \le \frac{1}{32}$. Then $|\mathscr{M} - \mathscr{M}'| \le d(G,G') \le \varepsilon$, so $B\setminus B'$ and $B'\setminus B$ consist of ξ whose eight links, before respectively after the move, satisfy $r < \max_a d(\ell_a, c_k) \le r+\varepsilon \le \frac{1}{32}$. On that shell, with every inside link within $\frac{1}{32}$ of I, Proposition 416 with radius $\frac{1}{32}$ and the proof of Proposition 424 show that the fibre is a graph over the non-path links with $J_8^{-1}\mathrm{vol}_M = |\det D_g V|^{-1} dz$, $|\det D_g V|^{-1} \le 828$. Since $e^{-\beta_W S_{\mathrm{dep}}} \le 1$ and Lemma 425(iv) identifies both disintegrations,

$$\int_{B\setminus B'} f\,d\xi + \int_{B'\setminus B} f'\,d\xi \le 2\cdot 828\cdot 7\cdot \frac{\varepsilon(r+\varepsilon)^2}{\sqrt2\,\pi},$$

since the Haar measure of the shell $\{s < d(\ell,c) \le s+\varepsilon\}$ is

$$\frac{2}{\pi}\int_{s/\sqrt2}^{(s+\varepsilon)/\sqrt2} \sin^2 t\,dt \le \frac{\varepsilon(s+\varepsilon)^2}{\sqrt2\,\pi}.$$

Divided by $Z \ge (\frac12 - 2.3\cdot10^{-6})e^{-72\beta_W}$ (Lemma 426) and with $r \le \frac{1}{32}$, this is at most $10.52\,e^{72\beta_W} d(e,e') + o(d(e,e'))$.

Adding the three parts,

$$\mathrm{TV}(\hat Q,\hat Q') \le \big(128.07\beta_W + 1.828K + 10.52\,\iota e^{72\beta_W}\big)\,d(e,e') + o(d(e,e')),$$

with $1.828\cdot 70.06 < 128.07$; multiplied by $16\sqrt2$, with $16\sqrt2\cdot 128.07 < 2898$, $16\sqrt2\cdot 1.828 < 41.37$ and $16\sqrt2\cdot 10.52 < 238.1$, this gives the constants. If every $u_a = I$ and $e = w_b$, then $D \equiv D' \equiv 1$ by Lemma 425, and (J) holds with $K = 0$; likewise in the coordinates $\xi_a = \ell_a g^{-1}$ if every $w_a = I$ and $e = u_b$, where Lemma 425 needs no bound on u, and the case $B = \mathrm{SU}(2)^7$ uses no bound on the inside links at all. For the last statement the case $B = \mathrm{SU}(2)^7$ uses Lemma 425 only, through the displacement $8L$ of the eight links, the ratio $(\frac{1+\lambda}{1-\lambda})^3$ of the bounds on D, which bounds $e^{|\Delta|}$ and the ratio of D'^{-1} to D^{-1}, and the change $\frac{48L}{\sqrt2} = 24\sqrt2 L$ of the action per unit move; with λ_u, L_u in place of λ, L the same three steps give the stated bound. □

The Jacobian D of Lemma 425(iii) jumps only where the maximal path changes. Write $f_x(\xi) = \delta(v_x, I)$ at $g = G(\xi)$, so that $m = \max_x f_x$. Call ξ *flat* if $m \notin (\frac12, 1)$ or the maximum is attained only at paths from points x'_c, and of *type a* if $\frac12 < m < 1$ and the maximum is attained at x_a only, $a \ne 0$. The two paths of index *a tie at the top* at ξ if $f_{x_a} = f_{x'_a} = m \in (\frac12, 1)$ and $f_y < m$ for every other path y.

Near a tie at the top of index a the map $g \mapsto F(g) = gE(g,\xi)$ has two branches. Put $f^{(1)}(g,\xi) = \delta(\tilde u_a \xi_a, I)$, which depends on g, and $f^{(2)}(\xi) = \delta(\xi_a w_a, I)$, which does not, and $Y_i = \phi(f^{(i)})N$, $F_i = g\exp Y_i$. Since ϕ decreases, $Y = Y_1$ where $f^{(1)} \ge f^{(2)}$ and $Y = Y_2$ where $f^{(2)} \ge f^{(1)}$, as long as the other paths stay below. At a tie ξ_0 with $g_0 = G(\xi_0)$ the two branches agree, and the proof of Lemma 425(iii) applies to each of them, so $D_g F_i$ is invertible; log is analytic on $\{\delta(v,I) < 1\}$ and ϕ is affine on $(\frac12, 1)$, so the implicit function theorem gives analytic maps G_i near ξ_0, also analytic in the inside links, with $G_i(\xi_0) = g_0$ and $F_i(G_i(\xi),\xi) = c_k$. Put $D_i = \det D_g F_i(G_i)$ and $h_i(\xi) = f^{(1)}(G_i(\xi),\xi) - f^{(2)}(\xi)$. Tangent vectors $\eta = (\eta_b)_{b\ne0}$ at ξ are derivatives of $\xi_b e^{s\eta_b}$, with $\|\eta\|^2 = \sum_b \|\eta_b\|^2$, and ∇h_2 is the gradient for this metric.

Lemma 428 (the ties)**.** *Let ξ_0 be a tie at the top of index a, and suppose $d(u_b, I) \le \frac{1}{32}$ for every $b \ne 0$; the links w_b are arbitrary. At (g_0,ξ_0) let $\ell = D_g f^{(1)}$, $A_2 = \Phi(\mathrm{ad}\,Y)\,\phi\,D_g N$ and $\kappa_0 = \ell((1+A_2)^{-1}N)$.*

(i) *Near ξ_0, h_1 and h_2 have the same sign; $G = G_1$ and $D = D_1$ where it is positive, and $G = G_2$ and $D = D_2$ where it is negative.*

(ii) *At ξ_0, $D_1 = D_2\,(1 - 2\kappa_0)$ and $|\kappa_0| < 0.04039$.*

(iii) *At ξ_0, $|\log D_1 - \log D_2| \le 1.9811\,|\nabla h_2|$.*

Proof. (i) For real τ near 0 let $\Gamma(\tau)$ be the solution near g_0 of

$$g\exp\big((\phi(f^{(2)}) - 2\tau)N\big) = c_k.$$

As in the proof of Lemma 425(ii),

$$d(\Gamma(\tau),\Gamma(\tau')) \le \lambda\,d(\Gamma(\tau),\Gamma(\tau')) + 2\|N\|\,|\tau - \tau'|,$$

so Γ is Lipschitz with constant $2\cdot 0.8945/(1-\lambda) < 1.989$. Since $\phi(t) = 2 - 2t$ on $(\frac12, 1)$, $\phi(f^{(2)}) - 2\tau = \phi(f^{(2)} + \tau)$, so $\Gamma(0) = G_2$ and $\Gamma(h_1) = G_1$. Let $k(\tau) = f^{(1)}(\Gamma(\tau),\xi) - f^{(2)} - \tau$. By the proof of Lemma 425(i), $f^{(1)}$ is σ-Lipschitz in g, with $\sigma = \frac{\sqrt2}{32}$, so $k(\tau') - k(\tau) \le -(1 - 1.989\,\sigma)(\tau' - \tau)$ for $\tau' > \tau$, and $1.989\,\sigma < 0.088$. As $k(0) = h_2$ and $k(h_1) = 0$, h_1 and h_2 have the same sign. If $h_1 > 0$, then $f^{(1)} > f^{(2)}$ at G_1, so $F = F_1$ near G_1 and $F(G_1) = c_k$; by uniqueness in Lemma 425(i), $G = G_1$ and $D = |\det D_g F_1| = D_1$, the determinant being positive since $\|A\| \le \lambda$. The case $h_2 < 0$ is the same with F_2.

(ii) At ξ_0, $G_1 = G_2 = g_0$ and $Y_1 = Y_2 = Y$, while $D_g Y_1 = \phi'(f^{(1)})N \otimes \ell + \phi\, D_g N = -2N \otimes \ell + \phi\, D_g N$ and $D_g Y_2 = \phi\, D_g N$. Since $[Y,N] = 0$, $\Phi(\mathrm{ad}\,Y)N = N$, so $1 + A_1 = (1 + A_2) - 2N \otimes \ell$, and the determinant of a rank-one perturbation gives $D_1 = D_2(1 - 2\ell((1+A_2)^{-1}N))$. The operator $\mathrm{ad}\,Y$ is normal with eigenvalues $0, \pm i\rho$, and $|\Phi(i\rho)| = |\sin\frac{\rho}{2}|/\frac{\rho}{2} \le 1$, so $\|\Phi(\mathrm{ad}\,Y)\| \le 1$. By the proof of Lemma 425(i), $\|D_g N\| \le \frac{7}{16}\Lambda\sigma < 0.02113$, $\|\ell\| \le \sigma$ and $\|N\| < 0.8945$, so $|\kappa_0| \le 0.044195 \cdot 0.8945/(1 - 0.02113) < 0.04039$.

(iii) Identify $\mathrm{SU}(2)$ with the unit quaternions, and write $\langle X, Y\rangle = -\mathrm{Tr}(XY)$ on $\mathfrak{su}(2)$ and $\Pi(z) = \frac{1}{2}(z - z^*)$ for the projection of 2×2 matrices to $\mathfrak{su}(2)$, orthogonal for $\mathrm{Re}\,\mathrm{Tr}(X^*Y)$. For $z \in \mathrm{SU}(2)$, $\delta(z,I)^2 = 4 - 2\,\mathrm{Re}\,\mathrm{Tr}\,z$, so the derivative of $\delta(ze^{s\eta}, I)$ at $s = 0$ is $-\mathrm{Re}\,\mathrm{Tr}(z\eta)/\delta = \langle \Pi(z), \eta\rangle/\delta$. At fixed g, with $z_1 = \tilde{u}_a \xi_a$ and $z_2 = w_a \xi_a$, and since $\mathrm{Re}\,\mathrm{Tr}(\xi_a e^{s\eta} w_a) = \mathrm{Re}\,\mathrm{Tr}(w_a \xi_a e^{s\eta})$, the derivatives of $f^{(1)}$ and $f^{(2)}$ along η_a are $p(\eta_a) = \langle \Pi(z_1), \eta_a\rangle/m$ and $q(\eta_a) = \langle \Pi(z_2), \eta_a\rangle/m$. If $z = \cos\theta + \sin\theta\, n$ with $n \in \mathfrak{su}(2)$, $n^2 = -1$, then $\Pi(z) = \sin\theta\, n$ and $\|n\| = \sqrt{2}$, so $|p| = \sqrt{2}\sin\theta/(2\sqrt{2}\sin\frac{\theta}{2}) = \cos\frac{\theta}{2} =: \gamma$, and $|q| = \gamma$ as z_2 has the same δ. As $m < 1$, $\theta < \alpha$, so $\gamma^2 > \cos^2\frac{\alpha}{2} = \frac{1}{2}(1 + \cos\alpha) = \frac{7}{8}$, since $\cos\alpha = 1 - 2\sin^2\frac{\alpha}{2} = \frac{3}{4}$. Put $\varpi = p - (1 - 2\kappa_0)q$. Since $|p| = |q| = \gamma$ and $|\kappa_0| < \frac{1}{2}$,

$$|\varpi| \ge \big|\,|p| - (1 - 2\kappa_0)|q|\,\big| = 2\gamma|\kappa_0|,$$

whatever the angle between p and q. Let η_a be the unit vector dual to $\varpi/|\varpi|$, and $\eta_a = 0$ if $\varpi = 0$; then $(p-q)(\eta_a) + 2\kappa_0 q(\eta_a) = \varpi(\eta_a) = |\varpi|$.

Differentiating $F_2(G_2(\xi), \xi) = c_k$, translated to the identity as in the proof of Lemma 425(iii), gives $(1 + A_2)\,dG_2(\eta) = -\Phi(\mathrm{ad}\,Y)(-2q(\eta_a)N + \phi\,\partial N(\eta))$, where ∂N is the derivative of N in ξ at fixed g. With $\ell' = \ell \circ (1 + A_2)^{-1}\Phi(\mathrm{ad}\,Y)$, so that $\ell'(N) = \kappa_0$,

$$\begin{aligned} dh_2(\eta) &= \ell(dG_2(\eta)) + (p - q)(\eta_a) \\ &= (p - q)(\eta_a) + 2\kappa_0 q(\eta_a) - \phi\,\ell'(\partial N(\eta)). \end{aligned}$$

In the translated form the differential of log at $v = e^X$ is $\psi(\mathrm{ad}\,X)$ with $\psi(z) = z/(1 - e^{-z})$, since $e^{-X}de^X = \Phi(-\mathrm{ad}\,X)\,dX$ and $\Phi(-z)\psi(z) = 1$; and $\xi_b e^{s\eta} w_b = \xi_b w_b e^{s\,\mathrm{Ad}(w_b^{-1})\eta}$. So $\partial N(\eta) = \frac{1}{16}\sum_b T_b \eta_b$ with $T_b = \psi(\mathrm{ad}\,X_b) + \psi(\mathrm{ad}\,X_b')\,\mathrm{Ad}(w_b^{-1})$, $X_b = \log v_{x_b}$, $X_b' = \log v_{x_b'}$. If X has eigenangle $\vartheta < \alpha$, $\mathrm{ad}\,X$ is normal with eigenvalues $0, \pm 2i\vartheta$, and $\psi(\pm 2i\vartheta) = \vartheta e^{\pm i\vartheta}/\sin\vartheta$; so $\|\psi(\mathrm{ad}\,X)\| \le \Lambda$, and the symmetric part of $\psi(\mathrm{ad}\,X)$, which is also that of its transpose $\psi(-\mathrm{ad}\,X)$, has eigenvalues 1 and $\vartheta\cot\vartheta \ge \alpha\cot\alpha = 3\alpha/\sqrt{7} > 0.8194$. Every path has $\delta < 1$ at a tie at the top, so $\|T_a\| \le 2\Lambda$. For $b \ne 0$, $\xi_b = \tilde{u}_b^{-1} e^{X_b}$ and $\xi_b w_b = e^{X_b'}$, so $w_b^{-1} = e^{-X_b'}\tilde{u}_b^{-1}e^{X_b}$ and $\mathrm{Ad}(w_b^{-1}) = e^{-\mathrm{ad}\,X_b'}\,\mathrm{Ad}(\tilde{u}_b^{-1})e^{\mathrm{ad}\,X_b}$; with $\psi(z)e^{-z} = \psi(-z)$ this gives

$$T_b = S_b\, e^{\mathrm{ad}\,X_b}, \qquad S_b = \psi(-\mathrm{ad}\,X_b) + \psi(-\mathrm{ad}\,X_b')\,\mathrm{Ad}(\tilde{u}_b^{-1}).$$

As $\|\mathrm{Ad}(\tilde{u}_b^{-1}) - 1\| = \|\mathrm{Ad}(u_b) - 1\| \le \sigma$ by the proof of Lemma 425(i),

$$\langle S_b\zeta, \zeta\rangle \ge (2 \cdot 0.8194 - 1.0927\,\sigma)\|\zeta\|^2 > 1.5905\,\|\zeta\|^2,$$

and $e^{\mathrm{ad}\,X_b}$ is orthogonal, so $|T_b^*\ell'| = |S_b^*\ell'| \ge 1.5905\,|\ell'|$ for every $b \ne 0$, whatever the link w_b, since $|S_b^*\ell'|\,|\ell'| \ge \langle S_b\ell'^\sharp, \ell'^\sharp\rangle$. Fix $c \ne 0, a$, take $\eta_b = 0$ for $b \ne a, c$, and $\eta_c = 0$ if $\ell' = 0$, otherwise η_c the vector dual to $-\ell'(T_a\eta_a)\,T_c^*\ell'/|T_c^*\ell'|^2$. Then $\ell'(\partial N(\eta)) = 0$, $\|\eta_c\| \le 2\Lambda/1.5905 < 1.3741$ and $\|\eta\| < (1 + 1.3741^2)^{1/2} < 1.6995$. Hence $dh_2(\eta) = |\varpi| \ge 2\gamma|\kappa_0|$, and with $\gamma > (\frac{7}{8})^{1/2} > 0.93541$, $2|\kappa_0| \le (1.6995/\gamma)|\nabla h_2| < 1.8169\,|\nabla h_2|$. Finally $|\log(1-x)| \le |x|/(1-|x|)$ for $|x| < 1$ and $2|\kappa_0| < 0.0808$ give $|\log D_1 - \log D_2| \le 2.1758\,|\kappa_0| \le 2.1758 \cdot 0.9085\,|\nabla h_2| < 1.9811\,|\nabla h_2|$. □

Near a tie at the top of index a the ties of index a are exactly the zeros of h_2, by (i). Let Z_a be the set of those ties at the top of index a at which $\nabla h_2 \ne 0$, a hypersurface of $\mathrm{SU}(2)^7$, and let H^{20} be its Hausdorff measure for the metric above, under which $\mathrm{SU}(2)$ has volume $V = 4\sqrt{2}\,\pi^2$. Put

$$\mathscr{A}_{\mathrm{tie}} = \sup V^{-7} H^{20}(Z_a),$$

the supremum over $a \ne 0$, over c_k, over links u_b within $\frac{1}{32}$ of I and over arbitrary links w_b.

Proposition 429 (the Jacobian across the ties)**.** *Let k, B and the move $e \to e'$ be as in Theorem 427. Then*

$$\int \big|\log D - \log D'\big|\,d\xi \le \big(53.96 + 15.134\,\mathscr{A}_{\mathrm{tie}}\big)\,d(e,e')$$

for $d(e,e') \le 0.08$, with $d\xi$ Haar measure on $\mathrm{SU}(2)^7$*, and* (J) *holds with $K \le (148.2 + 41.6\,\mathscr{A}_{\mathrm{tie}})\,e^{72\beta_W}$.*

Proof. Where the type is kept. On flat points $D_g Y = \phi(m)\,D_g N$, since $\phi'(m)D_g m = 0$ there: either $\phi'(m) = 0$, or the maximum sits at a path that does not depend on g. At points of type a, $D_g Y = -2N \otimes D_g f_{x_a} + \phi(m) D_g N$. With $\Phi(z) = (e^z - 1)/z$, the translated differential of F is $\mathrm{Ad}(E^{-1})(1 + A)$ with $A = \Phi(\mathrm{ad}\,Y)\,D_g Y$, so $D = \det(1 + A)$ and $\|A\| \le \lambda$ by Lemma 425(iii). Let ξ keep one type, or stay flat, for the parameters t of an interval of the move, at unit speed along the geodesic from e to e'. Then $\frac{d}{dt}\log D = \mathrm{tr}((1 + A)^{-1}\dot{A})$, of modulus at most $3\|\dot{A}\|/(1 - \lambda)$. We bound $\dot{A}$ with $\sigma = \frac{\sqrt{2}}{32} < 0.044195$, the norm $\|\mathrm{Ad}(u_a) - 1\| \le \sigma$, and $\sigma L < 0.09122$.

- By Lemma 425(ii), G moves at speed at most L, so each $\tilde{u}_c$ moves at speed at most σL, plus 1 for the moving link; every v_x, every f_x and m therefore move at speed at most $1 + \sigma L < 1.0913$, and Y at speed at most $\lambda L + (1 - \lambda)L = L$.
- $\|N\| \le \frac{7}{8}\sqrt{2}\,\alpha < 0.8945$, $\|D_g N\| \le \frac{7}{16}\Lambda\sigma < 0.02113$, $\|D_g f_{x_a}\| \le \sigma$, and $\|\dot{N}\| \le \frac{1}{16}\Lambda(1 + 7\sigma L) < 0.1120$.
- On $\mathfrak{su}(2)$, $\|\mathrm{ad}\,X\| = \sqrt{2}\|X\|$, so $\|\mathrm{ad}\,Y\| \le r = 1.2651$. The series of Φ gives $\|\Phi(\mathrm{ad}\,Y)\| \le 1$ and

$$\begin{aligned} \|\Phi(\mathrm{ad}\,Y) - \Phi(\mathrm{ad}\,Y')\| &\le \frac{re^r - e^r + 1}{r^2}\sqrt{2}\,\|Y - Y'\| \\ &\le 1.715\,\|Y - Y'\|. \end{aligned}$$

- $D_g f_{x_a} = \nabla\delta(v)\circ \mathrm{Ad}(\xi_a^{-1})(1-\mathrm{Ad}(\tilde u_a^{-1}))$ in translated form. At a point of type a, $\delta(v,I) = 2\sqrt2\sin\frac\theta2 \ge \frac12$, so $\sin\frac\theta2 \ge \frac{1}{4\sqrt2}$. The metric of SU(2) is that of the round sphere of radius $\sqrt2$, and the Hessian of the radial function δ has norm at most $\max\{\frac{1}{2\sqrt2}\sin\frac\theta2, \cos\theta/(2\sqrt2\sin\frac\theta2)\}$, at most $2\cos\theta \le 2(1-\frac{2}{32}) = \frac{15}{8}$. The translated derivative of the gradient adds $\frac12\|\mathrm{ad}\| \le 0.7072$. With $\|\frac{d}{dt}\mathrm{Ad}(\tilde u_a^{-1})\| \le \sqrt2(1+\sigma L)$, this gives $\|\frac{d}{dt} D_g f_{x_a}\| \le 2.5822\cdot 1.0913\,\sigma + 1.41422\cdot 1.0913 < 1.6679$.
- $D_g N = \frac{1}{16}\sum_c \mathrm{dlog}_{v_{x_c}}\circ \mathrm{Ad}(\xi_c^{-1})(1-\mathrm{Ad}(\tilde u_c^{-1}))$. In translated form $\mathrm{dlog}_v = \psi(\mathrm{ad}X)$, where $X = \log v$ and $\psi(z) = z/(1-e^{-z}) = 1 + \frac z2 + \frac{z^2}{12} - \frac{z^4}{720} + \dots$. With $\|\mathrm{ad}X\| \le 2\alpha < 1.4457$ the series gives Lipschitz constant $0.76\sqrt2\Lambda < 1.176$ in v. Hence $\|\frac{d}{dt} D_g N\| \le \frac{1}{16}(1.176\,\sigma + \sqrt2\Lambda)(1+7\sigma L) < 0.1636$.

Since $\phi' \in \{0,-2\}$ is constant along the interval and ϕ is 2-Lipschitz,

$$\Big\|\frac{d}{dt} D_g Y\Big\| \le 2\big(0.1120\,\sigma + 0.8945\cdot 1.6679\big) + 2\cdot 1.0913\cdot 0.02113 + 0.1636 < 3.204,$$

and $\|\dot A\| \le 1.715 L\lambda + 3.204 < 3.559$. So $|\frac{d}{dt}\log D| \le 3\cdot 3.559/(1-\lambda) < 11.87$ on the interval. The same bounds hold for $\log D_1$ and $\log D_2$ near a tie at the top, since D_1 has the form of type a and D_2 the flat form, and G_1, G_2 move at speed at most L by the proof of Lemma 425(ii).

The events. Call the following 77 conditions on ξ the *events*: $f_{x_a} = m \in \{\frac12, 1\}$; $f_{x_a} = f_{x_c'} = m \in [\frac12, 1]$ with $c \ne a$; and $f_{x_a} = f_{x_c} = m \in [\frac12, 1]$ with $c \ne a$, for $a, c \ne 0$. Every f_x moves at speed at most $1+\sigma L$, and at most one of them is moved directly. So for a difference h of two of them, or of one of them and a level, $|h| \le \omega\, d(e,e')$ before the move, if ξ meets the event $h = 0$ during the move, with $\omega = 1 + 2\sigma L < 1.1825$.

Fix the coordinates other than ξ_a and use $v = \tilde u_a(G(\xi))\xi_a$ in place of ξ_a. At fixed g, a unit move of ξ_a moves v_{x_a} and $v_{x_a'}$ at unit speed. As in Lemma 425(i), Y then moves at speed at most $2\cdot\frac78\sqrt2\,\alpha + \frac18\Lambda < 1.9256$, and G at speed at most $L_\xi = 1.9256/(1-\lambda) < 2.140$. Since $d(\tilde u\xi, \tilde u'\xi') $ differs from $d(\xi,\xi')$ by at most $d(\tilde u, \tilde u') \le \sigma L_\xi\, d(\xi,\xi')$, the map $\xi_a \mapsto v$ is a bi-Lipschitz bijection of SU(2), with constants 1 ± 0.0946. It is onto by invariance of domain, and its inverse multiplies Haar measure by at most $(1/0.9054)^3 < 1.348$.

Write $v = \exp(\theta\nu)$ with $\theta \in [0,\pi]$ and ν a unit direction. The Haar measure is $\frac{2}{\pi}\sin^2\theta\, d\theta$ times the uniform measure in ν. Along a ray of fixed ν:

- $f_{x_a} = 2\sqrt2\sin\frac\theta2$ increases at rate $\sqrt2\cos\frac\theta2 > 1.3028$ while $f_{x_a} \le 1.1$, that is $\theta \le 0.7990$;
- ξ_a moves at speed at most $\sqrt2/0.9054$, so f_{x_c}, $c \ne a$, which depends on ξ_a only through G, moves at rate at most $\sigma L_\xi\sqrt2/0.9054 < 0.1478$;
- $f_{x_c'}$, $c \ne a$, and the levels do not move.

For each event, $h = f_{x_a} - f_y$ or $f_{x_a} - \frac12$ or $f_{x_a} - 1$ therefore increases at rate at least 1.154 on $\{0.4 \le f_{x_a} \le 1.1\}$, and $\{|h| \le t\}$ meets each ray in an interval of length at most $2t/1.154$. With $\sin^2\theta \le 0.5146$ for $\theta \le 0.7990$, its Haar measure in v is at most $\frac{2}{\pi}\cdot 0.5146\cdot\frac{2}{1.154}t < 0.5678\,t$, and in ξ_a at most $0.766\,t$. For $d(e,e') \le 0.08$ every ξ that meets an event during the move lies in such a set with $t = \omega\, d(e,e')$. By Fubini these ξ have Haar measure at most $77\cdot 0.766\cdot 1.1825\, d(e,e') < 69.75\, d(e,e')$, and for them $|\log D - \log D'| \le 3\log\frac{1+\lambda}{1-\lambda} < 0.6034$.

The ties. Let ξ meet no event during the move, with D and D' defined at ξ, and let C be the set of parameters t at which ξ is a tie at the top of some index. Between two times of C the point ξ keeps one type or stays flat: the maximal path can change only at an event, at a tie at the top, or between two paths from points x_c', where D does not jump. At a time $t_0 \in C$ of index a, Lemma 428 applies with the inside links as parameters, and the times near t_0 in C are the zeros of the analytic function $t \mapsto h_2$; since no event occurs, C is closed. If C has an interior point, it is open by analyticity, so $C = [0, d(e,e')]$; otherwise C is finite. Where D is defined at a tie at the top, $\kappa_0 = 0$: if $\ell \ne 0$, both branches meet G, so $D_g F_1 = D_g F_2$, that is $N\otimes\ell = 0$ and $N = 0$. Hence $\log D = \log D_1 = \log D_2$ at the ends that lie in C. By Lemma 428(i) and the bound on the derivatives,

$$|\log D - \log D'| \le 11.87\, d(e,e') + \sum_t 1.9811\,|\nabla h_2|,$$

the sum over the times $t \in C$ at which h_2 changes sign, the terms with $\nabla h_2 = 0$ vanishing; so the sum runs over times t with $\xi \in Z_a(t)$, the set Z_a at the parameter t.

For one index a, the set $S = \{(t,\xi) : \xi \in Z_a(t)\}$ is locally the zero set of $H(t,\xi) = h_2$ with $\nabla_\xi H \ne 0$, a C^1 hypersurface of $[0, d(e,e')]\times \mathrm{SU}(2)^7$. The area formula for the projection $(t,\xi) \mapsto \xi$, of Jacobian $|\partial_t H|/|\nabla H|$ on S, and the coarea formula for $(t,\xi)\mapsto t$, of Jacobian $|\nabla_\xi H|/|\nabla H|$ on S [51, 3.2.20, 3.2.22], give

$$\int \sum_{t:\,\xi\in Z_a(t)} w\, d\mathrm{vol}(\xi) = \int_0^{d(e,e')}\int_{Z_a(t)} w\,\frac{|\partial_t H|}{|\nabla_\xi H|}\, dH^{20}\, dt$$

for $w \ge 0$. Here $H = f^{(1)}(G_2,\xi) - f^{(2)}$ changes at rate at most $1+\sigma L < 1.0913$, since G_2 moves at speed at most L, $f^{(1)}$ is σ-Lipschitz in g, and only the moving link enters directly. With $w = 1.9811|\nabla_\xi H|$ and $d\xi = V^{-7} d\mathrm{vol}$, the ties of index a contribute at most $1.9811\cdot 1.0913\,\mathscr{A}_{\mathrm{tie}}\, d(e,e')$, and the seven indices at most $15.134\,\mathscr{A}_{\mathrm{tie}}\, d(e,e')$.

Conclusion. Adding the three parts, with $11.87 + 0.6034\cdot 69.75 < 53.96$, gives the first bound. The density of $\hat Q$ is at most $1/(Z(1-\lambda)^3) \le 2.00002\, e^{72\beta_W}/0.7285$ by Lemma 426, and $2.00002\cdot 53.96/0.7285 < 148.2$, $2.00002\cdot 15.134/0.7285 < 41.6$. Since $|\log D - \log D'|$ is at most the sum of its increments along a subdivision of the geodesic into steps of length at most 0.08, and the bound on the density does not depend on the step, (J) holds with the stated K. □

Proposition 430 (the area of the ties). $\mathscr{A}_{\mathrm{tie}} < \infty$.

Proof. Identify $\mathrm{SU}(2)$ with the unit sphere of $\mathbb{R}^4$; the metric above is twice the Euclidean one, so H^{20} is 2^{10} times the Euclidean Hausdorff measure on $\mathrm{SU}(2)^7 \subset \mathbb{R}^{28}$. Every object of the construction is definable in the o-minimal structure $\mathbb{R}_{\mathrm{an}}$ of restricted analytic functions [47][40]: δ, ϕ and the maximum are semialgebraic; log on $\{\mathrm{Re}\,\mathrm{Tr}\, v \geq \frac{3}{2}\}$, which contains $\{\delta(v,I) \leq 1\}$, and exp on $\{\|X\| \leq 1\}$ are restricted analytic; N may be set to 0 where $m \geq 1$ without changing $Y = \phi(m)N$; and the graph of G is $\{g \exp Y = c_k\}$. The formula for dh_2 in the proof of Lemma 428 is a definable function of g, ξ and the parameters $P = (c_k, (u_b, w_b)_{b\neq 0})$, so $Z_a(P)$ is a definable family over a bounded set of parameters, of C^1 submanifolds of dimension 20. For an orthogonal projection ϖ of $\mathbb{R}^{28}$ onto $\mathbb{R}^{20}$ and $y \in \mathbb{R}^{20}$, the sets $Z_a(P) \cap \varpi^{-1}(y)$ form a definable family with parameters (P, ϖ, y), so those that are finite have at most $\mathfrak{b}$ points, with $\mathfrak{b}$ independent of (P, ϖ, y) [48, Ch. 3]; and for fixed (P, ϖ) the y with an infinite fibre form a definable set of dimension at most 19 [48, Ch. 4], which is Lebesgue-null. By the integralgeometric formula, $H^{20}(Z)$ equals a constant depending only on $(28, 20)$ times the average over ϖ of $\int \#(Z \cap \varpi^{-1}(y))\, dy$ [51, 2.10.15, 3.2.26], and $\varpi(Z)$ lies in the ball of radius $\sqrt{7}$. So $H^{20}(Z_a(P))$ is bounded uniformly in P and a. □

Theorem 431 (condition (J))**.** *Let k, B and the move be as in Theorem 427. Then* (J) *holds with $K = (148.2 + 41.6\,\mathscr{A}_{\mathrm{tie}})\, e^{72\beta_W}$, where $\mathscr{A}_{\mathrm{tie}} < \infty$ does not depend on k, c_k, the inside links or β_W. In particular* (J) *holds for B_r^{bad} on every configuration that the labels produce, and for $\mathrm{SU}(2)^7$ whenever the links u_a, $a \neq 0$, stay within $\frac{1}{32}$ of I along the geodesic, whatever the links w_a. For $B = \mathrm{SU}(2)^7$ and links w_a, $a \neq 0$, that stay within $\frac{1}{32}$ of I along the geodesic,* (J) *holds with the same K in the coordinates $\xi_a = \ell_a g^{-1}$ of Lemma 425, whatever the links u_a.*

Proof. Proposition 429 and Proposition 430, whose hypotheses bound only the links u_a; the case of B_r^{bad} by Proposition 424. For the last statement let $\mathfrak{i}$ send each private link ℓ_a to ℓ_a^{-1} and the pair (u_a, w_a) of inside links to (w_a^{-1}, u_a^{-1}), $a \neq 0$. It maps P_{2y} to P_{2y}^{-1} and the paths from x_a and x_a' to the inverses of the paths from x_a' and x_a. Since δ is invariant under inversion and $\log(P_{2y}P_x^{-1}) = -\mathrm{Ad}(P_{2y}) \log(P_{2y}^{-1}P_x)$, (35) gives $V_k^\sharp \circ \mathfrak{i} = (V_k^\sharp)^{-1}$. The map $\mathfrak{i}$ preserves Haar measure and the cost, and it maps the coordinates $\xi_a = \ell_a g^{-1}$ at the value c_k to the coordinates $\bar{g}^{-1}\bar{\ell}_a = \xi_a^{-1}$ of the inverted links $\bar{\ell} = \mathfrak{i}(\ell)$ at c_k^{-1}, with $V_k^\sharp = \exp(\chi N')\, g$ turned into $\bar{g} \exp(-\chi N')$; inversion is an isometry, so the two Jacobians D agree at corresponding points. The action enters Lemmas 425 and 426 and Proposition 429 only through the bounds $0 \leq S_{\mathrm{dep}} \leq 72$, which hold for every configuration. So (J) in the coordinates $\ell_a g^{-1}$ for (u, w) is (J) in the coordinates $g^{-1}\ell_a$ for (w^{-1}, u^{-1}), whose links of the first block are within $\frac{1}{32}$ of I, and the moving link keeps its distance. □

Corollary 432 (the first set with one block near I)**.** *Let $B = \mathrm{SU}(2)^7$ and let one link u_b or w_b, $b \neq 0$, move from e to e'. If the inside μ-links of one of the two blocks, the u_a or the w_a, $a \neq 0$, stay within $\frac{1}{32}$ of I along the geodesic, then, whatever the inside μ-links of the other block,*

$$W_1(Q, Q') \leq \big(16.52 + 2898\beta_W + 41.37K\big)\, d(e, e')$$

for the cost $\sum_a \delta$, with K of Theorem 431.

Proof. If the u_a stay within $\frac{1}{32}$ of I, this is Theorem 427 for $B = \mathrm{SU}(2)^7$ and Theorem 431. If the w_a do, then $\sigma_w \leq \frac{\sqrt{2}}{32}$, so $\lambda_w \leq \lambda$, $L_w \leq L$ and $R_w < 1.828$, and the last bound of Theorem 427, with (J) in the coordinates $\xi_a = \ell_a g^{-1}$ from Theorem 431, is at most $8L + 16\sqrt{2} \cdot 1.828\,(24\sqrt{2}L\beta_W + K)$ times $d(e, e')$; here $8L < 16.52$, $768 \cdot 1.828 \cdot 2.064 < 2898$ and $16\sqrt{2} \cdot 1.828 < 41.37$. □

The proof of Proposition 430 gives no value for $\mathscr{A}_{\mathrm{tie}}$: the number $\mathfrak{b}$ comes from o-minimality. The weight $|\log D_1 - \log D_2|$ in the flux cannot be dropped, since the near-ties need not be thin.

Proposition 433 (thick near-ties)**.** *Fix $a \neq 0$, let $u_a = w_a = u$ with $d(u, I) = s \leq \frac{1}{32}$, the other inside links within $\frac{1}{32}$ of I, and let $T(t)$ be the set of ξ with*

$$|f_{x_a} - f_{x_a'}| \leq t, \qquad f_{x_a} \geq m - t, \qquad \tfrac{1}{2} - t \leq f_{x_a} \leq 1 + t.$$

Then $T(t)$ contains the open set $U_0 = \{\xi : \frac{17}{32} < \delta(\xi_a, I) < \frac{31}{32},\ \delta(\xi_b, I) < \frac{15}{32}$ for $b \neq a\}$ for every $t \geq 2s$, for every c_k. For $s = 0$ every point of U_0 is a tie at the top of index a, and $\nabla h_2 = 0$ there.

Proof. Since $\delta(v, I) = \|v - I\|$ and $\|\tilde{u} - I\| = \|u - I\| \leq s$, $|f_{x_a} - \delta(\xi_a, I)| \leq s$ and $|f_{x_a'} - \delta(\xi_a, I)| \leq s$, using $\delta(\xi_a u, I) = \delta(u\xi_a, I)$; so on U_0 both lie in $(\frac{1}{2}, 1)$ and differ by at most $\|\tilde{u} - u\| \leq 2s$. For $b \neq a$, $f_{x_b}, f_{x_b'} \leq \delta(\xi_b, I) + \frac{1}{32} < \frac{1}{2}$, and the paths of offset zero give 0. So $m = \max(f_{x_a}, f_{x_a'})$ and $f_{x_a} \geq m - 2s$. For $s = 0$, $f_{x_a} = f_{x_a'} = \delta(\xi_a, I)$ on U_0, so h_2 vanishes on the open set U_0. □

Fig. 36 shows why the near-ties fill an open set.

So no bound $\mathrm{Haar}(T(t)) \leq Mt$ holds with M uniform in the inside links, nor one by $t|\log t|$: the Haar measure of $T(2s)$ is at least that of U_0 as $s \to 0$. At $s = 0$ the jump vanishes, since $\ell = 0$ when $u_a = I$.

The other inside links of the two blocks are not μ-links, do not enter $V_k^\sharp$, and lie in at most one plaquette with a private link; the proof of Lemma 415 applies to them unchanged and bounds the response to each of them by $8\beta_W$ for every B. So the response of the law of the eight private links to every inside link is bounded, with the constants of Theorem 427, as soon as the Jacobian condition (J) holds: for B_r^{bad} on every configuration that the labels produce, since there $d(u_a, I), d(w_a, I) \leq r$ by

[40] A structure on $\mathbb{R}$ is o-minimal when every definable subset of $\mathbb{R}$ is a finite union of points and intervals. In an o-minimal structure the number of connected components of the fibres of a definable family is bounded uniformly in the parameter [48, Ch. 3]; this uniformity is what makes the constant $\mathfrak{b}$ below independent of the configuration.

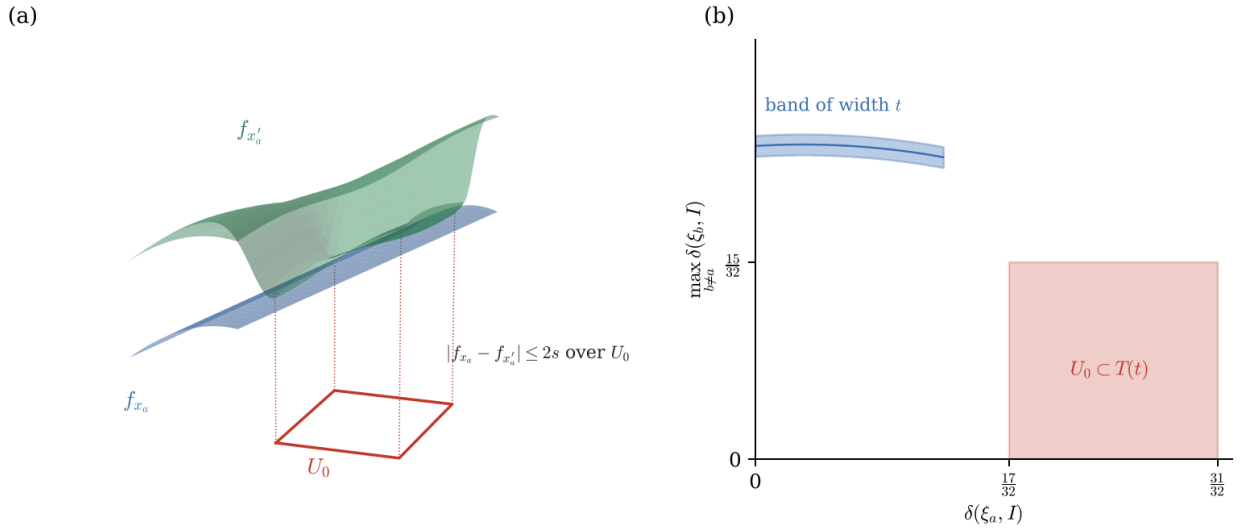


FIG. 36. Thick near-ties (Proposition 433), schematic. (a) The values f_{x_a} and $f_{x'_a}$ of the two paths of index a over two coordinates of ξ, for $u_a = w_a = u$ at distance s from I: over the open set U_0 they differ by at most $2s$, so the set $T(t)$ of near-ties contains U_0 for every $t \ge 2s$. (b) The set U_0 in the coordinates $\delta(\xi_a, I)$ and $\max_{b\neq a} \delta(\xi_b, I)$, beside a band of width t around a tie hypersurface, the shape that a bound $\mathrm{Haar}(T(t)) \le Mt$ would require.

Proposition 424, and for $\mathrm{SU}(2)^7$ on every configuration whose links u_a lie within $\frac{1}{32}$ of I. Theorem 431 proves (J) there for every w, in particular for B_r^{bad}, and in the coordinates $\ell_a g^{-1}$ when the w_a lie within $\frac{1}{32}$ of I, whatever the u_a; Corollary 432 bounds the response of the first set in both cases. By Lemma 425(iii), D lies in $[0.728, 1.332]$ for all inside links, so (J) asks only for its continuity in the mean. The derivative of $Y = \chi N$ in g contains $N \otimes \phi'(m) D_g m$, and $D_g m$ vanishes when the largest $\delta(v_x, I)$ is attained at a path from x'_a, which does not depend on g; so D jumps across the sets where the maximal path passes between a path from some x_a and another path while $\frac{1}{2} < m < 1$, and these sets move with the inside links. Off them D is a smooth function of the inside links. Condition (J) is the statement that the $\hat{Q}$-mass swept by these sets, weighted by the jumps of $\log D$, is of order $d(e, e')$. The last part of the theorem shows that the obstruction needs both blocks: if all the inside links of one of them are I, the law responds boundedly to every inside link of the other, without (J). For the sets B_ε of Theorem 423, whose inside links are within $\frac{1}{32}$ of I, Lemma 425 applies and the fibre is a graph over ξ; the bound fails there through the restricting set, whose boundary is not in the good shell and whose normaliser tends to zero with ε. For the first set the chart is available whenever $\sigma_u < 1/\kappa$ or $\sigma_w < 1/\kappa$, by the last part of Lemma 425, and the response is then bounded under (J) in that chart. A chart based at another path does not remove the condition: with $h = P_{x_b} = u_b \ell_b$ as base and $\xi_a = h^{-1} \ell_a$ $(a \neq b)$, $\xi_0 = h^{-1} g$, the paths are $v_{x_a} = \xi_0^{-1} (h^{-1} u_a h) \xi_a$, $v_{x'_a} = \xi_0^{-1} \xi_a w_a$, $v_{x_b} = \xi_0^{-1}$ and $v_{x'_b} = \xi_0^{-1} (h^{-1} u_b^{-1} h) w_b$, the equation $V_k^\sharp = c_k$ reads $h = c_k \exp(-\chi N)\, \xi_0^{-1}$, and h enters only through $\mathrm{Ad}(h)$ on the u_a, with the contraction constant $\kappa\sigma_u$ again; basing at $h = P_{x'_b}$ with $\xi_a = \ell_a h^{-1}$ leads in the same way to $\mathrm{Ad}(h)$ on the w_a and to $\kappa\sigma_w$. When $\sigma_u \ge 1/\kappa$ and $\sigma_w \ge 1/\kappa$ the argument of Lemma 425(i) gives no contraction in these charts, and for some such links no chart of this kind exists.

Proposition 434 (a fold of the relative charts)**.** *Fix $a \neq 0$, let $\mathbf{i} = \mathrm{diag}(i, -i)$ and $\mathbf{j} = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, and take the inside μ-links*

$$u_a = \mathbf{j}, \qquad w_a = \mathbf{j}\, e^{\mathbf{i}/10}, \qquad u_b = w_b = I \quad (b \neq 0, a),$$

so that $\sigma_u = \sigma_w = 2$. There are an open set $\mathscr{U}$ of values of the inside μ-links containing this one and open sets $U \subset \mathrm{SU}(2)^7$, $O \subset \mathrm{SU}(2)$ such that, for all inside μ-links in $\mathscr{U}$, $\xi \in U$ and $c_k \in O$, the equation $V_k^\sharp = c_k$ has at least two solutions g with $\ell_a = g \xi_a$. The same holds, with other open sets, in the coordinates $\xi_a = \ell_a g^{-1}$. So on an open set of inside links with $\sigma_u, \sigma_w \ge 1/\kappa$ the fibre $\{V_k^\sharp = c_k\}$ is a graph over ξ in neither coordinates, and neither chart of Lemma 425 exists.

Proof. Put $\mathbf{k} = \mathbf{ij}$, so that $\mathbf{j} e^{t\mathbf{i}} = e^{-t\mathbf{i}} \mathbf{j}$, $\mathbf{j} e^{t\mathbf{k}} = e^{-t\mathbf{k}} \mathbf{j}$ and $\mathbf{j}^2 = -I$, and $\delta(e^{t\mathbf{i}}, I) = 2\sqrt{2} \sin \frac{t}{2}$ for $t \in [0, \pi]$. As $\mathrm{Ad}(u_a)$ and $\mathrm{Ad}(w_a)$ are rotations by π, $\sigma_u = \sigma_w = 2$. Let $\theta = \frac{1}{2}$, $g_0 = I$, $\xi_a^0 = -\mathbf{j} e^{\theta \mathbf{i}}$ and $\xi_b^0 = e^{-2\mathbf{i}/5}$ for $b \neq 0, a$. Then $v_{x_a} = \tilde{u}_a \xi_a^0 = e^{\theta \mathbf{i}}$, $v_{x'_a} = \xi_a^0 w_a = e^{-2\mathbf{i}/5}$, $v_{x_b} = v_{x'_b} = e^{-2\mathbf{i}/5}$ for $b \neq 0, a$, and the two paths of offset zero are I. By $t - t^3/6 \le \sin t \le t$, $m = 2\sqrt{2} \sin \frac{1}{4} \in (0.6997, 0.7072)$, attained at x_a only, the other paths having $\delta \le 2\sqrt{2} \sin \frac{1}{5} < 0.566$. So near (g_0, ξ^0) and this configuration the maximum is attained at x_a only, $\frac{1}{2} < m < 1$, and $F(g, \xi) = g \exp(\phi(f_{x_a}) N)$ is analytic, with $Y = \chi N$, $\chi = 2 - 2m$, and

$$N = \tfrac{1}{16} \left(\theta - 13 \cdot \tfrac{2}{5}\right) \mathbf{i} = -\tfrac{47}{160}\, \mathbf{i}.$$

As in the proof of Lemma 428(ii), translated to the identity $D_g F = \mathrm{Ad}(E^{-1})(1 + A)$ with $A = \Phi(\mathrm{ad}\, Y)(-2N \otimes \ell + \chi D_g N)$,

$\ell = D_g f_{x_a}$. Only v_{x_a} depends on g, and along $g = e^{t\zeta}$ at fixed ξ^0 it is $v_{x_a}(t) = e^{-t\zeta}\mathbf{j}e^{t\zeta}\xi^0_a$. For $\zeta = \mathbf{j}$ it is constant, so $A\mathbf{j} = 0$. For $\zeta = \mathbf{i}$, $e^{-t\mathbf{i}}\mathbf{j}e^{t\mathbf{i}} = \mathbf{j}e^{2t\mathbf{i}}$ and $v_{x_a}(t) = e^{(\theta-2t)\mathbf{i}}$, so $f_{x_a} = 2\sqrt{2}\sin(\frac{\theta}{2}-t)$, $\ell(\mathbf{i}) = -2\sqrt{2}\cos\frac{1}{4}$, $D_g N(\mathbf{i}) = -\frac{1}{8}\mathbf{i}$, and, since $\Phi(\mathrm{ad}\,Y)$ fixes $\mathbf{i}$,

$$A\mathbf{i} = a\mathbf{i}, \qquad a = -\tfrac{47\sqrt{2}}{40}\cos\tfrac{1}{4} - \tfrac{\chi}{8} < -\tfrac{47\cdot 31\sqrt{2}}{40\cdot 32} < -1.6,$$

by $\cos\frac{1}{4} \geq 1 - \frac{1}{32}$. For $\zeta = \mathbf{k}$, $e^{-t\mathbf{k}}\mathbf{j}e^{t\mathbf{k}} = \mathbf{j}e^{2t\mathbf{k}}$ and $v_{x_a}(t) = e^{-2t\mathbf{k}}e^{\theta\mathbf{i}}$, whose translated derivative $-2\,\mathrm{Ad}(e^{-\theta\mathbf{i}})\mathbf{k}$ lies in the span of $\mathbf{j},\mathbf{k}$. This span is orthogonal to $\Pi(v_{x_a}) = \sin\theta\,\mathbf{i}$, so $\ell(\mathbf{k}) = 0$; it is invariant under $\psi(\mathrm{ad}(\theta\mathbf{i}))$, of norm $\theta/\sin\theta$ there, and under $\Phi(\mathrm{ad}\,Y)$, of norm at most 1. So $A\mathbf{k} = b\mathbf{j} + a'\mathbf{k}$ with $|a'| \leq \frac{\chi}{8}\cdot\frac{\theta}{\sin\theta} < \frac{1}{8}\cdot\frac{0.5}{0.479} < 1$. In the basis $\mathbf{i},\mathbf{j},\mathbf{k}$ the matrix of $1+A$ is upper triangular, and

$$\det(1+A) = (1+a)(1+a') < 0.$$

As $\mathrm{Ad}(E^{-1}) \in \mathrm{SO}(3)$, $\det D_g F < 0$ at (g_0, ξ^0).

Let $c^0 = F(g_0,\xi^0)$. By the inverse function theorem with parameters, applied to F as a function of g, ξ and the inside links, there are a closed ball $\bar{B}$ about g_0 with interior B and open sets $\mathscr{U}$, $U \ni \xi^0$, $O \ni c^0$ such that for inside links in $\mathscr{U}$, $\xi \in U$ and $c \in O$ the equation $F(g,\xi) = c$ has exactly one solution in $\bar{B}$, which lies in B, and $\det D_g F < 0$ there. For fixed ξ and inside links, $F_\xi(g) = g\exp Y(g,\xi)$ is a continuous map of $\mathrm{SU}(2)$ to itself, homotopic to the identity through $g\exp(sY(g,\xi))$, $0 \leq s \leq 1$, so of degree 1. If the solution in B were the only solution of $F_\xi = c$, the degree would be the local degree there, which is -1, since F_ξ is a local diffeomorphism there reversing the orientation [45, Ch. VIII, Sec. 4]. So $F_\xi = c$ has a second solution, outside $\bar{B}$.

For the coordinates $\xi_a = \ell_a g^{-1}$ let $\mathfrak{i}$ be the inversion of the proof of Theorem 431. It maps the configuration above to $u_a = w_a^{-1} = -e^{-\mathbf{i}/10}\mathbf{j}$, $w_a = u_a^{-1} = -\mathbf{j}$, the other links I. With $h = e^{-\mathbf{i}/20}\mathbf{k}$, $\mathrm{Ad}(h)\mathbf{i} = -\mathbf{i}$ and $\mathrm{Ad}(h)\mathbf{j} = -e^{-\mathbf{i}/10}\mathbf{j}$, so this is the configuration above conjugated by h. Conjugating the eight private links and the inside links by h commutes with $V^\sharp_k$, since δ and log are equivariant, and maps the coordinates $g^{-1}\ell_a$ to their conjugates; so the first part holds for the conjugated configuration, with $h\mathscr{U}h^{-1}$, hUh^{-1} and hOh^{-1}. The involution $\mathfrak{i}$ maps the coordinates $g^{-1}\ell_a$ at the value c_k to the coordinates $\ell_a g^{-1}$ at c_k^{-1}, which gives the statement in these coordinates. □

The proposition concerns the charts only. The law of the eight private links exists for every configuration, and its response to the inside links is neither bounded nor shown unbounded there; what fails is the description of that law as the image of a density in ξ under one map, on which Theorem 427 rests.

The jumps of D come from the two operations in the definition of $\chi_{y,\mu}$ before (35) that are not differentiable, the function ϕ and the maximum over the sites; that definition is made in this paper, and every result on the cut-off average is proved for it. What the cube condition uses can instead be taken from the site, in a form whose units do not overlap.

The sites of two consecutive bad pairs along μ share seven inside links, so the sites themselves do not partition the bad side. The following units do.

Definition 435. Fix a direction μ and a line of blocks $y_j = y_0 + je_\mu$, and join two bad blocks $y_j, y_{j'}$ of the line when $|j-j'| \leq 2$. For a connected set of bad blocks of the line, with smallest index j_0 and largest j_1, its *chain* $\mathscr{C}$ is the set of the following links: the eight private links of each pair $k_j = (y_j,\mu)$ with y_j bad; the active inside μ-links of the blocks y_j, $j_0 \leq j \leq j_1+1$; and the path links of the pairs k_j, $j_0 - 1 \leq j \leq j_1 + 1$, with y_j good. The pairs k_j, $j_0 - 1 \leq j \leq j_1 + 1$, are the pairs of the chain, *inner* when y_j is bad and *outer* otherwise. The cost on $\mathscr{C}$ is $\sum_{e\in\mathscr{C}}\delta$.

Proposition 436 (the law of a chain)**.** *Let $r \leq \frac{1}{32}$, let $\mathscr{C}$ be a chain with n_b inner pairs, and suppose that the non-path links of every outer pair k_j are within r of c_{k_j} and the active inside μ-links of the blocks y_{j_0-1} and y_{j_1+2} within r of I. Let E be a measurable set of values of the links of $\mathscr{C}$ contained in $\{d(e,I) \leq r\}$ for every inside link $e \in \mathscr{C}$ of a block y_j with y_{j-1} or y_j good.*

(i) *Distinct chains, of the same or of different lines and directions, are disjoint. Every link of $\mathscr{C}$ is based in a block at ℓ^∞ distance at most one from a bad block of its connected set, and the cut-off averages that contain a link of $\mathscr{C}$ are those of the pairs of the chain.*

(ii) *The law of $\mathscr{C}$ given the other links σ, the constraints $V^\sharp_{k_j} = c_{k_j}$ of the pairs of the chain and E is*

$$\frac{1}{Z}e^{-\beta_W S}\mathbf{1}_E J_{\mathscr{C}}^{-1}\mathrm{vol}_{M_{\mathscr{C}}}\prod_{k_j\ outer}\frac{\delta_{\Gamma_j}(d\ell_0^{(j)})}{|\det D_{\ell_0^{(j)}}V^\sharp_{k_j}|},$$

where $M_{\mathscr{C}}$ is the set where the constraints of the inner pairs hold, in the links of $\mathscr{C}$ other than the outer path links $\ell_0^{(j)}$, $J_{\mathscr{C}} \geq \sigma_0^{3n_b}$ almost everywhere, each Γ_j is the unique solution of Proposition 416 for k_j, and the determinants lie in $[(1-\kappa_0)^3,(1+\kappa_0)^3]$.

(iii) *If σ,σ' differ only at one link $e' \notin \mathscr{C}$, then for the law π of (ii) and $n = |\mathscr{C}|$,*

$$W_1\big(\pi(\cdot\mid\sigma),\pi(\cdot\mid\sigma')\big) \leq C_{e'}\,d(\sigma_{e'},\sigma'_{e'})$$

for the cost on $\mathscr{C}$, with $C_{e'} = nn_p(e')\beta_W$ if e' enters only the Wilson weight, $n_p(e') \leq 6$ the number of its plaquettes that contain a link of $\mathscr{C}$;

$$C_{e'} = \sqrt{2}n(5.793\beta_W + 64.94) + 1.1986$$

if e' is a non-path link of an outer pair; and

$$C_{e'} = \sqrt{2}n(2.543\beta_W + 32.47) + 0.5993$$

if e' is an active inside μ-link of y_{j_0-1} or y_{j_1+2}. The constants do not depend on the c_{k_j}, on E or on the rest of σ.

(iv) $n = 8n_b + p_\mu(j_1 - j_0 + 2) + (j_1 - j_0 + 3 - n_b) \le 7n_b + 8(j_1 - j_0) + 17 \le 23n_b + 1$. *For one inner pair and $p_\mu = 7$, $\mathscr{C}$ is the site $\mathscr{S}_k$, and (ii), (iii) are Lemma 419 and Theorem 421.*

Proof. (i) A chain for μ consists of μ-links. A private link belongs to one pair and an inside link to one block, so chains of different lines are disjoint. On one line two connected sets satisfy $j_0' \ge j_1 + 3$, so the pairs $j_0 - 1, \dots, j_1 + 1$ and the blocks $j_0, \dots, j_1 + 1$ of the first and those of the second are disjoint. The links of $\mathscr{C}$ are based in the blocks y_j, $j_0 - 1 \le j \le j_1 + 1$, and each such block is bad or adjacent to a bad block of the set, gaps having length one. A private link of (y, μ) enters only $V^\sharp_{y,\mu}$, and an inside μ-link of $B(y)$ enters only $V^\sharp_{y,\mu}$ and $V^\sharp_{y-e_\mu,\mu}$, since the path of $V^\sharp_{y',\nu}$ from a site x consists of (x, ν) and $(x + e_\nu, \nu)$.

(ii) The constraint of an outer pair k_j contains its path link, which enters no other constraint, its non-path links, fixed within r of c_{k_j}, and the inside μ-links of y_j and y_{j+1}, which are within r of I on E or by hypothesis, because y_j is good. So Proposition 416 applies and gives the atom and the determinant bounds, as in the proof of Lemma 419. The remaining constraints, of the inner pairs, form a Lipschitz map Φ into $\mathrm{SU}(2)^{n_b}$, differentiable off a null set by Lemma 371, and the coarea formula gives $J_{\mathscr{C}}^{-1} \mathrm{vol}_{M_{\mathscr{C}}}$ with $J_{\mathscr{C}} = \det(D\Phi D\Phi^*)^{1/2}$. The private links of an inner pair enter only its own constraint, so the part of $D\Phi D\Phi^*$ from the private links is block diagonal with blocks at least σ_0^2 by Lemma 371, and $D\Phi D\Phi^*$ dominates it.

(iii) This is the proof of Theorem 421 with the diameter $24 \cdot 2\sqrt{2}$ replaced by $n \cdot 2\sqrt{2}$. By (i) an exterior link enters the law through the Wilson weight, or as a non-path link of one outer pair, which moves one atom at speed at most 1.1986 and shares at most one plaquette with a link of $\mathscr{C}$, the path link of its pair, because the μ-links of a plaquette have a common μ-coordinate and by the argument of Lemma 415; or as an inside link of y_{j_0-1} or y_{j_1+2}, which moves one atom at speed at most 0.5993 and shares no plaquette with a link of $\mathscr{C}$, the other μ-links at its μ-coordinate being inside links of its own block. The density terms are those of Lemma 420, and the Wilson terms are $(1 + 6 \cdot 1.1986)/\sqrt{2} \le 5.793$ and $6 \cdot 0.5993/\sqrt{2} \le 2.543$ as there.

(iv) The chain has $8n_b$ private links of inner pairs, p_μ inside links in each of $j_1 - j_0 + 2$ blocks and one path link for each of the $j_1 - j_0 + 3 - n_b$ outer pairs; $p_\mu \le 7$, and consecutive bad blocks of the set differ by at most two in index, so $j_1 - j_0 \le 2(n_b - 1)$. □

The chains of all bad blocks are disjoint, contain the eight private links of every pair based at a bad block, and each responds to every link outside it with the bound of (iii), uniformly in the exterior. Where the cube condition calls for the response of the private links of a bad pair, the law of the chain of that pair can be used, with the constants of Theorem 421 in which 24 is replaced by $n \le 23n_b + 1$. The chains cost nothing in the counting of islands, except at the collar $s = 1$.

Corollary 437 (the chains and the islands)**.** *Let $s \ge 1$ be the collar.*

(i) *In Theorem 483, every link of a chain of a bad block of I_a is based at $d_a \le 1 \le s$ and lies in the class L_a, which is integrated in a_i; likewise for I_b. So the chains need no change of the joining distances of Theorem 483, Corollary 486 and the results built on them.*

(ii) *Let $s \ge 2$, and let every block at ℓ^∞ distance at most s from a window W be good. Then no link of a chain is freed in the law of Proposition 412 for W, and no constraint of a pair of K_W contains a link of a chain. So the chains and the windows of Theorem 465 have disjoint variables and constraints at the joining distance $2s + 1$ of that theorem, with $\Delta = (4s + 3)^4 - 1$, and its values for $s \ge 2$ stand, among them $\beta^*(10^{-8}, 5) = 5.401 \cdot 10^{21}$.*

(iii) *For $s = 1$ the statement of (ii) fails: if y is bad and every block at distance at most one from $y + 2e_\mu$ is good, the window $W = \{y + 2e_\mu\}$ is admissible, the outer pair $(y + e_\mu, \mu)$ of the chain of y enters W, and its path link is freed in the window law. With the collar $s = 2$ instead, $\Delta = 11^4 - 1$, $|\log q^*| = 25777.3$, $\beta^*(10^{-8}, 2) = 4.177 \cdot 10^{21}$ and $\beta^*(10^{-9}, 2) = 4.181 \cdot 10^{23}$.*

Proof. (i) By Proposition 436(i) every link of the chain is based within distance one of a bad block of its connected set; the bad blocks lie in $I_a \cup I_b$, whose distance exceeds $2s + 7 > 2$, so a connected set lies in one of them; and a link based at $d_a \le s$ is in L_a by the definition of the classes. (ii) A block of W is at distance at least $s + 1 \ge 3$ from every bad block, a pair entering W is based at distance at least $s \ge 2$, and the constraint of a pair of K_W contains links based in its own block and in the next one along its direction, both at distance at least $s \ge 2$. The links freed for W are based in W or are path links of pairs entering it. Links of chains are based at distance at most one. The rest is the proof of Theorem 465 unchanged. (iii) The window is admissible because y is at distance two from it; the numbers are those of Theorem 465 at $s = 2$. □

C. The projected average

The folds of Propositions 418 and 422 and the obstruction of Theorem 423 are zeros of $\det D_g V_k^\sharp$, the derivative of the cut-off average in its path link. They come from the two nonlinear parts of (35): the logarithm, whose differential L_Y grows like $\theta / \sin\theta$ as an eigenangle θ approaches π, and the cut-off, whose slope is 2. If the logarithm of each relative path is replaced by its orthogonal projection onto the Lie algebra, no cut-off is needed. The resulting coarse map has the same derivatives of order at most two as the cut-off average at every field whose sixteen paths agree, and its derivative in the path link is invertible at every field. This subsection proves the properties of this map that bear on the bad side; Remark 448 lists what is not proved for it. As in the rest of this section $G = \mathrm{SU}(2)$; Remark 446 gives the constants for other groups.

Let $\Pi_{\mathfrak{g}}$ be the orthogonal projection of the space of complex 2×2 matrices, with the real inner product $\mathrm{Re}\,\mathrm{Tr}(X^*Y)$, onto

$\mathfrak{g} = \mathfrak{su}(2)$.[41]

Definition 438. For a coarse link (y,μ) and the paths P_x of Sec. XII B, the *projected average* is

$$\widehat{V}_{y,\mu}(U) = P_{2y}\exp A_{y,\mu}(U),$$
$$A_{y,\mu}(U) = \frac{1}{16}\sum_{x\in B(y)} \Pi_{\mathfrak{g}}\big(P_{2y}^{-1}P_x\big).$$

For a pair $k=(y,\mu)$ in the tree gauge write $g=\ell_k$, $z=(\ell_a)_{a\neq 0}$, and $P_{x_a}=u_a\ell_a$, $P_{x'_a}=\ell_a w_a$ for the paths of nonzero offset, as in Lemma 425. The two paths of offset zero equal g and contribute nothing, so

$$A_k = \frac{1}{16}\sum_{a\neq 0}\Big(\Pi_{\mathfrak{g}}(g^{-1}u_a\ell_a)+\Pi_{\mathfrak{g}}(g^{-1}\ell_a w_a)\Big),$$

a function $A(g;z,u,w)$ of the path link, the non-path links and the inside links. Let $J(g;z,u,w)=|\det D_g\widehat{V}_k|$, the Jacobian between tangent spaces of the bi-invariant metric, as in (40).

Lemma 439. (i) *For $U\in\mathrm{SU}(2)$, $\Pi_{\mathfrak{g}}(U)=\frac12(U-U^{-1})$. The projection commutes with conjugation by $\mathrm{SU}(2)$, $\Pi_{\mathfrak{g}}(I)=0$, and $\|\Pi_{\mathfrak{g}}(U)-\Pi_{\mathfrak{g}}(U')\|\le\delta(U,U')\le d(U,U')$ for $U,U'\in\mathrm{SU}(2)$; in particular $\|\Pi_{\mathfrak{g}}(U)\|\le\delta(U,I)$.*

(ii) *For $X\in\mathfrak{g}$, $\|\Pi_{\mathfrak{g}}(e^X)-X\|\le\frac{1}{12}\|X\|^3$.*

(iii) *For $X\in\mathfrak{g}$, the operator $\mathrm{ad}X$ on $\mathfrak{g}$ has norm $\sqrt{2}\,\|X\|$, and the operator norm of the matrix X is $\|X\|/\sqrt{2}$.*

Proof. (i) For $U\in\mathrm{SU}(2)$, $U+U^*=(\mathrm{Tr}\,U)I$ with $\mathrm{Tr}\,U$ real, so $U=\frac12(\mathrm{Tr}\,U)I+Y$ with $Y=\frac12(U-U^*)\in\mathfrak{g}$. Since $\mathrm{Re}\,\mathrm{Tr}\,X=0$ for $X\in\mathfrak{g}$, I is orthogonal to $\mathfrak{g}$, and $\Pi_{\mathfrak{g}}(U)=Y$. Conjugation by $h\in\mathrm{SU}(2)$ is orthogonal for the inner product and maps $\mathfrak{g}$ onto itself, so it commutes with $\Pi_{\mathfrak{g}}$. As an orthogonal projection, $\Pi_{\mathfrak{g}}$ is a linear map of norm one, and δ is the Hilbert–Schmidt distance; a chord is not longer than an arc, so $\delta\le d$. (ii) In a basis with $X=\mathrm{diag}(i\theta,-i\theta)$, $\|X\|=\sqrt{2}\,|\theta|$ and $\Pi_{\mathfrak{g}}(e^X)=\mathrm{diag}(i\sin\theta,-i\sin\theta)$ by (i), so $\|\Pi_{\mathfrak{g}}(e^X)-X\|=\sqrt{2}\,|\theta-\sin\theta|\le\sqrt{2}\,|\theta|^3/6=\|X\|^3/12$. (iii) In the same basis $\mathrm{ad}X$ is skew with eigenvalues 0 and $\pm 2i\theta$, and X has the eigenvalues $\pm i\theta$. □

Proposition 440 (the projected average)**.** *The projected average is real-analytic on all fields, covariant under the gauge transformations at the block roots, and normalised.*

(i) *At a field with $\delta(P_x,P_{2y})\le\frac12$ for every $x\in B(y)$, with $X_x=\log(P_{2y}^{-1}P_x)$,*

$$d\big(\widehat{V}_{y,\mu},V^\sharp_{y,\mu}\big)\le\frac{1}{192}\sum_{x\in B(y)}\|X_x\|^3.$$

At every field at which the sixteen paths P_x agree, the derivatives of order at most two of $\widehat{V}_{y,\mu}$ and $V^\sharp_{y,\mu}$ coincide; in particular $\widehat{V}_{y,\mu}(e^{t\varphi})=I+tV_{y,\mu}(\varphi)+O(t^2)$, as in Theorem 353(i).

(ii) *The four inequalities of Theorem 353(ii) hold for $\widehat{V}$ with $c=1$, that is with the constants 896, 56, 1344 and 2696, for every field in the tree gauge.*

(iii) *If U equals I except on one link from $B(y)$ to $B(y+e_\mu)$, where it equals h, then $\widehat{V}_{y,\mu}(U)=h\exp(-\frac78\Pi_{\mathfrak{g}}(h))$ when the link is on the straight path and $\exp(\frac18\Pi_{\mathfrak{g}}(h))$ at the seven other positions, for every h. The slice windings of Proposition 351 are 1 at the straight path and 0 at the other positions.*

Proof. Products of matrices, $\Pi_{\mathfrak{g}}$ and exp are analytic, and $\widehat{V}\in G$ since $A\in\mathfrak{g}$. A gauge transformation equal to h_1 on $B(y)$ and to h_2 on $B(y+e_\mu)$ maps P_x to $h_1P_xh_2^{-1}$, as in the proof of Theorem 353, so $P_{2y}^{-1}P_x\mapsto h_2P_{2y}^{-1}P_xh_2^{-1}$, $A\mapsto h_2Ah_2^{-1}$ by Lemma 439(i), and $\widehat{V}\mapsto h_1\widehat{V}h_2^{-1}$. At $U=I$ every term of A is $\Pi_{\mathfrak{g}}(I)=0$.

(i) Under the hypothesis the cut-off is one, and $V^\sharp=P_{2y}e^N$ with $N=\frac{1}{16}\sum_x X_x$, by Theorem 353(i), while $A=\frac{1}{16}\sum_x\Pi_{\mathfrak{g}}(e^{X_x})$. By $d(e^A,e^N)\le\|A-N\|$ and Lemma 439(ii) the distance is at most $\frac{1}{16}\sum_x\frac{1}{12}\|X_x\|^3$. Near a field at which the paths agree the X_x are analytic functions of the links that vanish there, so the two analytic maps differ by $O(r^3)$ at distance r from it, and their Taylor expansions in a chart agree to order two. The last statement is Theorem 353(i).

(ii) In the proof of Theorem 353(ii), $\delta(V^\sharp,P)$ is bounded by $\frac{c}{4}(\sum_x\delta(P_x,P)^2)^{1/2}$, and $V^\sharp=vM$ with $\delta(M,I)$ at most the same quantity. For $\widehat{V}=Pe^A$, Lemma 352, Lemma 439(i) and the Cauchy–Schwarz inequality give

$$\delta(\widehat{V},P)=\delta(e^A,I)\le\|A\|\le\frac{1}{16}\sum_x\delta(P_x,P)\le\frac14\Big(\sum_x\delta(P_x,P)^2\Big)^{1/2},$$

which is that bound with $c=1$, and the rest of the proof is unchanged: $896/16=56$, $448+896=1344$ and $8+48\cdot 56=2696$.

(iii) At the straight path $P_{2y}=h$, the fourteen other paths are I, and $\Pi_{\mathfrak{g}}(h^{-1})=-\Pi_{\mathfrak{g}}(h)$; at another position $P_{2y}=I$ and two paths equal h. For $h=\mathrm{diag}(e^{iq},e^{-iq})$ the two values are $\mathrm{diag}(e^{i\psi},e^{-i\psi})$ with $\psi=q-\frac78\sin q$ and $\psi=\frac18\sin q$, loops of degree 1 and 0 in q. □

So $\widehat{V}$ is a continuous covariant extension of the block averages to all fields, with the slice windings that Proposition 351 demands; it departs from the transverse symmetry at third order at every field, where $V^\sharp$ departs from it only past the cut-off.

Theorem 441 (the global inverse)**.** *Let k be a pair, $c_k\in\mathrm{SU}(2)$, and let the non-path links z and the inside links u,w be arbitrary.*

[41] For $U\in\mathrm{SU}(2)$ this is the anti-Hermitian part $\frac12(U-U^{-1})$. Unlike the logarithm, it is defined and real-analytic on the whole group, which is why the projected average needs no restriction on the paths.

(i) *For all* g,g', $\|A(g)-A(g')\| \le \frac{7}{8}\delta(g,g')$, *and* $g \mapsto c_k\exp(-A(g))$ *is a contraction of* SU(2) *with constant* $\frac{7}{8}$ *for d. The equation* $\widehat{V}_k = c_k$ *has exactly one solution* $G = G(c_k;z,u,w)$*, the fixed point of this contraction, and G is real-analytic in* (c_k,z,u,w).

(ii) *For all* g,g',

$$\tfrac{1}{8}\,d(g,g') \le d\big(\widehat{V}_k(g),\widehat{V}_k(g')\big) \le \tfrac{15}{8}\,d(g,g'),$$

the singular values of $D_g\widehat{V}_k$ *lie in* $[\frac{1}{8},\frac{15}{8}]$ *at every g, and* $\frac{1}{512} \le J \le \frac{3375}{512}$. *The two bounds on J are attained when all fourteen relative paths* $g^{-1}P_x$ *of nonzero offset equal I, respectively* $-I$.

(iii) *For two configurations,*

$$d(G,G') \le 8\,d(c_k,c_k') + \sum_{a\neq 0} d(\ell_a,\ell_a') + \frac{1}{2}\sum_{a\neq 0}\big(d(u_a,u_a') + d(w_a,w_a')\big).$$

(iv) *Let one link* u_b *or* w_b, $b\neq 0$, *move along a unit-speed geodesic, with* c_k, *z and the other links fixed, and let* $J(t)$ *be the value of J at* $(G(t);z,u(t),w(t))$. *Then* $|\frac{d}{dt}\log J(t)| \le \frac{45}{4}\sqrt{2} < 15.91$. *If instead a non-path link* ℓ_b *moves, with* c_k, *u, w and the other non-path links fixed, the bound is* $\frac{45}{2}\sqrt{2} < 31.82$*; if* c_k *moves, with z, u, w fixed, it is* $\frac{315}{2}\sqrt{2} < 222.8$.

Proof. The derivative. Let $v_x = g^{-1}P_x$ for the fourteen paths of nonzero offset. Along $ge^{t\zeta}$, $\zeta\in\mathfrak{g}$, each v_x becomes $e^{-t\zeta}v_x$, so $A(ge^{t\zeta}) = A(g) + tB\zeta + O(t^2)$ with

$$B\zeta = -\frac{1}{16}\sum_x \Pi_{\mathfrak{g}}(\zeta v_x), \qquad \|B\| \le \frac{7}{8},$$

since $\|\zeta v_x\| = \|\zeta\|$. With $e^{-A}\frac{d}{dt}e^{A+tY}|_{t=0} = \int_0^1 \mathrm{Ad}(e^{-sA})Y\,ds$ and $R_A = \int_0^1 \mathrm{Ad}(e^{sA})\,ds$, of norm at most one,

$$\widehat{V}_k^{-1}\frac{d}{dt}\widehat{V}_k(ge^{t\zeta})\Big|_{t=0} = \mathrm{Ad}(e^{-A})\,D\zeta, \qquad D = I + R_AB.$$

As $\mathrm{Ad}(e^{-A})$ is a rotation of $\mathfrak{g}$, $J = |\det D|$; the maps $I+sR_AB$, $0\le s\le 1$, are invertible because $\|R_AB\| \le \frac{7}{8}$, so $\det D > 0$, and $\|D^{-1}\| \le (1-\frac{7}{8})^{-1} = 8$.

(i) By Lemma 439(i) and the invariance of δ under multiplication and inversion, $\|\Pi_{\mathfrak{g}}(g^{-1}P) - \Pi_{\mathfrak{g}}(g'^{-1}P)\| \le \delta(g^{-1}P, g'^{-1}P) = \delta(g,g')$ for each of the fourteen paths, which gives the first bound. With $d(e^X,e^Y) \le \|X-Y\|$ and $\delta\le d$, $d(c_ke^{-A(g)}, c_ke^{-A(g')}) \le \frac{7}{8}d(g,g')$, and SU(2) is complete, so the contraction has one fixed point; $\widehat{V}_k(g) = c_k$ is the equation $g = c_ke^{-A(g)}$. Since D is invertible at every point, the analytic implicit function theorem gives an analytic solution near every solution, which by uniqueness is G.

(ii) By bi-invariance, $d(ge^{A(g)}, g'e^{A(g)})$ equals $d(g,g')$, and $d(g'e^{A(g)}, g'e^{A(g')}) \le \|A(g)-A(g')\| \le \frac{7}{8}d(g,g')$; the triangle inequality gives both bounds, and along $ge^{t\zeta}$ they give $\frac{1}{8}\|\zeta\| \le \|D\zeta\| \le \frac{15}{8}\|\zeta\|$. If every v_x is I, respectively $-I$, then $A=0$, $R_A = I$ and $B = \mp\frac{7}{8}I$, so $J = 8^{-3}$, respectively $(\frac{15}{8})^3$; these values are taken, for instance, at $u=w=I$ and $\ell_a = \pm g$.

(iii) Let $T(g) = c_ke^{-A(g)}$ and T' be the map for the second configuration. Then $d(G,G') \le d(T(G),T(G')) + d(T(G'),T'(G'))$, the first term is at most $\frac{7}{8}d(G,G')$, and the second is at most $d(c_k,c_k') + \|A(G') - A'(G')\|$. A path of nonzero offset changes by at most $d(u_a,u_a') + d(\ell_a,\ell_a')$, respectively $d(\ell_a,\ell_a') + d(w_a,w_a')$, so $\|A(G')-A'(G')\|$ is at most $\frac{1}{16}$ times the sum of these quantities, and multiplying by 8 gives the bound.

(iv) Let $u_b(t) = u_be^{t\zeta'}$ with $\|\zeta'\| = 1$. By (i) and (iii), $G(t)$ is analytic and its left velocity $\eta = G^{-1}\dot G$ has $\|\eta\| \le \frac{1}{2}$. Then $\dot A = B\eta + \frac{1}{16}\Pi_{\mathfrak{g}}(G^{-1}u_b\zeta'\ell_b)$, so $\|\dot A\| \le \frac{7}{16} + \frac{1}{16} = \frac{1}{2}$. Next, $\frac{d}{dt}\mathrm{Ad}(e^{sA}) = \mathrm{ad}(W_s)\,\mathrm{Ad}(e^{sA})$ with $W_s = (\frac{d}{dt}e^{sA})e^{-sA} = s\int_0^1 \mathrm{Ad}(e^{rsA})\dot A\,dr$, so $\|W_s\| \le s\|\dot A\|$ and, by Lemma 439(iii), $\|\dot R_A\| \le \int_0^1 \sqrt{2}s\|\dot A\|\,ds \le \frac{1}{2\sqrt{2}}$. Further $\dot B\zeta = -\frac{1}{16}\sum_x \Pi_{\mathfrak{g}}(\zeta\dot v_x)$ with $\dot v_x = -\eta v_x$, except for the path from x_b, which has the additional term $G^{-1}u_b\zeta'\ell_b$; and $\|\zeta M\| \le \|\zeta\|\,\|M\|_{\mathrm{op}}$, $\|\eta v_x\|_{\mathrm{op}} = \|\eta\|/\sqrt{2}$ and $\|G^{-1}u_b\zeta'\ell_b\|_{\mathrm{op}} = 1/\sqrt{2}$ by Lemma 439(iii). Hence $\|\dot B\| \le \frac{1}{16}(\frac{14}{2\sqrt{2}} + \frac{1}{\sqrt{2}}) = \frac{1}{2\sqrt{2}}$, and $\dot D = \dot R_AB + R_A\dot B$ has

$$\|\dot D\| \le \frac{7}{8}\cdot\frac{1}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} = \frac{15}{16\sqrt{2}}.$$

Since $\det D > 0$, $\frac{d}{dt}\log J = \mathrm{Tr}(D^{-1}\dot D)$, at most $3\|D^{-1}\|\|\dot D\| \le 3\cdot 8\cdot\frac{15}{16\sqrt{2}} = \frac{45}{4}\sqrt{2}$ in modulus on the three-dimensional $\mathfrak{g}$. The link w_b is the same with the path from x_b'. If ℓ_b moves, $\|\eta\| \le 1$ by (iii) and two paths have an additional term of operator norm $1/\sqrt{2}$, so $\|\dot A\| \le \frac{7}{8} + \frac{2}{16} = 1$, $\|\dot R_A\| \le \frac{1}{\sqrt{2}}$, $\|\dot B\| \le \frac{1}{16}(\frac{14}{\sqrt{2}} + \frac{2}{\sqrt{2}}) = \frac{1}{\sqrt{2}}$, $\|\dot D\| \le \frac{15}{8\sqrt{2}}$, and the bound is $3\cdot 8\cdot\frac{15}{8\sqrt{2}} = \frac{45}{2}\sqrt{2}$. If c_k moves, $\|\eta\| \le 8$ by (iii) and no path has an additional term, so $\|\dot A\| \le 7$, $\|\dot R_A\| \le \frac{7}{\sqrt{2}}$, $\|\dot B\| \le \frac{14}{16}\cdot\frac{8}{\sqrt{2}} = \frac{7}{\sqrt{2}}$ and $\|\dot D\| \le \frac{105}{8\sqrt{2}}$, and the bound is $3\cdot 8\cdot\frac{105}{8\sqrt{2}} = \frac{315}{2}\sqrt{2}$. □

Fig. 37 draws the projected average and the contraction behind its global inverse.

Corollary 442 (the path-link law of the projected average)**.** *For every* c_k, *z, u, w and exterior* σ, *the law* (40) *with* $\widehat{V}_k$ *in place of* $V_k^\sharp$ *is the point mass at* $G(c_k;z,u,w)$, *and for two configurations with the same* c_k

$$W_1^d(\pi,\pi') \le \sum_{a\neq 0} d(\ell_a,\ell_a') + \frac{1}{2}\sum_{a\neq 0}\big(d(u_a,u_a') + d(w_a,w_a')\big),$$

with no dependence on β_W, *on* c_k *or on any other link, and no restriction on the size of any link.*

Proof. By Theorem 441(i) the sum in (40) has one term, whose weight cancels against Z; the bound is Theorem 441(iii). □

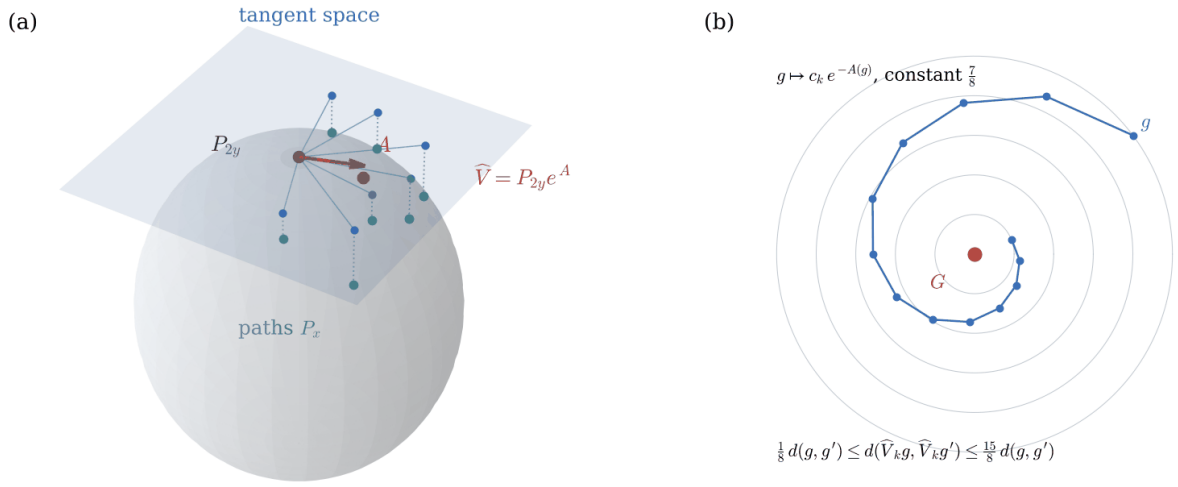


FIG. 37. The projected average of Definition 438, schematic, with a two-sphere standing for SU(2). (a) The paths P_x of a coarse link around the base path P_{2y}, their projections to the tangent space at P_{2y}, the mean A of the projections, and the value $\widehat{V} = P_{2y}e^{A}$. (b) The map $g \mapsto c_k\, e^{-A(g)}$ contracts with constant $\frac{7}{8}$, so its iterates converge to the unique solution G of $\widehat{V}_k = c_k$, and $\widehat{V}_k$ changes distances by a factor between $\frac{1}{8}$ and $\frac{15}{8}$ (Theorem 441).

So for the projected average the original form of item (i) of Sec. XIII V, a Lipschitz bound for the path-link law in the non-path links, holds at every field, with the constants 1 and $\frac{1}{2}$ in place of the constants 1.1986 and 0.5993 that Proposition 416 gives for $V^\sharp$ on the good side only; Propositions 414, 418 and 422 have no counterpart. The rest of Proposition 416 holds with no radius as well: if the non-path links and the inside links are within r of the reconstruction, the computation of Lemma 411, with $\|A_k\| \le \frac{7}{8}\max_x \delta(g^{-1}P_x, I)$ in place of the logarithmic bound, gives $d(G,c_k) \le 14r$ and $d(g^{-1}P_x, I) \le 16r$ for every r, and the density $1/J$ lies in $[(\frac{15}{8})^{-3}, 512]$. For the law of the eight private links let S_{dep} be as in Lemma 426, let dz be Haar measure on $\mathrm{SU}(2)^7$, $|B|$ the Haar measure of a set B, and write $G(z) = G(c_k; z, u, w)$.

Theorem 443 (the bad side of the projected average)**.** *Let k be a pair and $c_k \in \mathrm{SU}(2)$, fix every link other than the eight private links of k at arbitrary values, and let $B \subset \mathrm{SU}(2)^7$ be a measurable set of values of the non-path links with $|B| > 0$. Put*

$$\rho_B(z) = \mathbf{1}_B(z)\, e^{-\beta_W S_{\mathrm{dep}}(G(z),z)}\, J\big(G(z); z, u, w\big)^{-1},$$
$$Q_B(f) = \frac{1}{Z_B}\int f\big(G(z), z\big)\, \rho_B(z)\, dz, \qquad Z_B = \int \rho_B\, dz.$$

(i) *For every c_k, Q_B is the law of the eight private links given the other links, $\widehat{V}_k = c_k$ and $z \in B$, in the version of the disintegration that is continuous in c_k, and*

$$(\tfrac{8}{15})^3 e^{-72\beta_W}\,|B| \le Z_B \le 512\,|B|.$$

(ii) *If one inside link u_b or w_b, $b \neq 0$, moves from e to e', with c_k, B and every other link fixed, then for the cost*

$$\textstyle\sum_{a=0}^{7} \delta(\ell_a, \ell'_a)$$

$$W_1(Q_B, Q'_B) \le (24\beta_W + 180.5)\, d(e, e').$$

(iii) *If one link e' that is neither a private link of k nor one of the links u_a, w_a changes, then $W_1(Q_B, Q'_B) \le 8\beta_W\, d(\sigma_{e'}, \sigma'_{e'})$.*

(iv) *If c_k moves, with B and every link other than the private links of k fixed, then*

$$W_1(Q_B, Q'_B) \le (384\beta_W + 2528)\, d(c_k, c'_k).$$

The constants depend neither on B nor on $|B|$, nor on c_k or the values of the other links; no inside link needs to be near I.

Proof. (i) Let $\Phi(g,z) = (\widehat{V}_k(g; z, u, w), z)$. By Theorem 441(i), Φ is a bijection of $\mathrm{SU}(2)^8$, analytic with the analytic inverse $(c,z) \mapsto (G(c; z, u, w), z)$, and its Jacobian is J, the derivative being triangular with the identity in z. Haar measure is a constant multiple of the volume of the bi-invariant metric, so the change of variables gives, for bounded measurable F,

$$\int F(g,z)\, dg\, dz = \int dc \int F\big(G(c; z, u, w), z\big)\, J^{-1}\, dz,$$

with J at $(G(c; z, u, w); z, u, w)$. Taking F equal to f times the Wilson weight and $\mathbf{1}_B(z)$, and cancelling the plaquettes without a private link as in Lemma 426, the normalised inner integral is Q_B at $c = c_k$; it is weakly continuous in c by the continuity of G. Lemma 426 gives $0 \le S_{\mathrm{dep}} \le 72$, and Theorem 441(ii) gives $J \in [8^{-3}, (\frac{15}{8})^3]$.

(ii) Move e along a unit-speed geodesic $e(t)$, $0 \le t \le T = d(e, e')$, write $G_t(z)$, ρ_t, Q_t, and let ν_t be the law $\rho_t\, dz / Z_t$ on

B. The link e shares no plaquette with a private link, since the two μ-links of a plaquette have one μ-coordinate; so at fixed z, S_{dep} changes only through the path link $G_t(z)$, which moves at speed at most $\frac{1}{2}$ by Theorem 441(iii) and lies in six plaquettes, each changing at rate at most $\frac{1}{2}/\sqrt{2}$, by the proof of Lemma 415. With Theorem 441(iv),

$$|\partial_t \log \rho_t(z)| \le \omega := \frac{3}{\sqrt{2}}\beta_W + \frac{45}{4}\sqrt{2}$$

for every $z \in B$ and t. Let f be 1-Lipschitz for the cost, and put $\Phi_t(z) = (G_t(z), z)$. Then

$$Q_T(f) - Q_0(f) = (\nu_T - \nu_0)(f \circ \Phi_T) + \nu_0(f \circ \Phi_T - f \circ \Phi_0).$$

In the second term only the path link differs, by at most $\frac{T}{2}$ in d and so in δ. In the first, $\frac{d}{dt}\nu_t(\varphi) = \mathrm{Cov}_{\nu_t}(\varphi, \partial_t \log \rho_t)$ for the fixed bounded function $\varphi = f \circ \Phi_T$, by differentiation under the integral over the fixed set B, and $|\mathrm{Cov}(\varphi,\psi)| \le \frac{1}{4}\operatorname{osc}\varphi \operatorname{osc}\psi$; the oscillation of φ is at most the diameter $8 \cdot 2\sqrt{2}$ of the eight links for the cost, so the first term is at most $\frac{1}{4} \cdot 16\sqrt{2} \cdot 2\omega T = 8\sqrt{2}\,\omega T$. The normaliser Z_t cancels from the covariance. Hence $Q_T(f) - Q_0(f) \le (\frac{1}{2} + 8\sqrt{2}\,\omega)T = (24\beta_W + 180.5)T$, and the Kantorovich duality gives the bound.

(iii) The measure $\rho_B\,dz$ without its Wilson factor, and the map Φ, do not depend on e'; the link e' lies in at most one plaquette with a private link, by the proof of Lemma 415 and, for an inside link of another direction, by the remark after Theorem 427. The proof of Lemma 415 gives the bound.

(iv) As in (ii), with the path link moving at speed at most 8 by Theorem 441(iii), so that S_{dep} changes at rate at most $6 \cdot 8/\sqrt{2}$, and with the bound $\frac{315}{2}\sqrt{2}$ of Theorem 441(iv): $|\partial_t \log \rho_t| \le 24\sqrt{2}\beta_W + \frac{315}{2}\sqrt{2}$, and $8 + 8\sqrt{2}(24\sqrt{2}\beta_W + \frac{315}{2}\sqrt{2}) = 384\beta_W + 2528$. □

The theorem applies to the two sets $\mathrm{SU}(2)^7$ and B_r^{bad} of Proposition 424, defined with $c_k = \widehat{V}_k$, and to the sets B_ε of Theorem 423, at every ε; it needs neither the relative chart nor condition (J) of Theorem 427, and its constants have no factor $e^{72\beta_W}$. The reason is the following. The obstruction of Theorem 423 lives at a point where the projection of the fibre $\{V_k^\sharp = c_k\}$ to the non-path links folds, that is, where $\det D_g V_k^\sharp = 0$; as $\varepsilon \to 0$ the law over B_ε becomes the path-link law at the fold, which jumps. By Theorem 441(i) and (ii) the fibre of $\widehat{V}_k$ is, for every value of the inside links, the graph of $z \mapsto G(z)$ over the non-path links themselves. A set of non-path links held fixed while an inside link moves is therefore a fixed set of parameters of the fibre; only the density ρ_B and the position of the path link move, the first with a logarithmic derivative bounded at every point, the second at speed $\frac{1}{2}$. In the relative chart of Lemma 425 the fibre of $V_k^\sharp$ is a graph over $\xi_a = g^{-1}\ell_a$ instead, a fixed set B of non-path links is the moving set $\{\xi : G(\xi)\xi \in B\}$, and the terms with ι and K in Theorem 427 come from this motion and from the jumps of the Jacobian D. The bound of Theorem 443 grows linearly in β_W; it is a bound on the response of one pair at a fixed coarse field, not a contraction, and it does not give the cube condition (D).

On the whole torus the coarse fibre of the projected average has a global product parametrisation. On the periodic lattice of side $2L$, $L \ge 2$, in the tree gauge, with $\mathscr{B}$ blocks, the 49 active links of a block are the four path links ℓ_k of the pairs based at it and 45 other links, the 28 non-path private links and the 17 active links inside it; let z be the family of these $45\mathscr{B}$ links.

Proposition 444 (projected fibres). (i) *The map* $\Psi(U) = (\widehat{V}(U), z)$ *of the active links* U,

$$\Psi\colon \mathrm{SU}(2)^{49\mathscr{B}} \to \mathrm{SU}(2)^{4\mathscr{B}} \times \mathrm{SU}(2)^{45\mathscr{B}}$$

is an analytic diffeomorphism. Its inverse $(c,z) \mapsto \Gamma(c,z)$ *puts* $G_k = G(c_k; z, u, w)$ *of Theorem 441 on each path link, and its Jacobian is* $\prod_k J_k$, *with* J_k *the Jacobian of the pair* k.

(ii) *For every* c, *the law of the active links given* $\widehat{V} = c$, *in the version continuous in* c, *is the image under* $\Gamma(c,\cdot)$ *of*

$$\frac{1}{Z(c)}\, e^{-\beta_W S(\Gamma(c,z))} \prod_k J_k^{-1}\, dz,$$

whose density is positive at every z. *The factor of a plaquette depends on* c *and on links of* z *based in blocks at* ℓ^∞ *distance at most two from the block of its base point, and* J_k *on* c_k *and on links of* z *based in the two blocks of* k.

(iii) *Let* $H_c(z) = \beta_W S(\Gamma(c,z)) + \sum_k \log J_k$. *Along a unit-speed geodesic in one link of* z, $|\frac{d}{dt}H_c| \le 6\sqrt{2}\,\beta_W + \frac{45}{2}\sqrt{2}$.

Proof. (i) The paths of a pair (y', ν) consist of ν-links with base in $B(y')$ or $B(y' + e_\nu)$, and the path link of $k = (y, \mu)$ is a μ-link with $o_\mu = 1$ based in $B(y)$, a private link of k; so it enters only $\widehat{V}_k$. A non-path private link of k enters only $\widehat{V}_k$, and an active inside μ-link of $B(y)$ enters $\widehat{V}_{y,\mu}$ and $\widehat{V}_{y-e_\mu,\mu}$; since $L \ge 2$ these are two different pairs. For fixed z the map from the path links to $\widehat{V}$ is therefore the product over k of the maps $g \mapsto \widehat{V}_k(g; z, u, w)$, each a bijection with analytic inverse by Theorem 441(i). The map $(\ell, z) \mapsto (\widehat{V}, z)$ is a bijection with inverse Γ; its derivative is triangular, with the identity in z and the block diagonal part $(D_{\ell_k}\widehat{V}_k)_k$ in the path links, which gives the Jacobian.

(ii) The tree gauge has unit Jacobian, so the active links carry product Haar measure times the Wilson weight. The change of variables of (i), as in the proof of Theorem 443(i), gives the law, and the density is positive by Theorem 441(ii). A plaquette has its links based in blocks at distance at most one from the block of its base point; a path link among them is G_k, which depends on c_k and on links based in the two blocks of k, at distance at most one further. The Jacobian J_k depends on the same variables as G_k.

(iii) If the link of z is a non-path link of k, only G_k moves, at speed at most 1 by Theorem 441(iii); the two moving links lie in six plaquettes each, which changes S at rate at most $12/\sqrt{2} = 6\sqrt{2}$, and only J_k changes, at the rate $\frac{45}{2}\sqrt{2}$ of Theorem 441(iv). If it is an active inside link of $B(y)$ of direction μ, the path links of (y,μ) and $(y - e_\mu, \mu)$ move at speed at most $\frac{1}{2}$ each, so S changes at rate at most $(6+3+3)/\sqrt{2}$, and the two Jacobians at rate at most $\frac{45}{4}\sqrt{2}$ each. □

The coarse curvature of the projected average has a fine witness at every size of the fine plaquettes, with a smaller constant than Lemma 409. For a transverse offset $a \in \{0,1\}^4$, $a_\mu = 0$, let $|a|$ be the number of its nonzero coordinates and $|a|_{>\mu}$ the number of those with index above μ.

Proposition 445 (a fine witness for the projected average)**.** *Let P be the coarse plaquette at y in the plane $\{\mu,\nu\}$, $\mu<\nu$, let $\kappa \ge 0$, and suppose that $\delta(U_p, I) \le \kappa$ for every fine plaquette p whose vertices lie in $B(y)\cup B(y+e_\mu)\cup B(y+e_\nu)\cup B(y+e_\mu+e_\nu)$. Then $c=\widehat{V}(U)$ satisfies $\delta(c_{y',\mu'}, \ell_{y',\mu'}) \le \frac{9-\mu'}{2}\kappa$ for the four sides (y',μ') of P, and*

$$\delta(c_P, I) \le (22-\mu-\nu)\,\kappa \le 21\,\kappa.$$

Proof. Take the side $k=(y,\mu)$; the others are the same. Write $x_a = 2y+a$ and $x'_a = x_a + e_\mu$, so that $P_{x_a} = u_a \ell_a$ and $P_{x'_a} = \ell_a w_a$ with $u_a = U_{x_a,\mu}$ and $w_a = U_{x_a+2e_\mu,\mu}$, and $P_{x_0} = g = \ell_k$.

The paths from x_a. Let $\nu_1 < \cdots < \nu_m$ be the directions with $a_\nu = 1$, $m = |a|$, and $a^{(j)} = e_{\nu_1} + \cdots + e_{\nu_j}$. The ν_j-links at $x_{a^{(j-1)}}$ and at $x_{a^{(j-1)}} + 2e_\mu$ have offset $a^{(j-1)}$, whose coordinates of index ν_j and above vanish; they are tree links. So $P_{x_{a^{(j-1)}}} P^{-1}_{x_{a^{(j)}}}$ is the holonomy of the boundary of the rectangle formed by the plaquettes at $x_{a^{(j-1)}}$ and $x_{a^{(j-1)}} + e_\mu$ in the plane $\{\mu,\nu_j\}$, the product of their holonomies with one of them conjugated, and $\delta(P_{x_{a^{(j-1)}}}, P_{x_{a^{(j)}}}) \le 2\kappa$. Hence $\delta(P_{x_a}, g) \le 2|a|\kappa$.

The inside links. If $|a|_{>\mu} = 0$, u_a is a tree link. Otherwise let $\nu > \mu$ be the largest direction with $a_\nu = 1$. In the plaquette at $x_a - e_\nu$ in the plane $\{\mu,\nu\}$ the two ν-links have offsets $a - e_\nu$ and $a - e_\nu + e_\mu$, whose coordinates of index ν and above vanish, so they are tree links and $\delta(u_a, U_{x_a - e_\nu,\mu}) \le \kappa$. By induction $\delta(u_a, I) \le |a|_{>\mu}\kappa$, and in $B(y+e_\mu)$ likewise $\delta(w_a, I) \le |a|_{>\mu}\kappa$. Since $P_{x'_a} = u_a^{-1} P_{x_a} w_a$, $\delta(P_{x'_a}, g) \le 2(|a| + |a|_{>\mu})\kappa$.

The coarse link. By Lemma 439(i), $\|\Pi_{\mathfrak{g}}(g^{-1}P_x)\| \le \delta(P_x, g)$. Over the eight transverse offsets $\sum_a |a| = 12$ and $\sum_a |a|_{>\mu} = 4(3-\mu)$, so

$$\|A_k\| \le \frac{\kappa}{16}\sum_a \big(4|a| + 2|a|_{>\mu}\big) = \frac{9-\mu}{2}\,\kappa,$$

and $\delta(c_k, g) = \delta(e^{A_k}, I) \le \|A_k\|$ by Lemma 352. All plaquettes used lie in $B(y)\cup B(y+e_\mu)$.

The coarse plaquette. As in the proof of Lemma 409, the four path links around P form the holonomy ℓ_P of the square of side two at $2y$, and $\delta(\ell_P, I) \le 4\kappa$. Replacing the two path links of direction μ and the two of direction ν by the coarse links one at a time adds at most $(9-\mu)\kappa + (9-\nu)\kappa$, and $\mu + \nu \ge 1$. □

At the background of Proposition 408 the conclusion on the Hessian holds for $\widehat{V}$ as well: at $U = R(c)$ all sixteen paths of every pair agree, so by Proposition 440(i) the tangent space of the fibre of $\widehat{V}$, a submanifold by Theorem 441(ii), is that of the fibre of $V^\sharp$ and contains ξ; as every first derivative of S vanishes at U, the second derivative of $\beta_W S$ along any curve in the fibre with velocity ξ is $-\beta_W$. So the projected average also needs a coarse-curvature label, for which Proposition 445 gives the witness.

Remark 446. For a compact connected group $G \subset \mathrm{U}(N)$ with Lie algebra $\mathfrak{g}$, let $\Pi_{\mathfrak{g}}$ be the orthogonal projection of the complex $N\times N$ matrices onto $\mathfrak{g}$, which for $\mathrm{SU}(N)$, $N \ge 3$, is not the skew-Hermitian part. The proofs above use Lemma 439(iii) only for the constants; with $\|\mathrm{ad}X\| \le 2\|X\|$ and $\|X\|_{\mathrm{op}} \le \|X\|$ instead they give, with $s_p = 1 - \frac{1}{N}\mathrm{Re}\,\mathrm{Tr}\,U_p$: the contraction constant $\frac{7}{8}$, the responses 8, 1 and $\frac{1}{2}$, $J \in [8^{-\dim\mathfrak{g}}, (\frac{15}{8})^{\dim\mathfrak{g}}]$, the bounds $\frac{15}{2}\dim\mathfrak{g}$ and $15\dim\mathfrak{g}$ in Theorem 441(iv), the constant $\frac{1}{2} + 24\beta_W + 60\dim\mathfrak{g}\sqrt{N}$ in Theorem 443(ii), and $\frac{12}{\sqrt{N}}\beta_W + 15\dim\mathfrak{g}$ in Proposition 444(iii). Lemma 439(ii) becomes $\|\Pi_{\mathfrak{g}}(e^X) - X\| \le \frac{1}{6}\|X\|^2_{\mathrm{op}}\|X\|$, since the Hermitian part of e^X is orthogonal to $\mathfrak{g}$, and Proposition 445 holds unchanged.

Proposition 447 (the chart for the projected average)**.** *Let $G \subset \mathrm{U}(r)$ be compact and connected, $\Pi_{\mathfrak{g}}$ as in Remark 446, and replace $V^\sharp$ by $\widehat{V}$ in the fibre F and in the constrained law.*

(i) *Under the hypothesis of Lemma 360,*

$$\Big\| \widehat{V}_{y,\mu}(U)^{-1}\frac{d}{dt}\widehat{V}_{y,\mu}(Ue^{t\xi})\Big|_{t=0} - V_{y,\mu}(\tilde{\xi})\Big\| \le 166\,\kappa N_{y,\mu}(\xi).$$

(ii) *Proposition 364 holds with the same c_3 and c_6 for $\mathrm{SU}(2)$, and with $c_3 = \max\{1.5\cdot 10^4 r^2, \dim G\}$ and $c_6 = 6.81\cdot 10^5 \dim G\, c_3^2$ for general G.*

(iii) *Theorems 361, 363, 367 and 370 and Proposition 369 hold with these constants; in particular $\alpha_G = 2.5\cdot 10^{-4}$ and $c_9 = 1.2\cdot 10^4$.*

(iv) *For $\mathrm{SU}(2)$, Lemma 371 holds with $\sigma_0 = \frac{1}{8}$ at every field, Lemma 373 with $M = 512$, and Proposition 374 with the same ε_W.*

(v) *$\widehat{V}(R(c)) = c$, and Lemma 524 holds for $\widehat{V}$ with the same bound $4\kappa + 7700\kappa^2$.*

Proof. Hermitian matrices are orthogonal to $\mathfrak{g}$, so $\Pi_{\mathfrak{g}}(w) = \Pi_{\mathfrak{g}}(w-I)$ has norm at most $\delta(w,I)$ for unitary w, and for $Y \in \mathfrak{g}$, $\Pi_{\mathfrak{g}}(wY) - Y = \Pi_{\mathfrak{g}}((w-I)Y)$ has norm at most $\|w - I\|_{\mathrm{op}}\|Y\| \le \delta(w,I)\|Y\|$.

(i) Use the notation of the proof of Lemma 360(iv), with $A = A_{y,\mu}$ in place of M. From $\delta(w_x, I) \le 24\kappa$ and the two paths of offset zero, $\|A\| \le \frac{14}{16}24\kappa = 21\kappa$, and $\widehat{V}^{-1}\dot{\widehat{V}} = \mathrm{Ad}(e^{-A})\alpha + R\dot{A}$ with $R = \int_0^1 \mathrm{Ad}(e^{-sA})\,ds$, $\|R-1\| \le \|A\|$. Here $\dot{A} = \frac{1}{16}\sum_x \Pi_{\mathfrak{g}}(w_x Y_x)$ with $Y_x = \beta_x - \mathrm{Ad}(w_x^{-1})\alpha$, so $\|\dot{A}\| \le m_\beta + \|\alpha\|$, and $\Pi_{\mathfrak{g}}(w_x Y_x) - (\beta_x - \alpha) = (\Pi_{\mathfrak{g}}(w_x Y_x) - Y_x) + (\alpha - \mathrm{Ad}(w_x^{-1})\alpha)$. The four differences of that proof are at most $42\kappa\|\alpha\|$, $21\kappa(m_\beta + \|\alpha\|)$, $24\kappa m_\beta + 72\kappa\|\alpha\|$ and $42\kappa m_\xi$, and their sum is at most $(135 + 45\cdot 0.434 + 42/4)\kappa N_{y,\mu} < 166\kappa N_{y,\mu}$.

(ii) Follow the proof of Proposition 364. On unitary matrices $\Pi_{\mathfrak{g}}(w) = L(w)$ with $L(w) = P_{\mathbb{C}}(\frac{1}{2}(w - w^{-1}))$, $P_{\mathbb{C}}$ the orthogonal projection onto $\mathfrak{g}\otimes\mathbb{C}$ for the Hermitian inner product, which agrees with $\Pi_{\mathfrak{g}}$ on skew-Hermitian matrices; L is holomorphic on $\mathrm{GL}(r,\mathbb{C})$. Since $e^{X+Z} = e^X(I+F)$ gives $(I+F)^{-1} = I + F'$, w_x^{-1} has the same factorisation as w_x, and $\|L(w_x) - L(w'_x)\|_{\mathrm{op}} \le \|L(w_x) - L(w'_x)\| \le \sqrt{r}\,\psi$. The two

paths of offset zero have $w_x = I$, so the exponent moves by at most $d_L = \frac{7}{8}\sqrt{r}\,\psi$, which replaces d, and no domain condition on the sites is needed. At real points $\|A' - A_0\| \le \frac{7}{8}\cdot 3x_0$, so $\gamma_0 = x_0 + \frac{21}{8}x_0 < 0.0037$. For SU(2), $r = 2$: at $R = 0.07$, $\varphi < 0.0726$, $\psi < 0.2337$, $d_L < 0.2892$, $\eta < 0.4322$, $S < 0.767$, $6S/R^3 < 13417$ and $2S/R^2 < 314$; at $R = 10^{-3}$, $S/R < 4.8$; on $|z| \le 0.1$, $\gamma_0 + \eta < 0.71$. All are below $1.5\cdot 10^4$. For general G take the radii $R/\sqrt{r}$: by convexity $\varphi(R/\sqrt{r}) \le \varphi(R)$ and $\sqrt{r}\,\psi(R/\sqrt{r}) \le \psi(R)$, so $S(R/\sqrt{r})$ is at most the value above, and the Cauchy estimate loses $r^{m/2} \le r^{3/2}$. The derivatives of $\log j$ do not involve the average.

(iii) These proofs use the average in three places only: the derivative bound of Lemma 360(iv), whose constant 260κ is improved by (i); the constants c_3 and c_6; and, in the proof of Theorem 370(i), the form $V^\sharp = ve^M$ with $\|M\| \le 8\pi\kappa$. For $\widehat{V}$ the bound $\|A\| \le 21\kappa$ gives $\delta(v,v') \le 42\kappa$ and $\delta(U_e, U_e') \le 84\kappa < 92.3\kappa$.

(iv) The derivative in the eight private links satisfies $JJ^* \ge D_g\widehat{V}(D_g\widehat{V})^* \ge \frac{1}{64}$ by Theorem 441(ii). For fixed non-path links, $g \mapsto \widehat{V}$ is a diffeomorphism with Jacobian at least 8^{-3}, so the image of Haar measure has density at most 512, and so has its average over the non-path links. In the proof of Proposition 374, $\frac{7}{8}\Lambda$ becomes $\frac{7}{8}$, 1.957 becomes $\frac{15}{8}$, and $(512^4(\frac{15}{8})^{12})^{1/45} < 2.06 < 2.93$.

(v) At $R(c)$ every relative path is I. In the proof of Lemma 524, Lemma 439(ii) gives $\delta(\widehat{c}_k, c_k) \le \frac{14}{16}\cdot\frac{1}{12}a^3$ for each side, and replacing the four c_k by $\widehat{c}_k$ adds at most $\frac{7}{24}a^3 < 4100\kappa^3 < 10\kappa^2$ for $\kappa \le 10^{-8}$, which the bound $4\kappa + 7685\kappa^2$ of that proof absorbs. □

Remark 448 (what is proved for which map). The projected average is a different coarse map: it defines different coarse fields $c = \widehat{V}(U)$ and different conditional laws μ^c. Proved for it are Proposition 440, Theorems 441 and 443, Corollary 442, Propositions 444, 445 and 447, Lemma 415, whose proof applies unchanged, and the Hessian statement after Proposition 445. Through Proposition 447 this includes the chart of Theorems 361 and 363, the chart decay of Theorem 367 and Proposition 369, and Theorem 370. The results of Sec. XII E that concern only the Wilson measure and its plaquettes, Theorem 383, Corollary 396 and Proposition 397 among them, do not involve the coarse map. Not proved here for $\widehat{V}$: the composition across scales of Sec. XII C; the energy bounds of Sec. XII G, which on torus-valued fields use that $V^\sharp$ is the abelian average of the angles, while $\widehat{V}$ averages their sines; the labels, sites and chains of Secs. XIII A and XIII B, Proposition 413 among them; the islands, their threshold and the label covariance of Secs. XIII D to XIII O, including Theorems 526 and 529 and Proposition 525; and the one-step and multiscale results of Secs. XIII P to XIII T. Proposition 440(i) bounds the distance between the two maps; it does not transfer these results.

Remark 449 (the route to (D) with $\widehat{V}$). With $\widehat{V}$ in place of $V^\sharp$, the chart decay holds at $\alpha_G = 2.5\cdot 10^{-4}$ (Proposition 447), the good side holds at every field (Corollary 442), and the bad side of item (i) of Sec. XIII V holds for every restricting set, uniformly in the exterior and in the inside links, with no condition (Theorem 443). The statements that the cube condition (D) still needs are items (ii)–(iv) of Sec. XIII V with $\widehat{V}$ for $V^\sharp$:

(a) the island comparison: a covariance bound for every coarse field, uniform in the exterior, decaying at the rate $2\alpha_G$ without a floor, and the mixed differences in tree form;

(b) the island threshold at the scale of the cube;

(c) the effective action at every scale with summable one-step errors.

The results that the paper proves towards (a) and (b) for $V^\sharp$, and the energy bounds of Sec. XII G, are not re-derived for $\widehat{V}$ (Remark 448). For $V^\sharp$ the first set can have no chart of the relative kind (Proposition 434), so the bad side is available only through $\widehat{V}$. The labels and the islands use the following results on the coarse map. Theorem 353(ii) holds for $\widehat{V}$ by Proposition 440(ii). Parts (i)–(iii) of Lemma 360 do not involve the coarse map, and part (iv) is Proposition 447(i). Theorems 361, 363 and 367 and Propositions 364 and 369 are Proposition 447(ii), (iii). Lemma 409 is replaced by Proposition 445, with its own constant. Theorem 383 and Lemma 382 concern the Wilson measure only. Not available for $\widehat{V}$: Proposition 470; the fixed-point form of the path-link equation behind Lemma 411 and Propositions 416 and 412, which the proof of Theorem 483 differentiates (for $\widehat{V}$ the equation has one solution at every field, Theorem 441, but that form is not derived); the energy bounds of Sec. XII G, Lemma 401, Proposition 403, Theorem 404, Corollary 405 and Proposition 406; and the composition across scales, Lemma 354 and Theorem 355. Besides (a)–(c), the route through $\widehat{V}$ needs these, and the counting of the labels redone with the constant of Proposition 445. Neither (D) nor the mass gap is proved.

D. Counting bad labels

A label is bad when some fine plaquette in a fixed neighbourhood is bad: for the relative link labels this is the content of Theorem 353 and Lemma 360, and for the coarse-curvature labels it is Lemma 409. The chessboard estimate of Theorem 383 bounds the probability that a set of fine plaquettes is simultaneously bad by a product. The following lemma transfers such a bound to the labels, and the proposition after it shows that the loss it incurs is the true loss of this kind of argument.

Lemma 450 (coverage counting)**.** *Let a family of events E_y, indexed by a finite set, be such that each E_y forces at least one of a specified set Q_y of at most M sites to be bad, and suppose that each site lies in at most H of the sets Q_y and that* $\Pr(\text{all sites of } T \text{ bad}) \le p^{|T|}$ *for every finite set T of sites. Then for every nonempty set S of n labels and $k = \lceil n/H \rceil$,*

$$\Pr\Big(\bigcap_{y\in S} E_y\Big) \le \binom{Mn}{k} p^k \le (eMHp)^k,$$

and if $A = eMHp < 1$ then $\Pr(\bigcap_{y\in S} E_y) \le A^{n/H}$.

Proof. On the intersection, the union of the witness sets Q_y, $y \in S$, contains at least k distinct bad sites, since one site witnesses at most H labels; the union has at most Mn sites. A union bound over its k-subsets and the product bound for each give the first inequality, and $\binom{Mn}{k} \le (eMn/k)^k \le (eMH)^k$ since $n/k \le H$. No choice of witness is required to be measurable, and no independence is used. □

Proposition 451 (the exponent $1/H$ cannot be improved)**.** *In the setting of Lemma 450 let H labels share one site and let each be bad exactly when that site is bad, with probability p. Then* $\Pr(\text{all } H \text{ labels bad}) = p$*, so every bound of the form $q^{|S|}$ valid for this class of families has $q \ge p^{1/H}$.*

Proof. The event that all H labels are bad is the event that the one site is bad. □

For the labels of Sec. XIII A the witness sets are boxes of side 7 for the relative link labels, $M = 7^4$, $H = 4^4$, and of side 11 when the coarse-curvature labels are included, $M = 11^4$, $H = 6^4$, with p the chessboard bound of Theorem 383 at the threshold of Lemma 409. Lemma 450 then charges a bad island of n blocks at the rate $A^{n/H}$; the proposition shows that the denominator H is not an artefact of the union bound but the price of knowing only that each label has a witness, and that a better exponent needs joint information on the gauge field.

E. Islands and the environment

Two bad islands are separated by good blocks, and the question is how the weight of one island depends on the state of the other. The following lemma answers it in terms of a single quantity: the cross-ratio of a kernel on the good layer between them. It integrates arbitrary measures on the two islands, including hard constraints and sums over branches, and their masses do not enter.

Lemma 452 (integration through a common kernel)**.** *Let $K(x,y) > 0$ be measurable, let a_0, a_1 and b_0, b_1 be finite positive measures on the spaces of x and of y, and suppose that $Z_{ij} = \iint K\, da_i\, db_j$ is finite and positive for the four pairs $i,j \in \{0,1\}$. If*

$$e^{-\Delta} \le \frac{K(x_1,y_1)K(x_0,y_0)}{K(x_1,y_0)K(x_0,y_1)} \le e^{\Delta}$$

for all x_i in the support of a_i and y_j in the support of b_j, then

$$\left|\log Z_{11} + \log Z_{00} - \log Z_{10} - \log Z_{01}\right| \le \Delta.$$

Proof. Multiply the upper inequality by $K(x_1,y_0)K(x_0,y_1)$ and integrate against $da_1(x_1)\,da_0(x_0)\,db_1(y_1)\,db_0(y_0)$; by Tonelli the left side integrates to $Z_{11}Z_{00}$ and the right side to $e^{\Delta}Z_{10}Z_{01}$. The lower inequality gives the reverse bound, and the four numbers are positive, so logarithms may be taken. □

Corollary 453 (two labels)**.** *Let (A,B) be a pair of binary labels with joint law* $\Pr(A=i,B=j) = Z_{ij}/\sum Z$*, all $Z_{ij} > 0$, and let Δ bound the connected normaliser as in Lemma 452. Then*

$$\left|\Pr(B=1 \mid A=1) - \Pr(B=1 \mid A=0)\right| \le \tanh\frac{\Delta}{4},$$
$$\left|\operatorname{Cov}(A,B)\right| \le \frac{1}{4}\tanh\frac{\Delta}{4},$$

and both bounds are attained.

Proof. The conditional odds of $B=1$ given $A=1$ and given $A=0$ are Z_{11}/Z_{10} and Z_{01}/Z_{00}, whose logarithms differ by $\log(Z_{11}Z_{00}/Z_{10}Z_{01})$, of modulus at most Δ. For logistic probabilities $\sigma(u) = (1+e^{-u})^{-1}$ the difference $\sigma(u+s) - \sigma(u)$ is largest at $u = -s/2$ by the symmetry $\sigma(-u) = 1-\sigma(u)$, where it equals $\sigma(s/2) - \sigma(-s/2) = \tanh(s/4)$, and it increases in s. The covariance is $\Pr(A=1)\Pr(A=0)$ times the difference of the conditional probabilities, and $\Pr(A=1)\Pr(A=0) \le \frac{1}{4}$. The weights $Z_{11} = Z_{00} = e^{\Delta/2}$, $Z_{10} = Z_{01} = 1$ have $\Pr(A=1) = \frac{1}{2}$ and conditional probabilities $\sigma(\pm\Delta/2)$, so both bounds are equalities. □

When the kernel is the partition function of a calm layer, its cross-ratio is a connected second derivative of a log-normaliser, and that derivative is a covariance under the law of the layer.

Lemma 454 (connected normalisers)**.** *Let D be a fixed product of balls in the coordinates z of a window, let $\mathscr{U}(z;\eta,\zeta)$ be smooth in all variables, with η and ζ exterior parameters, and let $Z(\eta,\zeta) = \int_D e^{-\mathscr{U}}\,dz$. Then for directions u in η and v in ζ,*

$$\partial_u\partial_v \log Z = -\mathbb{E}\big[\partial_u\partial_v\mathscr{U}\big] + \operatorname{Cov}\big(\partial_u\mathscr{U}, \partial_v\mathscr{U}\big),$$

expectation and covariance under $e^{-\mathscr{U}}\,dz/Z$ on D. If $\mathscr{U}$ is a sum of local terms and the supports of η and ζ are farther apart than its range, the first term vanishes. If in addition, at every point of two segments $\eta_0 \to \eta_1$ and $\zeta_0 \to \zeta_1$, the law on D satisfies

$$\left|\operatorname{Cov}(F,H)\right| \le \mathscr{A}\,\|\nabla F\|\,\|\nabla H\|\,e^{-\alpha\,\operatorname{dist}(\operatorname{supp}\nabla F, \operatorname{supp}\nabla H)}$$

for all smooth F and H, then

$$\left|\log Z(\eta_1,\zeta_1) - \log Z(\eta_1,\zeta_0) - \log Z(\eta_0,\zeta_1) + \log Z(\eta_0,\zeta_0)\right| \le \mathscr{A} L_\eta L_\zeta\,\|\eta_1-\eta_0\|\,\|\zeta_1-\zeta_0\|\,e^{-\alpha d'},$$

where L_η, L_ζ bound the gradients in z of $\partial_u\mathscr{U}$ and $\partial_v\mathscr{U}$ per unit of u, v along the segments, and d' is a lower bound there for the distance between the supports of those gradients.

Proof. Differentiating under the integral sign on the fixed domain D gives $\partial_v \log Z = -\mathbb{E}[\partial_v\mathscr{U}]$ and then the identity, the covariance arising from the derivative of the normalised density. The left side of the integrated bound is the integral of $\partial_u\partial_v \log Z$ over the product of the two segments, in which the first term of the identity vanishes and the second is bounded by the covariance hypothesis. □

On a good window the law of Proposition 412, with the Hessian bound of Theorem 367 on the graph, supplies a covariance bound of the required form, so Lemma 454 controls the connected normaliser for continuous changes of the sources inside the chart, and Lemma 452 then integrates the hard island weights on the two sides. What the three statements do not supply is the kernel bound when a label change moves the domain D, enters an island, or adds a branch of Proposition 414; that is the open comparison of Sec. XIII V. The last result of this subsection shows that the comparison cannot be replaced by rarity, however strong.

Proposition 455 (rarity does not control the environment)**.** *Fix $q \in (0,1)$. Let S be a fair bit and, given S, let the labels B_x be independent Bernoulli variables with parameter $q/2$ if $S=0$ and q if $S=1$. Then $\Pr(B_x = 1\ \forall x \in T) \le q^{|T|}$ for every finite T, and for all distinct x, y,*

$$\operatorname{Cov}(B_x, B_y) = \frac{q^2}{16}.$$

If the configuration is a function of the labels, every covariance under the law conditioned on all labels vanishes, and the covariance of two observables is the covariance of their conditional expectations. More generally, for a binary label B with $p = \Pr(B=1 \mid \xi) \in (0,1)$ given the other labels ξ and $\Delta_F = \mu(F \mid B=1,\xi) - \mu(F \mid B=0,\xi)$,

$$\operatorname{Cov}(F, \mathbf{1}_B \mid \xi) = p(1-p)\Delta_F,$$
$$\operatorname{Cov}\big(\mathbb{E}[F \mid B,\xi], \mathbb{E}[H \mid B,\xi] \;\big|\; \xi\big) = p(1-p)\Delta_F\Delta_H.$$

Proof. Given S the labels are independent, so $\Pr(B_x = 1\ \forall x \in T) = \frac{1}{2}((q/2)^{|T|} + q^{|T|}) \le q^{|T|}$, and $\operatorname{Cov}(B_x, B_y) = \operatorname{Var}(\Pr(B_x = 1 \mid S)) = \frac{1}{4}(q - q/2)^2 = q^2/16$ by the conditional covariance formula. The two identities follow by expanding the expectations over the two values of B. □

The example satisfies every finite-intersection rarity bound at every order, yet its labels have a covariance with a positive floor at all distances. A labelling argument must therefore bound $\Delta_F\Delta_H$, the change of the conditional expectations under a label flip, and not only the probability of the flip: small p does not help when $\Delta_F\Delta_H$ does not decay, and normalising by p can cancel it. That bound is the connected normaliser of Lemma 452, which is where the island comparison enters.

F. Coupling cost and the comparison lemma

The proofs of Lemmas 15 and 665 take a coupling that minimises a weighted sum of expected displacements and derive, from its minimality, one inequality per site. The following lemma isolates this step; it needs neither stationarity of the optimal coupling under the local updates nor any property of the updates beyond the two stated ones, and its conclusion is exactly the recurrence that Lemma 665 iterates.

Lemma 456 (minimality gives the recurrence)**.** *Let I be a finite index set, let Π be a set of couplings of two probability measures on a product of metric spaces indexed by I, let $d_i(\pi) = \mathbb{E}_\pi \operatorname{dist}(x_i, x_i')$, let $w_i > 0$, and let $S \subset I$. Suppose $\pi \in \Pi$ minimises $\sum_i w_i d_i$ over Π, and that for each $i \notin S$ there is $\pi^{(i)} \in \Pi$ with $d_j(\pi^{(i)}) = d_j(\pi)$ for $j \ne i$ and $d_i(\pi^{(i)}) \le \sum_j C_{ij} d_j(\pi)$. Then $d_i(\pi) \le \sum_j C_{ij} d_j(\pi)$ for every $i \notin S$. If π is minimal only up to ε_i against $\pi^{(i)}$, then $d_i(\pi) \le \sum_j C_{ij} d_j(\pi) + \varepsilon_i / w_i$. Consequently, if $C_{ij} \ge 0$, $\sum_i w_i C_{ij} \le \theta w_j$ with $\theta < 1$ and $d_i(\pi) \le D$ for $i \in S$, then $\sum_{i \notin S} w_i d_i(\pi) \le \frac{\theta D}{1-\theta} \sum_{j \in S} w_j$.*

Proof. Since $\pi^{(i)}$ changes only the i-th displacement, minimality gives $w_i d_i(\pi) \le w_i d_i(\pi^{(i)}) + \varepsilon_i$, and the bound on $d_i(\pi^{(i)})$ gives the recurrence. The last statement is the arithmetic of Lemma 665: with $X = \sum_{i \notin S} w_i d_i$ and $Y = \sum_{i \in S} w_i d_i \le D \sum_{i \in S} w_i$, the recurrence and the column condition give $X \le \sum_j d_j \sum_i w_i C_{ij} \le \theta(X+Y)$, so $X \le \theta Y/(1-\theta)$. □

Lemma 457 (minimal couplings exist)**.** *Let μ, ν be probability measures on a compact metric space Ω and c a continuous cost on $\Omega \times \Omega$. The set of couplings of μ and ν is nonempty and compact in the weak topology, and c attains its minimum on it.*

Proof. The product $\mu \otimes \nu$ is a coupling; the marginal conditions are closed in the weak topology, and the probability measures on the compact space $\Omega \times \Omega$ form a compact set, so the couplings form a compact set on which the continuous functional $\pi \mapsto \int c\, d\pi$ attains its minimum. □

The displacements d_i are continuous costs on the compact product of copies of $\mathrm{SU}(2)$, so the minimal couplings of Lemma 15 and of the block comparison exist by Lemma 457, and the coupled update of one site is the choice of $\pi^{(i)}$ in Lemma 456. The recurrence, the column-weighted and row-weighted comparisons of Lemma 665, the decay with a floor of Lemma 667, and the two lemmas above are also written in Lean 4 in the archive, with Mathlib as the library; the specification of $\mathrm{SU}(2)$ enters only through the hypotheses of Lemma 456, which the proofs of Secs. V and X supply by hand.

The block response that Lemma 15 bounds can also be computed exactly in one direction, which measures the room that the coupling argument leaves.

Proposition 458 (the block response at zero field)**.** *Let μ_0 be the law of three unit quaternions (y_1, y_2, y_3) with density proportional to $\exp(\beta_W \operatorname{Re}(y_1 y_2 y_3))$ with respect to Haar measure, and h a unit quaternion. Then $f = \langle y_1, h\rangle + \langle \overline{y_2 y_3}, h\rangle$ is 1-Lipschitz for the sum of the three chordal distances and*

$$\operatorname{Cov}_{\mu_0}\big(f, \langle y_1, h\rangle\big) = \frac{1}{4}\Big(1 + \frac{I_2(\beta_W)}{I_1(\beta_W)}\Big)$$
$$= \frac{1}{4}\Big(1 + \frac{\beta_W}{4} - \frac{\beta_W^3}{96} + O(\beta_W^5)\Big),$$

which lies between 0.2718 and 0.27183 at $\beta_W = 0.351$. Hence the Kantorovich modulus of the block in the field at y_1 is at least this number, and $10\beta_W$ times it exceeds one for $\beta_W > 0.3667$.

Proof. Given y_1, the product $w = y_2 y_3$ is Haar distributed a priori, and the weight is $\exp(\beta_W \langle y_1, \bar{w}\rangle)$ since $\operatorname{Re}(pq) = \langle p, \bar{q}\rangle$

for unit quaternions. So $\bar w$ given y_1 follows the von Mises–Fisher law on S^3 with mean direction y_1, whose mean resultant is $\mathbb{E}[\bar w \mid y_1] = A(\beta_W)y_1$ with $A = I_2/I_1$, the ratio of the Bessel functions of orders $\frac{p}{2}$ and $\frac{p}{2}-1$ for $p=4$, and the marginal of y_1 is Haar. Multiplication and conjugation preserve the chordal distance, so f changes by at most the sum of the three displacements. Since y_1 is uniform, $\mathrm{Var}\langle y_1,h\rangle = \frac14$ and $\mathrm{Cov}(\langle \bar w,h\rangle,\langle y_1,h\rangle) = \mathbb{E}[A\langle y_1,h\rangle^2] = A/4$. The series of A follows from $I_\nu(x) = \sum_m (x/2)^{2m+\nu}/(m!\,\Gamma(m+\nu+1))$, and the derivative of $\mu_b(f)$ in the field b_1 at $b=0$ in the direction h is the covariance, which bounds the modulus below. By Lemma 680(ii) and (iv), $0.08721 \le A(0.351) \le 0.08731$, because $6.373201 \le 2.52453^2$ and $9.123201 \ge 3.02046^2$; this gives the two bounds at 0.351. The function $\frac52 \beta_W(1+A(\beta_W))$ increases, since A' is a variance by Lemma 680(i), and at $\beta_W = 0.3667$, where $0.3667^2 + \frac{25}{4} \le 2.52676^2$, Lemma 680(iv) gives $A \ge 0.3667/4.02676 > 0.09106 > \frac{4}{3.667} - 1$, so it exceeds one there. □

G. From one scale to the next

A multiscale version of Sec. XII C would eliminate the fine coordinates of a window scale by scale. The following statements say what one elimination step preserves and what it does not, for the quadratic forms that Theorem 367 controls at one scale. They are finite matrix lemmas; no all-scale construction is made here.

Lemma 459 (weighted Schur complements)**.** *Let $H = \left(\begin{smallmatrix} A & B \\ B^* & D \end{smallmatrix}\right)$ be positive definite in retained and eliminated coordinates, with $S = A - BD^{-1}B^*$ its constrained minimum form on the retained ones. Give both index sets a metric, and for a 1-Lipschitz function ϕ and $\alpha \ge 0$ write $\|M\|_\alpha$ for the supremum over ϕ of the operator norms of $e^{\alpha\phi}Me^{-\alpha\phi}$. If $\|A\|_\alpha \le a$, $\|B\|_\alpha \le b$ and $\|D^{-1}\|_\alpha \le g$, then $\|S\|_\alpha \le a + b^2 g$, and $|S_{ij}| \le (a+b^2g)e^{-\alpha\,\mathrm{dist}(i,j)}$. If H_0 has blocks A_0, B_0, D_0 with the same bounds and $\|A-A_0\|_\alpha, \|B-B_0\|_\alpha, \|D-D_0\|_\alpha \le \varepsilon$, then*

$$\|S - S_0\|_\alpha \le \varepsilon(1+bg)^2.$$

Proof. Conjugation by $e^{\alpha\phi}$ respects products and inverses, so the weighted norm is submultiplicative on compatible blocks, and $\|B^*\|_\alpha = \|B\|_\alpha$ because $-\phi$ is admissible; the bound on S follows from its formula. For the entry bound take $\phi(x) = \mathrm{dist}(x,j)$. For the difference, $S - S_0 = (A-A_0) - (B-B_0)D^{-1}B^* - B_0D^{-1}(B-B_0)^* - B_0(D^{-1}-D_0^{-1})B_0^*$ with $D^{-1} - D_0^{-1} = -D^{-1}(D-D_0)D_0^{-1}$, and the four terms are bounded by ε, εbg, εbg and $\varepsilon b^2 g^2$. □

Proposition 460 (what form comparison preserves)**.** *Let H_0, H be positive definite with $(1-\varepsilon)H_0 \le H \le (1+\varepsilon)H_0$, $0 \le \varepsilon < 1$, and for a surjective linear map Q let $S(v) = \inf_{Qx=v} x^{\mathsf T}Hx$ and S_0 likewise. Then $(1-\varepsilon)S_0 \le S \le (1+\varepsilon)S_0$. Form comparison does not preserve locality: on a path of n sites, $H_0 = I$ and $H = I + \varepsilon uu^{\mathsf T}$ with $u = (1,\dots,1)/\sqrt{n}$ satisfy the comparison, while $(H^{-1})_{1n} = -\varepsilon/((1+\varepsilon)n)$ decays in no exponential way. If instead, for $W = e^{\alpha\phi}$, $\|WH_0^{-1}W^{-1}\| \le G_0$ and $\|W(H-H_0)W^{-1}\| \le \varepsilon/G_0$, then $\|WH^{-1}W^{-1}\| \le G_0/(1-\varepsilon)$.*

Proof. The comparison of forms passes to infima over the same affine subspaces. The rank-one inverse is $H^{-1} = I - \frac{\varepsilon}{1+\varepsilon}uu^{\mathsf T}$, whose off-diagonal entries all equal $-\varepsilon/((1+\varepsilon)n)$. The last statement is the Neumann series $H^{-1} = \sum_m (-H_0^{-1}(H-H_0))^m H_0^{-1}$ in the weighted norm. □

Repeating the comparison over scales multiplies the factors $(1\pm\varepsilon_k)$, so the one-step errors must be summable, and Lemma 459 identifies the one-step amplification $(1+b_kg_k)^2$ of the locality norm. Both are conditions that an effective action built from Theorem 367 at every scale would have to meet; the theorem at one scale provides the bounds a, b, g for the first step only.

An error bound that is a multiple of the Euclidean form does not survive this repetition, because the block constraint enlarges it at every step.

Proposition 461 (absolute errors grow under the block map)**.** *Let U have plaquettes at most κ, let B be the derivative at U of the block averages of a window, as in Theorem 361, and $\mathscr{G} = BB^*$. For every positive semidefinite form N on the link coordinates, every $\varepsilon \ge 0$ and every w in the range of B,*

$$\inf_{BX=w}\big(N(X) + \varepsilon\|X\|^2\big) \ge \inf_{BX=w} N(X) + \frac{4\varepsilon}{1+2724\kappa}|w|^2.$$

Proof. The infimum of a sum is at least the sum of the infima, and $\inf_{BX=w}\|X\|^2 = \langle w, \mathscr{G}^{-1}w\rangle$, attained at $X = B^*\mathscr{G}^{-1}w$. By the proof of Theorem 361(iii), a row of $\mathscr{G}$ has its diagonal block of norm at most $\frac{3}{16} + 227\kappa$ and two further blocks of norm at most $\frac{1}{32} + 227\kappa$, so the Schur test gives $\|\mathscr{G}\| \le \frac14 + 681\kappa$ and $\mathscr{G}^{-1} \ge 4/(1+2724\kappa)$. □

A one-step bound of the form $|\mathrm{Hess} - H_{\mathrm{ref}}| \le \lambda\varepsilon\|\cdot\|^2$ on the fine links, which is the form in which Theorems 361 and 363 and Proposition 369 control every non-Gaussian term, is therefore met by $H_{\mathrm{ref}} + \lambda\varepsilon\|\cdot\|^2$, whose block infimum exceeds that of H_{ref} by $4\lambda\varepsilon(1+2724\kappa)^{-1}|w|^2$. Carried over k blocking steps with the Gaussian fibre gap independent of the scale (Proposition 335(iv)), such a bound certifies the gap of the fibre only while $4^k\varepsilon$ stays small against it, that is, for a number of scales of order $\log_4(1/\varepsilon)$: an absolute error is a mass term and is relevant in four dimensions. The comparison of Proposition 460 is relative and is preserved with factor one, so the one-step errors that an effective action needs are relative bounds by a gauge-invariant form; since the block action $\inf_{BX=w} Q_U(X)$ vanishes along the coarse gauge directions, such a bound requires the one-step error to vanish along them as well.

H. Spectral weight at positive time

Theorem 48 makes the nontriviality of the continuum limit equivalent to a ratio of lattice two-point functions staying away from zero. The following two lemmas show that a ratio at a fixed physical time controls the spectral weight of the renormalised observable, so that the ratio criterion is a tightness condition on its spectral measure and not on its unsmeared

moments, and the example after them shows that a mass gap by itself does not give nontriviality.

Lemma 462 (tightness at positive time)**.** *Let σ_a be positive measures on $[0,\infty)$ with Laplace transforms $C_a(t) = \int e^{-tE}d\sigma_a(E)$, let $s>0$ be fixed, and suppose $0<C_a(s/2)<\infty$ and $C_a(s)/C_a(s/2)\ge c>0$ uniformly in a. Let $d\nu_a = e^{-sE}d\sigma_a/C_a(s)$. Then*

$$\int E\,d\nu_a \le M_s = \frac{2}{s}\log\frac{1}{c} \le \frac{2}{esc}, \qquad \nu_a([R,\infty)) \le \frac{M_s}{R},$$
$$\frac{C_a(s+t)}{C_a(s)} \ge e^{-M_s t} \quad (t\ge 0).$$

Proof. The Laplace transform C_a is finite on $[s/2,\infty)$, smooth on $(s/2,\infty)$ with $C_a' = -\int E e^{-tE}d\sigma_a$, and $\log C_a$ is convex there by the Cauchy–Schwarz inequality $C_a(\frac{t+t'}{2})^2 \le C_a(t)C_a(t')$. A convex function has derivative at s at least the slope of its chord over $[s/2,s]$, so

$$\int E\,d\nu_a = -(\log C_a)'(s) \le \frac{2}{s}\log\frac{C_a(s/2)}{C_a(s)} \le \frac{2}{s}\log\frac{1}{c},$$

and $\log u \le u/e$ gives $M_s \le 2/(esc)$. Markov's inequality gives the tail, and convexity gives $\log C_a(s+t)-\log C_a(s) \ge t(\log C_a)'(s) \ge -M_s t$. □

When σ_a is the spectral measure of a centred observable O_a for a nonnegative Hamiltonian, ν_a is the spectral law of the unit vector $e^{-sH_a/2}O_a\Omega_a/C_a(s)^{1/2}$; the lemma keeps its spectral mass from escaping to infinite physical energy. The ratio at the two separations $s/2$ and s is the quantity that Proposition 51 bounds by slice correlations.

Lemma 463 (nonvanishing limits)**.** *If $\sigma_a([0,\infty)) \ge v>0$ and $\int E\,d\sigma_a \le M$ uniformly in a, then $C_a(t)\ge v e^{-Mt/v}$ for all $t\ge 0$, and every limit of the $C_a(t)$ is positive.*

Proof. With $v_a = \sigma_a([0,\infty))$, Jensen's inequality for the probability σ_a/v_a gives

$$C_a(t) \ge v_a \exp\Big(-\frac{t}{v_a}\int E\,d\sigma_a\Big) \ge v_a e^{-Mt/v_a},$$

and $x\mapsto xe^{-Mt/x}$ is increasing. □

Proposition 464 (a gap without a limit)**.** *Let X be the symmetric two-state Markov process with jump rate $m/2$ in each direction, sampled at times na, and let $O_a = aX$. The transfer matrix has eigenvalues 1 and e^{-ma}, so the gap in physical units is m for every a, while $C_a(t) = a^2e^{-mt}\to 0$ and every moment of O_a tends to zero. The sampled process is reflection positive, and the spectral measure σ_a is the atom $a^2\delta_m$, whose Laplace transform at $s/2$ and s has the ratio $e^{-ms/2}$, uniform in a; Lemma 462 applies, and the limit vanishes because $\sigma_a([0,\infty)) = a^2\to 0$, which Lemma 463 excludes by its hypothesis on the mass.*

Proof. The correlation $\mathbb{E}[X_0X_t] = e^{-mt}$ for the two-state process, and O_a scales it by a^2; the transfer matrix of the sampled chain is the matrix of its transition probabilities, with the stated eigenvalues. Reflection positivity of a stationary reversible Markov chain is standard. The rest is read off. □

The example separates the two conditions that Theorem 48 keeps apart: a uniform gap and a nonvanishing ratio. For a sequence of measures whose mass does not vanish, Lemma 463 converts a bound on the first moment at positive time, which Lemma 462 derives from one ratio, into a positive lower bound for the limit; the mass itself is the normalisation of Theorem 48.

I. The island threshold at one scale

The counting of Lemma 450 and the collar of Proposition 412 determine a coupling past which the bad blocks of one scale form islands whose entropy is beaten by their rarity. This subsection computes that coupling. Fix a radius r for the relative link labels and a radius η for the coarse plaquettes: a block y is *bad* when some non-path active link of a pair based at y is at distance more than r from its reconstruction value, or some coarse plaquette with corner y has $\delta(c_P,I)>\eta$, and *good* otherwise. Fix an integer collar $s\ge 1$: a window W is conditioned as in Proposition 412 when every block at ℓ^∞ distance at most s from W is good. Join two blocks when they are at ℓ^∞ distance at most $2s+1$, and call a maximal connected set of bad blocks an *island*; two islands are then separated by a layer of good blocks of width at least $2s+1$, whose middle blocks are at distance more than s from both, and the good layer of Lemma 452 is available in the middle of it. Each block has $\Delta = (4s+3)^4-1$ neighbours in this graph, and the witness sets of Lemma 450 are the boxes of side 11, $M=11^4$, $H=6^4$.

Theorem 465 (the island threshold)**.** *Let $G=\mathrm{SU}(2)$, let $\rho = \min\{r/43,\eta/92\}\le 10^{-8}$, and put*

$$\log q^* = -H\big(\log 2 + 2\log\Delta\big) - \log(eMH),$$
$$f(\beta) = 6(\log\beta + C_0) - \frac{\beta\rho^2}{16},$$

with C_0 as in Lemma 382. The equation $f(\beta)=\log q^*$ has a largest solution $\beta^* = \beta^*(\rho,s) > 96/\rho^2$, and for every $\beta_W \ge \beta^*$:

(i) *every block is bad with probability at most $A = eMH\,q_\rho(\beta_W) \le (2\Delta^2)^{-H}$;*

(ii) *for every block y and every $n\ge 1$,*

$$\mu_L\{y \text{ lies in an island of at least } n \text{ blocks}\} \le 2^{-n}\Delta^{-2},$$

and the expected number of islands of at least n blocks on a torus of L^4 blocks is at most $L^4 2^{-n}\Delta^{-2}/n$;

(iii) *if an estimate charges every bad block a factor $\Gamma\ge 1$ and is summed over the islands through a given block, the sum $\sum_{n\ge 1}\Delta^{2(n-1)}(\Gamma A^{1/H})^n$ is at most Δ^{-2} as soon as $\beta_W \ge \beta^*(\rho,s,\Gamma)$, the largest solution of $f(\beta)=\log q^* - H\log\Gamma$.*

In particular

$$\beta^*(10^{-8},1)\in[3.426,3.428]\cdot10^{21},$$
$$\beta^*(10^{-8},5)\in[5.400,5.402]\cdot10^{21},$$
$$\beta^*(10^{-9},1)\in[3.430,3.432]\cdot10^{23},$$
$$\beta^*(10^{-9},5)\in[5.404,5.406]\cdot10^{23},$$
$$\beta^*(10^{-8},5,10^{30})\in[1.972,1.974]\cdot10^{22};$$

in every case $\beta^*=16(|\log q^*|+H\log\Gamma)\rho^{-2}(1+\varepsilon)$ *with* $0<\varepsilon<2\cdot10^{-2}$*, and* $|\log q^*|$ *lies within* 10^{-3} *of* 21090.194 *for* $s=1$ *and of* 33424.872 *for* $s=5$.

Proof. Witnesses. Let y be bad because of a non-path link e of the pair $k=(y,\mu)$, and let κ be the largest defect $\delta(U_p,I)$ over the plaquettes with a vertex in a block at ℓ^∞ distance at most one from $B(y)$ or $B(y+e_\mu)$. If $\kappa\le10^{-8}$, Lemma 360(i) with $W=\{y,y+e_\mu\}$ gives $\delta(U_e,I)\le3\kappa$ when e lies inside a block, where $R_e(c)=I$, and $\delta(U_e,\ell_k)\le21\kappa$ when e leaves $B(y)$, where $R_e(c)=c_k$ and ℓ_k is the path link; the proof of Lemma 409 gives $\delta(c_k,\ell_k)\le7\pi\kappa$, so $\delta(U_e,R_e(c))\le(21+7\pi)\kappa<42.992\kappa$; since $d(g,h)\le(1+10^{-12})\delta(g,h)$ when $\delta(g,h)\le10^{-6}$, the geodesic distance of the label obeys $d(U_e,R_e(c))<43\kappa$. Since the label is bad, either $\kappa>10^{-8}\ge\rho$ or $43\kappa>r$, and in both cases some plaquette of that set has defect more than ρ. If y is bad because of a coarse plaquette with corner y, Lemma 409 gives in the same way a plaquette with defect more than $\min\{10^{-8},\eta/92\}\ge\rho$ among those with a vertex in a block at distance at most one from the four corner blocks. All these plaquettes lie in cubes whose corners are at ℓ^∞ distance at most five from $2y$, the box of side 11 of the fine lattice, and a cube lies in the box of at most 6^4 blocks. With the sites of Lemma 450 the unit cubes and $p=q_\rho(\beta_W)$, Theorem 383 gives the product bound, and the lemma gives $\mu_L(\text{all blocks of } S \text{ bad})\le A^{n/H}$ for every set S of n blocks when $A=eMHp<1$, and $\le A$ for one block.

The threshold. The function f is concave on $(0,\infty)$ with its maximum at $\beta=96/\rho^2$, where $f=6\log(96/\rho^2)+6C_0-6>0>\log q^*$; since $f(\beta)\to-\infty$, the equation $f=\log q^*$ has exactly two solutions and the larger one, β^*, exceeds $96/\rho^2$, with $f\le\log q^*$ on $[\beta^*,\infty)$. For $\beta_W\ge\beta^*$ this gives $q_\rho(\beta_W)\le e^{f(\beta_W)}\le q^*$, hence $A\le eMHq^*=(2\Delta^2)^{-H}$ and $\Delta^2A^{1/H}\le\frac12$, which is (i).

Islands. A connected set of n blocks containing y in the graph of the collar is recovered from a walk of length $2(n-1)$ around a spanning tree, each step to one of Δ neighbours, so there are at most $\Delta^{2(n-1)}$ of them. The island of y has at least n blocks only if some connected set of exactly n blocks containing y consists of bad blocks, and a given set of n blocks is bad with probability at most $A^{n/H}$, so

$$\mu_L\{\text{the island of } y \text{ has at least } n \text{ blocks}\}\le\Delta^{2(n-1)}A^{n/H}\le\Delta^{-2}2^{-n}.$$

An island of at least n blocks contains at least n blocks that lie in islands of at least n blocks, so the number of such islands is at most $1/n$ times the number of such blocks, whose expectation is at most $L^4\,2^{-n}\Delta^{-2}$; on a torus the number of neighbours of a block is at most Δ, which is all the count uses. For (iii), replace A by Γ^HA in the threshold: then $\Gamma\Delta^2A^{1/H}\le\frac12$ and the sum is at most $\Delta^{-2}\sum_n2^{-n}=\Delta^{-2}$. For the numbers, $|\log q^*|=H(\log2+2\log\Delta)+1+4\log11+4\log6$, with $\Delta=2400=2^5\cdot3\cdot5^2$ for $s=1$ and $\Delta=23^4-1$ for $s=5$. The series $2\operatorname{artanh}x=2\sum_kx^{2k+1}/(2k+1)$, with its tail bounded geometrically, at $x=\frac13,\frac15,\frac19,\frac{1}{21},\frac{1}{47}$ gives $\log2$, $\log\frac32$, $\log\frac54$, $\log\frac{11}{10}$ and $\log\frac{24}{23}$ to within 10^{-9}, hence $|\log q^*|$ to within 10^{-4}, and with $3.14159265<\pi<3.14159266$ it gives $C_0=\frac12+\frac23\log(24\pi^3)\in[4.90816,4.90817]$. The function $g(\beta)=\beta\rho^2/16-6(\log\beta+C_0)-|\log q^*|-H\log\Gamma$ increases for $\beta>96/\rho^2$ and β^* is its largest zero; at the two ends of each stated interval g is negative and positive, with margins above 0.5, so that $\log\beta$ to two decimals decides the signs. Since $\varepsilon=6(\log\beta^*+C_0)/(|\log q^*|+H\log\Gamma)$ and $\log\beta^*<55$, $0<\varepsilon<2\cdot10^{-2}$. □

The theorem gives the value asked for in item (iii) of the earlier versions of this section, but only at the block scale and in lattice units: at $\beta_W=5\cdot10^{21}$ the lattice spacing is far below every physical length, the correlation length of the theory in lattice units is astronomically larger than the side of a block, and the rarity of the islands at that scale says nothing yet about the islands of the effective theory at the scale of the cube, which is what condition (D) needs. The threshold therefore closes the entropy count of one scale, and leaves the comparison between scales to the effective action of Proposition 460 and Lemma 459. The numbers also show where the loss sits: the factor $16/\rho^2$ comes from the witness radius $\rho\le10^{-8}$ of Lemma 360, and the factor $|\log q^*|\approx2H\log\Delta$ from the exponent $1/H$, which Proposition 451 shows cannot be improved without joint information on the gauge field.

J. The island threshold at every scale

The threshold of Theorem 465 composes with the defect propagation of Theorem 388: the labels of the j-th iterate of the cut-off average have fine witnesses as well, in a larger box, so that the chessboard estimate and the coverage count apply at every scale up to the one that Corollary 389 reaches, divided by 256. Write $U^{(j)}$ for the j-th iterate in the tree gauge, an SU(2) field on the lattice of spacing $a_j=2^ja$, and call a block y of that lattice *bad at scale* j when it is bad for the field $U^{(j)}$ in the sense of Theorem 465, with the same radii r, η and $\rho=\min\{r/43,\eta/92\}\le10^{-8}$. Let $\mathfrak{B}_j$, η_j be as in Theorem 388 with $\eta_0=1.01\rho_0$ for a threshold $\rho_0\le\rho$, so that every plaquette of $\mathfrak{B}_0$ has defect more than ρ_0.

Lemma 466 (fine witnesses at scale j)**.** *Let* $j\ge1$ *and suppose that* $\eta_j\le\rho$*. If the block* y *is bad at scale* j*, then some fine plaquette* $p\in\mathfrak{B}_0$*, with defect more than* ρ_0*, has its corner* x *in* $2^{j+1}y+[-8\cdot2^j+2,9\cdot2^j-3]^4$*. The cubes of this box number* $M_j=(17\cdot2^j-4)^4$*, and a fine cube lies in the box of at most* H_j *blocks of scale* j*, with* $H_j=8^4$ *for* $j=1,2$ *and* $H_j=9^4$ *for* $j\ge3$.

Proof. Since $\eta_{i+1}\ge4\eta_i$, the hypothesis gives $\eta_i\le\rho\le10^{-8}$ for $i<j$, so Theorem 388(i) and (ii) apply up to scale j. The

witness paragraph of the proof of Theorem 465 is a statement about one field in the tree gauge and applies to $U^{(j)}$: a bad block y has a plaquette P of $U^{(j)}$ with corner $x' \in 2y+[-5,5]^4$ and $\delta(U_P^{(j)}, I) > \rho$. If $P \notin \mathfrak{B}_j$, Theorem 388(i) and Lemma 352 give $\delta(U_P^{(j)}, I) \le \|\Theta_P^{(j)}\| \le \eta_j \le \rho$; so $P \in \mathfrak{B}_j$, and by part (ii) some point of P is within ℓ^∞ distance $3(2^j-1)$, in fine units, of a plaquette $p \in \mathfrak{B}_0$ with corner x. In fine units the coordinates of P lie in $[2^j x_c', 2^j x_c' + 2^j]$ and those of p in $[x_c, x_c+1]$, so $x_c \in [2^j x_c' - 3\cdot 2^j + 2,\ 2^j x_c' + 4\cdot 2^j - 3]$, and the range of x_c' gives the box. It has $17\cdot 2^j - 4$ integer points per coordinate. A cube with corner x lies in the box of y exactly when $2^{j+1} y_c$ lies in an interval of length $17\cdot 2^j - 5$ for each coordinate, which contains at most $\lfloor 8.5 - 5\cdot 2^{-j-1} \rfloor + 1$ multiples of 2^{j+1}, that is 8 for $j = 1,2$ and 9 for $j \ge 3$. □

Theorem 467 (the island threshold at scale j)**.** *Let $G = \mathrm{SU}(2)$, $s \ge 1$, $j \ge 1$, $\rho = \min\{r/43, \eta/92\} \le 10^{-8}$, and put $\rho_0 = \rho/(1.01\cdot 1.023\cdot 4^j)$, $\eta_0 = 1.01\rho_0$,*

$$\log q_j^* = -H_j\big(\log 2 + 2\log\Delta\big) - \log(eM_jH_j),$$
$$f_j(\beta) = 6(\log\beta + C_0) - \frac{\beta\rho_0^2}{16},$$

with $\Delta = (4s+3)^4 - 1$ and M_j, H_j as in Lemma 466. Let β_j^* be the largest solution of $f_j(\beta) = \log q_j^*$. Then for every $\beta_W \ge \beta_j^*$:

(i) *$\eta_i \le 1.023\cdot 4^i\eta_0 \le \rho$ for $i \le j$, the iterates up to scale j satisfy Theorem 388(i) and (iii) with $E_j^\circ \le 1.53\,E_0^\circ$, and every block at scale j is bad with probability at most $A_j = eM_jH_jq_{\rho_0}(\beta_W) \le (2\Delta^2)^{-H_j}$;*

(ii) *for every block y at scale j and every $n \ge 1$, with the islands of Theorem 465 formed from the bad blocks of scale j,*

$$\mu_L\{y \text{ in an island of at least } n \text{ blocks}\} \le 2^{-n}\Delta^{-2},$$

and the expected number of such islands on the torus is at most $(L/2^{j+1})^4 2^{-n}\Delta^{-2}/n$; the weighted variant (iii) of Theorem 465 holds with $\log q_j^* - H_j\log\Gamma$.

The thresholds increase, $\beta_{j+1}^* > 16\beta_j^*$. Let $j_{\max}$ be the largest $j \ge 0$ with $\beta_W \ge \beta_j^*$, where $\beta_0^* = \beta^*(\rho, s)$ is the threshold of Theorem 465; the conclusions hold at every scale $j \le j_{\max}$. For $s = 5$ and $\rho = 10^{-8}$,

$$\beta_1^* \in [2.895, 2.897]\cdot 10^{23}, \quad \beta_2^* \in [4.633, 4.635]\cdot 10^{24},$$
$$\beta_3^* \in [1.185, 1.187]\cdot 10^{26},$$

and $16^{-j}\beta_j^* \le 2.90\cdot 10^{22}$ for $1 \le j \le 12$. Hence, for $\beta_1^* \le \beta_W < \beta_{12}^*$,

$$2^{j_{\max}} \ge \frac{1}{2}\Big(\frac{\beta_W}{2.90\cdot 10^{22}}\Big)^{1/4},$$

and for every $\beta_W \in [10^{22}, 10^{42}]$ the scale k_* of Corollary 389 satisfies $k_* - j_{\max} \in \{8,9\}$: the islands of the composed average are rare up to the scale $a_{k_*}/512$.

Proof. (i) $4^j\eta_0 = \rho/1.023 \le 10^{-8}$, so Theorem 388(iv) gives the bounds on η_i and E_j°, and $\eta_j \le \rho$ is the hypothesis of Lemma 466. That lemma makes the sets of Lemma 450 the boxes of M_j cubes with overlap H_j, and Theorem 383 at the threshold ρ_0 gives the product bound $p = q_{\rho_0}(\beta_W)$; the threshold paragraph of the proof of Theorem 465, with ρ_0, M_j, H_j, q_j^* in place of ρ, M, H, q^*, shows that $\beta_j^* > 96/\rho_0^2$ exists and that $q_{\rho_0}(\beta_W) \le q_j^*$ for $\beta_W \ge \beta_j^*$, hence $A_j \le (2\Delta^2)^{-H_j}$ and $\Delta^2 A_j^{1/H_j} \le \frac12$. (ii) is the island paragraph of that proof on the torus of $(L/2^{j+1})^4$ blocks of scale j, where a block has at most Δ neighbours.

The thresholds. Put $c = 1.01\cdot 1.023$, so that $16c^2 \le 17.0810278$ and $\rho_0 = \rho/(c4^j)$, and $g_j(\beta) = \beta\rho_0^2/16 - 6(\log\beta + C_0) - |\log q_j^*|$; g_j increases for $\beta > 96/\rho_0^2$, and β_j^* is its largest zero, so that $\beta_j^* \le \beta$ exactly when $g_j(\beta) \ge 0$, for $\beta > 96/\rho_0^2$. Since ρ_0^2 at $j+1$ is $\rho_0^2/16$ at j, $g_{j+1}(16\beta) = g_j(\beta) - 6\log 16 - (|\log q_{j+1}^*| - |\log q_j^*|)$, and $|\log q_j^*|$ increases with j because H_j and M_j do; at $\beta = \beta_j^*$ this is negative, so $\beta_{j+1}^* > 16\beta_j^*$. In particular $\beta_j^* \le \beta_W$ for every $j \le j_{\max}$, and if $\beta_W < \beta_{12}^*$ then $\beta_W < \beta_{j_{\max}+1}^* \le 16^{j_{\max}+1}\cdot 2.90\cdot 10^{22}$ by the bound proved below, which is the displayed bound.

The numbers. We have

$$|\log q_j^*| = H_j(\log 2 + 2\log\Delta) + 1 + 4\log(17\cdot 2^j - 4) + \log H_j$$

with $\Delta = 23^4 - 1$. The logarithms of the proof of Theorem 465, together with $\log 17 = 4\log 2 + 2\operatorname{artanh}\frac{1}{33}$, give $|\log q_1^*|$, $|\log q_2^*|$ and $\mathsf{A} = 6561(\log 2 + 2\log\Delta) + 1 + 8\log 3$ within 10^{-3} of 105605.899, 105608.929 and 169133.301. For $j \ge 3$, $|\log q_j^*| = \mathsf{A} + 4\log(17\cdot 2^j - 4)$, and $\mathsf{A} + 4\log 17 \le 169144.64$, so that

$$169152.831 \le |\log q_3^*| \le |\log q_j^*| \le 169144.64 + 4j\log 2.$$

With $C_0 \in [4.90816, 4.90817]$, at the ends of the three stated intervals g_1, g_2, g_3 are negative and positive, with margins above 18, so that $\log\beta$ to two decimals decides the signs. For $b > 96c^2/\rho^2$,

$$g_j(16^j b) = \frac{b\rho^2}{16c^2} - 6(\log b + C_0) - \big(|\log q_j^*| + 24j\log 2\big),$$

whose last bracket increases with j and is at most $169144.64 + 28j\log 2$ for $j \ge 3$. At $b = 2.90\cdot 10^{22}$ the three terms are at least 169779.0, at most 339.8 and, for $j \le 12$, at most 169377.6; so $g_j(16^j b) > 0$, that is $16^{-j}\beta_j^* \le 2.90\cdot 10^{22}$, for $1 \le j \le 12$.

The scale k_*. By its definition, $16^{k_*} \le \beta_W/D < 16^{k_*+1}$ with

$$D = 1.0201\cdot 96\cdot 10^8(\log\beta_W + C_1),$$

and $C_1 \in [6.48436, 6.48437]$ gives $5.595\cdot 10^{11} \le D \le 1.0106\cdot 10^{12}$ for $\beta_W \in [10^{22}, 10^{42}]$. At $b = 3\cdot 10^{22}$ the three terms of $g_j(16^j b)$ are at least 175633.4, at most 340.0 and, for $j \le 17$, at most 169474.6. At $b = 5.4\cdot 10^{21}$ the first is at most 31614.1, which is less than $|\log q_1^*|$. With $\beta_0^* \in [5.400, 5.402]\cdot 10^{21}$ this gives

$$5.4\cdot 10^{21} \le 16^{-j}\beta_j^* \le 3\cdot 10^{22} \qquad (0 \le j \le 17),$$

and $\beta_j^* > \beta_{17}^* > 10^{42}$ for $j > 17$. So, with $x = \log_{16}\beta_W$, $j_{\max} \geq 0$,

$$\lfloor x - \log_{16}(3\cdot 10^{22})\rfloor \leq j_{\max} \leq \lfloor x - \log_{16}(5.4\cdot 10^{21})\rfloor,$$

and $k_* = \lfloor x - \log_{16} D\rfloor$. Since $\lfloor a\rfloor - \lfloor b\rfloor$ is $\lfloor a-b\rfloor$ or $\lfloor a-b\rfloor + 1$,

$$\left\lfloor \log_{16}\frac{5.4\cdot 10^{21}}{D}\right\rfloor \leq k_* - j_{\max} \leq \left\lfloor \log_{16}\frac{3\cdot 10^{22}}{D}\right\rfloor + 1,$$

and $16^8 D \leq 4.35\cdot 10^{21}$, $16^9 D \geq 3.84\cdot 10^{22}$ give $8 \leq k_* - j_{\max} \leq 9$. □

The theorem settles the entropy count of the islands at every scale that the fine witnesses reach. The next proposition shows that they reach no further: a coarse plaquette can be large at scale j while every fine plaquette is small, because the cut-off average multiplies a constant curvature by four at every step.

Proposition 468 (constant curvature grows by four)**.** *Let $T = \frac{1}{\sqrt{2}}\operatorname{diag}(i,-i)$, let $L = 2^m$, $n \geq 1$ and $\theta = 2\sqrt{2}\pi n/L^2$, and let U be the field with $U_{x,1} = \exp(\theta x_0 T)$ for $0 \leq x_0 < L$, $U_{x,0} = \exp(-\theta L x_1 T)$ on the links with $x_0 = L-1$, and every other link I. Then $\Theta_p = \theta T$ on every plaquette of the plane $\{0,1\}$ and $\Theta_p = 0$ on the others, so that $\delta(U_p, I) = 2\sqrt{2}\sin(\theta/2\sqrt{2}) \leq \theta$ for every p, and if $4^{j+1}\theta \leq \frac{1}{12}$ then for every $i \leq j+1$*

$$\Theta_P^{(i)} = 4^i\theta\, T \quad \text{on the plane } \{0,1\},$$

$$\delta(U_P^{(i)}, I) = 2\sqrt{2}\sin\bigl(4^i\theta/2\sqrt{2}\bigr) \geq 4^i\theta\Bigl(1 - \frac{16^i\theta^2}{48}\Bigr).$$

Consequently, with $\eta_0 \geq \theta$ the set $\mathfrak{B}_0$ is empty, while every block is bad at scale j for the coarse label as soon as $4^{j+1}\theta(1-16^{j+1}\theta^2/48) > \eta$; for $\eta = 92\rho$ this is $4^j\theta > 23\rho$ up to a factor $1+10^{-3}$, whereas Theorem 467 needs $4^j\eta_0 \leq \rho/1.023$.

Proof. The field takes values in the torus $S = \exp(\mathbb{R}T)$, and the plaquette exponents are the abelian curl: on the plane $\{0,1\}$ every plaquette, including those of the column $x_0 = L-1$, has exponent θ, the seam links contributing $-\theta L x_1 + \theta L(x_1+1) - \theta(L-1) = \theta$ there and cancelling on the planes $\{0,2\}$, $\{0,3\}$; the plaquettes through $x_1 = L-1$ are consistent since $\exp(\theta L^2 T) = \exp(2\pi i n\cdot\operatorname{diag}(1,-1)) = I$. An S-valued gauge transformation puts U in the tree gauge without changing the plaquettes, and Theorem 355 applies with $\eta_G = \frac{1}{12}$: by Lemma 354, $\Theta_P^{(i+1)}$ is the mean over the sixteen offsets of the sum of the four plaquettes of a square of side two, which for a constant $\Theta^{(i)} = 4^i\theta T$ on the plane $\{0,1\}$ and 0 elsewhere is $4^{i+1}\theta T$ on that plane and 0 elsewhere, and the hypothesis (36) holds by induction while $4^i\theta \leq \frac{1}{12}$. The eigenvalues of T are $\pm i/\sqrt{2}$, so $\delta(\exp(tT), I) = 2\sqrt{2}|\sin(t/2\sqrt{2})|$, and $\sin u \geq u - u^3/6$. The coarse label of a block at scale j is the plaquette of $U^{(j+1)}$. □

Between the two conditions lies a factor $23.5 < 4^{2.3}$, so the fine witnesses of Lemma 466 stop within three scales of where any fine witness must stop. At the radius 10^{-8} the remaining six scales up to a_{k_*} are the ratio between the curvature margin 10^{-4} of Theorem 388(iv) and the witness radius of Lemma 360; Theorem 471 below removes that ratio and carries the count to $a_{k_*}/8$, and past a_{k_*} the rarity of a large coarse plaquette can no longer come from the fine scale, by the proposition, but must come from the coarse energy of Lemma 401, which is known for torus-valued fields only; through the chessboard estimate it cannot reach the scale of the cube either (Theorem 476). Theorem 467 is an annealed statement about the pushforward of μ_L: it says nothing about the law of $U^{(j)}$ given its labels, which is what the effective action of item (iv) below is for.

K. The witness radius for SU(2)

The thresholds above carry the factor $16/\rho^2$ of the witness radius $\rho \leq 10^{-8}$. For SU(2) the witness paragraph of Theorem 465 holds up to an explicit radius, which is sharp for Lemma 360(iv). Write $\angle g \in [0,\pi]$ for the angle of $g \in \mathrm{SU}(2)$, the bi-invariant geodesic distance to I on S^3, so that $\angle(gh) \leq \angle g + \angle h$, $\angle(hgh^{-1}) = \angle g$, $\delta(g,I) = 2\sqrt{2}\sin(\angle g/2) \leq \sqrt{2}\angle g$ and $\|\log g\| = \sqrt{2}\angle g$ for $\angle g < \pi$. Put $p(\kappa) = 2\arcsin(\kappa/2\sqrt{2})$, the largest angle of a plaquette with $\delta(U_p, I) \leq \kappa$, and

$$\psi_c = \tfrac{1}{3}\arcsin\tfrac{1}{4\sqrt{2}},$$

$$\kappa_c = 2\sqrt{2}\sin\tfrac{\psi_c}{2} \in [0.0837613, 0.0837617],$$

by $1.4142135 < \sqrt{2} < 1.4142136$, the arcsine series, whose partial sums are lower bounds and whose terms after y^5 add at most $y^7/(6(1-y^2))$, which give $0.05923686 \leq \psi_c \leq 0.0592371$, and $x - \frac{x^3}{6} \leq \sin x \leq x - \frac{x^3}{6} + \frac{x^5}{120}$; in particular $8/\kappa_c^2 < 1140.26$. Then $p(\kappa) < \psi_c$ exactly when $\kappa < \kappa_c$, and $\sqrt{2}\,p(\kappa) \leq 1.00015\,\kappa$ for $\kappa \leq \kappa_c$, since $\sqrt{2}\,p(\kappa)/\kappa = \frac{p/2}{\sin(p/2)}$ increases with κ.

Lemma 469 (the ladder)**.** *Let U be in the tree gauge, $v = P_{2y}$ the straight path of the pair (y,μ), and $x \in B(y)$ with $n_x \leq 3$ transverse coordinates of the offset equal to one. Then $v^{-1}P_x$ is a product of $2n_x$ conjugates of plaquette variables or their inverses, all with their vertices in $B(y)\cup B(y+e_\mu)$. If these plaquettes have $\delta(U_p, I) \leq \kappa$, then $\angle(v^{-1}P_x) \leq 2n_x p(\kappa)$, and every link e leaving $B(y)$ in the direction μ has $\angle(v^{-1}U_e) \leq 9p(\kappa)$.*

Proof. The tree path from $2y$ to x inside $B(y)$ raises the coordinates of the offset in increasing order of direction, so a step raising the coordinate α at z uses the tree link (z,α), and $(z+2e_\mu,\alpha)$, with the same offset in $B(y+e_\mu)$, is a tree link as well. A step with $\alpha = \mu$ changes nothing, since then (z,μ) and $(z+2e_\mu,\mu)$ are tree links and $P_z = U_{z+e_\mu,\mu} = P_{z+e_\mu}$. For $\alpha \neq \mu$ let p_1, p_2 be the plaquettes at z and $z+e_\mu$ in the plane $\{\mu,\alpha\}$, $g = U_{z+e_\mu,\alpha}$ and $z' = z + e_\alpha$. With the two tree links equal to I, $U_{p_1} = U_{z,\mu} g U_{z',\mu}^{-1}$ and $U_{p_2} = U_{z+e_\mu,\mu} U_{z'+e_\mu,\mu}^{-1} g^{-1}$, hence

$$P_z = U_{p_1}\bigl(U_{z',\mu} g^{-1} U_{p_2} g U_{z',\mu}^{-1}\bigr) P_{z'},$$

and each of the n_x transverse steps multiplies by two conjugated plaquettes. The angle is subadditive and conjugation invariant. For a leaving link $e=(x,\mu)$, $P_{x-e_\mu}=U_aU_e$ with U_a a link inside $B(y)$, whose angle is at most $3p$ by the chains of the proof of Lemma 360(i); so $\angle(v^{-1}U_e)\le\angle U_a+\angle(v^{-1}P_{x-e_\mu})\le 9p$. □

Proposition 470 (Lemma 360 for SU(2) to the sharp radius)**.** *Let $G=\mathrm{SU}(2)$.*

(i) *Parts (i)–(iii) of Lemma 360 hold for every $\kappa\ge 0$, and the constant 21 of (i) can be replaced by 9.*

(ii) *Part (iv) holds for every $\kappa<\kappa_c$, with $28.04\,\kappa$ in place of $260\,\kappa$.*

(iii) *For every $\kappa>\kappa_c$ there is a field with $\delta(U_p,I)\le\kappa$ for every plaquette at which the formula of (iv) is false.*

(iv) *For $\kappa<\kappa_c$, Lemma 409 holds with $\delta(c_P,I)\le 16.003\,\kappa$, and every non-path active link e of a pair at y has $\delta(U_e,R_e(c))\le d(U_e,R_e(c))\le 12.002\,\kappa$.*

Proof. (i) The proofs of (i)–(iii) use only the triangle inequality for δ, bi-invariance and $\|\mathrm{Ad}(g)X-X\|\le 2\delta(g,I)\|X\|$, none of which needs κ small; Lemma 469 gives 9κ for the leaving links.

(ii) By Lemma 469, $\delta(P_x,v)\le 2\sqrt{2}\sin(3p)$, which is below $\frac12$ exactly when $p<\psi_c$. Then the cut-off equals one near U and $V^\sharp=v\exp M$ with $M=\frac{1}{16}\sum_x\log w_x$, $w_x=v^{-1}P_x$. Since $\|\log w_x\|\le 2\sqrt{2}n_xp$ and $\sum_x n_x=24$, $\|M\|\le 3\sqrt{2}p$, so $\|\mathrm{ad}_{\log w_x}\|\le 12p$ and $\|\mathrm{ad}_M\|\le 6p$, the spectrum of ad_X being $\{0,\pm i\sqrt{2}\|X\|\}$. The four differences of the proof of Lemma 360(iv) are then bounded with $\|\mathrm{Ad}(g)-1\|=2\sin\angle g$ for $\angle g\le\frac{\pi}{2}$ in place of $2\delta(g,I)$, and with $\sup_{|z|\le a}|z/(1-e^{-z})-1|\le\hat\varphi(a)=\frac{a}{2}+0.09a^2$ for $a\le 1$ and $\sup_{|z|\le a}|(1-e^{-z})/z-1|\le\hat\psi(a)=\frac{a}{2}e^a$. The frame changes by e^{-M} and by w_x^{-1} cost at most $6p$ and $12p$ times $\|\alpha\|\le N_{y,\mu}$; the factor $(1-e^{-\mathrm{ad}_M})/\mathrm{ad}_M$ and the derivatives of the logarithms cost $\hat\psi(6p)(1+\hat\varphi(12p))+\hat\varphi(12p)$ times $m_\beta+\|\alpha\|\le(1+\frac{\sqrt3}{4})N_{y,\mu}$; and the terms ε_x, of norm at most $2\sin(9p)\|\xi_{x,\mu}\|$ for x_μ even and $2\sin(3p)\|\xi_{x,\mu}\|$ for x_μ odd, add $\frac{1}{16}\sqrt{8(18p)^2+8(6p)^2}N_{y,\mu}$ by Cauchy–Schwarz. The sum is $pG(p)$ with G increasing and p/κ increasing, so its supremum over $\kappa\le\kappa_c$, divided by κ, is at most $G(\psi_c)\psi_c/\kappa_c$. Since $\arcsin y\le y+y^3/(6(1-y^2))$ and $1/(4\sqrt{2})\le 0.1767767$, $\psi_c\le 0.059243$, and $\sin x\ge x-x^3/6$ gives $\kappa_c\ge\sqrt{2}\,\psi_c(1-\psi_c^2/24)\ge 1.414006\,\psi_c$. At $p=0.059243$ the Taylor series of the exponential with its tail bounded geometrically gives $e^{6p}\le 1.42686$, so $\hat\psi(6p)\le 0.253595$ and $\hat\varphi(12p)\le 0.400945$, so that $\hat\psi(6p)(1+\hat\varphi(12p))+\hat\varphi(12p)\le 0.756218$. With $1+\frac{\sqrt3}{4}\le 1.4330128$ and $\sqrt{2880}\le 53.66564$,

$$G(\psi_c)\le 18+\frac{53.66564}{16}+\frac{1.4330128\cdot 0.756218}{0.059243}\le 39.6461,$$

and the supremum is at most $39.6461/1.414006<28.04$.

(iii) Let $L=2^m$, $\varphi=2\pi n/L^2$ and $T=\frac{1}{\sqrt2}\,\mathrm{diag}(i,-i)$, and multiply the three torus-valued fields of Proposition 468 that have plaquette angle φ in the planes $\{\mu,\alpha\}$, $\alpha\ne\mu$, with a common orientation. Every plaquette of these planes has $\delta(U_p,I)=2\sqrt{2}\sin(\varphi/2)$ and every other plaquette is I. In the tree gauge the ladder identity reads $P_{z+e_\alpha}=U_{p_1}^{-1}U_{p_2}^{-1}P_z$, so for the offset $(1,1,1)$ the angle of $v^{-1}P_x$ is 6φ and $\delta(P_x,v)=2\sqrt{2}\sin 3\varphi$. If $\kappa>\kappa_c$, choose L and n with $\psi_c<\varphi\le p(\kappa)$: every plaquette has $\delta(U_p,I)\le\kappa$, the cut-off $\phi(\delta(P_x,v))=2-2\delta(P_x,v)$ is below one, and the logarithms $\log w_x$ are non-negative multiples of one element with mean of angle $3\varphi\ne 0$, so $V^\sharp\ne v\exp M$ and (iv) fails. For example, for $L=32$, $n=10$ one has $\kappa=0.0867614\ldots$, $\delta(P_x,v)=0.517715\ldots$ and cut-off $0.964570\ldots$; the argument does not use these values.

(iv) By (ii), $c_k=\ell_k e^{M_k}$ with $\angle e^{M_k}\le 3p$. A link leaving $B(y)$ has $\angle(c_k^{-1}U_e)\le 9p+3p$, and a link inside a block has $\angle U_e\le 3p$; so $\delta(U_e,R_e(c))\le d(U_e,R_e(c))\le 12\sqrt{2}\,p\le 12.002\,\kappa$. The holonomy of the square of side two at $2y$ is a product of four conjugated plaquettes, of angle at most $4p$, and replacing its four sides ℓ_k by c_k adds at most $4\cdot 3p$; so $\angle c_P\le 16p$ and $\delta(c_P,I)\le 2\sqrt{2}\sin 8p\le 16\sqrt{2}\,p\le 16.003\,\kappa$. □

Theorem 471 (the island count at radius 10^{-4})**.** *Let $G=\mathrm{SU}(2)$.*

(i) *Theorem 465 holds with the hypothesis $\rho\le 10^{-8}$ replaced by $\rho<\kappa_c$, and Lemma 466 and Theorem 467 hold with it replaced by $\rho\le 1.023\cdot 10^{-4}$.*

(ii) *At $\rho=10^{-4}$,*

$$\beta^*(10^{-4},1)\in[3.408,3.410]\cdot 10^{13},$$
$$\beta^*(10^{-4},5)\in[5.382,5.384]\cdot 10^{13},$$

and for $s=5$

$$\beta_1^*\in[2.892,2.894]\cdot 10^{15},\quad \beta_2^*\in[4.628,4.630]\cdot 10^{16},$$
$$\beta_3^*\in[1.185,1.186]\cdot 10^{18},$$

$16^{-j}\beta_j^*\le 2.90\cdot 10^{14}$ for $1\le j\le 20$, and for $\beta_1^*\le\beta_W<\beta_{20}^*$ the conclusions of Theorem 467 hold at every scale $j\le j_{\max}$ with $2^{j_{\max}}\ge\frac12(\beta_W/2.90\cdot 10^{14})^{1/4}$.

(iii) *For every $\beta_W\in[10^{22},10^{42}]$, $k_*-j_{\max}\in\{2,3\}$: the islands of the composed average are rare up to the scale $a_{k_*}/8$.*

Proof. (i) The witness paragraph of the proof of Theorem 465 uses the cap 10^{-8} only through Lemma 360(i) and the proof of Lemma 409. Let κ be the largest defect of the witness region. If $\kappa<\kappa_c$, Proposition 470(iv) gives $d(U_e,R_e(c))\le 12.002\kappa<43\kappa$ and $\delta(c_P,I)\le 16.003\kappa<92\kappa$, so a bad label forces $43\kappa>r\ge 43\rho$ or $92\kappa>\eta\ge 92\rho$; otherwise $\kappa\ge\kappa_c>\rho$. In both cases some plaquette has defect more than ρ, and the threshold and island paragraphs do not involve the cap. In Lemma 466 the cap enters only through the hypothesis $\eta_j\le\rho_G/1000$ of Theorem 388(i), (ii), with $\rho_G=1$, and in Theorem 467(i) through the hypothesis $4^j\eta_0\le 10^{-4}$ of Theorem 388(iv), which reads $\rho\le 1.023\cdot 10^{-4}$; then $\eta_i\le\rho\le 10^{-3}$ for $i\le j$. (ii), (iii) The proof of Theorem 467 applies at $\rho=10^{-4}$, with the same functions g_j and the same values

of $|\log q_j^*|$; for $s=1$, $|\log q^*| = 1296(\log 2 + 2\log 2400) + 1 + 4\log 11 + 4\log 6$ lies within 10^{-3} of 21090.194. At the ends of the five stated intervals g is negative and positive, with margins above 5. At $b = 2.90\cdot 10^{14}$ the three terms of $g_j(16^j b)$ are at least 169779.0, at most 229.3 and, for $j \le 20$, at most 169532.8, which gives $16^{-j}\beta_j^* \le 2.90\cdot 10^{14}$ for $1 \le j \le 20$ and (ii). At $b = 2.92\cdot 10^{14}$ they are at least 170949.9, at most 229.3 and, for $j \le 23$, at most 169591.1, and at $b = 2.88\cdot 10^{14}$ the first is at most $168608.2 < |\log q_3^*|$; so $2.88\cdot 10^{14} \le 16^{-j}\beta_j^* \le 2.92\cdot 10^{14}$ for $3 \le j \le 23$. Since $16^6 \cdot 2.92\cdot 10^{14} < 10^{22}$ and $16^{23}\cdot 2.88\cdot 10^{14} > 10^{42}$, $6 \le j_{\max} \le 22$ for $\beta_W \in [10^{22}, 10^{42}]$, and the floor argument of that proof gives

$$\left\lfloor \log_{16} \frac{2.88\cdot 10^{14}}{D} \right\rfloor \le k_* - j_{\max} \le \left\lfloor \log_{16} \frac{2.92\cdot 10^{14}}{D} \right\rfloor + 1;$$

with $16^2 D \le 2.59\cdot 10^{14}$ and $16^3 D \ge 2.29\cdot 10^{15}$ this is (iii). □

The two or three scales that remain are not a matter of radius. With $\beta_j^* \approx 16|\log q_j^*|(1.033\cdot 4^j)^2/\rho^2$ and $4^{k_*} \approx 10^{-4}/(1.01\rho(\beta))$,

$$4^{k_* - j_{\max}} \approx \frac{4.09\cdot 10^{-4}}{\rho}\Big(\frac{|\log q_j^*|}{96(\log\beta + C_0)}\Big)^{1/2},$$

which is about 20 at $\rho = 10^{-4}$, so the loss is the entropy exponent $|\log q_j^*| \approx 2H_j \log\Delta$ of Proposition 451. At this radius the labels have $r = 43\rho \le 4.4\cdot 10^{-3}$, beyond the hypothesis $r \le 10^{-4}$ of Lemma 411, and Proposition 416 supplies the good windows to radius $\frac{1}{32}$. The chart of Theorem 363 is a different matter: its proof bounds the second fundamental form of the fibre by $\|\nabla Q\|_\infty \|\dot w\|_1$ with a constant c_2 for the second derivative of the reconstruction, and no such constant is below the flat value of the next proposition.

Proposition 472 (the flat second derivative)**.** *At the flat background $U \equiv I$,*

$$D^2 c_k(0)[Y,Y] = \frac{1}{16}\sum_{x\in B(y)} [Y_{1x}, Y_{2x}],$$

where Y_{1x}, Y_{2x} are the variables of the two links of P_x, and $\sup\{\|D^2 c_k(0)[Y,Y]\| : N(Y) = 1\} = \frac{1}{16}$.

Proof. Expand to second order, by the Baker–Campbell–Hausdorff formula,

$$c = \log\big(e^{X_v} e^{M}\big), \qquad M = \frac{1}{16}\sum_x \log\big(e^{-X_v} e^{X_{1x}} e^{X_{2x}}\big) :$$

the terms $[X_v, X_{1x} + X_{2x}]$ from the two logarithms cancel, and $\frac{1}{32}\sum_x [X_{1x}, X_{2x}]$ remains, whose second derivative is the display. For $\mu = 0$ put $d = 7^{-1/2}$, $l = 14^{-1/2}$ and, for each of the seven nonzero transverse offsets, $Y = \frac{d}{2}T_1$ on the link of $B(y)$, lT_2 on the leaving link and $-\frac{d}{2}T_1$ on the link of $B(y+e_0)$, with $T_a = i\sigma_a/\sqrt{2}$ and zero elsewhere. Then $N(Y) = 1$ and, by $\|[T_1,T_2]\| = \sqrt{2}$, $\|D^2 c_k(0)[Y,Y]\| = \frac{7\sqrt{2}dl}{16} = \frac{1}{16}$. Conversely $\|[X,Y]\| \le \sqrt{2}\|X\|\|Y\|$ and $\sqrt{2}\|D\|\|L\| \le \frac{1}{2}\|D\|^2 + \|L\|^2$, with D the difference of the two inside variables and L the leaving one, give $\frac{1}{16}$ as the maximum. □

In the proof of Theorem 363(ii) the constant c_3 enters the term $9000 c_3 \kappa'$, $\kappa' = \kappa + 6s$, only as a bound for the second derivative of c, so by the proposition no admissible constant there is below $\frac{1}{16}$, and with $\frac{1}{16}$ that term alone exceeds $\frac{1}{195}$ once $\kappa' > 9.12\cdot 10^{-6}$. A chart at radius 10^{-4} needs a different estimate of the second fundamental form; no field is known at which the chart fails.

L. The ceiling of the chessboard estimate

The threshold of Theorem 465 is of order 10^{13} at the witness radius 10^{-4} of Theorem 471, and of order 10^{37} under the calmness of Corollary 486. This subsection measures what can be gained in them and what cannot. The count of connected sets of blocks can be sharpened, which lowers the threshold at one scale by a factor 1.7 at $s=1$ and 1.8 at $s=5$, and the union bound over connected sets cannot lower it by more than a further factor 1.39 at $s=1$ and 1.22 at $s=5$. The scale is a different matter. Every rarity bound that rests on the chessboard estimate in the form of Theorem 383 or Proposition 403, with fine or coarse witnesses, is below one only at scales whose blocks are smaller than the cube of the condition (D) by a fixed factor: at every coupling with the normalisation of Lemma 382, and at every coupling $\beta_W \ge 10^3$ with the exact normalisation. The witness radius at scale j is at most 4^{-j-1} times the label radius, by the constant curvature of Proposition 468, and the normalisation charges $\frac{9}{2}\log\beta_W$ per fine site at least.

Lemma 473 (connected sets of blocks)**.** *Let Γ be a graph in which every vertex has at most Δ neighbours, and let v be a vertex. The number of connected sets of n vertices that contain v is at most $\binom{\Delta n}{n-1} \le e^n \Delta^{n-1}$. In the graph of Theorem 465, on the torus of side $L/2$ in blocks, in which two blocks are joined at ℓ^∞ distance at most $R = 2s+1$, the number is at least λ_0^{n-1} with $\lambda_0 = R(2R+1)^3 = (2s+1)(4s+3)^3$ whenever $(n-1)R < L/2$.*

Proof. Order the neighbours of every vertex. For a connected set $S \ni v$ of n vertices, list S by a breadth-first search inside S from v: the list starts with v, and its vertices are processed in their order; processing w appends, in the order of the neighbours of w, those neighbours that lie in S and are not yet listed, and records a word of Δ letters in $\{0,1\}$ whose i-th letter is 1 exactly when the i-th neighbour of w is appended. Since S is connected, every vertex of S is listed, each vertex other than v is appended once, and the n words form a word of length Δn with exactly $n-1$ letters 1. Replaying the search recovers S from this word, so the number of sets is at most $\binom{\Delta n}{n-1}$. For $n \ge 2$, $k! \ge (k/e)^k$ and $(1+1/k)^k \le e$ with $k = n-1$ give

$$\binom{\Delta n}{n-1} \le \frac{(\Delta n)^{n-1}}{(n-1)!} \le \Big(\frac{e\Delta n}{n-1}\Big)^{n-1} \le e\,(e\Delta)^{n-1},$$

and for $n = 1$ the count is 1. For the lower bound let $y_0 = v$ and $y_k = y_{k-1} + d_k$ with $d_k \in \{1,\dots,R\}\times\{-R,\dots,R\}^3$, for $1 \le k < n$. Consecutive blocks are at distance at most R, so the set $\{y_0,\dots,y_{n-1}\}$ is connected. The first coordinate of $y_k - y_0$ lies in $[k, kR]$, and $(n-1)R < L/2$, so the n blocks are

distinct and the residues of these first coordinates order them; the set therefore determines the sequence (d_k), which takes $(R(2R+1)^3)^{n-1}$ values. □

Proposition 474 (the island threshold with the tree count)**.** *In the setting of Theorem 465 put*

$$\log q_T^* = -H\big(1+\log 2+\log\Delta\big)-\log(eMH),$$

and let $\beta_T^* = \beta_T^*(\rho,s)$ be the largest solution of $f(\beta)=\log q_T^*$; it exists and exceeds $96/\rho^2$. For every $\beta_W \ge \beta_T^*$:

(i) *every block is bad with probability at most $A = eMH\,q_\rho(\beta_W) \le (2e\Delta)^{-H}$;*

(ii) *for every block y and every $n \ge 1$,*

$$\mu_L\{y \text{ lies in an island of at least } n \text{ blocks}\} \le 2^{-n}\Delta^{-1},$$

and the expected number of islands of at least n blocks on a torus of N_b blocks is at most $N_b\,2^{-n}\Delta^{-1}/n$;

(iii) *if an estimate charges every bad block a factor $\Gamma \ge 1$, the sum $\sum_{n\ge1} e^n\Delta^{n-1}(\Gamma A^{1/H})^n$ is at most Δ^{-1} as soon as β_W is at least the largest solution of $f(\beta)=\log q_T^* - H\log\Gamma$.*

Conversely, for $(n-1)(2s+1)<L/2$ the union bound $\sum_S A^{n/H}$ over the connected sets S of n blocks that contain y is at least $\lambda_0^{-1}(\lambda_0 A^{1/H})^n$, with λ_0 as in Lemma 473, so it decreases geometrically in n only if $\lambda_0 A^{1/H}<1$, which requires

$$\beta_W > \frac{16}{\rho^2}\big(H\log\lambda_0+\log(eMH)\big).$$

At the radius of Theorem 471,

$$\beta_T^*(10^{-4},1)\in[2.001,2.003]\cdot10^{13},$$
$$\beta_T^*(10^{-4},5)\in[2.988,2.990]\cdot10^{13},$$

against $\beta^*(10^{-4},1)\in[3.408,3.410]\cdot10^{13}$ and $\beta^*(10^{-4},5)\in[5.382,5.384]\cdot10^{13}$ there, and the geometric decrease requires $\beta_W > 1.441\cdot10^{13}$ for $s=1$ and $\beta_W > 2.450\cdot10^{13}$ for $s=5$.

Proof. The witness and threshold paragraphs of the proof of Theorem 465 apply with q_T^* in place of q^*, since $\log q_T^* < 0 < f(96/\rho^2)$; they give $A \le eMHq_T^* = (2e\Delta)^{-H}$ and $\mu_L(\text{all blocks of } S \text{ bad}) \le A^{|S|/H}$. By Lemma 473 the island of y has at least n blocks with probability at most

$$e^n\Delta^{n-1}A^{n/H} = \Delta^{-1}\big(e\Delta A^{1/H}\big)^n \le \Delta^{-1}2^{-n},$$

and the count of islands is that of Theorem 465. In (iii), $\Gamma^H A \le (2e\Delta)^{-H}$, so the sum is at most $\Delta^{-1}\sum_n 2^{-n}$. For the converse, Lemma 473 gives at least λ_0^{n-1} sets, each charged $A^{n/H}$. If $\lambda_0 A^{1/H}<1$ then $A<1$, so $q_\rho(\beta_W)<1$, $\log A = \log(eMH)+f(\beta_W)$, and $\log A < -H\log\lambda_0$ reads

$$\frac{\beta_W\rho^2}{16} > H\log\lambda_0+\log(eMH)+6(\log\beta_W+C_0),$$

with $\log\beta_W + C_0 > 0$.

For the numbers, $|\log q_T^*| = H(1+\log2+\log\Delta)+1+4\log11+4\log6$, with $\Delta=2400$ for $s=1$ and $\Delta=23^4-1$ for $s=5$, $\log(23^4-1)=4\log23-\varepsilon$ with $0<\varepsilon<4\cdot10^{-6}$, and $\lambda_0 = 3\cdot7^3 = 1029$ for $s=1$ and $\lambda_0 = 11\cdot23^3 = 133837$ for $s=5$. The logarithms of the proof of Theorem 465, with $\log23 = 3\log2+\log3-\log\frac{24}{23}$ and $\log7 = 3\log2-2\operatorname{artanh}\frac{1}{15}$, give $|\log q_T^*|$ within 10^{-3} of 12299.136 and 18466.475, and $H\log\lambda_0+\log(eMH)$ within 10^{-3} of 9007.259 and 15316.232, for $s=1$ and $s=5$; the last two give the two lower bounds. The function $g_T(\beta) = \beta\rho^2/16 - 6(\log\beta+C_0) - |\log q_T^*|$ increases for $\beta > 96/\rho^2$ and β_T^* is its largest zero; at the two ends of each stated interval g_T is negative and positive, with margins above 5, so that $\log\beta$ to two decimals decides the signs. □

The same replacement applies at every scale: in Theorem 467 the exponent $H_j(\log2+2\log\Delta)$ of q_j^* becomes $H_j(1+\log2+\log\Delta)$, with the conclusions of part (ii) of the proposition. The gain is a constant factor in the coupling and therefore a quarter of its logarithm in the scale. The next results show that no constant factor of this kind reaches the scale of the cube.

Lemma 475 (the normalisation from below)**.** *For $\beta_W>0$ and $L\ge2$,*

$$Z_L \le \Big(\big(\tfrac{\pi}{2}\big)^{5/2}\beta_W^{-3/2}\Big)^{3L^3(L-1)},$$

so that

$$\log\frac{1}{Z_L} \ge \frac{9}{2}\Big(1-\frac{1}{L}\Big)L^4\big(\log\beta_W - c_I\big), \qquad c_I = \tfrac{5}{3}\log\tfrac{\pi}{2}.$$

In particular $c_I \in [0.75263, 0.75264]$.

Proof. Let F be the set of links $(x,0)$ with $x_0 \ne L-1$; each line of direction 0 meets F in a path, so F is a forest. Put $g_x = U_{(0,\vec x),0}\cdots U_{(x_0-1,\vec x),0}$ for $x=(x_0,\vec x)$ and $U'_{x,\mu} = g_x^{-1}U_{x,\mu}g_{x+e_\mu}$. Then $U'=I$ on F, and given the links of F the links U'_e, $e\notin F$, are independent and Haar distributed, since each is a fixed left and right translate of its own U_e. The weight $e^{-\beta_W S}$ is gauge invariant, so $Z_L = \int e^{-\beta_W S(U')}\,dU$ is the integral over the links off F with the links of F equal to I. For $\mu\in\{1,2,3\}$ and $x_0\le L-2$ the plaquette at x in the plane $\{0,\mu\}$ has holonomy $U'_{x+e_0,\mu}U'^{-1}_{x,\mu}$, its two links of direction 0 lying in F. Drop every other plaquette from S, which only increases the integrand. For fixed $\vec x$ and μ the links $U'_{(t,\vec x),\mu}$, $0\le t\le L-1$, enter only the $L-1$ factors $\exp(-\beta_W s(U'_{(t+1,\vec x),\mu}U'^{-1}_{(t,\vec x),\mu}))$; integrating over $t=0,1,\dots,L-2$ in turn, each integral equals

$$I(\beta_W) = \frac{2}{\pi}\int_0^\pi e^{-\beta_W(1-\cos t)}\sin^2 t\,dt,$$

by the invariance of Haar measure. So $Z_L \le I(\beta_W)^{3L^3(L-1)}$. On $[0,\pi]$, $1-\cos t = 2\sin^2\frac{t}{2} \ge \frac{2t^2}{\pi^2}$ and $\sin t \le t$, so

$$I(\beta_W) \le \frac{2}{\pi}\int_0^\infty t^2 e^{-2\beta_W t^2/\pi^2}dt = \frac{2}{\pi}\cdot\frac{\sqrt\pi}{4}\Big(\frac{\pi^2}{2\beta_W}\Big)^{3/2} = \Big(\frac{\pi}{2}\Big)^{5/2}\beta_W^{-3/2}.$$

Finally $\log\frac{\pi}{2} = \log\frac{3}{2} + \log\frac{\pi}{3}$, and the alternating series of $\log(1+x)$ at $x = \frac{\pi}{3} - 1 \in [0.0471975, 0.0471976]$ gives $\log\frac{\pi}{2} \in [0.4515827, 0.4515828]$. □

Together with Lemma 382, the lemma places the normalisation of the chessboard estimate between $\frac{9}{2}\log\beta_W$ and $6\log\beta_W$ per fine site, up to additive constants.[42] The constant curvature field bounds the witness radius. Put

$$\eta_c = 2\sqrt{2}\sin\frac{1}{24\sqrt{2}} \in [0.0833212, 0.0833213],$$

and for $0 < \eta < \eta_c$ let $\tilde\eta = 2\sqrt{2}\arcsin(\eta/2\sqrt{2})$, so that $\eta < \tilde\eta < \frac{1}{12}$ and, by $\arcsin y \le y + y^3/(6(1-y^2))$, $\tilde\eta \le 1.00015\,\eta$. Call a block y *η-bad at scale j* when some plaquette of $U^{(j+1)}$ with corner y has $\delta(U_P^{(j+1)}, I) > \eta$: this is the coarse-curvature part of the label of Sec. XIII J at scale j, and a block bad for that label is bad.

Theorem 476 (the ceiling of the chessboard estimate)**.** *Let $G = \mathrm{SU}(2)$, $\beta_W \ge 1$, $0 < \eta < \eta_c$, $j \ge 0$ and $\theta_j = 4^{-j-1}\tilde\eta$.*

(i) Witnesses. *Let $0 \le i \le j+1$ and $\rho_i > 0$, and suppose that on every torus Λ_L with $L = 2^m \ge L_0$, every field at which some block is η-bad at scale j has a plaquette P of $U^{(i)}$ with $\delta(U_P^{(i)}, I) > \rho_i$. Then $2\sqrt{2}\arcsin(\rho_i/2\sqrt{2}) \le 4^i\theta_j$; in particular $\rho_i \le 4^i\theta_j$, and a fine witness has $\rho_0 \le \theta_j$.*

(ii) Product bounds. *(a) If the hypothesis of (i) holds with $i = 0$, the weight $q_{\rho_0}(\beta_W)$ of Theorem 383 is below one only if*

$$16^{j+1} < \frac{\beta_W\tilde\eta^2}{96(\log\beta_W + C_0)}.$$

(b) Let $b = 2^k$, let E be an event determined by the links of the closed cube $[0,b]^4$ on which the plaquettes of that cube carry action at least A, let E_z be its reflections as in Proposition 403, and suppose that on every torus Λ_L with $L = 2^m \ge L_0$ every field at which a given block is η-bad at scale j lies in some E_z. Then the bound of Proposition 403 is below one, for a nonempty set Z, only if

$$16^{j+1} < \Big(1 + \frac{1}{b}\Big)^2 \frac{\beta_W\tilde\eta^2}{96(\log\beta_W + C_0)}.$$

(c) If in (a) and (b) the bound of Lemma 382 is replaced by $\log(1/Z_L)$ itself, the same holds with $96(\log\beta_W + C_0)$ replaced by $72(1-1/L)(\log\beta_W - c_I)$.

(iii) The cube. *Suppose that $D(n,\alpha)$ holds at β_W, with the chordal or the Hilbert–Schmidt distance, and that a bound of (a) or (b) is below one at scale j, or a bound of (c) with $\beta_W \ge 10^3$ and $L \ge 16$. Then*

$$\Big(\frac{n}{2^{j+1}}\Big)^4 > \frac{1.9268\,(1-\alpha)}{\tilde\eta^2},$$

and for (a) and (b) the constant 1.9268 can be replaced by 4.2069. Since $\tilde\eta < \frac{1}{12}$, $n > 4.08\,(1-\alpha)^{1/4}\,2^{j+1}$ in every case.

Proof. (i) Since $\tilde\eta < \frac{1}{12}$, there are $L = 2^m \ge L_0$ and integers n' for which $\theta = 2\sqrt{2}\pi n'/L^2$ satisfies $\theta_j < \theta$ and $4^{j+1}\theta \le \frac{1}{12}$, with θ as close to θ_j as desired. Let U be the field of Proposition 468 with this θ. By that proposition and its proof, for $i \le j+1$, $\Theta_P^{(i)} = 4^i\theta T$ on the plane $\{0,1\}$ and $\Theta_P^{(i)} = 0$ on the other planes, so every plaquette of $U^{(i)}$ has $\delta(U_P^{(i)}, I) \le 2\sqrt{2}\sin(4^i\theta/2\sqrt{2})$, while the plaquettes of $U^{(j+1)}$ in the plane $\{0,1\}$ have

$$\delta(U_P^{(j+1)}, I) = 2\sqrt{2}\sin\frac{4^{j+1}\theta}{2\sqrt{2}} > 2\sqrt{2}\sin\frac{\tilde\eta}{2\sqrt{2}} = \eta,$$

the sine increasing on $[0, \frac{1}{24\sqrt{2}}]$. So every block is η-bad at scale j, the hypothesis gives $\rho_i < 2\sqrt{2}\sin(4^i\theta/2\sqrt{2})$, and letting θ decrease to θ_j gives $\rho_i \le 2\sqrt{2}\sin(4^i\theta_j/2\sqrt{2})$, which is the first claim; $\sin x \le x$ gives the second.

(ii) (a) By its definition $q_{\rho_0}(\beta_W) < 1$ exactly when $\beta_W\rho_0^2/16 > 6(\log\beta_W + C_0)$. By (i), $\rho_0^2 \le \theta_j^2 = 16^{-j-1}\tilde\eta^2$, which gives the claim.

(b) Take U as in the proof of (i). By hypothesis $U \in E_z$ for some z. The reflections that carry $[0,b]^4$ to $z + [0,b]^4$ preserve $\mathrm{Tr}\,U_p$, so the reflected field lies in E and the plaquettes of $[0,b]^4$ carry the same action there as those of $z + [0,b]^4$ carry at U. The closed cube contains $b^2(b+1)^2$ plaquettes of the plane $\{0,1\}$, each with $s_p = \frac{1}{4}\delta(U_p, I)^2 = 2\sin^2(\theta/2\sqrt{2}) \le \theta^2/4$, and the other plaquettes have $s_p = 0$. Hence $A \le b^2(b+1)^2\theta^2/4$ for θ arbitrarily close to θ_j, so $A \le b^2(b+1)^2\theta_j^2/4$. The bound of Proposition 403 is $\exp(|Z|(6b^4(\log\beta_W + C_0) - \beta_W A/4))$, below one only if $\beta_W A/4 > 6b^4(\log\beta_W + C_0)$, that is, only if $\beta_W(1+1/b)^2\theta_j^2/16 > 6(\log\beta_W + C_0)$.

(c) Before Lemma 382 is used, the proofs of Theorem 383 and Proposition 403 bound the probabilities by $(e^{-\beta_W\rho_0^2L^4/16}/Z_L)^{|A|/L^4}$ and $(e^{-\beta_W(L/b)^4A/4}/Z_L)^{|Z|(b/L)^4}$. By Lemma 475 these are at least

$$\exp\Big(|A|\Big(\tfrac{9}{2}(1-\tfrac{1}{L})(\log\beta_W - c_I) - \tfrac{\beta_W\rho_0^2}{16}\Big)\Big),$$
$$\exp\Big(|Z|\Big(\tfrac{9}{2}(1-\tfrac{1}{L})b^4(\log\beta_W - c_I) - \tfrac{\beta_W A}{4}\Big)\Big),$$

and the arguments of (a) and (b) apply with $\frac{9}{2}(1-\frac{1}{L})(\log\beta_W - c_I)$ in place of $6(\log\beta_W + C_0)$.

(iii) Propositions 266 and 267, the latter with $N = 2$, give $n^4 \ge 2(1-\alpha)\beta_W/(5\ell(\beta_W))$ with $\ell(\beta_W)$ equal to $1 + \log 24 + \frac{3}{2}\log(9\beta_W)$ or to $1 + \frac{3}{2}\log(81\pi^2\beta_W) + 2\log\frac{12}{11}$. Both are at most $\frac{3}{2}x + 11.2$, $x = \log\beta_W$, since $1 + \log 24 + \frac{3}{2}\log 9 < 7.474$

[42] The chessboard estimate bounds the probability of a pattern of local events on a torus by the product of the probabilities of each event repeated by reflection over the whole torus [54]. The normalisation of each factor is a partition function of the whole torus, which is why the estimate pays for the fluctuations of every fine site.

and $1+\frac{3}{2}\log(81\pi^2)+2\log\frac{12}{11}<11.19989$. Write $c=1$ in (a) and $c=(1+1/b)^2\le 4$ in (b). In (a) and (b),

$$\Big(\frac{n}{2^{j+1}}\Big)^4 > \frac{192(1-\alpha)}{5c\,\tilde\eta^2}\cdot\frac{x+C_0}{\frac{3}{2}x+11.2},$$

and the last factor increases with x, because $11.2>\frac{3}{2}C_0$, so it is at least $C_0/11.2>0.43822$; with $c\le 4$ this gives 4.2069. In (c),

$$\Big(\frac{n}{2^{j+1}}\Big)^4 > \frac{144(1-\frac{1}{L})(1-\alpha)}{5c\,\tilde\eta^2}\cdot\frac{x-c_I}{\frac{3}{2}x+11.2},$$

the last factor increases with x, and at $x=\log 10^3=3\log 10\in[6.907755,6.907756]$ it is at least 0.28546; with $1-\frac{1}{L}\ge\frac{15}{16}$ and $c\le 4$ this gives $\frac{144\cdot 15\cdot 0.28546}{5\cdot 16\cdot 4}>1.9268$. Finally $144\cdot 1.9268>277.4$ and $277.4^{1/4}>4.08$. □

The hypotheses of (ii) are those of every rarity bound for the islands in this paper: Theorems 465, 467 and 471 use fine witnesses and Theorem 383, and a coarse energy as in Lemma 401 would be used through Proposition 403. A recursion that bounds the bad blocks of scale j through events of earlier scales, and those through Theorem 383 or Proposition 403, ends in events at some scale $i\le j+1$ that together cover the η-bad blocks, and (i) and (ii)(b) apply to them, however they are nested. The ceiling does not depend on the collar, on the count of connected sets, on the exponent $1/H$, or on the constants of the witness lemmas: part (ii) holds even when the bound need only be below one, which is far less than the threshold of Theorem 465 or of Proposition 474 asks.

Corollary 477 (the labels of the island theorems)**.** *Let $D(n,\alpha)$ hold at β_W, with $\alpha\le\frac{1}{2}$, and let a bound of Theorem 476(ii) be below one at scale j, under the conditions of part (iii).*

(i) *For the labels of Theorems 467 and 471, with $\eta=92\rho$ and $\rho\le 1.023\cdot 10^{-4}$, $n>12.14\,(1-\alpha)^{1/4}2^{j+1}>2^{j+4}$.*

(ii) *For coarse labels with $\eta\le 6.0\cdot 10^{-13}$, the size below which Theorem 526 needs the plaquettes of the background, which contains the calm labels of Corollary 486, $n>1.52\cdot 10^6(1-\alpha)^{1/4}2^{j+1}>2^{j+21}$.*

So the scale of the cube lies at least three, respectively twenty, dyadic scales above every scale that such a bound reaches.

Proof. Theorem 476(iii) applies, and $(1-\alpha)^{1/4}\ge 2^{-1/4}>0.8409$. (i) Here $\eta\le 0.0094116$, so $\tilde\eta\le 0.0094131$ and $1.9268/\tilde\eta^2>21745$; its fourth root exceeds 12.14, and $12.14\cdot 0.8409>8$. (ii) Here $\tilde\eta\le 6.0009\cdot 10^{-13}$, so $1.9268/\tilde\eta^2>5.35\cdot 10^{24}$; its fourth root exceeds $1.52\cdot 10^6$, and $1.52\cdot 10^6\cdot 0.8409>2^{20}$. □

Fig. 38 shows the dyadic scales that a chessboard bound reaches and the scale of the cube.

Remark 478 (what the ceiling leaves). The loss sits in the normalisation, not in the energy. In (ii)(b) the action that a coarse defect forces is at most $(1+1/b)^2b^4\theta_j^2/4$, which for b of the order 2^j does not decrease with j, in accordance with Lemma 401 for linear fields; the normalisation charges $\frac{9}{2}b^4(\log\beta_W-c_I)$ at least, which grows like 16^j. A bound that reaches the cube must compare the weight of the defects with Z_L without charging the fluctuation volume of the cube, as in the Gaussian theory. Theorem 404 is a bound of that kind; it gives the rarity at every scale for U(1) (Corollary 405), and for SU(2) its constant grows like 4^k on the gauge orbit of the vacuum (Proposition 406). The theorem says nothing about such bounds, nor about labels with $\eta\ge\eta_c$. At couplings of order one nothing of the kind of (ii)(a) exists at any scale: for $1\le\beta_W\le 10^3$ and every $\rho\le 1$, $q_\rho(\beta_W)=1$, since the convex function $\beta/16-6(\log\beta+C_0)$ is negative at $\beta=1$ and at $\beta=10^3$.

The fields that carry Theorem 476 are critical points of the Wilson action. Every first variation of the action vanishes at them, so a bound that removes the coarse defects detected by a first-order estimate, such as the force events of Theorem 404 and Proposition 407, and treats the remaining defects by the chessboard estimate, still meets the ceiling. On Λ_L write $S(U)=\sum_p s_p$, $s_p=1-\frac{1}{2}\operatorname{Tr}U_p$, as in Lemma 382, and for a link e and $Y\in\mathfrak{su}(2)$ let $U^{e,tY}$ be the field with U_e replaced by $U_e e^{tY}$. A field U is a *critical point* of S when $\partial_t S(U^{e,tY})|_{t=0}=0$ for every link e and every Y. A *field of directions* is a family $Y=(Y_e)$ of continuous maps from the fields on Λ_L to $\mathfrak{su}(2)$, and

$$D_Y S(U)=\partial_t S\big((U_e e^{tY_e(U)})_e\big)\big|_{t=0}.$$

The force D_rS of Theorem 404 is D_YS with the constant directions $Y_e=r_e$, and the force $D_r^F S$ of Proposition 407 is D_YS with $Y_e(U)=\operatorname{Ad}(h_y(U)^{-1})r_e$, which is continuous because $h_y(U)$ is a product of link variables.

Lemma 479 (constant curvature is critical)**.** *Every field U of Proposition 468 is a critical point of S. Consequently $D_YS(U)=0$ for every field of directions Y.*

Proof. The field takes values in the torus $\exp(\mathbb{R}T)$, every plaquette of the plane $\{0,1\}$ has $U_p=\exp(\theta T)$ with one orientation, and every other plaquette has $U_p=I$ (Proposition 468). Fix a link e of direction μ and $Y\in\mathfrak{su}(2)$, and write $Y=yT+Y_\perp$ with $y\in\mathbb{R}$ and $Y_\perp$ off-diagonal. As in the proof of Theorem 404(i), for a plaquette p through e the plaquette variable of $U^{e,tY}$ is $e^{\varepsilon_p t\operatorname{Ad}(h_p)Y}U_p$, where h_p is a product of link variables of U and $\varepsilon_p=+1$ if p traverses e forwards and -1 if backwards. Since h_p lies in the torus, $\operatorname{Ad}(h_p)Y=yT+Y'_\perp$ with $Y'_\perp$ off-diagonal, and $\operatorname{Tr}(Y'_\perp U_p)=0$ because U_p is diagonal. Hence

$$\partial_t s_p\big|_{t=0}=-\tfrac{1}{2}\varepsilon_p\, y\operatorname{Tr}(TU_p),$$

which depends on p only through ε_p and U_p. For each $\nu\ne\mu$ the link e lies in exactly two plaquettes of the plane $\{\mu,\nu\}$, on the two sides of e in the direction ν; with $U_p=U_1U_2U_3^{-1}U_4^{-1}$ one of them traverses e forwards and the other backwards, and the two have the same variable U_p, namely $\exp(\theta T)$ if $\{\mu,\nu\}=\{0,1\}$ and I otherwise. Their contributions cancel, and summing over the three planes gives $\partial_t S(U^{e,tY})|_{t=0}=0$. Finally S is a smooth function of the links, so by the chain rule $D_YS(U)=\sum_e\partial_t S(U^{e,tY_e(U)})|_{t=0}=0$. □

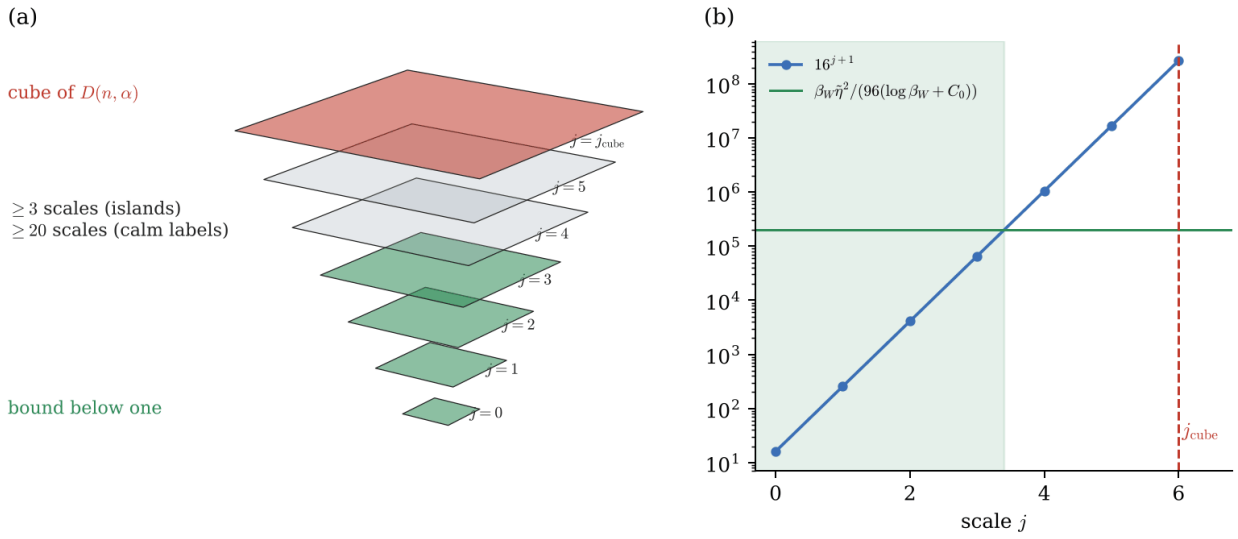


FIG. 38. The ceiling of the chessboard estimate (Theorem 476), schematic. (a) The dyadic scales as plates: a chessboard bound for the islands is below one only at the lower scales (green), and the cube of $D(n,\alpha)$ lies at least three scales higher for the labels of the island theorems and at least twenty for the calm labels (Corollary 477). (b) The condition of part (ii): the left side 16^{j+1} grows by a factor 16 per scale, the right side $\beta_W \tilde{\eta}^2/(96(\log\beta_W + C_0))$ does not depend on j, and they cross below the scale of the cube.

Theorem 480 (the ceiling at the critical points)**.** *Let $G = \mathrm{SU}(2)$, $\beta_W \geq 1$, $0 < \eta < \eta_c$ and $j \geq 0$, and for every torus Λ_L let $\mathcal{N}_L$ be an open set of fields on Λ_L that contains every critical point of S. Then Theorem 476 holds, with the same constants, when in the hypotheses of its parts* (i) *and* (ii) *the words "every field at which" are replaced by "almost every field of $\mathcal{N}_L$, for the product of the Haar measures, at which". So does Corollary 477.*

Proof. Take L, θ and the field U as in the proof of Theorem 476(i), so that $\theta_j < \theta$ and $4^{j+1}\theta \leq \frac{1}{12}$. By Lemma 479, $U \in \mathcal{N}_L$.

Continuity. For $i \leq j+1$ and every plaquette P of the lattice of spacing 2^i, the function $V \mapsto \delta(V_P^{(i)}, I)$ is continuous on the fields on Λ_L. Indeed $V^{(i)}$ is obtained by i steps, each of which puts the field into the tree gauge by the gauge transformation whose value at a site is the product of the links along the tree path from the root of its block, a continuous function of the field, and then applies the cut-off average, which is continuous by Theorem 353; the other choices of the tree gauge differ by transformations constant on the blocks, which conjugate the plaquette holonomies and leave $\delta(V_P^{(i)}, I)$ unchanged. The functions s_p are continuous as well.

A neighbourhood. At U every block is η-bad at scale j, with strict inequality, and every plaquette of $U^{(i)}$ has $\delta(U_P^{(i)}, I) \leq 2\sqrt{2}\sin(4^i\theta/2\sqrt{2})$; every plaquette of the plane $\{0,1\}$ has $s_p = 2\sin^2(\theta/2\sqrt{2}) \leq \theta^2/4$, and every other plaquette has $s_p = 0$. The torus has finitely many plaquettes at each scale, so for every $\varepsilon > 0$ there is an open neighbourhood $N_\varepsilon \subset \mathcal{N}_L$ of U on which every block is η-bad at scale j, every plaquette of $V^{(i)}$ has $\delta(V_P^{(i)}, I) < 2\sqrt{2}\sin(4^i\theta/2\sqrt{2}) + \varepsilon$ for $i \leq j+1$, and every fine plaquette has $s_p(V) < s_p(U) + \varepsilon$. A nonempty open set has positive Haar measure, so the weakened hypotheses apply to some field $V \in N_\varepsilon$.

(i) The field V has a plaquette of $V^{(i)}$ with $\delta(V_P^{(i)}, I) > \rho_i$, so $\rho_i < 2\sqrt{2}\sin(4^i\theta/2\sqrt{2}) + \varepsilon$. Letting $\varepsilon \to 0$ and then θ decrease to θ_j gives the conclusion of Theorem 476(i), and part (ii)(a) follows from it as there.

(ii)(b) The field V lies in some E_z, so its reflection lies in E, and the plaquettes of $[0,b]^4$ carry at the reflection the action that those of $z + [0,b]^4$ carry at V. The closed cube contains $b^2(b+1)^2$ plaquettes of each of the six planes, so

$$A \leq \frac{b^2(b+1)^2\theta^2}{4} + 6b^2(b+1)^2\varepsilon.$$

Letting $\varepsilon \to 0$ and θ decrease to θ_j gives $A \leq b^2(b+1)^2\theta_j^2/4$, the only property of E that the proof of Theorem 476(ii)(b) uses. Parts (ii)(c) and (iii) of that theorem, and Corollary 477, use only the inequalities of (i), (ii)(a) and (ii)(b). □

Corollary 481 (first-order estimates)**.** *Let $Y^1, \dots, Y^m$ be fields of directions on Λ_L, let $x_1, \dots, x_m > 0$, and put*

$$\mathcal{N} = \{U : |D_{Y^k} S(U)| < x_k \text{ for } k = 1, \dots, m\}.$$

(i) *$\mathcal{N}$ is open and contains every critical point of S.*

(ii) *If $0 < \eta < \eta_c$ and $2\sqrt{2}\pi < 4^{-j-1}L^2(\frac{1}{12} - \tilde{\eta})$, the set of fields in $\mathcal{N}$ at which a given block is η-bad at scale j has positive μ_L-measure. So the event that a block is η-bad is not contained, up to a μ_L-null set, in the union of the events $|D_{Y^k} S| \geq x_k$.*

(iii) *Every bound of the form*

$$\mu_L\{y \ \eta\text{-bad at scale } j\} \leq \sum_{k=1}^{m} \mu_L\big(|D_{Y^k} S| \geq x_k\big) + B,$$

in which B is a bound of Theorem 383 or Proposition 403 for events that cover almost every field of $\mathscr{N}$ at which y is η-bad, as in the hypotheses of Theorem 476, has $B<1$ only under the conditions of Theorem 476(ii)*, and then Theorem 476*(iii)* applies. This holds for the forces of Theorem 404 and Proposition 407 with any directions r.*

Proof. (i) Each $D_{Y^k}S(U)=dS(U)[(U_eY_e^k(U))_e]$ is continuous, since S is smooth and Y^k continuous, so $\mathscr{N}$ is open, and it vanishes at every critical point.

(ii) The interval $(\theta_j,4^{-j-1}/12]$ has length $4^{-j-1}(\frac{1}{12}-\tilde\eta)>2\sqrt{2}\pi/L^2$, so it contains $\theta=2\sqrt{2}\pi n/L^2$ for some integer $n\geq 1$. The proof of Theorem 480 with this θ and $\mathscr{N}_L=\mathscr{N}$ gives a nonempty open subset of $\mathscr{N}$ on which every block is η-bad at scale j. It has positive Haar measure, and μ_L has a positive density with respect to Haar measure.

(iii) This is Theorem 480 with $\mathscr{N}_L=\mathscr{N}$, which is admissible by (i). □

Remark 482 (what the critical points leave). At the fields of constant curvature the rarity of a coarse defect can only come from the size of the action, which is a second-order or global quantity there, and through the chessboard estimate that size, at most $(1+1/b)^2b^4\theta_j^2/4$ in a cube of side b, is compared with a normalisation of order $b^4\log\beta_W$. The first-order estimates themselves charge no fluctuation volume: Theorem 404 bounds the force events uniformly in the exterior with the exponent $\beta_W x^2/(2K_r)$, which contains no factor $\log\beta_W$ per site. What they lack near these fields is detection, by Corollary 481(ii). Criticality is not special to SU(2): the U(1) field with link angles φx_0 on the links of direction 1, $-\varphi Lx_1$ on the links of direction 0 with $x_0=L-1$, and 0 on the others, $\varphi=2\pi n/L^2<\pi$, has plaquette angle φ on every plaquette of the plane $\{0,1\}$ and 0 on the others, so the derivative of the U(1) action in the link angle e, $\sum_{p\ni e}\pm\sin\theta_p$, vanishes by the cancellation of the proof of Lemma 479. The force $\sum_p\sin\theta_p\,(da)_p$ of Corollary 405 therefore vanishes at it, so $\sum_p\theta_p(da)_p=0$, and by the identity of that corollary its coarse flux $(d_kV^{(k)}(\theta))_P$ is the term $2\pi\sum_p n_p(da)_p$. Theorems 476 and 480 say nothing about three kinds of bound: the chessboard estimate with the probability $\mu_L(\bigcap_z E_z)$ of the global event bounded otherwise than by its unnormalised weight $e^{-\beta_W(L/b)^4A/4}$ divided by Z_L, for instance by a map that preserves the Haar measures, acts at a finite distance and lowers the action on that event; bounds for the law of $U^{(j)}$ through an effective action, which is item (iv) of Sec. XIII V; and labels with $\eta\geq\eta_c$.

M. Two islands factorise through one kernel

Item (ii) of Sec. XIII V, in its original form, asks for the hypothesis of Lemma 452: a kernel on the good layer between two bad islands through which the weight of every pair of label patterns factors, a domain that the labels of the islands do not move, and a cross-ratio of the kernel controlled by Lemma 454. This subsection proves the first two unconditionally, for every exterior and every pair of patterns, once the layer is seven blocks wider than the collar, and reduces the third to one covariance bound for the law of the layer, which is the bound of Theorem 367(ii) in the coordinates of the labels.

The data. Let $G=\mathrm{SU}(2)$, fix a coarse field c with its reconstruction $R(c)$, the radii $r\leq 10^{-4}$ and η, and the labels of Sec. XIII I. Given c the coarse-plaquette part of every label is a constant, and the event that a block y is good is the intersection, over the non-path active links e of the pairs based at y, of the single-link events $G_e=\{d(U_e,R_e(c))\leq r\}$; these links are based in $B(y)$ and in the four blocks $B(y+e_\mu)$. Let Λ be a finite set of blocks, U^{ext} an arbitrary configuration on the active links based outside Λ, $I_a,I_b\subset\Lambda$ two disjoint sets of blocks, $s\geq 1$ the collar, and

$$w=\mathrm{dist}(I_a,I_b)-1$$

the number of block layers strictly between them, distances between blocks being ℓ^∞ distances of their coarse coordinates. Every block of $\Lambda\setminus(I_a\cup I_b)$ is good; on I_a a pattern ω_a^i, $i\in\{0,1\}$, is prescribed, with ω_a^0 the pattern in which every block is good, and likewise ω_b^j on I_b. The sector (i,j) is the event that the labels of Λ are these, and its weight is

$$Z_{ij}=\int\prod_e dU_e\,e^{-\beta_W S(U)}\prod_{k\in K_\Lambda}\delta\big(V_k^\sharp(U)-c_k\big)\,\mathbf{1}_{(i,j)}(U),$$

the product over the active links based in Λ, S the sum of the Wilson terms of the plaquettes containing one of them, K_Λ the pairs based in Λ, and the constraints understood through the coarea formula as in Proposition 412; the pairs entering Λ from outside have their path link in the exterior and their constraints are not imposed. Only ratios of the Z_{ij} are used.

The coordinates. Write every non-path active link as $U_e=R_e(c)e^{\zeta_e}$, $\zeta_e\in\mathfrak{g}$, so that on a good block the event G_e is the ball $\{\|\zeta_e\|\leq r\}$ and Haar measure is $j(\zeta_e)\,d\zeta_e$ with j smooth and positive on it. A pair all of whose non-path links lie in good blocks has, on the sector event, exactly one path link compatible with its constraint, by Lemma 411: $\ell_k=\Gamma_k$, a smooth function of its 23 non-path links, with cut-off one. Let $d_a(y)=\mathrm{dist}(y,I_a)$ and d_b likewise, and form the classes

- L_a: all active links based at $d_a\leq s$, path links included, all free;
- p_a: the path links of the pairs based at $d_a=s+1$, free;
- x: the non-path active links based at $d_a\in\{s+1,s+2\}$ and the 17 inside active links of every block at $d_a=s+3$;
- z: every other non-path active link based in Λ;
- L_b, p_b, y: the mirror classes for I_b,

and solve the path link of every pair based at $d_a\geq s+2$ and $d_b\geq s+2$. For $w\geq 2s+6$ a link at $d_a\leq s+3$ has $d_b\geq s+4$, so the classes are disjoint. Let D_x and D_y be the products of the r-balls in the coordinates ζ_e of x and of y; they are fixed by r and the geometry, since every block carrying an x- or y-link is good in all four sectors.

Theorem 483 (two islands and one kernel)**.** *Let $s \geq 1$, $r \leq 10^{-4}$ and $w \geq 2s+7$. There are finite positive measures a_i on D_x and b_j on D_y, $i,j \in \{0,1\}$, and a smooth function $K : D_x \times D_y \to (0,\infty)$ with*

$$Z_{ij} = \int_{D_x}\int_{D_y} K(x,y)\, a_i(dx)\, b_j(dy) \qquad (i,j \in \{0,1\}),$$

such that:

(i) *K does not depend on i or j; it depends on c, r, β_W, U^{ext} and on the labels of the blocks at distance at least $s+2$ from both islands only;*

(ii) *a_i depends on ω_a^i, on c, β_W and U^{ext}, and on nothing else; it carries every branch of the constraints of the pairs based at $d_a \leq s+1$, integrated, every label indicator that involves a link of L_a, and the coarea densities of the pairs at $d_a = s+2$; likewise b_j;*

(iii) *the supports of all four measures lie in D_x, D_y: the labels of the islands do not move the domain;*

(iv) *$K(x,y) = \int_{D_z} e^{-\mathscr{U}(z;x,y)} dz$, where D_z is the product of the r-balls of the z-coordinates and $\mathscr{U} = \beta_W S_K + \psi$ is smooth on a neighbourhood of $D_z \times D_x \times D_y$, S_K the sum of the Wilson terms assigned to K with the solved path links substituted, ψ the sum of $-\log j$ over the z-links and of $-\log J_k$ over the solved pairs based at distance at least $s+3$ from both islands, J_k the inverse Jacobian of Proposition 412;*

(v) *no term of $\mathscr{U}$ depends on both an x- and a y-coordinate, $\nabla_z \partial_x \mathscr{U}$ is supported on the z-links based at $d_a \leq s+4$ and $\nabla_z \partial_y \mathscr{U}$ on those at $d_b \leq s+4$, at coarse distance at least $w-2s-7$ and, for $w \geq 2s+8$, at fine distance at least $d' = 2(w-2s-8)$ between link midpoints;*

(vi) *$Z_{ij} > 0$ if and only if $a_i \neq 0$ and $b_j \neq 0$, that is, if and only if each pattern is realisable on its own side together with its collar; a_0 and b_0 are nonzero.*

Proof. Step 1: the factors. The weight of a sector is the product Haar measure times a product of factors of three kinds: the Wilson factor of a plaquette, whose four links are based in blocks at mutual distance at most one; the constraint of a pair $k = (y,\mu)$, which involves the 24 links of its paths, based in $B(y) \cup B(y+e_\mu)$; and the label indicator of a block, which for a good block is the product of the $\mathbf{1}_{G_e}$ over the non-path links of its pairs and for a bad block is one minus that product, all these links being based in $B(y)$ and $B(y+e_\mu)$.

Step 2: solving the far path links. A pair based at $d_a \geq s+2$ and $d_b \geq s+2$ has its non-path links based at distance at least $s+1$ from both islands, hence in good blocks, hence inside their balls on the sector event. Lemma 411 applies, the fibre of its constraint over its non-path links is the one point $\ell_k = \Gamma_k$, and since ℓ_k is private to k the integration of ℓ_k against the constraint replaces ℓ_k by Γ_k in every factor and multiplies the weight by J_k, as in Proposition 412; the cross derivatives between different pairs vanish, so the order is immaterial. After this step every factor is a function of the free coordinates and of U^{ext}, and a solved path link based at distance d depends on non-path links based at distances $d-1$, d, $d+1$.

Step 3: the assignment. We list what each factor depends on, distances being from I_a. A solved path link at $s+2$ depends on non-path links at $s+1, s+2, s+3$, those at $s+3$ being the second links of its paths, inside $B(y+e_\mu)$: all of them are x-coordinates. A solved path link at $d \geq s+3$ depends on x- and z-coordinates. Hence: a plaquette with a link at $d \leq s$ depends on L_a, p_a and x; one with its nearest link at $s+1$ on p_a and x; one with its nearest link at $s+2$ on x and z (its path links at $s+2$ are functions of x, its private links at $s+3$ are z, its path links at $s+3$ are functions of x and z); one with all links at $d \geq s+3$ on x and z. A pair based at $d \leq s$ has its path link in L_a and its non-path links in L_a and x; one at $s+1$ has its path link in p_a and its non-path links in L_a and x; one at $s+2$ is solved with J_k a function of x; one at $d \geq s+3$ is solved with J_k a function of x and z, of z alone for $d \geq s+5$. The bad-block indicators of I_a involve links at $d \leq 1 \leq s$, in L_a; the good single-link indicators are assigned to the class of their link. Call a factor an a-factor if it depends on a coordinate of $L_a \cup p_a$, a b-factor if on $L_b \cup p_b$, and a K-factor otherwise; factors depending on x alone go with a, on y alone with b. An a-factor depends only on L_a, p_a, x and U^{ext}, through links at $d_a \leq s+2$, which have $d_b \geq w+1-(s+2) \geq s+6$ for $w \geq 2s+7$, so it contains no coordinate of $L_b \cup p_b$ and no factor is both. The sector changes only the indicators of I_a and I_b, which are a- and b-factors, so the product Φ_K of the K-factors is the same in all four sectors.

Step 4: Tonelli. All factors are nonnegative and the Haar measures finite, so, with Φ_a^i the product of the a-factors in sector (i,j) and Φ_b^j that of the b-factors,

$$Z_{ij} = \int \Big[\int \Phi_a^i\, dL_a\, dp_a\Big]\Big[\int \Phi_K\, dz\Big]\Big[\int \Phi_b^j\, dL_b\, dp_b\Big] \times \prod_{e\in x} j\, d\zeta_e \prod_{e\in y} j\, d\zeta_e,$$

the inner integrals over L_a, p_a being the coarea integrals of the unsolved pairs, in which every branch of every constraint is integrated and none discarded, and the x- and y-integrals restricted to D_x and D_y because every x-link carries its indicator $\mathbf{1}_{G_e}$. Define $a_i(dx)$ as the first bracket times $\prod_{e\in x} j\, d\zeta_e$, $b_j(dy)$ likewise and K as the middle bracket with the j-factors of the z-links included. This is the factorisation; (i), (ii) and (iii) are read off Step 3.

Step 5: the kernel. Φ_K is a product of positive smooth functions of (z,x,y) (Wilson factors, Jacobians $J_k > 0$ since $\|DF_k\| < 0.9$, densities $j > 0$) and of the indicators of the z-balls; the solved Γ_k are smooth on a neighbourhood of the closed product of balls, the contraction of Lemma 411 still applying at radius $1.01r$. So $K = \int_{D_z} e^{-\mathscr{U}}\, dz$ with $\mathscr{U}$ smooth and finite near the compact set $D_z \times D_x \times D_y$, K is smooth and $K \geq |D_z| e^{-\sup \mathscr{U}} > 0$. This is (iv).

Step 6: supports. Two facts of the tree gauge are used. First, an inside link (a,ν), $o_\nu(a) = 0$, and a path link $(2y+e_\mu,\mu)$ lie in a common plaquette only if $a \in B(y) \cup B(y+e_\mu)$: as opposite sides they would have the same direction and bases differing by a unit vector e_ρ, $\rho \neq \mu$, forcing $o_\mu(a) = 1$; as adjacent sides the shared corner is $2y+e_\mu$ or $2y+2e_\mu$ (then a or, when $\nu = \mu$,

$a+e_\mu$ is that corner, so $a \in B(y) \cup B(y+e_\mu)$), or it is $a+e_\nu$ with $\nu \neq \mu$ equal to one of them, forcing $o_\nu(a)=1$. Second, if $d_a(y)=s+4$ and $d_a(y+e_\mu)=s+3$, then $d_a(y-e_\nu) \le s+4$ for every $\nu \neq \mu$, since some $a' \in I_a$ has $\|y+e_\mu-a'\|_\infty = s+3$, so $|y_\nu - a'_\nu| \le s+3$ for $\nu \neq \mu$. We claim that a K-term depending on an x-coordinate has all its links based at $d_a \le s+4$, one of them at $d_a \le s+3$, and its z-gradient supported at $d_a \le s+4$. A Jacobian J_k depends on x only when k is based at $s+3$, or at $s+4$ pointing inward, and then every link of k is at $d_a \le s+4$. A Wilson term depends on x only through an x-link (non-path at $s+2$, inside at $s+3$) or through a solved path link at $s+2$, at $s+3$, or inward at $s+4$. In the first two cases all links of the plaquette are at $d_a \le s+4$, and a path link of it at $s+4$ lies within one block of a link at $\le s+3$: if that link is inside at $s+3$ the path link points into its block by the first fact and depends on links at $s+3, s+4$; a plaquette cannot contain links at $s+2$ and at $s+4$; and a path link ℓ' at $s+4$ in a plaquette with a path link at $s+3$ cannot point outward to a block at $s+5$, since its base block is adjacent to a block at $s+3$, so ℓ' depends on links at $d_a \le s+4$. In the third case the plaquettes through an inward path link $(2y+e_\mu, \mu)$ with $d_a(y)=s+4$ lie in the planes (μ,ν) at $2y+e_\mu$ and at $2y+e_\mu-e_\nu$, with links based in $B(y)$, $B(y+e_\mu)$, $B(y-e_\nu)$, $B(y+e_\mu-e_\nu)$, all at $d_a \le s+4$ by the second fact, and their path links are within one block of $B(y+e_\mu)$ at $s+3$, so point to blocks at $d_a \le s+4$. This proves the claim, and its mirror for b. Hence $\nabla_z \partial_x \mathscr{U}$ lives at $d_a \le s+4$ and $\nabla_z \partial_y \mathscr{U}$ at $d_b \le s+4$; since $d_b \ge w+1-d_a$ their coarse distance is at least $w-2s-7$, and two links based in blocks at coarse distance $D \ge 1$ have midpoints at fine ℓ^∞ distance at least $2D-2$, which gives d'. A K-term depending on both x and y would contain a link at $d_a \le s+3$ and a link at $d_b \le s+3$ at mutual block distance at most one, so $w+1 \le 2s+7$; for $w \ge 2s+7$ there is none. This is (v).

Step 7: positivity. Since $K>0$ on $D_x \times D_y$ and the measures live there, $Z_{ij}>0$ exactly when $a_i(D_x)>0$ and $b_j(D_y)>0$; $a_i(D_x)$ is the constrained weight of ω_a^i with its two collars, which for the all-good pattern is positive by Proposition 412. This is (vi). □

The theorem uses only Lemma 411; it does not use β_W large, η, or Theorem 367. The three dangers named after Lemma 454 do not arise: the island labels change a_i and b_j only, the domain $D_x \times D_y \times D_z$ is fixed by the labels of the layer, which are good in every sector, and every branch of every bad pair is integrated inside a_i or b_j and never compared across sectors. The price is the width. Joining two bad blocks in Sec. XIII I at ℓ^∞ distance at most $2s+7$ instead of $2s+1$, so that two islands are separated by a layer of width at least $2s+7$, changes only the number of neighbours Δ of Theorem 465, from $(4s+3)^4-1$ to $(4s+15)^4-1$: $|\log q^*|$ becomes 31444.0 for $s=1$ and 37777.9 for $s=5$, and $\beta^*(10^{-8},1)=5.084\cdot 10^{21}$, $\beta^*(10^{-8},5)=6.097\cdot 10^{21}$, $\beta^*(10^{-9},1)=5.088\cdot 10^{23}$, by the same formula $\beta^* = 16|\log q^*|\rho^{-2}(1+\varepsilon)$.

The layer law. For $(x,y) \in D_x \times D_y$ let $\mu_{x,y}$ be the law $e^{-\mathscr{U}(z;x,y)}dz/K(x,y)$ on D_z. It is the law of Proposition 412 for the window of the blocks at distance at least $s+3$ from both islands, in the coordinates ζ of the labels, with the inside links of the blocks at distance $s+3$ frozen and the exterior given by (x,y,U^{ext}). The covariance bound of Theorem 367(ii) for this law reads: there are $\mathscr{A}>0$ and $\alpha>0$ such that for all (x,y), every φ on the z-links with $|\varphi(e)-\varphi(e')| \le d(e,e')$ and all smooth F, H near D_z,

$$|\mathrm{Cov}_{\mu_{x,y}}(F,H)| \le \mathscr{A}\, \big\| e^{-\alpha\varphi}\nabla_z F \big\|_{L^2(\mu_{x,y})} \big\| e^{\alpha\varphi}\nabla_z H \big\|_{L^2(\mu_{x,y})}. \tag{41}$$

Let $n_x = 45(N_{s+1}+N_{s+2})+17N_{s+3}$, with N_d the number of blocks of Λ at $d_a=d$, be the number of x-links, n_y likewise, and let Λ_2 be the largest operator norm of $\partial_x \nabla_z \mathscr{U}$ and $\partial_y \nabla_z \mathscr{U}$ on $D_z \times D_x \times D_y$; it is finite, of the form $\beta_W c_W + c_\psi$ with c_W, c_ψ depending on r and the geometry only.

Proposition 484 (the cross-ratio under the layer bound)**.** *Let $s \ge 1$, $r \le 10^{-4}$, $w \ge 2s+8$, and assume (41). Then for all $x_0, x_1 \in D_x$ and $y_0, y_1 \in D_y$,*

$$\Big|\log \frac{K(x_1,y_1)K(x_0,y_0)}{K(x_1,y_0)K(x_0,y_1)}\Big| \le \Delta,$$

$$\Delta := \mathscr{A}\Lambda_2^2 \|x_1-x_0\|\|y_1-y_0\| e^{-\alpha d'} \le 4\mathscr{A}\Lambda_2^2 r^2 \sqrt{n_x n_y}\, e^{-\alpha d'},$$

so that, for every pair of patterns with all $Z_{ij}>0$, $|\log Z_{11} + \log Z_{00} - \log Z_{10} - \log Z_{01}| \le \Delta$ and the two island labels have conditional covariance at most $\frac{1}{4}\tanh(\Delta/4)$ given the labels of the layer and U^{ext}. Moreover, for every bounded $F \ge 0$ depending on the links of $L_a \cup p_a \cup x$ only, with $\mathbb{E}_{ij}$ the expectation in sector (i,j),

$$e^{-\Delta} \le \frac{\mathbb{E}_{i1}[F]}{\mathbb{E}_{i0}[F]} \le e^{\Delta} \qquad (i=0,1)$$

whenever the four expectations are positive.

Proof. D_x and D_y are convex, so the segments $x(\sigma)$, $y(\tau)$ stay in them, and $\log K$ is smooth near $D_x \times D_y$ by Theorem 483(iv), so the cross-ratio is the double integral of $\partial_\sigma \partial_\tau \log K$ over the unit square. Lemma 454 with the fixed domain D_z gives $\partial_\sigma \partial_\tau \log K = -\mathbb{E}[\partial_\sigma \partial_\tau \mathscr{U}] + \mathrm{Cov}(\partial_\sigma \mathscr{U}, \partial_\tau \mathscr{U})$, and the first term vanishes by Theorem 483(v). Put $F = \partial_\sigma \mathscr{U}$, $H = \partial_\tau \mathscr{U}$: $\nabla_z F$ is supported on the z-links at $d_a \le s+4$, $\nabla_z H$ on those at $d_b \le s+4$, at distance at least d', with $\|\nabla_z F\| \le \Lambda_2\|x_1-x_0\|$ and $\|\nabla_z H\| \le \Lambda_2\|y_1-y_0\|$ pointwise. Apply (41) with $\varphi(e) = -\operatorname{dist}(e, \operatorname{supp}\nabla_z F)$, which is 1-Lipschitz, equal to 0 on the support of $\nabla_z F$ and at most $-d'$ on that of $\nabla_z H$; this gives the bound on the covariance uniformly in (σ,τ), and $\|x_1-x_0\| \le 2r\sqrt{n_x}$ since each coordinate is a vector of $\mathfrak{g}$ of norm at most r. The consequences are Lemma 452 applied to a_i, b_j, K and Corollary 453. For the last statement, $F\Phi_a^i$ is again a product of a-factors, so the factorisation holds with the measures Fa_i and a_i, the same K and the same b_j, and Lemma 452 for these gives $|\log Z(Fa_i,b_1) + \log Z(a_i,b_0) - \log Z(Fa_i,b_0) - \log Z(a_i,b_1)| \le \Delta$ with $Z(Fa_i,b_j)/Z(a_i,b_j) = \mathbb{E}_{ij}[F]$. □

The last bound is the quantity that Proposition 455 says a labelling argument must control: the change of the conditional expectation of an observable of one island under a flip of the label of the other, with the branches, the hard label constraints and the disintegration densities of both islands integrated, not compared. The hypothesis (41) is Theorem 367(ii) for the layer

law, but in other coordinates. The chart law of that theorem is written in the coordinates ξ of T_U, on the product of balls Box_r; the label coordinate of a non-path link is $\zeta_e = \xi_e + h_e(S\xi)$, so the label event $\prod_e\{\|\zeta_e\| \le r\}$ is a deformation of Box_r, convex in ζ and not in ξ, and the Hessian of the density in ζ acquires the term $\langle \nabla_\xi \bar H, D^2\xi\rangle$, which is not coordinate invariant. The next theorem carries the proof over to the coordinates ζ; it needs the layer calmer than Theorem 367 needs a chart, and a coupling four times larger.

Theorem 485 (the layer law obeys the chart bound)**.** *Let $G = \mathrm{SU}(2)$, $\lambda = \beta_W/4$, and let the data be those of Theorem 483. Let W' be the set of blocks of Λ at distance at least $s+3$ from both islands, and assume*

(a) *$r \le 8\cdot 10^{-5}$ and $4\cdot 10^5 c_3(\eta + 62r) \le \frac{1}{195}$, which holds when $\eta + 62r \le 6.0\cdot 10^{-13}$;*

(b) *$\lambda \ge 240\, c_6$, which holds when $\beta_W \ge 8.93\cdot 10^{17}$;*

(c) *every block outside Λ at distance at most two from W' is good and the constraints of its pairs hold; this is void when Λ contains the blocks at distance at most two from W'.*

Then (41) holds with $\mathscr{A} = 120/\lambda = 480/\beta_W$ and every $\alpha \le \alpha_G = 2.5\cdot 10^{-4}$, uniformly in $(x,y) \in D_x \times D_y$, in U^{ext}, in Λ and in the width w.

Proof. All constants without a name are those of the proofs of Theorems 363 and 367 and of Proposition 369, at $\vartheta = 0$ unless a weight is written: $\mathscr{G} = BB^* \ge \frac{1}{17}$, $\|B^*z\|_\infty \le 0.88\max_k\|z_k\|$, $\|R_k(X)\| \le 12c_3\|X\|_\infty^2$, the rows of $A-B$ of norm at most $7.35c_3 s$, $Y = B^*(AB^*)^{-1}$ with $\|Y\| \le 11.32$, $\bar P = S - Y(A-B)S$ with $\|\bar P\| \le 4.80$ and $\|\bar P\|_{1\to 1} \le 3.003$, $x_v = \bar P v$ and $x_{vv'} = -YD^2c(X)[x_v, x_{v'}]$ with $\|x_{vv'}\|_1 \le 1564.3\, c_3|v||v'|$. Put $s' = 32r$ for the chart radius.

Step 1: the background is calm. Let U be the point of the fibre whose non-path links in W' are $R(c)$ and whose path links there are the solutions $\Gamma_k(0)$ of Lemma 411. Take a plaquette p with a vertex in a block of W'^+. Its links are based in blocks at distance at most two from W', which are good, by the sector event inside Λ and by (c) outside it. A plaquette contains at most two path links, one of each of its directions: a path link $(2y + e_\mu, \mu)$ has exactly one odd coordinate at its base, and two such links of one direction on one plaquette would have to be opposite sides with bases differing by e_ρ, $\rho \ne \mu$, which gives one of them two odd coordinates. The plaquette of $R(c)$ at p is I unless both coordinates of the corner in its plane are odd, and it is then the coarse plaquette of a good block, so $\delta(R(c)_p, I) \le \eta$. Replacing the four links one at a time, with $\delta \le d$ (Lemma 352), bi-invariance and $d(\Gamma_k, c_k) \le 14r$, gives $\delta(U_p, I) \le \eta + 2\cdot 14r + 2r$. So the defect κ_0 of the background satisfies $\kappa_0 + s' \le \eta + 62r$, and (a) is the hypothesis $4\cdot 10^5 c_3(\kappa_0 + s') \le \frac{1}{195}$ of Theorem 363(iv) at radius s'; by Proposition 364 it implies $s' \le 1/(6000c_3)$, $\kappa_0 + 6s' \le 10^{-8}$ and $c_3 s' \le 1.29\cdot 10^{-8}$.

Step 2: the two coordinates. Let $\rho_1 = \frac{5}{4}r$, which is at most 10^{-4} by (a). For ζ with every $\|\zeta_e\| \le \rho_1$ let U' be the point of the fibre with non-path links $R_e(c)e^{\zeta_e}$, and $X = \log(U^{-1}U')$ link by link. On the non-path links $\|X_e\| \le \rho_1$; on a path link, by Lemma 411 at the radii r and ρ_1, $\|X_{d_k}\| \le 14r + 14\rho_1 = 31.5r$. So $t = \|X\|_\infty < s'$, and $BX = -R(X)$ with $\|R_k(X)\| \le 12c_3t^2$. With $z = \mathscr{G}^{-1}BX$ and $v = X - B^*z \in T_U$ we get $\max_k\|z_k\| \le 204c_3t^2 \le 8.2\cdot 10^{-5}r$ and $\|v\|_\infty \le t(1 + 180c_3t) \le 31.51r < s'$, and z is the fixed point that defines $h(v)$, so $U' = \Phi(v)$ with $\|h(v)\|_\infty \le 7.2\cdot 10^{-5}r$. Hence the map $\Psi(\xi) = \xi + g(\xi)$, $g(\xi) = h(S\xi)$ restricted to the non-path links, which sends the chart coordinate to the label coordinate, maps a subset of Box_{2r} onto $\{\|\zeta_e\| \le \rho_1\}$; it is injective there because two chart points with the same non-path links within ρ_1 of $R(c)$ have the same path links by Lemma 411. Its derivative is $1 + Dg$ with $Dg = -Y(A-B)S$ restricted to the non-path links, so

$$\|Dg\| \le 3.6\cdot 10^{-5}, \qquad \|Dg\|_{(0.197)} \le 7.0\cdot 10^{-5}, \qquad \|Dg\|_{1\to 1} \le 2.0\cdot 10^{-5},$$

from $\|S\|_{1\to 1} \le 1 + 2\cdot 8.01(\frac{1}{8} + 260\kappa_0)$, the rows of $A-B$, and the weighted bounds of Proposition 369. Thus Ψ is a diffeomorphism onto the ρ_1-balls, and $J = D\xi = (1+Dg)^{-1}$ has the norms $j_0 = 1 + 3.6\cdot 10^{-5}$, $j_\vartheta = 1 + 7.0\cdot 10^{-5}$ and $j_1 = 1 + 2.0\cdot 10^{-5}$ in the three norms.

Step 3: the density. By Theorem 363(iii) and the proof of Theorem 367, the constrained law on the chart has density $e^{-\bar H(\xi)}$ in ξ. By Step 2 and the change of variables $\zeta = \Psi(\xi)$, the law conditioned on the label event has density $e^{-\tilde U}\mathbf{1}_{\{\|\zeta_e\| \le r\}}$ with

$$\tilde U(\zeta) = \bar H(\xi(\zeta)) + \log\det\big(1 + Dg(\xi(\zeta))\big),$$

smooth on the ρ_1-balls. It is the law of Proposition 412, since the disintegration of a smooth positive density along a submersion is unique up to a factor constant on each fibre, and the layer law $\mu_{x,y}$ is this law with the inside links at distance $s+3$ frozen.

Step 4: the lower bound on the Hessian. By the chain rule, with $a = Jv$ and $M = 1 + Dg$,

$$\partial_v^2\tilde U = \mathrm{Hess}_\xi \bar H[a,a] + \big\langle \nabla_\xi \bar H, D^2\xi[v,v]\big\rangle + \mathrm{Tr}\big(M^{-1}M''\big) - \mathrm{Tr}\big((M^{-1}M')^2\big),$$

where $M' = D^2g[a,\cdot]$, $M'' = D^3g[a,a,\cdot] + D^2g[D^2\xi[v,v],\cdot]$ and $D^2\xi[v,v] = -JD^2g[a,a]$. The first term is at least $(\frac{\lambda}{30} - c_6)|a|^2$ by Theorem 363(ii), (iii), since $\frac{1}{26} - 4\cdot 10^5 c_3(\kappa_0 + s') \ge \frac{1}{30}$. For the second, the gradient of $\bar H$ in one link is the sum of $\lambda DQ[x_v]$, the derivative of the Haar densities and the derivatives of the two determinants; $|DQ[x_v]| \le 12(\kappa_0 + 6s')\|x_v\|_1$, the Haar part is at most $0.84\|x_v\|_1$, and the determinants give $\mathrm{Re}\,\mathrm{Tr}(A^*(AA^*)^{-1}A_v)$ and $\mathrm{Re}\,\mathrm{Tr}(\bar P(\bar P^*\bar P)^{-1}\bar P^*YA_v)$ with $A_v = D^2c(X)[x_v,\cdot]$, bounded by $5\|A_v\|_1$ and $11.32\|A_v\|_1$ since $AA^* \ge \frac{1}{25}$ and the second operator is Y followed by a projection; and $\|A_v\|_1 \le 2\dim G\, c_3\|x_v\|_1$ because each row of D^2c has rank at most $\dim G$ and each link enters two rows. So every component of $\nabla_\xi\bar H$ is at most $12\lambda(\kappa_0 + 6s')\cdot 3.003 + 296.5\, c_3$, and with $\|D^2\xi[v,v]\|_1 \le j_1 j_0^2\cdot 1564.3\, c_3|v|^2$ the second term is at most $(4.35\cdot 10^{-3}\lambda + 4.64\cdot 10^5 c_3^2)|v|^2$. The derivatives of g

are those of X: $D^2 g[a,\cdot] = -YA_a\bar P$ and

$$D^3 g[a,b,\cdot] = YA_aYA_b\bar P + YA_bYA_a\bar P - YD^3c[x_a,x_b,\cdot]\bar P - YD^2c[x_{ab},\cdot]\bar P,$$

restricted to the non-path links, and the Hilbert–Schmidt bound $\|A_a\|_2^2 \le 2\dim G\, c_3^2\|\bar P\|^2|a|^2$ together with $|\mathrm{Tr}(ST)| \le \|S\|\|T\|_1$ and $\|ST\|_1 \le \|S\|_2\|T\|_2$ bounds the two traces term by term by $(4.07+1.70+0.08+5.10+15.30)\cdot 10^5 c_3^2|v|^2$. Since $c_6 = 6.81\cdot 10^5 \dim G\, c_3^2$, the part of $\partial_v^2\tilde U$ that does not grow with λ, beyond the c_6 of the first term, is at most $1.512\,c_6|v|^2$, and by (b)

$$\mathrm{Hess}_\zeta \tilde U \ge \lambda\Big(\frac{1/30-1/240}{(1+3.6\cdot 10^{-5})^2} - 4.35\cdot 10^{-3} - \frac{1.512}{240}\Big) \ge 0.0185\,\lambda \ge \frac{\lambda}{60}.$$

Step 5: locality of the Hessian. The same four terms, with the vectors $p = wu$, $q = w^{-1}u'$ for $w = e^{\vartheta\varphi}$, $\vartheta = 0.197$, are bounded as in the paragraph "The Hessian" of the proof of Proposition 369: the weights of x_p and x_q on the links of one pair cancel up to $e^{2\vartheta}$, the conjugation of A_a by w costs at most e^{ϑ}, and the other factors are bounded in $\|\cdot\|_{(\vartheta)}$ by $\bar p \le 5.80$, $y \le 14.96$ and j_ϑ. With $\|\mathrm{Hess}_\xi \bar H\|_{(0.197)} \le 2068\lambda$ this gives

$$\|\mathrm{Hess}_\zeta \tilde U\|_{(0.197)} \le \lambda\Big(2068\, j_\vartheta^2 + 9.41\cdot 10^{-3} + \frac{5.88}{240}\Big) \le 2068.4\,\lambda.$$

Step 6: the covariance bound. The proof of Theorem 367(ii) now applies word for word with Box_r replaced by the product of the r-balls in ζ, both being products of balls in the coordinates in which the Hessian bounds of Steps 4 and 5 hold, and with the radii of the approximation $\frac32 r, \frac74 r, 2r$ replaced by $1.1r, 1.2r, 1.25r$, for which the convex penalty has Hessian at least $\frac{1}{22}$ outside $1.1r$. Lemma 368(ii) gives $\|\mathrm{Re}(\mathrm{Hess}\, H_n)^w - \mathrm{Hess}\, H_n\| \le \frac{\lambda}{120}$ for $\alpha \le 0.197(240\cdot 2068.4)^{-1/2} = 2.79\cdot 10^{-4}$, which contains α_G, and the weighted Helffer–Sjöstrand representation gives the bound with $\mathscr{A} = 2/m = 120/\lambda$. Freezing the inside links at distance $s+3$ restricts the Hessian to a principal submatrix, which keeps both bounds, and the argument runs on the free links with the same constants. □

Corollary 486 (two islands in a good region)**.** *Under the hypotheses of Theorems 483 and 485, with $w \ge 2s+8$, every pair of label patterns with all $Z_{ij} > 0$ satisfies*

$$\big|\log Z_{11} + \log Z_{00} - \log Z_{10} - \log Z_{01}\big| \le \Delta,$$
$$\Delta = \frac{1920}{\beta_W}\Lambda_2^2 r^2 \sqrt{n_x n_y}\, e^{-2\alpha_G(w-2s-8)},$$

the two island labels have conditional covariance at most $\frac14\tanh(\Delta/4)$, and every observable of one island and its collars changes by a factor in $[e^{-\Delta}, e^{\Delta}]$ when the label of the other island is flipped. In the normalisation $r = 43\rho$, $\eta = 92\rho$ of Theorem 465, hypothesis (a) of Theorem 485 holds for $\rho \le 2.17\cdot 10^{-16}$, where the island threshold with the joining distance $2s+8$ is $\beta^* = 1.122\cdot 10^{37}$ for $s = 1$ and $1.322\cdot 10^{37}$ for $s = 5$, and hypothesis (b) holds at every $\beta_W \ge \beta^*$.

Proof. Proposition 484 with $\mathscr{A} = 480/\beta_W$ from Theorem 485 and $d' = 2(w-2s-8)$. For the numbers, $\eta + 62r = (92 + 62\cdot 43)\rho = 2758\rho$, and the threshold is the largest root of the equation of Theorem 465 with $(4s+17)^4 - 1$ neighbours, the count for the joining distance $2s+8$, which makes every layer between two islands of width at least $2s+8$: $|\log q^*| = 32481.7$ for $s = 1$ and 38354.1 for $s = 5$. □

Lemma 487 (the mixed derivatives of the layer action)**.** *Let $G = \mathrm{SU}(2)$, let the data be those of Theorem 483 with $r \le 8\cdot 10^{-5}$, and let $\kappa \le 10^{-8}$ be such that $\delta(U_p, I) \le \kappa$ on $D_z \times D_x \times D_y$ for every plaquette p of S_K that contains the solved path link of a pair one of whose arguments, the non-path links of its paths, is a z-link. Then*

$$\Lambda_2 \le \beta_W c_W + c_\psi, \qquad c_W \le 101.11, \qquad c_\psi \le 4.94\cdot 10^6,$$

with c_W and c_ψ the explicit functions of r and κ given in (43). Under hypotheses (a) and (c) of Theorem 485 the condition holds with $\kappa = \eta + 30r \le 6.0\cdot 10^{-13}$, and

$$\Lambda_2 \le 99.0001\,\beta_W + 4.65\cdot 10^6.$$

Proof. Norms on $\mathfrak g$ are Hilbert–Schmidt norms, $\delta(g,h) = \|g-h\|$ on $\mathrm{SU}(2)$, and the coordinates are the ζ_e of Theorem 483, $U_e = R_e(c)e^{\zeta_e}$, with the reconstruction value $R_e(c) = I$ for a link inside a block and $R_e(c) = c_k$ for a private link of the pair k. For a function f of the coordinates and two coordinates $g \ne g'$ write $\|\partial_g\partial_{g'} f\|$ for the supremum of $|\partial_s\partial_t f(\zeta + sv_g + tw_{g'})|$ at $s = t = 0$ over $v, w \in \mathfrak g$ of norm one placed at g and g': it is the operator norm of the (g,g') block of the Hessian.

Step 1: reduction to row sums. Let $M = \partial_x\nabla_z\mathscr{U}$, with blocks M_{ef} for z-links e and x-links f. For vectors u on the z-links and v on the x-links, $|\langle u, Mv\rangle| \le \sum_{e,f}\|M_{ef}\|\|u_e\|\|v_f\|$, and the Cauchy–Schwarz inequality gives $\|M\| \le \max\{\max_e\sum_f\|M_{ef}\|, \max_f\sum_e\|M_{ef}\|\}$. A term $-\log j(\zeta_e)$ of ψ depends on one coordinate, so M_{ef} is the sum of the blocks of the Wilson terms $\beta_W W_p$, $W_p = 1 - \frac12\mathrm{Re}\,\mathrm{Tr}\,U_p$, of S_K and of the terms $-\log J_k = \log\det(1 - DF_k)$ of ψ, J_k the inverse Jacobian of Proposition 412 and DF_k the differential in X of the map F of Lemma 411 at the fixed point, that depend on e and on f. It therefore suffices to prove that for every x- or z-coordinate g

$$R_g = \sum_{g'}\sum_{\tau}\|\partial_g\partial_{g'}\tau\| \le \beta_W c_W + c_\psi,$$

g' running over the coordinates of the other class (z if g is an x-coordinate, x if g is a z-coordinate) and τ over the Wilson terms and the Jacobians that depend on g and g'; the bound for $\partial_y\nabla_z\mathscr{U}$ is the mirror statement.

Step 2: the geometry. (i) Every link lies in six plaquettes. (ii) The solved path link Γ_k of $k = (y,\mu)$ is a function of the 21 non-path links of the fourteen paths of nonzero offset (Lemma 411), its *arguments*: the seven non-path private links, each in two of these paths, and the fourteen links of direction μ in $B(y)$ and $B(y+e_\mu)$ other than $(2y,\mu)$ and $(2y+2e_\mu,\mu)$, each in one path, the tree links among them being fixed at I. A path of k consists of links of direction μ, so a non-path private link is an argument of its own pair only, and a link of direction μ

inside $B(y)$ of the two pairs (y,μ) and $(y-e_\mu,\mu)$. Let $K(g)$ be the set of solved pairs having g as an argument and n_g the number of paths of each containing g: $n_g=2$, $|K(g)|\le 1$ for a private link, $n_g=1$, $|K(g)|\le 2$ for an inside link. (iii) Let $\ell=(a,\mu)$, $a=2y+e_\mu$, be a path link and $\nu\neq\mu$. The plaquette at a in the plane $\{\mu,\nu\}$ has the further links (a,ν), of offset e_μ, inside $B(y)$; $(a+e_\mu,\nu)$, based at the root $2y+2e_\mu$, a tree link; and $(a+e_\nu,\mu)$, of offset $e_\mu+e_\nu$, a non-path private link of (y,μ). The plaquette at $a-e_\nu$ has the further links $(a-e_\nu,\mu)$ and $(a-e_\nu,\nu)$, both based at $2(y-e_\nu)+e_\mu+e_\nu$ and so non-path private links of $(y-e_\nu,\mu)$ and $(y-e_\nu,\nu)$, and $(a+e_\mu-e_\nu,\nu)=(2(y+e_\mu-e_\nu)+e_\nu,\nu)$, the path link of $(y+e_\mu-e_\nu,\nu)$. Hence each of the six plaquettes of a path link carries at most two non-path links, exactly three of them, those at $a-e_\nu$, carry a second path link, and no plaquette carries three path links. (iv) By (iii) a non-path link lies on a plaquette with a path link only if it is an inside link $(2y+e_\mu,\nu)$ with $\nu<\mu$, the only other case $\nu>\mu$ being a tree link, or a private link of offset $e_\mu+e_\nu$ and direction μ or ν. The first lies on one such plaquette, the one at $2y+e_\mu$ in the plane $\{\mu,\nu\}$, with one path link. The second, (b,μ) with $b=2y+e_\mu+e_\nu$, plays the role of $(a+e_\nu,\mu)$ for $a=2y+e_\mu$, on a plaquette with one path link, and the roles of $(a-e_\nu,\mu)$ for $a=2(y+e_\nu)+e_\mu$ and of $(a-e_\mu,\mu)$ for $a=2(y+e_\mu)+e_\nu$, which are one plaquette, the one at b in the plane $\{\mu,\nu\}$, with two path links. So the six plaquettes of a non-path link carry at most three path links in all.

Step 3: one Wilson term. Let $V_1,\dots,V_4$ be smooth functions of (s,t) with values in SU(2), $P=V_1V_2V_3V_4$, and $a_i=\|\partial_s V_i\|$, $b_i=\|\partial_t V_i\|$, $c_i=\|\partial_s\partial_t V_i\|$. Then

$$\left|\partial_s\partial_t \tfrac12 \operatorname{Re}\operatorname{Tr} P\right| \le \tfrac12\Big(\sum_i a_i\Big)\Big(\sum_j b_j\Big)+\tfrac12\,\delta(P,I)\sum_i c_i. \tag{42}$$

Indeed $\partial_s\partial_t P$ is the sum of the products with $\partial_s V_i$ and $\partial_t V_j$ in two places $i\neq j$, whose traces have modulus at most a_ib_j by $|\operatorname{Tr}(AB)|\le\|A\|\|B\|$ and unitarity of the other factors, and of the products with $\partial_s\partial_t V_i$ in place i, whose trace is $\operatorname{Tr}(\tilde P_i Q_i)$ with $Q_i=V_i^{-1}\partial_s\partial_t V_i$ and $\tilde P_i$ a conjugate of P. Differentiating $V_i^*V_i=I$ twice gives $Q_i+Q_i^*=-(\partial_sV_i)^*\partial_tV_i-(\partial_tV_i)^*\partial_sV_i$, so $|\operatorname{Re}\operatorname{Tr}Q_i|\le a_ib_i$, and $|\operatorname{Tr}((\tilde P_i-I)Q_i)|\le\delta(P,I)c_i$. The orientation of a link does not change a_i,b_i,c_i.

Step 4: first derivatives. For $W\in\mathfrak g$ the map $\operatorname{ad}W$ is skew for the trace form, with eigenvalues 0 and $\pm i\theta$, $\theta=\sqrt2\|W\|$: in a basis diagonalising $W=\operatorname{diag}(i\varphi,-i\varphi)$, $\|W\|=\sqrt2\varphi$, it multiplies the off-diagonal entries by $\pm2i\varphi$. A power series $f(\operatorname{ad}W)$ is therefore normal with norm $\max\{|f(0)|,|f(\pm i\theta)|\}$. For $\varpi(u)=(1-e^{-u})/u$ one has $\varpi(i\theta)=e^{-i\theta/2}\frac{\sin(\theta/2)}{\theta/2}$, so with $m(\theta)=\frac{\theta/2}{\sin(\theta/2)}$, increasing on $(0,2\pi)$,

$$\begin{gathered}\|\varpi(\operatorname{ad}W)\|=1,\quad \|\varpi(\operatorname{ad}W)^{-1}\|=m(\theta),\\ \|e^{\operatorname{ad}W}-1\|\le\theta,\quad \|\varpi(\operatorname{ad}W)-1\|\le 1-\tfrac{1}{m(\theta)}+\tfrac{\theta}{2},\\ \|\varpi(\operatorname{ad}W)^{-1}-1\|\le m(\theta)-1+m(\theta)\tfrac{\theta}{2}.\end{gathered}$$

The derivative of the exponential is $\frac{d}{dt}e^{W(t)}=e^W\varpi(\operatorname{ad}W)W'$. Hence a non-path link has $\|\partial_gU_g[v]\|=\|\varpi(\operatorname{ad}\zeta_g)v\|\le\|v\|$. Let $k=(y,\mu)$ be solved, $\Gamma_k=c_ke^X$. For a path x of nonzero offset, $c_k^{-1}P_x=e^{\alpha_x}e^{\beta_x}$, where (α_x,β_x) is $(\operatorname{Ad}(c_k^{-1})\zeta_u,\zeta_q)$ when $P_x=uq$ with u inside $B(y)$ and q private, and $(\zeta_q,\zeta_{u'})$ when $P_x=qu'$ with u' inside $B(y+e_\mu)$, a tree link contributing 0; so $\|\alpha_x\|,\|\beta_x\|\le r$. Put $Z_x=\log(e^{-X}e^{\alpha_x}e^{\beta_x})$; by Lemma 411, $\|X\|\le 14r$ and $\|Z_x\|\le 16r$. With $Y=e^{Z_x}$ and $Y^{-1}\partial Y=\varpi(\operatorname{ad}Z_x)\partial Z_x$,

$$\begin{aligned}\partial_X Z_x[h]&=-T_xh,\\ T_x&=\varpi(\operatorname{ad}Z_x)^{-1}\operatorname{Ad}(e^{-\beta_x}e^{-\alpha_x})\,\varpi(-\operatorname{ad}X),\\ \partial_\beta Z_x[w]&=\varpi(\operatorname{ad}Z_x)^{-1}\varpi(\operatorname{ad}\beta_x)w,\\ \partial_\alpha Z_x[v]&=\varpi(\operatorname{ad}Z_x)^{-1}\operatorname{Ad}(e^{-\beta_x})\varpi(\operatorname{ad}\alpha_x)v.\end{aligned}$$

With $\theta_Z=16\sqrt2 r$, $\theta_X=14\sqrt2 r$, $m_1=m(\theta_Z)$, the last two have norm at most m_1, and writing T_x-1 as $(\varpi(\operatorname{ad}Z_x)^{-1}-1)\operatorname{Ad}(\cdot)\varpi(\cdot)+(\operatorname{Ad}(\cdot)-1)\varpi(\cdot)+(\varpi(-\operatorname{ad}X)-1)$,

$$\|T_x-1\|\le\varepsilon_T=m_1-1+m_1\tfrac{\theta_Z}{2}+2\sqrt2 r+1-\tfrac{1}{m(\theta_X)}+\tfrac{\theta_X}{2}.$$

Since $F=-\frac{1}{16}\sum_x Z_x$ over the fourteen paths, $1-DF_k=\frac18-\frac{1}{16}\sum_x(T_x-1)$, so $1-DF_k$ is invertible with

$$\|(1-DF_k)^{-1}\|\le \mathsf m=\frac{8}{1-7\varepsilon_T},$$

and $\det(1-DF_k)>0$, every eigenvalue of $8(1-DF_k)$ lying within $7\varepsilon_T<1$ of 1. The implicit function theorem gives $\partial_gX=(1-DF_k)^{-1}\partial_gF$ and $\|\partial_gF\|\le n_gm_1/16$, and $\|\partial_g\Gamma_k[v]\|=\|e^X\varpi(\operatorname{ad}X)\partial_gX[v]\|\le\|\partial_gX[v]\|$, so

$$\|\partial_g\Gamma_k\|\le\|\partial_gX\|\le\gamma_g=\frac{n_gm_1}{2(1-7\varepsilon_T)}:$$

$\gamma_g=\gamma$ for a private link and $\gamma_g=\frac{\gamma}{2}$ for an inside link, with $\gamma=m_1/(1-7\varepsilon_T)$. In both cases $\gamma_g|K(g)|\le\gamma$, and the sum of $\gamma_{g'}$ over the arguments of one pair is at most $7\gamma+14\frac{\gamma}{2}=14\gamma$.

Step 5: higher derivatives. Let $G(X,\alpha,\beta)=\log(e^{-X}e^\alpha e^\beta)$ for $X,\alpha,\beta\in\mathfrak g\otimes\mathbb C$ with $\|X\|+\|\alpha\|+\|\beta\|\le\sigma=\frac12$. The Hilbert–Schmidt norm is submultiplicative, so $\|e^{-X}e^\alpha e^\beta-I\|\le e^\sigma-1<1$, the logarithmic series converges, G is holomorphic and agrees with the principal logarithm at real arguments, and $\|G\|\le M_G=-\log(2-e^{1/2})$. At a real point with $\|X\|+\|\alpha\|+\|\beta\|\le 16r$ and for unit vectors $v_1,\dots,v_j$ of $\mathfrak g$ placed in any of the three slots, $t\mapsto G(\cdot+\sum_i t_iv_i)$ is holomorphic on the polydisc $|t_i|\le\rho_j=(\frac12-16r)/j$, and the Cauchy estimates give

$$\|D^jG[v_1,\dots,v_j]\|\le B_j=M_G\rho_j^{-j}\qquad(j=2,3).$$

For g,g' arguments of k let $s_{gg'}\le1$ be the number of paths containing both. Differentiating $X=F(X,\zeta)$ twice and using $\|D_X^2F\|\le\frac{14}{16}B_2$, $\|D_XD_gF\|\le\frac{n_g}{16}B_2$, $\|D_gD_{g'}F\|\le\frac{s_{gg'}}{16}B_2$,

$$\begin{aligned}\|\partial_g\partial_{g'}X\|&\le\frac{\mathsf mB_2}{16}\big(14\gamma_g\gamma_{g'}+n_{g'}\gamma_g+n_g\gamma_{g'}+s_{gg'}\big)\\ &\le X_2=\frac{\mathsf mB_2}{16}(14\gamma^2+4\gamma+1).\end{aligned}$$

The second differential of exp at X has norm at most $e^{\|X\|}$ (each of the $n(n-1)$ terms of the second derivative of $X^n/n!$ has norm at most $\|X\|^{n-2}$), so $\|\partial_g\partial_{g'}\Gamma_k\| \le C_2 = e^{14r}\gamma^2 + X_2$. For $\mathscr{M} = 1 - DF_k$, the chain rule and the same bounds with B_3 for the third derivatives of F give

$$\|\partial_g \mathscr{M}\| \le M_1 = \frac{B_2}{16}(14\gamma+2),$$
$$\|\partial_g\partial_{g'}\mathscr{M}\| \le M_2 = \frac{B_3}{16}(14\gamma^2+4\gamma+1) + \frac{14}{16}B_2X_2,$$

the terms of $\partial_g\partial_{g'}\mathscr{M}$ being $D^3_X F[\partial_g X, \partial_{g'}X]$, $D^2_X D_{g'}F[\partial_g X]$, $D^2_X D_g F[\partial_{g'}X]$, $D^2_X F[\partial_g\partial_{g'}X]$ and $D_X D_g D_{g'} F$.

Step 6: the Wilson terms. For a plaquette p and a coordinate g let $A_g(p)$ be the sum over the links of p of the bounds of Step 4 for their derivatives in g: $A_g(p) = \mathbf{1}_{g\in p} + \gamma_g \#\{k \in K(g) : \Gamma_k \in p\}$. By (42), $\|\partial_g\partial_{g'}W_p\| \le \frac12 A_g(p)A_{g'}(p) + \frac12\delta(U_p,I)C_{gg'}(p)$, with $C_{gg'}(p) \le C_2$ times the number of solved path links of p with both g and g' as arguments. Put $S_g(p) = \sum_{g'\neq g} A_{g'}(p)$. A non-path link contributes at most one to $S_g(p)$ and a solved path link at most 14γ (Step 4), so $S_g(p) \le N + 14\gamma\Pi$ with N the number of non-path links of p other than g and Π the number of path links of p. If $g \in p$ then $N + \Pi \le 3$ and $S_g(p) \le 3 + 14\gamma\Pi$, and by Step 2(iv) the sum over the six plaquettes of g is at most $18 + 42\gamma$. If $\Gamma_k \in p$ then $N \le 2$ and $\Pi \le 2$, with $\Pi = 2$ on three of the six plaquettes of Γ_k (Step 2(iii)), so the sum over them is at most $12 + 14\gamma(6+3) = 12 + 126\gamma$. Hence

$$\sum_p A_g(p)S_g(p) \le 18 + 42\gamma + \gamma_g|K(g)|(12+126\gamma)$$
$$\le 18 + 54\gamma + 126\gamma^2.$$

The sum of the first part over all $g' \neq g$ bounds its sum over the other class. The second part of (42) enters only through a solved path link $\Gamma_k \in p$ with g and g' among its arguments; one of g, g' is a z-coordinate, so k has a z-link among its arguments and the hypothesis gives $\delta(U_p, I) \le \kappa$. There are at most $|K(g)| \le 2$ such pairs, six plaquettes for each and twenty coordinates g', so this part adds at most $\frac12\kappa\cdot 240 C_2$. The Wilson terms contribute at most $\beta_W c_W$ to R_g.

Step 7: the Jacobians. The term $\log\det\mathscr{M}$ of a solved pair depends on g only if $k \in K(g)$, and on $g' \neq g$ only if g' is one of the other twenty arguments. With $|\mathrm{Tr}A| \le 3\|A\|$ on $\mathfrak{g}$ and $\|\mathscr{M}^{-1}\| \le \mathsf{m}$,

$$\left|\partial_g\partial_{g'}\log\det\mathscr{M}\right| = \left|\mathrm{Tr}(\mathscr{M}^{-1}\partial_g\partial_{g'}\mathscr{M}) - \mathrm{Tr}(\mathscr{M}^{-1}\partial_{g'}\mathscr{M}\,\mathscr{M}^{-1}\partial_g\mathscr{M})\right| \le L = 3\mathsf{m}M_2 + 3\mathsf{m}^2M_1^2,$$

and the Jacobians contribute at most $c_\psi = 2\cdot 20\cdot L$ to R_g. Together,

$$c_W = 9 + 27\gamma + 63\gamma^2 + 120\kappa C_2, \qquad c_\psi = 40L. \tag{43}$$

Step 8: the numbers. Since $\sin x \ge x - x^3/6$, $m(\theta) \le (1-\theta^2/24)^{-1}$ and $1 - 1/m(\theta) \le \theta^2/24$; the series of $e^{1/2}$, whose tail after the term $k=8$ is at most twice its first term, gives $e^{1/2} \le 1.6487213$ and $M_G \le 1.046176$; and $e^{14r} \le (1-14r)^{-1}$ in C_2. Every constant of Steps 3 to 7 increases with r, and c_W with κ, so it suffices to take r and κ at their largest values; each bound below is computed from the rounded bounds before it, with $\sqrt{2} \le 1.4142136$. At $r = 8\cdot10^{-5}$: $\theta_Z \le 1.8102\cdot10^{-3}$, $\theta_X \le 1.5840\cdot10^{-3}$, $m_1 \le 1 + 1.4\cdot10^{-7}$, $1 - 1/m(\theta_X) \le 1.046\cdot10^{-7}$, $\varepsilon_T \le 1.924\cdot10^{-3}$, $\mathsf{m} \le 8.110$, $\gamma \le 1.01366$, $B_2 \le 16.825$, $B_3 \le 227.72$, $14\gamma^2 + 4\gamma + 1 \le 19.44$, $X_2 \le 165.8$, $C_2 \le 166.9$, $M_1 \le 17.03$, $M_2 \le 2718$ and $L \le 1.234\cdot10^5$; so for $r \le 8\cdot10^{-5}$ and $\kappa \le 10^{-8}$, $c_W \le 101.11$ and $c_\psi \le 4.94\cdot10^6$. Under hypothesis (a) of Theorem 485, $62r \le 6.0\cdot10^{-13}$, so $r \le 9.68\cdot10^{-15}$, and the same chain gives $m_1 \le 1 + 10^{-26}$, $\varepsilon_T \le 2.328\cdot10^{-13}$, $\gamma \le 1 + 1.7\cdot10^{-12}$, $\mathsf{m} \le 8(1 + 1.7\cdot10^{-12})$, $B_2 \le 16.73882$, $B_3 \le 225.975$, $14\gamma^2 + 4\gamma + 1 \le 19 + 10^{-10}$, $X_2 \le 159.019$, $C_2 \le 160.02$, $M_1 \le 16.7389$, $M_2 \le 2597.5$, $L \le 1.1614\cdot10^5$, and $9 + 27\gamma + 63\gamma^2 \le 99 + 4\cdot10^{-10}$. For the condition on κ, let $k = (y,\mu)$ be solved with a z-link among its arguments. A z-link is based in a block of W', so y or $y + e_\mu$ lies in W', and by Step 2(iii) the links of a plaquette p through Γ_k are based in blocks at distance at most one from $B(y)$, hence at distance at most two from W'; they are Γ_k, non-path links, a tree link, and possibly the path link of $k' = (y + e_\mu - e_\nu, \nu)$, both of whose blocks are at distance at most two from W'. These blocks are good, by the sector event inside Λ, since they lie at distance at least $s+1$ from both islands, and by (c) outside it, so every non-path link of p is within r of its reconstruction value. The path link of k' is within $14r$ of $c_{k'}$: if k' is based in Λ it is solved, since a pair based at distance at most $s+1$ from an island has its path link in $L_a \cup p_a$ or $L_b \cup p_b$ and its plaquettes are not K-factors, and otherwise Lemma 411 applies by (c). Step 1 of the proof of Theorem 485 gives $\delta(R(c)_p, I) \le \eta$ for these plaquettes, and replacing the links one at a time with bi-invariance and $\delta \le d$ (Lemma 352) gives $\delta(U_p, I) \le \eta + 2\cdot14r + 2r = \eta + 30r \le 6.0\cdot10^{-13}$. Hence $120\kappa C_2 \le 1.2\cdot10^{-8}$, $c_W \le 99.0001$ and $c_\psi \le 40L \le 4.65\cdot10^6$. □

With Lemma 487, $\Lambda_2^2 \le \beta_W^2(c_W + c_\psi/\beta_W)^2$ and the bound of Corollary 486 becomes

$$\Delta \le 1920\,\beta_W\left(c_W + \frac{c_\psi}{\beta_W}\right)^2 r^2\sqrt{n_x n_y}\,e^{-2\alpha_G(w-2s-8)},$$

which grows linearly in β_W; at $\beta_W = 1.13\cdot10^{37}$, just above β^*, $s=1$ and $r = 43\rho$, $\rho = 2.17\cdot10^{-16}$, where $\beta_W r^2 = 9.84\cdot10^8$, and for islands of one block, $n_x = n_y = 45(544+1776) + 17\cdot 4160 = 175120$, it reads $\Delta \le 3.25\cdot10^{21}e^{-2\alpha_G(w-10)}$. This is not small at the threshold: the bound is at most one exactly when $w \ge 99072$, at fixed r every factor e in β_W/β^* adds $1/(2\alpha_G) = 2000$ to this width, and islands of $|I_a|$ and $|I_b|$ blocks, for which $N_d \le |I_a|((2d+1)^4 - (2d-1)^4)$, add at most $1000\log(|I_a||I_b|)$.

The covariance therefore decays exponentially in the width of the good layer (Figure 39), at the rate of the chart, with the amplitude of Lemma 487, which is below one at the threshold only across good layers about 10^5 blocks wide. If further bad blocks lie in Λ at distance at least $s+4$ from both islands, the factorisation of Theorem 483 through K still holds as an identity of measures, with the constraints of the bad pairs of the layer kept as hard factors inside K, but $\mu_{x,y}$ is then no longer a good-window law, and the cross-ratio of K is the comparison

for the remaining islands: the corollary is the inductive step for two islands, and Sec. XIII N keeps those constraints out of the layer and performs the induction.

N. Many islands

The paragraph after Corollary 486 leaves the case in which the layer between two islands contains further islands. If the constraints of those islands are kept inside the kernel, the law of the layer carries hard constraints and branches, and Theorem 485 does not apply to it. This subsection keeps them out of the layer. Every island receives its own collar, the integral over the island and its collar becomes a positive measure on the collar coordinates of that island, as a_i and b_j in Theorem 483, and the layer between all the islands is a good window. The kernel of two islands is then a mixture, over the collar coordinates of the others, of good-window kernels. A mixture of kernels with small cross-ratios can have a large one, and Proposition 455 is such a mixture over one bit.[43] Theorem 491 bounds the additional cross-ratio by the product of the sensitivities of the two islands to the collars of the others, with no condition on the other islands, and by a sum over chains of islands under a Dobrushin condition on them.[44] The measures of the other islands enter only through their positivity.

The data. The data are those of Sec. XIII M with the following changes. Let $\mathscr{I} = \{a,b\} \cup \mathscr{Q}$ be a finite index set and $I_u \subset \Lambda$, $u \in \mathscr{I}$, pairwise disjoint sets of blocks with

$$w_{uv} = \operatorname{dist}(I_u, I_v) - 1 \geq 2s+7 \qquad (u \neq v),$$

which holds for distinct islands formed with the joining distance $2s+7$ of the paragraph after Theorem 483, and with $w_{uv} \geq 2s+8$ for the joining distance $2s+8$ of Corollary 486. Every block of $\Lambda \setminus \bigcup_u I_u$ is good; on I_a two patterns ω_a^0, ω_a^1 are prescribed, on I_b two patterns ω_b^0, ω_b^1, and on each I_q, $q \in \mathscr{Q}$, one pattern ω_q. The sector (i,j) is the event that the labels of Λ are these, with ω_a^i on I_a and ω_b^j on I_b, and its weight Z_{ij} is defined as in Sec. XIII M. With $d_u(y) = \operatorname{dist}(y, I_u)$, form for every $u \in \mathscr{I}$ the classes L_u, p_u, x_u of Sec. XIII M, with d_a replaced by d_u, let z be every other non-path active link based in Λ, and solve the path link of every pair based at $d_u \geq s+2$ for every u. A link at $d_u \leq s+3$ has $d_v \geq w_{uv} + 1 - (s+3) \geq s+5$ for $v \neq u$, so the classes are disjoint, and every block at $d_u \in \{s+1, s+2, s+3\}$ is good in all four sectors. Let D_u be the product of the r-balls in the coordinates ζ_e of x_u and D_z that of z, let $n_u = 45(N_{s+1}^u + N_{s+2}^u) + 17N_{s+3}^u$, with N_d^u the number of blocks of Λ at $d_u = d$, be the number of x_u-links, let Λ_2 be the largest operator norm of $\partial_{x_u} \nabla_z \mathscr{U}$ over $u \in \mathscr{I}$ and over $D_z \times \prod_u D_u$, and put

$$\Delta_{uv} = \frac{1920}{\beta_W} \Lambda_2^2 r^2 \sqrt{n_u n_v}\, e^{-2\alpha_G (w_{uv} - 2s - 8)_+} \qquad (u \neq v),$$

the quantity Δ of Corollary 486 for the pair I_u, I_v when $w_{uv} \geq 2s+8$.

Lemma 488 (many islands, one kernel)**.** *Let $s \geq 1$ and $r \leq 10^{-4}$. There are finite positive measures a_i on D_a, b_j on D_b, $i,j \in \{0,1\}$, and γ_q on D_q, $q \in \mathscr{Q}$, and a smooth function $\tilde{K} : \prod_{u \in \mathscr{I}} D_u \to (0,\infty)$ with*

$$Z_{ij} = \int \tilde{K}(x)\, a_i(dx_a)\, b_j(dx_b) \prod_{q \in \mathscr{Q}} \gamma_q(dx_q) \qquad (i,j \in \{0,1\}),$$

such that:

(i) *$\tilde{K}$ and every γ_q are the same in all four sectors; a_i depends on ω_a^i, c, β_W, U^{ext} only, and likewise b_j;*

(ii) *$\tilde{K}(x) = \int_{D_z} e^{-\mathscr{U}(z;x)} dz$ with $\mathscr{U} = \beta_W S_K + \psi$ as in Theorem 483(iv), smooth on a neighbourhood of $D_z \times \prod_u D_u$; no term of $\mathscr{U}$ depends on coordinates of x_u and of x_v for $u \neq v$, and $\nabla_z \partial_{x_u} \mathscr{U}$ is supported on the z-links based at $d_u \leq s+4$;*

(iii) *for $u \neq v$ the supports of $\nabla_z \partial_{x_u} \mathscr{U}$ and $\nabla_z \partial_{x_v} \mathscr{U}$ are at fine distance at least $d'_{uv} = 2(w_{uv} - 2s - 8)_+$ between link midpoints.*

Proof. The proof of Theorem 483 treats each island through its own distance function, and runs with one class per island. Steps 1 and 2 are unchanged, a pair being solved when it is based at distance at least $s+2$ from every island; its non-path links are then based at distance at least $s+1$ from every island, in good blocks. In Step 3 call a factor a u-factor if it depends on a coordinate of $L_u \cup p_u$, and assign the factors depending on x_u alone to u. A u-factor depends only on L_u, p_u, x_u and U^{ext}, through links at $d_u \leq s+2$, which have $d_v \geq w_{uv} + 1 - (s+2) \geq s+6$ for every $v \neq u$; so no factor is a u-factor and a v-factor, and the product Φ_K of the remaining factors, the K-factors, does not depend on the sector, which changes only the indicators of I_a and I_b. The indicators of the bad blocks of I_q involve links at $d_q \leq 1 \leq s$ and are q-factors. Step 4 with one bracket for each island gives the identity, with $\gamma_q(dx_q)$ the integral of the product of the q-factors over L_q and p_q times $\prod_{e \in x_q} j\, d\zeta_e$ on D_q, and a_i, b_j, $\tilde{K}$ as there; this is (i). Step 5 gives the first clause of (ii). Step 6 is a statement about one island at a time, so it gives the support of $\nabla_z \partial_{x_u} \mathscr{U}$, and a K-term depending on x_u and on x_v would contain a link at $d_u \leq s+3$ and a link at $d_v \leq s+3$ in blocks at distance at most one, so that $w_{uv} + 1 \leq 2s+7$, which is excluded. The two supports are at coarse distance at least $w_{uv} + 1 - 2(s+4) = w_{uv} - 2s - 7$, and links based in blocks at coarse distance $D \geq 1$ have midpoints at fine distance at least $2D-2$; this is (iii) for $w_{uv} \geq 2s+8$, and (iii) is trivial for $w_{uv} = 2s+7$. □

[43] The cross-ratio $Z_{11}Z_{00}/(Z_{10}Z_{01})$ of the four weights with the two islands in their two patterns measures how much the pattern of one island changes the odds of the pattern of the other: it is one exactly when the two islands are independent. A mixture of independent pairs need not be independent: if a hidden switch raises the chance of both patterns at once, the two islands are correlated however weakly each one depends on the switch.

[44] Dobrushin's condition asks that the conditional law of each island move by less than a fixed fraction $\theta < 1$ in total when the islands around it are changed; an influence then fades by the factor θ at each island it passes through, and the sum over chains converges.

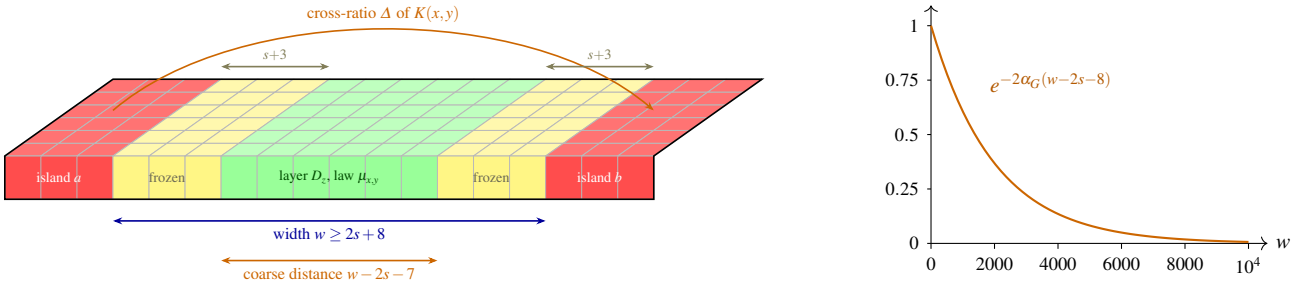


FIG. 39. Left: two bad islands at the ends of a slab, the frozen collars of width $s+3$ beside them, and the good layer between them, on which the layer law $\mu_{x,y}$ obeys the chart bound in the label coordinates (Theorem 485). The cross-ratio Δ of the kernel of Theorem 483 compares the two islands through the layer. Right: the factor $e^{-2\alpha_G(w-2s-8)}$ of Corollary 486 at $\alpha_G = 2.5\cdot 10^{-4}$; the amplitude in front of it is $1920\Lambda_2^2 r^2\sqrt{n_x n_y}/\beta_W$, of order $10^{-34}\Lambda_2^2 r^2\sqrt{n_x n_y}$ at the threshold $\beta_W = \beta^*$.

Lemma 489 (the layer between many islands)**.** *In addition, assume hypotheses (a)–(c) of Theorem 485, with W' the set of blocks of Λ at distance at least $s+3$ from every island. Then for every $x \in \prod_u D_u$ the law $\mu_x = e^{-\mathscr{U}(z;x)}dz/\tilde{K}(x)$ on D_z satisfies (41) with $\mathscr{A} = 480/\beta_W$ and $\alpha = \alpha_G$, and for $u \neq v$ and vectors h_u, h_v in the coordinates of x_u and x_v,*

$$\left|\partial_{h_u}\partial_{h_v}\log\tilde{K}(x)\right| \le \frac{480}{\beta_W}\Lambda_2^2\|h_u\|\|h_v\|e^{-2\alpha_G(w_{uv}-2s-8)_+}.$$

In particular the logarithm of the cross-ratio of $\tilde{K}$ in (x_u,x_v), between two points of D_u and two points of D_v with the other arguments fixed, has modulus at most Δ_{uv}.

Proof. The law μ_x is the law of Proposition 412 for the window W' in the coordinates of the labels, with the inside links of the blocks at distance $s+3$ from some island frozen and the exterior given by x and U^{ext}, by the identification in the paragraph on the layer law after Theorem 483, which uses only that the z-links are the free non-path links of W' and that the solved pairs are those of Lemma 488. The proof of Theorem 485 uses the islands only in Step 1, through the goodness of the blocks at distance at most two from W'; these are at distance at least $s+1$ from every island, hence good in all sectors inside Λ and good by (c) outside it. Steps 2 to 6 concern the window W' and are uniform in its exterior, and freezing the inside links at distance $s+3$ from any island restricts the Hessian to a principal submatrix. So (41) holds with the stated constants. For the derivative, Lemma 454 on the fixed domain D_z gives $\partial_{h_u}\partial_{h_v}\log\tilde{K} = -\mathbb{E}_{\mu_x}[\partial_{h_u}\partial_{h_v}\mathscr{U}] + \mathrm{Cov}_{\mu_x}(\partial_{h_u}\mathscr{U},\partial_{h_v}\mathscr{U})$, and the first term vanishes by Lemma 488(ii). As in the proof of Proposition 484, apply (41) with $\varphi(e) = -\mathrm{dist}(e,\mathrm{supp}\,\nabla_z\partial_{h_u}\mathscr{U})$, which vanishes on that support and is at most $-d'_{uv}$ on the support of $\nabla_z\partial_{h_v}\mathscr{U}$ by Lemma 488(iii), with $\|\nabla_z\partial_{h_u}\mathscr{U}\| \le \Lambda_2\|h_u\|$ and $\|\nabla_z\partial_{h_v}\mathscr{U}\| \le \Lambda_2\|h_v\|$ pointwise. For the cross-ratio, D_u and D_v are convex, the logarithm of the cross-ratio is the integral of the mixed derivative over the unit square, and two points of D_u differ by at most $2r\sqrt{n_u}$. □

The other islands now enter only through the measures γ_q, and the kernel of a and b is $K(x_a,x_b) = \int\tilde{K}(x)\prod_q\gamma_q(dx_q)$, an integral against the probability law on $\prod_q D_q$ proportional to $\tilde{K}(x_a,x_b,\cdot)\prod_q\gamma_q$. The next lemma bounds the covariances of that law in terms of the cross-ratios of its density alone.

Lemma 490 (covariances over the islands)**.** *Let $\mathscr{Q}$ be finite, X_q compact metric spaces, γ_q nonzero finite Borel measures on X_q, $X = \prod_q X_q$, $k : X \to (0,\infty)$ continuous, and $P = k\prod_q\gamma_q/\int k\prod_q\gamma_q$. For $\xi \in X$ and $\eta_q \in X_q$ let ξ^{η_q} be ξ with its q-coordinate replaced by η_q. For $q \neq q'$ let $\kappa_{qq'}$ bound $|\log k(\xi^{\eta_q\eta_{q'}}) + \log k(\xi) - \log k(\xi^{\eta_q}) - \log k(\xi^{\eta_{q'}})|$ for all $\xi,\eta_q,\eta_{q'}$, let $C_{qq'} = \tanh(\kappa_{qq'}/4)$ and $C_{qq} = 0$, and for continuous f on X let $\delta_q(f) = \sup\{|f(\xi^{\eta_q}) - f(\xi)|\}$. Then for continuous f,g:*

(i) $|\mathrm{Cov}_P(f,g)| \le \frac{1}{4}\sum_q\delta_q(f)\sum_q\delta_q(g)$;

(ii) *if $\sum_{q'}C_{qq'} \le \theta < 1$ for every q, then $D = \sum_{n\ge 0}C^n$ converges and*

$$|\mathrm{Cov}_P(f,g)| \le \frac{1}{4}\sum_{q,q'}\delta_q(f)D_{qq'}\delta_{q'}(g);$$

(iii) *if ℓ is a pseudometric on $\mathscr{Q}$, $\lambda \ge 0$ and $\sum_{q'}C_{qq'}e^{\lambda\ell(q,q')} \le \theta < 1$ for every q, then $0 \le D_{qq'} \le e^{-\lambda\ell(q,q')}/(1-\theta)$.*

Proof. Conditional laws. The law

$$P_q(d\eta \mid \xi) = \frac{k(\xi^\eta)\,\gamma_q(d\eta)}{\int k(\xi^{\eta'})\,\gamma_q(d\eta')}$$

on X_q depends on ξ through ξ_{-q} only, its density with respect to γ_q is continuous in (η,ξ), and P is invariant under resampling the q-coordinate from it.

Tilts. Let μ be a probability measure and $\mu' = h\mu/\mu(h)$ with h measurable and $0 < \inf h \le \sup h \le e^\kappa \inf h$. For a Borel set E with $0 < \mu(E) < 1$ the odds of E under μ' and under μ have the ratio $\frac{\mu(h\mathbf{1}_E)/\mu(E)}{\mu(h\mathbf{1}_{E^c})/\mu(E^c)} \in [e^{-\kappa},e^\kappa]$, so the logistic bound in the proof of Corollary 453 gives $|\mu'(E) - \mu(E)| \le \tanh(\kappa/4)$, and $\mu'(E) = \mu(E)$ when $\mu(E) \in \{0,1\}$. Hence $\|\mu' - \mu\|_{\mathrm{TV}} \le \tanh(\kappa/4)$. If ξ and ξ' differ only in the coordinate $q' \neq q$, the density of $P_q(\cdot \mid \xi')$ with respect to $P_q(\cdot \mid \xi)$ is proportional to $k(\xi'^\eta)/k(\xi^\eta)$, whose logarithm oscillates in η by at most $\kappa_{qq'}$, so $\|P_q(\cdot \mid \xi) - P_q(\cdot \mid \xi')\|_{\mathrm{TV}} \le C_{qq'}$.

Comparison. Let g be continuous, $t \in \mathbb{R}$, and $P^t = e^{tg}P/P(e^{tg})$, again of the form of the lemma; its conditional law at q is the tilt of $P_q(\cdot \mid \xi)$ by $e^{tg(\xi^\eta)}$, so $b_q = \sup_\xi\|P_q^t(\cdot \mid$

$\xi)-P_q(\cdot\mid\xi)\|_{\mathrm{TV}}\le\tanh(|t|\delta_q(g)/4)$. The couplings of P and P^t form a weakly compact set (Lemma 457), and $\Psi(\pi)=\sum_q\rho_q(\pi)$, $\rho_q(\pi)=\pi\{\xi_q\neq\xi'_q\}$, is weakly lower semicontinuous, since $\{\xi_q\neq\xi'_q\}$ is open; let π minimise it. For $q\in\mathscr{Q}$ replace (ξ_q,ξ'_q) by a sample of the maximal coupling of $P_q(\cdot\mid\xi)$ and $P^t_q(\cdot\mid\xi')$, which, with p,p' the two densities with respect to γ_q and $m=\int\min\{p,p'\}d\gamma_q$, draws $\eta=\eta'$ from $\min\{p,p'\}\gamma_q/m$ with probability m and otherwise, when $m<1$, η and η' independently from $(p-\min\{p,p'\})\gamma_q/(1-m)$ and $(p'-\min\{p,p'\})\gamma_q/(1-m)$; it is measurable in (ξ,ξ') by the continuity of the densities, and it gives $\{\eta\neq\eta'\}$ probability at most $1-m=\|P_q(\cdot\mid\xi)-P^t_q(\cdot\mid\xi')\|_{\mathrm{TV}}$. Both marginals are invariant, so the result $\pi^{(q)}$ is again a coupling; it changes only ρ_q, and changing the coordinates of ξ_{-q} to those of ξ'_{-q} one at a time,

$$\rho_q(\pi^{(q)})\le\mathbb{E}_\pi\sum_{q'}C_{qq'}\mathbf{1}_{\xi_{q'}\neq\xi'_{q'}}+b_q=\sum_{q'}C_{qq'}\rho_{q'}(\pi)+b_q.$$

By minimality $\rho_q(\pi)\le\rho_q(\pi^{(q)})$, which is the argument of Lemma 456 for the discrete metric, so $\rho\le C\rho+b$ componentwise. Under the hypothesis of (ii), $C^n\mathbf{1}\le\theta^n\mathbf{1}$ and $\rho\le\mathbf{1}$, so iterating gives $\rho\le\sum_{k<n}C^kb+\theta^n\mathbf{1}$ and, as $n\to\infty$, $\rho\le Db$. Changing the coordinates one at a time,

$$\begin{aligned}|P^t(f)-P(f)|&=\big|\mathbb{E}_\pi[f(\xi')-f(\xi)]\big|\\&\le\sum_q\delta_q(f)\rho_q(\pi)\le\sum_{q,q'}\delta_q(f)D_{qq'}b_{q'}.\end{aligned}$$

Covariances. The functions f and g are bounded, so the derivative of $P^t(f)$ at $t=0$ is $\mathrm{Cov}_P(f,g)$, and $b_{q'}\le|t|\delta_{q'}(g)/4$; dividing by $|t|$ and letting $t\to0$ gives (ii). For (i), $|\mathrm{Cov}_P(f,g)|\le(\mathrm{Var}_P f\,\mathrm{Var}_P g)^{1/2}$, a variance is at most a quarter of the squared oscillation, and changing the coordinates one at a time bounds the oscillation of f by $\sum_q\delta_q(f)$; the bound on the variance is Popoviciu's inequality [100]. Part (ii) is Föllmer's covariance estimate [53] under Dobrushin's condition [42, 43], with the coefficients $C_{qq'}$ read off from the cross-ratios of k. For (iii), by induction $(C^n)_{qq'}\le\theta^ne^{-\lambda\ell(q,q')}$: the case $n=0$ is $\ell(q,q)=0$, and $(C^{n+1})_{qq'}=\sum_kC_{qk}(C^n)_{kq'}\le\theta^ne^{-\lambda\ell(q,q')}\sum_kC_{qk}e^{\lambda\ell(q,k)}$ by the triangle inequality; summing over n gives the bound, and the hypothesis of (iii) contains that of (ii). □

Theorem 491 (many islands)**.** *Let $s\ge1$, let the data be those above, assume hypotheses (a)–(c) of Theorem 485 with W' as in Lemma 489, and suppose $Z_{ij}>0$ for $i,j\in\{0,1\}$. Put $S_u=\sum_{q\in\mathscr{Q}}\Delta_{uq}$ for $u\in\{a,b\}$, let C be the $\mathscr{Q}\times\mathscr{Q}$ matrix with $C_{qq}=0$ and $C_{qq'}=\tanh(\Delta_{qq'}/4)$, and let*

$$\Delta_*=\Delta_{ab}+\frac14S_aS_b;$$

when $\sum_{q'}C_{qq'}<1$ for every q, let instead, with $D=\sum_{n\ge0}C^n$,

$$\Delta_*=\Delta_{ab}+\frac14\min\Big\{S_aS_b,\ \sum_{q,q'\in\mathscr{Q}}\Delta_{aq}D_{qq'}\Delta_{q'b}\Big\}.$$

Then

$$\big|\log Z_{11}+\log Z_{00}-\log Z_{10}-\log Z_{01}\big|\le\Delta_*,$$

the two island labels, the pattern of I_a among ω^0_a,ω^1_a and that of I_b among ω^0_b,ω^1_b, have conditional covariance at most $\frac14\tanh(\Delta_*/4)$ given the labels of all other blocks of Λ, the coarse field and U^{ext}, and for every bounded measurable $F\ge0$ of the links of $L_a\cup p_a\cup x_a$,

$$e^{-\Delta_*}\le\frac{\mathbb{E}_{i1}[F]}{\mathbb{E}_{i0}[F]}\le e^{\Delta_*}\qquad(i=0,1)$$

whenever the four expectations are positive. For $\mathscr{Q}=\emptyset$, $\Delta_*=\Delta_{ab}$, which for $w_{ab}\ge2s+8$ is the bound of Corollary 486.

Proof. By Lemma 488, $Z_{ij}=\iint K\,da_i\,db_j$ with $K(x_a,x_b)=\int\tilde K(x)\prod_q\gamma_q(dx_q)$; $Z_{ij}>0$ forces every $\gamma_q\neq0$, and K is smooth and positive on $D_a\times D_b$. Fix $x^0_a,x^1_a\in D_a$, $x^0_b,x^1_b\in D_b$, and let $x_a(\vartheta)=x^0_a+\vartheta(x^1_a-x^0_a)$, $x_b(\varphi)$ likewise, which stay in the convex sets D_a, D_b. The logarithm of the cross-ratio of K at these points is $\int_0^1\int_0^1\partial_\vartheta\partial_\varphi\log K\,d\vartheta\,d\varphi$. With P the law on $\prod_qD_q$ proportional to $\tilde K(x_a(\vartheta),x_b(\varphi),\cdot)\prod_q\gamma_q$, differentiation under the integral sign gives

$$\partial_\vartheta\partial_\varphi\log K=\mathbb{E}_P\big[\partial_\vartheta\partial_\varphi\log\tilde K\big]+\mathrm{Cov}_P\big(\partial_\vartheta\log\tilde K,\partial_\varphi\log\tilde K\big).$$

By Lemma 489 with $h_a=x^1_a-x^0_a$, $h_b=x^1_b-x^0_b$, the first term has modulus at most Δ_{ab}. Apply Lemma 490 with $X_q=D_q$, $k=\tilde K(x_a(\vartheta),x_b(\varphi),\cdot)$, $f=\partial_\vartheta\log\tilde K$ and $g=\partial_\varphi\log\tilde K$. Changing x_q along the segment between two points of D_q and integrating the mixed derivative of Lemma 489 for the pair (a,q) gives $\delta_q(f)\le\Delta_{aq}$, likewise $\delta_q(g)\le\Delta_{bq}$, and the last statement of Lemma 489 for the pair (q,q') shows that $\kappa_{qq'}=\Delta_{qq'}$ is admissible, so the matrix of Lemma 490 is C. Parts (i) and (ii) of that lemma bound the covariance by $\Delta_*-\Delta_{ab}$, uniformly in (ϑ,φ). So the cross-ratio of K is at most e^{Δ_*} and at least $e^{-\Delta_*}$ on $D_a\times D_b$, and Lemma 452 gives the first bound. The joint law of the two labels given the others is $Z_{ij}/\sum Z$, so Corollary 453 gives the second. For the third, $F\Phi^i_a$ is a product of a-factors, so the identity holds with Fa_i in place of a_i and the same K and b_j, and Lemma 452 applied to Fa_i, a_i and b_1, b_0 gives it, since $Z(Fa_i,b_j)/Z(a_i,b_j)=\mathbb{E}_{ij}[F]$. □

The first form of Δ_* needs nothing of the other islands: their hard constraints, branches and masses are integrated in the γ_q and never compared. Its mediated term $\frac14S_aS_b$ does not decay with w_{ab} when other islands lie near both I_a and I_b, since it treats every collar coordinate as able to reach every other. The second form keeps the interaction between the islands, through the matrix C, and under a weighted form of the Dobrushin condition $\sum_{q'}\tanh(\Delta_{qq'}/4)<1$ it decays along chains of islands.

Corollary 492 (decay along chains of islands)**.** *Under the hypotheses of Theorem 491, let ℓ be the pseudometric on $\mathscr{I}$ given by the least value of $\sum_i(w_{u_iu_{i+1}}-2s-8)_+$ over the chains $u=u_0,u_1,\dots,u_k=v$ in $\mathscr{I}$, let $0\le\vartheta\le1$, and suppose*

$$\theta=\max_{q\in\mathscr{Q}}\sum_{q'\in\mathscr{Q}\setminus\{q\}}\tanh\frac{\Delta_{qq'}}{4}\,e^{2\vartheta\alpha_G\ell(q,q')}<1.$$

Then

$$\Delta_* \le \Delta_{ab} + \frac{e^{-2\vartheta\alpha_G \ell(a,b)}}{4(1-\theta)}\, T_a T_b, \qquad T_u = \sum_{q\in\mathscr{Q}} \Delta_{uq} e^{2\vartheta\alpha_G \ell(u,q)},$$

$$T_u \le \frac{1920}{\beta_W}\Lambda_2^2 r^2 \sqrt{n_u} \sum_{q\in\mathscr{Q}} \sqrt{n_q}\, e^{-2(1-\vartheta)\alpha_G (w_{uq}-2s-8)_+}.$$

Proof. ℓ is a pseudometric since it is a least sum over chains, and $\ell(u,v) \le (w_{uv}-2s-8)_+$. Lemma 490(iii) on $\mathscr{Q}$ with $\lambda = 2\vartheta\alpha_G$ gives $D_{qq'} \le e^{-\lambda\ell(q,q')}/(1-\theta)$ and the hypothesis of the second form of Δ_*, and $-\ell(q,q') \le -\ell(a,b)+\ell(a,q)+\ell(q',b)$ gives the first bound. The second is $\ell(u,q) \le (w_{uq}-2s-8)_+$ in each term. □

A chain crosses each island at no cost in ℓ: an island is a set on which nothing is known, and the corollary charges only the good layers between consecutive islands of the chain. With $\vartheta = 0$ the hypothesis is the Dobrushin condition $\sum_{q'} C_{qq'} \le \theta < 1$.

The sums S_u and T_u are sums over the islands near I_u, and the rarity of Theorem 465 controls them in the mean. For a bad block y let $|I(y)|$ be the number of blocks of its island, and $|I(y)| = 0$ for a good block.

Proposition 493 (the island sums in the mean)**.** *Form the islands with the joining distance $2s+8$, so that $w_{uv} \ge 2s+8$ for distinct islands, let $N_s = (4s+17)^4 - 1$ be the number of neighbours of a block, $\Gamma \ge 1$, and $\beta_W \ge \beta^*(\rho,s,\Gamma)$ of Theorem 465(iii) with N_s neighbours. Then:*

(i) *for every block y, $\mathbb{E}_{\mu_L}|I(y)| \le 2\Gamma(2\Gamma-1)^{-2}N_s^{-2}$;*

(ii) *for every set J of blocks of the torus and every $\kappa > 0$,*

$$\mathbb{E}_{\mu_L} \sum_{y\notin J} |I(y)| e^{-\kappa \operatorname{dist}(J,y)} \le \frac{2\Gamma}{(2\Gamma-1)^2 N_s^2}\, |J|\, \sigma(\kappa),$$

where, with $x = e^{-\kappa}$,

$$\sigma(\kappa) = \sum_{y\in\mathbb{Z}^4} e^{-\kappa\|y\|_\infty} = 1 + \frac{64x(1+4x+x^2)}{(1-x)^4} + \frac{16x}{(1-x)^2};$$

(iii) *in the setting of Theorem 491, when every I_q, $q\in\mathscr{Q}$, is an island, with $B' = \bigcup_q I_q$, $0 \le \vartheta \le 1$ and $\kappa = 2(1-\vartheta)\alpha_G$,*

$$T_u \le \frac{1920}{\beta_W}\Lambda_2^2 r^2 \sqrt{45\, n_u}\, (2s+7)^2 e^{\kappa(2s+9)} \times \sum_{y\in B'} |I(y)| e^{-\kappa \operatorname{dist}(I_u,y)}.$$

At $\kappa = 2\alpha_G = 5\cdot 10^{-4}$, $\sigma(\kappa) \le 6.151\cdot 10^{15}$, and at $\Gamma = 1$ the bound of (ii) is at most $3.253\cdot 10^5 |J|$ for $s=1$ and $3.503\cdot 10^3 |J|$ for $s=5$.

Proof. (i) The island paragraph of the proof of Theorem 465 gives $\mu_L\{|I(y)| = n\} \le N_s^{2(n-1)} A^{n/H}$, and its proof of (iii) gives $\Gamma N_s^2 A^{1/H} \le \frac{1}{2}$ at $\beta_W \ge \beta^*(\rho,s,\Gamma)$, so $\mu_L\{|I(y)| = n\} \le N_s^{-2}(2\Gamma)^{-n}$ and $\mathbb{E}|I(y)| \le N_s^{-2}\sum_n n(2\Gamma)^{-n} = 2\Gamma(2\Gamma-1)^{-2}N_s^{-2}$. (ii) By linearity it suffices that $\sum_y e^{-\kappa\operatorname{dist}(J,y)} \le |J|\sigma(\kappa)$; since $e^{-\kappa\operatorname{dist}(J,y)} \le \sum_{z\in J} e^{-\kappa\operatorname{dist}(z,y)}$, it suffices that $\sum_y e^{-\kappa\operatorname{dist}(z,y)} \le \sigma(\kappa)$ on the torus, which holds because each block of the torus has a preimage in $\mathbb{Z}^4$ at ℓ^∞ distance $\operatorname{dist}(z,y)$ from a preimage of z, distinct blocks having distinct preimages. The closed form counts the $(2k+1)^4 - (2k-1)^4 = 64k^3 + 16k$ points of $\mathbb{Z}^4$ at distance $k \ge 1$ and sums $\sum kx^k$ and $\sum k^3 x^k$. (iii) In the second bound of Corollary 492, $n_q \le 45(2s+7)^4 |I_q|$, since every x_q-link is based in a block at distance at most $s+3$ from I_q, so $\sqrt{n_q} \le \sqrt{45}(2s+7)^2|I_q|$; and $(w_{uq}-2s-8)_+ \ge \operatorname{dist}(I_u,I_q) - 2s - 9$, while $|I_q| e^{-\kappa\operatorname{dist}(I_u,I_q)}$ is the term of $\sum_{y\in I_q}|I(y)|e^{-\kappa\operatorname{dist}(I_u,y)}$ at a block of I_q nearest to I_u. For the numbers, $x \le 1$ and $1-x \ge \kappa - \kappa^2/2$ give $\sigma(\kappa) \le 1 + 384(\kappa-\kappa^2/2)^{-4} + 16(\kappa-\kappa^2/2)^{-2} \le 6.151\cdot 10^{15}$ at $\kappa = 5\cdot 10^{-4}$, and $N_1 = 21^4 - 1 = 194480$, $N_5 = 37^4 - 1 = 1874160$, so that $2\cdot 6.151\cdot 10^{15}/N_1^2 \le 3.253\cdot 10^5$ and $2\cdot 6.151\cdot 10^{15}/N_5^2 \le 3.503\cdot 10^3$. □

The proposition is annealed: it bounds the mean of the island sums under μ_L for a fixed set J, while Theorem 491 is a statement given all the labels, and it says nothing about the law of the islands near I_u given that I_u is an island. At $\Gamma = 1$ its bound exceeds one, because the weighted volume $\sigma(2\alpha_G) = 6.1\cdot 10^{15}$ of the decay length $1/(2\alpha_G) = 2000$ blocks of Theorem 485 exceeds N_s^2; for $\Gamma \ge 2$ the bound of (ii) is at most $|J|\sigma(\kappa)/(\Gamma N_s^2)$, and Theorem 465(iii) reaches every Γ at the coupling $\beta^*(\rho,s,\Gamma)$.

With Theorem 491 the comparison of item (ii) of Sec. XIII V holds with any number of islands, given the labels of all of them: the cross-ratio of two islands is at most Δ_*, explicit in the two-island quantities Δ_{uv} of Corollary 486, and the observables of one island change by a factor in $[e^{-\Delta_*}, e^{\Delta_*}]$ under a flip of the other. The prefactor Λ_2 is bounded by Lemma 487, whose proof uses only the plaquettes at one collar and applies to each island I_u when $r \le 8\cdot 10^{-5}$. Two things remain. The Dobrushin condition of the second form and the smallness of S_u or T_u are conditions on the configuration of the islands, which Proposition 493 controls only in the mean. And the covariance of the two labels with the other labels averaged out differs from the conditional covariance by the covariance of the conditional probabilities of the two labels given the others, which is the term of Proposition 455 and is not bounded here.

The first of them, the smallness of the island sums together with the Dobrushin condition, holds in every window of bounded size outside an event whose probability is at most the size of the window times the product of two rarities: that of one bad block and that of a second bad block or island. Two facts give this. The bound of Proposition 493(i) charges an island of n blocks the factor $A^{n/H}$, while Lemma 450 gives $A^{\lceil n/H\rceil}$, which is A for $n \le H$; with the ceiling, the mean mass of the island of a block is at most G_0 below, about H times the rarity of one bad block, instead of $2N_s^{-2}$. And two bad blocks at distance at least six have disjoint witness boxes, so that two islands near each other cost the product of two rarities. For two islands $I_u \ne I_v$ of the torus, formed with the joining distance

$2s+8$, put

$$\bar n_u = 45\,\big|\{y : \mathrm{dist}(y, I_u) \le s+3\}\big|,$$
$$\bar\Lambda_2 = 99.0001\,\beta_W + 4.65\cdot 10^6,$$
$$\bar\Delta_{uv} = \frac{1920}{\beta_W}\bar\Lambda_2^2 r^2 \sqrt{\bar n_u \bar n_v}\, e^{-2\alpha_G(w_{uv}-2s-8)},$$

and, for $0 \le \vartheta < 1$, $\bar T_u = \sum_v \bar\Delta_{uv} e^{2\vartheta\alpha_G(w_{uv}-2s-8)}$, the sum over all islands $I_v \ne I_u$ of the torus. These are functions of the labels alone. In the setting of Theorem 491, when the sets I_u are islands, $n_u \le \bar n_u$, Lemma 487, applied to each island as above, gives $\Lambda_2 \le \bar\Lambda_2$, and $\ell(u,v) \le w_{uv} - 2s - 8$; hence $\Delta_{uv} \le \bar\Delta_{uv}$, and $S_u \le \bar T_u$ and $T_u \le \bar T_u$ for the T_u of Corollary 492 with the same ϑ. Write W^{+t} for the set of blocks at distance at most t from a set W of blocks.

Theorem 494 (the island sums in a window)**.** *Let $G = \mathrm{SU}(2)$, $s \ge 1$, $2758\rho \le 6.0\cdot 10^{-13}$, $r = 43\rho$, $\eta = 92\rho$, form the islands of the torus with the joining distance $2s+8$, and let $N_s = (4s+17)^4 - 1$. Let β^* be the largest solution of the equation of Theorem 465 with N_s in place of Δ, $\log q^* = -H(\log 2 + 2\log N_s) - \log(eMH)$ its right side, C_0 that of Lemma 382, $\beta_W \ge \beta^*$, and*

$$G_0 = H\Big(Mq^* + \frac{A^* 2^{-H}}{1-2^{-H}}\Big), \qquad A^* = eMHq^*.$$

(i) *For every block y, $\mathbb{E}_{\mu_L}|I(y)| \le G_0$, and Proposition 493(ii) holds at $\Gamma = 1$ with $2\Gamma(2\Gamma-1)^{-2}N_s^{-2}$ replaced by G_0.*

(ii) *Let $0 \le \vartheta < 1$, $\kappa = 2(1-\vartheta)\alpha_G$, let $R \ge 2s+13$ be an integer, $\sigma > 0$, W a set of blocks,*

$$c_2 = \frac{1920}{\beta^*}\bar\Lambda_2(\beta^*)^2 r^2 \cdot 45(2s+12)^2(2s+7)^2 e^{\kappa(2s+14)},$$
$$\tau = \min\big\{\sigma(\kappa),\, e^{-\kappa R/2}\sigma(\kappa/2)\big\},$$

with $\bar\Lambda_2(\beta^*)$ the value of $\bar\Lambda_2$ at $\beta_W = \beta^*$ and $\sigma(\cdot)$ as in Proposition 493. Outside an event of μ_L-probability at most

$$\varepsilon = |W| Mq^* \Big(11^4(2R+1)^4 Mq^* + \frac{c_2 G_0 \tau}{\sigma}\Big),$$

every island I_u meeting W has ℓ^∞ diameter at most 5, hence at most 6^4 blocks, lies at distance more than R from every other island, and satisfies $\bar T_u \le \sigma$.

(iii) *Outside the same event, in the setting of Theorem 491 with I_a, I_b and every I_q islands meeting W: $S_a, S_b \le \sigma$ and $\Delta_* \le \Delta_{ab} + \frac14\sigma^2 \le \sigma + \frac14\sigma^2$; if $\sigma < 4$, the weighted Dobrushin condition of Corollary 492 holds with $\theta \le \sigma/4$, and*

$$\Delta_* \le \Delta_{ab} + \frac{e^{-2\vartheta\alpha_G\ell(a,b)}\sigma^2}{4-\sigma} \le \sigma + \frac{\sigma^2}{4-\sigma}.$$

At $\rho = 2.17\cdot 10^{-16}$ and $s = 1$: $\beta^* \in [1.1220, 1.1221]\cdot 10^{37}$, $|\log q^*| = 32481.67$, $Mq^* \le 10^{-14102.4}$, the bound A of Theorem 465(i) is at most $10^{-14098.8}$, $G_0 \le 10^{-14099.3}$, and the bound of (i) for Proposition 493(ii) at $\kappa = 2\alpha_G$ is $G_0\sigma(2\alpha_G)|J| \le 10^{-14083.5}|J|$, in place of $3.253\cdot 10^5|J|$. For $\vartheta = 0$, $c_2 \le 10^{22.13}$; with $R = 15$, $\varepsilon \le |W|(10^{-28194} + 10^{-28163.8}/\sigma)$, and with $R = 2\cdot 10^8$, $\varepsilon \le 10^{-28166}|W|$ for every $\sigma \in [10^{-1000}, 4)$; for $\vartheta = \frac12$ and $R = 4\cdot 10^8$, $\varepsilon \le 10^{-28165}|W|$ on the same range. At $s = 5$: $\beta^* \in [1.3216, 1.3217]\cdot 10^{37}$, $Mq^* \le 10^{-16652.7}$, $G_0 \le 10^{-16649.6}$, and with $\vartheta = 0$, $R = 2\cdot 10^8$, $\varepsilon \le 10^{-33267}|W|$ for $\sigma \in [10^{-1000}, 4)$.

Proof. Write $p = q_\rho(\beta_W)$ and B for the set of bad blocks, and let Q_y be the set of unit cubes with corner at ℓ^∞ distance at most five from $2y$. By the witness paragraph of the proof of Theorem 465, which applies since $\rho \le 10^{-8}$, a bad block y has a cube of Q_y on which the event of Theorem 383 holds; $|Q_y| \le M$, and a cube lies in at most H of the sets Q_y. For a set S of blocks put $Q_S = \bigcup_{y\in S} Q_y$.

Step 1: separated sets. Let $S_1, \dots, S_m$ be nonempty sets of blocks with $\mathrm{dist}(S_i, S_j) \ge 6$ for $i \ne j$, and $k_i = \lceil |S_i|/H \rceil$. Then

$$\mu_L\Big(\bigcup_i S_i \subset B\Big) \le \prod_i \binom{M|S_i|}{k_i} p^{k_i} \le \prod_i A^{k_i},$$

and the factor of a set of one block is Mp. Two blocks at distance at least six have a coordinate in which the corners $2y$ differ by at least twelve modulo the side of the torus, while a cube of $Q_y \cap Q_{y'}$ would put $2y$ and $2y'$ within ten of each other in every coordinate; so the sets Q_{S_i} are pairwise disjoint. On the event, Q_{S_i} contains at least k_i distinct bad cubes, since one cube witnesses at most H blocks; choosing k_i of them in each Q_{S_i} gives $\sum_i k_i$ distinct bad cubes, and the union bound over the choices with Theorem 383 gives the first inequality. The second is that of Lemma 450.

Step 2: the mass of an island. Let $u = N_s^{2H}A$, which is at most 2^{-H} by Theorem 465(i), and let T be a set of blocks at distance at least 6 from a block y, or empty. Let $n \ge 1$ and $k = \lfloor (n-1)/H \rfloor$. If $|I(y)| \ge n$, a spanning tree of $I(y)$ in the graph of the joining distance, cut at $kH+1$ vertices from y, gives a connected set $C \ni y$ of exactly $kH+1$ bad blocks; there are at most N_s^{2kH} such sets, since a walk around a spanning tree of C from y has $2kH$ steps. By Step 1, $\mu_L(T \subset B, C \subset B) \le \mu_L(T \subset B)' A^{k+1}$ when T is at distance at least 6 from C, with $\mu_L(T \subset B)'$ the factor of Step 1 for T, so

$$\mu_L\big(T \subset B, |I(y)| \ge n\big) \le \mu_L(T \subset B)' \min\{Mp, Au^k\}. \quad (44)$$

Summing over n in groups of H, with Mp for $n \le H$ and Au^k for $kH < n \le (k+1)H$,

$$\mathbb{E}\big[\mathbf{1}_{T\subset B}|I(y)|\big] \le \mu_L(T \subset B)' H\Big(Mp + \frac{Au}{1-u}\Big) \le \mu_L(T \subset B)' G_0,$$

since $p \le q^*$ and $A \le A^*$ by Step 6. With T empty, $\mathbb{E}|I(y)| \le G_0$. The proof of Proposition 493(ii) uses only this mean, which gives (i).

Step 3: no near pair. Let $\mathscr{G}$ be the event that no two bad blocks $y \in W^{+5}$ and y' satisfy $6 \le \mathrm{dist}(y, y') \le R$. On $\mathscr{G}$ let I_u be an island meeting W at y_0. Every block of I_u is within distance

5 of y_0: otherwise a path $y_0 = z_0, z_1, \dots$ in I_u with steps at most $2s+8$ has a first block z_k at distance more than 5 from y_0, and $6 \le \mathrm{dist}(y_0, z_k) \le 5 + 2s + 8 \le R$. So $I_u \subset W^{+5}$, two blocks of I_u are at distance at most $10 < R$, hence at most 5, and I_u lies in a box of side 6. If another island I_q had $\mathrm{dist}(I_u, I_q) \le R$, a pair realising this distance would consist of a bad block of W^{+5} and a bad block at distance between $2s+9 \ge 6$ and R. This proves the first two assertions of (ii) on $\mathscr{G}$.

Step 4: the island sums on $\mathscr{G}$. For $y \in B$ put

$$F(y) = \sum_{y'} |I(y')| e^{-\kappa \mathrm{dist}(y,y')},$$

the sum over $y' \in B \setminus I(y)$ with $\mathrm{dist}(y,y') > R$, and let $c_2(\beta_W)$ be c_2 with β_W in place of β^*. On $\mathscr{G}$, let I_u meet W at y. For an island $I_q \ne I_u$ let $y_q \in I_q$ realise $\mathrm{dist}(I_u, I_q)$; then $\mathrm{dist}(y, y_q) \le \mathrm{dist}(I_u, I_q) + 5$, $\mathrm{dist}(y, y_q) > R$ by Step 3, and $w_{uq} - 2s - 8 = \mathrm{dist}(I_u, I_q) - 2s - 9 \ge \mathrm{dist}(y, y_q) - 2s - 14$. The box of side $2s+12$ around I_u contains every block at distance at most $s+3$ from it, so $\bar n_u \le 45(2s+12)^4$, and $\bar n_q \le 45(2s+7)^4 |I_q|$ as in the proof of Proposition 493(iii). With $\sqrt{|I_q|} \le |I_q|$ and distinct islands giving distinct y_q,

$$\bar T_u = \frac{1920}{\beta_W} \bar\Lambda_2^2 r^2 \sqrt{\bar n_u} \sum_q \sqrt{\bar n_q}\, e^{-\kappa(w_{uq} - 2s - 8)} \le c_2(\beta_W) F(y).$$

Step 5: probabilities. By Step 1 with two single blocks, at most $11^4 |W|$ blocks of W^{+5} and at most $(2R+1)^4$ blocks within distance R of each,

$$\mu_L(\mathscr{G}^c) \le 11^4 (2R+1)^4 |W| (Mp)^2.$$

For $y \in W$, $\mathbb{E}[\mathbf{1}_{y \in B} F(y)]$ is a sum over y' with $\mathrm{dist}(y,y') > R$ of $e^{-\kappa \mathrm{dist}(y,y')}$ times a sum over the connected sets $C' \ni y'$ with $y \notin C'$ of $|C'| \mu_L\{y \in B, I(y') = C'\}$. On that event y and $I(y')$ lie in different islands, so y is at distance at least $2s+9 \ge 6$ from every subset of $I(y')$, and Step 2 with $T = \{y\}$, whose factor in Step 1 is Mp, bounds the sum over C' by $Mp G_0$. The sum over y' is at most $\sum_z e^{-\kappa \|z\|_\infty}$ over $z \in \mathbb{Z}^4$ with $\|z\|_\infty > R$, by the preimages of the proof of Proposition 493(ii), which is at most $\sigma(\kappa)$ and, by $e^{-\kappa\|z\|_\infty} \le e^{-\kappa R/2} e^{-\kappa \|z\|_\infty/2}$ there, at most $e^{-\kappa R/2} \sigma(\kappa/2)$. So $\mathbb{E}[\mathbf{1}_{y \in B} F(y)] \le Mp G_0 \tau$, and by Step 4, Markov's inequality and a union bound over W,

$$\mu_L\big(\mathscr{G} \cap \{\bar T_u > \sigma \text{ for some } I_u \text{ meeting } W\}\big) \le \frac{c_2(\beta_W)}{\sigma} |W| Mp G_0 \tau.$$

Step 6: the coupling. By Theorem 383, $p \le e^{f(\beta_W)}$ with f of Theorem 465, which decreases on $[96/\rho^2, \infty)$, so $p \le e^{f(\beta^*)} = q^*$. Since $\bar\Lambda_2 \ge 99.0001\,\beta_W$, the logarithmic derivative of $c_2(\beta) e^{f(\beta)}$ is at most $2/\beta - 1/\beta + 6/\beta - \rho^2/16$, which is negative for $\beta \ge \beta^*$ because $\beta^* \rho^2/16 = |\log q^*| + 6(\log \beta^* + C_0) > 7$. Hence $c_2(\beta_W) p \le c_2 q^*$, and Step 5 gives the bound ε on the complement of the event of (ii).

Step 7: the consequences. Outside that event, $\Delta_{ab} \le \bar\Delta_{ab} \le \bar T_a \le \sigma$, since $\bar\Delta_{ab}$ is a term of $\bar T_a$ with a factor at least one, and $S_a, S_b \le \sigma$, which gives the first form of Δ_* in Theorem 491. By $\tanh t \le t$ and $\ell(q,q') \le w_{qq'} - 2s - 8$, the sum of Corollary 492 at q is at most $\frac14 \bar T_q \le \frac14 \sigma$, so $\theta \le \sigma/4 < 1$, and the corollary with $T_a, T_b \le \sigma$ gives the last display.

The numbers. Here $H = 6^4$, $M = 11^4$ and $|\log q^*| = H(\log 2 + 2\log N_s) + 1 + 4\log 11 + 4\log 6$, with $N_1 = 21^4 - 1 = 2^4 \cdot 5 \cdot 11 \cdot 13 \cdot 17$ and $\log N_5 = 4\log 37 + \log(1 - 37^{-4})$, where $-x/(1-x) \le \log(1-x) \le -x$. The logarithms of the proof of Theorem 465, together with $\log \frac{13}{12} = 2\,\mathrm{artanh}\,\frac{1}{25}$, $\log \frac{17}{16} = 2\,\mathrm{artanh}\,\frac{1}{33}$ and $\log \frac{37}{36} = 2\,\mathrm{artanh}\,\frac{1}{73}$, give $|\log q^*|$ within 10^{-2} of 32481.67 for $s = 1$ and of 38354.07 for $s = 5$. The function $g(\beta) = \beta\rho^2/16 - 6(\log\beta + C_0) - |\log q^*|$ increases for $\beta > 96/\rho^2$, and at the ends of the two stated intervals it is negative and positive, with margins above 0.5. With $\log 10 = \log 2 + \log 5$, $\log_{10}(Mq^*) = (4\log 11 - |\log q^*|)/\log 10$ lies within 10^{-3} of -14102.446 for $s = 1$ and of -16652.797 for $s = 5$, and $\log_{10}(eH) \le 3.547$ gives $A \le A^* \le 10^{-14098.8}$ at $s = 1$. Since $2eH 2^{-H} \le 10^{-300}$, $G_0 \le HMq^*(1 + 10^{-300})$, and $\log_{10} H \le 3.1127$ gives the two bounds on G_0; with $\sigma(2\alpha_G) \le 6.151 \cdot 10^{15}$ from Proposition 493, $G_0 \sigma(2\alpha_G) \le 10^{-14083.5}$.

By Lemma 487, $\frac{1920}{\beta^*} \bar\Lambda_2(\beta^*)^2 r^2 = 1920 \beta^* r^2 (99.0001 + 4.65 \cdot 10^6/\beta^*)^2$, which increases with β^* and is therefore at most its value at the upper end of the interval for β^*: at most $10^{16.2645}$ for $s = 1$ and $10^{16.3356}$ for $s = 5$. The remaining factors of c_2 at $\kappa = 2\alpha_G = 5 \cdot 10^{-4}$ give $c_2 \le 10^{22.122}$ for $s = 1$ and $c_2 \le 10^{23.140}$ for $s = 5$, and c_2 is smaller at $\kappa = \alpha_G$. For the sums σ, write $x = 1 - v$; then $x(1 + 4x + x^2) = 6 - 12v + 7v^2 - v^3$, the function $(6 - 12v + 7v^2) v^{-4}$ decreases for $v > 0$, and $v \ge u = \kappa - \kappa^2/2$, so the closed form of Proposition 493 gives $\sigma(\kappa) \le 1 + 64(6 - 12u + 7u^2) u^{-4} + 16u^{-2}$, hence $\sigma(\alpha_G) \le 9.831 \cdot 10^{16}$ and $\sigma(\alpha_G/2) \le 1.573 \cdot 10^{18}$.

In ε, the first term has $\log_{10}(11^4 (2R+1)^4 (Mq^*)^2)$ at most -28194.76 for $R = 15$, -28166.31 for $R = 2 \cdot 10^8$ and -28165.11 for $R = 4 \cdot 10^8$ at $s = 1$, and -33267.01 for $R = 2 \cdot 10^8$ at $s = 5$. For $R = 15$, $\tau \le \sigma(2\alpha_G)$ and $Mq^* c_2 G_0 \tau \le 10^{-28163.86}$. For $R = 2 \cdot 10^8$ with $\kappa = 2\alpha_G$, and for $R = 4 \cdot 10^8$ with $\kappa = \alpha_G$, $\kappa R/2 = 5 \cdot 10^4$ and $e^{-5 \cdot 10^4} \le 10^{-21714}$, so $\tau \le 10^{-21714} \sigma(\kappa/2)$ and the second term is at most $10^{-48000} |W|$ for $\sigma \ge 10^{-1000}$, which the margins of the first term absorb. □

The bound is a union bound over the window and is linear in $|W|$; for $|W| \le 10^{28000}$ it is at most 10^{-166}. Its content lies in the windows where Theorem 465 does not exclude islands: the probability that W contains a bad block at all is at most $|W| A \le 10^{-14098.8} |W|$, so for $10^{14099} \le |W| \le 10^{28000}$ that theorem does not exclude bad blocks in W, and Theorem 494 shows that, outside probability 10^{-166}, every island they form is a cluster of diameter at most 5, at distance more than $2 \cdot 10^8$ from every other island, and meets the conditions of Theorem 491 and Corollary 492 with $\sigma = 10^{-1000}$. No coupling beyond the threshold β^* of the joining distance $2s+8$ and no factor $\Gamma > 1$ is needed: the weighted volume $\sigma(2\alpha_G) = 6.1 \cdot 10^{15}$ of the decay length is beaten by $G_0 \le 10^{-14099.3}$. The bound is void for windows of more than 10^{28166} blocks, and it is a statement about the labels under μ_L, not about the law of the labels near I_u given that I_u is an island; the covariance of two labels with the others averaged is bounded in the mean over the coarse field in Sec. XIII O, and for a fixed coarse field outside a set of

small probability by Proposition 503, but not uniformly in the coarse field.

O. The covariance of the labels

Theorem 491 compares two islands given the labels of all the others, and Theorem 494 controls the island sums in a window by a union bound. This subsection bounds the covariances with the labels averaged, on the whole torus and with no condition on the size of a window, in the mean over the coarse field. Three constructions do it. The islands are grouped: the islands near an observable form one group, the rest a second, and the mixed derivative of the kernel of Lemma 488 between the two groups is bounded by one application of (41), with no Dobrushin condition on the islands. Two independent copies of the labels are compared by an explicit involution, which exchanges the labels of the group of one observable between the copies; the weights of a configuration and of its image differ by the cross-ratio of the two groups. And a group that reaches far from its observable contains a chain of bad blocks in one copy or the other, whose probability is paid by the separated rarity of Theorem 494. For two functions of the labels near sets N_A and N_B the covariance is at most $2\varepsilon|N_A||N_B|$ times their oscillations, with ε the rarity of a bad block defined below, since each sees a bad block; the theorem multiplies this by a factor that decays at the rate $2\alpha_G$ of Theorem 485 down to a floor of order 10^{-7011}. For two local observables of the field the covariance under μ^c, in the mean over c, decays at the same rate down to a floor of order 10^{-14062}, and as $e^{-\sqrt{d}/2}$ when the joining distance of the groups grows with the distance.

Setting. Throughout, the hypotheses of Theorem 494 hold: $G=\mathrm{SU}(2)$, $s\ge 1$, $2758\rho\le 6.0\cdot 10^{-13}$, $r=43\rho$, $\eta=92\rho$, $\beta_W\ge\beta^*$, on the torus in the tree gauge, with Λ the set of all blocks and no exterior; hypotheses (a)–(c) of Theorem 485 then hold, (c) being void. For a coarse field c let μ^c be μ_L conditioned on $V^\sharp=c$, through the coarea formula as in Proposition 412, let P^c be the law of the labels under μ^c, and let $\mathbb{E}_c$ be the mean over the law of $V^\sharp$ under μ_L, so that $\mathbb{E}_c\mu^c=\mu_L$. A labelling ω assigns a label to every block, $B(\omega)$ is its set of bad blocks, ω_X its restriction to a set X of blocks, and $Z^c(\omega)$ the weight of the event that the labels are ω, defined as Z_{ij} in Sec. XIII M, so that $P^c(\omega)=Z^c(\omega)/\sum_{\omega'}Z^c(\omega')$. With f, M, q^* of Theorems 465 and 494 put

$$\varepsilon=Me^{f(\beta_W)}\le Mq^*,\qquad g_R=\sqrt{2}\,(2R+1)^4\varepsilon^{1/4},$$

so that, by Steps 1 and 6 of the proof of Theorem 494, $\mu_L(T\subset B)\le\varepsilon^{|T|}$ for every set T of blocks pairwise at distance at least 6. With $\bar\Lambda_2$ of Theorem 494 put

$$b_*=\frac{1920}{\beta_W}\bar\Lambda_2^2r^2,\qquad \nu_s=45(2s+9)^4.$$

For a set Y of blocks, its *components* are the classes of the joining distance $2s+8$ and its *R-clusters* those of the joining distance $R\ge 2s+8$; an R-cluster is a union of components. For disjoint sets X,X' of blocks put

$$\Delta(X,X')=b_*\nu_s|X|^{1/2}\Big(\sum_{y\in X'}e^{-4\alpha_G(\mathrm{dist}(y,X)-2s-9)_+}\Big)^{1/2},$$

and for integers R,t and $d>0$

$$M_{d,t}=b_*\nu_s e^{-2\alpha_G(d-t-2s-9)_+},$$
$$\Phi_R=b_*\nu_s\sigma(\alpha_G)\varepsilon^{1/2}e^{-\alpha_G(R-4s-17)},$$

with $\sigma(\kappa)=\sum_{y\in\mathbb{Z}^4}e^{-\kappa\|y\|_\infty}$ of Proposition 493; by the proof of its part (ii), $\sum_y e^{-\kappa\,\mathrm{dist}(W,y)}\le|W|\sigma(\kappa)$ for every set W of blocks.

Lemma 495 (two groups of islands)**.** *Let Y be a set of blocks, X a union of components of Y and $X'=Y\setminus X$. Let D_X be the product of the sets D_u of Sec. XIII N over the components $I_u\subset X$, and $D_{X'}$ likewise. There are a smooth function $\tilde K: D_X\times D_{X'}\to(0,\infty)$ and, for every labelling π of X and π' of X', finite positive measures a_π on D_X and $b_{\pi'}$ on $D_{X'}$, such that for all labellings ω with $B(\omega)\subset Y$:*

(i) $Z^c(\omega)=\iint\tilde K(x,x')\,a_{\omega_X}(dx)\,b_{\omega_{X'}}(dx')$;

(ii) *for all $x_0,x_1\in D_X$ and $x_0',x_1'\in D_{X'}$,*

$$\Big|\log\frac{\tilde K(x_1,x_1')\tilde K(x_0,x_0')}{\tilde K(x_1,x_0')\tilde K(x_0,x_1')}\Big|\le\Delta:=\min\{\Delta(X,X'),\Delta(X',X)\};$$

(iii) *for $Z^c(\omega)>0$ and bounded measurable F of the active links based in X and H of those based in X', there are measurable f on D_X, depending on F and ω_X only, and h on $D_{X'}$, depending on H and $\omega_{X'}$ only, with $|f|\le\|F\|$, $|h|\le\|H\|$, $\mathbb{E}_{\mu^c}[FH\mid\omega]=\int fh\,d\Pi_\omega$ and $\mathbb{E}_{\mu^c}[F\mid\omega]=\int f\,d\Pi_\omega$, where Π_ω is the probability law proportional to $\tilde K\,a_{\omega_X}\otimes b_{\omega_{X'}}$.*

Consequently, for $\omega,\tilde\omega,\omega'$ of positive weight with bad blocks in Y:

(a) *if $\tilde\omega$ agrees with ω on X, the conditional expectations of F given $\tilde\omega$ and given ω differ by at most $2\|F\|\tanh(\Delta/4)$;*

(b) $|\mathrm{Cov}_{\mu^c}(F,H\mid\omega)|\le\|F\|\|H\|\tanh(\Delta/4)$;

(c) *the labellings $\omega^\natural=(\omega_X',\omega_{X'})$ and $\omega'^\natural=(\omega_X,\omega_{X'}')$ have positive weight and*

$$e^{-\Delta}\le\frac{Z^c(\omega^\natural)Z^c(\omega'^\natural)}{Z^c(\omega)Z^c(\omega')}\le e^{\Delta}.$$

Proof. The kernel. Distinct components of Y are at distance at least $2s+9$, so Lemma 488 applies to the components I_u of Y, every block outside Y being good in every labelling considered. Its proof runs with any labelling of each I_u: the labels change only the u-factors, so the product of the K-factors, and with it $\tilde K(x)=\int_{D_z}e^{-\mathscr{U}(z;x)}dz$, does not depend on the labels of Y, while the measure γ_u obtained by integrating the u-factors over L_u and p_u depends on the labels of I_u, on c and on β_W only.

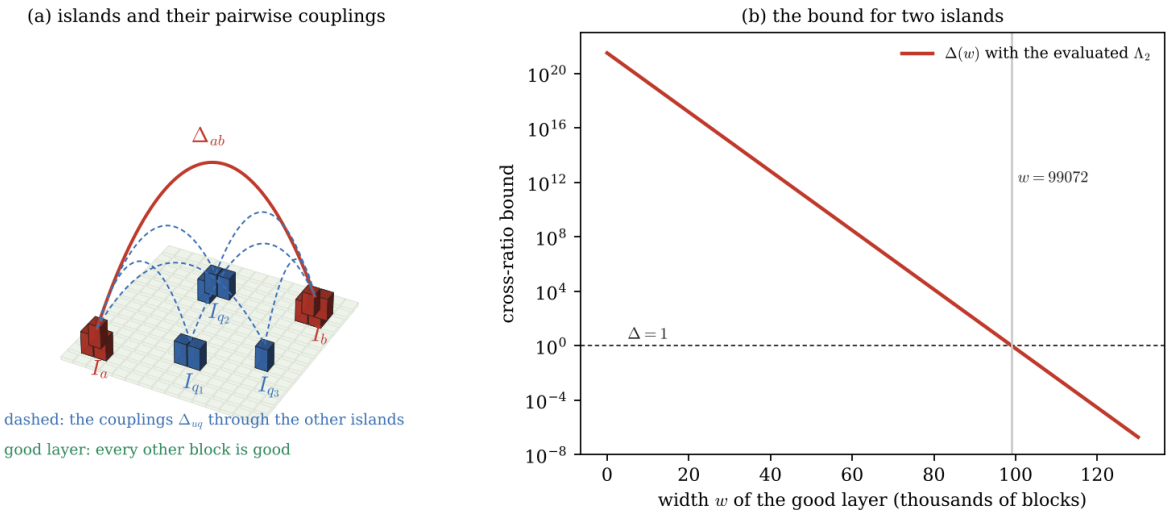


FIG. 40. (a) Islands of bad blocks (red: the two islands I_a, I_b compared; blue: other islands I_q) in a window whose remaining blocks are good. The solid arc is the two-island bound Δ_{ab} of Corollary 486; the dashed arcs are the bounds Δ_{uq} through which the other islands enter Δ_* in Theorem 491. (b) The bound Δ for two islands against the width w of the good layer between them, with Λ_2 from Lemma 487 at $\beta_W = 1.13 \cdot 10^{37}$; it falls below one at $w = 99072$.

Put $a_\pi = \bigotimes_{I_u \subset X} \gamma_u$ and $b_{\pi'} = \bigotimes_{I_u \subset X'} \gamma_u$ with the labels π, π'; this is (i). In the same construction, given x and ω, the families $L_u \cup p_u$ are independent over u, each with law proportional to the product of its u-factors, and independent of z. The active links based in X are at $d_u = 0$ for some $I_u \subset X$, hence in L_u; so $f(x) = \mathbb{E}[F \mid x, \omega]$ is the integral of F against the product of these laws over $I_u \subset X$, a function of x and ω_X, and likewise h; this is (iii).

The mixed derivative. Let h_X, $h_{X'}$ be vectors in the coordinates of $x \in D_X$ and $x' \in D_{X'}$ with $\|h_e\| \le 2r$ for each link. No term of $\mathscr{U}$ depends on coordinates of two components, by Lemma 488(ii), so Lemma 454 gives

$$\partial_{h_X} \partial_{h_{X'}} \log \tilde{K} = \mathrm{Cov}_{\mu_{x,x'}} \left(\partial_{h_X} \mathscr{U}, \partial_{h_{X'}} \mathscr{U} \right).$$

Let $v = \nabla_z \partial_{h_X} \mathscr{U}$ and $v' = \nabla_z \partial_{h_{X'}} \mathscr{U}$, supported on sets S and S' of z-links based at distance at most $s+4$ from X, respectively X'. The supports of distinct components lie in blocks at coarse distance at least 1, so the coordinates entering v_e belong to one component, and Step 1 of the proof of Lemma 487, which applies to each component, bounds $\sum_f \|\partial_e \partial_f \mathscr{U}\|$ over them by $\beta_W c_W + c_\psi \le \bar{\Lambda}_2$. Hence $\|v_e\| \le 2r\bar{\Lambda}_2$, and $\|v'_e\| \le 2r\bar{\Lambda}_2$. Apply (41), with $\mathscr{A} = 480/\beta_W$ and $\alpha = \alpha_G$ by Lemma 489, to $\varphi(e) = -\mathrm{dist}(e, S)$, the fine distance between link midpoints. A block carries 45 non-path active links, so $|S| \le \nu_s |X|$. A link $e \in S'$ is based in a block y with $\mathrm{dist}(y, y') \le s+4$ for some $y' \in X'$, and a link of S in a block y'' with $\mathrm{dist}(y'', X) \le s+4$; then $\mathrm{dist}(y, y'') \ge \mathrm{dist}(y', X) - 2s - 8$, and links based in blocks at coarse distance $D \ge 1$ have midpoints at fine distance at least $2D-2$, so $\mathrm{dist}(e, S) \ge 2(\mathrm{dist}(y', X) - 2s - 9)_+$. Summing over y', the $(2s+9)^4$ blocks y and their 45 links,

$$\left| \partial_{h_X} \partial_{h_{X'}} \log \tilde{K} \right| \le \frac{480}{\beta_W} (2r\bar{\Lambda}_2)^2 \left(\nu_s |X| \right)^{1/2} \times \left(\nu_s \sum_{y' \in X'} e^{-4\alpha_G (\mathrm{dist}(y', X) - 2s - 9)_+} \right)^{1/2},$$

which is $\Delta(X, X')$; with $\varphi = -\mathrm{dist}(\cdot, S')$ the same argument gives $\Delta(X', X)$. The sets D_X, $D_{X'}$ are convex, two of their points differ by at most $2r$ in each link, and the logarithm of the cross-ratio is the integral of the mixed derivative over the unit square; this is (ii).

The consequences. Let $\mu' = k\mu / \mu(k)$ with $\sup k \le e^\kappa \inf k$; the tilt paragraph of the proof of Lemma 490 gives $\|\mu' - \mu\|_{\mathrm{TV}} \le \tanh(\kappa/4)$. (a) Under Π_ω the law of x has density proportional to $k_\omega(x) = \int \tilde{K}(x, x') b_{\omega_{X'}}(dx')$ with respect to a_{ω_X}, and $a_{\tilde{\omega}_X} = a_{\omega_X}$, f is the same for both. Integrating (ii) against $b_{\tilde{\omega}_{X'}} \otimes b_{\omega_{X'}}$ shows that $k_{\tilde{\omega}} / k_\omega$ has logarithmic oscillation at most Δ, so the two laws differ by at most $\tanh(\Delta/4)$ in total variation, and $\mathrm{osc}\, f \le 2\|F\|$. (b) Given x, the law of x' under Π_ω is proportional to $\tilde{K}(x, \cdot) b_{\omega_{X'}}$, and by (ii) these laws differ by at most $\tanh(\Delta/4)$ for two values of x; so $m(x) = \mathbb{E}_{\Pi_\omega}[h \mid x]$ has $\mathrm{osc}\, m \le 2\|H\| \tanh(\Delta/4)$. By (iii), $\mathrm{Cov}_{\mu^c}(F, H \mid \omega) = \mathrm{Cov}_{\Pi_\omega}(f, h) = \mathrm{Cov}_{\Pi_\omega}(f, m)$, and a variance is at most a quarter of the squared oscillation (Popoviciu [100]), so $|\mathrm{Cov}(f, m)| \le \frac{1}{4} \mathrm{osc}\, f\, \mathrm{osc}\, m$. (c) The measures of ω and ω' are nonzero since their weights are positive, and $\tilde{K} > 0$; Lemma 452 with $a_{\omega_X}, a_{\omega'_X}$ and $b_{\omega_{X'}}, b_{\omega'_{X'}}$ gives the bound. □

The second construction is a union bound over chains in two copies. Let $Q^c = P^c \otimes P^c$ be the law of two independent labellings ω, ω' given c, and $\mathsf{D} = B(\omega) \cup B(\omega')$.

Lemma 496 (chains in two copies)**.** *Let N and N' be sets of blocks, $R \geq 6$ and $t \geq R$ integers with* $\operatorname{dist}(N,N') > t$ *and $g_R \leq \frac{1}{2}$, and $\mathscr{C}$ the union of the R-clusters of $\mathsf{D} \cup N \cup N'$ that meet N. Then*

$$\mathbb{E}_c\, Q^c\{\mathscr{C} \not\subset N^{+t}\} \leq 2|N^{+R}| g_R^{\lfloor t/R \rfloor},$$

and the same holds with $B(\omega)$ in place of D.

Proof. A chain. Let $\mathscr{C} \not\subset N^{+t}$. There is a path $y_0,\dots,y_n$ in $\mathsf{D}\cup N\cup N'$ with steps at most R, $y_0 \in N$ and $\operatorname{dist}(y_n,N) > t$; let i be its first index with $\operatorname{dist}(y_i,N) > t$. The blocks $y_0,\dots,y_{i-1}$ are at distance at most t from N, hence not in N', and if $j < i$ is the last index with $y_j \in N$, then $j < i-1$, since $\operatorname{dist}(y_i,N) > t \geq R$. So $y_{j+1},\dots,y_{i-1}$ is a sequence of blocks of D with steps at most R, beginning in N^{+R} and ending at distance more than $t-R$ from N. Choose such a sequence $z_1,\dots,z_m$ in D with m least. Then $\operatorname{dist}(z_k,z_l) > R$ for $|k-l| \geq 2$, since otherwise a shorter sequence exists, and $(m-1)R > t-2R$, so $m \geq \lfloor t/R \rfloor$.

Two copies. Let T be the set of the $\lceil m/2 \rceil$ blocks z_k with k odd, pairwise at distance more than $R \geq 6$. The event $T \subset \mathsf{D}$ is the union over $T_1 \subset T$ of $\{T_1 \subset B(\omega), T \setminus T_1 \subset B(\omega')\}$, and for each T_1, since $P^c \leq 1$ and by the Cauchy–Schwarz inequality, with $T_2 = T \setminus T_1$,

$$\mathbb{E}_c\big[P^c(T_1 \subset B)\, P^c(T_2 \subset B)\big] \leq \big(\mu_L(T_1 \subset B)\, \mu_L(T_2 \subset B)\big)^{1/2} \leq \varepsilon^{|T|/2}.$$

So $\mathbb{E}_c Q^c\{T \subset \mathsf{D}\} \leq (2\varepsilon^{1/2})^{\lceil m/2 \rceil} \leq (\sqrt{2}\,\varepsilon^{1/4})^m$, as $2\varepsilon^{1/2} \leq 1$. There are at most $|N^{+R}|(2R+1)^{4(m-1)}$ sequences of m blocks with steps at most R beginning in N^{+R}, so

$$\mathbb{E}_c Q^c\{\mathscr{C} \not\subset N^{+t}\} \leq |N^{+R}| \sum_{m \geq \lfloor t/R \rfloor} g_R^m \leq 2|N^{+R}| g_R^{\lfloor t/R \rfloor}.$$

With $B(\omega)$ in place of D the event is smaller. □

The third ingredient is the mean of Δ for a group confined near its observable. Let N, N', $R \geq 2s+9$, $t \geq R$ be as in Lemma 496, let $Y = \mathsf{B} \cup N \cup N'$ with B equal to D or to $B(\omega)$, let $\mathscr{C}$ be the union of the R-clusters of Y meeting N, and let k be 2 or 1 accordingly. On the event $\mathscr{C} \subset N^{+t}$, every $y \in Y \setminus \mathscr{C}$ lies in an R-cluster distinct from those of $\mathscr{C}$, so $\operatorname{dist}(y,\mathscr{C}) > R \geq 2s+9$, and $\operatorname{dist}(y,\mathscr{C}) \geq \operatorname{dist}(y,N^{+t})$; a block of N' has $\operatorname{dist}(y,\mathscr{C}) \geq \operatorname{dist}(N,N') - t$. For a block $y \in \mathsf{B} \setminus \mathscr{C}$ the exponent splits as

$$4\alpha_G(\operatorname{dist}(y,\mathscr{C}) - 2s - 9) \geq 2\alpha_G(R - 2s - 8) + 2\alpha_G(\operatorname{dist}(y,N^{+t}) - 2s - 9),$$

and with $\sqrt{\sum_y u_y} \leq \sum_y \sqrt{u_y}$,

$$\Big(\sum_{y \in Y \setminus \mathscr{C}} e^{-4\alpha_G(\operatorname{dist}(y,\mathscr{C}) - 2s - 9)_+}\Big)^{1/2} \leq |N'|^{1/2} e^{-2\alpha_G(\operatorname{dist}(N,N') - t - 2s - 9)_+} + e^{-\alpha_G(R-2s-8)} \sum_{y \in \mathsf{B}} e^{-\alpha_G(\operatorname{dist}(y,N^{+t}) - 2s - 9)}.$$

The mean of the last sum over $\mathbb{E}_c Q^c$, or $\mathbb{E}_c P^c$, is at most $k\varepsilon e^{\alpha_G(2s+9)} |N^{+t}| \sigma(\alpha_G)$, since $\mathbb{E}_c Q^c\{y \in \mathsf{D}\} \leq 2\mu_L(y \in B) \leq 2\varepsilon$ and $\mathbb{E}_c P^c\{y \in B\} \leq \varepsilon$. With $|\mathscr{C}| \leq |N^{+t}|$,

$$\mathbb{E}\big[\mathbf{1}_{\mathscr{C} \subset N^{+t}} \Delta(\mathscr{C}, Y \setminus \mathscr{C})\big] \leq |N^{+t}|^{1/2} |N'|^{1/2} M_{d',t} + k\varepsilon^{1/2} |N^{+t}|^{3/2} \Phi_R, \tag{45}$$

with $\mathbb{E}$ the mean over $\mathbb{E}_c Q^c$ or $\mathbb{E}_c P^c$ and $d' = \operatorname{dist}(N,N')$.

Theorem 497 (the label covariance)**.** *Let N_A, N_B be sets of blocks at distance $d \geq 6$, ϕ a function of the labels of N_A and ψ one of the labels of N_B. Then*

$$\mathbb{E}_c\big|\operatorname{Cov}_{P^c}(\phi,\psi)\big| \leq 2\varepsilon |N_A||N_B| \operatorname{osc}\phi \operatorname{osc}\psi.$$

If moreover $R \geq 2s+9$ and $t \geq 2R$ are integers with $d \geq t+6$ and $g_R \leq \frac{1}{2}$, then, with $V = |N_B^{+t}|$,

$$\mathbb{E}_c\big|\operatorname{Cov}_{P^c}(\phi,\psi)\big| \leq \operatorname{osc}\phi \operatorname{osc}\psi\, |N_A||N_B|\, \varepsilon \times \big(11^4 V^{1/2} M_{d-5,t} + 2V^{3/2}\Phi_R + 4(2R+1)^4 V g_R^{\lfloor t/R \rfloor - 2}\big).$$

Proof. Fix c. Let ω_0 be the labelling with every block good, and put $\hat\phi(\omega) = \phi(\omega) - \phi(\omega_0)$ and $\hat\psi(\omega) = \psi(\omega) - \psi(\omega_0)$. Then $|\hat\phi| \leq \operatorname{osc}\phi$, $\hat\phi(\omega_0) = 0$ unless $B(\omega) \cap N_A \neq \emptyset$, and likewise for $\hat\psi$ and N_B. Under Q^c, $\operatorname{Cov}_{P^c}(\phi,\psi) = \operatorname{Cov}_{P^c}(\hat\phi,\hat\psi) = \mathbb{E}_{Q^c} G$ with $G(\omega,\omega') = \hat\phi(\omega)(\hat\psi(\omega) - \hat\psi(\omega'))$, and

$$|G| \leq \operatorname{osc}\phi \operatorname{osc}\psi \sum_{z_A \in N_A, z_B \in N_B} J_{z_A z_B},$$

where $J_{z_A z_B} = \mathbf{1}[z_A, z_B \in B(\omega)] + \mathbf{1}[z_A \in B(\omega), z_B \in B(\omega')]$. As $\operatorname{dist}(z_A,z_B) \geq 6$, the mean of $J_{z_A z_B}$ over $\mathbb{E}_c Q^c$ is at most $\mu_L(z_A, z_B \in B) + \mathbb{E}_c[P^c(z_A \in B) P^c(z_B \in B)] \leq \varepsilon^2 + \varepsilon \leq 2\varepsilon$, by the Cauchy–Schwarz inequality and $P^c \leq 1$ in the second term; this is the first bound. For the second let $Y = \mathsf{D} \cup N_B$, $\mathscr{C}$ the union of its R-clusters meeting N_B, $\mathscr{G} = \{\mathscr{C} \subset N_B^{+t}\}$ and $\Delta_B = \Delta(\mathscr{C}, Y \setminus \mathscr{C})$.

The exchange. Define $\chi(\omega,\omega') = (\tilde\omega,\tilde\omega')$ by $\tilde\omega = \omega'$ and $\tilde\omega' = \omega$ on $\mathscr{C}$, and $\tilde\omega = \omega$, $\tilde\omega' = \omega'$ elsewhere. On every block the pair of labels is kept or exchanged, so D, Y, $\mathscr{C}$ and $\mathscr{G}$ are the same for (ω,ω') and its image, and χ is an involution. By Lemma 495(c) with $X = \mathscr{C}$, a union of components of Y, a pair of positive weight is mapped to one of positive weight, with

$$e^{-\Delta_B} \leq \frac{Q^c(\chi(\omega,\omega'))}{Q^c(\omega,\omega')} \leq e^{\Delta_B}.$$

On $\mathscr{G}$, $\mathscr{C} \cap N_A = \emptyset$, since $d > t$; so $\tilde\omega = \omega$ on N_A and $\tilde\omega = \omega'$, $\tilde\omega' = \omega$ on $N_B \subset \mathscr{C}$, and $G \circ \chi = -G$. Split $\mathbb{E}_{Q^c}[G\mathbf{1}_{\mathscr{G}}]$ into the orbits of χ. A fixed point has $G = 0$. An orbit $\{x, \chi x\}$ of two points contributes $(Q^c(x) - Q^c(\chi x))G(x)$, and $|1-\rho| \leq (1+\rho)\tanh(\Delta_B/2)$ for $\rho \in [e^{-\Delta_B}, e^{\Delta_B}]$. As Δ_B and $|G|$ are constant on the orbit, with $\tanh u \leq u$,

$$|\mathbb{E}_{Q^c} G| \leq \operatorname{osc}\phi \operatorname{osc}\psi \sum_{z_A, z_B} \mathbb{E}_{Q^c}\big[J_{z_A z_B}\big(\tfrac{1}{2}\mathbf{1}_{\mathscr{G}}\Delta_B + \mathbf{1}_{\mathscr{G}^c}\big)\big].$$

Fix $z_A \in N_A$ and $z_B \in N_B$; the two means are bounded in turn, after the mean $\mathbb{E}_c$.

The far islands. On $\mathscr{G}$, $|\mathscr{C}| \le V$ and $Y \setminus \mathscr{C} \subset \mathsf{D}$, each of its blocks at distance more than R from $\mathscr{C}$, hence from z_B. By $\sqrt{\sum_y u_y} \le \sum_y \sqrt{u_y}$,

$$\mathbf{1}_{\mathscr{G}} \Delta_B \le b_* \nu_s V^{1/2} \sum_{y \in \mathsf{D} \setminus \mathscr{C}} e^{-2\alpha_G(\operatorname{dist}(y,\mathscr{C})-2s-9)_+}.$$

The 11^4 blocks y at distance at most 5 from z_A have $\operatorname{dist}(y,\mathscr{C}) \ge \operatorname{dist}(z_A, N_B^{+t}) - 5 \ge d - t - 5$, so their terms add up to at most $11^4 V^{1/2} M_{d-5,t}$, whose product with $J_{z_A z_B}$ has mean at most $2\varepsilon \cdot 11^4 V^{1/2} M_{d-5,t}$. For every other block y with $\operatorname{dist}(y, z_B) > R$ the blocks z_A, z_B, y are pairwise at distance at least 6, and $J_{z_A z_B} \mathbf{1}[y \in \mathsf{D}]$ is at most a sum of four indicators of events $\{T_1 \subset B(\omega), T_2 \subset B(\omega')\}$ with $T_1 \cup T_2 = \{z_A, z_B, y\}$; as in the proof of Lemma 496 each has mean at most $\varepsilon^{3/2}$. As in the derivation of (45), $e^{-2\alpha_G(\operatorname{dist}(y,\mathscr{C})-2s-9)} \le e^{-\alpha_G(R-2s-8)} e^{-\alpha_G(\operatorname{dist}(y,N_B^{+t})-2s-9)}$, and summing over y with $\sigma(\alpha_G)$,

$$\tfrac{1}{2} \mathbb{E}_c \mathbb{E}_{Q^c}\big[J_{z_A z_B} \mathbf{1}_{\mathscr{G}} \Delta_B\big] \le \varepsilon\big(11^4 V^{1/2} M_{d-5,t} + 2V^{3/2} \Phi_R\big).$$

The chain. On $\mathscr{G}^c$, the proof of Lemma 496 with $N = N_B$ and $N' = \emptyset$ gives $m \ge \lfloor t/R \rfloor$ and blocks $z_1, \dots, z_m$ of D with steps at most R, $z_1 \in N_B^{+R}$, $\operatorname{dist}(z_k, z_l) > R$ for $|k-l| \ge 2$, $\operatorname{dist}(z_m, N_B) > t - R$ and m least. By minimality $\operatorname{dist}(z_k, N_B) \le t - R$ for $k < m$, so every z_k is within t of N_B and at distance at least $d - t \ge 6$ from z_A. The blocks z_k with k odd are pairwise at distance more than $R \ge 11$, so at most one of them is within 5 of z_B; removing it, or else the last odd one, leaves a set T of $\lceil m/2 \rceil - 1$ blocks with $\{z_A, z_B\} \cup T$ pairwise at distance at least 6. Each of the two terms of $J_{z_A z_B}$ times $\mathbf{1}\{T \subset \mathsf{D}\}$ is a sum over $T_1 \subset T$ of indicators of events $\{z_A \in B(\omega), T_1 \subset B(\omega)\}$ with z_B in one of the two copies and $T \setminus T_1 \subset B(\omega')$, so, as in Lemma 496,

$$\mathbb{E}_c \mathbb{E}_{Q^c}\big[J_{z_A z_B} \mathbf{1}\{T \subset \mathsf{D}\}\big] \le 2\varepsilon (2\varepsilon^{1/2})^{\lceil m/2 \rceil - 1} \le 2\varepsilon (\sqrt{2}\, \varepsilon^{1/4})^{m-2}.$$

With at most $V(2R+1)^{4(m-1)}$ sequences of length m,

$$\mathbb{E}_c \mathbb{E}_{Q^c}\big[J_{z_A z_B} \mathbf{1}_{\mathscr{G}^c}\big] \le 2\varepsilon (2R+1)^4 V \sum_{m \ge \lfloor t/R \rfloor} g_R^{m-2} \le 4\varepsilon (2R+1)^4 V g_R^{\lfloor t/R \rfloor - 2}.$$

Adding the two parts and summing over z_A and z_B gives the bound. □

The second bound is the first multiplied by half the factor in parentheses, which with V of order one is below 1 for d above about $1.2 \cdot 10^5$ (Corollary 499); its floor $2V^{3/2}\Phi_R$ is of order 10^{-7011}.

Comparing two copies along the clusters on which they may differ is the idea of the disagreement percolation of van den Berg and Maes [112]; here the two copies are independent, they are compared by an exchange whose weight is the cross-ratio of Lemma 495, and the clusters are those of the union of the two bad sets. For local observables of the field the conditional covariance given the labels is added, and the labels enter through conditional expectations, which change by Lemma 495(a) when the labels far from the observable change.

Theorem 498 (the fibre covariance)**.** *Let N_A, N_B be sets of blocks at distance d, F and H bounded measurable functions of the active links based in N_A and in N_B, and $R \ge 2s+9$, $t \ge R$ integers with $t < d$ and $g_R \le \frac{1}{2}$. Put $V = \max\{|N_A^{+t}|, |N_B^{+t}|\}$. Then*

$$\mathbb{E}_c\big|\operatorname{Cov}_{\mu^c}(F,H)\big| \le \|F\|\|H\| V\big(10 g_R^{\lfloor t/R \rfloor} + \tfrac{9}{4} M_{d,t} + \tfrac{17}{4} \varepsilon^{1/2} V^{1/2} \Phi_R\big),$$

and $\operatorname{Cov}_{\mu_L}(F,H)$ differs from the covariance under μ_L of $\mathbb{E}_{\mu^c} F$ and $\mathbb{E}_{\mu^c} H$, functions of c, by at most the same bound.

Proof. Fix c and write $\varphi(\omega) = \mathbb{E}_{\mu^c}[F \mid \omega]$, $\psi(\omega) = \mathbb{E}_{\mu^c}[H \mid \omega]$. Then

$$\operatorname{Cov}_{\mu^c}(F,H) = \mathbb{E}_{P^c} \operatorname{Cov}_{\mu^c}(F,H \mid \omega) + \operatorname{Cov}_{P^c}(\varphi, \psi).$$

Given the labels. Let $Y_1 = B(\omega) \cup N_A \cup N_B$, $\mathscr{C}_1$ the union of its R-clusters meeting N_A, and $\mathscr{G}_1 = \{\mathscr{C}_1 \subset N_A^{+t}\}$. On $\mathscr{G}_1$, $N_B \cap \mathscr{C}_1 = \emptyset$ since $d > t$, and Lemma 495(b) with $Y = Y_1$, $X = \mathscr{C}_1$ gives $|\operatorname{Cov}_{\mu^c}(F,H \mid \omega)| \le \frac{1}{4}\|F\|\|H\| \Delta(\mathscr{C}_1, Y_1 \setminus \mathscr{C}_1)$; off $\mathscr{G}_1$ the covariance is at most $\|F\|\|H\|$. By Lemma 496 and (45) with $k = 1$, $N = N_A$, $N' = N_B$, and $|N_A^{+R}|, |N_B| \le V$,

$$\mathbb{E}_c \mathbb{E}_{P^c} |\operatorname{Cov}_{\mu^c}(F,H \mid \omega)| \le \|F\|\|H\| V\big(2 g_R^{\lfloor t/R \rfloor} + \tfrac{1}{4} M_{d,t} + \tfrac{1}{4} \varepsilon^{1/2} V^{1/2} \Phi_R\big).$$

The labels. Let $Y = \mathsf{D} \cup N_A \cup N_B$, $\mathscr{C}_A$ and $\mathscr{C}_B$ the unions of its R-clusters meeting N_A and N_B, $\mathscr{G} = \{\mathscr{C}_A \subset N_A^{+t}, \mathscr{C}_B \subset N_B^{+t}\}$, $\Delta_A = \Delta(\mathscr{C}_A, Y \setminus \mathscr{C}_A)$ and $\Delta_B = \Delta(\mathscr{C}_B, Y \setminus \mathscr{C}_B)$. On $\mathscr{G}$, $\mathscr{C}_A \cap \mathscr{C}_B = \emptyset$, since an R-cluster in both would lie in N_A^{+t} and contain a block of N_B. Let $G = \varphi(\omega)(\psi(\omega) - \psi(\omega'))$, so that $\operatorname{Cov}_{P^c}(\varphi, \psi) = \mathbb{E}_{Q^c} G$ and $|G| \le 2\|F\|\|H\|$, and let χ exchange the labels on $\mathscr{C}_B$ as in the proof of Theorem 497; it preserves Y, $\mathscr{C}_A$, $\mathscr{C}_B$, $\mathscr{G}$, Δ_A, Δ_B, and changes the weight by a factor in $[e^{-\Delta_B}, e^{\Delta_B}]$. On $\mathscr{G}$, Lemma 495(a) applied with $X = \mathscr{C}_A$ to ω, $\tilde\omega$, which agree on $\mathscr{C}_A$, and with $X = \mathscr{C}_B$ to $\tilde\omega, \omega'$ and to $\tilde\omega', \omega$, which agree on $\mathscr{C}_B$, gives

$$|\varphi(\tilde\omega) - \varphi(\omega)| \le 2\|F\| \tanh(\Delta_A/4),$$
$$|\psi(\tilde\omega) - \psi(\omega')| \le 2\|H\| \tanh(\Delta_B/4),$$

and the same for $\psi(\tilde\omega') - \psi(\omega)$. Writing

$$G \circ \chi + G = \big(\varphi(\tilde\omega) - \varphi(\omega)\big)\big(\psi(\tilde\omega) - \psi(\tilde\omega')\big) + \varphi(\omega)\big(\psi(\tilde\omega) - \psi(\omega') - \psi(\tilde\omega') + \psi(\omega)\big)$$

gives $|G \circ \chi + G| \le 4\|F\|\|H\| (\tanh \frac{\Delta_A}{4} + \tanh \frac{\Delta_B}{4})$. With $w_x = Q^c(x) + Q^c(\chi x)$ and $w'_x = Q^c(x) - Q^c(\chi x)$, an orbit $\{x, \chi x\}$ of two points in $\mathscr{G}$ contributes

$$\tfrac{1}{2} w_x \big(G + G \circ \chi\big)(x) + \tfrac{1}{2} w'_x \big(G - G \circ \chi\big)(x),$$

with $|G - G \circ \chi| \le 4\|F\|\|H\|$ and $|w'_x| \le w_x \tanh(\Delta_B/2)$, and a fixed point contributes $\frac{1}{2} Q^c(x)(G + G \circ \chi)(x)$. With $\tanh u \le u$,

$$|\operatorname{Cov}_{P^c}(\varphi, \psi)| \le \|F\|\|H\| \big(2Q^c(\mathscr{G}^c) + \mathbb{E}_{Q^c}\big[\mathbf{1}_{\mathscr{G}}\big(\tfrac{1}{2}\Delta_A + \tfrac{3}{2}\Delta_B\big)\big]\big).$$

By Lemma 496 for N_A and for N_B, and (45) with $k=2$ for $(N,N')=(N_A,N_B)$ and (N_B,N_A),

$$\mathbb{E}_c|\mathrm{Cov}_{P^c}(\varphi,\psi)| \le \|F\|\|H\|V\big(8g_R^{\lfloor t/R\rfloor}+2M_{d,t}+4\varepsilon^{1/2}V^{1/2}\Phi_R\big).$$

Adding the two parts gives the bound. Finally $\mathrm{Cov}_{\mu_L}(F,H)=\mathbb{E}_c\,\mathrm{Cov}_{\mu^c}(F,H)+\mathrm{Cov}(\mathbb{E}_{\mu^c}F,\mathbb{E}_{\mu^c}H)$, the last covariance under the law of c. □

The bound has three terms. The first is the chance that the group of an observable reaches distance t; the second is the cross-ratio between the two groups through the good layer, at the rate $2\alpha_G$ of Theorem 485; the third is the cross-ratio between a group and the islands at distance more than R from it, in the mean, which is of order ε and does not decay with d. The numbers follow.

Corollary 499 (the covariances in numbers)**.** *At $\rho=2.17\cdot10^{-16}$, with $s\in\{1,5\}$, $\beta_W\ge\beta^*$, and V the volume of Theorem 497 or 498 for the stated t:*

(i) *for $d\ge6s+34$, with $R=2s+9$ and $t=R(\lceil d/(10^4R)\rceil+2)$,*

$$\mathbb{E}_c|\mathrm{Cov}_{P^c}(\phi,\psi)| \le \mathrm{osc}\,\phi\,\mathrm{osc}\,\psi\,|N_A||N_B|\,\varepsilon \times\Big(\frac{\beta_W}{\beta^*}10^{c_s}V^{1/2}e^{-2\alpha_G(1-10^{-4})d} +4(4s+19)^4Ve^{-\lambda_s d}+10^{-f_s}V^{3/2}\Big),$$

with $c_1=26.26$, $\lambda_1=0.0736$, $f_1=7011.8$, $c_5=27.29$, $\lambda_5=0.0503$, $f_5=8285.9$, and $V\le|N_B|(d/5000+12s+55)^4$;

(ii) *for $d\ge2s+10$, with $R=2s+9$ and $t=R\lceil d/(10^4R)\rceil$,*

$$\mathbb{E}_c|\mathrm{Cov}_{\mu^c}(F,H)| \le \|F\|\|H\|\Big(\frac{\beta_W}{\beta^*}10^{a_s}Ve^{-2\alpha_G(1-10^{-4})d} +10^{-a'_s}V^{3/2}\Big),$$

with $a_1=22.75$, $a'_1=14062.6$, $a_5=23.77$, $a'_5=16611.8$, and $V\le\max\{|N_A|,|N_B|\}(d/5000+4s+19)^4$;

(iii) *for $10^8\le d\le10^{100}$, with $R=\lceil2000\sqrt{d}\rceil$ and $t=\lfloor d/2\rfloor$,*

$$\mathbb{E}_c|\mathrm{Cov}_{\mu^c}(F,H)| \le 10^{-1965}\frac{\beta_W}{\beta^*}\|F\|\|H\|V^{3/2}e^{-\sqrt{d}/2},$$

with $V\le\max\{|N_A|,|N_B|\}(d+1)^4$.

Proof. Dependence on β_W. By Theorem 383 and Step 6 of the proof of Theorem 494, f decreases on $[\beta^*,\infty)$ and $f(\beta^*)=\log q^*$, so $\varepsilon\le Mq^*$ and g_R is at most its value at β^*. The logarithmic derivative of $b_*=1920r^2(99.0001\beta_W+4.65\cdot10^6)^2/\beta_W$ is at most $1/\beta_W$, so $b_*\le(\beta_W/\beta^*)b_*(\beta^*)$. As $f'(\beta)=6/\beta-\rho^2/16$, the logarithmic derivatives of $b_*\varepsilon^{1/2}$ and $b_*\varepsilon$ are at most $4/\beta_W-\rho^2/32$ and $7/\beta_W-\rho^2/16$, both negative since $\beta^*\rho^2/16>8$; so Φ_R and $\varepsilon^{1/2}\Phi_R$ are at most their values at β^*. At β^* of Theorem 494, $b_*\nu_s\le10^{22.0833}$ for $s=1$ and $10^{23.1039}$ for $s=5$, $\sigma(\alpha_G)\le9.831\cdot10^{16}$, and $-\log g_{2s+9}\ge8105.1$ for $s=1$ and 9571.0 for $s=5$. Indeed, the proof of that theorem gives $b_*(\beta^*)\le10^{16.2645}$ and $10^{16.3356}$, $\sigma(\alpha_G)\le9.831\cdot10^{16}$, $|\log q^*|$ within 10^{-2} of 32481.67 and 38354.07, and $\log_{10}(Mq^*)$ within 10^{-3} of -14102.446 and -16652.797; here $\nu_1=45\cdot11^4$, $\nu_5=45\cdot19^4$ with $\log19=\log18+2\,\mathrm{artanh}\,\frac{1}{37}$, and at $\varepsilon=Mq^*$

$$-\log g_R=\tfrac14\big(|\log q^*|-4\log11\big)-\tfrac12\log2-4\log(2R+1),$$

which at $2R+1=4s+19$, that is 23 and $39=3\cdot13$, lies within 10^{-2} of 8105.13 and 9571.12. Every number below is a decimal logarithm of a product of these factors, computed with $\log10=\log2+\log5$; and $b_*(\beta^*)\ge1920r^2\cdot99^2\beta^*>10^{16}$, so $10\le\frac94b_*\nu_s$.

(i) Theorem 497: $t\ge3R$, $t\le d/10^4+3R$ and $(6s+34)(1-10^{-4})\ge6s+33$, so $d\ge t+6$; $d-5-t-2s-9\ge(1-10^{-4})d-8s-41$, so $11^4M_{d-5,t}\le(\beta_W/\beta^*)10^{c_s}e^{-2\alpha_G(1-10^{-4})d}$ with $10^{c_s}\ge11^4b_*(\beta^*)\nu_se^{2\alpha_G(8s+41)}$; $\lfloor t/R\rfloor-2=\lceil d/(10^4R)\rceil$ and $-\log g_R/(10^4R)\ge\lambda_s$; $2\Phi_R\le10^{-f_s}$; and $2t+1\le d/5000+12s+55$. (ii) Theorem 498: $R\le t\le d/10^4+R<d$ for $d\ge2s+10$, and $d-t-2s-9\ge(1-10^{-4})d-2(2s+9)$. The chain term is at most $10e^{-\lambda d/(10^4R)}$ with $\lambda=-\log g_R$, and $\lambda/(10^4R)\ge2\alpha_G$ and $10\le\frac94b_*\nu_s$, so the first two terms are at most $(\beta_W/\beta^*)\frac92b_*(\beta^*)\nu_se^{4\alpha_G(2s+9)}e^{-2\alpha_G(1-10^{-4})d}V$; the third is $\frac{17}{4}b_*\nu_s\sigma(\alpha_G)\varepsilon e^{\alpha_G(2s+8)}V^{3/2}\le10^{-a'_s}V^{3/2}$. (iii) For $d\ge10^8$, $t\ge d/2-1\ge2000\sqrt{d}+1\ge R$, and $t/R\ge\sqrt{d}/4001\ge1$, so $\lfloor t/R\rfloor\ge\sqrt{d}/8002$; $R\le2000\cdot10^{50}+1$, where $-\log g_R\ge7623.9$ for $s=1$ and 9092.0 for $s=5$. The third term carries $e^{-\alpha_GR}\le e^{-\sqrt{d}/2}$, and $d-t-2s-9\ge d/2-2s-9$; the ratios of the chain and main terms to $e^{-\sqrt{d}/2}$ decrease in d on $[10^8,\infty)$, and $V\le V^{3/2}$.

The numbers, for $s=1$ and $s=5$. In (i),

$$\log_{10}\big(11^4b_*\nu_se^{2\alpha_G(8s+41)}\big)\le26.2596,\ 27.2871,$$
$$-\log g_R/(10^4R)\ge0.07368,\ 0.05037,$$
$$\log_{10}\big(2b_*\nu_s\sigma(\alpha_G)(Mq^*)^{1/2}e^{\alpha_G(2s+8)}\big)\le-7011.84,\ -8285.99,$$

which give c_s, λ_s and f_s. In (ii), $-\log g_R/(10^4R)\ge2\alpha_G$ and

$$\log_{10}\big(\tfrac92b_*\nu_se^{4\alpha_G(2s+9)}\big)\le22.742,\ 23.766,$$
$$\log_{10}\big(\tfrac{17}{4}b_*\nu_s\sigma(\alpha_G)Mq^*e^{\alpha_G(2s+8)}\big)\le-14062.73,\ -16612.06,$$

which give a_s and a'_s. In (iii), at $2R+1\le4\cdot10^{53}+3$ the display for $-\log g_R$ gives $\lambda=-\log g_R\ge7623.9$ and 9092.0, so that $\lambda/8002>\frac12$. The chain and main ratios are at most

$$10e^{-(\lambda/8002-1/2)\sqrt{d}},$$
$$\tfrac94b_*\nu_se^{-2\alpha_G(d/2-2s-9)+\sqrt{d}/2};$$

the first decreases in d, and the second for $\sqrt{d}>1/(4\alpha_G)=1000$. At $d=10^8$ the chain ratio is at most $10^{-1965.2}$ and

10^{-2761}, the main ratio at most $10^{-8663.4}$ and $10^{-8662.4}$, and the coefficient $\frac{17}{4}b_*\nu_s\sigma(\alpha_G)Mq^*e^{\alpha_G(4s+17)}$ of the third term at most $10^{-14062.7}$ and 10^{-16612}. □

The groups make the Dobrushin condition on the islands unnecessary in these bounds: two groups have one cross-ratio. Where that condition holds, as in Theorem 494(iii), part (ii) of Lemma 490 has a constructive proof, by the heat-bath sweep of Föllmer [53] under Dobrushin's matrix [42, 43], which needs neither the minimising coupling nor compactness.

Lemma 500 (the sweep)**.** *In the setting of Lemma 490, with f,g bounded measurable, suppose that $D=\sum_{n\ge0}C^n$ converges. Let $T_qu(\xi)=\int u(\xi^\eta)P_q(d\eta\mid\xi)$, let $q_1,q_2,\dots$ run through $\mathscr{Q}$ cyclically, $u_0=f$ and $u_k=T_{q_k}u_{k-1}$. Then*

$$\mathrm{Cov}_P(f,g)=\sum_{k\ge1}\mathbb{E}_P\big[\mathrm{Cov}_{q_k}(u_{k-1},g)\big],$$
$$\sum_{k:\,q_k=q}\delta_q(u_{k-1})\le\sum_{q'}\delta_{q'}(f)D_{q'q},$$

with Cov_q the covariance under $P_q(\cdot\mid\xi)$ in the coordinate q; hence $|\mathrm{Cov}_P(f,g)|\le\frac14\sum_{q,q'}\delta_{q'}(f)D_{q'q}\delta_q(g)$.

Proof. The law P is invariant under T_q, and T_qu does not depend on the coordinate ξ_q; so $\mathbb{E}_P[(T_qu)g]=\mathbb{E}_P[(T_qu)(T_qg)]$ and $\mathbb{E}_P[ug]=\mathbb{E}_P[(T_qu)(T_qg)]+\mathbb{E}_P\mathrm{Cov}_q(u,g)$, which gives $\mathrm{Cov}_P(u,g)-\mathrm{Cov}_P(T_qu,g)=\mathbb{E}_P\mathrm{Cov}_q(u,g)$. For $q'\ne q$, changing $\xi_{q'}$ changes T_qu by at most $\delta_{q'}(u)+C_{qq'}\delta_q(u)$: the integrand by $\delta_{q'}(u)$, and the law $P_q(\cdot\mid\xi)$ by $C_{qq'}$ in total variation, against a function of oscillation $\delta_q(u)$. So the vector $v_k=(\delta_q(u_k))_q$ obeys $v_k\le v_{k-1}-\tau_ke_{q_k}+\tau_kC_{q_k\cdot}$ with $\tau_k=(v_{k-1})_{q_k}$, and since $D\ge0$ and $D-CD=I$, $v_kD\le v_{k-1}D-\tau_ke_{q_k}$. Summing, the accumulated τ at each q is at most $(v_0D)_q$, the second claim. In particular $\sum_k\tau_k<\infty$; each $(v_n)_q$ is at most the sum of the τ_k since the last visit to q, as $C\le1$, so $v_n\to0$, $\operatorname{osc}u_n\le\sum_q(v_n)_q\to0$ and $\mathrm{Cov}_P(u_n,g)\to0$. This gives the identity, and $|\mathrm{Cov}_q(u,g)|\le\frac14\delta_q(u)\delta_q(g)$ by Popoviciu [100] gives the bound. □

Remark 501. Four things are not proved here. First, the bounds are means over the coarse field. For a fixed c the labels need not be rare: a coarse plaquette with $\delta(c_P,I)>\eta$ makes its blocks bad with P^c-probability one. The proofs use rarity only through $\mathbb{E}_cP^c(T\subset B)=\mu_L(T\subset B)$ and its two-copy form in Lemma 496; a bound for every c, or uniform in an exterior as (D) asks, is not given. Second, the floor. The chains are paid for at the joining distance R, while the islands beyond R enter Δ through $e^{-\alpha_GR}$, so with R fixed the bound stops at a floor, of order $\varepsilon^{3/2}$ in Theorem 497 and ε in Theorem 498, and with $R\propto\sqrt{d}$ it decays as $e^{-\sqrt{d}/2}$. Decay at the rate $2\alpha_G$ without a floor needs control of $\log Z^c$ as a function of the labels of three or more components: the mixed differences of order $k\ge3$ of $\log Z^c$ over the labels of k components, bounded by $C^ke^{-2\alpha_G\ell}$ with ℓ the length of the shortest tree joining them. The inputs give the differences of order two, and through Lemma 506 those of every order in terms of the diameter only; those of order three of $\log\tilde K$ are third cumulants of the layer law, which (41) does not bound, and $\tilde K$ and D_z depend on the set Y of components. Third, the covariance of $\mathbb{E}_{\mu^c}F$ and $\mathbb{E}_{\mu^c}H$ as functions of c is the multiscale part, item (iv) of Sec. XIII V. Fourth, the bounds are consistent with Proposition 455: in its mixture the labels are independent given the hidden bit, which plays the role of c, so the covariance under the fibre law vanishes and the covariance $q^2/16$ is that of the conditional means; the theorems above use the cross-ratio of Lemma 495, not rarity alone.

Remark 501 leaves open a covariance bound for the labels for a fixed coarse field, uniform in the exterior, decaying at the rate $2\alpha_G$ with no floor. The rest of this subsection determines what the inputs used above give for one law of the labels. The proof of Theorem 497 uses two properties of the laws P^c. For a law P on the labellings of Λ and $0<\varepsilon'<1$ consider:

($\mathrm{R}_{\varepsilon'}$) $P(T\subset B)\le\varepsilon'^{|T|}$ for every nonempty set T of blocks pairwise at distance at least 6;

(X) for every set Y of blocks, every union X of components of Y, $X'=Y\setminus X$, and all labellings ω,ω' of positive probability with $B(\omega),B(\omega')\subset Y$, the labellings $\omega^\natural=(\omega'_X,\omega_{X'})$ and $\omega'^\natural=(\omega_X,\omega'_{X'})$ have positive probability and

$$e^{-\Delta}\le\frac{P(\omega^\natural)P(\omega'^\natural)}{P(\omega)P(\omega')}\le e^{\Delta},$$

with $\Delta=\min\{\Delta(X,X'),\Delta(X',X)\}$.

For every coarse field c the law P^c has (X), by Lemma 495(c), and the law of the labels under μ_L has (R_ε).

Lemma 502 (one law)**.** *Let P be a law on the labellings of Λ with* (X) *and* ($\mathrm{R}_{\varepsilon^{1/2}}$)*, and let N_A,N_B,ϕ,ψ be as in Theorem 497. Then $|\mathrm{Cov}_P(\phi,\psi)|$ obeys both bounds of Theorem 497, with the same constants. In particular this holds for every law with* (X) *and* (R_ε).

Proof. Run the proof of Theorem 497 with P^c replaced by P, Q^c by $Q=P\otimes P$, and the means $\mathbb{E}_c$ omitted. The exchange and the orbit decomposition use (X) only. The mean over c enters through the probabilities, under Q, of events $\{T_1\subset B(\omega),T_2\subset B(\omega')\}$ with $T=T_1\cup T_2$ pairwise at distance at least 6, which the proof bounds by $\varepsilon^{|T|/2}$, and through $\mu_L(z_A,z_B\in B)\le\varepsilon^2$ in the first bound. Under Q the first probability is $P(T_1\subset B)P(T_2\subset B)\le\varepsilon^{|T|/2}$ by ($\mathrm{R}_{\varepsilon^{1/2}}$), with $P(\emptyset\subset B)=1$; it covers the means of $J_{z_Az_B}$, at most 2ε, of $J_{z_Az_B}\mathbf{1}[y\in\mathsf{D}]$, at most $4\varepsilon^{3/2}$, and of $J_{z_Az_B}\mathbf{1}\{T\subset\mathsf{D}\}$ along a chain, at most $2\varepsilon(2\varepsilon^{1/2})^{|T|}$. In the first bound the mean of $J_{z_Az_B}$ is at most $P(z_A,z_B\in B)+P(z_A\in B)P(z_B\in B)\le2\varepsilon$. Every other step is pointwise in the labellings. Finally $\varepsilon^{|T|}\le\varepsilon^{|T|/2}$. □

Proposition 503 (fixed coarse fields)**.** *Let $\mathscr{G}_\Lambda$ be the set of coarse fields c for which P^c has* ($\mathrm{R}_{\varepsilon^{1/2}}$)*. Then*

$$\mu_L\{V^\sharp\notin\mathscr{G}_\Lambda\}\le(1+\varepsilon^{1/2})^{|\Lambda|}-1,$$

which is at most $2|\Lambda|\varepsilon^{1/2}$ when $|\Lambda|\varepsilon^{1/2}\le1$, and for every $c\in\mathscr{G}_\Lambda$ the covariance $|\mathrm{Cov}_{P^c}(\phi,\psi)|$ obeys both bounds of Theorem 497. No coarse field with a coarse plaquette $\delta(c_P,I)>\eta$ lies in $\mathscr{G}_\Lambda$.

Proof. For a set T of blocks, $\mathbb{E}_c P^c(T \subset B) = \mu_L(T \subset B) \le \varepsilon^{|T|}$ when T is pairwise at distance at least 6, so by Markov's inequality the law of $V^\sharp$ gives the set $\{c : P^c(T \subset B) > \varepsilon^{|T|/2}\}$ mass at most $\varepsilon^{|T|/2}$. The union over the nonempty sets $T \subset \Lambda$ has mass at most $\sum_{T \ne \emptyset} \varepsilon^{|T|/2} = (1+\varepsilon^{1/2})^{|\Lambda|} - 1 \le e^{|\Lambda|\varepsilon^{1/2}} - 1$, and $e^x - 1 \le 2x$ on $[0,1]$. For $c \in \mathscr{G}_\Lambda$, P^c has (X) by Lemma 495(c), and Lemma 502 applies. A coarse plaquette with $\delta(c_P, I) > \eta$ makes its corner bad with P^c-probability one, and $1 > \varepsilon^{1/2}$. □

The floor of Theorem 497 comes from the joining distance R; with R of order $\sqrt{d}$ it becomes a stretched exponential, as for the fibre in Corollary 499(iii).

Corollary 504 (the labels, stretched)**.** *At $\rho = 2.17 \cdot 10^{-16}$, with $s \in \{1,5\}$, $\beta_W \ge \beta^*$ and $10^8 \le d \le 10^{100}$, both $\mathbb{E}_c|\mathrm{Cov}_{P^c}(\phi,\psi)|$ and, for every $c \in \mathscr{G}_\Lambda$, $|\mathrm{Cov}_{P^c}(\phi,\psi)|$ are at most*

$$\operatorname{osc}\phi \operatorname{osc}\psi\, |N_A||N_B|\, 10^{-u_s} V^{3/2} e^{-\sqrt{d}/20},$$

with $u_1 = 21114.2$, $u_5 = 24938.6$ and $V \le |N_B|(d+1)^4$.

Proof. Take $R = \lceil 200\sqrt{d} \rceil$ and $t = \lfloor d/2 \rfloor$ in Theorem 497, or in Lemma 502. For $d \ge 10^8$, $t \ge d/2 - 1 \ge 2R$, $d \ge t+6$, $2R+1 \le 401\sqrt{d}$ and $t/R \ge \sqrt{d}/401$, so $\lfloor t/R \rfloor - 2 \ge \sqrt{d}/401 - 3 > 0$; $R \le 200 \cdot 10^{50} + 1$, where $-\log g_R \ge 7633.1$ for $s = 1$ and 9101.2 for $s = 5$. The three terms of the bracket, multiplied by the factor ε in front, are bounded as follows. The far term $2V^{3/2}\Phi_R$ carries $e^{-\alpha_G R} \le e^{-\sqrt{d}/20}$, since $200\alpha_G = \frac{1}{20}$, with coefficient $2b_* \nu_s \sigma(\alpha_G)\varepsilon^{1/2} e^{\alpha_G(4s+17)}$. The main term has $d - 5 - t - 2s - 9 \ge d/2 - 2s - 14$, and its ratio to $e^{-\sqrt{d}/20}$, $11^4 b_* \nu_s e^{2\alpha_G(2s+14)} e^{-\alpha_G d + \sqrt{d}/20}$, decreases on $[10^8, \infty)$. The chain term is at most $4 \cdot 401^4 d^2 V e^{3\lambda} e^{-\lambda\sqrt{d}/401}$ with $\lambda = -\log g_R$, and its ratio to $e^{-\sqrt{d}/20}$ decreases on $[10^8, \infty)$ since $\lambda/401 > \frac{1}{20} + 4 \cdot 10^{-4}$. By the monotonicity in the proof of Corollary 499, εb_*, $\varepsilon^{3/2} b_*$ and g_R are at most their values at β^*. At β^* and $d = 10^8$, the coefficient of the far term times ε is at most $10^{-21114.22}$ for $s = 1$ and $10^{-24938.64}$ for $s = 5$, and the main and chain ratios are below 10^{-24000} and 10^{-86000}; with $V^{1/2}, V \le V^{3/2}$ and $2t + 1 \le d + 1$ this gives the bound.

The numbers come from the factors listed in the proof of Corollary 499. The display for $-\log g_R$ there at $2R + 1 \le 4 \cdot 10^{52} + 3$ gives $\lambda \ge 7633.1$ and 9101.2. With $\varepsilon \le Mq^*$, the decimal logarithm of $2b_* \nu_s \sigma(\alpha_G)(Mq^*)^{3/2} e^{\alpha_G(4s+17)}$ is at most -21114.28 for $s = 1$ and -24938.79 for $s = 5$; that of $11^4 b_* \nu_s e^{2\alpha_G(2s+14)} e^{-\alpha_G d + \sqrt{d}/20} Mq^*$ at $d = 10^8$ is at most -24716 and -27265; and the chain ratio times ε is at most $4 \cdot 401^4 d^2 e^{-\lambda(\sqrt{d}/401 - 3) + \sqrt{d}/20} Mq^*$, whose decimal logarithm at $d = 10^8$ is at most -86581 and -103119. □

The next theorem shows that no argument which uses only (X) and a rarity bound does better: for a fixed coarse field nothing crosses a bad component, and with rarity the decay is at best a stretched exponential.

Theorem 505 (what rarity and the group cross-ratios give)**.** *Let $s \ge 1$, α_G and $b_* \nu_s \ge 1$ be as above, and let $d \ge 2$ be an integer, on a torus of side larger than $2d$; let a and $b = a + de_1$ be two blocks.*

(i) *For every $J \ge 0$ there is a law P with* (X)*, under which the blocks $a + ie_1$, $0 < i < d$, are bad with probability one, such that*

$$\mathrm{Cov}_P\big(\mathbf{1}[a \in B], \mathbf{1}[b \in B]\big) = \tfrac{1}{4}\tanh\tfrac{J}{4};$$

in it every second difference of $\log P$ between the labels of two sets of blocks at distance at least $2s+9$, all other labels fixed, vanishes.

(ii) *Let $0 < \varepsilon' \le e^{-3}$, $\varkappa = \log(e/\varepsilon')$ and $h_* = \lceil \varkappa/(2\alpha_G) \rceil$, and suppose $h_* \ge 2s + 9$. If $d = m^2 h_*$ with an integer $m \ge 1$, there is a law P with* (X) *and* ($\mathrm{R}_{\varepsilon'}$) *such that*

$$\begin{aligned}\mathrm{Cov}_P\big(\mathbf{1}[a \in B], \mathbf{1}[b \in B]\big) &\ge \tfrac{1}{2} e^{-(2m+1)\varkappa - 2\alpha_G m} \\ &= \tfrac{1}{2} e^{-\varkappa} \exp\big(-(2\varkappa + 2\alpha_G)\sqrt{d/h_*}\big).\end{aligned}$$

In it the mixed difference of order $m+1$ of $\log P$ over the labels of the blocks $a + jmh_* e_1$, $0 \le j \le m$, equals $e^{-2\alpha_G d/m}$.

Consequently no bound

$$|\mathrm{Cov}_P(\phi,\psi)| \le C \operatorname{osc}\phi \operatorname{osc}\psi\, |N_A|\,|N_B|\, e^{-\kappa d}$$

with $C < \infty$ and $\kappa > 0$ holds for all laws with (X) *and* ($\mathrm{R}_{\varepsilon'}$)*; with $\kappa = 2\alpha_G$ and $C\varepsilon'$ in place of C it fails at $d = 9h_*$ for every $C < e^{3\varkappa - 6\alpha_G}/(2e)$.*

Proof. (i) Let $K = \{a + ie_1 : 0 < i < d\}$, a set connected at distance one. Let P give the labellings with $B = K \cup B_0$, $B_0 \subset \{a, b\}$, the weights $Z_{11} = Z_{00} = e^{J/2}$ and $Z_{10} = Z_{01} = 1$, the indices being the labels of a and b, and every other labelling the weight zero; by Corollary 453 the covariance is $\frac{1}{4}\tanh(J/4)$. Let Y, X, ω, ω' be as in (X). Then $K \subset Y$, and a and b, when they lie in Y, lie in the component of Y that contains K, since they are at distance one from K. If X contains that component, $\omega^\natural$ and $\omega'^\natural$ agree with ω' and ω at a and b; otherwise they agree with ω and ω' there. Every other block has the same label in all four labellings, so the ratio in (X) is one. Two sets at distance at least $2s+9$ off which every block is good contain K in one of them, and a and b, when bad, in the same one; so the label of only one of them varies, and the second difference vanishes.

(ii) Put $h = mh_*$, $y_j = a + jhe_1$ for $0 \le j \le m$, $S = \{y_0, \dots, y_m\}$, $k = m+1$, $p = \varepsilon'/e$ and $J = e^{-2\alpha_G h} \le 1$. Let P give the labelling with bad set $A \subset S$ the probability

$$P(A) = \frac{1}{N}\, p^{|A|}(1-p)^{k-|A|} e^{J\mathbf{1}[A=S]},$$

with $N = 1 + (e^J - 1)p^k$, and every other labelling the probability zero.

Rarity. For $T \subset S$ nonempty, $P(T \subset B) = (p^{|T|} + (e^J - 1)p^k)/N \le e^J p^{|T|} \le e^{1-|T|}\varepsilon'^{|T|} \le \varepsilon'^{|T|}$; for $T \not\subset S$ it vanishes.

The cross-ratios. With $\sigma_j = \mathbf{1}[y_j \in B]$, $\log P = \text{const} + |B| \log\frac{p}{1-p} + J\prod_j \sigma_j$. In (X) the bad sets of the four labellings lie in S, and $|B(\omega^\natural)| + |B(\omega'^\natural)| = |B(\omega)| + |B(\omega')|$.

If $S \not\subset Y$, some y_j is good in all four and the products vanish. If $S \subset X$ or $S \subset X'$, the two products on each side are the same. Otherwise, with x, x' the products over $S \cap X$ in ω, ω' and z, z' those over $S \cap X'$, the logarithm of the ratio is $J(x'z + xz' - xz - x'z') = -J(x - x')(z - z')$, of modulus at most J; and two consecutive blocks y_j, y_{j+1} lie on different sides, at distance h, so that both $\Delta(X, X')$ and $\Delta(X', X)$ are at least $b_* \nu_s e^{-2\alpha_G(h-2s-9)} \geq J$.

The covariance. $P(a \in B) = (p + (e^J - 1)p^k)/N$ and $P(a, b \in B) = (p^2 + (e^J - 1)p^k)/N$, so

$$\mathrm{Cov}_P\big(\mathbf{1}[a \in B], \mathbf{1}[b \in B]\big) = \frac{(e^J - 1)p^k(1-p)^2}{N^2} \geq \tfrac{1}{2} J p^k,$$

since $e^J - 1 \geq J$, $p \leq e^{-4}$ and $N \leq 1 + 2p^2$. As $2\alpha_G h_* \leq \varkappa + 2\alpha_G$, $J \geq e^{-m\varkappa - 2\alpha_G m}$, and $p^k = e^{-(m+1)\varkappa}$; with $m = \sqrt{d/h_*}$ this is the bound. The mixed difference of order k over $\sigma_0, \dots, \sigma_m$ of the linear part vanishes, as $k \geq 2$, and that of $J \prod_j \sigma_j$ is $J = e^{-2\alpha_G d/m}$.

The consequence. For fixed C and $\kappa > 0$ the lower bound of (ii) exceeds $Ce^{-\kappa d}$ for $d = m^2 h_*$ and m large. With $\kappa = 2\alpha_G$, $2\alpha_G d \geq m^2 \varkappa$ and $\varepsilon' = e^{1-\varkappa}$, so the ratio of the covariance to $C\varepsilon' e^{-2\alpha_G d}$ is at least $e^{(m^2-2m)\varkappa - 2\alpha_G m - 1}/(2C)$, which at $m = 3$ exceeds one for $C < e^{3\varkappa - 6\alpha_G}/(2e)$. □

At $\varepsilon' = 10^{-14102.4}$, the bound on ε for $s = 1$, $\varkappa \in [32472.97, 32472.98]$, $h_* = 64945953$, and the law of (ii) has covariance at least $\frac{1}{2} e^{-\varkappa} e^{-8.06\sqrt{d}}$ at $d = m^2 h_*$; for $s = 5$, $\varepsilon' = 10^{-16652.7}$, $h_* = 76690518$ and the exponent is $8.76\sqrt{d}$. By Lemma 502 every law with (X) and (R_ε) obeys the bound of Corollary 504. Within this class the decay is therefore a stretched exponential, with the coefficient of $\sqrt{d}$ between $\frac{1}{20}$ and 8.06 for $s = 1$; at $d = 9h_* \leq 5.85 \cdot 10^8$ a bound $C\varepsilon e^{-2\alpha_G d}$ already needs $C \geq 10^{42307}$. The laws of (ii) are made of one interaction among $m+1$ rare blocks at spacing d/m, which the group cross-ratios charge only through the spacing; a cluster expansion and a coupling that use only (X) and rarity are both subject to (ii).

The pair comparison of Theorem 491 is an input of another kind: it bounds the second difference of two islands with the labels of all the others fixed, in every such environment, and it controls the differences of every order.

Lemma 506 (differences of higher order)**.** *Let the data and hypotheses be those of Theorem 491, with the index set $\mathscr{I} = \mathscr{K} \cup \mathscr{Q}$, $|\mathscr{K}| = k \geq 2$, two patterns ω_u^0, ω_u^1 on I_u for $u \in \mathscr{K}$ and one pattern on I_q for $q \in \mathscr{Q}$. For $\varsigma \in \{0,1\}^{\mathscr{K}}$ let $Z(\varsigma)$ be the weight of the event that the labels of Λ are $\omega_u^{\varsigma_u}$ on I_u, $u \in \mathscr{K}$, the given patterns on the I_q and good elsewhere, and suppose $Z(\varsigma) > 0$ for all ς. Then*

$$\Big| \sum_{\varsigma} (-1)^{k - |\varsigma|} \log Z(\varsigma) \Big| \leq 2^{k-2} \min_{u \neq v \in \mathscr{K}} \Delta_*(u, v),$$

with $|\varsigma| = \sum_u \varsigma_u$ and $\Delta_*(u,v)$ the quantity Δ_* of Theorem 491 for $a = u$, $b = v$ and the sets I_w, $w \in \mathscr{I} \setminus \{u, v\}$, in the role of $\mathscr{Q}$.

Proof. The numbers Δ_{uv}, S_u, C and D of Theorem 491 depend on the sets I_w, on Λ_2 and on β_W, and not on the patterns, since $\tilde{K}$ and $\mathscr{U}$ are the same in all sectors by Lemma 488(i). Fix $u \neq v$ and, for $\varsigma' \in \{0,1\}^{\mathscr{K} \setminus \{u,v\}}$, let $D(\varsigma')$ be the second difference $\log Z_{11} + \log Z_{00} - \log Z_{10} - \log Z_{01}$ in $(\varsigma_u, \varsigma_v)$ with the other indices equal to ς'. These four weights are those of Theorem 491 with the sets I_w, $w \neq u, v$, carrying the patterns $\omega_w^{\varsigma'_w}$, so $|D(\varsigma')| \leq \Delta_*(u,v)$. The sum in the statement is $\sum_{\varsigma'} (-1)^{k-2-|\varsigma'|} D(\varsigma')$, which has 2^{k-2} terms. □

Under the weighted Dobrushin condition of Corollary 492 the right side is at most $2^{k-2}(\Delta_{uv} + e^{-2\vartheta\alpha_G \ell(u,v)} T_u T_v/(4(1-\theta)))$ for the pair u, v of largest $\ell(u,v)$, so the differences of order k decay with the diameter of the k sets in the pseudometric ℓ. In the laws of Theorem 505(ii) the pair y_0, y_m, with the blocks of S between them bad, has second difference $e^{-2\alpha_G d/m}$ at distance d, so a pair bound that decays with the distance as in Corollary 492 excludes them.

Remark 507. For a fixed coarse field the results above give the following. Proposition 503 gives the bounds of Theorem 497 for every coarse field outside a set of probability at most $2|\Lambda|\varepsilon^{1/2}$, and Corollary 504 decay as $e^{-\sqrt{d}/20}$ for $10^8 \leq d \leq 10^{100}$, with no floor in that range. Decay at the rate $2\alpha_G$ is not proved, for three reasons. First, by Theorem 505(ii) it does not follow from rarity and the group cross-ratios of Lemma 495, which are all that the proofs of Theorems 497 and 498 use. Second, Lemma 506 bounds the differences of order k by the diameter of the k sets, while a cluster expansion in the labels, through the criterion of Kotecký and Preiss [78], needs the length of the shortest tree joining them, as in Remark 501; with the diameter the sum over k islands in a ball of radius D carries the entropy $D^{4(k-1)}$, which the rarity pays only while εD^4 is small, and the pair bound decays only where the weighted Dobrushin condition holds. Third, uniformity in the coarse field and in the exterior is excluded for these inputs by Theorem 505(i): for a coarse field whose coarse plaquettes with $\delta(c_P, I) > \eta$ have corners joining a neighbourhood of N_A to one of N_B, the labels of N_A and N_B adjoin one component that is bad for every labelling, and neither Lemma 495 nor Theorem 491 bounds anything across it. A bound uniform in the coarse field therefore needs the response of the law across the interior of a bad component, which no result here gives, and a bound at the rate $2\alpha_G$ for the coarse fields of $\mathscr{G}_\Lambda$ needs the tree form of the differences of order three and more. The statement of item (ii) of Sec. XIII V is accordingly posed in the mean over the coarse field or for the coarse fields of $\mathscr{G}_\Lambda$.

The tree form named in Remarks 501 and 507 asks for a bound $C_0 C^k e^{-2\alpha_G \ell}$ on the mixed differences of order $k \geq 3$, with ℓ the length of the shortest tree joining the k sets. The next two propositions show that it follows neither from the second differences in every environment, which are all that Lemma 506 uses, nor from the layer bound (41), which is all that the proof of Theorem 491 uses of the layer law. The length of a tree whose vertices are blocks is the sum of the distances of its edges.

Proposition 508 (pair bounds give the diameter form)**.** *Let $s \geq 1$, α_G and $b_* \nu_s \geq 1$ be as above, $0 < \varepsilon' < 1$, $3 \leq k \leq 16$,*

$D \geq 2s+9$ an integer, and on a torus of side larger than $2D$ let S be a set of k vertices of a cube $a+\{0,D\}^4$, so that any two blocks of S are at distance D. Put $J = e^{-2\alpha_G(D-2s-9)}$. There is a law P with (X) *and* ($R_{\varepsilon'}$) *such that:*

(i) *every second difference of* $\log P$ *between the labels of two sets of blocks at distance t, all other labels fixed, vanishes unless both sets meet S, and then $t \leq D$ and its modulus is at most $J \leq e^{-2\alpha_G(t-2s-9)}$;*

(ii) *the mixed difference of order k of* $\log P$ *over the labels of the blocks of S equals J;*

(iii) *every tree whose vertices include S has length at least $kD/2$.*

So the bound of Lemma 506 by the second differences of a pair, in every environment, is attained up to its factor 2^{k-2}, while a bound $C_0C^k e^{-2\alpha_G \ell}$ fails for this law as soon as $\alpha_G(k-2)D > \log(C_0C^k) - 2\alpha_G(2s+9)$.

Proof. Put $p = \varepsilon'/e$ and let P give the labelling with bad set $A \subset S$ the probability $N^{-1}p^{|A|}(1-p)^{k-|A|}e^{J\mathbf{1}[A=S]}$, with $N = 1+(e^J-1)p^k$, and every other labelling the probability zero; this is the law of the proof of Theorem 505(ii) on the set S. For $T \subset S$ nonempty, $P(T \subset B) = (p^{|T|}+(e^J-1)p^k)/N \leq e^J p^{|T|} \leq e^{1-|T|}\varepsilon'^{|T|} \leq \varepsilon'^{|T|}$, as $J \leq 1$, and for $T \not\subset S$ it vanishes; this is ($R_{\varepsilon'}$). With $\sigma_y = \mathbf{1}[y \in B]$, on the labellings of positive probability $\log P = \text{const} + |B| \log \frac{p}{1-p} + J\prod_{y\in S}\sigma_y$. In (X) the computation of the proof of Theorem 505(ii) gives the ratio $e^{-J(x-x')(z-z')}$, which is one unless S meets both X and X'; then some $y \in S \cap X$ and $y' \in S \cap X'$ are at distance D, so both $\Delta(X,X')$ and $\Delta(X',X)$ are at least $b_* \nu_s e^{-2\alpha_G(D-2s-9)} \geq J$. (i) The linear part has no second differences, and the second difference of $J\prod_y \sigma_y$ between two sets is J times the product of the differences of the two partial products and of the fixed labels of the rest of S, so its modulus is at most J, and it vanishes unless both sets meet S; then $t \leq D$. (ii) The mixed difference of order $k \geq 2$ of the linear part vanishes, and that of $J\prod_y \sigma_y$ is J. (iii) A closed walk around a tree of length λ that starts at a block of S and traverses each edge twice has length 2λ and visits every block of S. Cut it at the first visits of the k blocks of S: it falls into k walks, each between two distinct blocks of S, so each of length at least D, and $2\lambda \geq kD$. For the last assertion, $J > C_0C^k e^{-\alpha_G kD}$ is the inequality stated, and $\ell \geq kD/2$. □

Proposition 509 (the layer bound and third cumulants)**.** *Let $\alpha > 0$, $D > 0$, and give three sites $1,2,3$ the distances $d(i,j) = D$ for $i \neq j$. For $x \in \mathbb{R}^3$ let μ_x be the probability measure on $\mathbb{R}^3$ with density proportional to $e^{-U(z;x)}$,*

$$U(z;x) = \tfrac{1}{2}|z|^2 + J\sin z_1 \sin z_2 \sin z_3 - x\cdot z, \qquad J = \tfrac{1}{6}e^{-\alpha D},$$

and let $K(x) = \int e^{-U(z;x)}\,dz$. Then:

(i) *for every x, every φ with $|\varphi(i)-\varphi(j)| \leq d(i,j)$ and all smooth F,H with bounded derivatives,*

$$|\mathrm{Cov}_{\mu_x}(F,H)| \leq 2\left\|e^{-\alpha\varphi}\nabla F\right\|_{L^2(\mu_x)}\left\|e^{\alpha\varphi}\nabla H\right\|_{L^2(\mu_x)},$$

which is (41) *with $\mathscr{A} = 2$ and the rate α;*

(ii) *$|\partial_{x_i}\partial_{x_j}\log K(x)| = |\mathrm{Cov}_{\mu_x}(z_i,z_j)| \leq 2e^{-\alpha D}$ for $i \neq j$ and every x;*

(iii) *$\partial_{x_1}\partial_{x_2}\partial_{x_3}\log K(0)$ is the third cumulant of z_1,z_2,z_3 under μ_0, of modulus at least $\frac{1}{30}e^{-\alpha D}$, while every tree joining the three sites has length at least $\frac{3}{2}D$.*

So (41)*, at every x and with the second derivatives of $\log K$ decaying at the rate α, gives no bound $Ce^{-\alpha\ell}$ on the third derivatives, with ℓ the length of the shortest joining tree: here $C \geq \frac{1}{30}e^{\alpha D/2}$ for every D.*

Proof. (i) Write $\mathrm{Hess}_z U = I + JM$, where the entries of M are $-\sin z_1 \sin z_2 \sin z_3$ on the diagonal and $\cos z_i \cos z_j \sin z_k$ off it, all of modulus at most one. With $w = e^{\alpha\varphi}$ the entries of wMw^{-1} have modulus at most $e^{\alpha D}$, so $\|JwMw^{-1}\| \leq 3Je^{\alpha D} = \frac{1}{2}$, and the same holds for M. The potential is smooth with bounded derivatives of order at least two and $\mathrm{Hess}_z U \geq \frac{1}{2}$, and the weighted Helffer–Sjöstrand representation in the proof of Theorem 367(ii) applies: $\mathrm{Re}\langle u, \mathscr{A}_w u\rangle \geq (1-\frac{1}{2})\|u\|^2$, so $\|\mathscr{A}_w^{-1}\| \leq 2$ and (i) follows. (ii) $\log K(x) - \log K(0) = \log \mathbb{E}_{\mu_0} e^{x\cdot z}$, so the derivatives of $\log K$ are the joint cumulants of z under μ_x; the second ones are covariances, and (i) with $F = z_i$, $H = z_j$ and $\varphi = -d(\cdot,i)$ gives $2e^{-\alpha D}$. (iii) $U(\cdot;0)$ is invariant under $(z_1,z_2) \mapsto (-z_1,-z_2)$ and $(z_1,z_3) \mapsto (-z_1,-z_3)$, so $\mathbb{E}_{\mu_0} z_i = 0$ and the third cumulant is $\mathbb{E}_{\mu_0} z_1z_2z_3$. Let γ_1 be the standard Gaussian measure on $\mathbb{R}$, $a_n = \int t\sin^n t\,d\gamma_1$ and $b_n = \int \sin^n t\,d\gamma_1$. Expanding $e^{-J\sin z_1 \sin z_2 \sin z_3}$, which is legitimate since the product is bounded,

$$\mathbb{E}_{\mu_0} z_1z_2z_3 = \frac{\sum_n (-J)^n a_n^3/n!}{\sum_n (-J)^n b_n^3/n!}.$$

By parity $a_n = 0$ for even n and $b_n = 0$ for odd n; $b_0 = 1$ and $b_n \geq 0$; $|a_n| \leq \int t^2 d\gamma_1 = 1$ for $n \geq 1$, since $|\sin t| \leq \min\{1,|t|\}$; and $a_1 = \int \cos t\,d\gamma_1 = e^{-1/2}$ by Gaussian integration by parts. So the numerator has modulus at least $Je^{-3/2} - (\sinh J - J)$ and the denominator lies in $[1,\cosh J]$. As $(2i+1)! \geq 6(2i-2)!$, $\sinh J - J \leq \frac{J^3}{6}\cosh J$, and with $J \leq \frac{1}{6}$, $\cosh J < 1.014$ and $e^{-3/2} > 0.2231$, as $e^{3/2} < 2.71829 \cdot 1.64873 < 4.482$,

$$|\mathbb{E}_{\mu_0} z_1z_2z_3| \geq \frac{J(0.2231-0.0047)}{1.014} > \frac{J}{5} = \frac{e^{-\alpha D}}{30}.$$

The bound on trees is Proposition 508(iii) with $k = 3$. □

Remark 510. The tree form of the differences of order three and more is therefore not proved, and Propositions 508 and 509 locate what it needs. In the law of Proposition 509 the third cumulant is carried by the term $J\sin z_1 \sin z_2 \sin z_3$, whose mixed third derivative couples three sites at mutual distance D with size $e^{-\alpha D}$. The differences of order three of $\log \tilde{K}$ are third cumulants of the layer law (Remark 501), and a proof of their tree form needs two inputs that (41) does not contain: a bound in tree form on the mixed derivatives of order three of $\mathscr{U}$ in the coordinates z and in the island coordinates, and a weighted representation of the third cumulants of $\mu_{x,y}$ through the Helffer–Sjöstrand operator and these derivatives, valid on the box D_z with its walls. The proof of (41) reaches the walls through

the approximations H_n of the proof of Theorem 367(ii), whose derivatives of order three grow with n, so that argument gives no bound on third cumulants uniform in n; Theorem 511 below avoids them at the cost of half the rate, and Theorem 515 at the full rate under a bound in tree form on the derivatives of order three of $\mathscr{U}$. With both inputs the differences of $\log\tilde{K}$ would still have to be carried to those of $\log Z^c$ through the island weights, and uniformity in the coarse field stays excluded by Theorem 505(i).

The proof of (41) uses one property of the layer law, a weighted lower bound on the Hessian of a smooth approximation, and this property bounds the third cumulants in tree form at half the rate of the covariances, with the walls of D_z included; it gives no tree form uniform in the order. Let E be a finite set with a metric d, and for $e \in E$ let $\mathbb{V}_e$ be a Euclidean space; put $\mathbb{V} = \prod_e \mathbb{V}_e$, write $z = (z_e)$, $u = (u_e)$ for its points, ∂_e for the gradient in z_e, and $|\cdot|$ for the Euclidean norm. For $\varphi : E \to \mathbb{R}$ and $a \geq 0$, $e^{a\varphi}$ is the linear map of $\mathbb{V}$ that multiplies u_e by $e^{a\varphi(e)}$. A probability law μ on $\mathbb{V}$ satisfies $(\mathrm{W}_{\mathscr{A},\alpha})$, with $\mathscr{A}, \alpha > 0$, if:

$(\mathrm{W}_{\mathscr{A},\alpha})$ there are smooth $H_n : \mathbb{V} \to \mathbb{R}$ with bounded derivatives of every order at least two such that the laws μ_n with density proportional to e^{-H_n} converge to μ in total variation, and for all n, $z, u \in \mathbb{V}$, $0 \leq a \leq \alpha$ and φ with $|\varphi(e) - \varphi(e')| \leq d(e,e')$,

$$\big\langle u, e^{a\varphi} \operatorname{Hess} H_n(z)\, e^{-a\varphi} u \big\rangle \geq \mathscr{A}^{-1} |u|^2.$$

In the proof of Theorem 367(ii), carried to the layer law in Step 6 of the proof of Theorem 485, H_n is the potential of the layer law composed with the retraction τ of that step plus n times its convex penalty, $\operatorname{Hess} H_n \geq \lambda/60$ and $\|\mathrm{Re}(\operatorname{Hess} H_n)^w - \operatorname{Hess} H_n\| \leq \lambda/120$ for $w = e^{\alpha\varphi}$, $\alpha \leq \alpha_G$; for real u the quadratic form of $w \operatorname{Hess} H_n w^{-1}$ is that of its real part. So every law μ_x of Lemma 489, with E the free z-links, d the fine distance between their midpoints and the coordinates z, satisfies $(\mathrm{W}_{\mathscr{A},\alpha_G})$ with $\mathscr{A} = 120/\lambda = 480/\beta_W$, and (41) is the consequence drawn from it there. The law μ_x of Proposition 509 satisfies $(\mathrm{W}_{2,\alpha})$ with $H_n = U(\cdot;x)$, by part (i) of its proof. For bounded functions $F_1, \dots, F_k$ write $\varkappa_\mu(F_1, \dots, F_k)$ for their joint cumulant under μ; for $k = 3$ it is $\mathbb{E}_\mu[(F_1 - \mathbb{E}_\mu F_1)(F_2 - \mathbb{E}_\mu F_2)(F_3 - \mathbb{E}_\mu F_3)]$.

Theorem 511 (third cumulants under the layer bound)**.** *Let μ satisfy $(\mathrm{W}_{\mathscr{A},\alpha})$ and $0 \leq a \leq \alpha$. Let F, G, H be smooth bounded functions on $\mathbb{V}$ with bounded derivatives of orders one and two, let S_F be the set of e with $\partial_e F \not\equiv 0$, and S_G, S_H likewise, all nonempty; put $d_{FG} = d(S_F, S_G)$, d_{FH}, d_{GH} likewise, $f_1 = \sup|\nabla F|$, g_1, h_1 likewise, and $g_2 = \sup\|\operatorname{Hess} G\|$, $h_2 = \sup\|\operatorname{Hess} H\|$, the suprema over $\mathbb{V}$. Then*

$$\begin{aligned} |\varkappa_\mu(F,G,H)| \leq{}& \mathscr{A}^2 f_1 g_1 h_2 e^{-a(d_{FH}+d_{GH})} \\ &+ \mathscr{A}^2 f_1 g_2 h_1 e^{-a(d_{FG}+d_{GH})} \\ &+ \mathscr{A}^{3/2} f_1 g_1 h_1 e^{-a(d_{FG}+d_{FH})/2}. \end{aligned}$$

In particular, with ℓ the sum of the two smallest of d_{FG}, d_{FH}, d_{GH}, which is at least the length of the shortest tree joining S_F, S_G and S_H,

$$|\varkappa_\mu(F,G,H)| \leq \big(\mathscr{A}^2 f_1 (g_1 h_2 + g_2 h_1) + \mathscr{A}^{3/2} f_1 g_1 h_1\big) e^{-a\ell/2}.$$

Proof. The representation. Fix n, write $\mathbb{E}$, Cov and $\|\cdot\|$ for the mean, the covariance and the norm of $L^2(\mu_n)$, let $\mathscr{L} = -\Delta + \nabla H_n \cdot \nabla$ and $\mathscr{M} = \mathscr{L} \otimes 1 + \operatorname{Hess} H_n$, acting on $\mathbb{V}$-valued functions. With $\varphi = 0$, $(\mathrm{W}_{\mathscr{A},\alpha})$ gives $\operatorname{Hess} H_n \geq \mathscr{A}^{-1}$. For K smooth and bounded with bounded derivatives, the Helffer–Sjöstrand theory [65, 69, 92] gives a smooth ϕ_K with $\mathscr{L}\phi_K = K - \mathbb{E}K$ whose gradient $v_K = \nabla\phi_K$ lies in the domain of $\mathscr{M}$ and satisfies $\mathscr{M} v_K = \nabla K$, the derivative of $\mathscr{L}\phi_K$ being $\mathscr{M}\nabla\phi_K$; integration by parts gives $\operatorname{Cov}(K,K') = \mathbb{E}[(\mathscr{L}\phi_K)K'] = \mathbb{E}\langle v_K, \nabla K'\rangle$ for every $K' \in H^1(\mu_n)$. Write $D^K_{e'e} = \partial_{e'} v_{K,e}$, the block (e',e) of $\operatorname{Hess}\phi_K$, so that $D^K_{ee'}$ is the transpose of $D^K_{e'e}$ and the two have the same Hilbert–Schmidt norm $\|\cdot\|$.

Two energy bounds. Let φ be 1-Lipschitz, $0 \leq a \leq \alpha$, $w = e^{a\varphi}$ and $u = w v_K$. As w is constant in z, it commutes with $\mathscr{L} \otimes 1$, and $\mathbb{E}\langle u, (\mathscr{L} \otimes 1)u\rangle = \mathbb{E}\sum_{e,e'} \|\partial_{e'} u_e\|^2$. So

$$\begin{aligned} \mathbb{E}\sum_{e,e'} \|\partial_{e'} u_e\|^2 + \mathscr{A}^{-1}\|u\|^2 &\leq \mathbb{E}\langle u, w\mathscr{M} w^{-1} u\rangle \\ &= \mathbb{E}\langle u, w\nabla K\rangle \leq \|u\|\,\|w\nabla K\|. \end{aligned}$$

Hence $\|u\| \leq \mathscr{A}\|w\nabla K\|$, and the first sum is at most $\max_{t\geq 0}(t\|w\nabla K\| - t^2/\mathscr{A}) = \frac{\mathscr{A}}{4}\|w\nabla K\|^2$. In blocks,

$$\begin{aligned} \mathbb{E}\sum_e e^{2a\varphi(e)} |v_{K,e}|^2 &\leq \mathscr{A}^2 \|e^{a\varphi}\nabla K\|^2, \\ \mathbb{E}\sum_{e,e'} e^{2a\varphi(e)} \|D^K_{e'e}\|^2 &\leq \frac{\mathscr{A}}{4}\|e^{a\varphi}\nabla K\|^2, \end{aligned}$$

and the second holds with $e^{2a\varphi(e')}$ in place of $e^{2a\varphi(e)}$, by the symmetry of $\|D^K_{e'e}\|$. As $2e^{a(s+t)} \leq e^{2as} + e^{2at}$,

$$\mathbb{E}\sum_{e,e'} e^{a(\varphi(e)+\varphi(e'))} \|D^K_{e'e}\|^2 \leq \frac{\mathscr{A}}{4}\|e^{a\varphi}\nabla K\|^2.$$

For $\varphi = d(\cdot, S_K)$, which vanishes on S_K, $\|e^{a\varphi}\nabla K\| \leq k_1 = \sup|\nabla K|$.

The cumulant. The third cumulant is $\operatorname{Cov}(F, GH) - \mathbb{E}G\operatorname{Cov}(F,H) - \mathbb{E}H\operatorname{Cov}(F,G)$, and the representation with $K = F$ turns it into

$$\begin{aligned} \varkappa_{\mu_n}(F,G,H) &= \operatorname{Cov}(G,X) + \operatorname{Cov}(H,Y), \\ X = \langle v_F, \nabla H\rangle, &\qquad Y = \langle v_F, \nabla G\rangle. \end{aligned}$$

By the energy bounds v_F and its derivatives are in $L^2(\mu_n)$, so X and Y are in $H^1(\mu_n)$, and $\operatorname{Cov}(G,X) = \mathbb{E}\langle v_G, \nabla X\rangle = T_1 + T_2$ with

$$\begin{aligned} T_1 &= \mathbb{E}\sum_{e,e'} \langle v_{G,e'}, (\operatorname{Hess} H)_{e'e} v_{F,e}\rangle, \\ T_2 &= \mathbb{E}\sum_{e'} \sum_{e \in S_H} \langle v_{G,e'}, D^F_{e'e}\partial_e H\rangle. \end{aligned}$$

The blocks of $\operatorname{Hess} H$ vanish unless $e, e' \in S_H$, so with P the restriction to the coordinates of S_H, $|T_1| \leq h_2 \|Pv_G\|\|Pv_F\|$. The first energy bound with $\varphi = d(\cdot, S_F)$, which is at least d_{FH} on S_H, gives $\|Pv_F\| \leq \mathscr{A} f_1 e^{-a d_{FH}}$, and likewise

$\|Pv_G\| \le \mathscr{A} g_1 e^{-a d_{GH}}$; so $|T_1| \le \mathscr{A}^2 f_1 g_1 h_2 e^{-a(d_{FH}+d_{GH})}$. In T_2, $|\sum_{e\in S_H} D^F_{e'e}\partial_e H| \le h_1(\sum_{e\in S_H}\|D^F_{e'e}\|^2)^{1/2}$, and with $\varphi = d(\cdot,S_F)$, $\psi = d(\cdot,S_G)$ the Cauchy–Schwarz inequality gives

$$|T_2| \le h_1\Big(\mathbb{E}\sum_{e'} e^{2a\psi(e')}|v_{G,e'}|^2\Big)^{1/2} \times \Big(\mathbb{E}\sum_{e'}\sum_{e\in S_H} e^{-2a\psi(e')}\|D^F_{e'e}\|^2\Big)^{1/2}.$$

The first factor is at most $\mathscr{A} g_1$. For $e\in S_H$, $\varphi(e)\ge d_{FH}$, and $2\psi(e')+\varphi(e')\ge d(e',S_G)+d(e',S_F)\ge d_{FG}$, so $e^{-2a\psi(e')} \le e^{-a(d_{FG}+d_{FH})}e^{a(\varphi(e)+\varphi(e'))}$, and the last energy bound makes the second factor at most $(\mathscr{A}/4)^{1/2} f_1 e^{-a(d_{FG}+d_{FH})/2}$. So $|T_2| \le \frac12 \mathscr{A}^{3/2} f_1 g_1 h_1 e^{-a(d_{FG}+d_{FH})/2}$. The same argument with G and H exchanged bounds $\mathrm{Cov}(H,Y)$ by $\mathscr{A}^2 f_1 h_1 g_2 e^{-a(d_{FG}+d_{GH})} + \frac12 \mathscr{A}^{3/2} f_1 g_1 h_1 e^{-a(d_{FH}+d_{FG})/2}$. The sum is the bound of the theorem for μ_n, with constants not depending on n.

The limit. As F,G,H are bounded and $\mu_n \to \mu$ in total variation, the means of F,G,H,FG,FH,GH,FGH under μ_n converge to those under μ, and so does the cumulant. Each of the three exponents is a or $a/2$ times the sum of two of d_{FG}, d_{FH}, d_{GH}, hence at least $a\ell/2$, and the two shorter edges form a tree joining the three sets. □

Corollary 512 (third differences of the kernel)**.** *In the setting of Lemma 489, let u,v,w be three distinct islands, h_u,h_v,h_w vectors in the coordinates of x_u,x_v,x_w, and ℓ_{uvw} the sum of the two smallest of $(w_{uv}-2s-8)_+$, $(w_{uw}-2s-8)_+$, $(w_{vw}-2s-8)_+$. For a vector h in the coordinates of one island let $F_h = (\partial_h \mathscr{U})\circ\tau_\delta$, where $\delta>0$ is such that $\mathscr{U}$ is smooth on the product of the $(r+\delta)$-balls of the z-links times $\prod_{u'} D_{u'}$, which exists by Lemma 488(ii) and the compactness of $D_z\times\prod_{u'}D_{u'}$, and τ_δ acts link by link as a smooth radial map $\zeta_e\mapsto \rho(\|\zeta_e\|)\zeta_e/\|\zeta_e\|$ with $\rho(t)=t$ for $t\le r$, $0\le\rho'\le 1$ and $\rho\le r+\delta/2$. Let $\hat\Lambda_2$, $\hat\Lambda_3$ be the suprema of $|\nabla_z F_h|$ and $\|\mathrm{Hess}_z F_h\|$ over all z, all $x\in\prod_{u'}D_{u'}$ and all h with $\|h\|\le 1$; they are finite, the derivatives of $\mathscr{U}$ being continuous on a compact set and those of τ_δ bounded. Then*

$$\partial_{h_u}\partial_{h_v}\partial_{h_w}\log\tilde K(x) = -\varkappa_{\mu_x}(\partial_{h_u}\mathscr{U},\partial_{h_v}\mathscr{U},\partial_{h_w}\mathscr{U}),$$

and, with $\mathscr{A} = 480/\beta_W$,

$$\big|\partial_{h_u}\partial_{h_v}\partial_{h_w}\log\tilde K(x)\big| \le \big(2\mathscr{A}^2\hat\Lambda_2^2\hat\Lambda_3 + \mathscr{A}^{3/2}\hat\Lambda_2^3\big)\|h_u\|\|h_v\|\|h_w\| e^{-\alpha_G \ell_{uvw}}.$$

Consequently the mixed difference of order three of $\log\tilde K$ in (x_u,x_v,x_w), between two points of each of D_u,D_v,D_w with the other arguments fixed, is at most this bound with $\|h_j\|$ replaced by $2rn_j^{1/2}$, n_j the number of x-links of island j.

Proof. The domain D_z does not depend on x and the integrand is smooth, so $\partial_h \log\tilde K = -\mathbb{E}_{\mu_x}\partial_h\mathscr{U}$ and $\partial_h\mathbb{E}_{\mu_x}\Phi = \mathbb{E}_{\mu_x}\partial_h\Phi - \mathrm{Cov}_{\mu_x}(\Phi,\partial_h\mathscr{U})$. No term of $\mathscr{U}$ depends on the coordinates of two islands (Lemma 488(ii)), so $\partial_{h_u}\partial_{h_v}\mathscr{U}$, $\partial_{h_u}\partial_{h_w}\mathscr{U}$ and $\partial_{h_v}\partial_{h_w}\mathscr{U}$ vanish; hence $\partial_{h_u}\partial_{h_v}\log\tilde K = \mathrm{Cov}_{\mu_x}(A,B)$ with $A=\partial_{h_u}\mathscr{U}$, $B=\partial_{h_v}\mathscr{U}$, and with $C=\partial_{h_w}\mathscr{U}$,

$$\partial_{h_w}\mathrm{Cov}_{\mu_x}(A,B) = -\mathrm{Cov}(AB,C)+\mathbb{E}A\,\mathrm{Cov}(B,C) + \mathbb{E}B\,\mathrm{Cov}(A,C) = -\varkappa_{\mu_x}(A,B,C).$$

The map τ_δ is the identity on D_z, so the cumulant $\varkappa_{\mu_x}(A,B,C)$ equals $\varkappa_{\mu_x}(F_{h_u},F_{h_v},F_{h_w})$; as τ_δ acts link by link, $\nabla_z F_{h_u}$ is supported on the support S_u of $\nabla_z\partial_{x_u}\mathscr{U}$, and the supports are at fine distance at least $d'_{uv}=2(w_{uv}-2s-8)_+$ by Lemma 488(iii). Apply Theorem 511 to μ_x, which satisfies $(\mathrm{W}_{\mathscr{A},\alpha_G})$, with $a=\alpha_G$, $f_1\le\hat\Lambda_2\|h_u\|$, $g_2\le\hat\Lambda_3\|h_v\|$ and so on; the exponents are at least α_G times half the sum of two of the d', that is at least $\alpha_G\ell_{uvw}$. For the difference, D_u, D_v, D_w are convex, two of their points differ by at most $2r$ in each link, and the difference is the integral of the third derivative over the unit cube. □

The rate in Theorem 511 is half the rate a of the covariance bound, and under $(\mathrm{W}_{\mathscr{A},\alpha})$ alone this is optimal: the law of Proposition 509 satisfies $(\mathrm{W}_{2,\alpha})$, its three sites have $\ell=2D$, and its third cumulant is at least $\frac{1}{30}e^{-\alpha D} = \frac{1}{30}e^{-\alpha\ell/2}$. For the orders above three $(\mathrm{W}_{\mathscr{A},\alpha})$ gives no tree form with a rate independent of the order.

Proposition 513 (the layer bound and cumulants of every order)**.** *Let $k\ge 3$, $\alpha>0$, $D>0$, and give k sites $1,\dots,k$ the distances $d(i,j)=D$ for $i\ne j$. For $x\in\mathbb{R}^k$ let μ_x be the law on $\mathbb{R}^k$ with density proportional to $e^{-U(z;x)}$,*

$$U(z;x) = \tfrac12|z|^2 + J\prod_{j=1}^k \sin z_j - x\cdot z, \qquad J = \tfrac12 e^{-k}e^{-\alpha D}.$$

Then μ_x satisfies $(\mathrm{W}_{2,\alpha})$ for every x, every tree whose vertices are the k sites has length $(k-1)D$, and

$$\big|\varkappa_{\mu_0}(z_1,\dots,z_k)\big| = \big|\partial_{x_1}\cdots\partial_{x_k}\log K(0)\big| \ge \tfrac13 e^{-3k/2}e^{-\alpha D},$$

with $K(x)=\int e^{-U(z;x)}dz$. So a bound $C_0C^k e^{-\kappa\ell}$ on the cumulants of order k of the laws satisfying $(\mathrm{W}_{2,\alpha})$, with ℓ the length of the shortest joining tree, holds for all D only if $\kappa\le\alpha/(k-1)$, and no $\kappa>0$ serves every order.

Proof. The condition. $\mathrm{Hess}_z U = I + JM$, where $M_{ii} = -\prod_j \sin z_j$ and $M_{ij} = \cos z_i\cos z_j\prod_{l\ne i,j}\sin z_l$ for $i\ne j$, all of modulus at most one. For φ with $|\varphi(i)-\varphi(j)|\le D$ and $0\le a\le\alpha$ the entries of $e^{a\varphi}Me^{-a\varphi}$ have modulus at most $e^{\alpha D}$, so its norm is at most $ke^{\alpha D}$ and $\|Je^{a\varphi}Me^{-a\varphi}\| \le \frac{k}{2}e^{-k}\le\frac12$. Hence the quadratic form of $e^{a\varphi}\mathrm{Hess}_z U e^{-a\varphi}$ is at least $\frac12|u|^2$; U is smooth with bounded derivatives of order at least two, and $H_n = U(\cdot;x)$ gives $(\mathrm{W}_{2,\alpha})$. A tree on k vertices has $k-1$ edges, each of length D.

The cumulant. As in Proposition 509(ii) the derivatives of $\log K$ at 0 are the joint cumulants of z under μ_0. For $i\ne j$ the map that changes the signs of z_i and z_j preserves $U(\cdot;0)$. If A is a nonempty proper subset of $\{1,\dots,k\}$, choose $i\in A$ and $j\notin A$; this map changes the sign of $\prod_{l\in A} z_l$, so $\mathbb{E}_{\mu_0}\prod_{l\in A} z_l = 0$. In the expansion of the joint cumulant over the partitions of $\{1,\dots,k\}$ every partition with two or more blocks has a proper

block, so $\varkappa_{\mu_0}(z_1,\dots,z_k)=\mathbb{E}_{\mu_0}z_1\cdots z_k$. With γ_1, a_n, b_n as in the proof of Proposition 509,

$$\mathbb{E}_{\mu_0}z_1\cdots z_k=\frac{\sum_n(-J)^n a_n^k/n!}{\sum_n(-J)^n b_n^k/n!},$$

the expansion being legitimate since the product of the sines is bounded. There $a_n=0$ for even n, $b_n=0$ for odd n, $b_0=1$, $0\le b_n\le 1$, $|a_n|\le 1$ for $n\ge 1$ and $a_1=e^{-1/2}$. So the denominator lies in $[1,\cosh J]$ and the numerator has modulus at least $Je^{-k/2}-(\sinh J-J)\ge Je^{-k/2}-\frac{J^3}{6}\cosh J$, the last step as in that proof. Here $J\le\frac12 e^{-3}<0.025$, so $\cosh J<1.001$, and $\frac{J^2}{6}\cosh J\le\frac{1.001}{24}e^{-2k}\le 0.0418e^{-4}e^{-k/2}<0.001e^{-k/2}$, as $e^{-2k}\le e^{-4}e^{-k/2}$ for $k\ge 3$ and $e^4>2.7^4>53$. Hence

$$|\mathbb{E}_{\mu_0}z_1\cdots z_k|\ge\frac{0.999}{1.001}Je^{-k/2}>0.99\,Je^{-k/2}>\tfrac13 e^{-3k/2}e^{-\alpha D}.$$

For the last assertion, the tree bound with $\ell=(k-1)D$ and this lower bound give $\frac13 e^{-3k/2}\le C_0C^k e^{-(\kappa(k-1)-\alpha)D}$ for every $D>0$, which forces $\kappa(k-1)\le\alpha$. □

Under $(\mathrm{W}_{\mathscr{A},\alpha})$ alone the half rate of Theorem 511 is optimal. It is lost in the off-diagonal blocks of $\operatorname{Hess}\phi_F$, and these decay in both indices at the full rate when, in addition, the derivatives of order three of the potential decay along trees. The walls of a product of balls do not interfere: a penalty for walls of one link at a time is a sum of functions of one link each, so its derivatives of order three lie in the diagonal blocks, which the energy bound of that proof already controls. Keep the notation E, d, $\mathbb{V}_e$, $\mathbb{V}$ of $(\mathrm{W}_{\mathscr{A},\alpha})$, and for $\mathscr{A}>0$ and $a\ge 0$ consider:

$(\mathrm{W}^+_{\mathscr{A},a})$ there are $r_e>0$ such that μ has density proportional to $e^{-U}\mathbf{1}_\Omega$, where $\Omega=\prod_e\{|z_e|\le r_e\}$ and U is smooth on a neighbourhood of Ω, and $\langle u,e^{a\varphi}\operatorname{Hess}U(z)\,e^{-a\varphi}u\rangle\ge\mathscr{A}^{-1}|u|^2$ for all $z\in\Omega$, $u\in\mathbb{V}$ and φ with $|\varphi(e)-\varphi(e')|\le d(e,e')$.

For such U let

$$\Xi_a(U)=\sup_{z\in\Omega}\max_{e\in E}\Big(\sum_{e'\ne e}e^{2ad(e,e')}\big\|(\operatorname{Hess}U(z))_{e'e}\big\|^2\Big)^{1/2},$$

with operator norms of the blocks. For $z\in\Omega$ and $e',e,e''\in E$ let $\|D^3U(z)_{e'ee''}\|$ be the largest Hilbert–Schmidt norm of the bilinear form $(b,c)\mapsto D^3U(z)[b,c,y]$ on $\mathbb{V}_{e'}\times\mathbb{V}_e$ over $y\in\mathbb{V}_{e''}$ with $|y|\le 1$, and let $\Theta_a(U)$ be the supremum over $z\in\Omega$ of the norm on ℓ^2 of the matrix with rows indexed by the pairs (e',e) with $e'\ne e$, columns indexed by $e''\in E$, and entries $e^{a(d(e',e'')+d(e,e''))}\|D^3U(z)_{e'ee''}\|$. For smooth F on $\mathbb{V}$ put

$$f_2^{(a)}=\sup_{z\in\mathbb{V}}\Big(\sum_{e'\ne e}e^{2ad(e,e')}\big\|(\operatorname{Hess}F(z))_{e'e}\big\|^2\Big)^{1/2},$$

with Hilbert–Schmidt norms. For $\emptyset\ne S\subset E$ let

$$T_S(e',e)=\min_{p\in E}\big(d(e',p)+d(e,p)+d(p,S)\big),$$

the least length of a star with centre in E joining e', e and S. For fixed e it is a 1-Lipschitz function of e', and conversely, as a minimum of such functions; and $T_S(e,e)=d(e,S)$, by the triangle inequality and $p=e$. Norms of blocks of matrix-valued functions are Hilbert–Schmidt norms unless said otherwise.

Lemma 514 (two-index decay)**.** *Let μ satisfy $(\mathrm{W}^+_{\mathscr{A},a})$, let $\varepsilon>0$ and $\mathscr{A}_\varepsilon=(1+\varepsilon)\mathscr{A}$. There are n_0 and smooth $H_n:\mathbb{V}\to\mathbb{R}$, $n\ge n_0$, with bounded derivatives of every order at least two, such that the laws μ_n with density proportional to e^{-H_n} converge to μ in total variation,*

$$\big\langle u,e^{a\varphi}\operatorname{Hess}H_n(z)\,e^{-a\varphi}u\big\rangle\ge\mathscr{A}_\varepsilon^{-1}|u|^2$$

for all $z,u\in\mathbb{V}$ and 1-Lipschitz φ, and the following holds. Let F be smooth and bounded with bounded derivatives of every order, S_F and f_1 as in Theorem 511, ϕ_F the solution of $\mathscr{L}\phi_F=F-\mathbb{E}F$ for μ_n of the proof of that theorem, $D^F=\operatorname{Hess}\phi_F$, and $\mathbb{E}$ the mean under μ_n. Then

$$\Big(\mathbb{E}\sum_e e^{2ad(e,S_F)}\|D^F_{ee}\|^2\Big)^{1/2}\le\Big(\frac{\mathscr{A}_\varepsilon}{4}\Big)^{1/2}f_1,$$

$$\Big(\mathbb{E}\sum_{e'\ne e}e^{2aT_{S_F}(e',e)}\|D^F_{e'e}\|^2\Big)^{1/2}\le\frac{\mathscr{A}_\varepsilon}{2}c_\varepsilon+\eta_n f_1,$$

$$c_\varepsilon=f_2^{(a)}+\big(\mathscr{A}_\varepsilon^{1/2}(\Xi_a(U)+\varepsilon)+\mathscr{A}_\varepsilon(\Theta_a(U)+\varepsilon)\big)f_1,$$

where η_n does not depend on F and $\eta_n\to 0$ as $n\to\infty$.

Proof. The approximation. Put $\Omega_t=\prod_e\{|z_e|\le r_e+t\}$ and fix $e_0\in E$. The quadratic form in $(\mathrm{W}^+_{\mathscr{A},a})$ does not change when a constant is added to φ, and the 1-Lipschitz φ with $\varphi(e_0)=0$ form a compact set; so its minimum $m(z)$ over these φ and over $|u|=1$ is continuous in z where U is smooth, and so are the maximum and the matrix norm whose suprema over Ω define $\Xi_a(U)$ and $\Theta_a(U)$. As Ω is compact, there is $\delta>0$ such that U is smooth near Ω_δ, $m\ge\mathscr{A}_\varepsilon^{-1}$ on Ω_δ, and the two functions are at most $\Xi_a(U)+\varepsilon$ and $\Theta_a(U)+\varepsilon$ on Ω_δ. Let τ act link by link as $\tau_e(\zeta)=\rho_e(|\zeta|)\zeta/|\zeta|$ with ρ_e smooth, $\rho_e(t)=t$ for $t\le r_e+\delta/2$, $0\le\rho_e'\le 1$, $\rho_e\le r_e+\delta$ and ρ_e constant on $[r_e+\delta,\infty)$; let v_e be smooth and convex on $[0,\infty)$, zero on $[0,r_e]$, with $v_e''>0$ on (r_e,∞) and $v_e''=1$ on $[r_e+\delta/4,\infty)$; and put $V(z)=\sum_e v_e(|z_e|)$ and $H_n=U\circ\tau+nV$. Then τ maps $\mathbb{V}$ into Ω_δ and is the identity on $\Omega_{\delta/2}$, and H_n is smooth with bounded derivatives of every order at least two. Moreover

$$\operatorname{Hess}H_n=D\tau^*\big(\operatorname{Hess}U\circ\tau\big)D\tau+B_0+n\operatorname{Hess}V,$$

where $D\tau$, B_0 and $\operatorname{Hess}V$ are block diagonal, the block e of B_0 is $\langle\nabla_eU(\tau(z)),D^2\tau_e(z_e)\rangle$, of norm at most a constant C_1, and at a link with $|z_e|<r_e+\delta/2$ one has $D\tau_e=1$, $B_{0,e}=0$ and $\operatorname{Hess}v_e(|z_e|)\ge 0$. At a link with $|z_e|=t\ge r_e+\delta/2$ the eigenvalues of $\operatorname{Hess}v_e(|z_e|)$ are $v_e''(t)=1$ and $v_e'(t)/t\ge(t-r_e-\delta/4)/t\ge c_0=\min_e\delta/(4r_e+2\delta)$. The block diagonal matrices commute with $e^{a\varphi}$, so with I the set of links with $|z_e|<r_e+\delta/2$, and since $\tau(z)\in\Omega_\delta$,

$$\begin{aligned}\big\langle u,e^{a\varphi}\operatorname{Hess}H_n e^{-a\varphi}u\big\rangle&\ge\mathscr{A}_\varepsilon^{-1}|D\tau\,u|^2+(nc_0-C_1)\sum_{e\notin I}|u_e|^2\\&\ge\mathscr{A}_\varepsilon^{-1}|u|^2\end{aligned}$$

for $n\ge n_0=(C_1+\mathscr{A}_\varepsilon^{-1})/c_0$. Finally $H_n=U$ on Ω, $nV\to\infty$ off Ω, and $e^{-H_n}\le e^{-U\circ\tau-V}$, which is integrable as $U\circ\tau$ is

bounded and V grows quadratically; by dominated convergence and Scheffé's lemma $\mu_n \to \mu$ in total variation, the boundary of Ω being a null set.

Regularity. Fix $n \ge n_0$. For K smooth and bounded with bounded derivatives of every order, let P_t be the semigroup of $\mathscr{L}$, $P_tK(z) = \mathbb{E}K(X_t^z)$ with $dX_t = -\nabla H_n(X_t)\,dt + \sqrt{2}\,dB_t$, $X_0 = z$. The first variation of the flow $z \mapsto X_t^z$ has norm at most $e^{-t/\mathscr{A}_\varepsilon}$, and the variation of order j solves a linear equation with the matrix $-\operatorname{Hess} H_n \le -\mathscr{A}_\varepsilon^{-1}$ and an inhomogeneity made of products of at least two variations of lower order, with the bounded derivatives of H_n as coefficients; by induction it is at most $C_{j,n}e^{-t/\mathscr{A}_\varepsilon}$. So the derivatives of P_tK of every order $j \ge 1$ are at most $C'_{j,n}e^{-t/\mathscr{A}_\varepsilon}$, and $|\nabla P_tK| \le k_1 e^{-t/\mathscr{A}_\varepsilon}$ with $k_1 = \sup|\nabla K|$, which is the gradient bound of [5]. Hence $\phi_K = \int_0^\infty (P_tK - \mathbb{E}K)\,dt$ has bounded derivatives of every order at least one, $|v_K| \le \mathscr{A}_\varepsilon k_1$ pointwise, and every integration by parts below is justified. The energy bounds of the proof of Theorem 511 use only the weighted lower bound on $\operatorname{Hess} H_n$, so they hold with $\mathscr{A}_\varepsilon$; their second one with $\varphi = d(\cdot, S_F)$, restricted to $e' = e$, is the first claim.

The equation. Write $v = v_F$, $D = D^F$ and $\mathscr{H} = \operatorname{Hess} H_n$. Differentiating the component e of $\mathscr{M}v = \nabla F$ in the coordinates of e', with $[\partial_{e'}, \mathscr{L}] = \sum_{e''} \mathscr{H}_{e'e''}\partial_{e''}$,

$$\mathscr{L}D_{e'e} + \sum_{e''} \mathscr{H}_{e'e''}D_{e''e} + \sum_{e''} D_{e'e''}\mathscr{H}_{e''e} + R_{e'e} = (\operatorname{Hess} F)_{e'e},$$

where $R_{e'e}$ is the bilinear form $(b,c) \mapsto \sum_{e''} D^3H_n[b, c, v_{e''}]$ on $\mathbb{V}_{e'} \times \mathbb{V}_e$. For $e' \ne e$ the term nV contributes nothing to $R_{e'e}$, being a sum of functions of one link each, so $R_{e'e}$ is formed with $U \circ \tau$ alone. Let D° be D with its diagonal blocks replaced by zero, and $\mathscr{O}$ the off-diagonal part of $\mathscr{H}$. In the equation for $e' \ne e$ the diagonal blocks of D enter only through $\mathscr{H}_{e'e}D_{ee}$ and $D_{e'e'}\mathscr{H}_{e'e}$, so

$$\mathscr{L}D^\circ_{e'e} + \big(\mathscr{H}D^\circ + D^\circ\mathscr{H}\big)_{e'e} = S_{e'e},$$
$$S_{e'e} = (\operatorname{Hess} F)_{e'e} - R_{e'e} - \mathscr{O}_{e'e}D_{ee} - D_{e'e'}\mathscr{O}_{e'e},$$

for $e' \ne e$, and put $S_{ee} = 0$.

Coercivity. Let $\theta(e',e) = e^{aT_{S_F}(e',e)}$ and $Q_{e'e} = \theta(e',e)D^\circ_{e'e}$. Multiply the equation by $\theta(e',e)$, pair it with $Q_{e'e}$ and sum over $e' \ne e$. As θ does not depend on z, the term $\mathscr{L}$ gives $\mathbb{E}\sum\|\nabla Q_{e'e}\|^2 \ge 0$. For fixed e, $\sum_{e'}\langle Q_{e'e}, \theta(e',e)(\mathscr{H}D^\circ)_{e'e}\rangle$ is a sum, over an orthonormal basis (b_j) of $\mathbb{V}_e$, of the forms $\langle q_j, e^{a\psi_e}\mathscr{H}e^{-a\psi_e}q_j\rangle$ with $q_j = (Q_{e'e}b_j)_{e'} \in \mathbb{V}$ and $\psi_e = T_{S_F}(\cdot, e)$, which is 1-Lipschitz; so it is at least $\mathscr{A}_\varepsilon^{-1}\sum_{e'}\|Q_{e'e}\|^2$. The same holds for $D^\circ\mathscr{H}$ with the roles of the two indices exchanged, as $\mathscr{H}$ is symmetric. Hence, with $\|\cdot\|$ also the norm of $L^2(\mu_n)$ of a matrix-valued function,

$$\frac{2}{\mathscr{A}_\varepsilon}\|Q\|^2 \le \mathbb{E}\sum_{e' \ne e}\langle Q_{e'e}, \theta(e',e)S_{e'e}\rangle \le \|Q\|\,\|\theta S\|,$$

so that $\|Q\| \le \frac{\mathscr{A}_\varepsilon}{2}\|\theta S\|$.

The sources. The blocks of $\operatorname{Hess} F$ vanish unless both indices lie in S_F, and then $T_{S_F}(e',e) \le d(e',e)$; so the first source contributes at most $f_2^{(a)}$ to $\|\theta S\|$. For the diagonal blocks, $T_{S_F}(e',e) \le d(e',e) + d(e,S_F)$ and $\le d(e',e) + d(e',S_F)$, taking $p = e$ and $p = e'$; and $\mathscr{O}_{e'e} = D\tau^*_{e'}(\operatorname{Hess} U)_{e'e}(\tau(z))D\tau_e$ with $\|D\tau\| \le 1$ and $\tau(z) \in \Omega_\delta$. So

$$\mathbb{E}\sum_{e' \ne e}\theta(e',e)^2\|\mathscr{O}_{e'e}D_{ee}\|^2 \le (\Xi_a(U)+\varepsilon)^2\,\mathbb{E}\sum_e e^{2ad(e,S_F)}\|D_{ee}\|^2,$$

which by the first claim is at most $(\Xi_a(U)+\varepsilon)^2\frac{\mathscr{A}_\varepsilon}{4}f_1^2$, and likewise for $D_{e'e'}\mathscr{O}_{e'e}$, the symmetry of $\mathscr{H}$ giving the same sums over the other index. These two sources contribute at most $\mathscr{A}_\varepsilon^{1/2}(\Xi_a(U)+\varepsilon)f_1$. For R, let I be the set of links with $|z_e| < r_e + \delta/2$, $R^{\mathrm{in}}_{e'e}$ the part of the sum over e'' with $e', e, e'' \in I$, and $R^{\mathrm{out}} = R - R^{\mathrm{in}}$. As τ acts link by link and is the identity near z_f for $f \in I$, the derivatives in R^{in} are those of U at $\tau(z) \in \Omega_\delta$, and $T_{S_F}(e',e) \le d(e',e'') + d(e,e'') + d(e'',S_F)$; as setting entries of a nonnegative matrix to zero does not increase its norm on ℓ^2,

$$\|\theta R^{\mathrm{in}}\|^2 \le (\Theta_a(U)+\varepsilon)^2\,\mathbb{E}\sum_{e''} e^{2ad(e'',S_F)}|v_{e''}|^2 \le (\Theta_a(U)+\varepsilon)^2\mathscr{A}_\varepsilon^2 f_1^2,$$

by the first energy bound. The part R^{out} vanishes on $\Omega^\circ = \prod_e\{|z_e| < r_e + \delta/2\}$, and off it $|\theta R^{\mathrm{out}}| \le C_2|v| \le C_2\mathscr{A}_\varepsilon f_1$, with C_2 depending on U, δ, a and E only; so $\|\theta R^{\mathrm{out}}\| \le C_2\mathscr{A}_\varepsilon f_1 \mu_n(\mathbb{V}\setminus\Omega^\circ)^{1/2}$. Since $\Omega \subset \Omega^\circ$ and $\mu_n \to \mu$ in total variation, the second claim follows with $\eta_n = \frac{1}{2}C_2\mathscr{A}_\varepsilon^2\mu_n(\mathbb{V}\setminus\Omega^\circ)^{1/2}$. □

Theorem 515 (third cumulants in tree form)**.** *Let μ satisfy $(\mathrm{W}^+_{\mathscr{A},a})$, let F, G, H be smooth and bounded with bounded derivatives of every order, with S_F, d_{FG}, f_1, g_2 and so on as in Theorem 511 and $f_2^{(a)}$ as above, and put $\Xi = \Xi_a(U)$, $\Theta = \Theta_a(U)$ and*

$$\ell = \min\{d_{FG} + d_{FH}, d_{FG} + d_{GH}, d_{FH} + d_{GH}, L_3\},$$
$$L_3 = \min_{p \in E}\big(d(p,S_F) + d(p,S_G) + d(p,S_H)\big).$$

Then

$$|\varkappa_\mu(F,G,H)| \le \Big(\mathscr{A}^2 f_1(g_1h_2 + g_2h_1) + \mathscr{A}^{3/2}f_1g_1h_1 + \mathscr{A}^2 g_1h_1\big(f_2^{(a)} + (\mathscr{A}^{1/2}\Xi + \mathscr{A}\Theta)f_1\big)\Big)e^{-a\ell}.$$

The number ℓ is at most the length of every tree with vertices in E that meets S_F, S_G and S_H; when the three sets are single links it is the length of the shortest such tree. It is at least three quarters of the sum of the two smallest of d_{FG}, d_{FH}, d_{GH}.

Proof. Take H_n as in Lemma 514 and write $\psi = d(\cdot, S_G)$, $\varphi = d(\cdot, S_F)$, $T = T_{S_F}$. As in the proof of Theorem 511, $\varkappa_{\mu_n}(F,G,H) = \operatorname{Cov}(G,X) + \operatorname{Cov}(H,Y)$ with $X = \langle v_F, \nabla H\rangle$, $Y = \langle v_F, \nabla G\rangle$, and $\operatorname{Cov}(G,X) = T_1 + T_2$ with T_1 bounded there by $\mathscr{A}_\varepsilon^2 f_1 g_1 h_2 e^{-a(d_{FH}+d_{GH})}$. Split $T_2 = T_2^{\mathrm{d}} + T_2^\circ$ into the terms with $e' = e$ and with $e' \ne e$. Since $|\partial_e H| \le h_1$ for every e,

$$|T_2^{\mathrm{d}}| \le h_1\Big(\mathbb{E}\sum_{e \in S_H}|v_{G,e}|^2\Big)^{1/2}\Big(\mathbb{E}\sum_{e \in S_H}\|D^F_{ee}\|^2\Big)^{1/2} \le \tfrac{1}{2}\mathscr{A}_\varepsilon^{3/2}f_1g_1h_1e^{-a(d_{FH}+d_{GH})},$$

by the first energy bound with $\psi \ge d_{GH}$ on S_H and the first claim of Lemma 514 with $\varphi \ge d_{FH}$ on S_H. For T_2°, take $e \in S_H$, any e' and p attaining $T(e',e)$; then $d(p,S_G) \le d(p,e') + \psi(e')$ and $d(p,S_H) \le d(p,e)$, so $L_3 \le \psi(e') + T(e',e)$ and $e^{-2a\psi(e')} \le e^{-2aL_3} e^{2aT(e',e)}$. The Cauchy–Schwarz step of the proof of Theorem 511 and the second claim of the lemma give

$$\begin{aligned} |T_2^\circ| &\le h_1 \Big(\mathbb{E} \sum_{e'} e^{2a\psi(e')} |v_{G,e'}|^2 \Big)^{1/2} \\ &\quad \times \Big(\mathbb{E} \sum_{e'} \sum_{e \in S_H, e \ne e'} e^{-2a\psi(e')} \|D^F_{e'e}\|^2 \Big)^{1/2} \\ &\le \mathscr{A}_\varepsilon g_1 h_1 e^{-aL_3} \Big(\frac{\mathscr{A}_\varepsilon}{2} c_\varepsilon + \eta_n f_1 \Big), \end{aligned}$$

with c_ε as in the lemma. The same argument with G and H exchanged bounds $\mathrm{Cov}(H,Y)$, with d_{FG} in place of d_{FH} in the first two terms and the same L_3. Every exponent is a times a quantity at least ℓ, so $|\varkappa_{\mu_n}(F,G,H)|$ is at most the bound of the theorem with $\mathscr{A}_\varepsilon$, $\Xi + \varepsilon$, $\Theta + \varepsilon$ in place of $\mathscr{A}$, Ξ, Θ, plus $2\mathscr{A}_\varepsilon \eta_n f_1 g_1 h_1$. As $n \to \infty$ the cumulants converge, as in that proof, and $\eta_n \to 0$; then let $\varepsilon \to 0$.

For the length, let a tree with vertices in E contain $f \in S_F$, $g \in S_G$ and $h \in S_H$. The union of its paths between f, g and h is a path, of length at least the sum of two of $d(f,g)$, $d(f,h)$, $d(g,h)$ by the triangle inequality, or has one vertex p of degree three, and then its length is at least $d(p,f) + d(p,g) + d(p,h) \ge L_3$. For single links the star with the best centre, or the path, attains ℓ. Finally $d(p,S_F) + d(p,S_G) \ge d_{FG}$ and its two analogues give $L_3 \ge \frac{1}{2}(d_{FG} + d_{FH} + d_{GH})$, which is at least three quarters of the sum of the two smallest, the largest being at least their mean. □

Corollary 516 (third differences of the kernel at the full rate)**.** *In the setting of Corollary 512, with its F_h, $\hat\Lambda_2$, $\hat\Lambda_3$ and $\mathscr{A} = 480/\beta_W$, let $\hat\Lambda_3'$ be the supremum of $f_2^{(\alpha_G)}$ for $F = F_h$ over all $x \in \prod_{u'} D_{u'}$ and all h with $\|h\| \le 1$, and $\hat\Theta$ the supremum of $\Theta_{\alpha_G}(\mathscr{U}(\cdot;x))$ over x, with $\Omega = D_z$; both are finite. Put $\omega_{uv} = (w_{uv} - 2s - 8)_+$ and*

$$\begin{aligned} \ell^c_{uvw} = \min\Big\{ &\omega_{uv} + \omega_{uw},\ \omega_{uv} + \omega_{vw},\ \omega_{uw} + \omega_{vw}, \\ &\min_y \sum_{j \in \{u,v,w\}} (d_j(y) - s - 5)_+ \Big\}, \end{aligned}$$

the minimum over the blocks y in which z-links are based. Then

$$\begin{aligned} \big| \partial_{h_u} \partial_{h_v} \partial_{h_w} \log \tilde K(x) \big| &\le C_3^* \|h_u\| \|h_v\| \|h_w\| e^{-2\alpha_G \ell^c_{uvw}}, \\ C_3^* &= 2\mathscr{A}^2 \hat\Lambda_2^2 \hat\Lambda_3 + \mathscr{A}^{3/2} \hat\Lambda_2^3 + \mathscr{A}^2 \hat\Lambda_2^2 \hat\Lambda_3' \\ &\quad + \mathscr{A}^2 \big(1.97 \cdot 10^4 \beta_W^{1/2} + \mathscr{A} \hat\Theta \big) \hat\Lambda_2^3, \end{aligned}$$

and the mixed difference of order three of $\log \tilde K$ in (x_u, x_v, x_w) is at most this bound with $\|h_j\|$ replaced by $2rn_j^{1/2}$. When $w_{uv}, w_{uw}, w_{vw} \ge 2s+8$, $\ell^c_{uvw} \ge \frac{3}{4}\ell_{uvw} - \frac{3}{2}$, with ℓ_{uvw} as in Corollary 512.

Proof. On D_z the potential $\mathscr{U}(\cdot;x)$ differs by a constant from the potential $\tilde U$ of Step 3 of the proof of Theorem 485, restricted to the free links, both being densities of μ_x; so on D_z its Hessian is a principal submatrix of $\mathrm{Hess}_\zeta \tilde U$, which by Steps 4 and 5 there is at least $\lambda/60$ and has $\|\cdot\|_{(0.197)} \le 2068.4\lambda$. By Lemma 368(ii), as in Step 6, μ_x satisfies $(\mathrm{W}^+_{\mathscr{A},a})$ with $\Omega = D_z$, $\mathscr{A} = 120/\lambda$ and every $a \le \alpha_G$. With $\varphi = d(\cdot,e)$ the column e of $e^{a\varphi} M e^{-a\varphi}$, for $M = \mathrm{Hess}_z \mathscr{U}$, has the blocks $e^{ad(e',e)} M_{e'e}$ and norm at most $\|M\|_{(a)}$; it has rank at most $\dim \mathfrak{g} = 3$, so the sum of the squared operator norms of its blocks is at most its squared Hilbert–Schmidt norm, at most $3\|M\|^2_{(a)}$. Hence $\Xi_{\alpha_G}(\mathscr{U}) \le \sqrt{3} \cdot 2068.4\lambda < 896\beta_W$, by Lemma 368(i), and $\mathscr{A}^{1/2}\Xi_{\alpha_G}(\mathscr{U}) < 1.97 \cdot 10^4 \beta_W^{1/2}$. The suprema $\hat\Lambda_3'$ and $\hat\Theta$ are finite because $\mathscr{U}$ is smooth on a neighbourhood of the compact set $D_z \times \prod_{u'} D_{u'}$ and τ_δ has bounded derivatives. The identity for the third derivative is that of Corollary 512. Apply Theorem 515 with $a = \alpha_G$ to F_{h_u}, F_{h_v}, F_{h_w}, with $f_1 \le \hat\Lambda_2 \|h_u\|$, $g_2 \le \hat\Lambda_3 \|h_v\|$, $f_2^{(\alpha_G)} \le \hat\Lambda_3' \|h_u\|$ and so on. The supports satisfy $d(S_u, S_v) \ge 2\omega_{uv}$ by Lemma 488(iii). A z-link p based in a block y is at coarse distance at least $d_j(y) - s - 4$ from the blocks of the links of S_j, which have $d_j \le s+4$, so $d(p,S_j) \ge 2(d_j(y) - s - 5)_+$, as links based in blocks at coarse distance $D \ge 1$ have midpoints at fine distance at least $2D-2$. Hence $\ell \ge 2\ell^c_{uvw}$ and $e^{-\alpha_G \ell} \le e^{-2\alpha_G \ell^c_{uvw}}$. The difference follows as in Corollary 512. For the last claim, $d_j(y) + d_{j'}(y) \ge w_{jj'} + 1$, so $\sum_j (d_j(y) - s - 5)_+ \ge \frac{1}{2}\sum (w_{jj'} + 1) - 3s - 15 = \frac{1}{2}(\omega_{uv} + \omega_{uw} + \omega_{vw}) - \frac{3}{2}$, and as in the proof of Theorem 515 half the sum of the three ω is at least $\frac{3}{4}\ell_{uvw}$. □

So at three islands the differences of order three of $\log \tilde K$ decay at the rate $2\alpha_G$ of the second differences, in the coarse length of the shortest joining tree. When the three widths are at least $2s+8$ the exponent is at least $\frac{3}{2}$ times that of Corollary 512 up to the additive constant $3\alpha_G$, and the bound is uniform in x and in the approximations; the walls of D_z enter only through the lower bound on the weighted Hessian. The condition $\Theta_a(U) < \infty$ uniformly in the geometry is what $(\mathrm{W}_{\mathscr{A},\alpha})$ lacks: for the potential of Proposition 509, the entry of the row $(1,2)$ and the column 3 of the matrix that defines Θ_a is $e^{2aD} J \cos z_1 \cos z_2 \cos z_3$, equal to $\frac{1}{6} e^{(2a-\alpha)D}$ at $z = 0$, so the quantity defined by the same formula is bounded in D only for $a \le \alpha/2$, the rate at which Theorem 511 is optimal. For the layer law it is finite by compactness, and its size is set by the finite range of $\mathscr{U}$: every term of $\mathscr{U}$ depends on the z-links of a bounded number of blocks at bounded mutual distance (Steps 1 to 3 of the proof of Theorem 483), so the entries vanish unless e', e, e'' lie within a fixed fine distance R of each other, and the Schur test gives $\hat\Theta \le N_R^{3/2} e^{2\alpha_G R} \Lambda_3$, with N_R the largest number of z-links within distance R of one z-link and Λ_3 the supremum of $\|D_z^3 \mathscr{U}_{e'ee''}\|$; Λ_3 is not computed here.

Remark 517. Theorem 511 and Corollary 512 bound the differences of order three of $\log \tilde K$ at the rate α_G in the coarse distance from the lower bound on the Hessian of the approximations H_n alone, and by the paragraph after Corollary 512 this half rate is all that $(\mathrm{W}_{\mathscr{A},\alpha})$ gives. Theorem 515 and Corollary 516 supply both inputs named in Remark 510 at the full rate $2\alpha_G$: the bound in tree form on the derivatives of order three of $\mathscr{U}$ is the finiteness of Θ_{α_G}, and the weighted repre-

sentation is the two-index decay of $\operatorname{Hess}\phi_F$ of Lemma 514, valid on D_z with its walls and uniform in n. Differentiating $\mathscr{M}v_F = \nabla F$ brings in the derivatives of order three of H_n; those of the penalty, which grow with n at the walls, lie in the diagonal blocks, which the energy bound controls with the weight $e^{2ad(e,S_F)}$, and the off-diagonal blocks satisfy a coercive equation whose sources involve only the off-diagonal blocks of $\operatorname{Hess}\mathscr{U}$, the diagonal blocks of $\operatorname{Hess}\phi_F$ and the derivatives of order three of $\mathscr{U}$. The rate is measured by the length of the shortest tree through one further link, which for three single links is at most the sum of the two shorter distances and can be smaller. For the orders $k \geq 4$ the representation needs derivatives of v_F of order two, which these bounds do not control, and Proposition 513 shows that $(\mathrm{W}_{\mathscr{A},\alpha})$ gives at most the rate $\alpha/(k-1)$. What finite range adds at these orders, and what remains between the bounds on $\log\tilde{K}$ and item (ii) of Sec. XIII V, is taken up next.

The law of Proposition 513 owes its large cumulants to a k-body term whose k sites are at mutual distance D, so its range grows with D. If every term of the potential has diameter at most a fixed R and the potential is smooth with bounded derivatives, the energy method of the proof of Theorem 511, applied to the derivatives of every order of the Helffer–Sjöstrand solutions, gives the tree form at the full rate for every order. The setting is that of $(\mathrm{W}_{\mathscr{A},\alpha})$. For nonempty sets $S_1,\dots,S_m \subset E$ let $\ell(S_1,\dots,S_m)$ be the least length of a tree whose vertices are $S_1,\dots,S_m$ and finitely many points of E, an edge between two vertices having the length of their distance; thus $\ell(S_1)=0$, $\ell(S_1,S_2)=d(S_1,S_2)$, and ℓ is at most the sum of the two smallest of $d(S_1,S_2)$, $d(S_1,S_3)$, $d(S_2,S_3)$ for $m=3$. Inside one of the sets the tree travels at no cost, as along the chains of Corollary 492. For a smooth function F on $\mathbb{V}$ and $j \geq 1$, $D^jF(z)$ is the tensor with the blocks $\partial_{e_1}\cdots\partial_{e_j}F(z) \in \mathbb{V}_{e_1}\otimes\cdots\otimes\mathbb{V}_{e_j}$, and $|\cdot|$ is the Euclidean norm of a tensor or of a block. A smooth H on $\mathbb{V}$ has range R if $\partial_{e_1}\cdots\partial_{e_i}H \equiv 0$ whenever $i \geq 2$ and $d(e_p,e_q) > R$ for some p,q; for $i \geq 3$ put

$$\Lambda_i(H) = \sup_z \max_{e_1\in E} \sum_{e_2,\dots,e_i\in E} \big|\partial_{e_1}\cdots\partial_{e_i}H(z)\big|.$$

Theorem 518 (cumulants of every order under finite range)**.** *Let $\mathscr{A},\alpha,R>0$ and $k\geq 2$. Let H be smooth on $\mathbb{V}$, with bounded derivatives of every order at least two and range R, such that $(\mathrm{W}_{\mathscr{A},\alpha})$ holds with $H_n = H$ for all n, and let μ have density proportional to e^{-H}. Let $0 \leq a \leq \alpha$, let $F_1,\dots,F_k$ be smooth bounded functions with bounded derivatives of every order, S_i the set of e with $\partial_e F_i \not\equiv 0$, all nonempty, $\ell = \ell(S_1,\dots,S_k)$,*

$$f = \max_{i\leq k}\max_{1\leq j\leq k-1}\sup_z |D^jF_i(z)|, \qquad \Lambda = \max_{2\leq i\leq k-1} e^{aiR}\Lambda_{i+1}(H),$$

with $\Lambda = 0$ for $k=2$. Then

$$|\varkappa_\mu(F_1,\dots,F_k)| \leq K_k f^k e^{-a\ell},$$

$$K_k = \big((k-1)!\big)^2 e^{k-2}\mathscr{A}^{k-1}(1+\mathscr{A}\Lambda)^{(k-1)(k-2)/2}.$$

In particular $K_2 = \mathscr{A}$ and $K_3 = 4e\mathscr{A}^2(1+\mathscr{A}\Lambda)$.

Proof. Helffer–Sjöstrand solutions. Let H' be smooth with bounded derivatives of order at least two and $\operatorname{Hess}H' \geq \mathscr{A}'^{-1}$, μ' its law, $\mathscr{L}' = -\Delta + \nabla H'\cdot\nabla$, and G smooth with bounded derivatives of every order. Let X_t^z solve $dX = -\nabla H'(X)\,dt + \sqrt{2}\,dB$, $X_0 = z$, and $P_tG(z) = \mathbb{E}G(X_t^z)$. The derivative J_t of X_t^z in z solves $\dot{J} = -\operatorname{Hess}H'(X_t)J$, so $|J_t| \leq e^{-t/\mathscr{A}'}$; the derivative of order $j \geq 2$ solves the same equation with a source made of products of at least two derivatives of lower order and of derivatives of H', so by induction and the Duhamel formula it is at most $C_j e^{-t/\mathscr{A}'}$. Hence every derivative of P_tG of order $j \geq 1$ is at most $C_j' e^{-t/\mathscr{A}'}$, and $|P_tG(z) - \mathbb{E}_{\mu'}G| \leq \sup|\nabla G| e^{-t/\mathscr{A}'}\mathbb{E}_{\mu'}|z-z'|$, by the coupling of X^z and $X^{z'}$ with the same noise and the invariance of μ'. So $\phi_G = \int_0^\infty (P_tG - \mathbb{E}_{\mu'}G)\,dt$ is smooth, its derivatives of every order $j \geq 1$ are bounded by constants depending only on $\mathscr{A}'$ and on the bounds of the derivatives of H' and G, and $\mathscr{L}'\phi_G = G - \mathbb{E}_{\mu'}G$; this is the solution of [65, 69, 92]. Integration by parts gives $\operatorname{Cov}_{\mu'}(G,F) = \mathbb{E}_{\mu'}\langle\nabla\phi_G,\nabla F\rangle$. For smooth F with bounded derivatives write $\Gamma'_F G = \langle\nabla\phi_G,\nabla F\rangle$, again smooth and bounded with bounded derivatives.

The representation. We show

$$\varkappa_{\mu'}(F_1,\dots,F_k) = \sum_\sigma \mathbb{E}_{\mu'}\big[\Gamma'_{F_{\sigma(k)}}\cdots\Gamma'_{F_{\sigma(2)}}F_1\big], \qquad (46)$$

the sum over the bijections σ of $\{2,\dots,k\}$, for every H' as above, by induction on k; for $k=2$ it is the covariance formula. Let $k \geq 3$, $\varepsilon_0 = (2\mathscr{A}'\sup\|\operatorname{Hess}F_k\|)^{-1}$, and for $|\varepsilon| < \varepsilon_0$ let μ^ε have density proportional to $e^{-H'+\varepsilon F_k}$, with ϕ^ε, Γ^ε, $\mathbb{E}_\varepsilon$ its objects; $\operatorname{Hess}(H'-\varepsilon F_k) \geq (2\mathscr{A}')^{-1}$, and the bounds above hold uniformly in ε. The joint cumulant is a mixed derivative of $\log\mathbb{E}e^{\sum_i t_iF_i}$ at $t=0$, so $\varkappa_{\mu'}(F_1,\dots,F_k)$ is the derivative at $\varepsilon = 0$ of $\varkappa_{\mu^\varepsilon}(F_1,\dots,F_{k-1})$, which by induction is a sum of $\mathbb{E}_\varepsilon$ of words in the Γ^ε. For X smooth with bounded derivatives, $\mathscr{L}_{\varepsilon'} = \mathscr{L}_\varepsilon - (\varepsilon'-\varepsilon)\nabla F_k\cdot\nabla$ gives

$$\mathscr{L}_{\varepsilon'}(\phi_X^{\varepsilon'} - \phi_X^\varepsilon) = (\varepsilon'-\varepsilon)\Gamma^\varepsilon_{F_k}X - (\mathbb{E}_{\varepsilon'}X - \mathbb{E}_\varepsilon X),$$

whose left side has mean zero under $\mu^{\varepsilon'}$; so $\nabla\phi_X^{\varepsilon'} - \nabla\phi_X^\varepsilon = (\varepsilon'-\varepsilon)\nabla\phi^{\varepsilon'}_{\Gamma^\varepsilon_{F_k}X}$. Applied twice, with the uniform bounds, this gives $\partial_\varepsilon\nabla\phi_X^\varepsilon = \nabla\phi^\varepsilon_{\Gamma^\varepsilon_{F_k}X}$, uniformly with all derivatives in z, and by linearity in X, for X_ε smooth in ε,

$$\partial_\varepsilon\big(\Gamma^\varepsilon_F X_\varepsilon\big) = \Gamma^\varepsilon_F\partial_\varepsilon X_\varepsilon + \Gamma^\varepsilon_F\Gamma^\varepsilon_{F_k}X_\varepsilon.$$

With $\partial_\varepsilon\mathbb{E}_\varepsilon X_\varepsilon = \mathbb{E}_\varepsilon\partial_\varepsilon X_\varepsilon + \operatorname{Cov}_\varepsilon(X_\varepsilon,F_k) = \mathbb{E}_\varepsilon[\partial_\varepsilon X_\varepsilon + \Gamma^\varepsilon_{F_k}X_\varepsilon]$ and $\partial_\varepsilon F_1 = 0$, the derivative of a word with $k-2$ letters is the sum of the $k-1$ words obtained by inserting Γ_{F_k} at each of its $k-1$ places, the outermost included, and each bijection of $\{2,\dots,k\}$ arises once. This is (46).

The weights. For a family $\mathscr{S} = (S_1,\dots,S_m)$ of nonempty subsets of E let $d_{\mathscr{S}}(e,e')$ be the least value of $\sum_i c(y_{i-1},y_i)$ over the chains $e = y_0,\dots,y_n = e'$, with $c(y,y') = 0$ when y,y' lie in a common S_l and $c(y,y') = d(y,y')$ otherwise; it is a pseudometric and $d_{\mathscr{S}} \leq d$. For $j \geq 0$ and $e_1,\dots,e_j \in E$ let $\psi_{\mathscr{S}}(e_1,\dots,e_j)$ be the least $d_{\mathscr{S}}$-length of a tree with vertices in E that contains $e_1,\dots,e_j$ and a point of each S_l. Then:

(a) $\psi_{\mathscr{S}}$ is symmetric, and 1-Lipschitz for d in each argument, the others fixed: add an edge;

(b) $\psi_{(S_1)}(e_1,\dots,e_j)=0$ when every $e_i\in S_1$;

(c) $\psi_{\mathscr{S}}(e_1,\dots,e_j)\le\psi_{\mathscr{S}}(g,e_{i+1},\dots,e_j)+\sum_{p\le i}d(e_p,g)$: add the edges from g to e_p;

(d) if $\mathscr{S}'$ is $\mathscr{S}$ with a set S added, $g\in S$ and $e_{i+1},\dots,e_j\in S$, then $\psi_{\mathscr{S}'}(e_1,\dots,e_j)\le\psi_{\mathscr{S}}(e_1,\dots,e_i,g)$: the optimal tree for the right side meets S at g, the edges from g to $e_{i+1},\dots,e_j$ have $d_{\mathscr{S}'}$-length zero, and $d_{\mathscr{S}'}\le d_{\mathscr{S}}$;

(e) every tree with vertices in E that meets each S_l has $d_{\mathscr{S}}$-length at least $\ell(S_1,\dots,S_m)$.

For (e) replace each edge by a chain that realises $d_{\mathscr{S}}$, each step of cost zero inside S_l by two edges of length zero through the vertex S_l, and join every S_l to a point of the tree in S_l by an edge of length zero: the result is a connected graph on $S_1,\dots,S_m$ and points of E whose edges are at least as long as the distances of their ends, and one of its spanning trees is admissible for ℓ.

Energy bounds of every order. From now on $\mathbb{E}$, Cov and $\|\cdot\|$ refer to μ, ϕ_G and Γ_F to $H'=H$, and $\|u\|^2=\mathbb{E}|u|^2$ for a tensor valued u. Let $\mathscr{S}$ be a family as above, G smooth and bounded with bounded derivatives, $\phi=\phi_G$, and for $j\ge1$ let w_j multiply the block $(e_1,\dots,e_j)$ by $e^{a\psi_{\mathscr{S}}(e_1,\dots,e_j)}$, $y_j=\|w_jD^jG\|$ and $c_j=\|w_jD^j\phi\|$. We show $c_1\le\mathscr{A}y_1$ and, for $2\le j\le k-1$,

$$c_j\le\frac{\mathscr{A}}{j}\Big(y_j+\sum_{i=2}^{j}\binom{j}{i}e^{aiR}\Lambda_{i+1}(H)\,c_{j-i+1}\Big).\qquad(47)$$

Differentiating $\mathscr{L}\phi=G-\mathbb{E}G$ in $z_{e_1},\dots,z_{e_j}$ and using the Leibniz rule on $\langle\nabla H,\nabla\phi\rangle$, with $u=D^j\phi$,

$$\mathscr{L}u+\sum_{p=1}^{j}\mathrm{Hess}_{(p)}\,u=D^jG-\sum_{|P|\ge2}R_P,$$

where $\mathrm{Hess}_{(p)}$ applies $\mathrm{Hess}\,H$ in the p-th index, P runs over the subsets of $\{1,\dots,j\}$ with at least two elements, and $(R_P)_{e_1\dots e_j}=\sum_g\partial_{e_P}\partial_gH\,\partial_{e_{P^c}}\partial_g\phi$, the contraction being in $\mathbb{V}_g$ and e_P the indices in P. Put $\tilde u=w_ju$, which is bounded since E is finite. The weight is constant in z, so $\mathbb{E}\langle\tilde u,w_j\mathscr{L}u\rangle=\mathbb{E}\langle\tilde u,\mathscr{L}\tilde u\rangle\ge0$. For fixed indices other than the p-th, $\psi_{\mathscr{S}}$ is a 1-Lipschitz function of the p-th by (a), so $(\mathrm{W}_{\mathscr{A},\alpha})$, applied to each component in an orthonormal basis of the other factors, gives $\langle\tilde u,w_j\,\mathrm{Hess}_{(p)}\,w_j^{-1}\tilde u\rangle\ge\mathscr{A}^{-1}|\tilde u|^2$. Hence

$$\frac{j}{\mathscr{A}}\|\tilde u\|^2\le\|\tilde u\|\Big(\|w_jD^jG\|+\sum_{|P|\ge2}\|w_jR_P\|\Big).$$

Let $|P|=i$. The block $\partial_{e_P}\partial_gH$ vanishes unless $d(e_p,g)\le R$ for $p\in P$, and then (c) gives $\psi_{\mathscr{S}}(e_1,\dots,e_j)\le\psi_{\mathscr{S}}(e_{P^c},g)+iR$. With $u'=D^{j-i+1}\phi$ and $w'=w_{j-i+1}$,

$$e^{a\psi_{\mathscr{S}}(e)}|(R_P)_e|\le e^{aiR}\sum_g M_{e_P,g}\big|(w'u')_{e_{P^c},g}\big|,$$

with $M_{e_P,g}=|\partial_{e_P}\partial_gH|$. By the symmetry of the blocks, the sums of M over g for fixed e_P and over e_P for fixed g are at most $\Lambda_{i+1}(H)$, so by Schur's test the matrix M has norm at most $\Lambda_{i+1}(H)$ on ℓ^2, and summing the squares over e_{P^c} gives $\|w_jR_P\|\le e^{aiR}\Lambda_{i+1}(H)c_{j-i+1}$. There are $\binom{j}{i}$ sets P of size i, and (47) follows; for $j=1$ there is no R_P.

Products. Let $\mathscr{S}'$ be $\mathscr{S}$ with $S=\{e:\partial_eF\not\equiv0\}$ added, F smooth and bounded with bounded derivatives, $Y=\Gamma_FG=\sum_{g\in S}\langle\partial_g\phi,\partial_gF\rangle$, and w'_j, $y'_j=\|w'_jD^jY\|$ the objects of $\mathscr{S}'$. Then

$$\partial_{e_1}\cdots\partial_{e_j}Y=\sum_B\sum_{g\in S}\big\langle\partial_{e_B}\partial_g\phi,\partial_{e_{B^c}}\partial_gF\big\rangle,$$

the sum over $B\subset\{1,\dots,j\}$. The term of B vanishes unless e_{B^c} lie in S, and then (d) gives $\psi_{\mathscr{S}'}(e_1,\dots,e_j)\le\psi_{\mathscr{S}}(e_B,g)$. By the Cauchy–Schwarz inequality in g and the sum over e_B and e_{B^c}, the term of B has weighted norm at most $\sup|D^{j-|B|+1}F|\,c_{|B|+1}$, so

$$y'_j\le\sum_{b=0}^{j}\binom{j}{b}\sup|D^{j-b+1}F|\,c_{b+1}.$$

Assembly. Fix σ, put $Y_1=F_1$ and $Y_m=\Gamma_{F_{\sigma(m)}}Y_{m-1}$, and let $\mathscr{S}_m=(S_1,S_{\sigma(2)},\dots,S_{\sigma(m)})$; the term of σ in (46) is $\mathbb{E}Y_k$. Write $\hat y_{m,j}$ for $y_j/j!$ of $G=Y_m$ and $\mathscr{S}_m$, $\hat Y_{m,j}=\max_{i\le j}\hat y_{m,i}$, and $\hat c_j=c_j/j!$. Dividing (47) by $j!$, with $\frac{j-i+1}{j}\le1$ and $\sum_{i\ge2}\frac{1}{i!}=e-2<1$,

$$\hat c_j\le\mathscr{A}\hat y_j+\mathscr{A}\Lambda\sum_{i=2}^{j}\frac{\hat c_{j-i+1}}{i!},$$

and induction on j gives $\hat c_j\le\mathscr{A}\hat Y_j(1+\mathscr{A}\Lambda)^{j-1}$ for $j\le k-1$, with $\hat Y_j=\max_{i\le j}\hat y_i$. The product bound divided by $j!$ gives $\hat y_{m,j}\le f\sum_{b\le j}\frac{b+1}{(j-b)!}\hat c_{b+1}$, with $\hat c$ for $G=Y_{m-1}$ and $\mathscr{S}_{m-1}$, so

$$\hat Y_{m,j}\le e(j+1)f\mathscr{A}(1+\mathscr{A}\Lambda)^j\hat Y_{m-1,j+1}$$

whenever $j+1\le k-m+1$. By (b), $\hat Y_{1,k-1}\le f$. Iterating from $m=k-1$, $j=1$ down to $m=2$, $j=k-2$,

$$\hat Y_{k-1,1}\le e^{k-2}(k-1)!\,f^{k-1}\mathscr{A}^{k-2}(1+\mathscr{A}\Lambda)^{(k-1)(k-2)/2},$$

since $\prod_{m=2}^{k-1}(k-m+1)=(k-1)!$ and $\sum_{m=2}^{k-1}(k-m)=\frac{(k-1)(k-2)}{2}$. Finally $\mathbb{E}Y_k=\sum_{g\in S_{\sigma(k)}}\mathbb{E}\langle\partial_g\phi,\partial_gF_{\sigma(k)}\rangle$ with $\phi=\phi_{Y_{k-1}}$, and for $g\in S_{\sigma(k)}$ a tree admissible for $\psi_{\mathscr{S}_{k-1}}(g)$ meets all k sets, so by (e) and $d_{\mathscr{S}_{k-1}}\ge d_{\mathscr{S}_k}$, $\psi_{\mathscr{S}_{k-1}}(g)\ge\ell$. The Cauchy–Schwarz inequality gives

$$|\mathbb{E}Y_k|\le e^{-a\ell}f\,c_1\le e^{-a\ell}f\mathscr{A}\hat Y_{k-1,1}.$$

Summing over the $(k-1)!$ bijections gives the bound. □

The rate is the rate a of the covariance bound, for every order; for $k=3$ it is the rate of Theorem 515, here without walls. The case $j=2$ of (47) is a two-index bound on $\mathrm{Hess}\,\phi_F$ of the kind asked for in Remark 517, with the weight $e^{a\psi(e,e')}$, $\psi(e,e')=\psi_{(S_F)}(e,e')$, in place of $e^{a(d(e,e')+d(e,S_F))}$. Finite range enters

only through (c): the source R_P of the equation for $D^j\phi$ is carried by indices within R of a common point. In the law of Proposition 513 the range is D, the factor e^{aiR} in Λ is e^{aiD}, and the bound of the theorem is consistent with that proposition.

Remark 519. Theorem 518 applies to the smooth approximations of the layer law and not to the layer law. Each term of $\mathscr{U}$ involves links based in blocks at mutual coarse distance at most three, by Steps 1 and 2 of the proof of Theorem 483; the retraction τ and the penalty $V=\sum_e v(\|\zeta_e\|)$ of Theorem 367(ii) act link by link. So the potential $H_{x,n}$ of the approximation $\mu_{x,n}$ of μ_x in Step 6 of the proof of Theorem 485 has a finite range R_0 fixed by the geometry, bounded derivatives of every order, and $(\mathrm{W}_{480/\beta_W,\alpha_G})$. For distinct islands $u_1,\dots,u_k$, unit vectors h_p in the coordinates of x_{u_p} and $\tilde K_n(x)=\int e^{-H_{x,n}(z)}dz$, no term of $\mathscr{U}$ depends on two islands, so as in Corollary 512

$$\partial_{h_1}\cdots\partial_{h_k}\log\tilde K_n=(-1)^k\varkappa_{\mu_{x,n}}(F_{h_1},\dots,F_{h_k}),$$

and the theorem bounds it by $K_k(n)f^k e^{-\alpha_G\ell}$, with ℓ the tree length of the supports S_{u_p} of $\nabla_z\partial_{x_{u_p}}\mathscr{U}$ in the fine distance; by Lemma 488(iii) any two of these supports are at fine distance at least $d'_{u_pu_q}$, so for $k=2$ the factor is $e^{-2\alpha_G(w_{u_1u_2}-2s-8)_+}$, the rate of Lemma 489. As $n\to\infty$, $\tilde K_n$ decreases to $\tilde K$, so the mixed differences of $\log\tilde K$ are limits of those of $\log\tilde K_n$, and a bound on $K_k(n)$ uniform in n would give the tree form of every order of the differences of $\log\tilde K$ at the rate α_G in the fine distance, the first input named in Remark 510. The constant depends on n through Λ only, and $\Lambda_3(H_{x,n})$ grows linearly in n, the penalty nV contributing n times the third derivatives of $v(\|\zeta_e\|)$, which do not vanish identically. The layer law itself has no smooth potential on $\mathbb{V}$: its density vanishes off D_z, and the walls of D_z are the one hypothesis of Theorem 518 that it does not meet. For $k=3$ the walls are handled by Lemma 514, which needs the energy bound only for $j=1$ and keeps the penalty in the diagonal blocks. Whether the differences of order $k\ge4$ of $\log\tilde K$ have a tree form with a rate independent of k is not decided here; a proof along these lines needs the energy bounds (47) for $j\ge2$ on the box D_z, with the boundary of the balls in place of the penalty.

The second step, from $\log\tilde K$ to the weights of the labels, integrates the coordinates of the other islands against $\tilde K$. Its derivatives in the island coordinates are given by the cumulants of that gas. In the setting of Theorem 491 let $\mathscr{K}\subset\mathscr{I}$ be a set of $k\ge2$ islands, $\mathscr{Q}'=\mathscr{I}\setminus\mathscr{K}$, suppose every $\gamma_q\ne0$, $q\in\mathscr{Q}'$, and put

$$Z_{\mathscr{K}}(x_{\mathscr{K}})=\int\tilde K(x_{\mathscr{K}},x_{\mathscr{Q}'})\prod_{q\in\mathscr{Q}'}\gamma_q(dx_q),$$

with $P_{x_{\mathscr{K}}}$ the law on $\prod_{q\in\mathscr{Q}'}D_q$ proportional to $\tilde K(x_{\mathscr{K}},\cdot)\prod_q\gamma_q$.

Lemma 520 (derivatives through the gas of islands)**.** *For vectors h_u in the coordinates of x_u, $u\in\mathscr{K}$, and $B\subset\mathscr{K}$, let $g_B=\big(\prod_{u\in B}\partial_{h_u}\big)\log\tilde K$. Then*

$$\prod_{u\in\mathscr{K}}\partial_{h_u}\log Z_{\mathscr{K}}=\sum_\pi\varkappa_{P_{x_{\mathscr{K}}}}\big((g_B)_{B\in\pi}\big),$$

the sum over the partitions π of $\mathscr{K}$, the cumulant having one argument for each block of π.

Proof. $\tilde K$ is smooth and positive on a neighbourhood of the compact set $\prod_u D_u$, so for smooth $G_1,\dots,G_m$ the cumulant $\varkappa_P(G_1,\dots,G_m)$ is the mixed derivative at $s=0$ of

$$\log\int e^{\sum_i s_iG_i}\tilde K\prod_q\gamma_q(dx_q),$$

and differentiating under the integral,

$$\partial_h\varkappa_P(G_1,\dots,G_m)=\sum_i\varkappa_P(G_1,\dots,\partial_hG_i,\dots,G_m)+\varkappa_P(G_1,\dots,G_m,\partial_h\log\tilde K).$$

For $u\notin B$, $\partial_{h_u}g_B=g_{B\cup\{u\}}$. Order $\mathscr{K}=\{u_1,\dots,u_k\}$; $\partial_{h_{u_1}}\log Z_{\mathscr{K}}=\mathbb{E}_P\,g_{\{u_1\}}$, and if the derivative in $u_1,\dots,u_m$ is the sum over the partitions of $\{u_1,\dots,u_m\}$, the rule adds u_{m+1} to one block or opens the block $\{u_{m+1}\}$, which gives each partition of $\{u_1,\dots,u_{m+1}\}$ once. □

For $k=2$ the identity is the formula in the proof of Theorem 491. The partition $\pi=\{\mathscr{K}\}$ gives the mean of a mixed derivative of $\log\tilde K$, to which a tree form for $\log\tilde K$ applies directly. The partitions into two blocks give covariances under P, which Lemma 490 bounds through the oscillations $\delta_q(g_B)$, themselves mixed differences of $\log\tilde K$ of order $|B|+1$, under the weighted Dobrushin condition of Corollary 492. The partitions into three or more blocks are joint cumulants of order at least three of the gas of islands, and no result of this paper bounds them; Lemma 490 concerns covariances only. A rate that does not depend on the strength of the pair interactions of the gas is excluded already for the covariances.

Proposition 521 (a chain of strongly coupled islands)**.** *Let $N\ge1$, $J>0$, $X_q=\{-1,1\}$ with γ_q the counting measure for $0\le q\le N$, and $k(\xi)=\exp(J\sum_{q<N}\xi_q\xi_{q+1})$. Then the mixed differences of $\log k$ of order at least three vanish, the second difference in $(\xi_q,\xi_{q'})$ is $\pm4J$ for $|q-q'|=1$ and 0 otherwise, the coefficients of Lemma 490 are $C_{qq'}=\tanh J$ for $|q-q'|=1$, and under $P=k\prod_q\gamma_q/\int k\prod_q\gamma_q$*

$$\mathrm{Cov}_P(\xi_0,\xi_N)=(\tanh J)^N.$$

Proof. $\log k$ is a sum of functions of two consecutive coordinates, so its mixed differences of order at least three vanish, as do those of order two in nonconsecutive coordinates; for consecutive ones the four values of $J\xi_q\xi_{q+1}$ give $4J$ in modulus, so $\kappa_{qq'}=4J$ and $C_{qq'}=\tanh J$. Put $\tau_q=\xi_{q-1}\xi_q$, $1\le q\le N$; the map $\xi\mapsto(\xi_0,\tau)$ is a bijection and $k=\prod_q e^{J\tau_q}$, so under P the τ_q and ξ_0 are independent, $\mathbb{E}\xi_0=0$ and $\mathbb{E}\tau_q=\tanh J$. As $\xi_0\xi_N=\prod_q\tau_q$, the covariance is $(\tanh J)^N$. □

With $J=\Delta/4$ the coefficients are those of Theorem 491 for islands whose pair cross-ratios are Δ, and $\Delta_{qq'}$ grows with $\sqrt{n_qn_{q'}}$, the sizes of the islands. The covariance decays along the chain at the rate $-\log\tanh J$ per island, which tends to zero as $J\to\infty$, while, for consecutive islands at distance t and every $\lambda>0$, the mixed differences of $\log k$ of every order obey the tree bound $4Je^{\lambda t}e^{-\lambda\ell}$. So a tree form for $\log\tilde K$, at any rate and of every order, does not carry over to $\log Z^c$ at

a rate independent of the configuration of the islands: the passage needs a condition that bounds the pair cross-ratios against the gaps, such as the weighted Dobrushin condition of Corollary 492, and, by Lemma 520, bounds on the joint cumulants of order up to k of the gas of islands under such a condition.

Remark 522. For item (ii) of Sec. XIII V the results give the following. For a law with smooth potential of finite range and the weighted Hessian bound, the cumulants of every order obey the tree bound at the rate of the covariances (Theorem 518); this answers, in that class, the question left by Proposition 513, which shows that the finite range is needed. For the layer law the order three holds at the full rate with the walls (Corollary 516), and the tree form of $\log \tilde{K}$ of order $k \geq 4$ is proved only for the smooth approximations, with constants that grow with the approximation (Remark 519). The passage to $\log Z^c$ reduces, by Lemma 520, to the decay of the joint cumulants of the gas of islands; their order two is given by Lemma 490 under the weighted Dobrushin condition, their orders three and more are not proved, and Proposition 521 shows that no rate uniform in the pair cross-ratios of the islands holds. Uniformity in the coarse field stays excluded by Theorem 505(i). The tree form of the differences of $\log Z^c$, and with it the rate $2\alpha_G$ without a floor, is therefore not proved for any coarse field.

For a fixed coarse field c the blocks that are bad for every labelling are the corners of the coarse plaquettes with $\delta(c_P, I) > \eta$. On them no label restricts the active links, and μ^c there is the constrained law of Sec. XII D, with the links outside as exterior. The rest of this subsection determines what the chart estimates of that section give for this law, in the SU(2) model and not in the class (X) + (R): a component on which the coarse field stays below the scale of the chart is crossed at the rate α_G, with no cost in its length beyond the mass of the constrained law off the chart, and that mass tends to one as the length grows at fixed β_W; above four times the chart threshold of the coarse curvature no point of the fibre carries a chart. The mass off the chart is a ratio of two weights of the fibre, and the next lemma records why its denominator has to be controlled.

Lemma 523 (conditioned weights)**.** *Let γ be a finite measure, D, G, T measurable sets with $T \cap G = \emptyset$, V a real function on D that is bounded above on $G \cap D$ and below on $T \cap D$, and $\nu = e^{-V} \mathbf{1}_D \gamma$.*

(i) *If $\gamma(G^c) \leq \delta < \gamma(D)$, then $\nu(G) > 0$ and*

$$\frac{\nu(T)}{\nu(G)} \leq e^{\sup_{G\cap D} V - \inf_{T\cap D} V} \frac{\delta}{\gamma(D) - \delta}.$$

(ii) *If γ has no atoms, $\gamma(G) > 0$ and $T = G^c$ with $\gamma(T) > 0$, then for every M there is $D \supset T$ with $\nu(T)/\nu(G) \geq M$ at $V = 0$.*

Proof. (i) The two weights obey

$$\nu(T) \leq e^{-\inf_{T\cap D} V} \gamma(T \cap D) \leq e^{-\inf_{T\cap D} V} \gamma(G^c),$$
$$\nu(G) \geq e^{-\sup_{G\cap D} V} \gamma(G \cap D) \geq e^{-\sup_{G\cap D} V} \big(\gamma(D) - \gamma(G^c)\big),$$

and the last factor is positive. (ii) An atomless measure takes every value between 0 and $\gamma(G)$ on subsets of G; take $D = T \cup G_\varepsilon$ with $G_\varepsilon \subset G$ and $\gamma(G_\varepsilon) = \varepsilon = \gamma(T)/M$. Then $\nu(T)/\nu(G) = \gamma(T)/\varepsilon = M$. □

A tail bound, the factor $e^{\sup V - \inf V} \delta$, therefore controls a conditioned probability only together with a lower bound on the mass $\gamma(D)$ of the conditioning set. For the fibres of the path map this mass is computed: the reference measure is product Haar measure on $G^{45\mathscr{B}}$ and Theorem 349 bounds $Z_\lambda(v)$ from both sides. For the fibres of the cut-off average the reference measure carries the coarea weights $|\det D_g V|^{-1}$, and these are not bounded: in Proposition 418 the weights of the branches $g_\pm(\lambda)$ tend to infinity as $\lambda \downarrow \gamma^*$ while the Wilson action of every branch lies in $[0, 12]$, and the law of the path link tends to δ_{g^*} for every β_W. So no bound on the conditioned probability of a branch in terms of the action alone holds uniformly in the coarse field.

For SU(2) the constant $4 + 28\pi$ of Lemma 409 improves to 4, up to a term of second order.

Lemma 524 (the sharp witness)**.** *Let $G = \mathrm{SU}(2)$. Under the hypotheses of Lemma 409,*

$$\delta(c_P, I) \leq 4\kappa + 7700\kappa^2.$$

Proof. Three facts about SU(2) with the Hilbert–Schmidt norm are used.

(a) For $X, Y \in \mathfrak{g}$, $\delta(e^X e^Y, e^{X+Y}) \leq \|X\|\|Y\|$. Indeed, with $g(t) = e^{-t(X+Y)} e^{tX} e^{tY}$, $g'(t) = e^{-t(X+Y)}[e^{tX}, Y] e^{tY}$ and $[e^{tX}, Y] = \int_0^t e^{uX}[X,Y] e^{(t-u)X}\, du$, so $\|g'(t)\| \leq t\|[X,Y]\| \leq 2t\|X\|\|Y\|$, and $\delta(e^X e^Y, e^{X+Y}) = \delta(g(1), I) \leq \int_0^1 \|g'\|$. Applying this three times, with bi-invariance of δ and the triangle inequality,

$$\delta\big(e^{X_1} e^{X_2} e^{X_3} e^{X_4}, e^{X_1+X_2+X_3+X_4}\big) \leq \sum_{i<j} \|X_i\|\|X_j\|.$$

(b) If $\delta(g, I) \leq 1$, then $\|\log g\| \leq \delta(g,I)(1 + \delta(g,I)^2)$. Let $e^{\pm i\psi}$, $0 \leq \psi \leq \pi$, be the eigenvalues of g and $t = \psi/2$. Then $\|\log g\| = 2\sqrt{2}t$, $\delta(g, I) = 2\sqrt{2}\sin t$, and $\delta(g, I) \leq 1$ gives $t < \pi/2$. Since $t - \sin t \leq t^3/6$ and $t \leq \frac{\pi}{2}\sin t$, $t \leq \sin t + \frac{\pi^3}{48}\sin^3 t \leq \sin t\,(1 + 8\sin^2 t)$.

(c) If $X \in \mathfrak{g}$ and $\|X\| < \sqrt{2}\pi$, then $\log e^X = X$, as the eigenvalues $\pm i\psi$ of X have $\sqrt{2}\psi = \|X\| < \sqrt{2}\pi$; and $\delta(e^X, I) \leq \|X\|$ by Lemma 352.

Let the sides $k = 1, \ldots, 4$ of P be the pairs (y, μ), $(y + e_\mu, \nu)$, $(y + e_\nu, \mu)$, (y, ν), with path links ℓ_k, and for $x \in B(y)$ let P_x^k be the path of the pair k from $x + o_k$, where $o_1 = o_4 = 0$, $o_2 = 2e_\mu$, $o_3 = 2e_\nu$. As in the proof of Lemma 409, $\delta(P_x^k, \ell_k) \leq 24\kappa < \frac{1}{2}$, so the cut-off of each side equals one, $Z_k^x = \log(\ell_k^{-1} P_x^k)$ is defined, and by (b) $\|Z_k^x\| \leq 24\kappa(1 + 576\kappa^2) \leq a = 24.001\,\kappa$. Then $c_k = \ell_k e^{\bar{Z}_k}$ with $\bar{Z}_k = \frac{1}{16}\sum_x Z_k^x$ and $\|\bar{Z}_k\| \leq a$.

The loop $H_x = P_x^1 P_x^2 (P_x^3)^{-1} (P_x^4)^{-1}$ is the holonomy around the square of side two at x in the plane $\{\mu, \nu\}$, whose four fine plaquettes have their vertices in the four corner blocks; as in the proof of Lemma 409, $\delta(H_x, I) \leq 4\kappa$. At $x = 2y$ every P_x^k is ℓ_k, so $\ell_P = \ell_1 \ell_2 \ell_3^{-1} \ell_4^{-1} = H_{2y}$ has $\delta(\ell_P, I) \leq 4\kappa$, and $Y = \log \ell_P$ has $\|Y\| \leq 4.001\,\kappa$ by (b).

Let $g_1=\ell_1$, $g_2=\ell_1\ell_2$, $g_3=\ell_1\ell_2\ell_3^{-1}$, and for $Z\in\mathfrak{g}^4$

$$\Psi(Z)=\ell_1 e^{Z_1}\ell_2 e^{Z_2}e^{-Z_3}\ell_3^{-1}e^{-Z_4}\ell_4^{-1},$$
$$\mathsf{L}(Z)=\mathrm{Ad}(g_1)Z_1+\mathrm{Ad}(g_2)(Z_2-Z_3)-\mathrm{Ad}(g_3)Z_4.$$

Moving the ℓ_k to the right,

$$\Psi(Z)=e^{\mathrm{Ad}(g_1)Z_1}e^{\mathrm{Ad}(g_2)Z_2}e^{-\mathrm{Ad}(g_2)Z_3}e^{-\mathrm{Ad}(g_3)Z_4}\ell_P,$$

and $H_x=\Psi(Z^x)$, $c_P=\Psi(\bar Z)$. The maps $\mathrm{Ad}(g_i)$ are isometries, so by (a), for Z with $\|Z_k\|\le a$,

$$\delta\big(\Psi(Z),e^{\mathsf{L}(Z)}\ell_P\big)\le 6a^2\le 3457\kappa^2,$$
$$\delta\big(e^{\mathsf{L}(Z)}\ell_P,e^{\mathsf{L}(Z)+Y}\big)\le 4a\cdot 4.001\kappa\le 385\kappa^2.$$

For every x this gives $\delta(e^{\mathsf{L}(Z^x)+Y},I)\le 4\kappa+3842\kappa^2$; as $\|\mathsf{L}(Z^x)+Y\|\le 4a+4.001\kappa<\sqrt{2}\pi$, (c) and (b) give $\|\mathsf{L}(Z^x)+Y\|\le 4\kappa+3843\kappa^2$, since $(4\kappa+3842\kappa^2)^3\le\kappa^2$. The average over x is $\mathsf{L}(\bar Z)+Y$, so $\delta(e^{\mathsf{L}(\bar Z)+Y},I)\le\|\mathsf{L}(\bar Z)+Y\|\le 4\kappa+3843\kappa^2$, and

$$\delta(c_P,I)\le(3457+385)\kappa^2+4\kappa+3843\kappa^2=4\kappa+7685\kappa^2.$$

□

Proposition 531 below shows that the constant 4 cannot be lowered.

Proposition 525 (coarse curvature and the chart)**.** *Let $G=\mathrm{SU}(2)$ and let c be a coarse field.*

(i) *$V^\sharp(R(c))=c$. A plaquette p of $R(c)$ in the plane $\{\mu,\nu\}$ has $R(c)_p=I$ unless both coordinates μ and ν of its corner are odd; then its corner lies in a block y and $R(c)_p=c_P$, with P the coarse plaquette at y in that plane.*

(ii) *Let W be a set of blocks that contains the four corner blocks of a coarse plaquette P with $\delta(c_P,I)>2.418\cdot10^{-12}$. Then every field U with $V_k^\sharp(U)=c_k$ for the four sides k of P has a plaquette with a vertex in a block of W^+ and $\delta(U_p,I)>6.044\cdot10^{-13}$. Consequently no background U of the fibre over W satisfies the hypothesis $4\cdot10^5c_3(\kappa+s)\le\frac{1}{195}$ of Theorem 363(iv) with the constant $c_3=1.5\cdot10^4\sqrt{2}$ of Proposition 364, for any $s\ge0$, and Theorems 363(iv) and 367 apply at no point of that fibre.*

Proof. (i) Every path of the pair k is the product of a link inside a block, equal to I, and a private link of k, equal to c_k, so every path and the path link equal c_k; in the formula $c_k=\ell_k\exp(\chi N)$ of the proof of Lemma 411 every logarithm in N vanishes, so $V_k^\sharp(R(c))=c_k$. Let p have corner x and links (x,μ), $(x+e_\mu,\nu)$, $(x+e_\nu,\mu)$, (x,ν). A link (z,μ) of $R(c)$ is I when z_μ is even, since it lies inside a block, and is $c_{y',\mu}$ when z_μ is odd, y' the block of z. If x_μ is even, both μ-links of p are I, and the two ν-links start at x and $x+e_\mu$, which lie in one block and have the same coordinate ν, so they are equal and $R(c)_p=I$; likewise if x_ν is even. If both are odd and $x\in B(y)$, the four links are $c_{y,\mu}$, $c_{y+e_\mu,\nu}$, $c_{y+e_\nu,\mu}$, $c_{y,\nu}$, since $x+e_\mu\in B(y+e_\mu)$ and $x+e_\nu\in B(y+e_\nu)$, and their product is c_P.

(ii) Put $\kappa=6.044\cdot10^{-13}\le10^{-8}$. If every plaquette with a vertex in one of the blocks at distance at most one from the four corner blocks had $\delta(U_p,I)\le\kappa$, Lemma 524 would give $\delta(c_P,I)\le4\kappa+7700\kappa^2<2.4177\cdot10^{-12}$, as $4\kappa=2.4176\cdot10^{-12}$ and $7700\kappa^2<3\cdot10^{-21}$. Those blocks lie in W^+. A background with plaquettes up to $\kappa'>\kappa$ on W^+ has $4\cdot10^5c_3\kappa'>4\cdot10^5c_3\kappa>\frac{1}{195}$, since $195\cdot4\cdot10^5\cdot1.5\cdot10^4\sqrt{2}>1.6546\cdot10^{12}$ gives $1/(195\cdot4\cdot10^5c_3)<6.044\cdot10^{-13}$. □

At $\rho=2.17\cdot10^{-16}$, $\eta=92\rho<2.0\cdot10^{-14}$. So the coarse plaquettes that make a block bad for every labelling range over three scales: up to $6.0\cdot10^{-13}$, thirty times η, part (i) gives a background with plaquettes below the chart threshold of Proposition 364 whenever the exterior near the window agrees with $R(c)$; above $2.418\cdot10^{-12}$, part (ii) excludes every background; in between, Proposition 531 gives backgrounds for some exteriors, and Proposition 536 gives them for the exterior $R(c)$ of the coarse fields c^θ up to $2.39\cdot10^{-12}$. Where a background exists, the constrained law of the component decays across it at the rate of the chart.

Theorem 526 (a bad component on the chart)**.** *Let $G=\mathrm{SU}(2)$, $\beta_W\ge2.3\cdot10^{17}$, let W be a set of blocks and U a field with $\delta(U_p,I)\le\kappa$ for every plaquette with a vertex in a block of W^+, and let $s>0$ with $s\le1/(6000c_3)$, $\kappa+6s\le10^{-8}$ and $\kappa+s\le6.0\cdot10^{-13}$. Let μ_W be the constrained law on the fibre F of U over W, $r=s/32$, Φ and Ω_r as in Theorem 367, and*

$$p=\mu_W\big(F\setminus\Phi(\Omega_r)\big).$$

For sets $X_1,X_2\subset A_W$ and bounded smooth functions $\tilde F$, $\tilde G$ of the links of X_1 and of X_2 whose left-invariant gradients have norm at most 1 in each link,

$$\big|\mathrm{Cov}_{\mu_W}(\tilde F,\tilde G)\big|\le\frac{4.8\cdot10^4}{\beta_W}\big(|X_1||X_2|\big)^{1/2}e^{-\alpha_G d(X_1,X_2)}+6p\,\|\tilde F\|_\infty\|\tilde G\|_\infty,$$

with $\alpha_G=2.5\cdot10^{-4}$ and d the distance of Theorem 367. The bound does not depend on W, on the exterior or on the coarse field beyond κ and p, and the labels of the blocks of W enter neither its hypotheses nor the bound.

Proof. The hypotheses are those of Theorem 363(iv): by Proposition 364, $\kappa+s\le6.0\cdot10^{-13}$ gives $4\cdot10^5c_3(\kappa+s)\le\frac{1}{195}$, and $\lambda=\beta_W/4\ge5.6\cdot10^{16}\ge60c_6$. Let $A=\Phi(\Omega_r)$ and $\mu_A=\mu_W(\cdot\mid A)$. By Theorem 367(iii) and Proposition 369, $|\mathrm{Cov}_{\mu_A}(\tilde F,\tilde G)|$ is at most the first term, as $c_9/\lambda=4.8\cdot10^4/\beta_W$. If $p=0$ this is the claim, and if $p=1$ the claim follows from $|\mathrm{Cov}|\le\|\tilde F\|_\infty\|\tilde G\|_\infty$. Otherwise, with $\mu_{A^c}=\mu_W(\cdot\mid F\setminus A)$ and $\mu_W=(1-p)\mu_A+p\mu_{A^c}$,

$$\mathrm{Cov}_{\mu_W}(\tilde F,\tilde G)=(1-p)\,\mathrm{Cov}_{\mu_A}(\tilde F,\tilde G)+p\,\mathrm{Cov}_{\mu_{A^c}}(\tilde F,\tilde G)+p(1-p)\big(\mu_A\tilde F-\mu_{A^c}\tilde F\big)\big(\mu_A\tilde G-\mu_{A^c}\tilde G\big),$$

so the difference from Cov_{μ_A} is at most $p\|\tilde F\|_\infty\|\tilde G\|_\infty$ twice, from the first two terms, plus $4p\|\tilde F\|_\infty\|\tilde G\|_\infty$. Nothing in Theorems 363 and 367 refers to the labels of the blocks of W. □

Remark 527. Theorem 526 bounds the response inside a bad component by the chart, with no cost growing with its length except through p, and leaves three things. First, the mass p. By Lemma 523 it is bounded by the excess action off the chart times the reference mass off the chart, divided by the reference mass of the chart in the fibre; the first factor is a Peierls bound, the reference masses carry the coarea weights, and Proposition 418 shows that these weights are not controlled by the action. In the other direction Theorem 529 bounds p from below: at fixed β_W it tends to one as the component grows, and it exceeds $1-e^{-5.5|W|}$ for $\beta_W \le 10^{24}$ (Corollary 530). In the mean over the coarse field $\mathbb{E}_c$ the mass off the chart is a probability under μ_L; for a fixed coarse field it is the open step of Remark 362(c) and Remark 365(a). Second, the coarse fields above $2.418\cdot 10^{-12}$, where by Proposition 525(ii) the chart applies at no point; there the fibre law near the component is that of a large field, which Proposition 418 shows to jump, and no estimate of this paper bounds its response. A bound uniform in the coarse field needs that regime. Third, the passage from μ_W to μ^c and to the labels. Under μ^c, given the links outside A_W, the averages of the pairs that enter W from outside are fixed as well, and they bind links of A_W; and the labels adjoining a component are functions of links outside its interior, so that their covariance across it is the cross-ratio of the weight of the interior as a function of its two ends. For a good layer Theorem 483 solves the path links of the entering pairs by Lemma 411, writes the weight through a kernel on a domain that the ends do not move, and Lemma 454 turns the cross-ratio into a covariance of the layer law. Both steps use non-path links within r of the reconstruction. For a bad component Proposition 532 moves the links that bind the entering averages to the exterior, so that the constrained law of the reduced fibre is the conditional law under μ^c, and Corollary 533 bounds the cross-ratio for ends that enter the action only; ends that enter the averages, on which the labels depend, are not treated (Remark 534).

The other two routes do not reach the interior of a component. A single-link Dobrushin or disagreement-percolation bound needs row sums below one, while the coefficient of Lemma 399 is $\frac{\pi^2}{8}\beta_W$ for each link that shares a plaquette with the given one, and on the fibre the Kantorovich response of the path-link law to its non-path links is infinite by Proposition 418(iv); blocks large enough to contract must carry the constraints of the cut-off average, which is the chart of Theorem 526. The tree form of the mixed differences of $\log Z^c$ asked for in Remark 501 concerns the labels of distinct components, which are separated by good layers; a component bad for every labelling lies inside one component together with the blocks it adjoins, so neither that bound nor a cluster expansion built on it reaches the response across it. In the law of Theorem 505(i) the second difference over the labels of a and b, all other labels fixed, is J, and a bound on it decaying with $\operatorname{dist}(a,b)$ is exactly the response across the component.

The bound of Theorem 526 contains the mass p of the constrained law off the product chart $\Phi(\Omega_r)$, and that chart asks every one of the $45|W|$ coordinates η_e, $e\in L'$, to lie in a ball of radius r. On the chart the law has a density whose Hessian is bounded above, so each coordinate leaves its ball with a conditional probability bounded below, and p grows with $|W|$. The next lemma is the elementary estimate behind this.

Lemma 528 (leaving a ball)**.** *Let E be a Euclidean space, $r>0$, $B_t=\{x\in E:|x|\le t\}$, and let f be a real function of class C^2 on a neighbourhood of B_{2r} with $\|\operatorname{Hess} f(x)\|\le\Lambda$ for $x\in B_{2r}$. Then*

$$\int_{B_{2r}\setminus B_r} e^{-f}\,dx \ge \frac{1}{3}e^{-5.5\Lambda r^2}\int_{B_{2r}} e^{-f}\,dx.$$

Proof. Choose a unit vector u with $\nabla f(0)\cdot u\le 0$. For $x\in B_{2r}$, $\nabla f(x)\cdot u\le \nabla f(0)\cdot u+\Lambda|x|\le\Lambda|x|$, and for $x,x+v\in B_{2r}$, the segment between them lying in the convex set B_{2r}, $f(x+v)\le f(x)+\nabla f(x)\cdot v+\frac12\Lambda|v|^2$. Put $O=B_{2r}\setminus B_r$, $S_+=\{x\in B_r: x\cdot u\ge 0\}$ and $S_-=B_r\setminus S_+$.

For $x\in S_+$, $|x+ru|^2=|x|^2+2rx\cdot u+r^2$ lies in $[r^2,4r^2]$ and exceeds r^2 unless $x=0$, so the translation $x\mapsto x+ru$ maps S_+ into O up to a null set, and $f(x+ru)\le f(x)+r\Lambda|x|+\frac12\Lambda r^2\le f(x)+1.5\Lambda r^2$. Hence $\int_O e^{-f}\ge e^{-1.5\Lambda r^2}\int_{S_+}e^{-f}$. The reflection $R(x)=x-2(x\cdot u)u$ maps S_- into S_+ and preserves Lebesgue measure; for $x\in S_-$, $R(x)=x+tu$ with $t=2|x\cdot u|\le 2r$, so $f(R(x))\le f(x)+t\Lambda r+\frac12\Lambda t^2\le f(x)+4\Lambda r^2$, and $\int_{S_+}e^{-f}\ge e^{-4\Lambda r^2}\int_{S_-}e^{-f}$. Therefore

$$\int_{B_r} e^{-f}\le\big(1+e^{4\Lambda r^2}\big)e^{1.5\Lambda r^2}\int_O e^{-f},$$

and $\int_O e^{-f}\big/\int_{B_{2r}}e^{-f}\ge(1+e^{1.5\Lambda r^2}+e^{5.5\Lambda r^2})^{-1}\ge\frac13 e^{-5.5\Lambda r^2}$. □

Theorem 529 (the mass off the chart)**.** *Under the hypotheses of Theorem 526,*

$$\begin{aligned}p&\ge 1-\Big(1-\frac13 e^{-2843.5\beta_W r^2}\Big)^{45|W|}\\&\ge 1-\exp\big(-15|W|e^{-10^{-24}\beta_W}\big).\end{aligned}$$

Proof. Write Box_t for the product over $e\in L'$ of the balls $\{\|\xi_e\|\le t\}$ in $\mathfrak{g}$, so that $\Omega_t=S(\mathrm{Box}_t)$ in the notation of the proof of Theorem 367. Since $S(\mathrm{Box}_{2r})\subset K_{30.2r}$ and $30.2r<s$, the paragraph on the density in that proof, which uses Theorem 363(iii) on K_s only, shows that the constrained law μ_W conditioned on $\Phi(\Omega_{2r})$ has, in the coordinates $\xi\in\mathfrak{g}^{L'}$, the density proportional to $e^{-\bar H}$ on Box_{2r}, with $\bar H$ smooth on the preimage under S of the interior of K_s, a neighbourhood of Box_{2r}. By Proposition 369 and Lemma 368(i), $\|\operatorname{Hess}\bar H\|\le\|\operatorname{Hess}\bar H\|_{(\vartheta)}\le 2068\lambda$ on Box_{2r}, with $\lambda=\beta_W/4$.

Let $a=\mu_W(\Phi(\Omega_{2r}))$. If $a=0$, then $p=1$. Otherwise let ν be the conditioned law in the coordinates ξ, enumerate $L'=\{e_1,\dots,e_N\}$ with $N=45|W|$, and let $A_i=\{\|\xi_{e_i}\|\le r\}$. Under ν, given the coordinates other than ξ_{e_i}, the coordinate ξ_{e_i} has the density proportional to e^{-f} on the ball of radius $2r$ in $\mathfrak{g}$, where f is $\bar H$ with the other coordinates fixed and $\|\operatorname{Hess} f\|\le 2068\lambda$. By Lemma 528 with $\Lambda=2068\lambda$, the conditional probability of A_i is at most $1-q_0$, with

$$q_0=\tfrac13 e^{-5.5\cdot 2068\lambda r^2}=\tfrac13 e^{-2843.5\beta_W r^2}.$$

The events $A_{i+1},\dots,A_N$ are functions of the other coordinates, so $\nu(A_i\cap\cdots\cap A_N)\le(1-q_0)\,\nu(A_{i+1}\cap\cdots\cap A_N)$, and by induction $\nu(\mathrm{Box}_r)\le(1-q_0)^N$. As Φ is injective, $\mu_W(\Phi(\Omega_r))=a\,\nu(\mathrm{Box}_r)\le(1-q_0)^N$, which is the first bound. For the second, $1-q\le e^{-q}$, and $\kappa+s\le 6.0\cdot10^{-13}$ gives $r\le 1.875\cdot10^{-14}$, so $2843.5r^2<10^{-24}$. □

Corollary 530. *Under the hypotheses of Theorem 526:*

(i) *if $\beta_W\le10^{24}$, then $p\ge1-e^{-5.5|W|}$; at $\beta_W=2.3\cdot10^{17}$, $p\ge1-e^{-14.99|W|}$;*

(ii) *the bound of Theorem 526 is smaller than the trivial bound $\|\tilde F\|_\infty\|\tilde G\|_\infty$ only if $\beta_W>10^{24}(4.40+\ln|W|)$;*

(iii) *for fixed β_W, $p\to1$ as $|W|\to\infty$.*

Proof. (i) For $\beta_W\le10^{24}$, $q_0\ge\frac13e^{-1}>0.1226$ and $45q_0>5.5$; at $\beta_W=2.3\cdot10^{17}$, $q_0\ge\frac13(1-2.3\cdot10^{-7})$ and $45q_0>14.99$. Use $(1-q_0)^N\le e^{-Nq_0}$. (ii) The bound contains $6p\|\tilde F\|_\infty\|\tilde G\|_\infty$, so it is smaller than the trivial bound only if $p<\frac16$. Then $e^{-45|W|q_0}\ge1-p>\frac56$, so $15|W|e^{-2843.5\beta_Wr^2}<\ln\frac65<0.1824$; as $2843.5r^2<10^{-24}$, this gives $10^{-24}\beta_W>\ln(15|W|/0.1824)>4.40+\ln|W|$. (iii) follows from the second bound of Theorem 529. □

Fig. 41 draws the bound of Theorem 529 and the couplings at which Theorem 526 improves on the trivial bound.

So the absence of a cost in the length in Theorem 526 is carried entirely by p: a product chart in all the coordinates of the component, which the Helffer–Sjöstrand argument of Theorem 367 uses, holds the whole component near the background, and a component of n blocks leaves it unless β_Wr^2 exceeds a multiple of $\ln n$. With the constants of Proposition 364 the radius r is at most $1.875\cdot10^{-14}$, so the threshold $2.3\cdot10^{17}$ of the theorem is seven orders of magnitude below the coupling at which p can be small for a single block. In the other direction Corollary 375 gives $p\le\varepsilon_W(K,\lambda)$ on every fibre through a field whose plaquettes near W are at most $K/(4\sqrt{n_P})$, with $s\ge1.24\cdot10^5K$, which needs λK^2 larger than a multiple of $|W|\ln(n_P/K)$; the necessary condition grows like $\ln|W|$ and the sufficient one like $|W|$.

The fibres to which the theorem applies include arbitrarily long bad components, and their coarse plaquettes come arbitrarily close to four times the bound $6.0\cdot10^{-13}$ on $\kappa+s$, which by Lemma 524 no such fibre exceeds by more than $7700\kappa^2$. Let $T_0=\mathrm{diag}(i,-i)\in\mathfrak g$, so that $\delta(e^{\theta T_0},I)=2\sqrt2|\sin(\theta/2)|$ for real θ.

Proposition 531 (a uniform field strength)**.** *Let $G=\mathrm{SU}(2)$, let the side $2L$ of the periodic lattice have $L\ge4$ even, let $\mu\ne\nu$ be two directions and $0<f\le7\cdot10^{-9}$. Let $g(t)=f\min\{t,2L-t\}$ for $t\in\{0,\dots,2L-1\}=\mathbb Z/2L\mathbb Z$, so that $g(t+1)-g(t)=f$ for $0\le t\le L-1$ and $g(t+1)-g(t)=-f$ for $L\le t\le2L-1$, let U^f have the links $U^f_{x,\mu}=e^{-g(x_\nu)T_0}$ and all other links I, and let $\tilde U$ be its image in the tree gauge under the gauge transformation that equals I at the corner of every block. Put $\kappa_f=2\sqrt2\sin(f/2)$.*

(i) *Every plaquette has $\delta(\tilde U_p,I)\le\kappa_f$.*

(ii) *The coarse field $c=V^\sharp(\tilde U)$ takes values in $\{e^{\theta T_0}\}$, and for every block y with $0\le y_\nu\le L/2-2$ the coarse plaquette P at y in the plane $\{\mu,\nu\}$ has $c_P=e^{4fT_0}$, so that*

$$\delta(c_P,I)=2\sqrt2\sin2f=4\cos\tfrac f2\cos f\;\kappa_f\ge4(1-f^2)\kappa_f.$$

(iii) *For every set W of blocks the fibre of $\tilde U$ over W satisfies the hypotheses of Theorem 526 with $U=\tilde U$ and $\kappa=\kappa_f$, for every $\beta_W\ge2.3\cdot10^{17}$ and every $s>0$ with $s\le1/(6000c_3)$, $\kappa_f+6s\le10^{-8}$ and $\kappa_f+s\le6.0\cdot10^{-13}$.*

In particular, for every $t\in(0,2.39\cdot10^{-12}]$ there are a coarse field with a coarse plaquette P of $\delta(c_P,I)=t$ and, for every W, an exterior whose fibre over W satisfies the hypotheses of Theorem 526 with $s=2.4\cdot10^{-15}$; and the constant 4 of Lemma 524 cannot be lowered.

Proof. (i) Every link of U^f lies in the torus $\{e^{\theta T_0}\}$, and we give it the real angle $\theta_{x,\mu}=-g(x_\nu)$, or 0. The real sum of a plaquette at x, $\theta_1+\theta_2-\theta_3-\theta_4$ in the notation of Sec. XII D, is $g(x_\nu+1)-g(x_\nu)=\pm f$ in the plane $\{\mu,\nu\}$, and 0 in every other plane, since a plaquette in the plane $\{\mu,\rho\}$, $\rho\ne\nu$, has its two μ-links at the same coordinate ν. The gauge transformation is $e^{\lambda_xT_0}$ with λ_x real, 0 at the corners, and fixed along the tree of each block by the condition that the real angles $\tilde\theta_{x,\rho}=\theta_{x,\rho}+\lambda_x-\lambda_{x+e_\rho}$ of the tree links vanish. The real sums of the plaquettes do not change, so every plaquette of $\tilde U$ is $e^{\pm fT_0}$ or I.

(ii) Let k be a pair, $v=P_{2y}$ its path link and $v_x=v^{-1}P_x$. By (i) and Lemma 360(i), which applies as $\kappa_f\le\sqrt2f\le10^{-8}$, $\delta(v_x,I)\le24\kappa_f<\frac12$, so the cut-off equals one, and $v_x=e^{\varphi_xT_0}$ with $\varphi_x=\tilde\theta(P_x)-\tilde\theta(v)$, where $\tilde\theta$ of a path is the sum of the real angles along it. The principal logarithm of v_x is φ'_xT_0 with $\varphi'_x\equiv\varphi_x$ modulo 2π. As functions of $f\in[0,7\cdot10^{-9}]$, φ_x is linear, since g and the gauge transformation are, and φ'_x is continuous, since v_x stays within $24\kappa_f<1$ of I, where the principal logarithm is continuous; their difference lies in $2\pi\mathbb Z$ and vanishes at $f=0$, so $\varphi'_x=\varphi_x$. Hence

$$c_k=v\exp\Big(\frac{1}{16}\sum_x\varphi_xT_0\Big)=\exp\Big(\frac{1}{16}\sum_x\tilde\theta(P_x)\,T_0\Big).$$

With the sides $k=1,\dots,4$ of P as in the proof of Proposition 525(i) and the sites $x\in B(y)$, $x+2e_\mu$, $x+2e_\nu$ of the same offset,

$$c_P=\exp\Big(\frac{1}{16}\sum_{x\in B(y)}\tilde\theta(\partial S_x)\,T_0\Big),$$

where ∂S_x is the boundary of the square of side two at x in the plane $\{\mu,\nu\}$. Its real angle is the sum of the real sums of its four plaquettes, whose coordinates ν lie in $\{2y_\nu,2y_\nu+1,2y_\nu+2\}\subset\{0,\dots,L-1\}$, so each is f and $c_P=e^{4fT_0}$. Finally $\sin2f=4\sin\frac f2\cos\frac f2\cos f$, and $\cos\frac f2\cos f\ge(1-\frac{f^2}{8})(1-\frac{f^2}{2})\ge1-f^2$.

(iii) $\tilde U$ lies in its own fibre and has every plaquette at most κ_f; the other hypotheses are the conditions on β_W and s.

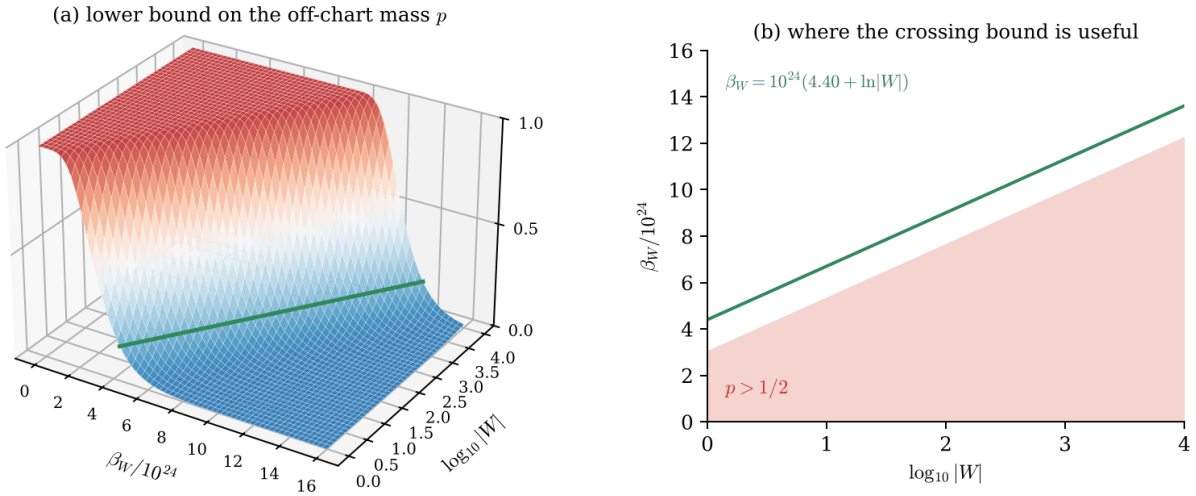


FIG. 41. The mass off the chart in a bad component W (Theorem 529). (a) The lower bound $1-\exp(-15|W|e^{-10^{-24}\beta_W})$ on p over $\beta_W/10^{24}$ and $\log_{10}|W|$. (b) The curve $\beta_W = 10^{24}(4.40+\ln|W|)$, above which the bound of Theorem 526 improves on the trivial one, and the region in which the lower bound on p exceeds $\frac{1}{2}$.

For the last statement let $f = \frac{1}{2}\arcsin(t/(2\sqrt{2}))$, so that $2\sqrt{2}\sin 2f = t$ and $f < 4.3\cdot 10^{-13}$. Then $\kappa_f \le t/(4(1-f^2)) < 5.976\cdot 10^{-13}$, so $s = 2.4\cdot 10^{-15}$ satisfies the conditions of (iii), as $1/(6000c_3) > 7.8\cdot 10^{-9}$. By (ii) the ratio $\delta(c_P, I)/\kappa_f$ tends to 4 as $f \to 0$, while $\kappa_f \le 10^{-8}$ and every plaquette of $\tilde{U}$ is at most κ_f. □

The blocks with $0 \le y_\nu \le L/2-1$ are corners of the coarse plaquettes of (ii), which exceed η once $2\sqrt{2}\sin 2f > \eta$, so they lie in one component bad for every labelling, of $L^4/2$ blocks. On these fibres Theorem 526 applies for every $|W|$, and Corollary 530 shows that its bound loses its content as $|W|$ grows.

Under μ^c the links of a window are not distributed by μ_W: the averages of the pairs that enter W from outside are fixed too, and they contain links of A_W. These links can be moved to the exterior. Let I_W be the set of active links (x,μ) inside a block $B(y)$, $y \in W$, with $y - e_\mu \notin W$, let $A_W^- = A_W \setminus I_W$, $L'^- = A_W^- \setminus \{d_k\}$, and for a field U let F^- be the set of fields that agree with U off A_W^- and have the same averages $V^\sharp_{y,\mu}$ as U for $y \in W$. The measure ν_c^- on F^- and the constrained law $\mu_W^- = e^{-\lambda Q^-}\nu_c^-/\nu_c^-(e^{-\lambda Q^-})$, with Q^- the sum of $\delta(U_p, I)^2$ over the plaquettes that contain a link of A_W^-, are defined as ν_c and μ_F before (39), with the links of I_W among the fixed links.

Proposition 532 (the window under μ^c)**.** *Let $G = \mathrm{SU}(2)$ and let W be a set of blocks.*

(i) *Every average $V^\sharp_{y',\mu}$ with $y' \notin W$ is a function of the links off A_W^-. For $\mathbb{E}_c$-almost every c, the constrained law μ_W^- on the fibre through a field is a version of the conditional law under μ^c of the links of A_W^- given the links off A_W^-.*

(ii) *Theorems 363, 367, 370(i), 526 and 529 and Propositions 364 and 369 hold for F^-, A_W^- and L'^- in place of F, A_W and L', with the same constants, and with $45|W|$ replaced by $|L'^-| \ge 28|W|$ in Theorem 529.*

Proof. (i) An inside link (x,μ) of a block $B(y)$ enters only the averages of (y,μ), as the first link of the path from x, and of $(y-e_\mu,\mu)$, as the second link of the path from $x-e_\mu$; a private link enters only the average of its pair. So $V^\sharp_{y',\mu}$, $y' \notin W$, depends on the links of $B(y')$ and the private links of (y',μ), which are not in A_W, and on inside links of $B(y'+e_\mu)$, which lie in I_W when $y'+e_\mu \in W$. In the tree gauge the active links have density proportional to $e^{-\beta_W S}$ with respect to product Haar measure h (proof of Theorem 370(ii)). Write the active links as (E,z) with z the links of A_W^-, and $C = (C_{\mathrm{out}}, C_W)$ for the averages of the pairs based off and in W, so that C_{out} is a function of E. The construction of (39), with the links of I_W fixed, gives $h(dz) = \int \nu^-_{E,c_W}(dz)\,dh(c_W)$, so

$$e^{-\beta_W S}h(dE)\,h(dz) = e^{-\beta_W S}h(dE)\int \nu^-_{E,c_W}(dz)\,dh(c_W),$$

and the conditional law of z given (E, C_W) is proportional to $e^{-\beta_W S}\nu^-_{E,C_W}$, which is μ_W^-, the plaquettes without a link of A_W^- giving a factor that does not depend on z. Since C_{out} is a function of E, conditioning on (E,C) is conditioning on (E,C_W), and by the tower property this is the conditional law under μ^c given E, for almost every c.

(ii) Let $J : \mathfrak{g}^{A_W^-} \to \mathfrak{g}^{A_W}$ be extension by zero. The fibre F^- is F with the links of I_W fixed, so ξ is tangent to F^- exactly when $J\xi$ is tangent to F, and on F^- the difference $Q - Q^-$ depends only on links off A_W^- and is constant. The window enters the proofs of the listed results only through the following objects, and each has a counterpart on F^- with the same bound.

(a) The derivative $DC_k = \tilde{B}_k + R_k$ of Lemma 360(iv), with $\|R_k\| \le 260\kappa$; on F^- it is $DC_kJ = \tilde{B}_kJ + R_kJ$, and $\|R_kJ\| \le \|R_k\|$, $\|\tilde{B}_kJ\| \le \|\tilde{B}_k\|$.

(b) The Gram matrix $\mathscr{G} = DCDC^*$. Its diagonal blocks are at least $\frac{1}{8} - 227\kappa$ because the squared coefficients of $V_{y,\mu}$ on the eight private links of (y,μ) add up to $\frac{1}{8}$; these links lie in A_W^-, so $\tilde{B}_k JJ^* \tilde{B}_k^* \geq \frac{1}{8}$ and the perturbation $2\|\tilde{B}_k J\| 260\kappa + (260\kappa)^2 \leq 227\kappa$ is the same. The upper bounds $\frac{3}{16} + 227\kappa$ on the diagonal blocks and $\frac{1}{32} + 227\kappa$ on the two off-diagonal blocks of a row bound sums of the norms of the contributions of single links, which only lose terms. So $\mathscr{G}^- = DCJJ^*DC^*$ has the block bounds from which the proofs of Theorem 363 and Proposition 369 derive $\mathscr{G} \geq \frac{1}{16} - 700\kappa$, $\|\mathscr{G}^{-1}\| \leq 17$ and the weighted norms of $\mathscr{G}^{-1}$.

(c) The derivative bounds with the constants c_3 and c_6 for c_k, Q and j, and $A = Dc(X)$: on vectors supported on A_W^- they hold as before, and AJ, DCJ have the same block sparsity.

(d) The section: $(S^-\xi)_e = \xi_e$ for $e \in L'^-$ and $(S^-\xi)_{d_k} = -B_{k,d_k}^{-1} \sum_{e \in A_W^- \setminus \{d_k\}} B_{k,e}\xi_e$, that is $JS^- = SJ$. The path link d_k lies in A_W^- and enters only V_k, and the bound 15.1 is a sum of norms over e that only loses terms.

The function $\bar{H}$ and the matrices P, $\bar{P}$ on F^- are defined by the same formulas from AJ, DCJ and S^-, and their bounds are derived in those proofs from (a)–(d) by the same computations. The distance d on L'^- is the restriction of that on L'. The proof of Theorem 370(i) applies to fields that agree off A_W^-, since these agree off A_W. Since DCJ is onto by (b), F^- has codimension $4|W|\dim G$ and $\dim T_U = |L'^-|\dim G$. The $28|W|$ private links other than the path links lie in L'^-. □

The weight of the interior as a function of two ends follows, for ends that enter only the action.

Corollary 533. *Let $G = \mathrm{SU}(2)$ and let E_a, E_b be disjoint sets of links off A_W^-, none of which enters an average $V_{y,\mu}^\sharp$ with $y \in W$, such that no plaquette contains a link of E_a and a link of E_b. Fix the other links off A_W^-, let $a_0, a_1 \in G^{E_a}$, $b_0, b_1 \in G^{E_b}$, $X_e, Y_e \in \mathfrak{g}$ with $a_{1,e} = a_{0,e}e^{X_e}$, $b_{1,e} = b_{0,e}e^{Y_e}$, and $x_a = \max_e \|X_e\|$, $x_b = \max_e \|Y_e\|$. Let $Z_{ij} = \nu_c^-(e^{-\lambda Q^-})$ on the fibre with the links a_i on E_a and b_j on E_b, and let N_a, N_b be the links of A_W^- that share a plaquette with a link of E_a, of E_b. Suppose that at the links $a_0e^{\sigma X}$, $b_0e^{\tau Y}$, for all $\sigma, \tau \in [0,1]$, the fibre satisfies the hypotheses of Theorem 526 with fixed κ and s, and let $\bar{p}$ bound its mass off the chart. Then*

$$\left|\log \frac{Z_{11}Z_{00}}{Z_{10}Z_{01}}\right| \leq 3.9 \cdot 10^6 \beta_W x_a x_b \big(|N_a||N_b|\big)^{1/2} e^{-\alpha_G d(N_a,N_b)} + 432\,\beta_W^2\,\bar{p}\,|E_a||E_b|x_a x_b.$$

Proof. Neither the fibre F^- nor ν_c^- depends on the links of $E_a \cup E_b$, since these enter no constraint, so $Z(\sigma,\tau) = \nu_c^-(e^{-\lambda Q^-})$ at the links $a_0e^{\sigma X}$, $b_0e^{\tau Y}$ is an integral over a fixed space, and differentiating under the integral, $\partial_\sigma\partial_\tau \log Z = \mathrm{Cov}(\phi_a, \phi_b)$ under μ_W^-, with $\phi_a = \partial_\sigma \lambda Q^-$ and $\phi_b = \partial_\tau \lambda Q^-$, because no plaquette contains links of both sets, so that $\partial_\sigma\partial_\tau Q^- = 0$. The function ϕ_a depends on the links of N_a. For a plaquette p, $\delta(U_p, I)^2 = 4 - 2\,\mathrm{Re}\,\mathrm{Tr}\,U_p$; moving a link e of p along X_e inserts X_e between unitary factors, and $\mathrm{Re}\,\mathrm{Tr}(VX) = \mathrm{Re}\,\mathrm{Tr}((V-I)X)$ for skew-Hermitian X, so the derivative is at most $2\delta(U_p,I)\|X_e\|$, and the mixed derivative along X_e and along a unit vector at another link of p is at most $2\|X_e\|$. A link lies in six plaquettes, and a plaquette with a link of A_W^- has at most three links of E_a; so the left-invariant gradient of ϕ_a has norm at most $6 \cdot 3 \cdot 2\lambda x_a = 36\lambda x_a$ in each link. As $\delta(U_p, I) \leq 2\sqrt{2}$, summing over the links $e \in E_a$ and the six plaquettes through each gives $\|\phi_a\|_\infty \leq 6|E_a| \cdot 2 \cdot 2\sqrt{2}\,\lambda x_a = 24\sqrt{2}\lambda |E_a| x_a$, and likewise for ϕ_b. By Proposition 532(ii), Theorem 526 applied to $\phi_a/(36\lambda x_a)$ and $\phi_b/(36\lambda x_b)$ gives

$$|\mathrm{Cov}(\phi_a,\phi_b)| \leq \frac{4.8 \cdot 10^4}{\beta_W}(36\lambda)^2 x_a x_b \times \big(|N_a||N_b|\big)^{1/2} e^{-\alpha_G d(N_a,N_b)} + 6\bar{p}\,(24\sqrt{2}\lambda)^2 |E_a||E_b| x_a x_b,$$

and with $\lambda = \beta_W/4$ the two coefficients are $81 \cdot 4.8 \cdot 10^4 \beta_W = 3.888 \cdot 10^6 \beta_W$ and $6 \cdot 1152 \beta_W^2/16 = 432\beta_W^2$. Integrate over $(\sigma,\tau) \in [0,1]^2$. □

The first term decays at the rate α_G in the distance between the ends, with no cost in the length of W; the second needs $\bar{p}$ small against β_W^{-2}, which by Theorem 529 requires $\beta_W r^2$ larger than a multiple of $\ln\beta_W + \ln|W|$.

Remark 534. What remains of the bad regions in item (ii) of Sec. XIII V:

(a) *The mass off the chart.* Theorem 529 shows that no bound on p uniform in the length of the component exists, for any coarse field and exterior to which Theorem 526 applies, and that p is close to one for $\beta_W \leq 10^{24}$. A bound on the response across a component uniform in its length therefore cannot pass through a single product chart of the whole component; it needs charts local in the component, around which the links far from the observables may leave the chart, that is a decomposition of the fibre into small-field and large-field regions with a convex chart on each small-field region. Theorem 535 shows that charts fixed by the exterior of their window put the window on the chart with conditional probability at most $0.033^{|W|}$ for $\beta_W \leq 10^{24}$, so that in such a decomposition the large-field region carries most of the mass. For fixed $|W|$, a bound on p for a fixed coarse field is Corollary 375, whose condition grows linearly in $|W|$, against the logarithmic necessary condition of Corollary 530(ii).

(b) *Larger coarse fields.* Above $2.418 \cdot 10^{-12}$ no field of the fibre carries a chart (Proposition 525(ii)). Between $6.0 \cdot 10^{-13}$ and $2.418 \cdot 10^{-12}$ the fibres of Proposition 531 carry backgrounds. For the exterior $R(c)$, whose plaquettes carry each coarse plaquette in one fine plaquette, a background would have to spread the coarse curvature over the plaquettes of the window, almost aligned and almost of equal size as in the proof of Lemma 524, while matching $R(c)$ off A_W. The exact condition is that the fibre over W of a field agreeing with $R(c)$ off A_W contain a field

U with $\delta(U_p,I)\le\kappa$ for every plaquette with a vertex in a block of W^+, for some $\kappa<6.0\cdot10^{-13}$; under it Theorems 526 and 529 apply with every $s\le6.0\cdot10^{-13}-\kappa$. It holds for the coarse fields c^θ of Proposition 536 up to $2.39\cdot10^{-12}$; for opposite fluxes at adjacent coarse plaquettes no field with values in a torus satisfies it above $1.8\cdot10^{-12}$; for a general coarse field it is not decided.

(c) *Ends in the averages.* Corollary 533 needs ends that enter no average of a pair of W. The links of I_W, and the inside links of a block $y+e_\mu\notin W$ in the direction μ for $y\in W$, enter those averages, and the labels of the blocks adjoining the component depend on such links; moving them moves the fibre, and Proposition 422 shows that off the chart the law then jumps. On the chart the dependence is smooth. The exact missing statement is the bound of Corollary 533 for ends E_a, E_b that contain such links, that is a bound on the mixed derivative of $\log\nu_c^-(e^{-\lambda Q^-})$ in two links that move the constraints, decaying in their distance. Lemma 537 places an end in one pair factor of ν_c^-, Proposition 538 shows that this factor jumps under small moves of the end at every value of the coarse link, and Remark 539 states the bound in the mean that is missing.

The three points of Remark 534 are taken in turn. The first concerns charts local in the component. Such a chart is attached to a window W, a set of blocks inside the component, and is fixed by the field outside the window; the next theorem bounds the conditional probability that the window lies on it, uniformly in the window, the exterior and the coarse field.

Theorem 535 (local charts)**.** *Let $G=\mathrm{SU}(2)$, $\beta_W\ge2.3\cdot10^{17}$, let c be a coarse field and W a set of blocks, and write E for the links off A_W^-. Suppose that for each E a field $U(E)$ is given in the reduced fibre $F^-(E)$ over W through E, with averages c_k for the pairs based in W, such that $F^-(E)$ satisfies the hypotheses of Theorem 526 at $U(E)$ with constants $\kappa(E)$, $s(E)$, as in Proposition 532*(ii)*, and let $\mathscr{A}$ be the event that the links of A_W^- lie in the chart $\Phi_E(\Omega_{r(E)})$ of $F^-(E)$ at $U(E)$, $r(E)=s(E)/32$, assumed measurable. Then for $\mathbb{E}_c$-almost every c*

$$\mu^c(\mathscr{A})\le\exp\Big(-\frac{28}{3}|W|e^{-10^{-24}\beta_W}\Big).$$

(i) *If $\beta_W\le10^{24}$, then $\mu^c(\mathscr{A})\le e^{-3.433|W|}<0.033^{|W|}$; at $\beta_W=2.3\cdot10^{17}$, $\mu^c(\mathscr{A})\le e^{-9.333|W|}$.*

(ii) *If $\beta_W\le10^{24}$ and $\mathscr{A}_1,\dots,\mathscr{A}_J$ are such events, given by fields $U_1(E),\dots,U_J(E)$, then the union of the $\mathscr{A}_j$ has μ^c-probability at most $J\,e^{-3.433|W|}$.*

(iii) *If $\mu^c(\mathscr{A}\mid E)\ge\frac12$ on a set of values of E of positive probability, then $\beta_W>2.6\cdot10^{24}$.*

Proof. By Proposition 532(i), for $\mathbb{E}_c$-almost every c the conditional law under μ^c of the links of A_W^- given E is the constrained law μ_W^- on $F^-(E)$. By Proposition 532(ii), Theorem 529 holds on $F^-(E)$ with $45|W|$ replaced by $|L'^-|\ge28|W|$, so that, with $q_0=\frac13e^{-2843.5\beta_W r(E)^2}$,

$$\begin{aligned}\mu^c(\mathscr{A}\mid E)&=\mu_W^-\big(\Phi_E(\Omega_{r(E)})\big)\\&\le(1-q_0)^{|L'^-|}\le e^{-28q_0|W|}.\end{aligned}$$

As $\kappa(E)+s(E)\le6.0\cdot10^{-13}$, $r(E)\le1.875\cdot10^{-14}$ and $2843.5\,r(E)^2<10^{-24}$, so $q_0\ge\frac13e^{-10^{-24}\beta_W}$; taking the mean over E gives the bound. (i) For $\beta_W\le10^{24}$, $q_0\ge\frac13e^{-1}>0.12262$ and $28\cdot0.12262>3.433$; as $e^3>20.08$ and $e^{0.433}>1.54$, $e^{3.433}>30.9>1/0.033$. At $\beta_W=2.3\cdot10^{17}$, $q_0\ge\frac13e^{-2.3\cdot10^{-7}}\ge\frac13(1-2.3\cdot10^{-7})$, and $28q_0>9.333$. (ii) is the union bound. (iii) On that set $e^{-28q_0}\ge e^{-28q_0|W|}\ge\frac12$, so $q_0\le\frac{\ln2}{28}$ and $e^{-10^{-24}\beta_W}\le\frac{3\ln2}{28}$. As $\ln2<0.69315$, $\frac{28}{3\ln2}>\frac{28}{2.07945}>13.465$, while $e^{2.6}<7.38906\cdot1.82212<13.4638$; so $10^{-24}\beta_W>2.6$. □

The bound holds for every window, so it holds block by block: whatever the exterior and the coarse field, a single block lies on a chart of this kind with conditional probability below 0.033 for $\beta_W\le10^{24}$, and below $\frac12$ only if $\beta_W>2.6\cdot10^{24}$, and $J\le15$ charts for each exterior cover it with probability below $\frac12$. A decomposition of the fibre of a component into small-field regions, each carrying such a chart, and a large-field region, as in Remark 534(a), has the large-field region carrying all but at most $J\,0.033^{|W'|}$ of the conditional mass of each window W'. The cost per block of a bound built on these charts is therefore not small at $\beta_W\le10^{24}$, and a bound uniform in the length must control the response on the large-field region, which is the regime of Proposition 418. The theorem does not cover charts whose background depends on the links of the window itself, such as a background chosen nearest to the configuration from a continuous family.

The second point is the existence of backgrounds for the exterior $R(c)$ between $6.0\cdot10^{-13}$ and $2.418\cdot10^{-12}$. In the next proposition the curvature of one coarse plaquette is spread over the fine plaquettes of its four corner blocks, with every link in the torus $\{e^{\varphi T_0}\}$. On such a field a link is given by a real angle, a plaquette at x in the plane $\{\mu,\nu\}$ has the real sum $\varphi_{x,\mu}+\varphi_{x+e_\mu,\nu}-\varphi_{x+e_\nu,\mu}-\varphi_{x,\nu}$, and a path the sum of its two angles.

Proposition 536 (spreading a coarse plaquette)**.** *Let $G=\mathrm{SU}(2)$, let the periodic lattice have side $2L$, let $\mu<\nu$ be two directions, $0<\theta\le1.7\cdot10^{-12}$, and let $M\ge4$ and $L\ge M+6$ be integers. Let $a(0)=0$, $a(1)=a(2)=a(3)=\theta$, $a(k)=\theta(M+3-k)/M$ for $3\le k\le M+3$ and $a(k)=0$ for $M+3\le k\le L-1$, indices modulo L. Let c^θ be the coarse field with $c^\theta_{y,\nu}=e^{a(y_\mu)T_0}$ for $y_\nu=0$ and every other coarse link I, and $W_\theta=\{y:y_\mu,y_\nu\in\{0,1\}\}$.*

(i) *The coarse plaquettes of c^θ other than I lie in the plane $\{\mu,\nu\}$ at blocks with $y_\nu=0$: $c^\theta_P=e^{\theta T_0}$ for $y_\mu=0$, so that $\delta(c^\theta_P,I)=t:=2\sqrt2\sin\frac\theta2$, and $c^\theta_P=e^{-(\theta/M)T_0}$ for $3\le y_\mu\le M+2$, with $\delta(c^\theta_P,I)\le\sqrt2\theta/M$. The set W_θ consists of the corner blocks of the first.*

(ii) *Let U^θ be the field with the angles of Table IV*(a) *on the links listed there, at every value of the coordinates other*

than μ and ν, and equal to $R(c^\theta)$ elsewhere. Then U^θ vanishes on the tree links, agrees with $R(c^\theta)$ off A_{W_θ}, has $V_k^\sharp(U^\theta) = c_k^\theta$ for every pair k based in W_θ, and every plaquette has $\delta(U_p^\theta, I) \le \kappa_\theta := 2\sqrt{2}\sin\frac{\theta}{8}$. So U^θ lies in the fibre over W_θ of $R(c^\theta)$.

(iii) *Let $U^{\theta,-}$ be the field with the angles of Table IV(b), and $R(c^\theta)$ elsewhere. Then $U^{\theta,-}$ vanishes on the tree links, agrees with $R(c^\theta)$ off $A^-_{W_\theta}$, has $V_k^\sharp(U^{\theta,-}) = c_k^\theta$ for every pair k, and every plaquette has $\delta(U_p^{\theta,-}, I) \le \kappa'_\theta := 2\sqrt{2}\sin\frac{\theta}{7}$. So for every set $W \supset W_\theta$, $U^{\theta,-}$ lies in the reduced fibre over W of $R(c^\theta)$, and in its fibre.*

(iv) *Let U be any field with values in the torus $\{e^{\varphi T_0}\}$, y a block, $W' = \{y + ie_\mu + je_\nu : 0 \le i \le 2,\ 0 \le j \le 1\}$, and suppose that $\delta(U_p, I) \le \kappa \le 10^{-8}$ for every plaquette with a vertex in a block of W'^+. Write $U_p = e^{a_p T_0}$ with $a_p \in (-\pi, \pi]$, and let S_x be the four plaquettes of the square of side two at x in the plane $\{\mu,\nu\}$. Then the coarse plaquettes P at y and P' at $y + e_\mu$ in that plane of $c = V^\sharp(U)$ satisfy*

$$c_P = \exp\Big(\frac{1}{16}\sum_{x\in B(y)}\sum_{p\in S_x} a_p T_0\Big),$$

$c_{P'}$ is given by the same formula with $B(y+e_\mu)$ in place of $B(y)$, and if $c_{P'} = c_P^{-1}$, then $\delta(c_P, I) \le 3\kappa$.

In particular, for every $t \in (0, 2.39\cdot 10^{-12}]$, with $\theta = 2\arcsin(t/(2\sqrt{2}))$, the fibre over W_θ of $R(c^\theta)$ contains a field at which it satisfies the hypotheses of Theorem 526 with $s = 2.4\cdot 10^{-15}$; and for every $t \in (0, 2.09\cdot 10^{-12}]$ and every $W \supset W_\theta$, the reduced fibre over W of $R(c^\theta)$ contains a field at which it satisfies them, in the sense of Proposition 532(ii), with the same s.

Proof. (i) A coarse plaquette at y in a plane $\{\rho,\sigma\}$ is $c_{y,\rho}c_{y+e_\rho,\sigma}c^{-1}_{y+e_\sigma,\rho}c^{-1}_{y,\sigma}$. Only the links in the direction ν with $y_\nu = 0$ differ from I, and two links in the direction ν based at blocks that differ in a coordinate other than μ and ν are equal; so only the plaquettes in the plane $\{\mu,\nu\}$ at $y_\nu = 0$ can differ from I, and there $c_P^\theta = e^{(a(y_\mu+1)-a(y_\mu))T_0}$. The increments of a are θ at $y_\mu = 0$, $-\theta/M$ for $3 \le y_\mu \le M+2$, and 0 otherwise, also at $y_\mu = L-1$, where $a(L-1) = a(0) = 0$ as $L-1 \ge M+3$. Finally $2\sqrt{2}\sin x \le 2\sqrt{2}x$.

(ii), (iii) We write φ for the angles of the field in question and φ^R for those of $R(c^\theta)$, all real. By Proposition 525(i) and its proof, φ^R is 0 on every link in the direction μ and on every link in the direction ν except (x,ν) with $x_\nu = 1$, where it is $a(y_\mu)$, $x \in B(y)$, and all sixteen paths of a pair (y,ρ) have the angle $\vartheta_{y,\rho}$ of $c^\theta_{y,\rho}$, which is $a(y_\mu)$ or 0.

Links. The field differs from $R(c^\theta)$ only on the links of the table with an entry different from that of $R(c^\theta)$, and the entries do not depend on the other two coordinates. These are links (x,ν) with $x_\nu = 1$ and $0 \le x_\mu \le 3$, which have $o_\nu = 1$ and are private links of pairs based in W_θ, and links (x,μ) with $0 \le x_\mu, x_\nu \le 3$, which are private links of such pairs when x_μ is odd; when x_μ is even the entry is nonzero only for x_ν odd, and then $o_\nu = 1$ with $\nu > \mu$, so the link is an active link inside a block of W_θ, not a tree link. In (b) the links with x_μ even have $x_\mu = 2$, in a block y with $y - e_\mu \in W_\theta$, so they are not in I_{W_θ}. Hence U^θ agrees with $R(c^\theta)$ off A_{W_θ} and $U^{\theta,-}$ off $A^-_{W_\theta}$, and both vanish on the tree links. For $W \supset W_\theta$, $A^-_{W_\theta} \subset A^-_W$: a link of A_{W_θ} in the direction ρ that lies in I_W is inside a block $y \in W_\theta$ with $y - e_\rho \notin W$, hence $y - e_\rho \notin W_\theta$, and so lies in I_{W_θ}.

Averages. Let $D_x = \varphi(P_x) - \vartheta_{y,\rho}$ for the paths P_x of a pair (y,ρ). Every entry differs from that of $R(c^\theta)$ by at most $\frac{2}{7}\theta$, so $|\varphi(P_x) - \varphi(P_{2y})| \le \frac{8}{7}\theta$ and $\delta(P_x, P_{2y}) = 2\sqrt{2}\sin\frac{|\varphi(P_x)-\varphi(P_{2y})|}{2} < 4\theta < \frac{1}{2}$. The cut-off of (35) equals one, $\rho_G = 1$, and as these differences are less than π the principal logarithm of $P_{2y}^{-1}P_x$ is $(\varphi(P_x) - \varphi(P_{2y}))T_0$. So

$$V^\sharp_{y,\rho} = \exp\Big(\Big(\vartheta_{y,\rho} + \frac{1}{16}\sum_{x\in B(y)} D_x\Big)T_0\Big),$$

and $V^\sharp_{y,\rho} = c^\theta_{y,\rho}$ when $\sum_x D_x = 0$. For $\rho \notin \{\mu,\nu\}$ every D_x vanishes. For $\rho = \nu$, $D_x \ne 0$ needs a link (x',ν) with $x'_\nu = 1$ on the path, so only the pairs with $y_\nu = 0$ are concerned,

TABLE IV. The fields of Proposition 536, in units of θ, at the sites with coordinates x_μ and x_ν: the angles of the links in the direction μ for $0 \le x_\nu \le 3$, of the links in the direction ν at $x_\nu = 1$, and the real sums of the plaquettes in the plane $\{\mu,\nu\}$ with corner x. Every other link has the angle of $R(c^\theta)$: 0 in the direction μ, and $a(y_\mu)/\theta$ for the links (x,ν) with $x_\nu = 1$, $x \in B(y)$, and 0 on every other link.

		x_μ					
	x_ν	-1	0	1	2	3	4
(a) U^θ							
links μ	3	0	$-\frac{1}{4}$	0	$-\frac{1}{4}$	$\frac{1}{8}$	0
	2	0	0	$\frac{1}{4}$	0	0	0
	1	0	$-\frac{1}{4}$	0	$-\frac{1}{4}$	$\frac{1}{8}$	0
	0	0	0	$\frac{1}{4}$	0	0	0
links ν	1	0	$-\frac{1}{4}$	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{5}{4}$	1
plaquettes	3	0	$-\frac{1}{4}$	0	$-\frac{1}{4}$	$\frac{1}{8}$	
	2	0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$-\frac{1}{8}$	
	1	$-\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$-\frac{1}{8}$	
	0	0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$-\frac{1}{8}$	
	-1	0	0	$-\frac{1}{4}$	0	0	
(b) $U^{\theta,-}$							
links μ	3	0	0	$-\frac{1}{14}$	$-\frac{2}{7}$	0	0
	2	0	0	$\frac{3}{14}$	0	$\frac{1}{7}$	0
	1	0	0	$-\frac{1}{14}$	$-\frac{2}{7}$	$\frac{1}{7}$	0
	0	0	0	$\frac{3}{14}$	0	0	0
links ν	1	0	$-\frac{1}{7}$	$\frac{1}{7}$	$\frac{5}{7}$	$\frac{9}{7}$	1
plaquettes	3	0	0	$-\frac{1}{14}$	$-\frac{2}{7}$	0	
	2	0	0	$\frac{2}{7}$	$\frac{2}{7}$	$\frac{1}{7}$	
	1	$-\frac{1}{7}$	$\frac{2}{7}$	$\frac{2}{7}$	$\frac{2}{7}$	$-\frac{2}{7}$	
	0	0	0	$\frac{2}{7}$	$\frac{2}{7}$	$-\frac{1}{7}$	
	-1	0	0	$-\frac{3}{14}$	0	0	

and by Lemma 329(iii) $\sum_x D_x = 8(d(2y_\mu)+d(2y_\mu+1))$, $d(m)$ the deviation of the link (x,ν) with $x_\mu = m$, $x_\nu = 1$. The deviations are $-\frac14,\frac14,-\frac14,\frac14$ in (a) and $-\frac17,\frac17,-\frac27,\frac27$ in (b) for $m = 0,1,2,3$, and 0 otherwise, so the sums vanish. For $\rho = \mu$, let $X(m)$ be the sum of the deviations of the two links (x,μ) with $x_\mu = m$ and $x_\nu \in \{2y_\nu, 2y_\nu+1\}$; by Lemma 329(iii), $\sum_x D_x = 4\big(X(2y_\mu)+2X(2y_\mu+1)+X(2y_\mu+2)\big)$. For $y_\nu \in \{0,1\}$ and $m = -1,\dots,4$, $X(m)$ is $0,-\frac14,\frac14,-\frac14,\frac18,0$ in (a) and $0,0,\frac17,-\frac27,\frac17,0$ in (b), and $X(m) = 0$ for other m or y_ν. The combination vanishes for $y_\mu \in \{0,1\}$ in (a), and for every y in (b).

Plaquettes. A plaquette in a plane other than $\{\mu,\nu\}$ has two parallel links that differ only in a coordinate other than μ and ν and two links in a direction other than μ and ν; the first two have equal angles and the other two the angle 0, so its real sum is 0. In the plane $\{\mu,\nu\}$ the real sum at x is read off the table; the table lists every plaquette that contains a link at which the field differs from $R(c^\theta)$, and all other plaquettes are those of $R(c^\theta)$, whose real sums are 0 or $a(y_\mu+1)-a(y_\mu) \in \{0,-\theta/M\}$ with $|\theta/M| \le \frac{\theta}{4}$, by Proposition 525(i). So every real sum has modulus at most $\frac{\theta}{4}$ in (a) and $\frac{2\theta}{7}$ in (b), and $\delta(U_p,I) = 2\sqrt2\sin\frac{|\cdot|}{2}$ gives the bounds.

(iv) For a pair k based in W' let $v = P_{2y}$ be its path link. By Lemma 360(i), $\delta(v^{-1}P_x,I) \le 24\kappa < \frac12$, so the cut-off is one and $c_k = v\exp(\frac{1}{16}\sum_x \psi_x T_0)$, where $v^{-1}P_x = e^{\psi_x T_0}$ with $\psi_x \in (-\pi,\pi]$ and $|\psi_x| \le 27\kappa$, since $2\sqrt2\sin\frac{|\psi|}{2} \ge \frac{2\sqrt2}{\pi}|\psi|$. Choose real angles φ_k of the path links. Then $c_k = e^{\gamma_k T_0}$ with $\gamma_k = \varphi_k + \frac{1}{16}\sum_x \psi_x$, and with the sides $k = 1,\dots,4$ of P and signs $\varepsilon_k = 1,1,-1,-1$, $c_P = e^{\Theta T_0}$ with

$$\Theta = \sum_k \varepsilon_k\gamma_k = \frac{1}{16}\sum_{x\in B(y)}\Theta_x, \qquad \Theta_x = \sum_k \varepsilon_k\big(\varphi_k + \psi^{(k)}_{x_k}\big),$$

where $x_k \in \{x, x+2e_\mu, x+2e_\nu, x\}$ is the site of the same offset as x in the block of side k. The term $\varphi_k + \psi^{(k)}_{x_k}$ is an angle of the path $P^{(k)}_{x_k}$, and on the torus the product of the four paths around the square of side two at x is the product of the four plaquettes of S_x, so $\Theta_x \equiv \sigma_x := \sum_{p\in S_x} a_p$ modulo 2π. Now $|a_p| \le 1.12\kappa$, since $2\sqrt2\sin\frac{|a|}{2} \ge \frac{2\sqrt2}{\pi}|a|$, so $|(\Theta_x-\sigma_x)-(\Theta_{2y}-\sigma_{2y})| \le 4\cdot 27\kappa + 8\cdot 1.12\kappa < 2\pi$, as $\psi^{(k)}_{2y} = 0$. Hence $\Theta_x - \sigma_x$ is the same multiple of 2π for all x, and Θ differs from $\frac{1}{16}\sum_x\sigma_x$ by a multiple of 2π, which gives the formula for c_P; the same holds for P'. Let Θ_P, $\Theta_{P'}$ be the two exponents. A plaquette at the corner z, $z_\mu = 2y_\mu + i$ and $z_\nu = 2y_\nu + j$, lies in S_x for $n(i)n(j)$ sites x of $B(y)$ with the coordinates of z other than μ and ν, and for $n(i-2)n(j)$ sites of $B(y+e_\mu)$, where $n(0) = n(2) = 1$, $n(1) = 2$ and $n = 0$ otherwise. So

$$\Theta_P - \Theta_{P'} = \frac{1}{16}\sum_z w_z a_{p_z},$$
$$\sum_z |w_z| = (1+2+0+2+1)(1+2+1)\cdot 4 = 96,$$

the factor 4 counting the offsets in the other two directions. If $c_{P'} = c_P^{-1}$, then $\Theta_P + \Theta_{P'} \in 2\pi\mathbb{Z}$, and as $|\Theta_P|, |\Theta_{P'}| \le 4\cdot 1.12\kappa < \pi$ it vanishes. So $2|\Theta_P| \le 6\alpha$ with $\alpha = \max|a_p|$, and as $\sin 3x \le 3\sin x$ for $0 \le x \le \frac{\pi}{6}$,

$$\delta(c_P,I) = 2\sqrt2\sin\frac{|\Theta_P|}{2} \le 2\sqrt2\sin\frac{3\alpha}{2} \le 3\cdot 2\sqrt2\sin\frac{\alpha}{2} \le 3\kappa.$$

For the last statement, $\sin x \ge x(1-x^2/6)$ gives $\theta < 1.7\cdot 10^{-12}$ for $t \le 2.39\cdot 10^{-12}$. In (ii), $\sin\frac{\theta}{2} = 4\sin\frac{\theta}{8}\cos\frac{\theta}{8}\cos\frac{\theta}{4}$, so $\kappa_\theta = t/(4\cos\frac{\theta}{8}\cos\frac{\theta}{4}) \le t/(4(1-\theta^2)) < 5.976\cdot 10^{-13}$. In (iii), $t \ge \sqrt2\theta(1-\theta^2/24)$, so $\kappa'_\theta \le \frac{2\sqrt2}{7}\theta \le \frac27 t(1+\theta^2/12) < 5.972\cdot 10^{-13}$ for $t \le 2.09\cdot 10^{-12}$. Every plaquette is below $\kappa_\theta \le 10^{-8}$, respectively κ'_θ, so the plaquette condition holds on W^+ for every W; and $s = 2.4\cdot 10^{-15}$ satisfies $s \le 1/(6000c_3)$, $\kappa + 6s \le 10^{-8}$ and $\kappa + s \le 6.0\cdot 10^{-13}$. □

The fields of (ii) and (iii) spread the coarse curvature θ over the plaquettes of the corner blocks with the largest real sum $\frac{\theta}{4}$ and $\frac{2\theta}{7}$; by Lemma 524 no field does better than $\frac{\theta}{4}$ up to a term of order θ^2, so (ii) is sharp up to that term. The gap between $2.39\cdot 10^{-12}$ and $2.418\cdot 10^{-12}$ is the rounding of the chart threshold $6.044\cdot 10^{-13}$ of Proposition 525(ii) to $6.0\cdot 10^{-13}$ in Theorem 526: for every threshold κ_* in place of $6.0\cdot 10^{-13}$, (ii) gives a background whenever $\delta(c_P,I) < 4\kappa_*(1-\theta^2)$ and Lemma 524 excludes one above $4\kappa_* + 7700\kappa_*^2$. Part (iv) shows that the factor depends on the neighbouring coarse plaquettes: for a coarse field with opposite fluxes at two adjacent coarse plaquettes and $\delta(c_P,I) > 1.8\cdot 10^{-12}$, no field with values in a torus has every plaquette near the six blocks below $6.0\cdot 10^{-13}$.

The third point is the ends that enter the averages. They meet the fibre measure in one factor.

Lemma 537 (ends and pairs)**.** *Let W be a set of blocks and e an active link off A^-_W that enters an average $V^\sharp_{y,\rho}$ with $y \in W$. Then $e = (x,\rho)$ with x_ρ even, and either*

(a) *$x \in B(y)$, $y - e_\rho \notin W$, $e \in I_W$, and e is the first link of the path P_x, whose second link is the private link $(x+e_\rho,\rho)$ of (y,ρ); or*

(b) *$x \in B(y+e_\rho)$, $y + e_\rho \notin W$, and e is the second link of P_{x-e_ρ}, whose first link is the private link $(x-e_\rho,\rho)$ of (y,ρ).*

In either case (y,ρ) is the only pair based in W whose average contains e, e lies on one of its sixteen paths, and among the factors of ν_c^- over the pairs based in W only the factor of (y,ρ) depends on e.

Proof. A private link enters only the average of its own pair, and the private links of pairs based in W lie in A^-_W; so e is an inside link (x,ρ), x_ρ even, of a block $B(y')$. As in the proof of Proposition 532(i), it enters only the averages of (y',ρ), as the first link of P_x, and of $(y'-e_\rho,\rho)$, as the second link of P_{x-e_ρ}. If $y' \in W$, then $e \in A_W \setminus A^-_W = I_W$, so $y'-e_\rho \notin W$ and $y = y'$, which is (a). If $y' \notin W$, the pair is $(y'-e_\rho,\rho)$, which is (b). The other link of the path has offset 1 in the direction ρ and joins $B(y)$ to $B(y+e_\rho)$, so it is a private link of (y,ρ). Given the inside links and the links off A^-_W, the averages of the pairs based in W depend on disjoint sets of private links, and

ν_c^- is the product of the Haar measure of the inside links in A_W^- and of one factor for each pair, built from its average as before (39); only the factor of (y,ρ) contains e. □

Proposition 538 (an end at the fold)**.** *Let $G=\mathrm{SU}(2)$, $\mu\le 2$, let W be a set of blocks, $y\in W$ with $y-e_\mu\notin W$, $k=(y,\mu)$, and $c_k\in\mathrm{SU}(2)$. There are values of the links of the pair k and of the links inside $B(y)$ and $B(y+e_\mu)$, with an active inside link u_b of $B(y)$ in the direction μ, which lies in I_W and is an end of case* (a) *of Lemma 537, moved along a one-parameter family $u_b(t)$, such that for every $\beta_W\ge 0$, every value of the other links and every $\varepsilon>0$ there are $t<0<t'$ with $d(u_b(t),u_b(t'))<\varepsilon$ and*

$$W_1^{\delta}\big(\pi^{(t)},\pi^{(t')}\big)\ge 0.126,$$

where $\pi^{(t)}$ is the law (40) of the path link of k given its seven non-path links, at the value c_k.

Proof. Take the configuration of Proposition 422, at the value c_{γ^*}, with its moved link $u_b(t)$, and let $h=c_k c_{\gamma^*}^{-1}$. Let Γ_h multiply every private link of k on the left by h and conjugate every link of $B(y)$ in the direction μ by h; it fixes the tree links, which are I. A path of k from a site with $o_\mu=0$ is $u\ell$, with u in $B(y)$ and ℓ private, and $(huh^{-1})(h\ell)=hu\ell$; a path from a site with $o_\mu=1$ is ℓw, with w in $B(y+e_\mu)$, and becomes $h\ell w$. So Γ_h multiplies every path of k on the left by h, the distances $\delta(P_x,P_{2y})$ and the logarithms in (35) do not change, and $V_k^\sharp\circ\Gamma_h=hV_k^\sharp$. As a function of the path link g, the preimages of c_k after Γ_h are the points hg with g a preimage of c_{γ^*} before, and since left translation is an isometry of the bi-invariant metric, $|\det D_g V|$ is the same at hg as at g. The action $S(g;\cdot)$ of the path link is a sum of nonnegative multiples of $\delta(U_p,I)^2\le 8$ over the plaquettes through it, so $0\le S\le S_{\max}$ with $S_{\max}$ independent of the configuration. Hence, by Proposition 422, for $0<|t|<\tau$ the preimages are $hg_L(t)$, whose weight is at most $1/|\det D_g V(g_L(t))|$, which tends to $1/|\det D_g V(g_L^*)|<\infty$, and for $t<0$ also $hg_\pm(t)$, whose weights are at least $e^{-\beta_W S_{\max}}/|f'(q_\pm(t);t)|\to\infty$ as $t\uparrow 0$, with the function f of the proof of that proposition. As there, $\pi^{(t)}\to\delta_{hg^*}$ as $t\uparrow 0$ and $\pi^{(t)}\to\delta_{hg_L^*}$ as $t\downarrow 0$, and $\delta(hg^*,hg_L^*)=\delta(g^*,g_L^*)\ge 0.1268$. Kantorovich duality with $\delta(\cdot,hg^*)$ gives the bound, and $d(hu_b(t)h^{-1},hu_b(t')h^{-1})=d(u_b(t),u_b(t'))$. The link u_b lies in $B(y)$, has $o_\mu=0$, is active, and $y-e_\mu\notin W$, so it lies in I_W. □

By Lemma 537 the dependence of the weight of the interior on an end is that of one pair factor, and by Proposition 538 that factor, conditioned on the non-path links, jumps under arbitrarily small moves of the end at every value of the coarse link and every β_W. In the same configuration the normalisation Z of (40) is bounded for small $t>0$ and tends to infinity as $t\uparrow 0$, so the derivative of $\log Z$ in the end, at fixed non-path links, is not bounded.

Remark 539. What remains of the three points.

(a) *Length.* By Theorem 535, charts fixed by the exterior of their window carry less than half of the conditional mass of each block for $\beta_W\le 2.6\cdot 10^{24}$, and less than 0.033 of it for $\beta_W\le 10^{24}$. A bound uniform in the length needs the response on the large-field region of the fibre, or charts whose background moves with the window links, together with a bound on the mass of their union; neither is available.

(b) *Coarse fields.* For the coarse fields c^θ the exterior $R(c)$ carries a background up to $2.39\cdot 10^{-12}$, and the reduced fibre up to $2.09\cdot 10^{-12}$ for every window containing the component (Proposition 536). For a general coarse field between $6.0\cdot 10^{-13}$ and $2.418\cdot 10^{-12}$ with exterior $R(c)$, and for the reduced fibre between $2.09\cdot 10^{-12}$ and $2.418\cdot 10^{-12}$, the existence of a background is not decided; by Proposition 536(iv) a background with values in a torus does not exist above $1.8\cdot 10^{-12}$ for opposite fluxes at adjacent coarse plaquettes.

(c) *Ends.* A bound of the form of Corollary 533 for ends in the averages cannot hold at fixed non-path links, by Proposition 538. The missing input is a bound, uniform in c_k, on the derivative in an end of the logarithm of its pair factor after integration over the seven non-path links, where the coarea formula keeps the normalisation finite, together with its locality in the distance between two ends.

The three points of Remark 539 are taken up below without a chart. The pair factors of the fibre measure are within $3.1\cdot 10^{-4}$ of Haar measure in total mass, whatever the links that bind them; this bounds the weight of the interior as a function of an end that enters an average, after the non-path links are integrated, and gives a decay of the constrained law along a tube of blocks at every β_W. The coarse fields with opposite fluxes at adjacent coarse plaquettes are decided up to a rounding gap.

Lemma 540 (the pair factors are almost Haar)**.** *Let $G=\mathrm{SU}(2)$ and $q=h(\{v:\delta(v,I)\le 1\})$.*

(i) *$q=(\alpha-\frac{3\sqrt{7}}{16})/\pi<0.07447$ and $q^7<1.3\cdot 10^{-8}$. For every pair k, every c_k and every value of the links other than its eight private links, the density (38) satisfies*

$$1-1.3\cdot 10^{-8}\le 1-q^7\le\rho(c_k)\le 1+23640\,q^7<1+3.1\cdot 10^{-4}.$$

(ii) *For every c, $(1-1.3\cdot 10^{-8})^{4|W|}\le\nu_c(F_c)\le(1+3.1\cdot 10^{-4})^{4|W|}$, and the same bounds hold for the mass of ν_c^-.*

(iii) *Let E be a set of links off A_W^-, K_E the set of pairs based in W whose average contains a link of E, and n_E the number of plaquettes that contain a link of A_W^- and either a link of E or a private link of a pair of K_E. Fix the other links off A_W^-, and let $Z(e)=\nu_c^-(e^{-\lambda Q^-})$ with the value e on E. For any two values e,e',*

$$\Big|\log\frac{Z(e)}{Z(e')}\Big|\le 2\beta_W n_E+3.2\cdot 10^{-4}|K_E|.$$

Proof. (i) By Lemma 678(iii), $\delta(v,I)\le 1$ exactly when the eigenangle of v is at most α, and $\cos\alpha=1-2\sin^2\frac{\alpha}{2}=\frac{3}{4}$, $\sin\alpha=\frac{\sqrt{7}}{4}$; Lemma 678(ii) gives the formula for q. Since

$\cos x \le 1-\frac{x^2}{2}+\frac{x^4}{24}$, $\cos 0.73 \le 1-0.26645+0.011833 < \frac{3}{4}$, so $\alpha < 0.73$; and $\sqrt{7} > 2.6457$ gives $\frac{3\sqrt{7}}{16} > 0.49606$. Hence $q < 0.23394/\pi < 0.07447$, and $0.07447^2 < 0.005546$, $0.005546^3 < 1.706\cdot 10^{-7}$, $1.706\cdot 10^{-7}\cdot 0.07447 < 1.3\cdot 10^{-8}$.

Use the regions and coordinates of the proof of Lemma 373. On R_0 the average is the path link, so the first term of (38) is the Haar measure of the set of $(\ell_a)_{a\neq 0}$ with $(c_k,\ell)\in R_0$. Its complement lies in the set where $\delta(v_{x_a},I)\le 1$ for $a=1,\dots,7$, and with $g=c_k$ fixed the maps $\ell_a \mapsto v_{x_a}=c_k^{-1}u_a\ell_a$ carry Haar measure to Haar measure independently, so the complement has measure at most q^7 and the first term lies in $[1-q^7,1]$. In the term of R_1 the coordinates $h_1,\dots,h_7$ are independent and Haar distributed, the integrand vanishes unless $\max_a \delta(h_a,I)<1$, which has measure at most q^7, and the inner sum is at most $(1-\frac{7}{8}\Lambda)^{-3}$; the same holds for R_2 with $h'_a=v_{x'_a}$. As $1-\frac{7}{8}\Lambda > 0.0439$ and $0.0439^3 > 8.46\cdot 10^{-5}$, $(1-\frac{7}{8}\Lambda)^{-3} < 11820$, and $23640\cdot 1.3\cdot 10^{-8} < 3.1\cdot 10^{-4}$.

(ii) The measure ν_c is the product of Haar measure on the links inside the blocks and of one factor for each pair, of total mass $\rho(c_k)$ at every value of the inside links, and likewise ν_c^-.

(iii) By Lemma 537 and the construction before (39), ν_c^- is the product of Haar measure on the inside links of A_W^-, of the factors ν_k of the pairs $k\notin K_E$, which do not depend on e, and of the factors $\nu_k^{(e)}$, $k\in K_E$, each a measure on the private links of k of total mass $\rho(c_k)$. Write $\lambda Q^- = \beta_W(S_E+S_{\rm r})$, with S_E the sum of $1-\frac{1}{2}\operatorname{Re}\operatorname{Tr}U_p$ over the n_E plaquettes and $S_{\rm r}$ over the other plaquettes that contain a link of A_W^-; $S_{\rm r}$ depends neither on e nor on the private links of the pairs of K_E, and $0\le S_E\le 2n_E$. Integrating the private links of the pairs of K_E first,

$$e^{-2\beta_W n_E}\int \prod_{k\in K_E}\rho(c_k)\,d\Xi \le Z(e) \le \int \prod_{k\in K_E}\rho(c_k)\,d\Xi,$$

with $d\Xi = e^{-\beta_W S_{\rm r}}$ times the product of Haar measure on the inside links and the factors ν_k, $k\notin K_E$, which does not depend on e. By (i) the product of the $\rho(c_k)$ lies in $[(1-1.3\cdot 10^{-8})^{|K_E|},(1+3.1\cdot 10^{-4})^{|K_E|}]$ at every value of the inside links and of e, and $\log(1+3.1\cdot 10^{-4})-\log(1-1.3\cdot 10^{-8}) < 3.2\cdot 10^{-4}$. □

For an end of Proposition 538, $|K_E|=1$: the law of the path link given the non-path links jumps under arbitrarily small moves of the end, while after the non-path links are integrated the pair factor changes its mass by less than $3.2\cdot 10^{-4}$ under every move, since the cut-off acts only where seven paths lie within distance one of the path link, a set of Haar measure at most q^7 in the coordinates of each region. The bound in (iii) depends on β_W only through the Wilson weight of the plaquettes through the end and through the private links of its pair. The same splitting along a tube of blocks gives decay at every β_W.

Fix a direction ρ and a set W of blocks for which some value j_0 of the coordinate ρ satisfies $y_\rho \neq j_0$ for every $y\in W$. Number the other values of y_ρ by $1,\dots,L-1$ in their cyclic order starting after j_0, and let $\Sigma_i=\{y\in W : y_\rho = i\}$ be the slabs of W, K_i the pairs (y,ρ) with $y\in\Sigma_i$, and

$$D_i = |\Sigma_i|\big(72\beta_W + 3.2\cdot 10^{-4}\big).$$

Every link of A_W^- has its base in a block of W; we say that it lies in the slab of that block.

Theorem 541 (decay along a tube)**.** *Let $G=\mathrm{SU}(2)$, $\beta_W\ge 0$, let W, ρ and the slabs be as above, and let $1\le a$, $a+2\le b\le L-1$. Let $\tilde F$ be a bounded measurable function of the links of A_W^- in the slabs $1,\dots,a$ and $\tilde G$ one of the links of A_W^- in the slabs $b,\dots,L-1$. Then*

$$\big|\mathrm{Cov}_{\mu_W^-}(\tilde F,\tilde G)\big| \le 2\|\tilde F\|_\infty\|\tilde G\|_\infty \prod_{i=a+1}^{b-1}\big(1-e^{-D_i}\big),$$

for every coarse field and every exterior, and with no condition on the plaquettes. The same bound holds for μ_W, with A_W in place of A_W^-.

Proof. Let C_i be the set of private links of the pairs of K_i, which are the links (x,ρ) with $o_\rho=1$ and x in a block of Σ_i, and S_i the other links of A_W^- in the slab i.

Plaquettes. Let p be a plaquette in a plane $\{\rho,\tau\}$ with corner z; its links in the direction ρ are based at z and $z+e_\tau$, and those in the direction τ at z and $z+e_\rho$. The blocks of z and $z+e_\tau$ have the same coordinate ρ, say i. If z_ρ is odd, the two links in the direction ρ have $o_\rho=1$, so those of them in A_W^- lie in C_i, the link at z lies in the slab i and the link at $z+e_\rho$ in the slab $i+1$; neither is in a set C_j, as their direction is $\tau\neq\rho$. If z_ρ is even, $z+e_\rho$ lies in the block of z, all four links lie in the slab i, and none lies in a set C_j. A plaquette in a plane without ρ has its four bases in blocks with the same coordinate ρ and no link in a set C_j. So a plaquette either contains links of exactly one set C_i, and then its other links of A_W^- lie in $S_i\cup S_{i+1}$, or has its links of A_W^- in one set S_i.

Pairs. The average of a pair (y,σ), $y\in\Sigma_i$, depends on its private links, on the inside links of $B(y)$ in the direction σ, and on those of $B(y+e_\sigma)$ in the direction σ. For $\sigma\neq\rho$ all of them lie in S_i or off A_W^-. For $\sigma=\rho$ the private links lie in C_i, the links of $B(y)$ in S_i or in I_W, and the links of $B(y+e_\rho)$ in S_{i+1} or off A_W.

Integrating the crossing links. For each pair k based in W let ν_k be its factor of ν_c^-, a measure on its private links of mass $\rho(c_k)$ that depends on the other links of its paths. Let P_i be the set of plaquettes that contain a link of C_i, so $|P_i|\le 36|K_i|$ by Lemma 426, and

$$\Psi_i = \int \prod_{k\in K_i}\nu_k(d\ell^k)\exp\Big(-\beta_W\sum_{p\in P_i}\big(1-\tfrac{1}{2}\operatorname{Re}\operatorname{Tr}U_p\big)\Big).$$

By the two structural facts, Ψ_i is a function of the links of S_i and S_{i+1}, and $\nu_c^-(e^{-\lambda Q^-}\cdot)$, with the links of all sets C_i integrated, has the form

$$\prod_i \Phi_i(S_i)\,m_i(dS_i)\prod_i\Psi_i(S_i,S_{i+1}),$$

where m_i is the product of Haar measure on the inside links of S_i and of the factors ν_k of the pairs (y,σ), $y\in\Sigma_i$, $\sigma\neq\rho$, and

Φ_i is the Wilson weight of the plaquettes whose links of A_W^- lie in S_i only; Ψ_{L-1} depends on S_{L-1} only, since the blocks $y+e_\rho$, $y\in\Sigma_{L-1}$, have $y_\rho=j_0$. Every term of the sum lies in $[0,2]$, so Lemma 540(i) gives, at every value of S_i and S_{i+1},

$$(1-1.3\cdot10^{-8})^{|K_i|}e^{-72\beta_W|K_i|}\le\Psi_i\le(1+3.1\cdot10^{-4})^{|K_i|},$$

and as $|K_i|=|\Sigma_i|$, the ratio of the two bounds is at most e^{D_i}.

The chain. Put $Z_0=(C_1,S_1,\dots,C_a,S_a,S_{a+1})$, $Z_j=S_{a+1+j}$ for $1\le j\le b-a-2$, and let Z_{b-a-1} consist of the links of the slabs $b,\dots,L-1$, with $C_{b-1},\dots,C_{a+1}$ integrated as above and $C_1,\dots,C_a$ kept in Z_0. The links of C_a depend on S_{a+1} through ν_k and P_a, which lie in Z_0. So μ_W^-, restricted to $(Z_0,\dots,Z_{b-a-1})$, has the form

$$\frac{1}{\mathscr{Z}}\Gamma_0(dZ_0)\prod_{j=0}^{b-a-2}\Psi_{a+1+j}\big(S_{a+1+j},S_{a+2+j}\big)\prod_{j\ge1}\Gamma_j(dZ_j)$$

with finite measures Γ_j, where S_b is part of Z_{b-a-1} and $C_b,\dots,C_{L-1}$ are kept in Z_{b-a-1}. This is a Markov chain: given Z_j, the law of Z_{j+1} is

$$K_j(Z_j,dz)=\frac{\Psi_{a+1+j}(s,z)\,H_{j+1}(z)\,\Gamma_{j+1}(dz)}{\int\Psi_{a+1+j}(s,\cdot)\,H_{j+1}\,d\Gamma_{j+1}},$$

with s the part S_{a+1+j} of Z_j and H_{j+1} the integral of the factors beyond Z_{j+1}. With $m\le\Psi_{a+1+j}\le M$ and $M/m\le e^{D_{a+1+j}}$, $K_j(Z_j,\cdot)\ge e^{-D_{a+1+j}}\pi_{j+1}$ for the probability π_{j+1} proportional to $H_{j+1}\Gamma_{j+1}$, so $\|K_j(z,\cdot)-K_j(z',\cdot)\|_{\mathrm{TV}}\le1-e^{-D_{a+1+j}}$ for all z,z'. Let $g(Z_0)$ be the conditional expectation of $\tilde G$ given Z_0, obtained by applying $K_0,\dots,K_{b-a-2}$ to $\tilde G$. Each kernel multiplies the oscillation by at most its coefficient, so $\operatorname{osc}g\le2\|\tilde G\|_\infty\prod_{i=a+1}^{b-1}(1-e^{-D_i})$, and

$$|\mathrm{Cov}(\tilde F,\tilde G)|=\big|\mathbb{E}\big[\tilde F\,(g-\mathbb{E}g)\big]\big|\le\|\tilde F\|_\infty\operatorname{osc}g,$$

since $\tilde F$ is a function of Z_0. For μ_W the links of I_W are free as well; each is an inside link of a block of some Σ_i and is added to S_i, and the two structural facts and the rest of the proof are unchanged. □

Corollary 542 (ends at the two sides of a tube)**.** *Let $G=\mathrm{SU}(2)$, $\beta_W\ge0$, and let W, ρ, the slabs, a and b be as in Theorem 541. Let E_a, E_b be disjoint sets of links off A_W^-, which may enter averages of pairs based in W, such that every link of A_W^- that shares a plaquette with a link of E_a, or is a private link of a pair based in W whose average contains a link of E_a, lies in the slabs $1,\dots,a$, and likewise for E_b with the slabs $b,\dots,L-1$. Fix the other links off A_W^-, let a_0,a_1 be values on E_a, b_0,b_1 values on E_b, and $Z_{ij}=\nu_c^-(e^{-\lambda Q^-})$ with a_i on E_a and b_j on E_b. Then*

$$\Big|\log\frac{Z_{11}Z_{00}}{Z_{10}Z_{01}}\Big|\le\big(e^{2D_a}-1\big)\prod_{i=a+1}^{b-1}\big(1-e^{-D_i}\big).$$

Proof. Use the decomposition of the proof of Theorem 541. A factor Φ_i, m_i or Ψ_i depends on the value on E_a only through a plaquette that contains a link of E_a and a link of A_W^-, or through a pair whose average contains a link of E_a. By the hypothesis the links of A_W^- involved lie in the slabs $1,\dots,a$, so the factor has $i\le a$, a plaquette of P_i containing a link of C_i. Likewise the value on E_b enters only factors with $i\ge b$; in particular $\Psi_{a+1},\dots,\Psi_{b-1}$, the measure m_{a+1} and the weight Φ_{a+1} depend on neither. Integrating everything in Z_0 except S_{a+1} gives

$$Z_{ij}=\int\mathscr{A}_{a_i}(s)\,\Phi_{a+1}(s)\,m_{a+1}(ds)\,R_{b_j}(s),$$

where $\mathscr{A}_{a_i}(s)$ is the integral of the factors of the slabs $1,\dots,a$ with $S_{a+1}=s$ and $R_{b_j}(s)$ that of the factors beyond S_{a+1}. The variable s enters $\mathscr{A}_{a_i}$ only through $\Psi_a(S_a,s)$, whose bounds in the proof of Theorem 541 hold at every value of the ends, so $\mathscr{A}_{a_i}(s)/\mathscr{A}_{a_i}(s')\le e^{D_a}$ for all s,s'. Let $\Phi=\mathscr{A}_{a_1}/\mathscr{A}_{a_0}$, so that $\sup\Phi\le e^{2D_a}\inf\Phi$, and let π_j be the law of S_{a+1} under the measure with a_0 and b_j. Then $Z_{1j}/Z_{0j}=\pi_j(\Phi)$. Under that measure the chain $Z_{b-a-1},\dots,Z_0$ run backwards has kernels that do not depend on b_j, given by the same formula with the factors before Z_j in place of H, and with the same minorisation; so $\|\pi_1-\pi_0\|_{\mathrm{TV}}\le\theta:=\prod_{i=a+1}^{b-1}(1-e^{-D_i})$, the passage from Z_0 to S_{a+1} not increasing the distance. Hence $|\pi_1(\Phi)-\pi_0(\Phi)|\le\theta(\sup\Phi-\inf\Phi)$, both values are at least $\inf\Phi$, and $|\log x-\log x'|\le|x-x'|/\min(x,x')$ gives the bound. □

Remark 543. Theorem 541 and Corollary 542 need no chart and no bound on the plaquettes, and they allow ends that enter averages; the price is the rate. At $\beta_W=0$, $1-e^{-D_i}\le3.2\cdot10^{-4}|\Sigma_i|$, and the constrained law of a tube of thin slabs is close to a product; for $\beta_W|\Sigma_i|\le\frac{1}{72}$ and $|\Sigma_i|\le100$ each factor is at most $1-e^{-1.04}<0.65$. For $\beta_W\ge2.3\cdot10^{17}$, the range of Theorem 526, every nonempty slab has $D_i\ge72\beta_W>1.6\cdot10^{19}$, and as $\prod_i(1-x_i)\ge1-\sum_i x_i$ the product is at least $\frac12$ for every tube of fewer than $\frac12e^{1.6\cdot10^{19}}$ slabs; the bound then has no content on any lattice that the rest of the paper uses. The rate is the minorisation constant of the kernel across one interface, which uses only the oscillation of $\log\Psi_i$; the coarea factors contribute $3.2\cdot10^{-4}$ per pair to it, and the Wilson weight $72\beta_W$ per pair. A rate that does not degrade with β_W needs information on where in the fibre the law concentrates, which is what the chart of Theorem 526 supplies, at the cost of the mass p off the chart. The projected average does not improve the rate: by Theorem 443(i) its pair factor has mass in $[(\frac{8}{15})^3e^{-72\beta_W},512]$ times that of Haar measure, so the oscillation per pair is $72\beta_W+3\log\frac{15}{8}+\log512>72\beta_W+8$, and the Lipschitz constant $24\beta_W+180.5$ of Theorem 443(ii) exceeds one at every β_W, so it gives no contraction across an interface either.

The ends that enter the averages are taken up once more, through Lipschitz bounds in the end in place of oscillations. Fix a pair $k=(y,\rho)$ based in W and an end e of Lemma 537. In case (a), $e=u_b$ for some $b\ne0$, and e enters only the path $P_{x_b}=u_b\ell_b$ of k; in case (b), $e=w_b$, and e enters only $P_{x'_b}=\ell_bw_b$. The *links of the block of e* are the links u_a, $a\ne0$, in case (a), and w_a, $a\ne0$, in case (b); they lie in I_W, respectively off A_W, so they are fixed links of the fibre, and e is one of them.

For a measurable function $W \ge 0$ of the eight private links $\ell = (\ell_0, \dots, \ell_7)$ that does not depend on e, and a value c_k, put

$$Z_k^W(c_k; e) = \int W \, d\nu_k,$$

where ν_k is the factor of k at the value c_k, built before (39) from (38). For $W = 1$ this is $\rho(c_k)$, and for $W = e^{-\beta_W S_{\mathrm{dep}}}$, with S_{dep} the sum of $1 - \frac{1}{2}\operatorname{Re}\operatorname{Tr} U_p$ over the 36 plaquettes that contain a private link of k (Lemma 426), it is the integrated pair factor. No plaquette contains e and a private link of k: both are ρ-links, the two ρ-links of a plaquette have the same coordinate ρ, and this coordinate is even for e and odd for a private link. On R_0 the average is the path link, so the part of the fibre where $m > 1$ is $\{\ell_0 = c_k\} \times R_0(e)$, where $R_0(e)$ is the set of $(\ell_a)_{a \ne 0}$ at which $m > 1$ for $\ell_0 = c_k$, and there ν_k is Haar measure of the seven non-path links.

Lemma 544 (the boundary $\{m = 1\}$)**.** *Let $G = \mathrm{SU}(2)$, $r_0 = \sqrt{2}\,\alpha$ and $B(p) = \{v : d(v,p) \le r_0\}$, and fix c_k and every link other than the private links of k and the end.*

(i) *The complement of $R_0(e)$ is $\prod_{a \ne 0} A_a$, where $A_a = B(u_a^{-1} c_k) \cap B(c_k w_a^{-1})$. For $W = 1$ the part of $\rho(c_k)$ on R_0 is $1 - \prod_{a \ne 0} \lambda(s_a)$, where $s_a = d(c_k^{-1} u_a c_k, w_a)$ and $\lambda(s) = h(B(I) \cap B(z))$ for any z with $d(I,z) = s$; in particular $\lambda \le q$.*

(ii) *For every move $e \to e'$ of the end and every bounded measurable $W \ge 0$,*

$$\Big| \int_{R_0(e)} W(c_k, \cdot)\, dh^{\otimes 7} - \int_{R_0(e')} W(c_k, \cdot)\, dh^{\otimes 7} \Big| \le \frac{7}{8\sqrt{2}\,\pi} q^6 \|W\|_\infty d(e,e') < 3.4 \cdot 10^{-8} \|W\|_\infty d(e,e').$$

Proof. (i) By Lemma 678(iii), $\delta(v,I) \le 1$ exactly when $d(v,I) \le r_0$. At $\ell_0 = c_k$ the paths of offset zero give $v = I$, and for $a \ne 0$, by bi-invariance, $d(c_k^{-1} u_a \ell_a, I) \le r_0$ exactly when $\ell_a \in B(u_a^{-1} c_k)$, and $d(c_k^{-1} \ell_a w_a, I) \le r_0$ exactly when $\ell_a \in B(c_k w_a^{-1})$. So $m \le 1$ exactly when $\ell_a \in A_a$ for every $a \ne 0$. The two centres are at distance $d(u_a^{-1} c_k, c_k w_a^{-1}) = d(c_k^{-1} u_a^{-1} c_k, w_a^{-1}) = s_a$, inversion being an isometry. The isometries of the round sphere act transitively on pairs of points at a given distance and preserve h, so $h(A_a) = \lambda(s_a) \le h(B(I)) = q$, and the seven conditions bear on distinct coordinates.

(ii) In case (a) only A_b depends on e, through the centre $p = e^{-1} c_k$, so the symmetric difference of $R_0(e)$ and $R_0(e')$ is $\prod_{a \ne 0, b} A_a$ times the symmetric difference of $A_b(e)$ and $A_b(e')$, which lies in $B(p) \triangle B(p')$, $p' = e'^{-1} c_k$. Let γ be a geodesic of unit speed from e to e', of length $s = d(e,e')$, and $p_t = \gamma(t)^{-1} c_k$. If $x \in B(p) \setminus B(p')$, the continuous function $t \mapsto d(x, p_t)$ is at most r_0 at $t = 0$ and larger at $t = s$, so x lies on the sphere $\partial B(p_t)$ for some t; likewise for $B(p') \setminus B(p)$. Since $\partial B(p_t) = p_t \Sigma$ with $\Sigma = \partial B(I)$, the set $B(p) \triangle B(p')$ lies in the image of $\Phi(t, \sigma) = \gamma(t)^{-1} c_k \sigma$ on $[0,s] \times \Sigma$. For fixed t the map $\sigma \mapsto \Phi(t,\sigma)$ is an isometry and $t \mapsto \Phi(t,\sigma)$ has unit speed, so by the area formula the image has volume at most $s H^2(\Sigma)$. By Lemma 678, $\mathrm{SU}(2)$ is the unit sphere $S^3 \subset \mathbb{R}^4$, with d equal to $\sqrt{2}$ times the angle and h the normalised surface measure; in these units Σ is the two-sphere at angle α from I, of area $4\pi \sin^2 \alpha = \frac{7\pi}{4}$, as $\sin \alpha = \frac{\sqrt{7}}{4}$, and the curve sweeps the angle $s/\sqrt{2}$. So

$$h\big(B(p) \triangle B(p')\big) \le \frac{s}{\sqrt{2}} \cdot \frac{7\pi/4}{2\pi^2} = \frac{7s}{8\sqrt{2}\,\pi},$$

and the symmetric difference of $R_0(e)$ and $R_0(e')$ has measure at most q^6 times this. In case (b) the moving centre is $c_k w_b^{-1}$, and the same argument applies with $\Phi(t,\sigma) = \sigma c_k \gamma(t)^{-1}$. Finally $8\sqrt{2}\,\pi > 35.54$, $\frac{7}{35.54} < 0.19697$, and $q^6 < 1.706 \cdot 10^{-7}$ by the proof of Lemma 540(i). □

So the motion of the boundary $\{m = 1\}$ costs at most $3.4 \cdot 10^{-8}$ per unit move of the end, for every coarse value and every configuration. The law of the coarse value itself moves by a bounded amount in the Kantorovich distance, with no condition at all.

Proposition 545 (the coarse value moves Lipschitz)**.** *Let $G = \mathrm{SU}(2)$ and fix every link other than the private links of k and the end, with no condition on them. Let $W \ge 0$ be an integrable function of the private links that does not depend on e. Then $\int Z_k^W(c;e)\, dh(c) = \int W \, dh^{\otimes 8}$ for every e, and for every function φ on $\mathrm{SU}(2)$ that is 1-Lipschitz for d and every move $e \to e'$,*

$$\Big| \int \varphi(c) \big(Z_k^W(c;e) - Z_k^W(c;e') \big)\, dh(c) \Big| \le 1.858\, d(e,e') \int W \, dh^{\otimes 8}.$$

Proof. The measures ν_k at the values $c \in \mathrm{SU}(2)$ disintegrate Haar measure on the eight private links along $V_k^\sharp$, by the proof of Lemma 373, so $\int \varphi(c) Z_k^W(c;e)\, dh(c) = \int \varphi(V_k^\sharp(\ell;e)) W(\ell)\, dh^{\otimes 8}(\ell)$; $\varphi = 1$ gives the first claim. Fix ℓ and move e along a geodesic of unit speed. Write $V_k^\sharp = \ell_0 e^Y$ with $Y = \chi N$. Only the path that contains e moves, and its v_x moves at unit speed. Since $\delta(\cdot, I)$ is 1-Lipschitz for d, m moves at speed at most 1 and $\chi = \phi(m)$ at most 2. On the open set of times at which $\chi > 0$, every v_x has $\delta(v_x, I) < 1$, so $\|N\| \le \frac{14}{16}\sqrt{2}\,\alpha$, the two paths of offset zero giving 0, and N moves at speed at most $\frac{1}{16}\Lambda$; hence Y moves at speed at most $\frac{7}{4}\sqrt{2}\,\alpha + \frac{1}{16}\Lambda < 1.858$, by $\alpha < 0.7228$ and $\Lambda < 1.0927$. Where $\chi = 0$, $Y = 0$, and Y is continuous, so Y is 1.858-Lipschitz along the move and $d(V_k^\sharp(\ell;e), V_k^\sharp(\ell;e')) \le \|Y - Y'\| \le 1.858\, d(e,e')$. Integrate against W. □

Corollary 546 (the derivative in the mean over the coarse link)**.** *In the setting of Lemma 540(iii) with $E = \{e\}$, an end of the pair k, write $Z(e; c_k)$ for $Z(e)$ as a function of the coarse value c_k, the other coarse links being fixed. For every positive Lipschitz function ψ on $\mathrm{SU}(2)$ and every move $e \to e'$,*

$$\Big| \log \frac{\int \psi Z(e;\cdot)\, dh}{\int \psi Z(e';\cdot)\, dh} \Big| \le \Big(3\sqrt{2}\,\beta_W + 1.858\, \frac{\operatorname{Lip}\psi}{\min \psi} \Big) d(e,e'),$$

uniformly in the other links off A_W^- and in the other coarse links.

Proof. Write $\lambda Q^- = \beta_W(S_{\mathrm{dep}} + S_e + S_{\mathrm{r}})$, with S_e the sum of $1 - \frac{1}{2}\operatorname{Re}\operatorname{Tr} U_p$ over the plaquettes that contain e and a link of A_W^-, none of which contains a private link of k, and S_{r} the sum over the other plaquettes that contain a link of A_W^-; S_{r} depends neither on e nor on the private links of k. By Lemma 537 only the factor ν_k of ν_c^- depends on e, and only it depends on c_k. So, with ζ the links of A_W^- other than the private links of k,

$$Z(e;c_k) = \int d\Xi(\zeta)\, e^{-\beta_W S_e(e,\zeta)}\, Z_k^{W_\zeta}(c_k;e),$$
$$W_\zeta = e^{-\beta_W S_{\mathrm{dep}}(\cdot,\zeta)},$$

where ν_k in $Z_k^{W_\zeta}$ is taken at the inside links in ζ, and $d\Xi$ does not depend on e or c_k. By Proposition 545 applied to $\psi/\operatorname{Lip}\psi$, the integral $I_\zeta(e) = \int \psi Z_k^{W_\zeta}(\cdot;e)\,dh$ satisfies $|I_\zeta(e) - I_\zeta(e')| \le 1.858 \operatorname{Lip}\psi\, d(e,e') \int W_\zeta\, dh^{\otimes 8}$, and $\int W_\zeta\, dh^{\otimes 8} = \int Z_k^{W_\zeta}(\cdot;e)\,dh \le I_\zeta(e)/\min\psi$. The link e lies in six plaquettes, and a move of e by $d(e,e')$ changes each $1 - \frac{1}{2}\operatorname{Re}\operatorname{Tr} U_p$ by at most $d(e,e')/\sqrt{2}$, by the proof of Lemma 415. So the integrand at e' is at most $e^{(3\sqrt{2}\beta_W + 1.858 \operatorname{Lip}\psi/\min\psi) d(e,e')}$ times the integrand at e, for every ζ; integrate and exchange e and e'. □

For a fixed coarse value the derivative needs the part of the pair factor on $R_1 \cup R_2$, and this is where the relative chart of Lemma 425 enters. Its hypothesis bears on the links of the block of the end only.

Lemma 547 (the derivative on the relative chart)**.** *Let $G = \mathrm{SU}(2)$ and $\beta_W \ge 0$, and let the end e move along a geodesic to e' while the links of the block of e stay within $\frac{1}{32}$ of I; the other links are arbitrary. Let $W(\ell) = \int e^{-\beta_W S_{\mathrm{dep}}(\ell,\zeta)}\, \omega(d\zeta)$ for a nonzero finite measure ω on the values ζ of the links other than the private links of k. Take Z_k^W in the version $\int W(\Gamma(\xi))\, D(\xi)^{-1} d\xi$ of Lemma 425*(iv)*, with $\xi_a = g^{-1}\ell_a$ and $\Gamma(\xi) = (G(\xi), G(\xi)\xi)$ in case* (a)*, and $\xi_a = \ell_a g^{-1}$ and $\Gamma(\xi) = (G(\xi), \xi G(\xi))$ in case* (b)*; it agrees with $\int W\, d\nu_k$ for almost every c_k. Then for every c_k,*

$$\left|\log \frac{Z_k^W(c_k;e)}{Z_k^W(c_k;e')}\right| \le L_*\, d(e,e'),$$
$$L_* = 128.07\,\beta_W + \left(180.4 + 50.6\,\mathscr{A}_{\mathrm{tie}}\right) e^{72\beta_W},$$

with $\mathscr{A}_{\mathrm{tie}} < \infty$ of Proposition 430.

Proof. Case (a). By Lemma 425, applied along the move, the equation $V_k^\sharp = c_k$ has for every ξ the unique solution $G(\xi)$, and $Z = Z_k^W(c_k;e) = \int f\, d\xi$ with $f = (W \circ \Gamma)/D$. Let e' lie at distance $\delta \le 0.08$ along the geodesic, with G', Γ', D', f', Z'.

The weight. By Lemma 425(ii), $d(G,G') \le L\delta$ with $L < 2.064$, and the eight links of $\Gamma(\xi)$, which are G and $G\xi_a$, move by $d(G,G')$ each. Each lies in six plaquettes of S_{dep}, and a link that moves by ε changes $1 - \frac{1}{2}\operatorname{Re}\operatorname{Tr} U_p$ by at most $\varepsilon/\sqrt{2}$, by the proof of Lemma 415; so for every ζ, $|S_{\mathrm{dep}}(\Gamma'(\xi),\zeta) - S_{\mathrm{dep}}(\Gamma(\xi),\zeta)| \le 24\sqrt{2}L\delta < 70.06\,\delta$, and $|W\circ\Gamma' - W\circ\Gamma| \le (e^{70.06\beta_W\delta} - 1)\, W \circ \Gamma$.

The Jacobian. By Lemma 425(iii), D and D' lie in $[(1-\lambda)^3, (1+\lambda)^3]$, the ratio of whose ends is less than 1.828. So $D'^{-1} \le 1.828\, D^{-1}$, and with $\Delta = \log D - \log D'$, $|D^{-1} - D'^{-1}| = D^{-1}|1 - e^{\Delta}| \le 1.828\, D^{-1}|\Delta|$. Hence

$$\int |f - f'|\, d\xi \le 1.828\left(e^{70.06\beta_W\delta} - 1\right) Z + 1.828 \int f\,|\Delta|\, d\xi.$$

Since $0 \le S_{\mathrm{dep}} \le 72$, $\sup W \le e^{72\beta_W} \inf W$, so $\sup f \le 1.828\, e^{72\beta_W} \inf f \le 1.828\, e^{72\beta_W} Z$, $d\xi$ being a probability. By Proposition 429, whose hypotheses for $B = \mathrm{SU}(2)^7$ are those of the lemma (Theorem 431), $\int |\Delta|\, d\xi \le (53.96 + 15.134\,\mathscr{A}_{\mathrm{tie}})\,\delta$. So $|Z' - Z| \le \varepsilon(\delta) Z$ with

$$\varepsilon(\delta) = 1.828\left(e^{70.06\beta_W\delta} - 1\right) + 1.828^2 e^{72\beta_W}\left(53.96 + 15.134\,\mathscr{A}_{\mathrm{tie}}\right)\delta.$$

Divide the geodesic into n equal steps; then $|\log Z(e) - \log Z(e')| \le -n\log(1 - \varepsilon(d(e,e')/n))$, which tends to $(1.828 \cdot 70.06\,\beta_W + 1.828^2 e^{72\beta_W}(53.96 + 15.134\,\mathscr{A}_{\mathrm{tie}}))\, d(e,e')$. Finally $1.828 \cdot 70.06 < 128.07$, $1.828^2 < 3.3416$, $3.3416 \cdot 53.96 < 180.4$ and $3.3416 \cdot 15.134 < 50.6$.

Case (b). The last part of Lemma 425 gives (i)–(iv) in the coordinates $\xi_a = \ell_a g^{-1}$ with $\lambda_w \le \lambda$ and $L_w \le L$, the eight private links G and $\xi_a G$ moving by $d(G,G')$. The inversion i of the proof of Theorem 431 maps the configuration to one of case (a) whose links of the block of the moving link are within $\frac{1}{32}$ of I, preserves Haar measure and D, and so gives the same bound on $\int |\Delta|\, d\xi$. The rest is unchanged. □

The links of the block of e are the inside ρ-links of $B(y)$ in case (a) and of $B(y + e_\rho)$ in case (b), and these are non-path active links of the pair $(y - e_\rho, \rho)$, respectively $(y + e_\rho, \rho)$, based at the block across the end. The reconstruction $R(c)$ is I on the links inside the blocks, so the hypothesis of Lemma 547 holds whenever the block across the end, $y - e_\rho$ in case (a) and $y + e_\rho$ in case (b), is good for labels of radius $r \le \frac{1}{32}$ at the two values of the end; the labels of Sec. XIII M have $r \le 10^{-4}$.

Corollary 548 (the derivative in an end)**.** *In the setting of Lemma 540*(iii) *with $E = \{e\}$, suppose that the links of the block of e stay within $\frac{1}{32}$ of I along the geodesic from e to e'. Then, with the factor of k in the version of Lemma 547, for every coarse field*

$$\left|\log \frac{Z(e)}{Z(e')}\right| \le \left(132.32\,\beta_W + \left(180.4 + 50.6\,\mathscr{A}_{\mathrm{tie}}\right) e^{72\beta_W}\right) d(e,e'),$$

uniformly in the other links off A_W^-.

Proof. Use the decomposition of the proof of Corollary 546, apply Lemma 547 with $\omega = \delta_\zeta$ at each ζ, and bound the plaquettes through e as there; $L_* + 3\sqrt{2}\,\beta_W$ has the stated constants, as $128.07 + 3\sqrt{2} < 132.32$. □

The same bound replaces the oscillation of the ends in Corollary 542.

Corollary 549 (mixed differences along a tube)**.** *Let $G=\mathrm{SU}(2)$, $\beta_W\ge 0$, and let W, ρ, the slabs, a, b, E_a, E_b, a_0,a_1,b_0,b_1 and Z_{ij} be as in Corollary 542. Suppose that for every link e of E_a that enters an average of a pair based in W, every link of the block of e lies within $\frac{1}{32}$ of I, at each of its values if it belongs to $E_a\cup E_b$. Put $L_e=3\sqrt{2}\,\beta_W+L_*$ for these links of E_a, $L_e=3\sqrt{2}\,\beta_W$ for the links of E_a that enter no such average, and $\Lambda_a=\sum_{e\in E_a}L_e\,d(a_{0,e},a_{1,e})$. Then, with the factors of the pairs in the version of Lemma 547,*

$$\Big|\log\frac{Z_{11}Z_{00}}{Z_{10}Z_{01}}\Big|\le\big(e^{2\Lambda_a}-1\big)\prod_{i=a+1}^{b-1}\big(1-e^{-D_i}\big).$$

If E_b satisfies the same hypothesis, and Λ_b is defined in the same way, then also

$$\Big|\log\frac{Z_{11}Z_{00}}{Z_{10}Z_{01}}\Big|\le\big(e^{2\Lambda_a}-1\big)\big(e^{2\Lambda_b}-1\big)\prod_{i=a+1}^{b-2}\big(1-e^{-D_i}\big).$$

Proof. In the proof of Corollary 542, $\mathscr{A}_{a_i}(s)$ is the integral of the factors of the slabs $1,\dots,a$ with $S_{a+1}=s$; it contains the private links of every pair whose average contains a link of E_a, and the links of A_W^- in every plaquette through a link of E_a. Move the links of E_a from a_0 to a_1 one at a time along geodesics; the links of the blocks stay within $\frac{1}{32}$ of I, since the geodesic ball of radius $\frac{1}{32}$ is convex. A link e enters the integrand through the plaquettes that contain it, each of which changes by at most $d/\sqrt{2}$, and, if it enters an average, through the factor ν_k of the one pair k whose average contains it (Lemma 537), whose private links enter the integrand otherwise only through S_{dep}. As in the proof of Corollary 548, Lemma 547 with $\omega=\delta_\zeta$ at every value ζ of the other variables gives $e^{-\Lambda_a}\le\mathscr{A}_{a_1}(s)/\mathscr{A}_{a_0}(s)\le e^{\Lambda_a}$ for every s. So the function $\Phi=\mathscr{A}_{a_1}/\mathscr{A}_{a_0}$ of that proof satisfies $\sup\Phi\le e^{2\Lambda_a}\inf\Phi$, and the rest of that proof gives the first bound.

For the second, let ι_j be the law of S_{b-1} under the measure with a_0 on E_a and b_j on E_b. Its density is proportional to $\mathscr{L}(s)\mathscr{R}_{b_j}(s)$, where $\mathscr{L}(s)$ is the integral of the factors of the slabs $1,\dots,b-1$ with $S_{b-1}=s$, and $\mathscr{R}_{b_j}(s)$ that of $\Psi_{b-1}(s,\cdot)$ and of the factors of the slabs $b,\dots,L-1$. Only $\mathscr{R}_{b_j}$ depends on b_j, and it integrates the private links of every pair whose average contains a link of E_b; the argument above gives $e^{-\Lambda_b}\le\mathscr{R}_{b_1}(s)/\mathscr{R}_{b_0}(s)\le e^{\Lambda_b}$ for every s, so $d\iota_1/d\iota_0\le e^{2\Lambda_b}$ and $\|\iota_1-\iota_0\|_{\mathrm{TV}}\le e^{2\Lambda_b}-1$. Given S_{i+1}, for $a+1\le i\le b-2$, the law of S_i under either measure is the backward kernel of the proof of Corollary 542, which does not depend on b_j and contracts the total variation distance by the factor $1-e^{-D_i}$. Hence the laws π_j of S_{a+1} satisfy $\|\pi_1-\pi_0\|_{\mathrm{TV}}\le(e^{2\Lambda_b}-1)\prod_{i=a+1}^{b-2}(1-e^{-D_i})$, and $|\pi_1(\Phi)-\pi_0(\Phi)|\le\|\pi_1-\pi_0\|_{\mathrm{TV}}(\sup\Phi-\inf\Phi)$ ends the proof as before. □

For ends that enter no average the hypothesis is empty and $L_e=3\sqrt{2}\,\beta_W$; the second bound is then a mixed difference in the form of Corollary 533, without a chart and without a bound on the plaquettes, at the rate of Theorem 541. For ends in the averages the constant L_* contains $e^{72\beta_W}$: the jumps of $\log D$ across the ties are integrated against Haar measure in Proposition 429, and the weight $e^{-\beta_W S_{\mathrm{dep}}}$ can exceed its mean by that factor on the set they sweep. Neither L_* nor the rate of Theorem 541 has content in the range $\beta_W\ge 2.3\cdot 10^{17}$ of Theorem 526 (Remark 543).

The factor $e^{72\beta_W}$ in L_* comes from bounding the weight $f=(W\circ\Gamma)/D$ of Lemma 547 by its supremum on the sets swept by the events and the ties. Along the curves used below the logarithm of $W\circ\Gamma$ is Lipschitz with a constant linear in β_W, and integral-geometric trace inequalities replace the supremum by the integral. The ties are counted by a crossing number.

Write $\mathscr{P}$ for the set of parameters $P=(c_k,(u_b,w_b)_{b\ne 0})$ with $d(u_b,I)\le\frac{1}{32}$ for every $b\ne 0$, and $Z_a(P)$ for the set Z_a of Proposition 429 at P. For $b\ne 0$, $\xi\in\mathrm{SU}(2)^7$ and a unit vector $\eta\in\mathfrak{su}(2)$ let $C_{b,\xi,\eta}$ be the orbit of ξ under the flow Θ_s that replaces ξ_b by $\xi_b e^{s\eta}$ and fixes the other coordinates. Since η has the eigenvalues $\pm i/\sqrt{2}$, the orbit is a great circle in the factor b, of length $T_0=2\sqrt{2}\,\pi$, run at unit speed, and Θ_s preserves Haar measure.

Definition 550. The *crossing number* $\mathfrak{b}_{\mathrm{tie}}$ is the supremum of $\#(Z_a(P)\cap C_{b,\xi,\eta})$ over $a,b\ne 0$, $P\in\mathscr{P}$, $\xi\in\mathrm{SU}(2)^7$ and unit vectors η, among the intersections that are finite.

Lemma 551 (crossings)**.** (i) $\mathfrak{b}_{\mathrm{tie}}<\infty$. (ii) *For fixed a, b, P and η, the set of ξ for which $Z_a(P)\cap C_{b,\xi,\eta}$ is infinite is Haar-null.*

Proof. (i) By the proof of Proposition 430, $Z_a(P)$ is a definable family over $\mathscr{P}$ in $\mathbb{R}_{\mathrm{an}}$. With $\eta=n/\sqrt{2}$, $n^2=-1$, $\xi_b e^{s\eta}=\xi_b(\cos\frac{s}{\sqrt{2}}+\sin\frac{s}{\sqrt{2}}\,n)$, and sin and cos on $[0,2\pi]$ are restricted analytic, so the sets $\{s\in[0,T_0):\Theta_s\xi\in Z_a(P)\}$ form a definable family of subsets of $\mathbb{R}$ with the parameters (a,b,P,ξ,η). By uniform finiteness [48, Ch. 3] there is an N such that every finite member has at most N points.

(ii) An infinite definable subset of $[0,T_0)$ contains an open interval. On that interval the orbit lies in the C^1 hypersurface $Z=Z_a(P)$, so its velocity, the left-invariant field η in the factor b, is tangent to Z there, that is $\langle\nu_b,\eta\rangle=0$, where $\nu=(\nu_c)_{c\ne 0}$ is a unit normal of Z written in the left-invariant frame. So ξ lies in $\Psi(Z^0\times[0,T_0])$, where $Z^0=\{z\in Z:\langle\nu_b(z),\eta\rangle=0\}$ and $\Psi(z,s)=\Theta_s z$. The map Ψ is C^1 on $Z\times\mathbb{R}$, and its Jacobian is $|\langle\nu_b,\eta\rangle|$: each Θ_s is an isometry, and its differential maps the field η at z to the field η at $\Theta_s z$, as $\mathrm{Ad}(e^{-s\eta})\eta=\eta$. By the area formula [51, 3.2.20] the image of $Z^0\times[0,T_0]$ is null. □

Lemma 552 (traces on curves)**.** *Let γ be a curve of length T run at unit speed and $w>0$ a function on it such that $w\circ\gamma$ is Lipschitz and $|(\log w\circ\gamma)'|\le K$ almost everywhere. If γ is closed, then $\sup w\circ\gamma\le(\frac{1}{T}+\frac{K}{2})\int_0^T w(\gamma(s))\,ds$; if γ is a segment, then $\sup w\circ\gamma\le(\frac{1}{T}+K)\int_0^T w(\gamma(s))\,ds$.*

Proof. For s_0,s let $\Gamma(s,s_0)$ be the shorter arc of a closed curve between them, or the interval between them on a segment. Then $w(\gamma(s_0))\le w(\gamma(s))+\int_{\Gamma(s,s_0)}|(w\circ\gamma)'|\le w(\gamma(s))+K\int_{\Gamma(s,s_0)}w\circ\gamma$. Average over s: a point u lies in $\Gamma(s,s_0)$ for a set of s of measure at most $\frac{T}{2}$ on a closed curve and at most T on a segment. □

Proposition 553 (the ties against a weight)**.** *Let $a\ne 0$, $P\in\mathscr{P}$, $Z=Z_a(P)$, and let $w>0$ be a Lipschitz function on* $\mathrm{SU}(2)^7$

with $|\frac{d}{ds}\log w(\Theta_s\xi)| \le K$ *almost everywhere, for every* b, ξ *and unit* η. *Then*

$$V^{-7}\int_Z w\,dH^{20} \le 14\,\mathfrak{b}_{\mathrm{tie}}\Big(\frac{1}{T_0}+\frac{K}{2}\Big)\int w\,d\xi.$$

Proof. Fix b and η, and let $\Psi(z,s)=\Theta_s z$ on $Z\times[0,T_0)$, with Jacobian $|\langle \nu_b,\eta\rangle|$ as in the proof of Lemma 551. A point x has one preimage (z,s) for each $z\in Z\cap C_{b,x,\eta}$, the flow having period T_0. The area formula with the function $(z,s)\mapsto w(z)$ gives

$$T_0\int_Z w\,|\langle \nu_b,\eta\rangle|\,dH^{20} = \int \sum_{z\in Z\cap C_{b,x,\eta}} w(z)\,d\mathrm{vol}(x).$$

For almost every x the sum has at most $\mathfrak{b}_{\mathrm{tie}}$ terms, by Lemma 551, and by Lemma 552 it is at most $\mathfrak{b}_{\mathrm{tie}}(\frac{1}{T_0}+\frac{K}{2})\int_0^{T_0} w(\Theta_s x)\,ds$. Since Θ_s preserves Haar measure,

$$\int\!\!\int_0^{T_0} w(\Theta_s x)\,ds\,d\mathrm{vol}(x) = T_0\int w\,d\mathrm{vol},$$

so

$$\int_Z w\,|\langle \nu_b,\eta\rangle|\,dH^{20} \le \mathfrak{b}_{\mathrm{tie}}\Big(\frac{1}{T_0}+\frac{K}{2}\Big)\int w\,d\mathrm{vol}.$$

The mean of $|\langle \nu_b,\eta\rangle|$ over the unit sphere of $\mathfrak{su}(2)\cong\mathbb{R}^3$ is $\frac12|\nu_b|$, and $\sum_b|\nu_b|\ge(\sum_b|\nu_b|^2)^{1/2}=1$. Average over η, sum over the seven values of b and divide by V^7. □

With $w=1$ and $K=0$ this is an explicit form of Proposition 430.

Corollary 554. $\mathscr{A}_{\mathrm{tie}} \le 14\,\mathfrak{b}_{\mathrm{tie}}/T_0 < 1.5756\,\mathfrak{b}_{\mathrm{tie}}$, *so that* $L_* \le 128.07\,\beta_W + (180.4+79.73\,\mathfrak{b}_{\mathrm{tie}})e^{72\beta_W}$.

Proof. $14/(2\sqrt2\,\pi) < 1.57556$ and $50.6\cdot 1.5756 < 79.73$. □

Proposition 555 (the Jacobian across the ties against the weight)**.** *Let* $B=\mathrm{SU}(2)^7$, *let the links* u_a, $a\ne 0$, *and the move of one link* u_b *or* w_b *be as in Theorem 427, take the coordinates* $\xi_a = g^{-1}\ell_a$, *and let* W *be as in Lemma 547. Let* e' *lie at distance* $\delta \le 0.08$ *from* e *along the geodesic, and let* $f=(W\circ\Gamma)/D$ *and* $\Delta=\log D-\log D'$. *Then*

$$\int f\,|\Delta|\,d\xi \le C(\beta_W)\,\delta\int f\,d\xi$$

with

$$C(\beta_W) = 3934 + 242610\,\beta_W + \mathfrak{b}_{\mathrm{tie}}\big(43.59+14888\,\beta_W\big).$$

Proof. Write $w=W\circ\Gamma$. A private link lies in six plaquettes of S_{dep}, and a link that moves by ε changes $1-\frac12\mathrm{Re}\,\mathrm{Tr}\,U_p$ by at most $\varepsilon/\sqrt2$; so for every ζ the ratio of $e^{-\beta_W S_{\mathrm{dep}}(\ell,\zeta)}$ at two values of the eight links is at most $e^{3\sqrt2\beta_W\sum_j d(\ell_j,\ell'_j)}$, and the same holds for W. By Lemma 425(iii), D takes values in $[D_-,D_+]$ with $D_+/D_-<1.828$, so $w/D_+ \le f \le w/D_-$ almost everywhere. We follow the proof of Proposition 429, whose three parts bound $|\Delta|$ pointwise.

Where the type is kept. Off the ξ that meet an event, $|\Delta| \le 11.87\,\delta + \sum_t 1.9811|\nabla h_2|$, and the first term contributes $11.87\,\delta\int f\,d\xi$.

The events. Fix an event of index a, the coordinates ξ_c, $c\ne a$, and the coordinate $v=\tilde u_a(G(\xi))\xi_a$ of the proof of Proposition 429, with $t=\omega\delta$, $\omega<1.1825$. Write $v=\exp(\theta\nu)$. The ξ that meet the event during the move lie in $\{|h|\le t\}$ and in $\{0.4\le f_{x_a}\le 1.1\}$, and $f_{x_a}=2\sqrt2\sin\frac{\theta}{2}$. So on each ray of fixed ν they form a set I_ν of length at most $2t/1.154 < 1.7332\,t$ inside the interval $J=\{\theta : 0.4\le 2\sqrt2\sin\frac{\theta}{2}\le 1.1\}=[\theta_-,\theta_+]$, with $\sin^2\theta_- = 4\cdot\frac{1}{50}\cdot\frac{49}{50}=0.0784$, $\theta_+<0.7990$ and $|J|>0.515099$. Along the ray v moves at speed $\sqrt2$ per unit of θ, ξ_a at speed at most $\sqrt2/0.9054<1.5620$, G at speed at most $2.140\cdot 1.5620<3.3427$, and the eight links G, $G\xi_c$ together at speed at most $8\cdot 3.3427+1.5620$; so $|\frac{d}{d\theta}\log w| \le K_\theta = 3\sqrt2(8\cdot 3.3427+1.5620)\beta_W < 120.09\,\beta_W$. By Lemma 552 on the segment J, with Haar density $\frac{2}{\pi}\sin^2\theta$ and $\sin^2\theta\le 0.5146$ on J,

$$\int_{I_\nu} w\,\tfrac{2}{\pi}\sin^2\theta\,d\theta \le \tfrac{2}{\pi}0.5146\,|I_\nu|\Big(\frac{1}{|J|}+K_\theta\Big)\int_J w\,d\theta$$
$$\le \frac{0.5146}{0.0784}\,1.7332\,t\,(1.9414+K_\theta)\int_J w\,\tfrac{2}{\pi}\sin^2\theta\,d\theta.$$

Integrating over ν, and passing between v and ξ_a with the factors 1.348 and $1.0946^3<1.3115$ of the bi-Lipschitz map $\xi_a\mapsto v$, the set S of ξ that meet the event satisfies $\int_S w\,d\xi \le 20.113\,t\,(1.9414+K_\theta)\int w\,d\xi$, and hence, as $D_+/D_-<1.828$ and $1.828\cdot 20.113<36.77$,

$$\int_S f\,d\xi \le 36.77\,t\,(1.9414+K_\theta)\int f\,d\xi.$$

On S, $|\Delta|<0.6034$. The 77 events therefore contribute at most $77\cdot 0.6034\cdot 36.77\cdot 1.1825\,\delta(1.9414+K_\theta)\int f < 2020.2\,\delta(1.9414+120.09\,\beta_W)\int f$.

The ties. By the area and coarea formulas of the proof of Proposition 429, applied to the function $1.9811|\nabla_\xi H|\,w/D_-$, which dominates $1.9811|\nabla_\xi H|f$ almost everywhere,

$$\int f\sum_{t:\,\xi\in Z_a(t)} 1.9811|\nabla h_2|\,d\xi$$
$$\le \frac{1.9811\cdot 1.0913}{D_-}\int_0^\delta V^{-7}\int_{Z_a(t)} w\,dH^{20}\,dt.$$

Along the flow Θ_s the coordinate ξ_b moves at unit speed, G at speed at most 2.140 by the proof of Proposition 429, and the eight links together at speed at most $8\cdot 2.140+1$; so w satisfies the hypothesis of Proposition 553 with $K=3\sqrt2\cdot 18.12\,\beta_W<76.877\,\beta_W$, at every parameter t of the move. With $\int w\,d\xi \le D_+\int f\,d\xi$ the ties of index a contribute at most

$$1.9811\cdot 1.0913\cdot 1.828\cdot 14\,\mathfrak{b}_{\mathrm{tie}}\Big(\frac{1}{T_0}+38.4385\,\beta_W\Big)\delta\int f\,d\xi,$$

and the seven indices at most $\mathfrak{b}_{\mathrm{tie}}(43.59+14888\,\beta_W)\,\delta\int f$, by $7\cdot 1.9811\cdot 1.0913<15.134$, $25.592/T_0<2.8802$ and $25.592\cdot 38.4385<983.72$.

Conclusion. $11.87+2020.2\cdot 1.9414<3934$ and $2020.2\cdot 120.09<242610$. □

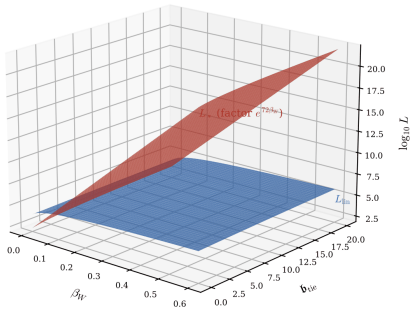


FIG. 42. The derivative in an end (Theorem 556): $\log_{10}$ of $L_* = 128.07\,\beta_W + (180.4 + 50.6\,\mathscr{A}_{\rm tie})e^{72\beta_W}$, with $\mathscr{A}_{\rm tie} < 1.5756\,\mathfrak{b}_{\rm tie}$ (red), and of $L_{\rm lin} = 7192 + 443620\,\beta_W + \mathfrak{b}_{\rm tie}(79.69 + 27216\,\beta_W)$ (blue), over β_W and the tie length $\mathfrak{b}_{\rm tie}$.

Theorem 556 (the derivative on the chart, linear in β_W)**.** *In the setting of Lemma 547, for every c_k,*

$$\left|\log \frac{Z_k^W(c_k;e)}{Z_k^W(c_k;e')}\right| \le L_\natural\, d(e,e'), \qquad L_\natural = \min(L_*, L_{\rm lin}),$$
$$L_{\rm lin} = 7192 + 443620\,\beta_W + \mathfrak{b}_{\rm tie}\big(79.69 + 27216\,\beta_W\big).$$

Proof. For $\delta \le 0.08$ the proof of Lemma 547 gives

$$\int |f - f'|\,d\xi \le 1.828\big(e^{70.06\beta_W\delta} - 1\big)Z + 1.828\int f|\Delta|\,d\xi.$$

Bound the last integral by Proposition 555 in place of $\sup f$, so that $|Z' - Z| \le \varepsilon(\delta)Z$ with $\varepsilon(\delta) = 1.828(e^{70.06\beta_W\delta} - 1) + 1.828\,C(\beta_W)\delta$. Subdividing the geodesic as there, $|\log Z(e) - \log Z(e')| \le (128.07\,\beta_W + 1.828\,C(\beta_W))\,d(e,e')$, and $1.828 \cdot 3934 < 7192$, $128.07 + 1.828 \cdot 242610 < 443620$, $1.828 \cdot 43.59 < 79.69$, $1.828 \cdot 14888 < 27216$. In case (b) the inversion $\mathfrak{i}$ of the proof of Lemma 547 maps the configuration to one of case (a); it preserves Haar measure, D and $\int f\,d\xi$, and maps W to $W \circ \mathfrak{i}$. The proof of Proposition 555 uses W only through the bound $e^{3\sqrt{2}\beta_W \sum_j d(\ell_j,\ell'_j)}$ on its ratio at two values of the eight links, which $W \circ \mathfrak{i}$ satisfies, inversion being an isometry. So Proposition 555 holds in case (b) with the same constant. □

The constant $L_{\rm lin}$ is linear in β_W. For $\beta_W \ge 0.08$ its part free of $\mathfrak{b}_{\rm tie}$ is smaller than the part of L_* free of $\mathscr{A}_{\rm tie}$, and in the range $\beta_W \ge 2.3 \cdot 10^{17}$ of Theorem 526 it replaces a factor $e^{72\beta_W}$ by a factor of order β_W.[45] Fig. 42 compares the two constants.

The hypothesis of Lemma 547 bears on the links of the block of the end. It may bear on the other block instead.

[45] At $\beta_W = 1$ and $\mathfrak{b}_{\rm tie} = 1$, $L_{\rm lin} = 478107.69$, while the part $180.4\,e^{72}$ of L_* alone exceeds $3 \cdot 10^{33}$. At $\beta_W = 0.08$ the two parts free of the ties are 42681.6 and more than $5 \cdot 10^4$.

Lemma 557 (the other block)**.** *Lemma 547 and Theorem 556 hold, with the same constants, when the inside ρ-links of the other block of the pair, the w_a in case* (a) *and the u_a in case* (b)*, $a \ne 0$, stay within $\frac{1}{32}$ of I along the move, the links of the block of the end being arbitrary. The version of Z_k^W is then taken in the coordinates $\xi_a = \ell_a g^{-1}$ in case* (a) *and $\xi_a = g^{-1}\ell_a$ in case* (b)*.*

Proof. Case (a), $e = u_b$. By the last part of Lemma 425, with σ_w in place of σ_u, (i)–(iv) hold for every u in the coordinates $\xi_a = \ell_a g^{-1}$, with $\lambda_w \le \lambda$ and $L_w \le L$, and (ii) applies to the move of u_b; the eight private links G, $\xi_a G$ move by $d(G,G')$ each. By the last statement of Theorem 431, Proposition 429 holds in these coordinates, through the inversion $\mathfrak{i}$, which maps the configuration to one whose first block w^{-1} is within $\frac{1}{32}$ of I and in which the moving link u_b^{-1} is a link of the second block. The proof of Proposition 555 transfers through $\mathfrak{i}$ in the same way. The rest of the proofs of Lemma 547 and Theorem 556 is unchanged. Case (b), $e = w_b$ with the u_a near I, is Lemma 425 itself in the coordinates $\xi_a = g^{-1}\ell_a$, whose part (ii) covers a move of w_b, and Proposition 429 as stated. □

So Corollaries 548 and 549 hold with L_* replaced by $L_\natural$, and in Corollary 548 with the constant $L_\natural + 3\sqrt{2}\,\beta_W$, which is at most $7192 + 443625\,\beta_W + \mathfrak{b}_{\rm tie}(79.69 + 27216\,\beta_W)$. Both also hold when, for each end in an average, the inside ρ-links of the other block of its pair lie off A_W^- and within $\frac{1}{32}$ of I at each of their values, in place of the links of the block of the end: the proofs apply Lemma 547 at fixed values of the integrated links, and Lemma 557 needs the other block fixed.

The derivative in an end is uniform in the coarse value on the whole pair fibre, on and off the window chart of Theorem 526, since the relative chart of Lemma 425 covers the fibre of the pair. Integrating the private links of the end pairs before applying Theorem 526 gives the analogue of Corollary 533 for ends in the averages, at the rate α_G.

Theorem 558 (two ends in the averages)**.** *Let $G = \mathrm{SU}(2)$, $\beta_W \ge 2.3 \cdot 10^{17}$, W a set of blocks, and E_a, E_b disjoint sets of links off A_W^-. Let K_a, K_b be the sets of pairs based in W whose average contains a link of E_a, of E_b, and P_a, P_b their private links. Suppose:*

(H1) *$K_a \cap K_b = \emptyset$, and no plaquette contains a link of $E_a \cup P_a$ and a link of $E_b \cup P_b$;*

(H2) *for every pair $(y,\rho) \in K_a \cup K_b$, neither $y - e_\rho$ nor $y + e_\rho$ lies in W;*

(H3) *for every pair of $K_a \cup K_b$, the inside ρ-links of one of its two blocks lie within $\frac{1}{32}$ of I at all the values below.*

Fix the other links off A_W^-, and let $a_0, a_1 \in G^{E_a}$, $b_0, b_1 \in G^{E_b}$, the links being moved one at a time along geodesics, and let $Z_{ij} = \nu_c^-(e^{-\lambda Q^-})$ with a_i on E_a, b_j on E_b, and the factors of the pairs of $K_a \cup K_b$ in the versions of Lemmas 547 and 557. Put $L_e = 3\sqrt{2}\,\beta_W + L_\natural$ for the links that enter an average of a pair based in W and $L_e = 3\sqrt{2}\,\beta_W$ for the others, $L'_e = \beta_W(12.39 + 8.486\,L_e)$, d_e the length of the move of e, and

$$\Lambda_a = \sum_{e \in E_a} L_e d_e, \qquad \Lambda'_a = \sum_{e \in E_a} L'_e d_e,$$

and likewise Λ_b, Λ'_b. Let N_a, N_b be the links of A_W^- outside $P_a \cup P_b$ that share a plaquette with a link of $E_a \cup P_a$, of $E_b \cup P_b$. Suppose that at all the intermediate values of the ends the fibre satisfies the hypotheses of Theorem 526, as in Proposition 532(ii)*, with fixed κ and s, and let $\bar p$ bound its mass off the chart. Then*

$$\Big|\log\frac{Z_{11}Z_{00}}{Z_{10}Z_{01}}\Big| \le \frac{4.8\cdot 10^4}{\beta_W}\,\Lambda'_a\Lambda'_b\big(|N_a||N_b|\big)^{1/2}e^{-\alpha_G d(N_a,N_b)} + 6\bar p\,\Lambda_a\Lambda_b.$$

Proof. Factorisation. Let $\mathscr{P}_a$ be the plaquettes of Q^- that contain a link of $E_a \cup P_a$, and likewise $\mathscr{P}_b$; by (H1) they are disjoint. By Lemma 537 only the factors of the pairs of K_a depend on E_a. By (H2) the inside ρ-links of the two blocks of a pair of $K_a \cup K_b$ lie in I_W or off A_W, so they are fixed links of the fibre, and by (H3) each such pair has a relative chart (Lemmas 425 and 557). Let ζ be the links of A_W^- outside $P_a \cup P_b$, and $\xi^a = (\xi^k)_{k\in K_a}$ the chart coordinates of the pairs of K_a. Then

$$Z(a,b) = \int \Xi(d\zeta)\,\mathscr{A}_a(\zeta)\,\mathscr{B}_b(\zeta),$$
$$\mathscr{A}_a(\zeta) = \int A(\xi^a;\zeta,a)\,d\xi^a,$$

where $A = \prod_{k\in K_a} D_k^{-1}\exp(-\beta_W\sum_{p\in\mathscr{P}_a}(1-\frac12\operatorname{Re}\operatorname{Tr}U_p))$, the links of P_a being $\Gamma_k(\xi^k)$, $\mathscr{B}_b$ is defined in the same way, and the measure Ξ, which collects the other pair factors, the Haar measure of the inside links in A_W^- and the other plaquettes, depends neither on a nor on b. Neither Γ_k nor D_k depends on ζ, so $\mathscr{A}_a$ is a smooth function of the links of N_a, and

$$\frac{Z_{11}Z_{00}}{Z_{10}Z_{01}} = \frac{\mu(R_aR_b)}{\mu(R_a)\,\mu(R_b)}, \qquad R_a = \frac{\mathscr{A}_{a_1}}{\mathscr{A}_{a_0}}, \quad R_b = \frac{\mathscr{B}_{b_1}}{\mathscr{B}_{b_0}},$$

where μ is the law of ζ under the constrained law μ_W^- at (a_0,b_0).

One step. Move one end $e \in E_a$ of a pair $k \in K_a$ by $\delta \le 0.08$ along its geodesic, and let $\varepsilon_e(\delta)$ be such that $\int|A_{e'}-A_e|\,d\xi^a \le \varepsilon_e(\delta)\int A_e\,d\xi^a$. The proofs of Lemma 547 and Theorem 556, applied at fixed $\xi^{k'}$, $k'\ne k$, and ζ, with ω a multiple of the point mass at these values, together with the six plaquettes through e, none of which contains a private link of k, give $\varepsilon_e(\delta) \le e^{3\sqrt2\beta_W\delta}(1+\varepsilon(\delta))-1$, with $\varepsilon(\delta)$ of those proofs, so $\varepsilon_e(\delta) \le L_e\delta + o(\delta)$; for an end that enters no average, D and Γ do not move and $\varepsilon_e(\delta) \le e^{3\sqrt2\beta_W\delta}-1$. In particular $|\log\mathscr{A}_{e'} - \log\mathscr{A}_e| \le L_e\delta + o(\delta)$. Differentiating under the integral, for a link z of N_a,

$$\nabla_z\log\mathscr{A}_e = -\beta_W E^e\big[h_e\big], \qquad h_e = \nabla_z\sum_{p\in\mathscr{P}_a}\big(1-\tfrac12\operatorname{Re}\operatorname{Tr}U_p\big),$$

with E^e the probability with density $A_e/\int A_e$ in ξ^a. The link z lies in at most six plaquettes. For $U_p = MzN$ the left-invariant gradient of $1-\frac12\operatorname{Re}\operatorname{Tr}U_p$ in z is $\frac12\Pi(NMz)$ up to sign, of norm at most $\frac{1}{\sqrt2}$, and it changes by at most $\frac12 d(M,M') + \frac12 d(N,N')$ when the other links move; so $\|h_e\| \le 3\sqrt2$. In the step only e and the eight links of k move, these by at most $L\delta$, $L < 2.064$; a plaquette contains at most two ρ-links, and none contains e and a private link of k, so each plaquette through z has at most two moving links, which move by at most $L\delta$ each, and $\sup\|h_{e'} - h_e\| \le 6L\delta$. The two densities differ in L^1 by at most $2\varepsilon_e(\delta)$. Hence

$$\big\|\nabla_z\log\mathscr{A}_{e'} - \nabla_z\log\mathscr{A}_e\big\| \le \beta_W\big(6L\delta + 3\sqrt2\cdot 2\varepsilon_e(\delta)\big).$$

Summing over a subdivision of the moves and letting the step tend to 0, $|\log R_a| \le \Lambda_a$ and $\|\nabla_z\log R_a\| \le \Lambda'_a$ for every z, by $6L < 12.39$ and $6\sqrt2 < 8.486$; the same holds for every part of the moves, with the corresponding parts of Λ_a and Λ'_a, and for R_b.

Cells. Parametrise the moves of E_a by $\sigma \in [0,1]$ and of E_b by $\tau\in[0,1]$, and cut each interval into n cells on which every end moves by at most $1/n$ of its total move. For the cell (i,j) let $R_a^{(i)}$, $R_b^{(j)}$ be the ratios above between the ends of the cell, $\Lambda_a^{(i)}$, $\Lambda_a^{\prime(i)}$, $\Lambda_b^{(j)}$, $\Lambda_b^{\prime(j)}$ the parts of the sums, and μ_{ij} the law of ζ at the corner (σ_i,τ_j). The logarithm of the cross-ratio is the sum over the cells of $\log(1+x_{ij})$ with $x_{ij} = \operatorname{Cov}_{\mu_{ij}}(R_a^{(i)},R_b^{(j)})/(\mu_{ij}(R_a^{(i)})\mu_{ij}(R_b^{(j)}))$. The functions $R_a^{(i)}$ depend on the links of N_a only, are bounded by $e^{\Lambda_a^{(i)}}$, have gradient at most $e^{\Lambda_a^{(i)}}\Lambda_a^{\prime(i)}$ in each link, satisfy $\|R_a^{(i)}-1\|_\infty \le e^{\Lambda_a^{(i)}}-1$ and $\mu_{ij}(R_a^{(i)}) \ge e^{-\Lambda_a^{(i)}}$, and likewise for $R_b^{(j)}$. A covariance does not change when constants are subtracted, so Theorem 526, through Proposition 532(ii), gives

$$|x_{ij}| \le e^{2\Lambda_a^{(i)}+2\Lambda_b^{(j)}}\Big(\frac{4.8\cdot10^4}{\beta_W}\Lambda_a^{\prime(i)}\Lambda_b^{\prime(j)} \times\big(|N_a||N_b|\big)^{1/2}e^{-\alpha_G d(N_a,N_b)} + 6\bar p\,\Lambda_a^{(i)}\Lambda_b^{(j)}\Big),$$

by $e^x-1 \le xe^x$. With $|\log(1+x)| \le |x|/(1-|x|)$, the sums of $\Lambda_a^{(i)}\Lambda_b^{(j)}$ and of $\Lambda_a^{\prime(i)}\Lambda_b^{\prime(j)}$ over the cells being $\Lambda_a\Lambda_b$ and $\Lambda'_a\Lambda'_b$, and $\max_{i,j}(\Lambda_a^{(i)}+\Lambda_b^{(j)}+|x_{ij}|) \to 0$ as $n\to\infty$, the claim follows. □

The rate α_G does not depend on β_W, and the prefactor is a polynomial in β_W and $\mathfrak{b}_{\text{tie}}$, of degree three in β_W. As in Corollary 533, the second term needs the mass off the chart small, against $\Lambda_a\Lambda_b$, which is of order β_W^2 times the moves; by Theorem 529 this fails for long components at $\beta_W \le 10^{24}$.

Remark 559. What remains of the ends after Theorems 556 and 558.

(a) *Both blocks far from I.* Lemma 557 needs the inside ρ-links of one of the two blocks of the pair within $\frac{1}{32}$ of I. When both blocks have links far from I, the fibre of the pair is a graph over ξ in neither coordinates on an open set of configurations (Proposition 434), the proof of Lemma 428 uses $\|\operatorname{Ad}(u_b)-1\| \le \frac{\sqrt2}{32}$, and the derivative uniform in c_k is not proved. No configuration with an unbounded derivative is known either.

(b) *The crossing number.* $\mathfrak{b}_{\text{tie}}$ is finite by o-minimality and is not computed, so $L_\natural$ and the bound $1.5756\,\mathfrak{b}_{\text{tie}}$ on $\mathscr{A}_{\text{tie}}$ are explicit up to this one integer.

(c) *The rate off the chart.* Theorem 558 decays at the rate α_G up to the term $6\bar{p}\Lambda_a\Lambda_b$, and needs the blocks next to each end pair outside W (H2), so that the links that enter the averages besides the ends are fixed; a pair whose second block lies in W has random inside links, and the ratio R_a then depends on them through the chart and the Jacobian. A decay between two ends at a rate independent of β_W off the chart, or without (H2), is not proved; along a tube the rate of Theorem 541 remains the only one without a chart.

The last part of the section concerns the coarse fields with opposite fluxes at adjacent coarse plaquettes, for an arbitrary field. Let P and P' be the coarse plaquettes at y and $y+e_\mu$ in a plane $\{\mu,\nu\}$ and $W' = \{y+ie_\mu+je_\nu : 0\le i\le 2,\ 0\le j\le 1\}$.

Proposition 560 (a coarse dipole)**.** *Let $G=\mathrm{SU}(2)$.*

(i) *Let U be a field with $\delta(U_p,I)\le\kappa\le 10^{-8}$ for every plaquette with a vertex in a block of W'^{+}, and $c=V^\sharp(U)$. Then*

$$\left\|\log c_P - \mathrm{Ad}(c_{y,\mu})\log c_{P'}\right\| \le 6\kappa + 1.7\cdot 10^4\kappa^2.$$

(ii) *If moreover $c_{y,\mu}c_{P'}c_{y,\mu}^{-1} = c_P^{-1}$, then $\delta(c_P,I)\le 3\kappa + 8500\kappa^2$. Hence, if a coarse field has this property and $\delta(c_P,I) > 1.8001\cdot 10^{-12}$, no field of the fibre over any $W\supset W'$, for any exterior, satisfies the hypotheses of Theorem 526, and for $\delta(c_P,I) > 1.8133\cdot 10^{-12}$ none satisfies those of Theorem 363*(iv).

(iii) *Let $L\ge 6$, $\mu<\nu$, $0<\theta\le 1.7\cdot 10^{-12}$, let $c^{\theta,d}$ be the coarse field with $c^{\theta,d}_{y,\nu} = e^{\theta T_0}$ for $y_\mu=1$, $y_\nu=0$ and every other coarse link I, and $W^d = \{y : 0\le y_\mu\le 2,\ 0\le y_\nu\le 1\}$. Its coarse plaquettes other than I are $e^{\theta T_0}$ at $y_\mu=0$ and $e^{-\theta T_0}$ at $y_\mu=1$, in the plane $\{\mu,\nu\}$ with $y_\nu=0$, so that the hypothesis of* (ii) *holds at $y=0$ with $t := \delta(c_P,I) = 2\sqrt{2}\sin\frac{\theta}{2}$. The field $U^{\theta,d}$ with the angles of Table V on the links listed there, at every value of the other two coordinates, and equal to $R(c^{\theta,d})$ elsewhere, lies in the fibre over W^d of $R(c^{\theta,d})$, and every plaquette has $\delta(U^{\theta,d}_p,I)\le\kappa_d := 2\sqrt{2}\sin\frac{\theta}{6}$.*

In particular, for $t\le 1.79\cdot 10^{-12}$ the fibre over W^d of $R(c^{\theta,d})$ contains a field at which it satisfies the hypotheses of Theorem 526 with $s = 2.4\cdot 10^{-15}$, and for $t > 1.8001\cdot 10^{-12}$ no fibre over a set containing W^d does, whatever the exterior.

Proof. (i) We use the facts (a)–(c) and the notation of the proof of Lemma 524, and one more fact.

(d) Let $X,A\in\mathfrak{g}$ have norms at most 1, and suppose that $\varepsilon = \delta(e^X,e^A)+\|X\|\|A\|\le 1$. Then $\|X-A\|\le\varepsilon(1+\varepsilon^2)$. Indeed $\delta(e^{-A}e^X,e^{X-A})\le\|A\|\|X\|$ by (a), so $\delta(e^{X-A},I)\le\varepsilon$, and $\|X-A\| < \sqrt{2}\pi$, so (c) and (b) give the claim.

The hypothesis of Lemma 409 holds for P and for P', whose corner blocks and their neighbours lie in W'^{+}. Let $\ell_1,\dots,\ell_4$ be the path links of the sides of P, ℓ_5 and ℓ_6 those of $(y+e_\mu,\mu)$ and $(y+2e_\mu,\nu)$, and define on the blocks $y+ie_\mu+je_\nu$ of W', indexed by (i,j),

$$t_{00}=I,\ t_{10}=\ell_1,\ t_{01}=\ell_4,\ t_{11}=\ell_1\ell_2,\ t_{20}=\ell_1\ell_5,\ t_{21}=\ell_1\ell_5\ell_6.$$

TABLE V. The field $U^{\theta,d}$ of Proposition 560(iii), in units of θ, at the sites with coordinates x_μ and x_ν: the angles of the links in the direction μ for $0\le x_\nu\le 3$, of the links in the direction ν at $x_\nu=1$, and the real sums of the plaquettes in the plane $\{\mu,\nu\}$ with corner x. Every other link has the angle of $R(c^{\theta,d})$: 1 for the links (x,ν) with $x_\nu=1$ and $x_\mu\in\{2,3\}$, and 0 on every other link.

		x_μ						
	x_ν	-1	0	1	2	3	4	5
links μ	3	0	$-\frac{1}{3}$	$-\frac{1}{12}$	0	$\frac{1}{12}$	$\frac{1}{3}$	$-\frac{1}{12}$
	2	0	0	$\frac{1}{4}$	0	$-\frac{1}{4}$	0	$-\frac{1}{12}$
	1	0	$-\frac{1}{3}$	$-\frac{1}{12}$	0	$\frac{1}{12}$	$\frac{1}{3}$	$-\frac{1}{12}$
	0	0	0	$\frac{1}{4}$	0	$-\frac{1}{4}$	0	$-\frac{1}{12}$
links ν	1	0	$-\frac{1}{3}$	$\frac{1}{3}$	1	1	$\frac{1}{3}$	$-\frac{1}{3}$
plaquettes	3	0	$-\frac{1}{3}$	$-\frac{1}{12}$	0	$\frac{1}{12}$	$\frac{1}{3}$	$-\frac{1}{12}$
	2	0	$\frac{1}{3}$	$\frac{1}{3}$	0	$-\frac{1}{3}$	$-\frac{1}{3}$	0
	1	$-\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0	$-\frac{1}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$
	0	0	$\frac{1}{3}$	$\frac{1}{3}$	0	$-\frac{1}{3}$	$-\frac{1}{3}$	0
	-1	0	0	$-\frac{1}{4}$	0	$\frac{1}{4}$	0	$\frac{1}{12}$

Step 1. In the proof of Lemma 524, $\delta(c_P,e^{\mathsf{L}(\bar{Z})+Y})\le 3842\kappa^2$ and $\delta(H_x,e^{\mathsf{L}(Z^x)+Y})\le 3842\kappa^2$, the exponents have norm at most 4.001κ, and so have $\log c_P$ and $\log H_x$, by (b) and $\delta(c_P,I)\le 4\kappa+7700\kappa^2$, $\delta(H_x,I)\le 4\kappa$. By (d), with $3842+4.001^2 < 3858.01$, each of $\log c_P$ and $\log H_x$ is within $3858.1\kappa^2$ of its exponent; as $\mathsf{L}(\bar{Z})+Y$ is the mean of the $\mathsf{L}(Z^x)+Y$,

$$\left\|\log c_P - \frac{1}{16}\sum_{x\in B(y)}\log H_x\right\| \le 7717\kappa^2.$$

Step 2. Let p_1,p_2,p_3,p_4 be the plaquettes of S_x at x, $x+e_\mu$, $x+e_\nu$, $x+e_\mu+e_\nu$. Writing out the links,

$$H_x = \left(g_2U_{p_2}g_2^{-1}\right)\left(g_4U_{p_4}g_4^{-1}\right)U_{p_1}\left(g_3U_{p_3}g_3^{-1}\right),$$

with $g_1=I$, $g_2=U_{x,\mu}$, $g_3=U_{x,\nu}$, $g_4=U_{x,\mu}U_{x+e_\mu,\nu}$: in the product the four inner links of the square cancel in pairs. With $A_r=\mathrm{Ad}(g_r)\log U_{p_r}$, $\|A_r\|\le 1.0001\kappa$ by (b), (a) gives $\delta(H_x,e^{A_1+\cdots+A_4})\le 6.002\kappa^2$, and (d) gives $\|\log H_x-\sum_r A_r\|\le 22.01\kappa^2$.

Step 3. Let $z(p)$ be the block of the corner of a plaquette p. By Lemma 360(i) applied to W', an inside link is within 3κ of I and a link that leaves a block $y'\in W'$ in the direction σ within 21κ of the path link of (y',σ); the links $U_{x,\mu}$, $U_{x,\nu}$, $U_{x+e_\mu,\nu}$ are of one of these kinds, according to the offsets of x, and comparing with the definition of the t_{ij} gives $\delta(g_r,t_{z(p_r)})\le 42\kappa$ for $x\in B(y)$. As $\|\mathrm{Ad}(g)X-\mathrm{Ad}(g')X\|\le 2\delta(g,g')\|X\|$, replacing each g_r by $t_{z(p_r)}$ costs at most $4\cdot 2\cdot 42\cdot 1.0001\kappa^2 < 336.04\kappa^2$. With Steps 1 and 2,

$$\Lambda_P = \frac{1}{16}\sum_{x\in B(y)}\sum_{p\in S_x}\mathrm{Ad}(t_{z(p)})\log U_p,$$

$$\|\log c_P - \Lambda_P\| \le 8076\kappa^2.$$

For P' the same steps, with the base point $y+e_\mu$ and the transports $t'_{ij}=\ell_1^{-1}t_{ij}$, give $\|\log c_{P'}-\Lambda_{P'}\|\le 8076\kappa^2$ with $\Lambda_{P'}$

formed with $B(y+e_\mu)$ and the t', so that $\mathrm{Ad}(\ell_1)\Lambda_{P'}$ is formed with the same transports $t_{z(p)}$ as Λ_P.

Step 4. As in the proof of Proposition 536(iv), a plaquette at the corner z enters Λ_P with the multiplicity $n(i)n(j)$ and $\mathrm{Ad}(\ell_1)\Lambda_{P'}$ with $n(i-2)n(j)$, where $z_\mu = 2y_\mu + i$, $z_\nu = 2y_\nu + j$, so

$$\Lambda_P - \mathrm{Ad}(\ell_1)\Lambda_{P'} = \frac{1}{16}\sum_z w_z \,\mathrm{Ad}(t_{z(p_z)}) \log U_{p_z},$$
$$\sum_z |w_z| = 96,$$

and its norm is at most $6\kappa(1+\kappa^2)$, as $\|\log U_p\| \le \kappa(1+\kappa^2)$ by (b). Finally $c_{y,\mu} = \ell_1 e^{\bar{Z}_1}$ with $\|\bar{Z}_1\| \le 24.001\kappa$, so $\delta(c_{y,\mu}, \ell_1) \le 24.001\kappa$ by (c), and $\|(\mathrm{Ad}(c_{y,\mu}) - \mathrm{Ad}(\ell_1))\log c_{P'}\| \le 48.002\kappa \cdot 4.001\kappa < 192.1\kappa^2$. Altogether the left side of (i) is at most $6\kappa + 6\kappa^3 + (2\cdot 8076 + 192.1)\kappa^2 \le 6\kappa + 1.7\cdot 10^4 \kappa^2$.

(ii) The principal logarithm commutes with conjugation, so $\mathrm{Ad}(c_{y,\mu})\log c_{P'} = \log c_P^{-1} = -\log c_P$, both norms being below $\sqrt{2}\pi$. Then (i) gives $2\|\log c_P\| \le 6\kappa + 1.7\cdot 10^4\kappa^2$, and $\delta(c_P, I) \le \|\log c_P\|$ by (c). The pairs that enter (i) are based in W', so (i) applies to every field of a fibre over $W \supset W'$. If such a field satisfied the hypotheses of Theorem 526, its plaquettes on $W^+ \supset W'^+$ would be at most $\kappa < 6.0\cdot 10^{-13}$, and $\delta(c_P, I) \le 1.8\cdot 10^{-12} + 8500\cdot 3.6\cdot 10^{-25} < 1.8001\cdot 10^{-12}$. The hypothesis of Theorem 363(iv) needs plaquettes at most $\kappa < 6.044\cdot 10^{-13}$, by the proof of Proposition 525(ii), and then $\delta(c_P, I) < 1.8132\cdot 10^{-12} + 3.2\cdot 10^{-21} < 1.8133\cdot 10^{-12}$.

(iii) The coarse plaquettes are computed as in Proposition 536(i), with $a(1) = \theta$ and $a = 0$ elsewhere; $c_{0,\mu} = I$ and $c_{P'} = c_P^{-1}$. We write φ for the angles of $U^{\theta,d}$ and φ^R for those of $R(c^{\theta,d})$, which by Proposition 525(i) are θ on the links (x,ν) with $x_\nu = 1$ and x in a block with $y_\mu = 1$, $y_\nu = 0$, and 0 elsewhere.

Links. The links where $\varphi \ne \varphi^R$ are the links (x,ν) with $x_\nu = 1$, $x_\mu \in \{0,1,4,5\}$, which have $o_\nu = 1$ and are private links of pairs based in W^d; the links (x,μ) with $x_\mu \in \{1,3,5\}$, $0 \le x_\nu \le 3$, private links of such pairs; and the links (x,μ) with $x_\mu \in \{0,4\}$ and $x_\nu \in \{1,3\}$, which have $o_\mu = 0$ and $o_\nu = 1$ with $\nu > \mu$ and are active links inside blocks of W^d. So $U^{\theta,d}$ agrees with $R(c^{\theta,d})$ off A_{W^d} and vanishes on the tree links.

Averages. Every entry differs from that of $R(c^{\theta,d})$ by at most $\frac{\theta}{3}$, so for every pair (y,σ) the angles of its paths differ by at most $\frac{4\theta}{3}$, $\delta(P_x, P_{2y}) < 4\theta < \frac{1}{2}$, and as in the proof of Proposition 536, $V^\sharp_{y,\sigma} = c^{\theta,d}_{y,\sigma}$ when $\sum_x D_x = 0$, D_x the deviation of the angle of P_x from that of the coarse link. For $\sigma \notin \{\mu,\nu\}$ every D_x vanishes. For $\sigma = \nu$ only the pairs with $y_\nu = 0$ meet a changed link, and $\sum_x D_x = 8(d(2y_\mu) + d(2y_\mu + 1))$, where the deviations $d(m)$ of the links (x,ν), $x_\nu = 1$, $x_\mu = m$, are $-\frac{1}{3}, \frac{1}{3}, 0, 0, \frac{1}{3}, -\frac{1}{3}$ for $m = 0,\dots,5$ and 0 otherwise. For $\sigma = \mu$ and $y_\nu \in \{0,1\}$, $\sum_x D_x = 4(X(2y_\mu) + 2X(2y_\mu + 1) + X(2y_\mu + 2))$, where $X(m)$, the sum of the deviations of the two links (x,μ) with $x_\mu = m$ and $x_\nu \in \{2y_\nu, 2y_\nu + 1\}$, is $-\frac{1}{3}, \frac{1}{6}, 0, -\frac{1}{6}, \frac{1}{3}, -\frac{1}{6}$ for $m = 0,\dots,5$ and 0 otherwise; for other y_ν the pair meets no changed link. The sums vanish for $y_\mu \in \{0,1,2\}$, that is for every pair based in W^d.

Plaquettes. As in the proof of Proposition 536, a plaquette in a plane other than $\{\mu,\nu\}$ has real sum 0. In the plane $\{\mu,\nu\}$ the table lists every plaquette that contains a changed link, and every plaquette with real sum $\pm\theta$ in $R(c^{\theta,d})$, those at the corners $x_\mu \in \{1,3\}$, $x_\nu = 1$; all others are I. The largest real sum is $\frac{\theta}{3}$, which gives κ_d.

For the last statement, $\sin x \ge x(1 - x^2/6)$ gives $\theta < 1.7\cdot 10^{-12}$ for $t \le 1.79\cdot 10^{-12}$, and $\sin 3x = \sin x\,(3 - 4\sin^2 x)$ gives $\kappa_d = t/(3 - 4\sin^2\frac{\theta}{6}) \le \frac{t}{3}(1+\theta^2) < 5.967\cdot 10^{-13}$, so $s = 2.4\cdot 10^{-15}$ satisfies $s \le 1/(6000c_3)$, $\kappa_d + 6s \le 10^{-8}$ and $\kappa_d + s \le 6.0\cdot 10^{-13}$, as in the proof of Proposition 536. The other half is (ii). □

So the threshold of the coarse plaquette at which a background exists is not a function of the size of the coarse plaquette alone. Between $1.8001\cdot 10^{-12}$ and $2.39\cdot 10^{-12}$ the coarse fields c^θ of Proposition 536 have a background with exterior $R(c)$, while the coarse fields with opposite fluxes at adjacent coarse plaquettes have none for any exterior; part (ii) removes the restriction to fields with values in a torus in Proposition 536(iv). The factor 3 of the dipole, against 4 for a single coarse plaquette, comes from the weights $\sum_z |w_z| = 96$ of the difference.

Remark 561. What these results leave open.

(a) *Ends.* Lemma 540(iii) and Corollary 542 bound the weight of the interior as a function of ends that enter averages, after integration over the non-path links, uniformly in the coarse links and in the moves of the ends. For the derivative in an end, Lemma 544 bounds the motion of the boundary $\{m = 1\}$ by $3.4\cdot 10^{-8}$ per unit move at every configuration; Proposition 545 and Corollary 546 bound the derivative after the coarse link is averaged against a positive Lipschitz weight, with no condition on the other links; and Lemma 547 and Corollary 548 bound it uniformly in c_k when the links of the block of the end lie within $\frac{1}{32}$ of I, in particular when the block across the end is good. Theorem 556 replaces the constant $128.07\beta_W + (180.4 + 50.6\mathscr{A}_{\mathrm{tie}})e^{72\beta_W}$ by one linear in β_W, Corollary 554 gives $\mathscr{A}_{\mathrm{tie}} < 1.5756\,\mathfrak{b}_{\mathrm{tie}}$ with the crossing number of Definition 550, and Lemma 557 allows the condition on the other block of the pair in place of the block of the end. Corollary 549 gives mixed differences in two ends that decay along a tube at the rate of Theorem 541, and Theorem 558 gives, on the chart of Theorem 526, mixed differences in two ends in the averages that decay at the rate α_G, independent of β_W, up to a term proportional to the mass off the chart. Three things are not proved (Remark 559): the derivative uniform in c_k when both blocks of the pair have links far from I, where the relative chart can fold (Proposition 434); the value of $\mathfrak{b}_{\mathrm{tie}}$; and a decay between two ends at a rate independent of β_W off the chart, or for pairs whose second block lies in W. The rate of Theorem 541 is the oscillation of the Wilson weight across one interface (Remark 543).

(b) *Rate.* For $\beta_W \ge 2.3\cdot 10^{17}$ the decay across a bad component beyond the chart of Theorem 526 is not proved;

Remark 543 locates the loss in the oscillation of the Wilson weight across one interface.

(c) *Coarse fields.* With exterior $R(c)$, a background exists up to $1.79\cdot10^{-12}$ for the dipole $c^{\theta,d}$ and up to $2.39\cdot10^{-12}$ for c^{θ}, and none exists above $1.8001\cdot10^{-12}$ for any coarse field with opposite fluxes at adjacent coarse plaquettes. For a general coarse field between $6.0\cdot10^{-13}$ and $2.418\cdot10^{-12}$ the existence is not decided.

P. The one-step error in gauge-invariant form

Proposition 461 requires the one-step error to be a relative bound by a gauge-invariant form. This subsection constructs such a bound at the trivial background on a box of blocks, from an absolute bound on the gauge complement taken as a hypothesis. The error of the covariant Hessian of a gauge-invariant induced action vanishes on the coarse gauge directions and is at most a multiple ε' of the coarse Wilson form. On a box of side n, ε' is of order n^2 times the absolute error, and this order cannot be improved.

Setting. On the coarse lattice of side L let $W=\{y: a_\nu\le y_\nu\le a_\nu+n-1\}$ be a box of n^4 blocks, $n+2\le L$, $K=\{(y,\mu): y\in W\}$ and $W^\circ=\{y\in W: y-e_\mu\in W \text{ for every } \mu\}$. As in Theorem 367, the links off A_W are fixed and the averages of the pairs that enter W from outside are integrated. For $w\in\mathfrak{g}^K$ put $|w|^2=\sum_k\|w_k\|^2$, let $Q_1(w)$ be the coarse plaquette form of w extended by zero off K, and put

$$S_W(w)=\inf\big\{Q(X): X\in\mathfrak{g}^{A_W},\ V_k(X)=w_k\ (k\in K)\big\}.$$

At $U=I$, $2\lambda Q_1$ is the Hessian at I of the coarse Wilson action λQ_1 with the links off K fixed. $2\lambda S_W$ is the Hessian of the Gaussian block-spin step, the minimum of the Hessian $2\lambda Q$ of the fine action at given averages. The *coarse gauge group* $\Gamma_W=G^{W^\circ}$ acts on G^K by $(h\cdot V)_{y,\mu}=h_yV_{y,\mu}h^{-1}_{y+e_\mu}$, with $h_y=I$ for $y\notin W^\circ$. This is the action on the cut-off averages of the fine gauge transformations that are constant on blocks and equal to I off the blocks of W°. These transformations preserve the tree gauge by Lemma 329(ii), and they fix the links off A_W: a link with base in a block $y'\notin W$ ends in y' or $y'+e_\mu$, and neither lies in W°. They also preserve the Wilson action and Haar measure, and the cut-off average is covariant (Theorem 353). For $V_0\in G^K$ and $\eta\in\mathfrak{g}^{W^\circ}$, extended by zero, put

$$(d_{V_0}\eta)_{y,\mu}=\mathrm{Ad}(V^{-1}_{0,y,\mu})\eta_y-\eta_{y+e_\mu},$$

let $\mathscr{O}_{V_0}=d_{V_0}(\mathfrak{g}^{W^\circ})$ be the gauge directions at V_0 and Π_{V_0} the orthogonal projection of $\mathfrak{g}^K$ onto $\mathscr{O}^\perp_{V_0}$. At $V_0=I$ write d, $\mathscr{O}$ and Π.

Lemma 562 (the window off the gauge directions)**.** *Let W be the box above.*

(i) *The map d is injective on $\mathfrak{g}^{W^\circ}$, and $\mathscr{O}=\ker Q_1=\ker S_W$.*

(ii) *For every $w\in\mathfrak{g}^K$, $|\Pi w|^2\le C_nQ_1(w)$ with*

$$C_n=1+\frac{1}{4\sin^2(\pi/2n)}\le 1+\frac{n^2}{4}.$$

(iii) *$c_*Q_1\le S_W\le 4Q_1$ on $\mathfrak{g}^K$, with $c_*=(1+\sqrt{21/8})^{-2}>0.1456$.*

(iv) *There is $w\in\mathscr{O}^\perp\setminus\{0\}$ with $|w|^2=C_n^-Q_1(w)$, where $C_n^-=(12\sin^2(\pi/(2n+2)))^{-1}\ge(n+1)^2/(3\pi^2)$.*

(v) *If W is the whole coarse torus, the fields that are constant in each direction lie in $\mathscr{O}^\perp\cap\ker Q_1\cap\ker S_W$, so no bound of the form* (ii) *holds.*

Proof. Let $\Lambda=\prod_\nu[a_\nu,a_\nu+n]$, with boundary $\partial\Lambda$. The interior sites of Λ form W°. Call $(y,\mu)\in K$ *inner* if $y_\nu>a_\nu$ for every $\nu\ne\mu$, and a *face link* otherwise. The inner links are the links of Λ not contained in $\partial\Lambda$; call a plaquette of Λ not contained in $\partial\Lambda$ inner. The links of $\partial\Lambda$ that are not face links have base outside W, so they are fixed, and w vanishes on them.

(ii) The cochain complex of Λ relative to $\partial\Lambda$, with values in $\mathfrak{g}$, has the interior sites, inner links and inner plaquettes as cells of degrees $0,1,2$. On these, $d_0=d$, and d_1 is the curvature on the inner plaquettes of a field on the inner links extended by zero. This complex is the fourfold tensor product of the complex of $[0,n]$ relative to its endpoints. In that complex $C^0=\mathbb{R}^{n-1}$, $C^1=\mathbb{R}^n$ and d is the difference operator. With the product inner products, the Laplacian $\Delta=dd^*+d^*d$ of the product is the sum of the Laplacians of the factors. On C^0, d^*d is the Dirichlet Laplacian, with eigenvalues $4\sin^2(\pi j/2n)$ for $1\le j\le n-1$. On C^1, dd^* has the same nonzero eigenvalues and one zero eigenvalue. A 1-cochain of Λ is a sum of products with exactly one factor of degree one, so $\Delta_1\ge\gamma_n=12\sin^2(\pi/2n)$. On $\ker d_0^*=\mathscr{O}^\perp$ this gives $|d_1u|^2\ge\gamma_n|u|^2$.

Write $w=u+f$, with u on the inner links and f on the face links. Since $\mathscr{O}$ lies on the inner links, $\Pi w=\Pi u+f$ and $|\Pi w|^2=|\Pi u|^2+|f|^2$. Let $f_k\ne0$ at a face link $k=(y,\mu)$, and let $\nu\ne\mu$ be the least direction with $y_\nu=a_\nu$. The plaquette at $y-e_\nu$ in the plane $\{\mu,\nu\}$ has k as its only link in K, since its other three links have base outside W. So its curvature is $\pm f_k$, and distinct face links give distinct such plaquettes, none of them inner. Consider next an inner plaquette P in the plane $\{\mu,\nu\}$ at z. It has $z_\rho>a_\rho$ for $\rho\notin\{\mu,\nu\}$, so its face links are among (z,μ), if $z_\nu=a_\nu$, and (z,ν), if $z_\mu=a_\mu$. Thus P contains at most two face links, and a face link lies in at most one inner plaquette. Hence $(d_1u)_P^2\le3\big((dw)_P^2+\sum_{k\in P}f_k^2\big)$, where the sum runs over the face links of P. Since $d_1\Pi u=d_1u$,

$$\begin{aligned}|\Pi w|^2&\le\gamma_n^{-1}|d_1u|^2+|f|^2\\&\le\frac{3}{\gamma_n}\Big(\sum_{P\text{ inner}}(dw)_P^2+|f|^2\Big)+|f|^2\le C_nQ_1(w),\end{aligned}$$

because the inner plaquettes and the plaquettes of the face links are distinct terms of $Q_1(w)$. Finally $\sin x\ge2x/\pi$ on $[0,\pi/2]$ gives $C_n\le1+n^2/4$.

(iii) Let $X\in\mathfrak{g}^{A_W}$ with $V(X)=w$ on K. Extended by zero, X vanishes on the tree links, and its averages are w on K, zero on the coarse links with both blocks outside W, and $v'_{y-e_\mu,\mu}=S^\mu_y/16$ on a pair entering $y\in W$ from $y-e_\mu\notin W$, by Lemma 329(iii). By Proposition 335(i), $Q_1(w+v')\le Q(X)$.

The sum S_y^μ has at most seven active links, and every link inside a block enters one such sum. So $\sum|v'|^2 \le \frac{7}{256}I \le \frac{7}{32}Q(X)$, by the bound $I \le 8Q_{\rm in}$ in the proof of Theorem 330, which uses only the tree links. A coarse plaquette contains at most two entering pairs, since (z,μ) and $(z+e_\mu,\nu)$ cannot both enter, nor can (z,ν) and $(z+e_\nu,\mu)$. A coarse link lies in six plaquettes, so $Q_1(v') \le 12\sum|v'|^2 \le \frac{21}{8}Q(X)$. By the triangle inequality for $Q_1^{1/2}$, $Q_1(w)^{1/2} \le (1+\sqrt{21/8})Q(X)^{1/2}$, and the infimum over X gives $S_W \ge c_* Q_1$. For the upper bound, the field $\varphi^*(w)$ of Proposition 335(ii), with w extended by zero, lies in $\mathfrak{g}^{A_W}$, has averages w on K and $Q(\varphi^*(w)) = 4Q_1(w)$.

(i) If $d\eta = 0$, follow the links of direction 0 from $y \in W^\circ$ to the first site outside W°, where η vanishes; so $\eta_y = 0$. Extended by zero, $d\eta$ is exact on the whole lattice, so $Q_1(d\eta) = 0$, and $\ker Q_1 \subset \mathscr{O}$ by (ii). The fine field $\varphi = d(\eta \circ b)$, with $b(x)$ the block of x, vanishes on the tree links and has $Q(\varphi) = 0$. It has $V(\varphi) = d\eta$ by Lemma 329(ii). It lies in $\mathfrak{g}^{A_W}$, since a link on which it is nonzero meets a block of W° and so has base in W. Hence $S_W(d\eta) = 0$, and $\ker S_W \subset \ker Q_1$ by (iii).

(iv) Let f be the product of sines on $\{a_\nu, \dots, a_\nu + n - 1\}^3$, $\nu = 1,2,3$, whose eigenvalue for the Dirichlet Laplacian of the cube is $12\sin^2(\pi/(2n+2))$. Let $w_{y,0} = f(y_1,y_2,y_3)Z$ for a unit $Z \in \mathfrak{g}$, and $w = 0$ in the other directions. The sum over y_0 telescopes, so $\langle w, d\eta\rangle = 0$ for every η. The curvature of a plaquette in the plane $\{0,\nu\}$ is a difference of f along e_ν, the plaquettes in the other planes vanish, and for each of the n values of y_0 the sum is the Dirichlet form of f. Hence $Q_1(w)/|w|^2 = 12\sin^2(\pi/(2n+2)) \le 3\pi^2/(n+1)^2$.

(v) Then $W^\circ = W$ and every coarse link is in K. A constant field has $dw = 0$, and $\langle w, d\eta\rangle = \sum_y \sum_\mu \langle w_\mu, \eta_y - \eta_{y+e_\mu}\rangle = 0$. The field $\varphi^*(w)$ of Proposition 335(ii) has $Q = 4Q_1(w) = 0$, so $S_W(w) = 0$. □

The gauge directions at a background V_0 near I are close to those at I, and the flat form keeps its gap off them.

Lemma 563 (the window at a curved background)**.** *Let $V_0 \in G^K$ with $\|\mathrm{Ad}(V_{0,k}^{-1}) - 1\| \le \theta$ for every $k \in K$, and put $s_n = 2\sin(\pi/2n)$. If $\theta < s_n$, then every $w \in \mathscr{O}_{V_0}^\perp$ satisfies*

$$|w - \Pi w| \le \frac{\theta}{s_n}|w|, \qquad |w|^2 \le \Big(1 - \frac{\theta}{s_n}\Big)^{-2} C_n Q_1(w).$$

Proof. Write $d_{V_0} = d + R$ with $(R\eta)_{y,\mu} = (\mathrm{Ad}(V_{0,y,\mu}^{-1}) - 1)\eta_y$. Every site of W° is the base of the four links (y,μ), all in K, so $|R\eta|^2 \le 4\theta^2|\eta|^2$ and $\|R^*\| \le 2\theta$. The form $|d\eta|^2$ is the Dirichlet form of the box Λ of the proof of Lemma 562 relative to its boundary, the sum of four one-dimensional Dirichlet Laplacians, so $d^*d \ge 4s_n^2$ and $\|d(d^*d)^{-1}\| = \|(d^*d)^{-1/2}\| \le (2s_n)^{-1}$. For $w \perp \mathscr{O}_{V_0}$ we have $d_{V_0}^* w = 0$, that is $d^*w = -R^*w$, and $w - \Pi w = d(d^*d)^{-1}d^*w = -d(d^*d)^{-1}R^*w$. Hence $|w - \Pi w| \le \theta|w|/s_n$, so $|w| \le (1-\theta/s_n)^{-1}|\Pi w|$, and Lemma 562(ii) gives the second bound. □

For a smooth function f near $V_0 \in G^K$ write $\mathrm{Hess}\, f(0)$ and $\nabla_k f(0)$ for the Hessian and the gradient in the coordinates $\omega \mapsto f(V_0 e^\omega)$, $e^\omega = (e^{\omega_k})_k$.

Lemma 564 (invariant functions and the slice)**.** *Let f be smooth and Γ_W-invariant near $V_0 \in G^K$.*

(i) *For $w \in \mathfrak{g}^K$ and $\eta \in \mathfrak{g}^{W^\circ}$, with $a_k = \mathrm{Ad}(V_{0,k}^{-1})\eta_y$ and $b_k = \eta_{y+e_\mu}$ at $k = (y,\mu)$,*

$$\mathrm{Hess}\, f(0)[w, d_{V_0}\eta] = \frac{1}{2}\sum_k \langle \nabla_k f(0), [w_k, a_k + b_k]\rangle.$$

(ii) *d_{V_0} is injective, and $\Upsilon(s,\eta) = e^\eta \cdot (V_0 e^s)$, on $\mathscr{O}_{V_0}^\perp \times \mathfrak{g}^{W^\circ}$, is a diffeomorphism from a neighbourhood of 0 onto a neighbourhood of V_0, with $f(\Upsilon(s,\eta)) = f(V_0 e^s)$. The* covariant Hessian

$$\mathscr{H} f(w,w') = \mathrm{Hess}(f\circ\Upsilon)(0)\big[D\Upsilon(0)^{-1}w, D\Upsilon(0)^{-1}w'\big]$$

equals $\mathrm{Hess}\, f(0)[\Pi_{V_0}w, \Pi_{V_0}w']$. *It vanishes if w or w' lies in $\mathscr{O}_{V_0}$, and it equals* $\mathrm{Hess}\, f(0)$ *if $\nabla f(0) = 0$.*

Proof. (i) For the action of $e^{t\eta}$,

$$(e^{t\eta}\cdot V_0 e^\omega)_k = V_{0,k} e^{\omega_k} e^{t\,\mathrm{Ad}(e^{-\omega_k})a_k} e^{-tb_k},$$

and $\frac{d}{dt}\log(e^\omega e^{t\xi})|_{t=0} = \Phi(\mathrm{ad}_\omega)\xi$ with $\Phi(z) = z/(1-e^{-z})$. So the orbit through $V_0 e^\omega$ has the tangent

$$\zeta(\omega)_k = \Phi(\mathrm{ad}_{\omega_k})\big(\mathrm{Ad}(e^{-\omega_k})a_k - b_k\big),$$

with $\zeta(0) = d_{V_0}\eta$ and $D\zeta(0)w = -\frac{1}{2}([w_k, a_k + b_k])_k$. Invariance gives $Df(\omega)[\zeta(\omega)] = 0$ near 0. Differentiating along w at 0 gives $\mathrm{Hess}\, f(0)[w,\zeta(0)] = -Df(0)[D\zeta(0)w]$.

(ii) Injectivity follows as in Lemma 562(i), with $\eta_y = \mathrm{Ad}(V_{0,k})\eta_{y+e_\mu}$ along the links k of direction 0. $D\Upsilon(0)(\sigma,\eta) = \sigma + d_{V_0}\eta$ is an isomorphism, and the inverse function theorem applies. Invariance gives $f\circ\Upsilon(s,\eta) = f(V_0 e^s)$, whose Hessian at 0 is $\mathrm{Hess}\, f(0)$ on the s-variables and zero in η. Since $D\Upsilon(0)^{-1}w = (\Pi_{V_0}w, \eta)$ with $d_{V_0}\eta = w - \Pi_{V_0}w$, the formula follows. If $\nabla f(0) = 0$, (i) shows that $\mathrm{Hess}\, f(0)$ vanishes on $\mathfrak{g}^K \times \mathscr{O}_{V_0}$, so $\mathrm{Hess}\, f(0)[w,w'] = \mathrm{Hess}\, f(0)[\Pi_{V_0}w, \Pi_{V_0}w']$. □

At a point that is not critical, $\mathrm{Hess}\, f(0)$ does not vanish on the gauge directions. The Hessian in exponential coordinates of an invariant function, and with it any absolute bound on it, is therefore not gauge-invariant; $\mathscr{H}f$ is the gauge-invariant object. Let $j(\omega) = \det\big((1-e^{-\mathrm{ad}_\omega})/\mathrm{ad}_\omega\big)$ be the density of Haar measure in exponential coordinates. Since ad_ω is skew, $\log j(\omega) = \frac{1}{24}\mathrm{Tr}(\mathrm{ad}_\omega^2) + O(\|\omega\|^4)$. Put

$$\delta_G = \frac{1}{12}\sup_{\|x\|=1}|\mathrm{Tr}(\mathrm{ad}_x^2)| \le \frac{\dim G}{3},$$

which is $\frac{1}{3}$ for SU(2), where $\|\mathrm{ad}_x\|_{HS} = 2\|x\|$.

Proposition 565 (relative one-step bound)**.** *Let $U = I$, so that $V_0 = I$. Let A be smooth near $0 \in \mathfrak{g}^K$ and suppose that, for some $\varepsilon \ge 0$,*

$$\big|\mathrm{Hess}\, A(0)[\sigma,\sigma] - 2\lambda S_W(\sigma)\big| \le \lambda\varepsilon|\sigma|^2 \quad \text{for } \sigma \in \mathscr{O}^\perp. \tag{H}$$

Let $A^{\rm gi}$ be the Γ_W-invariant function with $A^{\rm gi}(\Upsilon(s,\eta)) = A(s) + \sum_k \log j(s_k)$, and let $E = \mathscr{H}A^{\rm gi} - 2\lambda S_W$ as bilinear forms. Then:

(i) $E(w,g)=0$ *for every* $w\in\mathfrak{g}^K$ *and* $g\in\mathscr{O}$*;*

(ii) *for every* $w\in\mathfrak{g}^K$*,*

$$|E(w,w)|\le(\lambda\varepsilon+\delta_G)|\Pi w|^2\le(\lambda\varepsilon+\delta_G)C_nQ_1(w)$$
$$\le\varepsilon'\cdot2\lambda S_W(w),\qquad\varepsilon'=\Big(\varepsilon+\frac{\delta_G}{\lambda}\Big)\frac{C_n}{2c_*},$$

so that $(1-\varepsilon')2\lambda S_W\le\mathscr{H}A^{\mathrm{gi}}\le(1+\varepsilon')2\lambda S_W$*;*

(iii) *the factor* C_n *is necessary up to a constant. The* Γ_W*-invariant* F *with* $F(\Upsilon(s,\eta))=\lambda S_W(s)+\frac12\lambda\varepsilon|s|^2$ *has* $\operatorname{Hess}F(0)=2\lambda S_W+\lambda\varepsilon|\Pi\cdot|^2$ *on* $\mathscr{O}^\perp$ *and* $\mathscr{H}F-2\lambda S_W=\lambda\varepsilon|\Pi\cdot|^2$*. At the field* w *of Lemma 562*(iv) *this is at least* $\frac18\varepsilon C_n^-\cdot2\lambda S_W(w)$*.*

Proof. (i) $\mathscr{H}A^{\mathrm{gi}}(w,g)=0$ by Lemma 564(ii). S_W is positive semidefinite with $g\in\ker S_W$ by Lemma 562(i), so $S_W(w,g)=0$ by Cauchy–Schwarz. (ii) By Lemma 564(ii), $\mathscr{H}A^{\mathrm{gi}}(w,w)$ is the Hessian at 0 of $s\mapsto A(s)+\sum_k\log j(s_k)$ at Πw, that is, $\operatorname{Hess}A(0)[\Pi w,\Pi w]+\frac{1}{12}\sum_k\operatorname{Tr}(\operatorname{ad}^2_{(\Pi w)_k})$. Also $S_W(w)=S_W(\Pi w)$, by (i). Hypothesis (H) at $\sigma=\Pi w$ gives the first inequality. Lemma 562(ii) gives the second, and Lemma 562(iii), $Q_1\le S_W/c_*$, gives the third. (iii) The Hessian of $s\mapsto\lambda S_W(s)+\frac12\lambda\varepsilon|s|^2$ is $2\lambda S_W+\lambda\varepsilon|\cdot|^2$. That w lies in $\mathscr{O}^\perp$ and has $|w|^2=C_n^-Q_1(w)\ge\frac14C_n^-S_W(w)$, by Lemma 562(iii). □

Loop remainders. Hypothesis (H) is an absolute bound, and Proposition 565(iii) shows that it costs the factor C_n. A remainder built from Wilson loops that are filled by plaquettes obeys a relative bound instead, at every flat background, and then C_n falls on the Haar term alone. Let $G\subset\mathrm{U}(N)$, with $\langle X,Y\rangle=-\operatorname{Tr}(XY)$ on $\mathfrak{g}$, and put

$$s(g)=1-\frac1N\operatorname{Re}\operatorname{Tr}g=\frac{1}{2N}\|I-g\|_{HS}^2\qquad(g\in\mathrm{U}(N)),$$

which is invariant under conjugation and inversion. For a coarse path C let V_C be its holonomy, the ordered product of its links. An *ordered filling* of a closed path C is an identity

$$V_C=\prod_{j=1}^{A_C}h_jV_{p_j}^{\varepsilon_j}h_j^{-1},\qquad\varepsilon_j=\pm1,\tag{48}$$

valid for every field, in which each h_j is the holonomy of a path and V_{p_j} the holonomy around a plaquette p_j read from one of its corners. Let m_{Cp} be the number of j with $p_j=p$. This is the lattice form of the non-abelian Stokes theorem of Bralić [29].[46]

Lemma 566 (filled loops)**.** (i) *For* $g_1,\dots,g_A\in\mathrm{U}(N)$*,*

$$s(g_1\cdots g_A)\le A\sum_{j=1}^{A}s(g_j).$$

(ii) *Every closed path of length* ℓ *in* $\mathbb{Z}^4$ *has an ordered filling by at most* $\ell(\ell-1)/2$ *plaquettes, all in the smallest box that contains the path.*

(iii) *If* C *has an ordered filling, then* $s(V_C)\le A_C\sum_p m_{Cp}\,s(V_p)$ *for every field* V*.*

Proof. (i) Since $I-g_1\cdots g_A=\sum_j g_1\cdots g_{j-1}(I-g_j)$ and the Hilbert–Schmidt norm is unitarily invariant,

$$\|I-g_1\cdots g_A\|_{HS}\le\sum_j\|I-g_j\|_{HS}.$$

Square, apply Cauchy–Schwarz and divide by $2N$.

(ii) Write C as a word of unit steps from its base point. Suppose two consecutive steps u, from x to $x+e$, and v, from $x+e$ to $x+e+f$, point in different directions, with $e,f\in\{\pm e_\mu\}$. Let v' be the step from x to $x+f$ and u' the step from $x+f$ to $x+e+f$. The loop $uv(u')^{-1}(v')^{-1}$ is the boundary of a plaquette p read from its corner x, so $V_uV_v=V_p^{\pm1}V_{v'}V_{u'}$. If P is the part of C before u and C' is C with uv replaced by $v'u'$, then

$$V_C=\big(V_PV_p^{\pm1}V_P^{-1}\big)V_{C'}.$$

Each exchange of two adjacent steps of different directions therefore puts one factor of the form (48) in front. Sort the steps by their direction in this way, keeping the order of the steps of one direction. The number of exchanges is the number of pairs of steps out of order, at most $\ell(\ell-1)/2$. The sorted path is a succession of runs along single lines. Since C is closed, each run returns to its starting point, and a closed path on a line has holonomy I: it contains a step followed by its reverse, which cancel. So V_C is the product of the factors. The new corner $x+f$ has each coordinate equal to the coordinate of x or of $x+e+f$, so every path in the process, and every plaquette, stays in the smallest box containing C.

(iii) Apply (i) to (48); conjugation and inversion do not change s. □

At $V_0\in G^K$, with the links off K fixed, write $D_pw=\frac{d}{dt}V_p(V_0e^{tw})|_{t=0}$ for the derivative of the holonomy of p. If V_0 is flat on p, that is $V_{0,p}=I$, then $D_pw\in\mathfrak{g}$, and changing the corner from which p is read conjugates D_pw and leaves $\|D_pw\|$ unchanged. At $V_0=I$, $D_pw=\pm(dw)_p$, so $\sum_p\|D_pw\|^2=Q_1(w)$.

Proposition 567 (loop remainders at flat backgrounds)**.** *Let* $R(V)=\sum_C r_C\,s(V_C)$*, a sum over closed coarse paths* C*, each with a chosen ordered filling, such that*

$$\mathscr{L}=\sup_p\sum_C|r_C|A_Cm_{Cp}<\infty$$

and the series converges with its first two derivatives near V_0*.*

(i) $|R(V)|\le\mathscr{L}\sum_p s(V_p)$ *for every field* V*, including large fields.*

(ii) *If* V_0 *is flat on every plaquette of the fillings, then* $\nabla R(0)=0$ *and, in the coordinates* V_0e^{ω}*,*

$$\big|\operatorname{Hess}R(0)[w,w]\big|\le\frac{\mathscr{L}}{N}\sum_p\|D_pw\|^2\qquad(w\in\mathfrak{g}^K).$$

[46] In words: the holonomy of a loop is a product of plaquette holonomies, each carried to the base point along a path. The order of the factors matters, the field does not.

(iii) *Under (ii),* $\operatorname{Hess} R(0)[w, d_{V_0}\eta] = 0$ *for every* w *and every* $\eta \in \mathfrak{g}^{W^\circ}$.

Proof. (i) By Lemma 566(iii),

$$|R(V)| \le \sum_C |r_C| A_C \sum_p m_{Cp} s(V_p),$$

and the order of the nonnegative sums may be exchanged.

(ii) For a C^2 path $g(t)$ in $\mathrm{U}(N)$ with $g(0) = I$, differentiating $s(g) = \frac{1}{2N}\|I - g\|_{HS}^2$ twice gives

$$\frac{d}{dt} s(g(t))\Big|_0 = 0, \qquad \frac{d^2}{dt^2} s(g(t))\Big|_0 = \frac{1}{N}\|\dot g(0)\|_{HS}^2.$$

Along $V(t) = V_0 e^{tw}$ every factor of (48) equals I at $t = 0$, so $V_C(0) = I$, and the derivative of the product is the sum of the derivatives of the factors. Since $V_{p_j}(0) = I$, the derivatives of h_j and of h_j^{-1} cancel, and

$$\dot V_C(0) = \sum_{j=1}^{A_C} \varepsilon_j h_j(0) D_{p_j} w\, h_j(0)^{-1}.$$

Hence

$$\frac{d^2}{dt^2} s(V_C(t))\Big|_0 = \frac{1}{N}\|\dot V_C(0)\|^2 \le \frac{1}{N}\Big(\sum_j \|D_{p_j} w\|\Big)^2 \le \frac{A_C}{N}\sum_p m_{Cp}\|D_p w\|^2.$$

In the coordinates ω the path is the line $\omega = tw$, so this second derivative is $\operatorname{Hess} s(V_C)(0)[w,w]$, and the first derivative vanishes. Summing with the weights $|r_C|$ gives (ii).

(iii) R is Γ_W-invariant: a gauge transformation conjugates V_C by its value at the base point. Since $\nabla R(0) = 0$, Lemma 564(i) gives (iii). □

The flat backgrounds include the pure gauges $V_{0,y,\mu} = g_y g_{y+e_\mu}^{-1}$, whose links may be far from I.

Corollary 568 (one step with a loop remainder)**.** *Let* $U = I$*, so that* $V_0 = I$*, and let* $A = A_0 + R$ *near* 0*, where* $\operatorname{Hess} A_0(0)[\sigma,\sigma] = 2\lambda S_W(\sigma)$ *for* $\sigma \in \mathscr{O}^\perp$ *and* R *is as in Proposition 567, with all its paths and fillings in the box* Λ*, and* $\mathscr{L} \le N\lambda\varepsilon$*. Then the error* E *of Proposition 565 vanishes on* $\mathfrak{g}^K \times \mathscr{O}$ *and*

$$|E(w,w)| \le \lambda\varepsilon Q_1(w) + \delta_G|\Pi w|^2 \le \varepsilon'' \cdot 2\lambda S_W(w),$$
$$\varepsilon'' = \Big(\varepsilon + \frac{\delta_G C_n}{\lambda}\Big)\frac{1}{2c_*}.$$

For $\mathrm{SU}(2)$*,* $\lambda = 10^{100}$ *and* $\varepsilon = 1.63\cdot 10^{-9}$*,* $\varepsilon'' \le 5.60\cdot 10^{-9}$ *for* $n \le 10^{40}$ *and* $\varepsilon'' \le 0.1$ *for* $n \le 9.2\cdot 10^{49}$*, against* $n \le 8455$ *under (H).*

Proof. Part (i) of Proposition 565 does not use (H). As in its part (ii),

$$E(w,w) = \operatorname{Hess} R(0)[\Pi w, \Pi w] + \frac{1}{12}\sum_k \operatorname{Tr}\big(\operatorname{ad}^2_{(\Pi w)_k}\big),$$

since the term of A_0 is $2\lambda S_W(\Pi w) = 2\lambda S_W(w)$. Proposition 567(ii) at $V_0 = I$ bounds the first term by $\lambda\varepsilon Q_1(\Pi w)$, and $Q_1(\Pi w) = Q_1(w)$ because Q_1 is positive semidefinite and vanishes on $\mathscr{O}$ (Lemma 562(i)). The second term is at most $\delta_G|\Pi w|^2 \le \delta_G C_n Q_1(w)$ by Lemma 562(ii), and $Q_1 \le S_W/c_*$ by Lemma 562(iii). The numbers follow from $2c_* > 0.29131$ and $C_n \le \frac{5}{4} + n^2/\pi^2$. □

Proposition 565(iii) and Corollary 568 do not conflict. The function F there has the error $\frac{1}{2}\lambda\varepsilon|\sigma|^2$ in the slice variable, a sum over links that is not a function of loops. The Hessian of a loop remainder at a flat background is a sum of squares of combinations of the curvature derivatives $D_p w$, so Q_1 itself controls it.

Remark 569 (what the loop bound does not give). The bound has three limits.

(a) *Winding loops.* On the whole coarse torus of side L, a loop winding around the torus has no filling. Take the field with $V_{y,\mu} = I$ for $\mu \ne 0$ and

$$V_{y,0} = g = \operatorname{diag}\big(e^{i\pi/2L}, e^{-i\pi/2L}\big) \in \mathrm{SU}(2).$$

Every plaquette holonomy is I, while the loop along direction 0 has holonomy $g^L = \operatorname{diag}(i,-i)$, and $s(g^L) = 1$. No bound of $s(V_C)$ by a multiple of the plaquette action holds for it. In the box Λ every closed path has a filling, by Lemma 566(ii).

(b) *Values are not Hessians.* For $\mathrm{SU}(2)$, $m \in 4\mathbb{N}$ and $g(\theta) = \operatorname{diag}(e^{i\theta}, e^{-i\theta})$, the function $R_m(g(\theta)) = \varepsilon(1 - \cos\theta)\cos(m\theta)$ is a class function, smooth because $\cos(m\theta) = T_m(\frac{1}{2}\operatorname{Tr} g)$ with T_m the Chebyshev polynomial. It satisfies $|R_m| \le \varepsilon s$ everywhere, while

$$\frac{d^2}{d\theta^2} R_m(g(\theta))\Big|_{\theta=\pi/2} = -\varepsilon m^2,$$

since $\cos\frac{m\pi}{2} = 1$ and $\sin\frac{m\pi}{2} = 0$. Composed with one plaquette holonomy it is a gauge-invariant action term with a relative bound on its values and an unbounded Hessian at a background where that plaquette has holonomy $g(\pi/2)$. A relative bound on the values of the remainder therefore gives no Hessian bound at curved backgrounds; part (ii) of Proposition 567 uses flatness through $V_C(0) = I$.

(c) *The representation.* The induced action is not shown here to be a series of filled loops with small $\mathscr{L}$. Its determinants and coarea factors are not of this form as they stand, and the Haar term δ_G keeps the factor C_n.

Curved backgrounds. Proposition 567(ii) uses flatness through $V_C(0) = I$, and Remark 569(b) shows that a bound on values cannot replace it. A bound on the coefficients can. At a background whose plaquettes lie within κ of I, the Hessian of a loop remainder is bounded by a covariant plaquette form plus a Euclidean term of order κ (Proposition 572). On the window the Euclidean term costs a factor of order κn^2 (Lemma 571), and this order is attained (Proposition 573). Measured by the action of the background on the window, the

loss is of order $\|\kappa\|_2$ for every n, and this order is attained as well (Proposition 577, Corollary 578). Norms of matrices in $\mathrm{U}(N)$ are operator norms in this part.

For a path γ and $w \in \mathfrak{g}^K$, extended by zero, let

$$Z_\gamma(w) = \frac{d}{dt}\Big|_{t=0} V_\gamma(V_0)^{-1} V_\gamma(V_0 e^{tw}) \in \mathfrak{g}$$

be the derivative of its holonomy at V_0, trivialised on the left. If γ makes the steps $k_1,\dots,k_\ell$ with orientations $\varepsilon_i = \pm 1$, then $(V_k e^{tw_k})^{-1} = V_k^{-1} e^{-t\,\mathrm{Ad}(V_k)w_k}$ gives

$$V_\gamma(V_0 e^{tw}) = V_\gamma(V_0)\, e^{tY_1}\cdots e^{tY_\ell}, \qquad Z_\gamma(w) = \sum_i Y_i, \tag{49}$$

where $Y_i = \mathrm{Ad}(s_i^{-1})w_{k_i}$ if $\varepsilon_i = 1$, $Y_i = -\mathrm{Ad}(s_i^{-1}V_{0,k_i})w_{k_i}$ if $\varepsilon_i = -1$, and s_i is the holonomy at V_0 of the part of γ after its i-th step. In particular $\|Y_i\| = \|w_{k_i}\|$. For a plaquette p in the plane $\{\mu,\nu\}$, $\mu<\nu$, with base point x, Z_p is taken along (x,μ), $(x+e_\mu,\nu)$, $(x+e_\nu,\mu)^{-1}$, $(x,\nu)^{-1}$, and

$$Q^{V_0}(w) = \sum_p \|Z_p(w)\|^2$$

is the *covariant plaquette form* at V_0. If $V_{0,p} = I$, then $Z_p(w) = D_p w$; in particular $Q^I = Q_1$. A filling (48) is in *normal form* if every V_{p_j} is read from the base point of p_j in this order. Reading p from another corner, or in the other direction, conjugates V_p by the holonomy of at most two of its links and may invert it, so every ordered filling is brought to normal form by lengthening each h_j by at most two links. For a closed path C with a filling in normal form let ℓ_C be its length, n_{Ck} the number of its steps along the link k, $T_C = \sum_j \ell(h_j)$ the total length of the paths h_j, and t_{Ck} the number of their steps along k.

Lemma 570 (a filled loop at a curved background)**.** *Let C be a closed path with an ordered filling in normal form, and let $\|I - V_{0,p_j}\| \le \kappa$ for every j. In the coordinates $V_0 e^{\omega}$, for every $w \in \mathfrak{g}^K$:*

(i) *with $Z_C = Z_C(w)$,*

$$\mathrm{Hess}\, s(V_C)(0)[w,w] = -\frac{1}{N}\,\mathrm{Re}\,\mathrm{Tr}\Big(V_C(V_0)\Big(Z_C^2 + \sum_{i<i'}[Y_i,Y_{i'}]\Big)\Big);$$

(ii) $\|Z_C\| \le \sum_j \|Z_{p_j}\| + 2\kappa \sum_j \|Z_{h_j}\|$;

(iii) *the second derivative differs from $\frac{1}{N}\|Z_C\|^2$ by*

$$\Big|\mathrm{Hess}\, s(V_C)(0)[w,w] - \frac{1}{N}\|Z_C\|^2\Big| \le \frac{A_C\kappa}{N}\Big(\|Z_C\|^2 + \ell_C \sum_k n_{Ck}\|w_k\|^2\Big);$$

(iv) *with $a_C = (1+\kappa)(1+A_C\kappa)$,*

$$\big|\mathrm{Hess}\, s(V_C)(0)[w,w]\big| \le \frac{a_C A_C}{N}\sum_p m_{Cp}\|Z_p\|^2 + \frac{\kappa}{N}\sum_k \big(A_C \ell_C n_{Ck} + 4 a_C T_C t_{Ck}\big)\|w_k\|^2.$$

Proof. (i) By (49), $V_C(V_0 e^{tw}) = V_C(V_0)E(t)$ with $E(t) = e^{tY_1}\cdots e^{tY_\ell}$, so $E(0) = I$ and

$$E''(0) = \sum_i Y_i^2 + 2\sum_{i<i'} Y_i Y_{i'} = Z_C^2 + \sum_{i<i'}[Y_i, Y_{i'}].$$

Differentiate $s(g) = 1 - \frac{1}{N}\mathrm{Re}\,\mathrm{Tr}\, g$ twice.

(ii) The identity (48) holds for every field, so it may be differentiated. Let $Z(g)$ denote the left-trivialised derivative of a holonomy g. Then $Z(h^{-1}) = -\mathrm{Ad}(h)Z(h)$ and

$$Z(G_1\cdots G_A) = \sum_j \mathrm{Ad}\big((G_{j+1}\cdots G_A)^{-1}\big)Z(G_j).$$

For $G = hVh^{-1}$ these give

$$Z(G) = \mathrm{Ad}(h)Z(V) + \mathrm{Ad}(h)\big(\mathrm{Ad}(V^{-1}) - 1\big)Z(h).$$

Take $G_j = h_j V_{p_j}^{\varepsilon_j} h_j^{-1}$. Then $\|Z(V_{p_j}^{\varepsilon_j})\| = \|Z_{p_j}\|$, and $\|\mathrm{Ad}(g) - 1\| \le 2\|I - g\|$ for unitary g, since $\mathrm{Ad}(g)x - x = (g-I)xg^{-1} + x(g^{-1} - I)$. As Ad is isometric, (ii) follows.

(iii) Put $M = V_C(V_0) - I$. Each factor of (48) has $\|I - G_j\| = \|I - V_{0,p_j}\| \le \kappa$, so the telescoping sum in the proof of Lemma 566(i) gives $\|M\| \le A_C\kappa$. Since $-\mathrm{Tr}\, Z_C^2 = \|Z_C\|^2$ and $\mathrm{Tr}[Y,Y'] = 0$, part (i) gives

$$\mathrm{Hess}\, s(V_C)(0)[w,w] - \frac{1}{N}\|Z_C\|^2 = -\frac{1}{N}\,\mathrm{Re}\,\mathrm{Tr}\Big(M\Big(Z_C^2 + \sum_{i<i'}[Y_i,Y_{i'}]\Big)\Big).$$

With $|\mathrm{Tr}(MX)| \le \|M\|\,\|X\|_1$, $\|Z_C^2\|_1 = \|Z_C\|^2$, $\|[Y,Y']\|_1 \le 2\|Y\|\|Y'\|$ and

$$2\sum_{i<i'}\|Y_i\|\|Y_{i'}\| \le \Big(\sum_i \|w_{k_i}\|\Big)^2 \le \ell_C \sum_k n_{Ck}\|w_k\|^2,$$

(iii) follows.

(iv) By (ii) and $(a+b)^2 \le (1+\kappa)a^2 + (1+\kappa^{-1})b^2$,

$$\|Z_C\|^2 \le (1+\kappa)A_C \sum_p m_{Cp}\|Z_p\|^2 + 4\kappa(1+\kappa)\Big(\sum_j \|Z_{h_j}\|\Big)^2,$$

and $\big(\sum_j \|Z_{h_j}\|\big)^2 \le \big(\sum_k t_{Ck}\|w_k\|\big)^2 \le T_C \sum_k t_{Ck}\|w_k\|^2$; for $\kappa = 0$ the first bound follows from (ii) directly. Insert this in (iii). □

The Euclidean term is converted with the gap of Q^{V_0} off the gauge directions, which depends on the curvature alone.

Lemma 571 (the window at small curvature)**.** *Let $\Lambda' = \prod_\nu [a_\nu - 1, a_\nu + n]$, and suppose that $\|I - V_{0,p}\| \le \kappa$ for every plaquette p with its corners in Λ', the links off K included. Put $\delta = 3(n+1)\kappa$ and $\rho_0 = C_n^{1/2}/(1 - 2\delta/s_n)$, and suppose $2\delta < s_n$ and $\gamma_\kappa = \sqrt{432}\,\delta\rho_0 < 1$. Then*

$$|u|^2 \le \rho_n(\kappa)^2\, Q^{V_0}(u) \quad (u \in \mathscr{O}_{V_0}^{\perp}), \qquad \rho_n(\kappa) = \frac{\rho_0}{1 - \gamma_\kappa}.$$

If $\kappa(n+2)^2 \le 10^{-2}$, the hypotheses hold and $\rho_n(\kappa)^2 \le 2.31\, C_n$.

Proof. A gauge transformation $g \in G^{\Lambda'}$ acts on the links of Λ' by $(gV)_{y,\mu} = g_y V_{y,\mu} g_{y+e_\mu}^{-1}$. Let $(Tw)_{y,\mu} = \mathrm{Ad}(g_{y+e_\mu}) w_{y,\mu}$, an isometry of $\mathfrak{g}^K$. Then $(gV_0)e^{Tw} = g(V_0 e^w)$, so Z_p at gV_0 in the direction Tw is $\mathrm{Ad}(g_x)Z_p(w)$, x the base point of p, and $Q^{gV_0}(Tw) = Q^{V_0}(w)$. Also $d_{gV_0}(\mathrm{Ad}(g)\eta) = T d_{V_0}\eta$, so T maps $\mathscr{O}^\perp_{V_0}$ onto $\mathscr{O}^\perp_{gV_0}$, and the plaquette holonomies are conjugated. The statement for gV_0 therefore gives it for V_0.

The links (x,μ) of Λ' with $x_\nu = a_\nu - 1$ for every $\nu < \mu$ form a spanning tree of Λ', so g can be chosen with $gV_0 = I$ on them; write V_0 for gV_0. Let (x,μ) be another link of Λ', and $\nu < \mu$ the least direction with $x_\nu \ge a_\nu$. The plaquette p in the plane $\{\nu,\mu\}$ with base point $x - e_\nu$ has its corners in Λ', and its two links of direction ν lie in the tree, since their coordinates below ν are those of x. So $V_{0,p} = V_{0,x,\mu} V_{0,x-e_\nu,\mu}^{-1}$ and $\|I - V_{0,x,\mu}\| \le \kappa + \|I - V_{0,x-e_\nu,\mu}\|$. Induction on $\sum_{\nu<\mu}(x_\nu - a_\nu + 1) \le \mu(n+1)$ gives $\|I - V_{0,k}\| \le \delta$ for every link k of Λ'.

Then $\|\mathrm{Ad}(V_{0,k}^{-1}) - 1\| \le 2\delta < s_n$, and Lemma 563 gives $|u| \le \rho_0 Q_1(u)^{1/2}$ on $\mathscr{O}^\perp_{V_0}$. By (49), $Z_p(u) - (du)_p = \sum_i \pm(\mathrm{Ad}(g_i) - 1)u_{k_i}$, where g_i is a holonomy of three, two, two and one links of p for $i = 1,\dots,4$. Hence $\|Z_p(u) - (du)_p\| \le 2\delta(3b_1 + 2b_2 + 2b_3 + b_4)$ with $b_i = \|u_{k_i}\|$, and its square is at most $72\delta^2 \sum_i b_i^2$. A link lies in six plaquettes, so

$$Q_1(u)^{1/2} \le Q^{V_0}(u)^{1/2} + \sqrt{432}\,\delta|u|,$$
$$|u| \le \rho_0 Q^{V_0}(u)^{1/2} + \gamma_\kappa |u|.$$

For the numbers, $s_n \ge 2/n$ and $C_n \le 1 + n^2/4 \le (n+2)^2/4$ give $2\delta/s_n \le 3n(n+1)\kappa \le 0.03$ and $\gamma_\kappa \le \frac{3\sqrt{432}}{2(1-0.03)}\kappa(n+2)^2 \le 0.3215$, so $\rho_n(\kappa)^2 \le C_n/(0.97^2 \cdot 0.6785^2) \le 2.31\,C_n$. □

Proposition 572 (loop remainders at curved backgrounds)**.** *Let $R(V) = \sum_C r_C s(V_C)$, with fillings in normal form, converge with its first two derivatives near V_0. Let $\|I - V_{0,p}\| \le \kappa$ for every plaquette of the fillings, and put $a_C = (1+\kappa)(1 + A_C\kappa)$,*

$$\mathscr{L}_\kappa = \sup_p \sum_C |r_C|\, a_C A_C m_{Cp},$$
$$\mathscr{M}_\kappa = \sup_k \sum_C |r_C| \big(A_C \ell_C n_{Ck} + 4 a_C T_C t_{Ck}\big).$$

(i) *For every $w \in \mathfrak{g}^K$,*

$$\big|\mathrm{Hess}\,R(0)[w,w]\big| \le \frac{\mathscr{L}_\kappa}{N} Q^{V_0}(w) + \frac{\kappa \mathscr{M}_\kappa}{N}|w|^2.$$

(ii) *R is Γ_W-invariant, and $|\mathscr{H}R(w,w)| \le \frac{\mathscr{L}_\kappa}{N} Q^{V_0}(\Pi_{V_0} w) + \frac{\kappa\mathscr{M}_\kappa}{N}|\Pi_{V_0} w|^2$.*

(iii) *Under the hypotheses of Lemma 571,*

$$|\mathscr{H}R(w,w)| \le \frac{\mathscr{L}_\kappa + \kappa\rho_n(\kappa)^2\mathscr{M}_\kappa}{N} Q^{V_0}(\Pi_{V_0} w),$$

and if $\kappa(n+2)^2 \le 10^{-2}$ the factor is at most $(\mathscr{L}_\kappa + 0.58\,\kappa(n+2)^2\mathscr{M}_\kappa)/N$.

Proof. (i) Sum Lemma 570(iv) with the weights $|r_C|$ and exchange the nonnegative sums. (ii) A gauge transformation in Γ_W fixes the links off K and conjugates V_C, so R is invariant, and Lemma 564(ii) gives $\mathscr{H}R(w,w) = \mathrm{Hess}\,R(0)[\Pi_{V_0}w, \Pi_{V_0}w]$. (iii) Lemma 571 at $u = \Pi_{V_0} w$, with $2.31\,C_n \le 0.58(n+2)^2$. □

The background V_0 need not be a critical point of R or of the induced action: the gradient terms of Lemma 564(i) are removed by the projection, and the bound holds uniformly on the set of fields with plaquettes within κ of I, without third derivatives. At $\kappa = 0$, $\mathscr{L}_0 = \mathscr{L}$ and (i) is Proposition 567(ii). For the plaquette sum $R = r\sum_p s(V_p)$, $\mathscr{L}_\kappa = (1+\kappa)^2|r|$ and $\mathscr{M}_\kappa = 24|r|$, so (iii) gives the factor $(1+\kappa)^2 + 14\kappa(n+2)^2$ times $|r|/N$. At $V_0 = I$, $Q^I \circ \Pi = Q_1 \le S_W/c_*$. At a curved background the Gaussian reference is $2\lambda S_W^U \circ \Pi_{V_0}$ (Remark 584(d)), and its comparison with $Q^{V_0} \circ \Pi_{V_0}$ is not proved here. The order κn^2 of the loss in (iii) cannot be improved.

Proposition 573 (the loss κn^2 is attained)**.** *Let $G = \mathrm{SU}(2)$, $a_\nu = 1$, $X_j = i\sigma_j/\sqrt{2}$, $0 < \alpha \le \pi/4$ and $\varsigma = \pm 1$. On the links of Λ' let $V_{0,x,0} = A = \cos\alpha + i\sigma_1 \sin\alpha$, $V_{0,x,1} = B = \cos\alpha + \varsigma i\sigma_2 \sin\alpha$ and $V_{0,x,2} = V_{0,x,3} = I$. The plaquettes in the plane $\{0,1\}$ have holonomy $ABA^{-1}B^{-1}$, with $\|I - ABA^{-1}B^{-1}\| = 2\sin^2\alpha = \kappa$, and the others have holonomy I. Let $\phi(j) = \sin(\pi j/(n+1))$ for $1 \le j \le n$, $\phi(j) = 0$ otherwise, and for $y \in W$*

$$u_{y,0} = \phi(y_1)\phi(y_2)\phi(y_3)X_1, \qquad u_{y,1} = \phi(y_0)\phi(y_2)\phi(y_3)X_2,$$

with $u = 0$ on the other links. Let $P = \sum_p s(V_p)$ over the plaquettes of Λ'. Then $u \in \mathscr{O}^\perp_{V_0}$, $\mathscr{H}P(u,u) = \mathrm{Hess}\,P(0)[u,u]$, and for $n \ge 9$ and $\sin^2\alpha \le 10^{-2}$:

(i) *if $\varsigma = 1$ and $\kappa(n+1)^2 \le 2$, then $\mathscr{H}P(u,u) \ge \big(1 + 0.037\,\kappa(n+1)^2\big)\frac{1}{2}Q^{V_0}(u)$;*

(ii) *if $\varsigma = -1$ and $\kappa(n+1)^2 \ge 31$, then $\mathscr{H}P(u,u) < 0$.*

Proof. Put $s = \sin\alpha$, $c = \cos\alpha$, $x = \pi/(2n+2)$ and $\Psi = ((n+1)/2)^2$. Then $\sum_j \phi(j)^2 = (n+1)/2$, $\sum_j \phi(j)\phi(j+1) = \cos(2x)(n+1)/2$ and $\sum_j \phi(j) = \cot x$.

$\mathrm{Ad}(A)$ is the rotation by 2α about X_1 and $\mathrm{Ad}(B)$ the rotation by $2\varsigma\alpha$ about X_2; in particular $\mathrm{Ad}(A)X_1 = X_1$ and $\mathrm{Ad}(B)X_2 = X_2$. The adjoint of d_{V_0} is $(d_{V_0}^* u)_y = \sum_\mu (\mathrm{Ad}(V_{0,y,\mu})u_{y,\mu} - u_{y-e_\mu,\mu})$, so for $y \in W^\circ$

$$(d_{V_0}^* u)_y = (u_{y,0} - u_{y-e_0,0}) + (u_{y,1} - u_{y-e_1,1}) = 0,$$

because $u_{\cdot,0}$ does not depend on y_0 nor $u_{\cdot,1}$ on y_1 in W. So $\Pi_{V_0} u = u$, and $\mathscr{H}P(u,u) = \mathrm{Hess}\,P(0)[u,u]$ by Lemma 564(ii). The real part of the commutator $ABA^{-1}B^{-1}$ of unit quaternions with orthogonal axes is $1 - 2s^4$, so its eigenvalues $e^{\pm i\vartheta}$ have $1 - \cos\vartheta = 2s^4$ and $\|I - ABA^{-1}B^{-1}\| = (2 - 2\cos\vartheta)^{1/2} = 2s^2$.

In the plane $\{0,\nu\}$, $\nu \ge 2$, with coefficients f, f' of u on $(z,0)$, $(z+e_\nu,0)$, the holonomy along $V_0 e^{tu}$ is $A e^{t(f-f')X_1} A^{-1}$. So $Z_p = (du)_p$, and the Hessian of $s(V_p)$ is $\frac{1}{2}(f-f')^2 = \frac{1}{2}\|(du)_p\|^2$. The same holds in the planes $\{1,\nu\}$, and u vanishes in the plane $\{2,3\}$. In the plane $\{0,1\}$ let f, g', f', g be

the coefficients of u on $(z,0)$, $(z+e_0,1)$, $(z+e_1,0)$, $(z,1)$, and $F=f+f'$, $G=g+g'$. By (49),

$$Y_1 = f\,\mathrm{Ad}(BAB^{-1})X_1, \quad Y_2 = g'\,\mathrm{Ad}(BA)X_2,$$
$$Y_3 = -f'\,\mathrm{Ad}(B)X_1, \quad Y_4 = -gX_2.$$

In the basis X_1,X_2,X_3, with $c_2=\cos 2\alpha$, $s_2=\sin 2\alpha$, these are $f(c_2^2+c_2s_2^2,-\varsigma s_2^2,\varsigma s_2c_2(1-c_2))$, $g'(\varsigma s_2^2,c_2,-s_2c_2)$, $-f'(c_2,0,\varsigma s_2)$ and $-g(0,1,0)$. Their inner products give

$$\|Z_p\|^2 = (f-f')^2+(g-g')^2+4s^2(ff'+gg')+8\varsigma s^2c^2fg.$$

Write $ABA^{-1}B^{-1}=(1-2s^4)I+\hat{C}$ with $\hat{C}\in\mathfrak{g}$; its coordinates are $2\sqrt{2}s^2(sc,-\varsigma sc,-\varsigma c^2)$. Since $Z^2=-\frac{1}{2}\|Z\|^2I$ on $\mathfrak{su}(2)$, Lemma 570(i) gives

$$\mathrm{Hess}\,s(V_p)(0)[u,u] = \left(\tfrac{1}{2}-s^4\right)\|Z_p\|^2+\tfrac{1}{2}\Big\langle \hat{C}, \sum_{i<i'}[Y_i,Y_{i'}]\Big\rangle,$$

and the six triple products $\langle\hat{C},[Y_i,Y_{i'}]\rangle=-\sqrt{2}\det(\hat{C},Y_i,Y_{i'})$ add up to

$$\mathrm{Hess}\,s(V_p)(0)[u,u] = \tfrac{1}{2}\|Z_p\|^2 - s^4(F^2+G^2) + 2\varsigma s^2c^2(FG-2fg).$$

Both are identities between polynomials in c, s, f, f', g, g' modulo $c^2+s^2=1$.

Over the plaquettes of the plane $\{0,1\}$, $\sum ff'=\sum gg'=\frac{1}{2}n(n+1)\cos(2x)\Psi$, $\sum fg=\cot^2x\,\Psi$, $\sum FG=\sigma^2\Psi$ with $\sigma=\sum_{j=1}^n(\phi(j)+\phi(j+1))=2\cot x-\sin 2x$, and $\sum(F^2+G^2)=4n(n+1)\cos^2x\,\Psi$. Also $|u|^2=n(n+1)\Psi$, and as in Lemma 562(iv), $Q_1(u)=12\sin^2x\,|u|^2$. Hence

$$Q^{V_0}(u) = Q_1(u)+\Psi s^2\big(4n(n+1)\cos 2x + 8\varsigma c^2\cot^2x\big),$$
$$\mathrm{Hess}\,P(0)[u,u] = \tfrac{1}{2}Q^{V_0}(u) - 4s^4n(n+1)\cos^2x\,\Psi + 2\varsigma s^2c^2(\sigma^2-2\cot^2x)\Psi.$$

From $\tan x\ge x$ and $\cos x\ge 1-x^2/2$, $x^{-2}-1\le\cot^2x\le x^{-2}$; and $\sigma^2-2\cot^2x=2\cot^2x-8\cos^2x+\sin^2 2x\ge 2x^{-2}-10$. Also $\frac{1}{2}Q_1(u)=6\sin^2x\,n(n+1)\Psi\le\frac{3}{2}\pi^2\Psi$. Put $t=s^2(n+1)^2=\frac{1}{2}\kappa(n+1)^2$, so that $s^2x^{-2}=4t/\pi^2$, and use $n\ge 9$ and $s^2\le 10^{-2}$.

(i) Here

$$\mathrm{Hess}\,P(0)[u,u]-\tfrac{1}{2}Q^{V_0}(u) \ge \Psi t\Big(c^2\Big(\frac{16}{\pi^2}-\frac{20}{(n+1)^2}\Big)-4s^2\Big)\ge 1.3669\,\Psi t,$$
$$\tfrac{1}{2}Q^{V_0}(u)\le\Psi\Big(\frac{3\pi^2}{2}+t\Big(2+\frac{16}{\pi^2}\Big)\Big)\le 18.43\,\Psi$$

for $t\le 1$. The ratio is at least $0.0741\,t\ge 0.037\,\kappa(n+1)^2$.

(ii) Here, dropping the term in s^4,

$$\mathrm{Hess}\,P(0)[u,u]\le\Psi\Big(\frac{3\pi^2}{2}+t\Big(2-\frac{32c^2}{\pi^2}\Big)+24s^2c^2\Big) \le\Psi\Big(\frac{3\pi^2}{2}-0.9698t\Big),$$

which is negative for $2t\ge 31$. The constants follow from $c^2\ge 0.99$, $s^2\le t/100$ and $9.8696<\pi^2<9.8697$: in (i), $0.99(1.62112-0.2)-0.04\ge 1.3669$ and $14.805+3.622\le 18.43$; in (ii), $2-31.68/9.8697+0.24\le -0.9698$ and $14.805/0.9698<15.27$. □

For $R=rP$ with $r>0$, part (i) gives $\mathscr{H}R(u,u)\ge(1+0.037\,\kappa(n+1)^2)\frac{\mathscr{L}}{N}Q^{V_0}(\Pi_{V_0}u)$, against the factor $(1+\kappa)^2+14\kappa(n+2)^2$ of Proposition 572(iii): the loss is of order κn^2 in both. Part (ii) says more. Once $\kappa(n+1)^2\ge 31$, the covariant Hessian of the coarse plaquette action has a negative direction off the gauge directions, so it obeys no comparison with a positive semidefinite gauge-invariant form there (Fig. 43). This is the lattice form of the unstable mode of Nielsen and Olesen [93] in a constant chromomagnetic field. The comparison asked for in Remark 584(d) must therefore lose a multiple of $\kappa|\cdot|^2$, and on a window of side n it is a relative bound only while κn^2 is small.

Curvature in the action. The loss κn^2 of Proposition 572(iii) comes from the supremum κ of the curvature and from the gap C_n of the window in ℓ^2. In four dimensions the Euclidean term can be paid instead with the ℓ^2 norm of the curvature, the square root of the Wilson action of the background, and an ℓ^4 bound on the field, which holds on the window with a constant independent of n. For a plaquette p write $\kappa_p=\|I-V_{0,p}\|$, the operator norm, which does not depend on the corner from which p is read or on its orientation, and put

$$\|\kappa\|_2=\Big(\sum_p\kappa_p^2\Big)^{1/2}\le\Big(2N\sum_p s(V_{0,p})\Big)^{1/2},$$

the sums over the plaquettes with their corners in the box $\Lambda'=\prod_\nu[a_\nu-1,a_\nu+n]$ of Lemma 571; the inequality is $\kappa_p\le\|I-V_{0,p}\|_{HS}$. For $w\in\mathfrak{g}^K$ put $|w|_4=(\sum_k\|w_k\|^4)^{1/4}$. As in the proof of Lemma 562, a link $(y,\mu)\in K$ is inner when $y_\nu>a_\nu$ for every $\nu\ne\mu$, and a face link otherwise.

The ℓ^4 bound rests on an inequality of Gagliardo type. For finite sets $X_1,\dots,X_d$, $d\ge 2$, and $x\in\prod_i X_i$, let $\hat{x}_i$ be x with its i-th coordinate removed. If $F_i\ge 0$ are functions of $\hat{x}_i$, then

$$\sum_x\prod_{i=1}^d F_i(\hat{x}_i)^{\frac{1}{d-1}}\le\prod_{i=1}^d\Big(\sum_{\hat{x}_i}F_i(\hat{x}_i)\Big)^{\frac{1}{d-1}}. \tag{50}$$

For $d=2$ both sides equal $\sum F_1\sum F_2$. For $d\ge 3$, sum first over x_d, on which F_d does not depend. Hölder's inequality with $d-1$ factors bounds $\sum_{x_d}\prod_{i<d}F_i^{1/(d-1)}$ by $\prod_{i<d}G_i^{1/(d-1)}$, where $G_i=\sum_{x_d}F_i$ does not depend on x_i or x_d. Hölder's inequality with the exponents $d-1$ and $(d-1)/(d-2)$ in the remaining variables, and (50) for $d-1$ applied to the G_i, give the claim, since $\sum G_i=\sum F_i$.

Lemma 574 (a Sobolev inequality on the window)**.** *Let $n\ge 2$, fix μ, and let B_μ be the set of y with $a_\mu\le y_\mu\le a_\mu+n-1$ and $a_\nu+1\le y_\nu\le a_\nu+n-1$ for $\nu\ne\mu$. For $f\colon B_\mu\to[0,\infty)$, extended by zero, let $E_\mu(f)$ be the sum of $(f(y+e_\nu)-f(y))^2$ over the pairs $(y,y+e_\nu)$, $\nu\ne\mu$, that meet B_μ, and over the pairs $(y,y+e_\mu)$ in B_μ. Then*

$$\Big(\sum_y f(y)^4\Big)^{1/2}\le c_S\,E_\mu(f),\qquad c_S=\Big(\frac{27}{32}\Big)^{1/2}<0.9186.$$

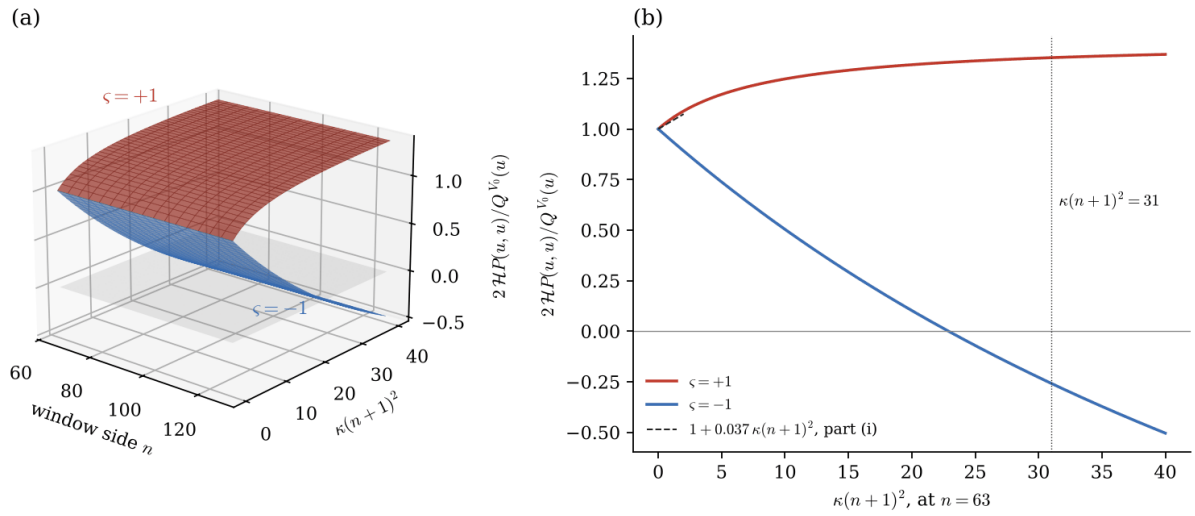


FIG. 43. The ratio of the covariant Hessian of the plaquette action to half the covariant plaquette form at the field u and the background of Proposition 573, from the closed forms in its proof. (a) The ratio over the window side n and $\kappa(n+1)^2$, for the two orientations $\varsigma = \pm 1$. (b) The section $n = 63$: for $\varsigma = +1$ the ratio exceeds one, above the bound of part (i), dashed; for $\varsigma = -1$ it changes sign near $\kappa(n+1)^2 = 23$, below the threshold 31 of part (ii).

Proof. Put $g = f^3$, $\Phi = (\sum f^4)^{1/2}$, a_ν^2 for the part of $E_\mu(f)$ in the direction ν, and $b = a_\mu$. In a direction $\nu \neq \mu$ a line of B_μ has $n-1$ points and g vanishes at its two ends outside B_μ. Summing the differences of g from either end gives $g(y) \leq F_\nu(\hat{y}_\nu) = \frac{1}{2} h_\nu(\hat{y}_\nu)$, where h_ν is the sum of $|g(z+e_\nu) - g(z)|$ over the n pairs of the line. In the direction μ a line has n points, and $g(y) \leq g(z) + h_\mu$ for every z on it, where h_μ is the sum over its $n-1$ pairs; averaging over z gives $g(y) \leq F_\mu(\hat{y}_\mu) = \frac{1}{n}\sum_z g(z) + h_\mu(\hat{y}_\mu)$.

For $s,t \geq 0$, $|s^3 - t^3| \leq \frac{3}{2}|s-t|(s^2+t^2)$, and each point of a line is an end of at most two of its pairs. By Cauchy–Schwarz and $(s^2+t^2)^2 \leq 2(s^4+t^4)$, the sum of $|g(z+e_\nu) - g(z)|$ over all pairs in the direction ν is at most

$$\frac{3}{2} a_\nu \Big(\sum 2\big(f(z)^4 + f(z+e_\nu)^4 \big) \Big)^{1/2} \leq 3 a_\nu \Phi.$$

Fix $\nu_0 \neq \mu$. The Dirichlet Laplacian on a line of $n-1$ points has least eigenvalue $4\sin^2(\pi/2n) \geq 4/n^2$, so $\sum f^2 \leq \frac{n^2}{4} a_{\nu_0}^2$, and $\frac{1}{n}\sum g \leq \frac{1}{n}(\sum f^2)^{1/2}\Phi \leq \frac{1}{2} a_{\nu_0}\Phi$. Hence $\sum F_\nu \leq \frac{3}{2} a_\nu \Phi$ for $\nu \neq \mu$ and $\sum F_\mu \leq (\frac{1}{2} a_{\nu_0} + 3b)\Phi$. Since $f^4 = g^{4/3} \leq \prod_\rho F_\rho^{1/3}$, (50) with $d = 4$ gives

$$\Phi^2 \leq \Phi^{4/3} \Big(\frac{27}{8} a_{\nu_0} a_{\nu_1} a_{\nu_2} \big(\tfrac{1}{2} a_{\nu_0} + 3b \big) \Big)^{1/3},$$

where $\{\nu_0, \nu_1, \nu_2\}$ are the directions other than μ. Let $E = E_\mu(f)$. Then $a_{\nu_0}^2 a_{\nu_1} a_{\nu_2} \leq \frac{1}{2} a_{\nu_0}^2 (a_{\nu_1}^2 + a_{\nu_2}^2) \leq \frac{1}{8} E^2$ and $a_{\nu_0} a_{\nu_1} a_{\nu_2} b \leq (E/4)^2$, so $\Phi^2 \leq \frac{27}{8}(\frac{1}{16} + \frac{3}{16}) E^2 = \frac{27}{32} E^2$. □

For $v \in \mathfrak{g}^K$ and x with $(x,\mu) \in K$, let

$$(D_\nu v_\mu)(x) = v_{x+e_\nu,\mu} - \mathrm{Ad}\big(V_{0,x+e_\mu,\nu}^{-1} \big) v_{x,\mu},$$

with v extended by zero off K. The term $v_{x,\mu}$ is the coordinate of $V_{0,x,\mu} e^{v_{x,\mu}}$ and transforms at $x + e_\mu$, so $D_\nu v_\mu$ compares two values at $x + e_\mu + e_\nu$; it is the covariant difference of v_μ in the direction ν. Since Ad is isometric,

$$\big| \|v_{x+e_\nu,\mu}\| - \|v_{x,\mu}\| \big| \leq \|(D_\nu v_\mu)(x)\|. \tag{51}$$

Lemma 575 (the covariant Gaffney inequality)**.** *Let $n \geq 2$ and let $v \in \mathfrak{g}^K$ vanish on the face links. Let E_μ be as in Lemma 574, $f_\mu(y) = \|v_{y,\mu}\|$, and for $k = (y,\mu)$ let $c_k = \sum_{\nu \neq \mu} \kappa_{p(y;\mu,\nu)}$ if $y \in W^\circ$ and $c_k = 0$ otherwise, where $p(y;\mu,\nu)$ is the plaquette at y in the plane $\{\mu,\nu\}$. Then*

$$\sum_\mu E_\mu(f_\mu) \leq \sum_{P \text{ inner}} \|Z_P(v)\|^2 + |d_{V_0}^* v|^2 + 2\sum_k c_k \|v_k\|^2,$$

and $\sum_k c_k^2 \leq 6\|\kappa\|_2^2$.

Proof. The support of v is $\bigcup_\mu B_\mu \times \{\mu\}$. For $\nu \neq \mu$ the pairs $(x, x+e_\nu)$ of E_μ are those with x the base of an inner plaquette in the plane $\{\mu,\nu\}$, and the pairs $(y - e_\mu, y)$ of E_μ in the direction μ are those with $y \in W^\circ$. By (51), $\sum_\mu E_\mu(f_\mu)$ is at most the covariant energy $\mathscr{E}(v)$, the sum of $\|(D_\nu v_\mu)(x)\|^2$ over the same pairs.

Let P be the plaquette at x in the plane $\{\mu,\nu\}$, $\mu < \nu$, with links $k_1 = (x,\mu)$, $k_2 = (x+e_\mu,\nu)$, $k_3 = (x+e_\nu,\mu)$, $k_4 = (x,\nu)$, and write $V_i = V_{0,k_i}$. By (49),

$$Z_P(v) = \mathrm{Ad}(V_4 V_3)\big(\mathrm{Ad}(V_2^{-1}) v_{k_1} + v_{k_2} - v_{k_3} - \mathrm{Ad}(V_3^{-1}) v_{k_4} \big),$$

so $\|Z_P(v)\| = \|(D_\nu v_\mu)(x) - (D_\mu v_\nu)(x)\|$. Also $(d_{V_0}^* v)_y = \sum_\mu \mathrm{Ad}(V_{0,y,\mu})(D_\mu v_\mu)(y - e_\mu)$ for $y \in W^\circ$. Expanding the squares,

$$\sum_{P \text{ inner}} \|Z_P(v)\|^2 + |d_{V_0}^* v|^2 = \mathscr{E}(v) + 2\sum_{\mu<\nu} (\Sigma_{\mu\nu} - \Sigma'_{\mu\nu}),$$

where, with $a(x)=v_{x,\mu}$, $b(x)=v_{x,\nu}$, $\alpha_y=\mathrm{Ad}(V_{0,y,\mu})$ and $\beta_y=\mathrm{Ad}(V_{0,y,\nu})$,

$$\Sigma_{\mu\nu}=\sum_{y\in W^\circ}\big\langle \alpha_y a(y)-a(y-e_\mu),\beta_y b(y)-b(y-e_\nu)\big\rangle,$$
$$\Sigma'_{\mu\nu}=\sum_{P\ \mathrm{inner}}\big\langle (D_\nu v_\mu)(x),(D_\mu v_\nu)(x)\big\rangle.$$

Each sum has four terms per index. Since $\alpha_{x+e_\nu}=\mathrm{Ad}(V_3)$ and $\beta_{x+e_\mu}=\mathrm{Ad}(V_2)$, three terms of Σ equal three terms of Σ', with the same signs:

$$\langle \alpha_y a(y),b(y-e_\nu)\rangle=\langle a(x+e_\nu),\alpha_y^{-1}b(x)\rangle,$$
$$\langle a(y-e_\mu),\beta_y b(y)\rangle=\langle \beta_y^{-1}a(x),b(x+e_\mu)\rangle,$$
$$\langle a(y-e_\mu),b(y-e_\nu)\rangle=\langle a(x+e_\nu),b(x+e_\mu)\rangle,$$

at $y=x+e_\nu$, $y=x+e_\mu$ and $y=x+e_\mu+e_\nu$ respectively. Each of these terms vanishes unless both its links are inner, and two inner links in these positions force $y\in W^\circ$ and P inner. For instance (y,μ) and $(y-e_\nu,\nu)$ inner give $a_\rho<y_\rho<a_\rho+n$ for $\rho\notin\{\mu,\nu\}$ from either link, $y_\nu\ge a_\nu+1$ and $y_\nu\le a_\nu+n-1$ from the first, and $a_\mu<y_\mu<a_\mu+n$ from the second. So these three kinds of terms cancel, and what remains is

$$\Sigma_{\mu\nu}-\Sigma'_{\mu\nu}=\sum_{x\in W^\circ}\big\langle\big(\mathrm{Ad}(V_4^{-1}V_1)-\mathrm{Ad}(V_3V_2^{-1})\big)a(x),b(x)\big\rangle.$$

Here $(V_3V_2^{-1})^{-1}V_4^{-1}V_1=V_1^{-1}V_{0,P}V_1$, so the operator has norm $\|\mathrm{Ad}(V_1^{-1}V_{0,P}V_1)-1\|\le 2\kappa_P$, and $2|\Sigma_{\mu\nu}-\Sigma'_{\mu\nu}|\le 2\sum_{x\in W^\circ}\kappa_P(\|a(x)\|^2+\|b(x)\|^2)$. Summed over $\mu<\nu$ this is $2\sum_k c_k\|v_k\|^2$. Finally $c_k^2\le 3\sum_{\nu\ne\mu}\kappa^2_{p(y;\mu,\nu)}$, and the plaquette $p(y;\mu,\nu)$ enters c_k for the two links (y,μ), (y,ν), so $\sum_k c_k^2\le 6\|\kappa\|_2^2$. □

Lemma 576 (the window in ℓ^4)**.** *Let $n\ge 1$, $\theta=\frac{9}{2}\|\kappa\|_2<1$ and $u\in\mathscr{O}_{V_0}^\perp$. Then*

$$|u|_4^2\le\Big(1+\frac{3c_S}{1-\theta}\Big)Q^{V_0}(u).$$

In particular $|u|_4^2\le c_4Q^{V_0}(u)$ with $c_4=1+\frac{9}{4}\sqrt{6}<6.512$ if $\|\kappa\|_2\le\frac{1}{9}$.

Proof. Write $u=v+f$ with v on the inner links and f on the face links. If $n=1$ there are no inner links, and $|u|_4^2\le|u|^2=|f|^2\le Q^{V_0}(u)$ by the face-link plaquettes below. Let $n\ge 2$. A link (y,μ) on which some $d_{V_0}\eta$ is nonzero has $y\in W^\circ$ or $y+e_\mu\in W^\circ$, and is inner, so $\mathscr{O}_{V_0}$ lies on the inner links, $v\perp\mathscr{O}_{V_0}$ and $d^*_{V_0}v=0$.

As in the proof of Lemma 562(ii), each face link k is the only link in K of a plaquette that is not inner, distinct for distinct k, and for it $\|Z_p(u)\|=\|f_k\|$ by (49). An inner plaquette P contains at most two face links, and a face link lies in at most one inner plaquette, so $\|Z_P(v)\|^2\le 3(\|Z_P(u)\|^2+\sum_{k\in P}\|f_k\|^2)$, the sum over the face links of P. With $A=\sum_{P\ \mathrm{inner}}\|Z_P(u)\|^2$ and $B=|f|^2$, this gives $\sum_{P\ \mathrm{inner}}\|Z_P(v)\|^2\le 3(A+B)$ and $A+B\le Q^{V_0}(u)$.

Let $\Phi_\mu=(\sum_y\|v_{y,\mu}\|^4)^{1/2}$. Then $|v|_4^2=(\sum_\mu\Phi_\mu^2)^{1/2}\le\sum_\mu\Phi_\mu$, and Lemmas 574 and 575, with Cauchy–Schwarz in $\sum_k c_k\|v_k\|^2$, give

$$|v|_4^2\le c_S\big(3(A+B)+2\sqrt{6}\,\|\kappa\|_2|v|_4^2\big).$$

Since $2\sqrt{6}c_S=\frac{9}{2}$, $(1-\theta)|v|_4^2\le 3c_S(A+B)$. The supports of v and f are disjoint, so $|u|_4^2\le|v|_4^2+|f|_4^2$, and $|f|_4^2\le|f|^2=B$. Hence $|u|_4^2\le\frac{3c_S}{1-\theta}(A+B)+B$. For $\theta\le\frac{1}{2}$, $6c_S=\frac{9}{4}\sqrt{6}<5.512$. □

For a closed path C with a filling in normal form, as in Lemma 570, let $\kappa_C=\sum_p m_{Cp}\kappa_p$ and $a_C=(1+\kappa_C)^2$, let $\ell(h_j)$ be the length of the path h_j and t_{jk} the number of its steps along k. For a loop sum $R=\sum_C r_C s(V_C)$ put

$$\mathscr{L}^\sharp=\sup_p\sum_C|r_C|a_CA_Cm_{Cp},$$
$$\mathscr{K}(k,p)=\sum_C|r_C|\Big(m_{Cp}\ell_Cn_{Ck}+4a_C\sum_{j:p_j=p}\ell(h_j)t_{jk}\Big),$$
$$\mathscr{N}=\Big(\sup_k\sum_p\mathscr{K}(k,p)\Big)^{1/2}\Big(\sup_p\sum_k\mathscr{K}(k,p)\Big)^{1/2}.$$

Proposition 577 (loop remainders in the action)**.** *Let $R(V)=\sum_C r_C s(V_C)$, with paths and fillings in normal form in Λ', converge with its first two derivatives near V_0, with $\mathscr{L}^\sharp,\mathscr{N}<\infty$.*

(i) *For every $w\in\mathfrak{g}^K$,*

$$\big|\mathrm{Hess}\,R(0)[w,w]\big|\le\frac{\mathscr{L}^\sharp}{N}Q^{V_0}(w)+\frac{\mathscr{N}\|\kappa\|_2}{N}|w|_4^2.$$

(ii) *If $\|\kappa\|_2\le\frac{1}{9}$, then for every $w\in\mathfrak{g}^K$*

$$|\mathscr{H}R(w,w)|\le\frac{\mathscr{L}^\sharp+c_4\mathscr{N}\|\kappa\|_2}{N}Q^{V_0}(\Pi_{V_0}w).$$

(iii) *For the plaquette sum $R=r\sum_p s(V_p)$ over the plaquettes with their corners in Λ', $\mathscr{L}^\sharp\le|r|(1+\kappa_\infty)^2$ with $\kappa_\infty=\max_p\kappa_p$, and $\mathscr{N}=8\sqrt{6}|r|$. If $\|\kappa\|_2\le\frac{1}{9}$, the factor in* (ii) *is at most $(1+130\|\kappa\|_2)|r|/N$, for every n.*

Proof. (i) Let C be one of the paths and $G_j=h_jV_{p_j}^{\varepsilon_j}h_j^{-1}$ the factors of its filling at V_0, with $\|I-G_j\|=\kappa_{p_j}$. The telescoping sum in the proof of Lemma 566(i) gives $\|I-V_C(V_0)\|\le\kappa_C$, and the proof of Lemma 570(iii) with this bound gives

$$\Big|\mathrm{Hess}\,s(V_C)(0)[w,w]-\frac{1}{N}\|Z_C\|^2\Big|\le\frac{\kappa_C}{N}\Big(\|Z_C\|^2+\ell_C\sum_k n_{Ck}\|w_k\|^2\Big).$$

The proof of Lemma 570(ii) gives

$$\|Z_C\|\le\sum_j\|Z_{p_j}\|+2\sum_j\kappa_{p_j}\|Z_{h_j}\|.$$

By Cauchy–Schwarz, $\big(\sum_j \|Z_{p_j}\|\big)^2 \le A_C \sum_p m_{Cp}\|Z_p\|^2$. The norm $\|Z_{h_j}\|$ is at most the sum of $\|w_k\|$ over the steps of h_j, by (49), so

$$\Big(\sum_j \kappa_{p_j}\|Z_{h_j}\|\Big)^2 \le \kappa_C \sum_j \kappa_{p_j}\|Z_{h_j}\|^2 \le \kappa_C \sum_j \kappa_{p_j}\ell(h_j)\sum_k t_{jk}\|w_k\|^2.$$

With $(s+t)^2 \le (1+\kappa_C)s^2 + (1+\kappa_C^{-1})t^2$, and directly if $\kappa_C = 0$,

$$\|Z_C\|^2 \le (1+\kappa_C)A_C\sum_p m_{Cp}\|Z_p\|^2 + 4(1+\kappa_C)\sum_j \kappa_{p_j}\ell(h_j)\sum_k t_{jk}\|w_k\|^2.$$

Hence

$$N\big|\mathrm{Hess}\, s(V_C)(0)[w,w]\big| \le a_C A_C \sum_p m_{Cp}\|Z_p\|^2 + \sum_k \beta_k^C \|w_k\|^2,$$

where

$$\beta_k^C = \sum_p \kappa_p \Big(m_{Cp}\ell_C n_{Ck} + 4a_C \sum_{j:p_j=p} \ell(h_j) t_{jk}\Big).$$

Summing with the weights $|r_C|$,

$$N\big|\mathrm{Hess}\, R(0)[w,w]\big| \le \mathscr{L}^\sharp Q^{V_0}(w) + \sum_k \beta_k \|w_k\|^2,$$

with $\beta_k = \sum_p \mathscr{K}(k,p)\kappa_p$. By Cauchy–Schwarz,

$$\beta_k^2 \le \sum_p \mathscr{K}(k,p)\sum_p \mathscr{K}(k,p)\kappa_p^2,$$

so $\sum_k \beta_k^2 \le \mathscr{N}^2\|\kappa\|_2^2$, and the last sum is at most $\mathscr{N}\|\kappa\|_2|w|_4^2$.

(ii) As in Proposition 572(ii), $\mathscr{H}R(w,w)$ is the Hessian of R at 0 in the direction $\Pi_{V_0}w$. This field lies in $\mathscr{O}_{V_0}^\perp$, and Lemma 576 applies to it.

(iii) Here every path is a plaquette p read from its base point, with $A_C = 1$, $m_{Cp} = 1$, $\ell_C = 4$, $n_{Ck} = 1$ for the four links k of p and no paths h_j. So $\mathscr{K}(k,p) = 4|r|$ if $k \in p$ and 0 otherwise. A link lies in six plaquettes and a plaquette has four links, so $\mathscr{N} = (24\cdot 16)^{1/2}|r| = 8\sqrt{6}|r|$. Since $\kappa_\infty \le \|\kappa\|_2 \le \frac{1}{9}$, $(1+\kappa_\infty)^2 \le 1 + \frac{19}{9}\|\kappa\|_2$, and $8\sqrt{6}\,c_4 = 8\sqrt{6}+108 < 127.6$. □

Since $\|\kappa\|_2^2 \le 6(n+1)^2(n+2)^2\kappa_\infty^2$, the hypothesis of Lemma 571 with $\kappa_\infty(n+2)^2 \le 10^{-2}$ gives $\|\kappa\|_2 \le 0.025$, and part (iii) then gives a factor $1 + O(\kappa_\infty n^2)$, the order of Proposition 572(iii) with a larger constant. When the curvature is concentrated the factor does not grow with n. If $V_0 = I$ except on one link $k_0 \in K$, where $V_{0,k_0} = g$, then the six plaquettes of k_0 have $\kappa_p = \|I-g\|$ and the others are flat, so $\|\kappa\|_2 = \sqrt{6}\|I-g\|$. Part (iii) gives the factor $1 + 319\|I-g\|$ on every window that contains k_0, against $(1+\kappa)^2 + 14\kappa(n+2)^2$ in Proposition 572(iii). The order of the loss in the action cannot be improved, and the hypothesis that the action of the background on the window is small cannot be dropped.

Corollary 578 (the loss is attained in the action)**.** *Let $G = \mathrm{SU}(2)$, and let V_0, u and P be as in Proposition 573. Then $\|\kappa\|_2 = \kappa(n+1)(n+2)$.*

(i) *If $n \ge 9$, $\varsigma = 1$ and $\|\kappa\|_2 \le 2$, then $\mathscr{H}P(u,u) \ge (1 + 0.0336\|\kappa\|_2)\frac{1}{2}Q^{V_0}(u)$.*

(ii) *For every $n \ge 39$ there is a background with $\|\kappa\|_2 \le 31.8$ at which $\mathscr{H}P$ is negative at $u \in \mathscr{O}_{V_0}^\perp$.*

Proof. The plaquettes with their corners in $\Lambda' = [0,n+1]^4$ in the plane $\{0,1\}$ have bases with $x_0, x_1 \in [0,n]$ and $x_2, x_3 \in [0,n+1]$, so there are $(n+1)^2(n+2)^2$ of them, each with $\kappa_p = \kappa$, and the others are flat. (i) Here $\kappa \le 2/110$, so $\sin^2\alpha = \kappa/2 \le 10^{-2}$, and $\kappa(n+1)^2 = \|\kappa\|_2(n+1)/(n+2) \le 2$. Proposition 573(i) gives the factor $1 + 0.037\,\kappa(n+1)^2 \ge 1 + 0.037\cdot\frac{10}{11}\|\kappa\|_2$. (ii) Take $\varsigma = -1$ and $\sin^2\alpha = 15.5/(n+1)^2 \le 15.5/1600 < 10^{-2}$, so that $\kappa(n+1)^2 = 31$; Proposition 573(ii) applies, and $\|\kappa\|_2 = 31(n+2)/(n+1) \le 31\cdot\frac{41}{40} < 31.8$. □

So the threshold of the unstable mode is a bound on the action of the background on the window, independent of n, and the condition $\|\kappa\|_2 \le \frac{1}{9}$ of Proposition 577 is necessary up to the constant. The calm windows of Corollary 380 satisfy both forms of the condition, because their curvature is scaled with the size of the window.

Corollary 579 (calm windows)**.** *Let $G = \mathrm{SU}(2)$, $0 < K \le 10^{-8}$, let W be a box of n^4 blocks, n_P as in Proposition 374 and $\kappa_f = K/(4\sqrt{n_P})$. Let U be a fine field with $\delta(U_p, I) \le \kappa_f$ for every fine plaquette p with a vertex in a block at ℓ^∞ distance at most two from W, and let V_0 be its cut-off averages on the links of Λ'. Then $\kappa_p \le 2.683K/n^2$ for every coarse plaquette with its corners in Λ', so that*

$$\kappa_\infty(n+2)^2 \le 24.2\,K, \qquad \|\kappa\|_2 \le 39.5\,K.$$

In particular Proposition 572(iii) and Proposition 577(ii) hold on W with a factor $\mathscr{L}_\kappa + 14.1K\mathscr{M}_\kappa$, respectively $\mathscr{L}^\sharp + 257.3K\mathscr{N}$, for every n.

Proof. A coarse plaquette P with its corners in Λ' has its four blocks at ℓ^∞ distance at most one from W, and the blocks at distance at most one from these are at distance at most two from W. So Lemma 409 applies with $\kappa_f \le 10^{-8}$ and gives $\kappa_P \le \delta(V_{0,P}, I) \le 92\kappa_f = 23K/\sqrt{n_P}$. As in the proof of Proposition 397, $n_P \ge 73.5|W| = 73.5\,n^4$, and $\sqrt{73.5} > 8.573$, $23/8.573 < 2.683$. Then $(n+2)^2/n^2 \le 9$ gives $24.15K$, and $\|\kappa\|_2 \le \sqrt{6}(n+1)(n+2)\kappa_\infty$ with $(n+1)(n+2)/n^2 \le 6$ gives $\sqrt{6}\cdot 6\cdot 2.683K < 39.44K$. Both bounds are far below the hypotheses $\kappa_\infty(n+2)^2 \le 10^{-2}$ and $\|\kappa\|_2 \le \frac{1}{9}$. The factors follow from Proposition 572(iii), $0.58\cdot 24.2 < 14.1$, and from $c_4 \cdot 39.5 < 257.3$. □

On calm windows the loss κn^2 is therefore already of order K, uniformly in n, and the bound in the action adds no strength there. Its use lies in the hypothesis and in the count of windows. The action is additive over windows, and the sup norm scaled by n^2 is not.

Corollary 580 (windows that fail)**.** *Let the coarse torus have side $L = mn$ with $m, n \geq 2$, tiled by the boxes $W_i = a_i + [0, n-1]^4$, $a_i \in n\mathbb{Z}^4$, with the boxes Λ'_i, and let $\|\kappa\|_{2,i}$ be $\|\kappa\|_2$ on Λ'_i. For every coarse field V_0 and $\eta > 0$,*

$$\#\{i : \|\kappa\|_{2,i} > \eta\} \leq \frac{32N}{\eta^2} \sum_P s(V_{0,P}),$$

the sum over all coarse plaquettes.

Proof. In each coordinate a plaquette occupies one value x or two values $x, x+1$, and lies in $[a-1, a+n]$ only if a is among $n+2$ consecutive integers. For $n \geq 2$ these contain at most two multiples of n, and $L \geq 2n \geq n+2$ keeps them distinct modulo L. So a plaquette has its corners in at most 2^4 of the boxes Λ'_i, and $\sum_i \|\kappa\|_{2,i}^2 \leq 16 \sum_P \kappa_P^2 \leq 32N \sum_P s(V_{0,P})$. □

With $\eta = \frac{1}{9}$, all but $2592N \sum_P s(V_{0,P})$ windows of the tiling satisfy Proposition 577(ii) with a factor at most $\mathscr{L}^\sharp + 0.73\mathscr{N}$, whatever n. The supremum condition counts differently. Let $V_0 = I$ except on the link $a_i + (0, e_0)$ of each W_i, where $V_0 = g$ with $\|I - g\| = 2\eta'(n+2)^{-2}$. Links of distinct windows differ by at least $n \geq 2$ in a coordinate and share no plaquette, so every window has $\kappa_\infty (n+2)^2 = 2\eta' > \eta'$, while $\sum_P \kappa_P^2 = 24 m^4 \eta'^2 (n+2)^{-4}$. A curved plaquette with its corners in Λ'_i contains a curved link whose base, a corner, lies in Λ'_i; the coordinates of that base minus a_i lie in $\{0, n\}$, so there are at most 16 such links and $\|\kappa\|_{2,i} \leq \sqrt{96}\,\|I - g\| = 8\sqrt{6}\,\eta'(n+2)^{-2}$. All m^4 windows fail the condition $\kappa_\infty (n+2)^2 \leq \eta'$, and none fails $\|\kappa\|_{2,i} \leq \frac{1}{9}$ once $(n+2)^2 \geq 72\sqrt{6}\,\eta'$.

Remark 581 (what the iteration needs). The one-step factors $1 + 130\|\kappa^{(j)}\|_2$ of Proposition 577(iii) at the backgrounds of the scales $j = 1, 2, \ldots$ have a bounded product exactly when $\sum_j \|\kappa^{(j)}\|_2 < \infty$. On calm windows at the levels K_j this holds if $\sum_j K_j < \infty$, by Corollary 579. Three inputs are not proved here. First, Corollary 579 uses the supremum form of Lemma 409; a bound of the coarse action $\sum_P \kappa_P^2$ by the fine action near the window, with a constant at most one, would let the action bound $Q_W \leq \frac{3}{4}\kappa^2$ of the calm completions (Corollary 380) pass from scale to scale, and it is not proved. Second, the comparison of the Gaussian reference $S_W^U \circ \Pi_{V_0}$ with $Q^{V_0} \circ \Pi_{V_0}$ of Remark 584(d) remains open. Third, the Haar term keeps the factor C_n in every loop representation (Proposition 582(iv)), since it is a multiple of $|\Pi_{V_0} w|^2$ and not of $|\Pi_{V_0} w|_4^2$; Lemma 576 does not remove it.

The loop representation itself is the remaining question. For SU(2) the class of loop sums is closed under products and dense, while a loop sum has no a priori bound on its weights.

Proposition 582 (loop sums for SU(2))**.** *Let $G = \mathrm{SU}(2)$.*

(i) *Let C, C' be closed paths based at x, x', γ a path from x to x', and $C \vee C'$, $C \vee C'^{-1}$ the closed paths $C\gamma C'\gamma^{-1}$ and $C\gamma C'^{-1}\gamma^{-1}$. Then*

$$s(V_C)s(V_{C'}) = s(V_C) + s(V_{C'}) - \tfrac{1}{2}s(V_{C \vee C'}) - \tfrac{1}{2}s(V_{C \vee C'^{-1}}),$$

and ordered fillings of C and C' give ordered fillings of $C \vee C'^{\pm 1}$ with $A_C + A_{C'}$ plaquettes and multiplicities $m_{Cp} + m_{C'p}$.

(ii) *On a box Λ' with all its links variable, every continuous function invariant under the gauge group $G^{\Lambda'}$ is a uniform limit of functions $c + \sum_C r_C s(V_C)$, finite sums over closed paths in Λ'.*

(iii) *Let φ be a class function with $\varphi(\mathrm{diag}(e^{i\theta}, e^{-i\theta})) = \sum_{m \geq 0} c_m \cos m\theta$ and $\sum_m m^2 |c_m| < \infty$, and C a closed path with an ordered filling. Then*

$$\varphi(V_C) = \varphi(I) - \sum_{m \geq 1} c_m\, s(V_{C^m}),$$

where C^m is C traversed m times with m copies of its filling. Its contribution to $\mathscr{L}$ is $\sum_m m^2 |c_m| \sup_p A_C m_{Cp}$. The function R_m of Remark 569(b) on one plaquette has $\mathscr{L} = (2m^2+1)\varepsilon$, although $|R_m| \leq \varepsilon s$.

(iv) *If R is a loop sum as in Proposition 567, with paths and fillings in Λ, and $|\mathscr{H}R(w,w) + \sigma|\Pi w|^2| \leq \tau Q_1(w)$ at $V_0 = I$ for every w, with $\sigma, \tau \geq 0$, then $\mathscr{L} \geq N(\sigma C_n^- - \tau)$. The Haar term is $-\frac{1}{3}|\Pi w|^2$ for $\mathrm{SU}(2)$, so a loop sum that contains it has $\mathscr{L} \geq 2(\frac{(n+1)^2}{9\pi^2} - \tau)$.*

Proof. (i) For $g, h \in \mathrm{SU}(2)$, $h + h^{-1} = (\mathrm{Tr}\, h) I$, so $\mathrm{Tr}\, g\, \mathrm{Tr}\, h = \mathrm{Tr}\, gh + \mathrm{Tr}\, gh^{-1}$. Apply it to $g = V_C$ and $h = V_\gamma V_{C'} V_\gamma^{-1}$, whose traces are real, and substitute $\mathrm{Tr} = 2(1 - s)$. The product of the fillings of V_C and of $V_\gamma V_{C'}^{\pm 1} V_\gamma^{-1}$, the latter conjugated by V_γ and for $V_{C'}^{-1}$ inverted and reversed, is a filling of $C \vee C'^{\pm 1}$.

(ii) Each orbit of $G^{\Lambda'}$ meets the fields equal to I on the spanning tree τ of the proof of Lemma 571, in an orbit of the constant gauge transformations. Fix a site o. For a link $k \notin \tau$ let C_k run from o along τ to the start of k, along k, and back along τ; at such fields $V_{C_k} = V_k$. Let V, V' be two such fields with $s(V_C) = s(V'_C)$ for every closed path C. Every word in the V_k is the holonomy of a closed path based at o, and the trace is real on SU(2), so the unitary representations ρ, ρ' of the free group on $\{k \notin \tau\}$ with $\rho(k) = V_k$, $\rho'(k) = V'_k$ have the same character. They are direct sums of irreducible representations. If $E_1, \ldots, E_r$ are the pairwise inequivalent irreducible summands of $\rho \oplus \rho'$, the theorems of Burnside and Jacobson give an element of the group algebra acting as the identity on E_i and as zero on E_j, $j \neq i$; its trace under ρ and under ρ' is $\dim E_i$ times the multiplicity of E_i. So ρ and ρ' are equivalent. The unitary factor U of the polar decomposition of an intertwiner intertwines, and $U(\det U)^{-1/2} \in \mathrm{SU}(2)$, as a constant gauge transformation, maps V to V'. Thus the functions $s(V_C)$ separate the orbits. With the constants they span an algebra by (i), and the Stone–Weierstrass theorem on the compact space of orbits gives (ii). This is the lattice form of the reconstruction theorem of Giles [58].

(iii) At $g = \mathrm{diag}(e^{i\theta}, e^{-i\theta})$, $s(g^m) = 1 - \cos m\theta$, so $\varphi(g) = \sum_m c_m - \sum_{m \geq 1} c_m s(g^m)$; both sides are class functions, and $V_{C^m} = V_C^m$. The repeated filling has mA_C factors and multiplicities $m\, m_{Cp}$. The derivatives of $s(g^m)$ of orders one and two are of order m and m^2, so the series converges with two derivatives. For R_m, $(1 - \cos\theta)\cos m\theta = \cos m\theta - \frac{1}{2}\cos(m+1)\theta - \frac{1}{2}\cos(m-1)\theta$, so $R_m(V_p) = -\varepsilon s(V_p^m) + \frac{\varepsilon}{2} s(V_p^{m+1}) + \frac{\varepsilon}{2} s(V_p^{m-1})$, and $\mathscr{L} = \varepsilon(m^2 + \frac{1}{2}(m+1)^2 + \frac{1}{2}(m-1)^2)$.

(iv) Lemma 564(ii) and part (ii) of Proposition 567 give $|\mathscr{H}R(w,w)| \le \frac{\mathscr{L}}{N} Q_1(\Pi w) = \frac{\mathscr{L}}{N} Q_1(w)$. At the field w of Lemma 562(iv), $\Pi w = w$ and $|w|^2 = C_n^- Q_1(w)$, so $\sigma C_n^- Q_1(w) \le (\frac{\mathscr{L}}{N} + \tau) Q_1(w)$. For SU(2), $\frac{1}{12}\operatorname{Tr}(\operatorname{ad}_x^2) = -\frac{1}{3}\|x\|^2$, and $C_n^- \ge (n+1)^2/(3\pi^2)$. □

Remark 583 (what the representation leaves open). Proposition 582(ii) represents every invariant function by loop sums in the sense of uniform approximation only. That controls neither the values relative to $\sum_p s(V_p)$ nor the Hessians, and the weights $\mathscr{L}_\kappa$, $\mathscr{M}_\kappa$ of the approximants may diverge. They belong to a representation and not to the function: by conjugation invariance $s(V_{\gamma C \gamma^{-1}}) = s(V_C)$ while T and t grow with γ, and $\mathscr{M}_\kappa$ can be taken as an infimum over representations. That the exact induced action of one step, at strong or at weak coupling, is a loop sum with finite $\mathscr{L}_\kappa$ and $\mathscr{M}_\kappa$ is not proved. It would follow from a convergent expansion of the induced action in polynomials in Wilson loops whose weights beat the growth in part (i), where a product of k loop functions becomes 2^{k-1} loop functions whose fillings have the sum of the areas. The character expansion at strong coupling and the expansion of Bałaban [12] at weak coupling are of this kind, and neither is carried out here for the cut-off averages. For $\mathrm{SU}(N)$, $N \ge 3$, the identity in (i) has no analogue linear in traces, and products of traces would have to be admitted. Part (iv) shows that C_n stays on the Haar term in any loop representation, as in Corollary 568.

The induced action. Let $e^{-A(\omega)}$ be the density in ω of the cut-off averages of a window under the Wilson measure restricted by a cut-off. The small-field restriction is an example: it is a soft box in the coordinates of the chart of Theorem 367, which is not Γ_W-invariant. Then $e^{-A^{\mathrm{gi}}}$ is the density with respect to Haar measure on G^K of the averages under the measure whose cut-off on the fibre over $\Upsilon(s,\eta)$ is the transport by the block-constant gauge transformation e^{η} of the cut-off on the fibre over e^s. The transport maps fibre to fibre (Theorem 353) and preserves the Wilson weight and Haar measure, so the fibre integrals agree. On the slice $\mathscr{O}^\perp$ the two cut-offs coincide. A cut-off defined through gauge-invariant quantities gives $A^{\mathrm{gi}} = A + \sum_k \log j$ everywhere.

Remark 584 (inputs and what is missing). Proposition 565 rests on one input and leaves the following open.

(a) *Input.* The bound (H) on the Hessian of the induced action at one point is an absolute bound, of the kind by which Theorems 361 and 363 control the non-Gaussian terms, and only its restriction to $\mathscr{O}^\perp$ is used. It is not proved in this paper. Corollary 568 replaces it by a loop representation of the remainder, under which C_n falls on the Haar term alone. As an illustration that no result uses, for SU(2), $\lambda = 10^{100}$ and $\varepsilon = 1.63 \cdot 10^{-9}$, interval arithmetic gives $\varepsilon' \le 8.40 \cdot 10^{-9}$ at $n = 2$, $1.52 \cdot 10^{-8}$ at $n = 4$, $1.52 \cdot 10^{-7}$ at $n = 16$ and $5.95 \cdot 10^{-4}$ at $n = 1024$, and $\varepsilon' \le 0.1$ for every $n \le 8455$.

(b) *The cut-off.* The construction replaces a non-invariant cut-off off the slice by its gauge transport. Bałaban defines the small-field regions by gauge-invariant conditions on plaquettes [7, 12, 40]. For such a cut-off the transport is not needed.

(c) *Points that are not critical.* $\mathscr{H}$ is taken at one point. An iteration needs the bound as a function on a neighbourhood. There the gradient terms of Lemma 564(i) and the third derivatives enter, and Proposition 565 says nothing about them. Bałaban expands around the minimiser of the action with given averages, the background field, at which the first-order terms vanish [6, 40].

(d) *Curved backgrounds.* Lemma 564 holds at every V_0. The Gaussian reference $S_W^U(w) = \inf\{Q_U(X) : BX = w\}$ need not vanish on $\mathscr{O}_{V_0}$, where B is the derivative of the cut-off averages at U. The fine gauge variation X_η of e^η has $BX_\eta = d_{V_0}\eta$ and $D_p(X_\eta) = (1 - \operatorname{Ad} U_p)\eta_{y(p)}$, with $y(p)$ the block of the base of p. Hence $S_W^U(d_{V_0}\eta) \le 384\kappa^2|\eta|^2$ when $\delta(U_p, I) \le \kappa$, where 96 plaquettes have their base in a block. The reference for an invariant action is $S_W^U \circ \Pi_{V_0}$. By Lemma 563 its gap on $\mathscr{O}_{V_0}^\perp$ follows from a comparison $S_W^U \ge c_1 Q_1 - c_2\kappa|\cdot|^2$ there, with the gap $c_1(1-\theta/s_n)^2 C_n^{-1} - c_2\kappa$; that comparison is not proved here.

(e) *The action.* The induced action is not a plaquette action. Its Hessian is compared with $2\lambda S_W$, not with $2\lambda Q_1$, and Theorem 367 does not apply to it as it stands at the next scale. The large fields, which Bałaban treats separately [12], are removed by the cut-off and not estimated.

Remark 585 (relative errors over scales). The proof of Proposition 460 uses only the order of forms, so it holds for positive semidefinite forms. A bound $(1 \pm \varepsilon')2\lambda S_W$ on the covariant Hessian therefore passes to the block infimum of the next step with factor one. Proposition 461 does not apply to it. That proposition concerns an error bounded by the Euclidean form, which does not vanish on $\mathscr{O}$. The block infimum turns that error into a term at least $4\lambda\varepsilon(1+2724\kappa)^{-1}|w|^2$, also where S_W vanishes.

The two statements agree. A window of side n in blocks of one scale has side $n/2$ in the blocks of the next, and $C_n/C_{n/2} \to 4$. Converting an absolute error at one scale costs the factor C_n. Carrying it one step as an absolute error multiplies it by at least four, after which converting costs $C_{n/2}$. Both routes have the same order, and the factor four of Proposition 461 is the loss of the gap of the window.

With windows of side $n_k = n_1 2^{1-k}$ at the scales $k = 1, \dots, K$, $\sum_k C_{n_k} \le K + n_1^2/3$, so

$$\sum_k \varepsilon_k' \le \sup_k \Big(\varepsilon_k + \frac{\delta_G}{\lambda_k}\Big) \frac{K + n_1^2/3}{2c_*}.$$

The part $n_1^2/3$ is the conversion cost. It is the absolute error carried to the last scale, $4^{K-1} \approx n_1^2$, so it is finite for a fixed window but not uniform in the volume. Removing it needs a gauge-invariant localisation of the one-step bound, which is part of Bałaban's analysis [12]. For a remainder that is a sum of gauge-invariant terms on boxes, Theorem 657 gives that localisation, with the gap of each box in place of C_n, and

by Proposition 659 the factor C_n of the Haar term comes from stating the hypothesis on the chart density. For a remainder that is a series of filled Wilson loops, Corollary 568 removes it from ε_k and leaves it on the Haar term δ_G/λ_k. The part K is marginal. The Haar term δ_G/λ_k is of order g_k^2, and $\sum_k g_k^2$ diverges logarithmically along the one-loop coupling by Remark 345(c). The order g^2 of the one-step error must therefore be absorbed into the running coupling and the normalisation of the field. Only a remainder of order g^4 can be carried as ε_k'. For $n_1 = 2^{12}$, $K = 13$, $\varepsilon_k = 1.63 \cdot 10^{-9}$ and $\lambda_k = 10^{100}$, the values of C_{n_k} give $\sum_k \varepsilon_k' \le 0.0127$, against $4^{12}\varepsilon_1 \le 0.028$ for the absolute error of the first scale carried to the last. No effective action is constructed here beyond the first scale, and item (iv) of Sec. XIII V remains open.

Q. Every compact simple group

The constructions of this section were written for SU(2). This subsection records which of them hold for a general group, and with which constants. Let G be a compact connected group whose Lie algebra $\mathfrak{g}$ is simple, with a faithful unitary representation in which $G \subset \mathrm{U}(N)$, the metric $\langle X, Y\rangle = -\mathrm{Tr}(XY)$ on $\mathfrak{g}$, the least Ricci curvature κ_G of Theorem 5, $d = \dim G$ and $K = \kappa_G/(d-1)$. The distance $\delta(g,h) = \|g-h\|$ is the Hilbert–Schmidt distance, $d(g,h)$ the geodesic distance of G, the Wilson weight is $\exp(-\beta_W \sum_p s_p)$ with

$$s_p = 1 - \tfrac{1}{N}\,\mathrm{Re}\,\mathrm{Tr}\,U_p = \frac{\delta(U_p, I)^2}{2N}, \qquad \beta_W = N^2\beta,$$

which is the normalisation of this section for $N = 2$, and ρ_G is the radius of Sec. XII. Put $I_d(x) = \int_0^x \sin^{d-1} u\,du$.

Lemma 586 (spectral constants)**.** *Let $X, Y \in \mathfrak{g}$ and $U \in G$.*

(i) *The operator $\mathrm{ad}X$ on $\mathfrak{g}$ is skew, with spectrum in $i[-\sqrt{2}\|X\|, \sqrt{2}\|X\|]$.*

(ii) *$-\mathrm{Tr}\big((\mathrm{ad}X)^2\big) = 4\kappa_G\|X\|^2$; in particular $\delta_G = \kappa_G/3$ in Proposition 565.*

(iii) *With $s(V) = 1 - \frac{1}{N}\,\mathrm{Re}\,\mathrm{Tr}\,V$, the derivative of $s(Ue^{tX})$ at $t = 0$ has modulus at most $\|X\|/\sqrt{N}$; on $\mathrm{U}(N)$, $\delta(g,h) \le 2\sqrt{N}$.*

(iv) *Let j be the density of Haar measure in exponential coordinates,*

$$j(X) = \det\big((1 - e^{-\mathrm{ad}X})/\mathrm{ad}X\big).$$

If $\|X\| \le 10^{-3}$, then

$$|\partial_Y \log j(X)| \le \tfrac{\kappa_G}{3}(1 + 10^{-6})\|X\|\|Y\|.$$

Proof. (i) Invariance of the trace makes $\mathrm{ad}X$ skew. On $\mathfrak{u}(N)$, in a basis in which $X = \mathrm{diag}(i\lambda_1, \dots, i\lambda_N)$, $\mathrm{ad}X$ has eigenvalues $i(\lambda_j - \lambda_k)$, and $\mathfrak{g}$ is an invariant subspace; $(\lambda_j - \lambda_k)^2 \le 2(\lambda_j^2 + \lambda_k^2) \le 2\|X\|^2$. (ii) $\mathrm{Ric}(X,X) = -\frac{1}{4}\mathrm{Tr}((\mathrm{ad}X)^2)$ is an Ad-invariant symmetric form. Since $\mathfrak{g} \otimes \mathbb{C}$ is simple, the adjoint representation is irreducible over $\mathbb{C}$ and every invariant symmetric form is a multiple of $\langle\cdot,\cdot\rangle$, so $\mathrm{Ric} = \kappa_G\langle\cdot,\cdot\rangle$. Then $\delta_G = \frac{1}{12}\cdot 4\kappa_G$. (iii) The derivative is $-\frac{1}{N}\,\mathrm{Re}\,\mathrm{Tr}(UX)$. Since $\overline{\mathrm{Tr}(UX)} = -\mathrm{Tr}(U^*X)$, $\mathrm{Re}\,\mathrm{Tr}(UX) = \frac{1}{2}\mathrm{Tr}((U - U^*)X)$, and $\|U - U^*\|^2 = 4\sum_j \sin^2\theta_j \le 4N$ for the eigenangles θ_j of U. The diameter bound is $\|g-h\|^2 = 2N - 2\,\mathrm{Re}\,\mathrm{Tr}(g^*h) \le 4N$. (iv) $\mathrm{Tr}\,\mathrm{ad}X = 0$, so $\log j(X) = \mathrm{Tr}\,\Phi(\mathrm{ad}X)$ with $\Phi(z) = \log(\sinh(z/2)/(z/2))$, and $\partial_Y \log j(X) = \mathrm{Tr}(\Phi'(\mathrm{ad}X)\,\mathrm{ad}Y)$. For real y, $\Phi'(iy) = i(\frac{1}{y} - \frac{1}{2}\cot\frac{y}{2})$ and $0 \le \frac{1}{y} - \frac{1}{2}\cot\frac{y}{2} \le \frac{y}{12}(1 + \frac{y^2}{50})$ for $0 \le y \le 1$; by (i) the spectral radius of $\mathrm{ad}X$ is at most $\sqrt{2}\cdot 10^{-3}$, so $\|\Phi'(\mathrm{ad}X)\|_{HS} \le \frac{1}{12}(1 + 10^{-6})\|\mathrm{ad}X\|_{HS}$, and (ii) together with the Cauchy–Schwarz inequality gives the bound. □

For SU(2) the spectrum in (i) is $\{0, \pm i\sqrt{2}\|X\|\}$, and the estimates of Lemma 411, Proposition 416, Lemma 420 and Step 4 of Lemma 487 that use the spectrum of an operator $\mathrm{ad}W$ use only that it lies in $i[-\sqrt{2}\|W\|, \sqrt{2}\|W\|]$: the bounds $\|L_X\| \le \theta/\sin\theta$, $\|E_Y\| \le 1$, $\|\varpi(\mathrm{ad}W)^{-1}\| \le m(\theta)$, the divided-difference bounds and $\kappa_0 < 0.8935$ hold for G unchanged. The Hilbert–Schmidt norm $\|\mathrm{ad}W\|_{HS} = 2\|W\|$ becomes $2\sqrt{\kappa_G}\|W\|$, a trace over $\mathfrak{g}$ is bounded by d times the operator norm instead of 3 times it, and the pair $(\|X\|/\sqrt{2}, 2\sqrt{2})$ of (iii) becomes $(\|X\|/\sqrt{N}, 2\sqrt{N})$.

Lemma 587 (the logarithm radius of a stabiliser)**.** *Let t be a tensor in $(\mathbb{C}^N)^{\otimes a} \otimes (\mathbb{C}^{N*})^{\otimes b}$ with $m = a + b \ge 1$, and let G be the identity component of the stabiliser of t in $\mathrm{U}(N)$. If $g \in G$ and $\delta(g, I) < 2\sin\frac{\pi}{2m}$, the principal logarithm of g lies in $\mathfrak{g}$.*

Proof. Let $e^{i\theta_j}$, $|\theta_j| \le \pi$, be the eigenvalues of g and $X = \log g$. Since $4\sin^2(\theta_j/2) \le \delta(g,I)^2 < 4\sin^2\frac{\pi}{2m}$, every $|\theta_j| < \pi/m$. The induced action of e^{sX} on the tensor space is e^{sD}, where the derivation D induced by X is skew-Hermitian with eigenvalues i times sums of m terms $\pm\theta_j$, so its spectrum lies in $i(-\pi, \pi)$. Decompose $t = \sum_\mu t_\mu$ over the eigenspaces of D. From $e^D t = t$, $e^{i\mu}t_\mu = t_\mu$ with $|\mu| < \pi$, so $t_\mu = 0$ for $\mu \ne 0$ and $Dt = 0$. The one-parameter group e^{sX} then fixes t and lies in its identity component G, so $X \in \mathfrak{g}$. □

$\mathrm{SO}(N)$ and $\mathrm{Sp}(n)$ are the identity components of the stabilisers of a symmetric and of a skew form, $m = 2$ and $2\sin\frac{\pi}{4} > 1$, so $\rho_G = 1$. $G_2 \subset \mathrm{U}(7)$, $F_4 \subset \mathrm{U}(26)$, $E_6 \subset \mathrm{U}(27)$ and $E_8 \subset \mathrm{U}(248)$ are the identity components of the stabilisers of tensors of degree at most three: the three-form of the octonions, the quadratic and cubic invariants of the traceless exceptional Jordan algebra, its cubic determinant, and the Lie bracket [17]. Each $\rho_G < 1$ is admissible for them, for instance $\rho_G = 0.99$. $E_7 \subset \mathrm{U}(56)$ is the identity component of the stabiliser of the symplectic and the quartic form of the Freudenthal triple system [17], $m = 4$, so $\rho_G = 0.76 < 2\sin\frac{\pi}{8}$ is admissible. For $\mathrm{SU}(N)$, $\rho_G = \min\{1, 5/\sqrt{N}\}$.

Lemma 588 (small balls)**.** *For $r > 0$, the Haar measure of $\{g \in G : d(g,I) \le r\}$ is at least $I_d(\sqrt{K}r)/I_d(\pi)$, with equality for* SU(2)*. If $\sqrt{K}r \le \frac{1}{4}$, it is at least $(\frac{95}{96})^{d-1}(\sqrt{K}r)^d/(d\,I_d(\pi))$.*

Proof. The Ricci curvature is at least $(d-1)K$, so the diameter is at most $\pi/\sqrt{K}$ by Myers' theorem, and the relative volume comparison of Bishop and Gromov [98] bounds the ratio of the volume of the ball of radius r to that of the ball of radius $\pi/\sqrt{K}$, which is G, from below by the same ratio on the sphere of curvature K and dimension d, $I_d(\sqrt{K}r)/I_d(\pi)$. For SU(2) the group is that sphere, of radius $\sqrt{2}$. The second bound follows from $\sin u \ge u(1-u^2/6) \ge \frac{95}{96}u$ on $[0,\frac14]$. □

Lemma 588 takes the place of the angle density $\frac{2}{\pi}\sin^2\theta$ used in Lemma 382. With it, put

$$C_0(G) = \frac12 + \frac23\Big(\log\big(d\,I_d(\pi)\big) + \frac{d}{2}\log\frac{16}{KN} - (d-1)\log\frac{95}{96}\Big).$$

Lemma 589. *For $\beta_W \ge KN$, the normalising constant of the Wilson measure of G on Λ_L satisfies*

$$-\log Z_L \le 6L^4\Big(\frac{d}{3}\log\beta_W + C_0(G)\Big),$$

and $\mathbb{E}S \le 6L^4(\frac{d}{3}\log\beta_W + C_0(G))/\beta_W$. If $KN \le 16$, then $C_0(G) > 0$.

Proof. Put $r = \sqrt{N}/(4\sqrt{\beta_W})$, so that $\sqrt{K}r \le \frac14$. If every link is within geodesic distance r of I, every plaquette has $\delta(U_p, I) \le d(U_p, I) \le 4r$ and $s_p \le 8r^2/N = 1/(2\beta_W)$, so $\beta_W S \le 3L^4$. By Lemma 588 the $4L^4$ links do so with probability at least

$$\Big(\big(\tfrac{95}{96}\big)^{d-1}\big(\tfrac{KN}{16\beta_W}\big)^{d/2}/\big(d\,I_d(\pi)\big)\Big)^{4L^4},$$

which gives the first bound; the second follows from the concavity of $-\log Z_L$ in the coupling, as in Lemma 382. The integrals satisfy $I_d I_{d+1} = 2\pi/d$, by induction from $I_{d+2} = \frac{d}{d+1}I_d$, and $I_d \ge I_{d+1}$, so $d\,I_d(\pi) \ge \sqrt{2\pi d} > 1$; the other terms of $C_0(G)$ are nonnegative when $KN \le 16$. □

For SU(2), $C_0(G) = 4.32001$, below the constant 4.90817 of Lemma 382. For SU(N), $KN = N^2/(2(N^2-2)) \le 1$; for every group of Table I, $KN \le 1$, so the lemma holds at every $\beta_W \ge 1$.

Theorem 590. *For $\beta_W \ge KN$, $0 < \rho \le 1$ and every set A of unit cubes, Theorem 383 and Corollary 384 hold for the Wilson measure of G with*

$$q_\rho^G(\beta_W) = \min\Big\{1, \exp\Big(2d\log\beta_W + 6C_0(G) - \frac{\beta_W\rho^2}{8N}\Big)\Big\}.$$

Proof. Reflection in a hyperplane through sites preserves Haar measure on G and maps $\mathrm{Re}\,\mathrm{Tr}\,U_p$ to $\mathrm{Re}\,\mathrm{Tr}\,U_p$ or $\mathrm{Re}\,\mathrm{Tr}\,U_p^{-1} = \mathrm{Re}\,\mathrm{Tr}\,U_p$, so the chessboard estimate holds as in Theorem 383. A bad plaquette has $s_p > \rho^2/(2N)$, and if every cube is bad there are at least $L^4/4$ bad plaquettes, so $S \ge \rho^2 L^4/(8N)$. Lemma 589 bounds Z_L. The corollary uses only the product bound. □

Theorem 591 (Section XIII for a compact simple group)**.** *Let $G \subset \mathrm{U}(N)$ be as above, with $\kappa_G \le 2500$ in (iii). Replace* SU(2) *by G, the hypothesis $\rho \le 10^{-8}$ by $\rho \le 10^{-8}\rho_G$, and C_0, q_ρ by $C_0(G)$, q_ρ^G. Then:*

(i) *Lemmas 409, 411, 450, 466, Theorems 465 and 467* (i), (ii) *hold with*

$$f_G(\beta) = 2d\log\beta + 6C_0(G) - \frac{\beta\rho^2}{8N}$$

in place of f (and ρ_0 in place of ρ at scale j), and with $\beta^* > 16dN/\rho^2$ in place of $\beta^* > 96/\rho^2$. The numbers M, H, Δ, M_j, H_j and $\log q^*$ do not depend on G, and

$$\beta_G^* = \frac{8N|\log q^*|}{\rho^2}(1+\varepsilon_G), \quad \varepsilon_G = \frac{2d\log\beta_G^* + 6C_0(G)}{|\log q^*|}.$$

(ii) *Theorems 363, 367, Propositions 364 and 369 hold as stated, since they are stated for every compact $G \subset \mathrm{U}(r)$: $c_3 = \max\{1.5\cdot10^4\sqrt{N}, d\}$, $c_6 = 6.81\cdot10^5 d\,c_3^2$, $\alpha_G = 2.5\cdot10^{-4}$.*

(iii) *Theorem 485 holds with $\lambda = \beta_W/(2N)$, hypothesis (b) read as $\beta_W \ge 480Nc_6$, and $\mathscr{A} = 120/\lambda = 240N/\beta_W$. Corollary 486 holds with*

$$\Delta = \frac{960N}{\beta_W}\Lambda_2^2 r^2\sqrt{n_x n_y}\,e^{-2\alpha_G(w-2s-8)},$$

and its hypothesis (a) holds for $\rho \le 1/(2758\cdot7.8\cdot10^7 c_3)$.

(iv) *Lemma 487 holds with $\frac{2}{N}c_W$ and $\frac{d}{3}c_\psi$ in place of c_W and c_ψ; under hypotheses (a) and (c) of Theorem 485,*

$$\Lambda_2 \le \frac{198.0002}{N}\beta_W + \frac{4.65\cdot10^6}{3}d.$$

(v) *Proposition 416 holds with the density bounds $(1+\kappa_0)^{-d}$ and $(1-\kappa_0)^{-d}$, and with radius $\rho_G/32$ when $\rho_G < 1$; Lemma 419 holds with the Jacobian bound σ_0^d; Lemma 420 holds with $64.94\cdot\frac{d}{3}\sqrt{\kappa_G}$ and $32.47\cdot\frac{d}{3}\sqrt{\kappa_G}$; Theorem 421 holds with $24n_p\beta_W$, $197\beta_W + a_G$ and $87\beta_W + b_G$, where*

$$a_G = 1558.56\sqrt{N}\,\tfrac{d}{3}\sqrt{\kappa_G} + 1.1986,$$
$$b_G = 779.28\sqrt{N}\,\tfrac{d}{3}\sqrt{\kappa_G} + 0.5993.$$

(vi) *Theorem 491, Corollary 492, Proposition 493 and Theorem 494 hold with the threshold β_G^* of (i), the condition $2758\rho \le 1/(7.8\cdot10^7 c_3)$, $\frac{960N}{\beta_W}$ in place of $\frac{1920}{\beta_W}$, and $\bar\Lambda_2 = \frac{198.0002}{N}\beta_W + \frac{4.65\cdot10^6}{3}d$.*

(vii) *The statements of Subsections XIII D to XIII H, other than Proposition 458, of Subsection XIII N and of Subsection XIII P hold for G as stated, with $\delta_G = \kappa_G/3$.*

For SU(3), $d=8$, $N=3$, $\kappa_G=\frac{3}{2}$, $\rho_G=1$, $C_0(G)\in[10.44720,10.44721]$, $f_G(\beta)=16\log\beta+6C_0(G)-\beta\rho^2/24$, *and*

$$\beta^*(10^{-8},1)\in[5.268,5.270]\cdot10^{21},$$
$$\beta^*(10^{-8},5)\in[8.230,8.232]\cdot10^{21},$$
$$\beta^*(10^{-9},1)\in[5.285,5.287]\cdot10^{23},$$
$$\beta^*(10^{-9},5)\in[8.247,8.249]\cdot10^{23},$$
$$\beta^*(10^{-8},5,10^{30})\in[2.971,2.973]\cdot10^{22},$$

with $0.026<\varepsilon_G<0.045$ *in the first four. At scale* j, *for* $s=5$ *and* $\rho=10^{-8}$, $\beta_1^*\in[4.367,4.368]\cdot10^{23}$, $\beta_2^*\in[6.990,6.992]\cdot10^{24}$, $\beta_3^*\in[1.785,1.787]\cdot10^{26}$ *and* $16^{-j}\beta_j^*\le4.38\cdot10^{22}$ *for* $1\le j\le12$. *In (iii) for* SU(3), $c_3\le25980.8$, $c_6\le3.678\cdot10^{15}$, *(b) holds for* $\beta_W\ge5.30\cdot10^{18}$, *(a) for* $\rho\le1.789\cdot10^{-16}$, *and at* $\rho=1.78\cdot10^{-16}$ *the threshold with the joining distance* $2s+8$ *lies in* $[2.569,2.571]\cdot10^{37}$ *for* $s=1$ *and in* $[3.014,3.016]\cdot10^{37}$ *for* $s=5$, *where (b) holds;* $\Lambda_2\le66.0001\beta_W+1.24\cdot10^7$, *and the constant* c_2 *of Theorem 494 at* $s=1$, $\vartheta=0$ *is at most* $10^{22.14}$.

Proof. (i) The witness paragraph of the proof of Theorem 465 uses Lemma 360, which is stated for every G with $\kappa\le10^{-8}\rho_G$; Lemma 352, a statement on U(N); and $d(g,h)\le(1+10^{-12})\delta(g,h)$ for $\delta(g,h)\le10^{-6}$, which holds on G because there $\delta\le\rho_G$, the principal logarithm of $g^{-1}h$ lies in $\mathfrak{g}$ and $d(g,h)=\|\log(g^{-1}h)\|$. In Lemma 409 the constants are those of Lemma 352 and the triangle inequality. In Lemma 411 the cut-off is nonzero only where $\max_x\delta(v_x,I)<\rho_G$, so the logarithms lie in $\mathfrak{g}$, and the contraction bound uses the spectrum of $\mathrm{ad}X$ only through Lemma 586(i). Lemmas 450 and 466 count cubes; Theorem 388, used in the latter, is stated for every G. The threshold paragraph applies to f_G, which is concave with maximum $2d\log(16dN/\rho^2)+6C_0(G)-2d>0>\log q^*$ at $16dN/\rho^2$, by Lemma 589 and $\rho\le10^{-8}$. The identity for ε_G is $f_G(\beta_G^*)=\log q^*$. (ii) There is nothing to prove. (iii) In the proof of Theorem 485 the Wilson action enters as $\lambda\|\cdot\|^2$ with $\lambda=\beta_W/(2N)$, the dimension enters through $\|A_v\|_1\le2dc_3\|x_v\|_1$ and $\|A_a\|_2^2\le2dc_3^2\|\bar P\|^2|a|^2$, which are absorbed in c_6, and the Haar part is the logarithmic gradient of j at $\|X_e\|\le10^{-3}$, at most $\frac{\kappa_G}{3}(1+10^{-6})10^{-3}\le0.84$ per unit of $\|x_v\|_1$ by Lemma 586(iv) and $\kappa_G\le2500$. The rest of the proof is a statement about the chart constants of (ii). The corollary is Proposition 484 with $4\mathscr{A}=960N/\beta_W$, and $\eta+62r=2758\rho$. (iv) The Wilson term of a plaquette is $\beta_W(1-\frac{1}{N}\mathrm{Re}\,\mathrm{Tr}\,U_p)$, so every bound of (42) carries $\frac{1}{N}$ in place of $\frac{1}{2}$; Steps 2 to 6 use the spectrum of $\mathrm{ad}W$ through Lemma 586(i) only. In Step 7, $|\mathrm{Tr}A|\le d\|A\|$ on $\mathfrak{g}$ replaces $|\mathrm{Tr}A|\le3\|A\|$. (v) In the proof of Proposition 416 the density of a linear map T with $\|T-I\|\le\kappa_0$ on $\mathfrak{g}$ lies between $(1-\kappa_0)^d$ and $(1+\kappa_0)^d$; the cut-off is inactive below $\rho_G/2$, which gives the radius $\rho_G/32$. Lemma 371 is stated with ρ_G, and $JJ^*\ge\sigma_0^2$ on the d-dimensional $\mathfrak{g}$ gives $\sqrt{\det JJ^*}\ge\sigma_0^d$. In the proof of Lemma 420, $\mathrm{tr}(T^{-1}\dot T)$ is at most $d\|T^{-1}\|\|\dot T\|$ and $\|\mathrm{ad}W\|_{HS}=2\sqrt{\kappa_G}\|W\|$ by Lemma 586(ii), while the spectra lie in the segments used there by Lemma 586(i). In the proof of Theorem 421 the oscillation of a site function is at most $24\cdot2\sqrt{N}$ and $|\dot s_p|\le v/\sqrt{N}$ by Lemma 586(iii), so the first term is at most $24\sqrt{N}\sup|\partial_t\log h_t|$; the factors $\sqrt{N}$ cancel in the Wilson part, which gives $24n_p\beta_W$, $24\cdot8.1916\beta_W<197\beta_W$ and $24\cdot6\cdot0.5993\beta_W<87\beta_W$, and remain in the density part, which gives a_G and b_G. For SU(2), $a_G<2206$ and $b_G<1103$, the constants of the theorem. (vi) These statements use the preceding ones through the constants listed, and the threshold through q^G. (vii) These statements concern the label variables, the coupling of labels, the island combinatorics and the quadratic forms of a box. The group enters them only through the results of Sec. XII, stated for every G, through the constants of (iii) and (iv), and through δ_G in Subsection XIII P, which is $\kappa_G/3$ by Lemma 586(ii).

The numbers for SU(3). Here $K=\kappa_G/7=\frac{3}{14}$, $KN=\frac{9}{14}$ and $I_8(\pi)=\int_0^\pi\sin^7u\,du=\frac{32}{35}$, so $d\,I_d(\pi)=\frac{256}{35}$, and the factorisations $\frac{256}{35}(\frac{224}{9})^4(\frac{96}{95})^7=2^{63}3^{-1}5^{-8}7^3\,19^{-7}$ and $7^2=2\cdot5^2\cdot\frac{49}{50}$, $19=2^5\cdot3\cdot5^{-1}\cdot\frac{95}{96}$ give

$$C_0(G)=\tfrac{1}{2}+\tfrac{2}{3}\big(\tfrac{59}{2}\log2-8\log3+2\log5-\tfrac{3}{2}\log\tfrac{50}{49}+7\log\tfrac{96}{95}\big).$$

The logarithms of the proof of Theorem 465, with $\log\frac{50}{49}=2\,\mathrm{artanh}\,\frac{1}{99}$ and $\log\frac{96}{95}=2\,\mathrm{artanh}\,\frac{1}{191}$, give $C_0(G)$ within 10^{-7} of 10.4472096. The function $g_G(\beta)=\beta\rho^2/24-16\log\beta-6C_0(G)-|\log q^*|-H\log\Gamma$ increases for $\beta>384/\rho^2$, and β^* is its largest zero. With $|\log q^*|$ within 10^{-3} of 21090.194 and 33424.872 (Theorem 465), at the ends of each of the five stated intervals g_G is negative and positive, with margins above 2, so that $\log\beta$ to two decimals decides the signs; $\varepsilon_G=(16\log\beta_G^*+6C_0(G))/|\log q^*|$ increases with β_G^*, and its values at the ends of the first four intervals lie between 0.02603 and 0.04442. At scale j, $\rho_0=\rho/(c4^j)$ with $c=1.01\cdot1.023$ and $24c^2\le25.6215417$, and $|\log q_j^*|$ is as in the proof of Theorem 467: within 10^{-3} of 105605.899 and 105608.929 for $j=1,2$, and at most $169144.64+4j\log2$ for $j\ge3$. The same sign check, with margins above 11, gives the three intervals for β_1^*, β_2^*, β_3^*. For $b>384c^2/\rho^2$,

$$g_{G,j}(16^jb)=\frac{b\rho^2}{24c^2}-16\log b-6C_0(G)-\big(|\log q_j^*|+64j\log2\big),$$

and at $b=4.38\cdot10^{22}$ the first term is at least 170949.9, the next two at most 834.15 and 62.684, and the bracket, for $j\le12$, at most 105697.7 when $j\le2$ and $169144.64+68j\log2\le169710.3$ when $j\ge3$; so $g_{G,j}(16^jb)>0$ and $16^{-j}\beta_j^*\le4.38\cdot10^{22}$. In (iii), $c_3=1.5\cdot10^4\sqrt{3}\le25980.8$, $c_6=6.81\cdot10^5\cdot8\cdot6.75\cdot10^8=3.6774\cdot10^{15}$, $480\cdot3\cdot c_6=5.295456\cdot10^{18}$ and $2758\cdot7.8\cdot10^7\cdot25980.8\le5.5891\cdot10^{15}$, which give (b) and (a). For the joining distance $2s+8$, $|\log q^*|$ lies within 10^{-2} of 32481.67 for $s=1$ and of 38354.07 for $s=5$ (Theorem 494), and the sign check at $\rho=1.78\cdot10^{-16}$, with margins above 7, gives the two intervals. In (iv), $\frac{198.0002}{3}\le66.0001$ and $\frac{4.65\cdot10^6\cdot8}{3}=1.24\cdot10^7$. Finally, with $960N=2880$, $r=43\rho$

and $\kappa = 2\alpha_G = 5\cdot 10^{-4}$,

$$c_2 = 2880\,\beta^* r^2\Big(66.0001 + \frac{1.24\cdot 10^7}{\beta^*}\Big)^2 \cdot 45\cdot 14^2\cdot 9^2 e^{0.008},$$

which increases with β^* and is at most $10^{22.134}$ at $\beta^* = 2.571\cdot 10^{37}$. □

Remark 592. Four parts of this section do not transfer. Subsection XIII K (Lemma 469, Propositions 470 and 468, Theorem 471) measures frames by the angle of a unit quaternion; on SU(N), $N \geq 3$, the operator-norm angle is subadditive and conjugation invariant, and the constants of the subsection have not been computed for it. Proposition 418 uses every preimage of the average on SU(2), and Proposition 458 is a computation with unit quaternions. The examples of Propositions 408, 413 and 414 are stated on SU(2) and are not claimed for other groups.

Remark 593. For SU(3) the plaquette blocks reach $\beta_W \leq 0.52$ (Theorem 7), in place of 0.348 for SU(2). For other groups the block constant (15) needs the modulus $\Lambda(r)$ of (8), a gap of the one-link measure and bounds on the Hessian of $\log Z$, and none of them has been certified. The modulus that holds for every G does not suffice: the one-link measure ν_b satisfies the Bakry–Émery condition with constant $\kappa_G - \|b\|_{\mathrm{op}} \geq \kappa_G - r$, so $\Lambda(r) \leq 1/(\kappa_G - r)$; with this bound the right side of (8) is below one only for $N\beta < \kappa_G/18$, and, since $K_3 \geq s \geq \Lambda(r)$, every value of α_3 computed from it is at least $10N\beta/(\kappa_G - 6N\beta)$, below one only for $N\beta < \kappa_G/16$. Both ranges lie inside $N\beta < 2\kappa_G/29$ of Theorem 5.

R. One step of the block map: the large fields

This subsection carries out, for $G = \mathrm{SU}(2)$, the large-field part of one step of the block map $U \mapsto c = V^\sharp(U)$ on the whole torus. It proves three things. A set of rough blocks is suppressed by $e^{-\lambda\kappa^2}$ per block, either absolutely or in the mean. The weight of a coarse field is an exact sum over large-field regions of kernels with this suppression. The coarse action lies between $\lambda/(1.65\cdot 10^6)$ and 8λ times the coarse curvature, up to a volume term. It also states which bound the next step would need, and how far Theorem 471 supplies it. The small-field part and its locality are the subject of item (iv) of Sec. XIII V and are not treated here. The lattice counts and the numerical constants are proved by hand. The one exception is the value of the coupling $(M_W)_{AB}$ of Theorem 607, which is computed in Remark 608 and used by no proof; the statements that the calm weight is not local are conditional on $(M_W)_{AB} \neq 0$.

Throughout, Λ_L is the torus of side $L = 2^m \geq 32$ in the tree gauge, T is the coarse torus of the blocks $B(y)$, $|T| = L^4/16$, $\lambda = \beta_W/4$, so that the Wilson weight is $e^{-\lambda Q}$ with $Q = \sum_p \delta(U_p, I)^2$, and ν_c is the fibre measure of Sec. XII E with $W = T$. For a coarse field c write c_P for the holonomy of a coarse plaquette and $Q_1(c) = \sum_P \delta(c_P, I)^2$.

Definition 594. The plaquette at x in the plane $\{\mu,\nu\}$ is *owned* by the block containing x; $\Pi(y)$ is the set of plaquettes owned by y, and $\Pi(X) = \bigcup_{y\in X}\Pi(y)$. For $\kappa > 0$ a block y is *calm* if $\delta(U_p, I) \leq \kappa$ for every $p \in \Pi(y)$, and *rough* otherwise; $R(U)$ is the set of rough blocks. For a block y let I_y be the 17 active links inside $B(y)$, $P_{y,\mu}$ the eight private links of the pair (y,μ), and $\Xi_y = (I_y, P_{y,0}, \dots, P_{y,3})$.

Lemma 595. (i) *Every plaquette has exactly one owner, and* $|\Pi(y)| = 96$.

(ii) *Every* $p \in \Pi(y)$ *in the plane* $\{\mu,\nu\}$ *contains an active link with base in* $B(y)$*, and each of its links has its base in* $B(y)$*,* $B(y+e_\mu)$ *or* $B(y+e_\nu)$.

(iii) *The variables* Ξ_y*,* $y \in T$*, are the active links, each once. The factor* $f_y = \exp(-\lambda\sum_{p\in\Pi(y)}\delta(U_p,I)^2)$ *and the indicator* r_y *of* $\{y$ *rough*$\}$ *are functions of* Ξ_y *and* Ξ_{y+e_μ}*,* $0 \leq \mu \leq 3$*, and* $e^{-\lambda Q} = \prod_y f_y$.

Proof. (i) A plaquette has one corner, and each of the 16 sites of a block is the corner of six plaquettes. (ii) Let o be the offset of the corner x. If $o_\mu = 1$ the link (x,μ) leaves $B(y)$ and is private, and likewise if $o_\nu = 1$. If $o_\mu = o_\nu = 0$ the four vertices lie in $B(y)$ and the plaquette is a cycle in the block, which the fifteen tree links, a tree by Lemma 329(i), cannot contain; so one of its links, all based in $B(y)$, is active. The links of p are based at x, $x+e_\mu$ and $x+e_\nu$. (iii) A link based in $B(y)$ is a tree link, an active link inside $B(y)$, or a private link of a pair (y,μ), by Lemma 329(i); the rest follows from (ii). □

A computer enumeration of the 96 plaquettes of a block agrees with (i) and (ii); the proof does not use it.

Suppression. For a set X of blocks let A_X, Q_X, ν_c^X, π_E and $Q_*(E)$ be as in Sec. XII E with $W = X$, so that ν_c^X lives on the fibre of X with the links off A_X fixed.

Theorem 596. *Let* $G = \mathrm{SU}(2)$*,* $\kappa > 0$*,* $\lambda > 0$ *and let* X *be a set of blocks.*

(i) *For every* c *and every choice of the links off* A_X*,*

$$\int \mathbf{1}\{X \subset R(U)\}\, e^{-\lambda Q_X}\, d\nu_c^X \leq M^{4|X|} e^{-\lambda\kappa^2|X|},$$

with $M < 2.3621\cdot 10^4$ *the constant of Lemma 373.*

(ii) *Let* $\kappa \leq 1$ *and* $0 < \tau \leq 1$*. If the exterior* E *of* X *has* $Q_*(E) \leq (1-\tau)\kappa^2|X|$*, then*

$$\pi_E\{X \subset R\} \leq \Big(\frac{5.9\cdot 10^{14}}{(\tau\kappa^2)^3}\Big)^{49|X|} e^{-3\lambda\tau\kappa^2|X|/4}.$$

(iii) *If* $\beta_W \geq 1$ *and* $\kappa \leq 1$*, then* $\mu_L\{X \subset R\} \leq (16\, q_\kappa(\beta_W))^{|X|}$*, with* q_κ *as in Theorem 383.*

Proof. On $\{X \subset R\}$ each block of X owns a plaquette with $\delta(U_p, I) > \kappa$; these plaquettes are distinct and, by Lemma 595(ii), contain links of A_X. Hence $Q_X > \kappa^2|X|$ there.

(i) The integrand is at most $e^{-\lambda\kappa^2|X|}$, and ν_c^X has mass at most $M^{4|X|}$ by (39).

(ii) With $t = \tau\kappa^2|X|$, $\{X \subset R\} \subset \{Q_X > Q_*(E) + t\}$. The proof of Proposition 379 uses $t \leq 1$ only to have $\rho =$

$t/(90.6 n_P) < 1.5\cdot 10^{-4}$. Each of the $49|X|$ links of A_X lies in six plaquettes and a plaquette contains at most four of them, so $n_P \geq 73.5|X|$ and $\rho \leq \tau\kappa^2/6659 < 1.51\cdot 10^{-4}$, which still gives $16\sqrt{2}\rho + 16\rho^2 < 22.63\rho$. The proof therefore gives the bound $(14\cdot 90.6^3 n_P^3/t^3)^{49|X|} e^{-3\lambda t/4}$, and $n_P \leq 384|X|$ with $14\cdot 90.6^3\cdot 384^3 < 5.9\cdot 10^{14}$.

(iii) The plaquette at x lies in the unit cube $\mathscr{C}_x$, so a rough block y lies in $\bigcup_{x\in B(y)} B_x$. Expanding the intersection over $y\in X$ of these unions gives at most $16^{|X|}$ intersections $\bigcap_{y\in X} B_{x(y)}$ with $x(y)\in B(y)$, each over $|X|$ distinct cubes, and Theorem 383 bounds each by $q_\kappa^{|X|}$. □

Part (i) is uniform in the exterior and in c, but it is an absolute bound: it does not compare rough with calm completions. The comparison (ii) needs its hypothesis. For the frustrated exterior described after Corollary 389, a plaquette through the link e is bad with conditional probability tending to one. No conditional bound uniform in the exterior holds.

Coarse curvature. For a coarse plaquette P let $N(P)$ be the fine plaquettes with a vertex in one of the 144 blocks at ℓ^∞ distance at most one from the four blocks of P, the witness region of Lemma 409, and let N_{ov} be the largest number of witness regions that contain one fine plaquette. Then $N_{\mathrm{ov}} = 1441$, and 2310 witness regions meet $\Pi(y)$. Indeed, let P have the blocks z, $z+e_\mu$, $z+e_\nu$, $z+e_\mu+e_\nu$, with $B(z) = 2z + \{0,1\}^4$. The 144 blocks fill the box of fine sites w with $w_i - 2z_i \in [-2,5]$ for $i\in\{\mu,\nu\}$ and $w_i - 2z_i \in [-2,3]$ otherwise, and the plaquette at x in the plane $\{\alpha,\beta\}$ has a vertex in it exactly when, for each i, x_i lies in the i-th interval, enlarged by one site to the left when $i\in\{\alpha,\beta\}$. For fixed x the number of admissible z_i is 4 for $i\in\{\mu,\nu\}$ and 3 otherwise, plus one when $i\in\{\alpha,\beta\}$ and x_i is odd. With x_α, x_β odd the six planes $\{\mu,\nu\}$ give $5\cdot 5\cdot 3\cdot 3$, four times $5\cdot 4\cdot 4\cdot 3$ and 4^4, which sum to $225+960+256 = 1441$, and no parity gives more. For the second count take $y=0$: the plaquettes of $\Pi(0)$ have corners in $\{0,1\}^4$ and any plane. For each i the admissible z_i are those with $z_i\in[-2,1]$ for $i\in\{\mu,\nu\}$ and $z_i\in[-1,1]$ otherwise, and $z_i = 2$, which needs $x_i = 1$ and $i\in\{\alpha,\beta\}$; so z is admissible exactly when at most two coordinates equal 2. Of the $5\cdot 5\cdot 4\cdot 4 = 400$ points of the enlarged box, $8+6+1 = 15$ have three or four coordinates equal to 2, and the six planes give $6\cdot 385 = 2310$.

Lemma 597. *Let $G = \mathrm{SU}(2)$, $C_E = 8/\kappa_c^2 < 1140.26$ and $C_* = 1441\, C_E < 1.6432\cdot 10^6$. For every field U in the tree gauge, $c = V^\sharp(U)$ and every coarse plaquette P,*

$$\delta(c_P, I)^2 \leq C_E \max_{p\in N(P)} \delta(U_p, I)^2,$$

and if $\kappa < \kappa_c$ and $\delta(c_P,I) > 16.003\,\kappa$, some $p\in N(P)$ has $\delta(U_p,I) > \kappa$. Consequently $Q_1(c) \leq C_*\, Q(U)$, and

$$Q_1^{>}(c) := \sum_{\delta(c_P,I)>16.003\kappa} \delta(c_P,I)^2 \leq C_* \sum_{\delta(U_p,I)>\kappa} \delta(U_p,I)^2.$$

In particular $Q(U) \geq r^2/1140.26$ whenever some coarse plaquette has $\delta(c_P,I)\geq r$.

Proof. Let m be the maximum over $N(P)$. If $m < \kappa_c$, then Proposition 470(iv), applied with every $\kappa\in(m,\kappa_c)$, gives $\delta(c_P,I) \leq 16.003\, m$, and $16.003^2 < 256.1 < C_E$. If $m\geq\kappa_c$, then $\delta(c_P,I)\leq 2\sqrt{2}$, the diameter of SU(2), so $\delta(c_P,I)^2 \leq 8 \leq C_E m^2$. The second statement is the same proposition. Summing $\delta(c_P,I)^2 \leq C_E \sum_{p\in N(P)} \delta(U_p,I)^2$ over P counts each fine plaquette at most N_{ov} times; for $Q_1^>$ keep in each sum only the plaquettes above κ, among which the maximum lies. □

The last statement is the first step of Proposition 402 for SU(2), with 1140.26 in place of 1154; it is a bound for one step only.

The coarse action. Since $h = \int \nu_c\, dh(c)$, the law of $c = V^\sharp(U)$ under the Wilson measure is $Z_L^{-1} e^{-A_1(c)}\, dh(c)$ with

$$e^{-A_1(c)} = Z(c) = \nu_c\big(e^{-\lambda Q}\big).$$

Let $S_1(c) = \inf_{F_c} Q$. In the reconstruction $R(c)$ of Sec. XII B, with $c_{y,\mu}$ on the private links of (y,μ) and I on the links inside the blocks, every path of $V^\sharp_{y,\mu}$ has holonomy $c_{y,\mu}$. So $v_x = I$ at every site, $R(c)$ is regular, and $V^\sharp(R(c)) = c$. A plaquette at x in the plane $\{\mu,\nu\}$ of $R(c)$ is c_P when $o_\mu = o_\nu = 1$ and I otherwise, so $Q(R(c)) = 4Q_1(c)$.

Theorem 598. *Let $G = \mathrm{SU}(2)$. For every coarse field c:*

(i) *$A_1(c) \geq \lambda S_1(c) - 4|T|\log M$, and $S_1(c) \geq Q_1(c)/C_*$;*

(ii) *for $\lambda \geq 100$, $\eta > 0$ and every regular $U\in F_c$, $A_1(c) \leq (1+\eta)\lambda Q(U) + c_1(\lambda,\eta)|T|$, where*

$$c_1(\lambda,\eta) = 12\log 1.957 + 45\log 14 + 67.5\big(1+\log(11520(1+\eta^{-1})\lambda)\big);$$

in particular $A_1(c) \leq 8\lambda Q_1(c) + c_1(\lambda,1)|T|$.

Proof. (i) $Z(c) \leq \nu_c(F_c) e^{-\lambda S_1(c)}$ and $\nu_c(F_c) \leq M^{4|T|}$ by (39); then Lemma 597.

(ii) Let $\rho^2 = 135/(2\cdot 96\cdot 8100(1+\eta^{-1})\lambda)$, so $\rho \leq 9.32\cdot 10^{-4}$ and $46\rho < 0.043$. The proof of Proposition 377 with $W = T$ uses $t\leq 1$ only through this bound on 46ρ. It gives a set $\Gamma(D)\subset F_c$ with $\nu_c(\Gamma(D)) \geq 1.957^{-12|T|}(\rho^3/14)^{45|T|}$ on which each of the $n_P = 96|T|$ plaquettes moves by at most 90ρ. There $\delta(U'_p,I) \leq \delta(U_p,I) + 90\rho$, so

$$Q(U') \leq (1+\eta)Q(U) + (1+\eta^{-1})\,96|T|\,8100\rho^2.$$

Hence $A_1(c) \leq (1+\eta)\lambda Q(U) + |T| g(\rho)$ with $g(\rho) = (1+\eta^{-1})777600\lambda\rho^2 + 135\log\rho^{-1} + 12\log 1.957 + 45\log 14$, and the chosen ρ minimises g, with value $c_1(\lambda,\eta)$. For the last statement take $U = R(c)$ and $\eta = 1$. □

So the coarse action is λ times a quantity between Q_1/C_* and $8Q_1$, up to $O(\log\lambda)$ per block; the volume term $67.5\log\lambda$ is the Gaussian count of the 135 real fluctuation coordinates of a block. Neither constant is near one. For example $c_1(10^{12},1) < 2737.5$ and $4\log M < 40.35$.

The large-field representation. Write $s_y = 1 - r_y$. For $X \subset T$ let $X_1, \dots, X_n$ be its components for ℓ^∞ adjacency of blocks, so that distinct components are at ℓ^∞ distance at least two. For a set Y of blocks let $\partial^- Y$ be the set of blocks $z \notin Y$ with $z + e_\mu \in Y$ for some μ; a block of $\partial^- X_i$ lies off X. By the definition of ν_c,

$$\nu_c(d\Xi) = \prod_y h(dI_y) \prod_{y,\mu} \nu_c^{y,\mu}(dP_{y,\mu} \mid I_y, I_{y+e_\mu}),$$

where $\nu_c^{y,\mu}$ is the measure of (38) on the private links of the pair, of mass at most M.

Let $\partial^= X$ be the set of blocks $w \notin X \cup \partial^- X$ with $w + e_\mu + e_\nu \in X$ for some $\mu \neq \nu$, and put $\hat{X} = X \cup \partial^- X \cup \partial^= X$. Let $\mathscr{P}_X$ be the set of pairs from X together with the pairs (z,μ) with $z \notin X$ and $z + e_\mu \in X$, and let $\mathscr{V}_X$ be the links inside the blocks of X together with the private links of the pairs of $\mathscr{P}_X$: these are the variables that the kernel of X in Theorem 600 integrates.

Lemma 599. (i) *The factors f_y and r_y are functions of I_y, of the $P_{y,\mu}$, and of I_{y+e_ν} and $P_{y+e_\nu,\mu}$ for $\mu \neq \nu$. The average $V^\sharp_{y,\mu}$ is a function of $P_{y,\mu}$, I_y and I_{y+e_μ}.*

(ii) *The factor f_w involves a link of $\mathscr{V}_X$ if and only if $w \in \hat{X}$.*

(iii) *Let the components X_i of X be at ℓ^∞ distance at least two. Then $\hat{X}$ is the disjoint union of the $\hat{X}_i$, each block of $\hat{X}_i$ lies within ℓ^∞ distance one of X_i, and for $j \neq i$ no factor f_w or r_w with $w \in \hat{X}_i$ involves a link of $\mathscr{V}_{X_j}$.*

(iv) *Every plaquette that contains a link of $\mathscr{V}_X$ is owned by a block of $\hat{X}$ and has a vertex in a block of X. There are at most $216|X|$ such plaquettes, every plaquette of $\Pi(X)$ is one of them, and $|\mathscr{P}_X| \le 8|X|$.*

Proof. (i) Let p be the plaquette at $x \in B(y)$ in the plane $\{\mu,\nu\}$. By Lemma 595(ii) its links are based in $B(y)$, $B(y+e_\mu)$ or $B(y+e_\nu)$. The one based in $B(y+e_\nu)$ is $(x+e_\nu,\mu)$, whose offset in the direction μ is that of x: it is a link inside $B(y+e_\nu)$ or a tree link if $o_\mu = 0$, and a private link of $(y+e_\nu,\mu)$ if $o_\mu = 1$. The links based in $B(y)$ are inside $B(y)$ or private to pairs from y. The average is the mean over the paths $(x,\mu),(x+e_\mu,\mu)$, $x \in B(y)$, of Lemma 329(iii). (ii) By (i), f_w involves a variable of Ξ_y with $y \in X$ if and only if $w = y$ or $w + e_\nu = y$, that is $w \in X \cup \partial^- X$. It involves a private link of a pair (z,μ) with $z \notin X$, $z + e_\mu \in X$ if and only if $w = z$ or $w = z - e_\nu$ with $\nu \neq \mu$; the first gives $w \in \partial^- X$ and the second $w + e_\mu + e_\nu \in X$. (iii) Every block of $\hat{X}_i$ has the form x, $x - e_\mu$ or $x - e_\mu - e_\nu$ with $x \in X_i$, so it lies within distance one of X_i, and a block of $\hat{X}_i \cap \hat{X}_j$ would give blocks of X_i and X_j at distance at most one. For the same reason a block of $\partial^- X_i$ or $\partial^= X_i$ lies in no X_j and in no $\partial^- X_j$, so $\hat{X} = \bigcup_i \hat{X}_i$. By (ii) for X_j, a factor that involves a link of $\mathscr{V}_{X_j}$ belongs to a block of $\hat{X}_j$, which is not in $\hat{X}_i$. (iv) A link of $\mathscr{V}_X$ is based in a block of X, or is a private link (x,μ) of a pair (z,μ) with $z + e_\mu \in X$, which ends at $x + e_\mu \in B(z+e_\mu)$. The owner of a plaquette that contains it involves it, so it lies in $\hat{X}$ by (ii). Since $\mathscr{V}_X \subset \bigcup_{x \in X} \mathscr{V}_{\{x\}}$, the count is at most $|X|$ times the number of plaquettes with a vertex in one block $B(y) = 2y + \{0,1\}^4$. The plaquette at c in the plane $\{\alpha,\beta\}$ has a vertex in $B(y)$ exactly when $c_i - 2y_i \in \{0,1\}$ for $i \notin \{\alpha,\beta\}$ and $c_i - 2y_i \in \{-1,0,1\}$ for $i \in \{\alpha,\beta\}$, so there are $6 \cdot 2^2 \cdot 3^2 = 216$ of them. There are $4|X|$ pairs from X, and $(z,\mu) \mapsto (z + e_\mu, \mu)$ is injective on the others. The plaquettes of $\Pi(X)$ contain an active link with base in a block of X, by Lemma 595(ii). □

A computer enumeration agrees with (i), and with (ii)–(iv) for 300 random ℓ^∞-connected sets of up to seven blocks and for 300 random pairs of such sets at distance at least two; the proof does not use it.

Theorem 600. *Let $G = \mathrm{SU}(2)$ and $\kappa > 0$.*

(i) *For every c, $Z(c) = \sum_{X \subset T} Z_X(c)$ with*

$$Z_X(c) = \int \nu^{\mathrm{out}}_{c,X}(d\Xi) \prod_{y \notin \hat{X}} f_y s_y \prod_i K_{X_i}(c;\Xi),$$

with $\hat{X} = \bigcup_i \hat{X}_i$ as in Lemma 599. Here $\nu^{\mathrm{out}}_{c,X}$ is the product of the factors of ν_c for the links inside the blocks off X and for the pairs with both blocks off X. The kernel K_{X_i} is the integral of $\prod_{y \in X_i} f_y r_y \prod_{z \in \hat{X}_i \setminus X_i} f_z s_z$ against the factors of ν_c for the links inside X_i, the pairs from X_i and the pairs (z,μ) with $z \in \partial^- X_i$ and $z + e_\mu \in X_i$. It depends on the integration variable of $\nu^{\mathrm{out}}_{c,X}$ only through the blocks at ℓ^∞ distance one from X_i. No two kernels share a factor or an integration variable; two kernels may share boundary arguments, the inside links of a block between them, which the common outer integral integrates.

(ii) *For $\kappa < \kappa_c$, uniformly in the outer variables,*

$$0 \le K_{X_i} \le M^{8|X_i|} \exp\Big(-\lambda \max\Big\{\kappa^2 |X_i|, \frac{Q^{(i)}(c)}{C_*}\Big\}\Big),$$

where $Q^{(i)}(c)$ is the sum of $\delta(c_P, I)^2 > (16.003\kappa)^2$ over the coarse plaquettes P with $\emptyset \neq N(P) \cap \Pi(X) \subset \Pi(X_i)$.

(iii) *With $N_\lambda = e^{-c_1(\lambda,1)}$, $K_{X_i} \le N_\lambda^{|X_i|} e^{-a(\lambda,\kappa)|X_i|}$, where*

$$a(\lambda,\kappa) = \lambda\kappa^2 - 8\log M - c_1(\lambda,1),$$

and $Z(c) \ge N_\lambda^{|T|} e^{-8\lambda Q_1(c)}$ for $\lambda \ge 100$.

Proof. (i) Expand $\prod_y (f_y r_y + f_y s_y)$ and group the terms by $X = \{y : r_y\}$; then apply Fubini to the nonnegative integrands. By Lemma 595(iii), f_y and r_y involve Ξ_y and Ξ_{y+e_μ}, and $\nu_c^{y,\mu}$ involves $P_{y,\mu}$, I_y and I_{y+e_μ}. By Lemma 599(ii) and (iii), the factors that involve a variable integrated by the kernel of X_i are those of the blocks of $\hat{X}_i$, and none of them involves a variable integrated by the kernel of X_j, $j \neq i$. The measure of a pair into X_i involves its private links and the inside links of its two blocks, and a pair is into or from at most one component. (ii) The measures in K_{X_i} have mass at most M for each of the $4|X_i|$ pairs from X_i and the at most $4|X_i|$ pairs into it, and h is a probability. The integrand is at most $e^{-\lambda Q_{\Pi(X_i)}}$ and vanishes unless every block of X_i is rough. Then $Q_{\Pi(X_i)} >$

$\kappa^2|X_i|$, and every fine plaquette above κ is owned by a rough block. So for the P counted in $Q^{(i)}$ the plaquettes above κ in $N(P)$ lie in $\Pi(X_i)$, and Lemma 597 gives $Q^{(i)}(c) \le C_* Q_{\Pi(X_i)}$. (iii) The first bound is (ii) with the definition of a; the second is Theorem 598(ii) with $U = R(c)$. □

The representation is exact, and every kernel carries $e^{-\lambda\kappa^2}$ per block, uniformly in its boundary values and in c. When the coarse field is curved its kernel also carries $e^{-\lambda Q^{(i)}/C_*}$. The small-field factor $\prod f_y s_y$, however, is not a product of local factors once the outer variables are integrated. The comparison of a kernel with the calm weight S_{X_i}, the same integral with s_y in place of r_y on X_i, would give the relative activities K_{X_i}/S_{X_i}. It fails where $S_{X_i} = 0 < K_{X_i}$, that is, where the boundary values admit no calm completion; Theorem 596(ii) bounds the ratio only under its hypothesis.

Theorem 601. *Call two nonempty ℓ^∞-connected sets of blocks compatible when their ℓ^∞ distance is at least two, and let w be activities with $|w(X)| \le e^{-9|X|}$.*

(i) *The criterion of Kotecký and Preiss [78] holds with $a(X) = |X|/10$. So $\log\sum\prod_i w(X_i)$, the sum over families of pairwise compatible sets, is the absolutely convergent sum of the connected clusters. The clusters incompatible with a given block contribute at most $1/10$ in absolute value, and the logarithm is at most $|T|/10$ in modulus.*

(ii) *The rate of Theorem 600(iii) satisfies $a(\lambda,\kappa) \ge 9$ for $\lambda \ge \max\{\lambda^*(\kappa), 100\}$, with $\beta^* = 4\lambda^*$ given below.*

κ	β^*	$\lambda^*\kappa^2$
0.08	$1.13125\cdot10^6$	1810
10^{-2}	$8.404\cdot10^7$	2101
10^{-4}	$1.0964\cdot10^{12}$	2741
10^{-8}	$1.604\cdot10^{20}$	4010

Proof. (i) A connected set of n blocks through a given block is recovered from a walk of length $2(n-1)$ around a spanning tree, with 80 neighbours at each step, as in Corollary 384; so there are at most 6400^{n-1}. A set incompatible with X contains one of the at most $81|X|$ blocks within distance one of X. Hence

$$\sum_{X' \not\sim X} |w(X')| e^{|X'|/10} \le 81|X| \frac{u}{1-6400u},$$

with $u = e^{-8.9}$. Since $8.9 > \log 7290 = \log 2 + 6\log 3 + \log 5$, $u < 1/7290$, and the right side, which increases with u, is below $81|X|/(7290-6400) < 0.092|X| < |X|/10$. This is the hypothesis of the theorem of Kotecký and Preiss, whose conclusion is the statement.

(ii) Write $L_p = \log p$. With its tail bounded geometrically, the series $\log\frac{1+x}{1-x} = 2\sum_{k\ge0}\frac{x^{2k+1}}{2k+1}$ at $x = \frac13, \frac15, \frac19$ and $\frac{1}{99}$ gives $\log 2$, $\log\frac32$, $\log\frac54$ and $\log\frac{50}{49}$ to within 10^{-10}, hence

$$0.69314718 < L_2 < 0.69314719,$$
$$1.09861228 < L_3 < 1.09861229,$$
$$1.60943791 < L_5 < 1.60943792,$$
$$1.94591014 < L_7 < 1.94591015,$$

and $\log 10 = L_2 + L_5$. By Lemma 373, $\log M < \log(2.4\cdot10^4) = 6L_2 + L_3 + 3L_5$, and $\log 1.957 < \log 1.96 = 2L_7 - 2L_5$. With $\log 14 = L_2 + L_7$ and $\log 23040 = 9L_2 + 2L_3 + L_5$, for $\lambda \ge 100$

$$8\log M + c_1(\lambda,1) \le K_0 + 67.5\log\lambda,$$

where $K_0 = 67.5 + 700.5L_2 + 143L_3 + 67.5L_5 + 69L_7 < 953.06$. Put $t = \lambda\kappa^2$. Then $a(\lambda,\kappa) \ge t - 67.5\log t - K_0 - 67.5\log\kappa^{-2}$, whose right side increases with t for $t > 67.5$. So $a \ge 9$ for $\lambda \ge \max\{\lambda^*, 100\}$ once $t^* = \lambda^*\kappa^2$ satisfies

$$t^* - 67.5\log t^* \ge 962.06 + 67.5\log\kappa^{-2}.$$

In the four rows of the table $\log\kappa^{-2}$ equals $4L_5 - 2L_2$, $4\log 10$, $8\log 10$ and $16\log 10$, and, by $\log(1+x) \le x$, $\log t^*$ is at most $\log 1800 + \frac{1}{180}$, $\log 2100 + \frac{1}{2100}$, $\log 2744$ and $\log 4000 + \frac{1}{400}$, with $1800 = 2^3 3^2 5^2$, $2100 = 2^2\cdot3\cdot5^2\cdot7$, $2744 = 2^3 7^3$ and $4000 = 2^5 5^3$. With these bounds the left side exceeds the right by more than 0.6 in each row. □

Theorem 601 applies to the large fields once the small-field part is in local form. Then each region costs its kernel divided by the calm weight of the same region, and $a \ge 9$ beats the entropy of connected regions if that calm weight is at least N_λ per block. This lower bound on the calm weight with boundary values is not proved. It is the small-field cluster expansion of item (iv) of Sec. XIII V, and it must have the relative and local form required by Propositions 460 and 461 and Lemma 459.

The small-field part. The rest of this subsection treats the factor $\prod f_y s_y$. It proves three things. First, a kernel is at most $e^{-a_2|X|}$ times the calm weight of its region whenever the boundary values admit a completion that is calm at a smaller threshold κ'. Second, the large-field sum is exactly the calm weight times a mean, under the calm law, of products of these ratios, and the polymer sums of Theorem 601 bound it once the ratios are small. Third, to leading order in λ the logarithm of the calm weight is $-\lambda$ times the Gaussian block-spin form, so that the calm weight is a product of local factors to within $e^{\pm o(\lambda)}$ only if this form does not couple distant coarse links; its coefficient at two coarse links three blocks apart is a rational number $(M_W)_{AB}$, which a computation not used in the proofs finds nonzero. What remains open is stated at the end.

The calm weight. Fix $X \ne \emptyset$ and the outer variables of K_X. Let λ_X be the measure of K_X, the product of h on the links inside the blocks of X and of the measures ν_c^k of the pairs $k \in \mathscr{P}_X$, of mass at most $M^{8|X|}$. The *calm weight* is

$$S_X = \int \prod_{y\in\hat X} f_y s_y \, d\lambda_X,$$

the integral of K_X with s_y in place of r_y on X. A plaquette of $\Pi(\hat X)$ is *moved* if it contains a link of $\mathscr{V}_X$ and *fixed* otherwise; Q_X^{mov} and Q_X^{fix} are the sums of $\delta(U_p, I)^2$ over the moved and the fixed plaquettes, and Q_X^{fix} is a function of the outer variables. A *completion* is a value of $\mathscr{V}_X$ at which the averages of the pairs of $\mathscr{P}_X$ equal c. Put

$$c_2(\lambda,\eta) = 24\log 1.957 + 73\log 14 + 109.5\big(1 + \log(15979(1+\eta^{-1})\lambda)\big).$$

Theorem 602. *Let $G=\mathrm{SU}(2)$, $0<\kappa'<\kappa$, $\eta>0$, and $\lambda\geq 100$ with $\lambda(\kappa-\kappa')^2\geq 1$. Suppose that the fixed plaquettes of $\Pi(\hat X)$ are at most κ, which holds whenever $K_X>0$, and that there is a completion U^0 with*

(a) *$\delta(v_x,I)\leq\frac14$ at every site of every pair of $\mathscr{P}_X$;*

(b) *$\delta(U^0_p,I)\leq\kappa'$ for every moved plaquette.*

Then

$$S_X\geq e^{-c_2(\lambda,\eta)|X|}e^{-\lambda Q_X^{\mathrm{fix}}}e^{-(1+\eta)\lambda Q_X^{\mathrm{mov}}(U^0)},$$

and

$$\frac{K_X}{S_X}\leq\exp\Big(-\big(\lambda\kappa^2-8\log M-c_2(\lambda,\eta)\big)|X|+(1+\eta)\lambda Q_X^{\mathrm{mov}}(U^0)\Big)\leq e^{-a_2|X|},$$

where $a_2(\lambda,\kappa)=\lambda\kappa^2\big(1-216(1+\eta)\kappa'^2/\kappa^2\big)-8\log M-c_2(\lambda,\eta)$.

Proof. Let $\rho^2=1/(15978.08(1+\eta^{-1})\lambda)$. Then $\rho\leq 7.92\cdot10^{-4}$, and $90\rho\leq 0.712\lambda^{-1/2}<\kappa-\kappa'$. Let L' be the links of $\mathscr{V}_X$ other than the path links of the pairs of $\mathscr{P}_X$, $17|X|+7|\mathscr{P}_X|\leq 73|X|$ in number, and D the set of $z\in G^{L'}$ with $d(z_e,U^0_e)\leq\rho$ for every $e\in L'$. For $z\in D$ and a pair k, each site v_x of k is ℓ^{-1} times a product of at most two links of L' and outer links, ℓ the path link. As in the proof of Proposition 374, with $R=44\rho$ the sites lie within $46\rho<0.043$ of their values at U^0, so $\delta(v_x,I)<\frac12$ by (a), the cut-off equals one, and the contraction argument of that proof solves each path link in the ball of radius R about its value at U^0. This gives an injective map Γ from D into the completions, near whose image the averages are smooth with $|\det\partial_{d_k}V_k^\sharp|\leq 1.957^3$. The coarea form of ν_c^k stated after (39), applied to each pair of $\mathscr{P}_X$ with the inside links of its blocks given, shows that λ_X restricted to $\Gamma(D)$ is the image of $h^{\otimes L'}$ on D divided by $\prod_k|\det\partial_{d_k}V_k^\sharp|$. Hence

$$\lambda_X\big(\Gamma(D)\big)\geq 1.957^{-3|\mathscr{P}_X|}\Big(\frac{\rho^3}{14}\Big)^{|L'|}\geq 1.957^{-24|X|}\Big(\frac{\rho^3}{14}\Big)^{73|X|}.$$

A plaquette has at most two path links, so on $\Gamma(D)$ each moved plaquette moves by at most 90ρ, as in that proof, and stays at most $\kappa'+90\rho<\kappa$; the fixed ones are at most κ. Every block of $\hat X$ is therefore calm on $\Gamma(D)$, and by Lemma 599(iv),

$$Q_X^{\mathrm{mov}}\leq(1+\eta)Q_X^{\mathrm{mov}}(U^0)+(1+\eta^{-1})\,8100\rho^2\cdot216|X|.$$

So $S_X\geq e^{-g|X|}e^{-\lambda Q_X^{\mathrm{fix}}}e^{-(1+\eta)\lambda Q_X^{\mathrm{mov}}(U^0)}$ with $g=24\log1.957+73\log14+219\log\rho^{-1}+(1+\eta^{-1})1749600\lambda\rho^2$. With the chosen ρ the last term is 109.5 and $g\leq c_2(\lambda,\eta)$, as $2\cdot216\cdot8100/219<15978.09$.

The integrand of K_X is at most $e^{-\lambda Q_X^{\mathrm{fix}}}e^{-\lambda Q_X^{\mathrm{mov}}}$ and vanishes unless every block of X is rough. Then each block of X owns a plaquette above κ, these are distinct, and they are moved by Lemma 599(iv), so $Q_X^{\mathrm{mov}}>\kappa^2|X|$. With λ_X of mass at most $M^{8|X|}$, $K_X\leq M^{8|X|}e^{-\lambda Q_X^{\mathrm{fix}}}e^{-\lambda\kappa^2|X|}$. Dividing gives the first bound, and (b) with Lemma 599(iv) gives $Q_X^{\mathrm{mov}}(U^0)\leq 216\kappa'^2|X|$. □

So the calm weight with given boundary values is at least e^{-c_2} per block, times the Gaussian factor of the completion, whenever the boundary values admit a completion calm at κ'; the constant c_2 takes the place of c_1 because K_X integrates up to 73 rather than 45 links per block. With $\kappa'=\kappa/30$ and $\eta=1$, $a_2=0.52\lambda\kappa^2-8\log M-c_2(\lambda,1)$, and $a_2\geq9$, the rate that Theorem 601 needs, holds for $\lambda\geq\lambda^{**}(\kappa)$, with $\beta^{**}=4\lambda^{**}$ as follows:

κ	β^{**}	$\lambda^{**}\kappa^2$
0.08	$3.6625\cdot10^6$	5860
10^{-2}	$2.7064\cdot10^8$	6766
10^{-4}	$3.504\cdot10^{12}$	8760
10^{-8}	$5.0872\cdot10^{20}$	12718

Indeed, as in the proof of Theorem 601(ii), now with $2\cdot15979<32000=2^85^3$, $8\log M+c_2(\lambda,1)\leq K_2+109.5\log\lambda$ for $\lambda\geq100$, where $K_2=109.5+997L_2+8L_3+304.5L_5+121L_7<1534.89$, and $0.52t-109.5\log t$ increases for $t>211$. In the four rows, with $t=\lambda^{**}\kappa^2$, $\log t$ is at most $\log5832+\frac{7}{1458}$, $\log6750+\frac{8}{3375}$, $\log8750+\frac{1}{875}$ and $\log12600+\frac{59}{6300}$, where $5832=2^33^6$, $6750=2\cdot3^35^3$, $8750=2\cdot5^47$ and $12600=2^33^25^27$, and $0.52t-109.5\log t$ exceeds $1543.89+109.5\log\kappa^{-2}$ by more than 0.14. At these couplings $\rho\leq(\kappa-\kappa')/90$ and $\rho\leq7.92\cdot10^{-4}$. The same construction bounds the calm weight of the whole torus.

Corollary 603. *Let $G=\mathrm{SU}(2)$, $0<\kappa'<\kappa<\kappa_c$, $\eta>0$ and $\lambda\geq100$ with $\lambda(\kappa-\kappa')^2\geq1$, and let $Z_\emptyset(c)=\nu_c(\prod_y f_ys_y)$ be the calm weight of c. If $\delta(c_P,I)\leq\kappa'$ for every coarse plaquette, then*

$$Z_\emptyset(c)\geq e^{-c_1(\lambda,\eta)|T|}e^{-4(1+\eta)\lambda Q_1(c)},$$

in particular $Z_\emptyset(c)\geq N_\lambda^{|T|}e^{-8\lambda Q_1(c)}$. If $\delta(c_P,I)>16.003\kappa$ for some P, then $Z_\emptyset(c)=0$.

Proof. The plaquettes of the regular field $R(c)$ are the c_P and I. In the proof of Theorem 598(ii) with $U=R(c)$, $\rho\leq(11520\lambda)^{-1/2}$, so the plaquettes move by at most $90\rho<\kappa-\kappa'$ on $\Gamma(D)$, every block is calm there, and the bound of that proof holds for $Z_\emptyset$. If $\delta(c_P,I)>16.003\kappa$, every field of F_c has a plaquette above κ by Lemma 597, so $\prod_y s_y=0$ on F_c. □

So the class of coarse fields for which the calm weight can serve as a reference lies between the fields with all coarse plaquettes at most κ' and those with all at most 16.003κ. For $Z_\emptyset(c)>0$ let $\mu_c^\circ=Z_\emptyset(c)^{-1}\prod_y f_ys_y\,\nu_c$ be the calm law, and for each X and each value of its outer variables with $S_X>0$ let $\rho_X=K_X/S_X$.

Proposition 604. *Let $G=\mathrm{SU}(2)$, $\kappa>0$ and $Z_\emptyset(c)>0$.*

(i) *$Z(c)=Z_\emptyset(c)\,\Xi(c)+Z_\perp(c)$, where*

$$\Xi(c)=\sum_{X\subset T}\mathbb{E}_{\mu_c^\circ}\Big[\prod_i\rho_{X_i}\Big],$$

the X_i being the components of X, ρ_{X_i} is defined μ_c°-almost surely, and $Z_\perp(c)\geq0$ is the sum over X of the integrals of Theorem 600(i), in the form of Lemma 599(iii),

restricted to the outer variables at which some S_{X_i} vanishes.

(ii) *Suppose that for every nonempty ℓ^∞-connected X, $\rho_X \le e^{-9|X|}$ holds μ_c°-almost surely. Then $\Xi(c) = \mathbb{E}_{\mu_c^\circ}[e^\Lambda]$, where Λ is the sum of the connected clusters of the activities ρ_X given by Theorem 601(i). Each cluster depends on the field only within distance one of its support, the clusters incompatible with a given block add up to at most $1/10$, $0 \le \Lambda \le |T|/10$, and*

$$A_1(c) = -\log Z_\emptyset(c) - \log \mathbb{E}_{\mu_c^\circ}\big[e^\Lambda\big] - \log\Big(1 + \frac{Z_\perp(c)}{Z_\emptyset(c)\Xi(c)}\Big),$$

with $0 \le \log \mathbb{E}_{\mu_c^\circ}[e^\Lambda] \le |T|/10$.

Proof. (i) Fix X. The integrand $\prod_y f_y s_y$ of $Z_\emptyset$ factorises as the integrand of Z_X does in Lemma 599(iii), with s_y in place of r_y on X, so by Fubini the law of the outer variables of X under μ_c° is $Z_\emptyset^{-1}\Phi_X \prod_i S_{X_i}\, \nu_{c,X}^{\mathrm{out}}$, with Φ_X the outer product. This law gives no mass to $\{S_{X_i} = 0\}$, and

$$Z_X = \int \nu_{c,X}^{\mathrm{out}} \Phi_X \prod_i K_{X_i}\big(\mathbf{1}\{\text{all } S_{X_i} > 0\} + \mathbf{1}\{\text{some } S_{X_i} = 0\}\big).$$

The first part is $Z_\emptyset\, \mathbb{E}_{\mu_c^\circ}[\prod_i \rho_{X_i}]$; sum over X. (ii) For a field U let $\rho_X(U)$ be ρ_X at the outer variables of X in U. On a set of full μ_c°-measure all these finitely many values are at most $e^{-9|X|}$, and there $\sum_X \prod_i \rho_{X_i}(U) = e^{\Lambda(U)}$ by Theorem 601(i), with the stated bounds; $\Lambda \ge 0$ because the sum contains the empty family and $\rho \ge 0$. The mean of $\prod_i \rho_{X_i}$ is the same whether taken on the outer variables of X or on the field, which gives $\Xi = \mathbb{E}[e^\Lambda]$, and $A_1 = -\log Z$. □

Theorem 602 gives the hypothesis of (ii) at every field U that is regular near X and whose moved plaquettes are at most $\kappa' = \kappa/30$, with U itself as the completion. It does not give it at the calm fields with a plaquette near X between κ' and κ, and it says nothing about $Z_\perp$.

Remark 605. The comparison of K_X with S_X needs a margin between the threshold of the boundary values and that of the kernel, because a completion must stay calm when it is moved by 90ρ and because its energy Q_X^{mov} must be small against $\kappa^2|X|$. For a calm field with plaquettes near X in the shell $(\kappa', \kappa]$, the moved energy of every completion can be of order $216\kappa^2|X|$, and then the bound of Theorem 602 on K_X/S_X exceeds one. Putting the blocks of the shell into the large-field region removes them from the calm law, but their kernels then carry only $e^{-\lambda\kappa'^2}$, and the comparison moves to a smaller threshold $\kappa'' < \kappa'$ with a new shell $(\kappa'', \kappa']$. A bound on the calm law of the shell, uniform in c and in the exterior, would end this regress; it is not proved.

Remark 606. The decoupling of $\log \mathbb{E}_{\mu_c^\circ}[e^\Lambda]$ in c and the bound on the shell both ask for decorrelation of the calm law. Its structure is as follows. By Lemma 599(i), ν_c is $\prod_y h(dI_y)$ times the pair kernels $\nu_c^{y,\mu}(dP_{y,\mu} \mid I_y, I_{y+e_\mu})$, each supported on $\{V^\sharp_{y,\mu} = c_{y,\mu}\}$ with mass the density of Lemma 373 at $c_{y,\mu}$, a function of I_y and I_{y+e_μ}. So μ_c° is a Gibbs measure for the variables Ξ_y of the blocks, with the factors $f_y s_y$, which by Lemma 599(i) couple Ξ_y to Ξ_{y+e_ν}, and the pair kernels, which couple I_y to I_{y+e_μ}; all couplings have range one. Large λ does not make this a high-temperature system. In the linearised law, the Gaussian measure proportional to $e^{-\lambda Q(\varphi)}$ on an affine fibre $\{V(\varphi) = \gamma\}$ of real fields, the conditional law of the variables of one block given the others is Gaussian with a mean that is an affine function of the others independent of λ, and a covariance equal to λ^{-1} times a matrix independent of λ. Hence the interdependence coefficients of Dobrushin's condition for the Wasserstein distance, which for Gaussian laws with equal covariances are the norms of the derivatives of these means, do not depend on λ. Theorem 367(iii) has the same scaling: for functions λg_1 and λg_2 with g_1, g_2 as there, it gives $c_9\lambda|X_1||X_2|e^{-\alpha_G d(X_1,X_2)}$, which is below one only for $d(X_1,X_2) \ge \alpha_G^{-1}\log(c_9\lambda|X_1||X_2|)$; the logarithm of a calm weight is of this form to leading order, by Theorem 607 below. The input that (ii) of Proposition 604 needs for locality in c is the decay of the truncated correlations of all orders of the cluster terms of Λ under μ_c°, for every c in the class of Corollary 603, the analogue for the calm law of the differences of order three and more in Remark 501. Theorem 367, (41) and Theorem 485 give covariances of order two only.

The Gaussian part of the calm weight. Let W be a finite set of blocks and fix every link off A_W to I. With the notation of Sec. XII E, let F_c be the fibre of $c = (c_k)_{k\in W\times\{0,\dots,3\}}$, ν_c its measure, Q_W the sum over the n_P plaquettes that contain a link of A_W, and

$$Z_W^\circ(c) = \int \prod_p \mathbf{1}\{\delta(U_p, I) \le \kappa\}\, e^{-\lambda Q_W}\, d\nu_c,$$

the product over the same n_P plaquettes, the calm weight of W. For real $\gamma \in \mathbb{R}^{W\times\{0,\dots,3\}}$ let

$$q_W(\gamma) = \min\Big\{\sum_p (d\varphi)_p^2 : \varphi \in \mathbb{R}^{A_W},\ V_k(\varphi) = \gamma_k \text{ for all } k\Big\},$$

with $\varphi = 0$ off A_W and V_k the linear average of Definition 328. The constraints are onto, since the path link of k enters V_k alone, with coefficient $\frac{1}{8}$, and Theorem 330 makes $\sum_p (d\varphi)_p^2$ positive definite on $\{V(\varphi) = 0\}$, so $q_W(\gamma) = \gamma^{\mathrm{T}} M_W \gamma$ for a positive semidefinite matrix M_W. For $\mathfrak{g}$-valued w, $\sum_a q_W(w^a)$ over the components in an orthonormal basis of $\mathfrak{g}$ is the form S_W of Sec. XIII P, the minimum of the fine Hessian at given averages, when W is a box.

Theorem 607. *Let $G = \mathrm{SU}(2)$, $\kappa > 0$, $W = \{0, e_0, 2e_0, 3e_0\}$, $A = (0,1)$ and $B = (3e_0, 1)$, $\tau \in \mathfrak{g}$ with $\|\tau\| = 1$, and for $\varepsilon > 0$ and $s,t \in \{0,1\}$ let c_ε^{st} equal $e^{\varepsilon s\tau}$ on A, $e^{\varepsilon t\tau}$ on B and I on the other coarse links of W. Write $\Delta F = \sum_{s,t}(-1)^{s+t} F(c_\varepsilon^{st})$.*

(i) *$(M_W)_{AB}$ is a rational number.*

(ii) *There is $\varepsilon_0 > 0$ such that for $0 < \varepsilon \le \varepsilon_0$*

$$\liminf_{\lambda\to\infty} \frac{1}{\lambda}\big|\Delta \log Z_W^\circ\big| \ge \varepsilon^2 |(M_W)_{AB}|.$$

(iii) *For such ε there is λ_0 such that for $\lambda \ge \lambda_0$ and every family of real functions ϕ_j on $G^{W\times\{0,\dots,3\}}$, each of which does not depend on c_A or does not depend on c_B,*

$$\max_{s,t}\Big|\log Z^\circ_W(c^{st}_\varepsilon) - \sum_j \phi_j(c^{st}_\varepsilon)\Big| \ge \frac{\lambda\varepsilon^2|(M_W)_{AB}|}{5}.$$

In particular, if $(M_W)_{AB}$ is not zero and λ exceeds $\max\{\lambda_0, 2/(\varepsilon^2|(M_W)_{AB}|)\}$, then at these four coarse fields Z°_W is not of the form

$$\prod_{y\in W}\phi_y(c)\sum\prod_i w(X_i),$$

with $\phi_y > 0$ a function of the $c_{y',\mu}$ with $\ell^\infty(y',y)\le 1$ and activities $|w(X)| \le e^{-9|X|}$ on the connected subsets of W, compatible as in Theorem 601.

Proof. (i) The minimiser of $q_W(\gamma)$ solves the linear system of its Lagrange conditions, which has 196 unknowns, 16 multipliers and the 605 plaquettes, and $(M_W)_{AB} = \varphi_A^{\mathrm T} G \varphi_B$ with φ_k the minimiser for $\gamma = e_k$ and G the matrix of $\sum_p (d\varphi)_p^2$. The system has rational coefficients and a unique solution, so φ_A, φ_B and $(M_W)_{AB}$ are rational.

(ii) All constants C below depend on W only. *Two-sided bounds.* Since $\nu_c(F_c) \le M^{4|W|}$, $Z^\circ_W(c) \le M^{16} e^{-\lambda q(c)}$ with $q(c) = \inf_{F_c} Q_W$. If $U \in F_c$ is regular with all n_P plaquettes at most $\kappa - 90\rho$, $\rho \le 9.32\cdot 10^{-4}$, the construction of the proof of Proposition 377 gives

$$Z^\circ_W(c) \ge 1.957^{-48}\Big(\frac{\rho^3}{14}\Big)^{180} e^{-\lambda((1+\eta)Q_W(U)+(1+\eta^{-1})8100\rho^2 n_P)}.$$

Take $\eta = \lambda^{-1/2}$ and $\rho = \lambda^{-1}$. Then

$$q(c) \le \liminf_{\lambda\to\infty}\frac{-\log Z^\circ_W(c)}{\lambda} \le \limsup_{\lambda\to\infty}\frac{-\log Z^\circ_W(c)}{\lambda} \le Q_W(U).$$

Expansion of q. Write fields near I as $U = e^X$, $X \in \mathfrak{g}^{A_W}$, $|X| = \max_e \|X_e\|$, and let $\gamma^{st} = (s\,\delta_{kA} + t\,\delta_{kB})_k \tau$, so that $c^{st}_\varepsilon = e^{\varepsilon\gamma^{st}}$ and the form $q^{\mathrm{lin}}(\gamma) = \sum_a q_W(\gamma^a)$ has $q^{\mathrm{lin}}(\gamma^{st}) = s(M_W)_{AA} + 2st(M_W)_{AB} + t(M_W)_{BB}$. Near I every cut-off equals one and $V^\sharp$ and Q_W are real analytic. By the formula for the derivative in Lemma 360(iv) at $U = I$, and by the Baker–Campbell–Hausdorff formula with $\delta(e^Y, I)^2 = \|Y\|^2 + O(\|Y\|^4)$,

$$\log V_k^\sharp(e^X) = V_k(X) + O(|X|^2),$$

$$Q_W(e^X) = \sum_p \|(dX)_p\|^2 + O(|X|^3).$$

The path link d_k enters $V_k^\sharp$ alone, with derivative $\frac18$ at I, so the implicit function theorem solves $V^\sharp(e^X) = c$ for the path links near I. *Upper bound.* Let X^* be the minimiser of $q^{\mathrm{lin}}(\varepsilon\gamma^{st})$, $|X^*| \le C\varepsilon$, and change its path links by $O(\varepsilon^2)$ so that $U_\varepsilon = e^X \in F_c$, $c = c^{st}_\varepsilon$. Then $Q_W(U_\varepsilon) \le \varepsilon^2 q^{\mathrm{lin}}(\gamma^{st}) + C\varepsilon^3$, U_ε is regular, and its plaquettes are at most $C\varepsilon$. *Lower bound.* Let $U \in F_c$ with $Q_W(U) \le q(c) + \varepsilon^3 \le C\varepsilon^2$. Every plaquette with a vertex in a block of W^+ is one of the n_P, at most $C\varepsilon$, or consists of links off A_W and equals I. For ε small Lemma 360(i) puts the active links inside the blocks within $C\varepsilon$ of I and the links leaving $B(y)$ in the direction μ within $C\varepsilon$ of the path link v, and by (iv) of that lemma $c_k = ve^N$ with $\|N\| \le C\varepsilon$, so v is within $C\varepsilon$ of I. By Lemma 352, $U = e^X$ with $|X| \le C\varepsilon$. Then $V(X) = \varepsilon\gamma^{st} + R$ with $|R| \le C\varepsilon^2$, and since $q^{\mathrm{lin}}(a+b) \ge q^{\mathrm{lin}}(a) + 2\langle a, (M_W\otimes 1)b\rangle$,

$$Q_W(U) \ge q^{\mathrm{lin}}(\varepsilon\gamma^{st} + R) - C\varepsilon^3 \ge \varepsilon^2 q^{\mathrm{lin}}(\gamma^{st}) - C\varepsilon^3.$$

So $|q(c^{st}_\varepsilon) - \varepsilon^2 q^{\mathrm{lin}}(\gamma^{st})| \le C\varepsilon^3$, and for $\varepsilon \le \varepsilon_1$, where ε_1 depends on κ and W, the field U_ε is admissible in the two-sided bound. Hence every limit point of $-\lambda^{-1}\Delta\log Z^\circ_W$ lies within $4C\varepsilon^3$ of $\Delta(\varepsilon^2 q^{\mathrm{lin}}) = 2\varepsilon^2(M_W)_{AB}$, and $\varepsilon_0 = \min\{\varepsilon_1, |(M_W)_{AB}|/(4C)\}$ gives (ii) if $(M_W)_{AB} \ne 0$; otherwise (ii) holds with $\varepsilon_0 = \varepsilon_1$, its right side being zero.

(iii) $\Delta\phi_j = 0$ for every j, since a function that does not depend on c_A takes the same value at c^{1t} and c^{0t}, and likewise for c_B. So $|\Delta\log Z^\circ_W| = |\Delta E| \le 4\max|E|$ for the difference E, and by (ii) $|\Delta\log Z^\circ_W| \ge \frac45\lambda\varepsilon^2|(M_W)_{AB}|$ for $\lambda \ge \lambda_0$. A factor ϕ_y that depends on c_A and on c_B would have $\ell^\infty(y,0) \le 1$ and $\ell^\infty(y, 3e_0) \le 1$, which is impossible. The proof of Theorem 601(i) applies to the connected subsets of W and bounds the logarithm of the polymer sum by $|W|/10$ in modulus; $Z^\circ_W > 0$ by the lower bound. A representation of this form would therefore give $\lambda\varepsilon^2|(M_W)_{AB}|/5 \le 2/5$, that is, $\lambda \le 2/(\varepsilon^2|(M_W)_{AB}|)$. □

Remark 608 (a computation, not used by any result). Solving the system of the proof of Theorem 607(i) in exact rational arithmetic gives $(M_W)_{AB} = -0.035264626\ldots$, so that the hypothesis $(M_W)_{AB} \ne 0$ of the last assertion of Theorem 607(iii) holds by this computation; no proof in this paper uses it, and no hand proof of $(M_W)_{AB} \ne 0$ is given. For the rows $W_n = \{0,\dots,(n-1)e_0\}$ and the end links $A = (0,1)$, $B = ((n-1)e_0, 1)$, the same computation gives $(M_{W_n})_{AA} = 20.0796\ldots$ for $n \ge 4$ and

n	2	3	4	5	6	7
$10^{n-2}(M_{W_n})_{AB}$	−3.32	3.35	−3.53	3.88	−4.47	5.40

so in this computation the coupling decays by a factor between 8 and 10 per block and does not vanish.

By the argument of Theorem 607, a product of factors of finite range r reproduces the calm weight of W_n, $n-1 > 2r$, only up to an error of order $\lambda\varepsilon^2|(M_{W_n})_{AB}|$. To leading order in λ the calm weight is $e^{-\lambda q}$ with q the Gaussian block-spin form, whose kernel decays at a rate independent of λ with amplitude proportional to λ. A local expansion of the small-field part therefore has to keep this form and expand around it, as the Gaussian small-field expansion of Bałaban does; a product of factors of range one times a polymer sum with activities $e^{-9|X|}$, which is the form (H1) below asks for, fails for the calm weight of W at every $\lambda > \max\{\lambda_0, 2/(\varepsilon^2|(M_W)_{AB}|)\}$ when $(M_W)_{AB} \ne 0$, by Theorem 607(iii).

So the small-field part of one step stands as follows. The kernels are local, with the second layer $\partial^= X$ (Lemma 599).

The calm weight of a region with boundary values is at least e^{-c_2} per block times the Gaussian factor of a completion calm at κ', and the relative activity is then at most $e^{-a_2|X|}$ (Theorem 602). The large-field sum is exactly $Z_\emptyset \Xi + Z_\perp$, and under the uniform bound on the relative activities the large fields change A_1 by at most $|T|/10$ (Proposition 604). Not proved are that bound at calm fields with plaquettes in the shell $(\kappa', \kappa]$ (Remark 605), a bound on $Z_\perp$, and the locality of $-\log Z_\emptyset(c)$ and of $\log \mathbb{E}_{\mu_c^\circ}[e^\Lambda]$ in c. For the first of these the form of the answer is fixed by Theorem 607: to leading order it is the Gaussian block-spin form with amplitude λ, and, if $(M_W)_{AB} \neq 0$, no product of local factors.

Locality, the shell and the remainder. The three questions left open above are taken up in turn. For the locality of the calm weight, the form $e^{-\lambda q^{\mathrm{lin}}(\log c)}$ times a polymer sum with local activities is impossible at $\lambda > 0.548$, because the calm weight is invariant under coarse gauge transformations and the Gaussian block-spin form of $\log c$ is not; the linearised propagator with the block-spin constraint decays at an explicit rate, which is the input that an expansion around a gauge-invariant leading term would use. For the shell, in the linearisation of the model the relative activities are not bounded under the calm law, at any λ and κ, and for SU(2) they are not bounded at any λ and $\kappa \leq 10^{-8}$. For the remainder, $Z_\perp(I) > 0$ for SU(2) at every λ and every $\kappa \leq 10^{-8}$, and $Z_\perp$ is bounded in the mean over c by the probability of a rough block.

For a gauge field $g \in G^T$ on the coarse torus put $(c^g)_{y,\mu} = g_y c_{y,\mu} g_{y+e_\mu}^{-1}$. The *Gaussian block-spin form of the torus* is $q_T(\gamma) = \min\{Q(\varphi) : V(\varphi) = \gamma\}$ for real γ on the coarse links, the minimum over real fields on the active links of Λ_L; for $\mathfrak{g}$-valued w put $q^{\mathrm{lin}}(w) = \sum_a q_T(w^a)$, with the components in the orthonormal basis $\tau_a = i\sigma_a/\sqrt{2}$ of $\mathfrak{su}(2)$. As for q_W, q_T is a positive semidefinite quadratic form, by Theorem 330 on the periodic lattice.

Lemma 609. *Let $G = \mathrm{SU}(2)$, $\kappa > 0$ and $g \in G^T$. Then $Z(c^g) = Z(c)$ and $Z_\emptyset(c^g) = Z_\emptyset(c)$ for every c, and $q_T(\gamma + d\chi) = q_T(\gamma)$ for real γ and χ, where $(d\chi)_{y,\mu} = \chi_y - \chi_{y+e_\mu}$.*

Proof. Let Φ be the fine gauge transformation equal to g_y at every site of $B(y)$. It fixes the tree links, so it preserves the tree gauge, and it preserves Q, every indicator s_y and r_y, and product Haar measure. The cut-off average is covariant (Theorem 353), so Φ maps F_c onto F_{c^g}. In the proof of Lemma 373 for the pair (y,μ), Φ replaces the path link ℓ_0 by $g_y \ell_0 g_{y+e_\mu}^{-1}$, the links inside $B(y)$ by their conjugates by g_y and those of $B(y+e_\mu)$ by their conjugates by g_{y+e_μ}, so every site v_x is conjugated by g_{y+e_μ}. The regions R_0, R_1, R_2, which are defined by the norms $\delta(v_x, I)$, are therefore preserved, the coordinates h_a are conjugated, which preserves Haar measure, and the derivative of $V^\sharp$ is conjugated by $\mathrm{Ad}(g_y)$, so $|\det DV^\sharp|$ is unchanged. Hence the image of ν_c under Φ is ν_{c^g}, and $Z = \nu_c(e^{-\lambda Q})$ and $Z_\emptyset = \nu_c(\prod_y f_y s_y)$ are invariant. For q_T, Lemma 329(ii) with h equal to χ_y on $B(y)$ maps $\{V = \gamma\}$ onto $\{V = \gamma + d\chi\}$ and preserves Q. □

Theorem 610. *Let $G = \mathrm{SU}(2)$, $\lambda > 0$, $\kappa > 0$, and let F be Z or $Z_\emptyset$. Let y be a block, $z = y + e_0$, $0 < \phi \leq \pi/2$, and let $g_y = \exp(\sqrt{2}\phi\tau_1)$, $g_z = \exp(\sqrt{2}\phi\tau_2)$ and $g_w = I$ otherwise. Suppose that at $c = I$ and at $c = I^g$*

$$F(c) = e^{-\lambda q^{\mathrm{lin}}(\log c) - E|T|} \prod_y \phi_y(c) \sum \prod_i w(X_i; c),$$

where log *is the principal logarithm on each coarse link, E is a constant, each $\phi_y > 0$ is invariant under coarse gauge transformations, and the sum runs over the families of pairwise compatible connected sets as in Theorem 601, with activities $|w(X;c)| \leq e^{-9|X|}$ that depend on c only through the $c_{w',\mu}$ with $\ell^\infty(w', X) \leq 1$. Then $\lambda \leq 8.1/|\eta(\phi)|^2 \leq 4.05/\sin^4\phi$, where $\cos\psi = \cos^2\phi$, $0 < \psi \leq \pi/2$, and*

$$|\eta(\phi)|^2 = 4\Big(\frac{\psi \sin\phi\cos\phi}{\sin\psi} - \phi\Big)^2 + \frac{2\psi^2 \sin^4\phi}{\sin^2\psi}.$$

At $\phi = \pi/2$, $|\eta|^2 = 3\pi^2/2$ and $\lambda \leq 5.4/\pi^2 < 0.5472$. The field I^g is flat, $\delta((I^g)_P, I) = 0$ for every coarse plaquette, and its links have $\delta(c_k, I) \leq 2\sin\phi$.

Proof. Write $g_y = a + b \cdot i\sigma$ as unit quaternions. The links of $c = I^g$ are g_y on (y,μ), $\mu \neq 0$, and g_y^{-1} on $(y - e_\mu, \mu)$; g_z on (z,μ) and g_z^{-1} on $(z - e_\mu, \mu)$, $\mu \neq 0$; g_z on $(z,0)$, g_y^{-1} on $(y - e_0, 0)$, and $g_y g_z^{-1}$ on $k = (y,0)$; I elsewhere. Their bases form the set S of the nine blocks $y - e_0$, y, z, $y - e_\mu$ and $z - e_\mu$ ($\mu = 1,2,3$). With χ equal to $\log g_y = \sqrt{2}\phi\tau_1$ at y, $\log g_z = \sqrt{2}\phi\tau_2$ at z and 0 elsewhere, $\log(I^g) = d\chi + \eta\,\delta_k$ with $\eta = \log(g_y g_z^{-1}) - \chi_y + \chi_z$. The quaternion of $g_y g_z^{-1}$ is $\cos^2\phi + (\sin\phi\cos\phi, -\sin\phi\cos\phi, \sin^2\phi)\cdot i\sigma$, whose principal logarithm is $\sqrt{2}\psi$ times the unit vector of its vector part, in the coordinates of the τ_a; this gives $|\eta|^2$, and at $\phi = \pi/2$, $g_y = i\sigma_1$, $g_z = i\sigma_2$, $g_y g_z^{-1} = i\sigma_3$ and $\eta = \frac{\pi}{\sqrt{2}}(-\tau_1 + \tau_2 + \tau_3)$. As $\psi \geq \sin\psi$, the second term of $|\eta|^2$ is at least $2\sin^4\phi$; and $5.4/\pi^2 < 0.5472$ since $\pi^2 > 9.8696$. By Lemma 609 and since q^{lin} is a sum of positive semidefinite forms, $q^{\mathrm{lin}}(\log I^g) = q^{\mathrm{lin}}(\eta\,\delta_k)$. By the proof of Proposition 335(i), which applies on the torus, $q_T \geq Q_1$, and δ_k lies in six coarse plaquettes, so $q^{\mathrm{lin}}(\eta\,\delta_k) \geq 6|\eta|^2$.

By Lemma 609, $F(I^g) = F(I)$, and the ϕ_y and E cancel, so $\lambda q^{\mathrm{lin}}(\eta\,\delta_k) = \log\Sigma(I^g) - \log\Sigma(I)$ with Σ the polymer sum. By Theorem 601(i), $\log\Sigma$ is the sum of the connected clusters, a cluster depends on c only through the links with base within distance one of its support, and the clusters incompatible with a given block add up to at most $\frac{1}{10}$ in modulus. A cluster whose support does not meet the set $N_1(S)$ of blocks within distance one of S takes the same value at I and I^g; a cluster that meets it is incompatible with a block of $N_1(S)$. Count $N_1(S)$ by the level a of a block $w = y + ae_0 + s$, s spatial. The blocks of S have $s = 0$ at level -1 and $s \in D = \{0, -e_1, -e_2, -e_3\}$ at levels 0 and 1, and the spatial points within distance one of D are the 27 points of $\{-1,0,1\}^3$ and the $3 \cdot 9$ points with one coordinate -2 and the others in $\{-1,0,1\}$. So level -2 contributes the 27 points within distance one of $s = 0$, each of the levels -1, 0, 1, 2 contributes 54, and $|N_1(S)| \leq 27 + 4 \cdot 54 = 243$. Hence $6\lambda|\eta|^2 \leq 2 \cdot 243/10 = 48.6$. The plaquettes of I^g are conjugates of those of I, and $\delta(g_y g_z^{-1}, I) = 2\sin\phi$ is the largest link. □

The following values are evaluations of the closed form for $|\eta(\phi)|^2$, rounded to the digits shown, those of the last column upwards. The conclusions below use only the row $\phi=\pi/2$, whose entries are exact, and the bound $4.05/\sin^4\phi$.

ϕ	$\lvert\eta(\phi)\rvert^2$	$\max_k \delta(c_k,I)$	$\lambda \le$
$\pi/2$	14.8044	2	0.5472
0.5	0.13191	0.9589	61.41
0.1	$2.0044\cdot10^{-4}$	0.1997	$4.0411\cdot10^{4}$
0.05	$1.2507\cdot10^{-5}$	0.09996	$6.4765\cdot10^{5}$

So at every κ the representation fails for $\lambda>0.5472$ at a flat coarse field, and, among coarse fields with links within 0.1 of I, for $\lambda>6.5\cdot10^5$: at $\phi=0.05$ the links are within $2\sin\phi<0.1$ of I, and $\sin\phi>\phi-\phi^3/6>0.04997$, so $\lambda\le 4.05/0.04997^4<6.5\cdot10^5$. This is below $\lambda^{**}(0.08)<9.16\cdot10^5$. The defect is the failure of log to intertwine the coarse gauge action: $q^{\mathrm{lin}}(\log c)$ is invariant only under gauge transformations that commute with c. A representation of the required kind needs a leading term invariant under the coarse gauge group, such as $\lambda S_1(c)$ with $S_1(c)=\inf_{F_c}Q$; by Theorem 607 it is not local if $(M_W)_{AB}\neq0$, and its Hessian at $c=I$ is 2λ times the form q^{lin}. Whether $Z_\emptyset(c)e^{\lambda S_1(c)}$, or the same with another invariant form of this Hessian, is a polymer sum with local activities is not decided here.

The block-spin propagator. The expansion of such a form around a minimiser uses the propagator of the fine field with the block-spin constraint. In the linearisation it decays at a rate that the following theorem makes explicit. For a pair $k=(y,\mu)$ label its eight private links by $a\in\{0,1\}^3$, the offsets in the three directions $\nu\neq\mu$, and for $w\in\{0,1\}^3\setminus\{0\}$ put $\xi_{k,w}=8^{-1/2}\sum_a(-1)^{a\cdot w}\varphi_{k,a}$. By Lemma 329(iii), on $\{V=0\}$ the mean of the eight is $m_k=-\frac{1}{16}(S^\mu_y+S^\mu_{y+e_\mu})$, so

$$\varphi_{k,a}=8^{-1/2}\sum_w(-1)^{a\cdot w}\xi_{k,w}+m_k,$$

and the coordinates $v=(\varphi_e)_{e\text{ inside}}\oplus(\xi_{k,w})$, with 17 inside links and 28 Walsh coordinates per block, define a linear bijection $P:v\mapsto\varphi$ onto $\{V=0\}$. Let H be the matrix of $v\mapsto Q(Pv)$, and for a coordinate, a fine link or a coarse link let its block be the block of its base. Write $d(\cdot,\cdot)$ for the ℓ^∞ distance of the blocks on the coarse torus. Let L_0 map γ to the field equal to γ_k on the eight private links of k and to zero elsewhere, as in Proposition 335(ii), and let A and B be the matrices of $Q(L_0\gamma)=\gamma^{\mathrm T}A\gamma$ and of the cross term, so that $Q(L_0\gamma+Pv)=\gamma^{\mathrm T}A\gamma+2\gamma^{\mathrm T}Bv+v^{\mathrm T}Hv$.

Theorem 611. *On the periodic lattice of side $L\ge32$ let*

$$\bar S=168+42\sqrt2,\qquad \bar b=168+336\sqrt2,\qquad p_1=\frac{7}{2\sqrt2}+\frac78,$$

so that $\bar S<227.40$ and $\bar b<643.18$, and for $\alpha>0$ let $c_\alpha=\frac{1}{52}-(\cosh3\alpha-1)\bar S$.

(i) *$H_{ij}=0$ for $d(i,j)\ge4$, the sum $\sum_j|H_{ij}|$ is at most $\bar S$ for every i, and $v^{\mathrm T}Hv\ge\frac{1}{52}|v|^2$ for every v.*

(ii) *If $0<\alpha\le\frac{1}{250}$, then $c_\alpha>0$ and $|(H^{-1})_{ij}|\le c_\alpha^{-1}e^{-\alpha d(i,j)}$ for all coordinates i and j.*

(iii) *For real γ the Gaussian probability measure proportional to $e^{-\lambda Q(\varphi)}$ on the affine space $\{V(\varphi)=\gamma\}$ of fields in the tree gauge has as mean the minimiser $\varphi^*(\gamma)=L_0\gamma-PH^{-1}B^{\mathrm T}\gamma$ of $q_T(\gamma)$ and as covariance $C=(2\lambda)^{-1}PH^{-1}P^{\mathrm T}$, and for $0<\alpha\le\frac{1}{250}$*

$$|C_{ee'}|\le\frac{e^{2\alpha}p_1^2}{2\lambda c_\alpha}e^{-\alpha d(e,e')}.$$

(iv) *$q_T(\gamma)=\gamma^{\mathrm T}M\gamma$ with $M=A-BH^{-1}B^{\mathrm T}$; the rows of A and B have ℓ^1 norms at most 192 and $\bar b$, $A_{kk'}=0$ for $d(k,k')\ge2$ and $B_{ki}=0$ for $d(k,i)\ge3$; and for $d(k,k')\ge2$ and $0<\alpha\le\frac{1}{250}$, $|M_{kk'}|\le e^{4\alpha}\bar b^2c_\alpha^{-1}e^{-\alpha d(k,k')}$.*

At $\alpha=\frac{1}{500}$, $c_\alpha>0.0151$, and the three constants are below 66.3, $746.2/(2\lambda)$ and $2.762\cdot10^7$.

Proof. (i) Write e_i for the coordinate vectors of v. For an inside link $e=(x,\mu)$ of $B(b)$ the coordinate enters only S^μ_b, so by the formula for m_k the field Pe_e is 1 on e and $-\frac{1}{16}$ on the eight private links of each of the pairs (b,μ) and $(b-e_\mu,\mu)$; its ℓ^1 norm is 2. For a Walsh coordinate $\xi_{k,w}$, $k=(b,\mu)$, the field is $\pm8^{-1/2}$ on the eight private links of k, of ℓ^1 norm $2\sqrt2$. In both cases the links have base in b or $b-e_\mu$. A row of P at an inside link has the single entry 1; at a private link of a pair (b,μ) it has 7 Walsh entries $\pm8^{-1/2}$ and the $2(8-2^\mu)$ active links of S^μ_b and $S^\mu_{b+e_\mu}$ with $-\frac{1}{16}$, so its ℓ^1 norm is at most p_1, the value at $\mu=0$. A plaquette has four links and a link lies in six plaquettes, so $\sum_p|(dPe_i)_p|\le6\cdot2\sqrt2$ and $\sum_j|(dPe_j)_p|\le\sum_{l\in p}\sum_j|P_{lj}|\le4p_1$. As $H_{ij}=\sum_p(dPe_i)_p(dPe_j)_p$,

$$\sum_j|H_{ij}|\le\sum_p|(dPe_i)_p|\sum_j|(dPe_j)_p|\le48\sqrt2\,p_1=\bar S.$$

If $H_{ij}\neq0$, a plaquette contains a link of Pe_i and a link of Pe_j; the bases of two links of a plaquette lie in blocks at distance at most one, and the links of Pe_i have base within distance one of the block of i, so $d(i,j)\le3$. Finally $v^{\mathrm T}Hv=Q(Pv)\ge\frac{1}{52}|Pv|^2$ by Theorem 330, whose proof uses only the plaquettes inside a block and those joining two neighbouring blocks and holds verbatim on the periodic lattice, and $|Pv|^2=\sum_{\text{inside}}\varphi_e^2+\sum_k(\sum_w\xi_{k,w}^2+8m_k^2)\ge|v|^2$, since the characters $a\mapsto(-1)^{a\cdot w}$, $w\neq0$, are orthogonal and orthogonal to the constants.

(ii) Since $\cosh u-1=\sum_{m\ge1}u^{2m}/(2m)!\le\frac{u^2}{2}\cosh u$, so that $\cosh u\le(1-\frac{u^2}{2})^{-1}$ for $u^2<2$, at $\alpha=\frac{1}{250}$ one has $(\cosh3\alpha-1)\bar S\le\frac{0.012^2}{2}\cdot1.0001\cdot227.40<0.01638<\frac{1}{52}$, and c_α decreases in α. Fix j, put $f_i=d(i,j)$ and $H_\alpha=e^{\alpha f}He^{-\alpha f}$. For real v, by the symmetry of H,

$$\begin{aligned}v^{\mathrm T}H_\alpha v&=\sum_{i,l}H_{il}\cosh\big(\alpha(f_i-f_l)\big)v_iv_l\\&=v^{\mathrm T}Hv+\sum_{1\le d(i,l)\le3}H_{il}\big(\cosh(\alpha(f_i-f_l))-1\big)v_iv_l,\end{aligned}$$

since $f_i = f_l$ when $d(i,l) = 0$ and $H_{il} = 0$ when $d(i,l) \geq 4$. As $|f_i - f_l| \leq d(i,l) \leq 3$, the last sum is at most $(\cosh 3\alpha - 1)\bar{S}|v|^2$ in modulus, by the Schur test for the symmetric matrix of its coefficients and (i). So $|H_\alpha v| \geq c_\alpha |v|$, H_α is invertible with $\|H_\alpha^{-1}\| \leq c_\alpha^{-1}$, and its entry (i,j) is $e^{\alpha d(i,j)}(H^{-1})_{ij}$.

(iii) $V(L_0\gamma) = \gamma$ by Proposition 335(ii), so the fields with $V = \gamma$ are the $L_0\gamma + Pv$, and the exponent is the quadratic form above in v; completing the square gives the mean and the covariance $(2\lambda H)^{-1}$ of v. By (i) a row of P has ℓ^1 norm at most p_1, and $P_{ei} \neq 0$ only if the block of i is the block of e or its neighbour in the direction of e, so the coordinates i, l that enter $C_{ee'}$ have $d(i,l) \geq d(e,e') - 2$. Then apply (ii).

(iv) Minimising over v gives M. The field $L_0\delta_k$ is 1 on the eight private links of k, which have base in the block of k and belong to no other pair, so $\sum_p |(dL_0\delta_k)_p| \leq 48$ and $\sum_{k'} |(dL_0\delta_{k'})_p| \leq 4$. As in (i), $A_{kk'} = \sum_p (dL_0\delta_k)_p (dL_0\delta_{k'})_p$ and $B_{ki} = \sum_p (dL_0\delta_k)_p (dPe_i)_p$ give the row bounds $48 \cdot 4 = 192$ and $48 \cdot 4p_1 = \bar{b}$ and the ranges one and two. Then $|(BH^{-1}B^{\mathrm{T}})_{kk'}| \leq \sum_{i,l} |B_{ki}||(H^{-1})_{il}||B_{k'l}|$ with $d(i,l) \geq d(k,k') - 4$, and (ii) gives the bound.

At $\alpha = \frac{1}{500}$, $(\cosh 3\alpha - 1)\bar{S} \leq \frac{0.006^2}{2} \cdot 1.0001 \cdot 227.40 < 0.004094$, so $c_\alpha > \frac{1}{52} - 0.004094 > 0.0151$; with $p_1^2 = \frac{49}{8} + \frac{49}{8\sqrt{2}} + \frac{49}{64} < 11.222$, $e^{0.004} < 1.00402$, $e^{0.008} < 1.00807$ (from $e^u \leq (1-u)^{-1}$) and $\bar{b}^2 < 413{,}681$ the constants follow. □

Remark 612. A computation on the lattice of side 12, which no statement here uses, gives the entries of H, A and B: all three have range one, the largest sum of $|H_{ij}|$ over $d(i,j) = 1$ is $S_1 = 20.45495\ldots$, and the rows of A and B have ℓ^1 norms at most 96 and 77.456. With these values the argument of (ii), with $\cosh\alpha$ in place of $\cosh 3\alpha$ and S_1 in place of $\bar{S}$ and with the constant $\frac{537}{5000}$ of Remark 331 in place of $\frac{1}{52}$, gives decay for $\alpha < 0.10243$, and at $\alpha = 0.08$ constants 23.87, $314.3/(2\lambda)$ and $1.680 \cdot 10^5$. Both rates are far from the decay of $(M_{W_n})_{AB}$ by a factor 8 to 10 per block, which corresponds to $\alpha \approx 2.1$; the loss comes from bounding the off-diagonal part of H by its row sums.

Remark 613. Bałaban's renormalisation group for lattice gauge theories [6–12] expands the small-field part around the minimiser of the action at given block averages: he treats this variational problem and the background fields it defines in [9], the propagators in such a background in [8], and the generation of the effective actions in a small field approximation and their cluster expansions in [10, 11]. They are cited for the method only. His averaging operations are not the cut-off average $V^\sharp$, and no proof here uses his results. By Theorems 607 and 610 the small-field part of the present step calls for the same structure: a leading term of amplitude λ, nonlocal if $(M_W)_{AB} \neq 0$, which cannot be $\lambda q^{\mathrm{lin}}(\log c)$, and an expansion around it. Theorem 611 is the linear propagator bound of such an expansion at the trivial background; the background field of $V^\sharp$ and the expansion around it are not carried out.

The shell in the linearised model. In the *linear model* the links are real: φ is a real field on the active links of the torus, $Q(\varphi) = \sum_p (d\varphi)_p^2$, a block y is calm when $|(d\varphi)_p| \leq \kappa$ for every $p \in \Pi(y)$, the coarse field is $\gamma = V(\varphi)$, and the fibre $\{V = \gamma\}$ carries Lebesgue measure in the coordinates given by the values on the links other than the path links. The statements of Lemma 599 concern only which links a factor involves, and hold verbatim. For $X = \{y\}$ and $\gamma = 0$, the free links of $\mathscr{V}_X$, the links of $\mathscr{V}_X$ other than the path links of the pairs of $\mathscr{P}_X$, are coordinates $u \in \mathbb{R}^{73}$ of the completions of given outer variables o, the remaining links. Let $S_X^{\mathrm{lin}}(o)$ and $K_X^{\mathrm{lin}}(o)$ be the integrals over u of $\prod_{w \in \hat{X}} f_w s_w$ and of $f_y r_y \prod_{w \in \hat{X} \setminus X} f_w s_w$, with $f_w = \exp(-\lambda \sum_{p \in \Pi(w)} (d\varphi)_p^2)$. The calm law is proportional to $\prod_w f_w s_w$ on $\{V = 0\}$. As in the proof of Proposition 604, the law of o under it has density proportional to $\Phi_X(o) S_X^{\mathrm{lin}}(o)$ with respect to Lebesgue measure in the free outer links, $\Phi_X = \prod_{w \notin \hat{X}} f_w s_w$, and $\rho_X^{\mathrm{lin}} = K_X^{\mathrm{lin}} / S_X^{\mathrm{lin}}$ is defined almost surely.

Theorem 614. *In the linear model on a torus of coarse side at least* 8*, with $\gamma = 0$, $X = \{y\}$, every $\kappa > 0$ and every $\lambda > 0$, the ratio ρ_X^{lin} is not essentially bounded under the calm law: for every R the calm law gives positive probability to $\{\rho_X^{\mathrm{lin}} > R\}$. In particular the hypothesis of Proposition 604(ii) fails in the linear model at every λ and κ.*

Proof. The map $\varphi \mapsto \kappa\varphi$ replaces κ by 1 and λ by $\lambda\kappa^2$, so let $\kappa = 1$. Let $x = 2y + (1,1,1,0)$. The three faces of the unit cube at x in the directions 0, 1, 2 that contain x are owned by y; write F_1, F_2, F_3 for $d\varphi$ on the faces in the planes $\{0,1\}$, $\{0,2\}$, $\{1,2\}$. Every link of the cube lies in two of its faces with opposite signs, so

$$F_1 - F_2 + F_3 = (d\varphi)_{x+e_2,01} - (d\varphi)_{x+e_1,02} + (d\varphi)_{x+e_0,12} =: R,$$

and the three faces on the right are owned by $y + e_2$, $y + e_1$, $y + e_0 \notin \hat{X}$ and contain no link of $\mathscr{V}_X$, so $R = R(o)$ is a linear function of o.

The field φ_1. For $\varepsilon \in \{0,1\}^3 \times \{0\}$ and $\varepsilon_\mu = 0$ the edge $(x + \varepsilon, \mu)$ of the cube has offset one in the direction μ, so it is a private link of the pair $(y + \varepsilon, \mu)$, and distinct edges lie in distinct pairs. Let φ_1 take the values

$$-\tfrac{1}{2},\ \tfrac{1}{2},\ \tfrac{1}{2},\ -\tfrac{1}{2},\ -\tfrac{1}{2},\ \tfrac{1}{2}$$

on the edges $(x+e_1,0)$, $(x+e_2,0)$, $(x+e_0,1)$, $(x+e_2,1)$, $(x+e_0,2)$, $(x+e_1,2)$, minus one seventh of that value on the other seven private links of the pair of each of these edges, and zero elsewhere. Private links are active, the sum over the private links of every pair is zero and no inside link is used, so $V(\varphi_1) = 0$ by Lemma 329(iii). The six pairs are those of the blocks $y + e_0$, $y + e_1$, $y + e_2$, none of which lies in $\hat{X}$, so φ_1 vanishes on $\mathscr{V}_X$. A pair contains one edge of the cube, so φ_1 vanishes on the other six edges, and

$$\begin{aligned} F_1 &= \varphi_1(x+e_0,1) - \varphi_1(x+e_1,0) = 1, \\ F_2 &= \varphi_1(x+e_0,2) - \varphi_1(x+e_2,0) = -1, \\ F_3 &= \varphi_1(x+e_1,2) - \varphi_1(x+e_2,1) = 1, \end{aligned}$$

and the outer faces take the values $\varphi_1(x+e_2,0) - \varphi_1(x+e_2,1) = 1$, $\varphi_1(x+e_1,0) - \varphi_1(x+e_1,2) = -1$ and $\varphi_1(x+e_0,1) - \varphi_1(x+e_0,2) = 1$. Two edges of the cube lie in a common plaquette only if it is a face of the cube, so every other plaquette contains at most one link with $|\varphi_1| = \frac{1}{2}$ and three with $|\varphi_1| \leq \frac{1}{14}$, and $|d\varphi_1| \leq \frac{5}{7} = 1 - \delta_0$, $\delta_0 = \frac{2}{7}$.

Let $w \in \mathbb{R}^{73}$ lower $(x,1)$ by one; to keep $V_{y,1}$, the path link $(2y+e_1,1)$, also a private link of $(y,1)$, rises by one. The link $(x,1)$ lies in F_1 with the sign $-$ and in F_3 with the sign $+$, the path link lies in none of the six faces, and the two links share no plaquette, so w changes (F_1,F_2,F_3) by $(1,0,-1)$ and every plaquette by at most 1. Lowering $(x,2)$ in the same way changes (F_1,F_2) by $(0,1)$, so the map from the free links of $\mathscr{V}_X$ to (F_1,F_2) has rank two. Let o_t be the outer part of $(1-t)\varphi_1$, $0<t\le\delta_0/4$; its inner part u_1 is zero.

Upper bound on S_X^{lin}. If all blocks of $\hat{X}$ are calm, then $F_1\le 1$, $F_2\ge -1$ and $F_3 = R-F_1+F_2\le 1$, so (F_1,F_2) lies in the square $[R-2,1]\times[-1,2-R]$ of side $3-R$. The map $u\mapsto(F_1,F_2)$ is affine with linear part of rank two, and $Q(\varphi(o,u)) = u^{\mathrm{T}}Hu + \ell_o(u) + \text{const}$ with H positive definite, since a change of u is a field in $\{V=0\}$ to which Theorem 330 applies. The image of the finite measure $e^{-\lambda Q}du$ under $u\mapsto(F_1,F_2)$ is Gaussian, with mass at most $(\pi/\lambda)^{73/2}\det H^{-1/2}$ and a nondegenerate covariance that does not depend on o, so its density is at most a constant C_λ and $S_X^{\mathrm{lin}}(o)\le C_\lambda(3-R(o))^2$. Let B_t be the ball of outer variables about o_t on which $|R(o)-R(o_t)|\le t$; there $3-R(o)\le 4t$ and $S_X^{\mathrm{lin}}\le 16C_\lambda t^2$.

Lower bound on K_X^{lin}. At $(o_t, u_1+\frac{\delta_0}{2}w)$, $F_1 = 1-t+\frac{\delta_0}{2}\ge 1+\frac{\delta_0}{4}$, so y is rough, and every plaquette owned by a block of $\hat{X}\setminus X$ is at most $(1-t)(1-\delta_0)+\frac{\delta_0}{2}\le 1-\frac{\delta_0}{2}$. All plaquettes are linear in (o,u), so there is $r>0$, independent of t, such that these inequalities hold with $\frac{\delta_0}{8}$ and $\frac{\delta_0}{4}$ on the ball of radius r about this point, and Q is bounded there by a constant Q_0. Hence $K_X^{\mathrm{lin}}(o)\ge c_r e^{-\lambda Q_0} =: k_0>0$ for o within r of o_t, with c_r the volume of the ball of radius r in $\mathbb{R}^{73}$.

Positive probability. At (o_t,u_1) every plaquette other than the six faces has modulus at most $(1-t)(1-\delta_0)$ and the six have modulus $1-t$, so all blocks are calm with a margin. For o near o_t the outer blocks stay calm, $\Phi_X(o)>0$, and the completions near u_1 stay calm, $S_X^{\mathrm{lin}}(o)>0$. So the calm law gives positive probability to every small ball about o_t, and on such a ball inside B_t and within r of o_t, $\rho_X^{\mathrm{lin}}\ge k_0/(16C_\lambda t^2)$. Let $t\to 0$. □

Remark 615. The mechanism is the conditioning: ρ_X compares two parts of the mass of the completions, and its denominator is the mass of the calm ones, which the Bianchi identity of a cube with three outer faces near their threshold squeezes into a corner of area $(3-R)^2/2$, while the rough completions keep a mass of order one. Weak coupling does not help, since the same holds at every λ; and the bound of Theorem 602 does not apply, since at these boundary values every completion has a face at least $1-3t$, above κ'. In SU(2) the Bianchi identity of the cube holds exactly up to conjugation, and ρ_X is not essentially bounded under the calm law either, at every λ and every $\kappa\le 10^{-8}$ (Theorem 617 below). The hypothesis of Proposition 604(ii) can therefore hold at best in a form that excludes such boundary values, for example in the mean under the calm law, and Proposition 604 (ii) is not proved in such a form for SU(2). In the linear model the mean form holds, with an explicit constant (Corollary 627).

The shell for SU(2). The Bianchi identity of a cube holds in SU(2) exactly, up to conjugation, and the calm condition is invariant under conjugation. This makes the construction of Theorem 614 work for SU(2) at every λ.

For $g\in\mathrm{SU}(2)$ let $\theta(g)\in[0,\pi]$ be its angle, so that the eigenvalues of g are $e^{\pm i\theta(g)}$. Then $\delta(g,I) = 2\sqrt{2}\sin(\theta(g)/2)$ and $d(g,I) = \|\log g\| = \sqrt{2}\,\theta(g)$. On the unit quaternions $\theta(g)$ is the geodesic distance from 1 to g, and left and right multiplications are isometries, so

$$\theta(gh)\le\theta(g)+\theta(h),\qquad \theta(g^{-1}) = \theta(kgk^{-1}) = \theta(g),$$

and under h the angle has the density $\frac{2}{\pi}\sin^2\theta$ on $[0,\pi]$. Put $\theta_\kappa = 2\arcsin(\kappa/(2\sqrt{2}))$, so that $\delta(g,I)\le\kappa$ if and only if $\theta(g)\le\theta_\kappa$. For a field U, a site x and $\mu\ne\nu$ let $F^x_{\mu\nu} = U_{x,\mu}U_{x+e_\mu,\nu}U^{-1}_{x+e_\nu,\mu}U^{-1}_{x,\nu}$ be the holonomy of the plaquette at x in the plane $\{\mu,\nu\}$, based at x.

Lemma 616. *For every field U and every site x,*

$$F^x_{01}F^x_{12}F^x_{20} = U_{x,0}\,H\,U^{-1}_{x,0},$$

where $H = U_{x+e_0,1}U^{-1}_{x+e_1,0}U_{x+e_1,2}U^{-1}_{x+e_2,1}U_{x+e_2,0}U^{-1}_{x+e_0,2}$ is the holonomy of the hexagon of the unit cube at x in the directions $0,1,2$ that avoids x and the opposite corner. If $\delta(F^x_{\mu\nu},I)\le\kappa$ for the three plaquettes, then $\theta(H)\le 3\theta_\kappa$ and $\theta(F^x_{01}),\theta(F^x_{12})\ge\theta(H)-2\theta_\kappa$.

Proof. In the product the factors $U^{-1}_{x,1}U_{x,1}$ and $U^{-1}_{x,2}U_{x,2}$ cancel, which leaves $U_{x,0}HU^{-1}_{x,0}$. Hence $\theta(H) = \theta(F^x_{01}F^x_{12}F^x_{20})$, which is at most the sum of the three angles and at most $\theta(F^x_{01})+2\theta_\kappa$; the same holds for F^x_{12}. □

In the abelian case the identity is $F_1-F_2+F_3 = R$ of the proof of Theorem 614, R being the circulation of the hexagon. The faces at x enter only through their angles, so the commutators that separate SU(2) from the linear model do not weaken the constraint, and its extremal case $\theta(H) = 3\theta_\kappa$ is attained by abelian fields with three aligned faces.

Theorem 617. *Let $G=\mathrm{SU}(2)$, $0<\kappa\le 10^{-8}$, $\lambda>0$, $c=I$ and $X=\{y\}$. There is $c_0>0$ such that for every $0<t\le\frac{1}{16}$ the calm law μ_I° gives positive probability to $\{\rho_X\ge c_0t^{-2}\}$. So ρ_X is not essentially bounded under the calm law, and the hypothesis of Proposition 604*(ii) *fails for* SU(2) *at every $\lambda>0$ and every $\kappa\le 10^{-8}$.*

Proof. Setting. Let $x = 2y+(1,1,1,0)$ and $l_\mu = (x,\mu)$ for $\mu = 0,1,2$. As in the proof of Theorem 614, the edge $(x+\varepsilon,\mu)$ of the cube at x, $\varepsilon\in\{0,1\}^3\times\{0\}$, $\varepsilon_\mu = 0$, is a private link of the pair $(y+\varepsilon,\mu)$. So l_μ is a private link of $(y,\mu)\in\mathscr{P}_X$, and not its path link $(2y+e_\mu,\mu)$; the six links $(x+e_\nu,\mu)$ of H are private to pairs whose two blocks lie off X, so they are not in $\mathscr{V}_X$, and H is a function $H(o)$ of the outer variables o, the links off $\mathscr{V}_X$. The three plaquettes at x are owned by y, and $F^x_{01} = l_0Al_1^{-1}$, $F^x_{12} = l_1Bl_2^{-1}$ with A, B products of outer links. Here $X=\{y\}$ has $|\mathscr{P}_X| = 8$, $|\hat{X}| = 11$, and L', the links of $\mathscr{V}_X$ other than the eight path links, has 73 links. On the geodesically convex ball $N_0 = \{v:\delta(v,I)\le\frac{1}{100}\}$ the derivative of log at e^X is $\Phi(\pm\mathrm{ad}_X)$ in the trivialisations of the proof of Lemma 360(iv), of norm at most $1+2\|X\|\le 1.021$, since

$\|X\| \le \frac{\pi}{300}$ by Lemma 352; so $\|\log v - \log v'\| \le 1.021\, d(v,v')$ on N_0.

Step 1: the calm completions. Let o have $\Phi_X(o) > 0$ and $S_X(o) > 0$, and put $s(o) = 3\theta_\kappa - \theta(H(o))$. We show $s(o) \ge 0$ and

$$S_X(o) \le C_1 \theta_\kappa^4 s(o)^2, \qquad C_1 = \frac{4 \cdot 9^{24}}{\pi^2}.$$

There is a completion U'' at which every block of $\hat X$ is calm, and the blocks off $\hat X$ are calm because $\Phi_X(o) > 0$; so every plaquette is at most κ, and $s(o) \ge 0$ by Lemma 616. As in the proof of Theorem 621, Lemma 360(i), (iv) with $c = I$ put every link of U'' within $\Delta = (21 + 8\pi)\kappa < 46.14\kappa$ of I, every path link v within $d(v,I) \le 8\pi\kappa$, and so, by Lemma 352, every link within $d \le \Delta_1 = 50\kappa$ of I. These bounds hold for the outer links, which all completions share.

Let D_* be the set of $z \in G^{L'}$ with $d(z_e, I) \le \Delta_1$ for every e, and B_* the ball $d(\ell, I) \le R_* = 45\Delta_1$. For $z \in D_*$, a pair k and $\ell \in B_*$ on its path link, one site of k equals I, one is a single link of L' or an outer link, and the other fourteen are ℓ^{-1} times a product of two such links; so all lie within $47\Delta_1 < \frac{1}{100}$ of I, the cut-off equals one, and $V_k^\sharp = \ell e^{N_k(\ell)}$ with N_k the mean of the logarithms. The map $\Psi(\ell) = e^{-N_k(\ell)}$ satisfies $d(\Psi(\ell), I) \le \|N_k\| \le \frac{1.021}{16}(14 \cdot 47 + 1)\Delta_1 < 42.1\Delta_1 < R_*$ and, since fourteen sites move by $d(\ell,\ell')$, $d(\Psi(\ell), \Psi(\ell')) \le \frac{14}{16} 1.021\, d(\ell,\ell')$. So Ψ has a unique fixed point $\psi_k(z) \in B_*$, which gives an injective map Γ_* from D_* into the completions. Every completion with its free links in D_* and its path links in B_* is $\Gamma_*(z)$, since its path link is a fixed point of Ψ in B_*; by the previous paragraph this includes every calm completion. Near $\Gamma_*(D_*)$ the averages are smooth. As in the proof of Proposition 374, the right trivialised derivative of $V_k^\sharp$ in its path link is $\mathrm{Ad}(\ell)T$ with $T = 1 - E_{N_k} \frac{1}{16} \sum_x L_{X_x}$, the sum over the fourteen sites that move, E_N the derivative of the exponential map at N and L_{X_x} that of the logarithm at $v_x = e^{X_x}$. Since every $\|X_x\|$ and $\|N_k\|$ is below $2.4 \cdot 10^{-5}$, the bounds $|\Phi(z) - 1|, |\Psi(z) - 1| \le |z|$ of the proof of Lemma 360(iv) give $\|E_{N_k} L_{X_x} - 1\| < 10^{-4}$, so $\|T - \frac{1}{8}\| < 10^{-4}$ and $|\det \partial_{d_k} V_k^\sharp| \ge 9^{-3}$. As in the proof of Theorem 602, λ_X restricted to $\Gamma_*(D_*)$ is the image of $h^{\otimes L'}$ on D_* divided by $\prod_k |\det \partial_{d_k} V_k^\sharp|$, and the integrand of S_X is at most one and vanishes unless the completion is calm. By Lemma 616, at a calm completion F_{01}^x and F_{12}^x lie in the shell $E = \{g : \theta_\kappa - s(o) \le \theta(g) \le \theta_\kappa\}$, so

$$S_X(o) \le 9^{24} h^{\otimes L'}\big(\{z : F_{01}^x \in E,\ F_{12}^x \in E\}\big).$$

For fixed values of the other links, $(l_0, l_1, l_2) \mapsto (F_{01}^x, F_{12}^x, l_1)$ is a bijection of G^3 that preserves $h^{\otimes 3}$, since for fixed l_1 the maps $l_0 \mapsto l_0 A l_1^{-1}$ and $l_2 \mapsto l_1 B l_2^{-1}$ preserve h. So the measure on the right is at most $h(E)^2$, and $h(E) \le \frac{2}{\pi}\theta_\kappa^2 s(o)$.

Step 2: the rough completions. Let φ_1, w and $\delta_0 = \frac{2}{7}$ be as in the proof of Theorem 614, w being the field equal to -1 on l_1, to 1 on the path link $(2y + e_1, 1)$ and to 0 elsewhere. For $0 < t \le \frac{1}{16}$ let U_t and U_t' equal

$$\exp\big(\theta_\kappa(1-t)\varphi_1\, i\sigma_3\big), \qquad \exp\big(\theta_\kappa((1-t)\varphi_1 + w/7)\, i\sigma_3\big)$$

on every active link. These fields are abelian, the angle of a plaquette is θ_κ times the modulus of the linear plaquette, and every link has $d \le \sqrt{2} \cdot \frac{9}{14}\theta_\kappa$, as $|\varphi_1| \le \frac{1}{2}$ and $|w| \le 1$. Their sites lie within $3\theta_\kappa$ of I, so they are regular, and $V^\sharp = \exp(\theta_\kappa V(\cdot)\, i\sigma_3) = I$ since $V(\varphi_1) = V(w) = 0$. Since φ_1 vanishes on $\mathscr{V}_X$ and w off it, U_t and U_t' are completions of the same outer variables o_t, with $\theta(H(o_t)) = 3(1-t)\theta_\kappa$, as $(F_1, F_2, F_3) = (1, -1, 1)$. Every plaquette of U_t has angle at most $(1-t)\theta_\kappa$. At U_t' the plaquette at x in the plane $\{0,1\}$ has angle $((1-t) + \frac{1}{7})\theta_\kappa \ge \frac{121}{112}\theta_\kappa$, and the plaquettes owned by the blocks of $\hat X \setminus X$, which are not faces of the cube, have angle at most $(\frac{5}{7} + \frac{1}{7})\theta_\kappa$.

Let $\rho_0 = \theta_\kappa/1000$ and let o be outer variables with $d(o_e, (o_t)_e) \le \rho_0$ for every outer link e. Let z' and ℓ_k' be the free and the path links of U_t', D the set of $z \in G^{L'}$ with $d(z_e, z_e') \le \rho_0$, and B_k the ball of radius $R = 44\rho_0$ about ℓ_k'. At (o, z, ℓ) with $\ell \in B_k$ fourteen sites of k lie within $R + 2\rho_0$ of their values at U_t' and one within ρ_0, all in N_0. With $\Psi(\ell) = e^{-N_k(\ell)}$ formed at (o, z), $d(\Psi(\ell), \ell_k') \le \frac{1.021}{16}(14 \cdot 46 + 1)\rho_0 < 41.2\rho_0 < R$, and Ψ is a contraction as in Step 1. Its fixed points give an injective map Γ_o from D into the completions of o, near whose image the averages are smooth with $|\det \partial_{d_k} V_k^\sharp| \le 1$, by the bound on T of Step 1. As in the proof of Theorem 602,

$$\lambda_X\big(\Gamma_o(D)\big) \ge \Big(\frac{\rho_0^3}{14}\Big)^{73}.$$

A plaquette has at most two path links, so on $\Gamma_o(D)$ each plaquette lies within $d \le 2R + 2\rho_0 = 90\rho_0$ of its value at U_t', and its angle moves by less than $64\rho_0 = 0.064\,\theta_\kappa$. Hence y is rough there, $\frac{121}{112} - 0.064 > 1$, and the blocks of $\hat X \setminus X$ are calm, $\frac{6}{7} + 0.064 < 1$; with $f_w \ge e^{-768\lambda}$ for each of the eleven blocks of $\hat X$,

$$K_X(o) \ge k_0 = \Big(\frac{\rho_0^3}{14}\Big)^{73} e^{-8448\lambda}.$$

Step 3: positive probability. Let $\rho_1 = t\theta_\kappa/3000$. The construction in the proof of Theorem 598(ii), with ρ_1 in place of the ρ chosen there and the regular field $U_t \in F_I$, gives a set $\Gamma_t \subset F_I$ with $\nu_I(\Gamma_t) > 0$ on which every link lies within $d \le 44\rho_1$ of its value at U_t and every plaquette within $90\rho_1$. On Γ_t every plaquette has angle at most $(1 - t + 0.022t)\theta_\kappa < \theta_\kappa$, so every block is calm and $\Phi_X > 0$ at the outer variables o of the field; these satisfy $d(o_e, (o_t)_e) \le 44\rho_1 \le \rho_0$, as $t \le \frac{1}{16}$; and $\theta(H(o)) \ge 3(1-t)\theta_\kappa - 6 \cdot 44\rho_1/\sqrt{2}$, so $s(o) \le 3.07\, t\, \theta_\kappa$. By Steps 1 and 2, wherever $S_X(o) > 0$,

$$\rho_X = \frac{K_X(o)}{S_X(o)} \ge \frac{k_0}{C_1 \theta_\kappa^4 (3.07 t \theta_\kappa)^2} \ge \frac{c_0}{t^2}, \qquad c_0 = \frac{k_0 \pi^2}{37.7 \cdot 9^{24} \theta_\kappa^6}.$$

The integrand $\prod_y f_y s_y$ of μ_I° is positive on Γ_t, so $\mu_I^\circ(\Gamma_t) > 0$, and by the proof of Proposition 604(i) the calm law gives no mass to $\{S_X = 0\}$. So $\mu_I^\circ\{\rho_X \ge c_0 t^{-2}\} \ge \mu_I^\circ(\Gamma_t) > 0$. □

Remark 618. (a) The obstruction lies at the trivial coarse field and holds at every λ; the boundary values that cause it have every plaquette calm, the three plaquettes at x in the shell $(\kappa/30, \kappa]$ and close to κ, and the hexagon close

to its largest possible angle $3\theta_\kappa$. They squeeze the two faces F^x_{01}, F^x_{12} of every calm completion into a shell of angular width $s(o)$, while the rough completions keep a mass bounded below.

(b) Since $\rho_X \ge 0$ and the calm law of the outer variables has density proportional to $\Phi_X S_X$, the mean of a product of relative activities is $\mathbb{E}_{\mu_c^\circ}[\prod_i \rho_{X_i}] \le Z_X(c)/Z_\emptyset(c) = \mu_c\{R = X\}/\mu_c\{R = \emptyset\}$, with equality when $Z_\perp$ receives nothing from X. A bound in the mean is therefore a bound on the constrained law at a fixed coarse field, which Theorem 596 gives only absolutely or under its hypothesis. These means do not factor over the components of X, so they are not activities to which Theorem 601 applies. In the Gaussian layer the mean bound of Corollary 627 rests on the Gaussian correlation inequality, for which no $\mathrm{SU}(2)$ form is proved here.

(c) Lemma 616 holds for every condition that confines the three plaquettes at x to an angle θ_1: the completions vanish when $\theta(H) > 3\theta_1$ and are confined to the shell $\theta \ge \theta(H) - 2\theta_1$ otherwise. Moving the threshold to $\kappa'' \le 10^{-8}$, as in Remark 605, reproduces the configuration of the proof with κ'' in place of κ. For $\kappa > 10^{-8}$ Step 1 is not proved, since Lemma 360 needs $\kappa \le 10^{-8}$.

The remainder $Z_\perp$. Let $\mu_c = Z(c)^{-1} e^{-\lambda Q} \nu_c$ be the constrained law on F_c, and let $\mathbb{P}$ be the law of c under the Wilson measure μ_L, with density $Z(c)/Z_L$ with respect to h.

Proposition 619. *Let $G = \mathrm{SU}(2)$, $\kappa > 0$ and $Z_\emptyset(c) > 0$.*

(i) *The following hold:*

$$\frac{Z(c)}{Z_\emptyset(c)} = \Xi(c) + \frac{Z_\perp(c)}{Z_\emptyset(c)}, \qquad \frac{Z_\emptyset(c)}{Z(c)} = \mu_c\{R = \emptyset\},$$

and $0 \le Z_\perp(c) \le Z(c) - Z_\emptyset(c) = Z(c)\,\mu_c\{R \ne \emptyset\}$.

(ii) *For $\beta_W \ge 1$ and $\kappa \le 1$, $\int Z_\perp/Z\, d\mathbb{P} \le \mu_L\{R \ne \emptyset\} \le 16|T| q_\kappa(\beta_W)$.*

Proof. (i) The first identity is Proposition 604(i) divided by $Z_\emptyset$, and $Z_\emptyset = \nu_c(e^{-\lambda Q}\mathbf{1}\{R = \emptyset\})$. By Theorem 600(i), $Z = Z_\emptyset + \sum_{X \ne \emptyset} Z_X$, and $Z_\perp$ is a sum of parts of the nonnegative Z_X, $X \ne \emptyset$. (ii) $\int \mu_c\{R \ne \emptyset\}\, d\mathbb{P} = \mu_L\{R \ne \emptyset\}$ since $h = \int \nu_c\, dh(c)$, and Theorem 596 (iii) with $X = \{y\}$ bounds each $\mu_L\{y \in R\}$. □

The next theorem shows that $Z_\perp$ is not zero: there are boundary values that admit a rough completion and no calm one, with positive measure. For a unit quaternion $g = a + b \cdot i\sigma$ write $b(g) \in \mathbb{R}^3$ for its vector part; then $\delta(g, I)^2 = 4(1-a)$ and $|b(g)| \le \delta(g,I)/\sqrt{2}$.

Lemma 620. *Let $g_1, \dots, g_4 \in \mathrm{SU}(2)$ with $\delta(g_i, I) \le \Delta \le 1$. Then*

$$\big|b(g_1 g_2 g_3^{-1} g_4^{-1}) - b(g_1) - b(g_2) + b(g_3) + b(g_4)\big| \le 3\Delta^2 + 3.54\Delta^3.$$

Proof. $b(g^{-1}) = -b(g)$, and $b(gh) = a_g b(h) + a_h b(g) - b(g) \times b(h)$. Let P_j be the product of the first j factors and ε_j the error after j factors. Then $\delta(P_j, I) \le j\Delta$, so $|b(P_j)| \le j\Delta/\sqrt{2}$ and $1 - a(P_j) \le j^2\Delta^2/4$, and with g the next factor,

$$\begin{aligned}\varepsilon_{j+1} &\le \varepsilon_j + (1 - a_g)|b(P_j)| + (1 - a(P_j))|b(g)| \\ &\quad + |b(P_j)||b(g)| \\ &\le \varepsilon_j + \frac{(j + j^2)\Delta^3}{4\sqrt{2}} + \frac{j\Delta^2}{2}.\end{aligned}$$

Summing over $j = 1, 2, 3$ with $\varepsilon_1 = 0$ gives $3\Delta^2 + \frac{20}{4\sqrt{2}}\Delta^3$. □

Theorem 621. *Let $G = \mathrm{SU}(2)$, $0 < \kappa \le 10^{-8}$ and $\lambda > 0$. Then $Z_\perp(I) > 0$: for $X = \{y\}$ the outer variables at which $S_X = 0 < K_X$ and $\Phi_X > 0$ have positive $\nu^{\mathrm{out}}_{I,X}$-measure. In the linear model the same holds for every $\kappa > 0$ and $\lambda > 0$.*

Proof. Let $x = 2y + (1,1,1,1)$, and let the cubes $\mathscr{A}$ and $\mathscr{B}$ at x be those in the directions $0,1,2$ and $0,1,3$. Each has three faces at x, owned by y, and three opposite faces, owned by $y + e_2$, $y + e_1$, $y + e_0$ and by $y + e_3$, $y + e_1$, $y + e_0$, none in $\hat{X}$, which contain no link of $\mathscr{V}_X$. The face p_0 at x in the plane $\{0,1\}$ belongs to both. *The field φ_2.* Let $\sigma = \frac{11}{20}$, $\tau = \frac{1}{4}$, and let $C = x + \{0,1\}^4$. As for φ_1, the edge $(x + \varepsilon, \mu)$ of C is a private link of the pair $(y + \varepsilon, \mu)$, and distinct edges lie in distinct pairs. Let φ_2 take the value σ on $(x + e_2, 0)$, $(x + e_1, 2)$, $(x + e_3, 1)$, $(x + e_0, 3)$; $-\sigma$ on $(x + e_2, 1)$, $(x + e_0, 2)$, $(x + e_3, 0)$, $(x + e_1, 3)$; $\mp\frac{2}{3}\sigma$ on $(x + e_0 + e_2, 1)$, $(x + e_0 + e_3, 1)$; $\mp\frac{1}{3}\sigma$ on $(x + e_0 + e_1, 2)$, $(x + e_0 + e_1, 3)$; τ on $(x + e_1 + e_3, 2)$, $(x + e_0 + e_2, 3)$; $-\tau$ on $(x + e_1 + e_2, 3)$, $(x + e_0 + e_3, 2)$; minus one seventh of that value on the other seven private links of the pair of each of these sixteen edges; and zero elsewhere. The exchange of the coordinates 2 and 3 about x maps φ_2 to $-\varphi_2$. As for φ_1, $V(\varphi_2) = 0$, φ_2 vanishes on $\mathscr{V}_X$, since its pairs are those of blocks $y + \varepsilon$ with $\varepsilon \ne 0$ and $\varepsilon \ge 0$, and on a 2-face of C it is given by the sixteen values, the other edges of C carrying zero. Write $[a; \mu\nu]$ for the plaquette at $x + a$ in the plane $\{\mu, \nu\}$. The 2-faces of C not at x are the eighteen $[e_a; \mu\nu]$ with $a \notin \{\mu, \nu\}$ and $[e_a + e_b; \mu\nu]$ with $\{a, b, \mu, \nu\} = \{0, 1, 2, 3\}$, owned by $y + e_a$ and $y + e_a + e_b$. Adding the values on their edges gives

$[e_2; 01]$	$\frac{4}{3}\sigma$	$[e_3; 01]$	$-\frac{4}{3}\sigma$	$[e_1; 23]$	$2\sigma - 2\tau$
$[e_1; 02]$	$-\frac{4}{3}\sigma$	$[e_1; 03]$	$\frac{4}{3}\sigma$	$[e_0; 23]$	$2\tau - 2\sigma$
$[e_0; 12]$	$\frac{4}{3}\sigma$	$[e_0; 13]$	$-\frac{4}{3}\sigma$	$[e_0 + e_1; 23]$	$-\frac{2}{3}\sigma$
$[e_2; 03]$	$\sigma + \tau$	$[e_3; 02]$	$-\sigma - \tau$	$[e_0 + e_3; 12]$	$\frac{2}{3}\sigma + \tau$
$[e_2; 13]$	$-\sigma - \tau$	$[e_3; 12]$	$\sigma + \tau$	$[e_0 + e_2; 13]$	$-\frac{2}{3}\sigma - \tau$

and the faces $[e_1 + e_2; 03]$, $[e_1 + e_3; 02]$, $[e_2 + e_3; 01]$ contain one of the sixteen edges or none. A plaquette that is not a 2-face of C contains at most one edge of C, so at most one link with $|\varphi_2| > \frac{\sigma}{7}$, and has $|d\varphi_2| \le \frac{10}{7}\sigma = \frac{11}{14}$. Every entry of the table has modulus at most $\sigma + \tau$, so every plaquette not owned by y has $|d\varphi_2| \le m = \sigma + \tau = \frac{4}{5}$. The first three rows of the first two columns are the outer faces of $\mathscr{A}$ and $\mathscr{B}$, so, with the signs of the Bianchi identity as in the proof of Theorem 614, $R_{\mathscr{A}} = 4\sigma = \frac{11}{5}$ and $R_{\mathscr{B}} = -\frac{11}{5}$.

Linear model. As in the proof of Theorem 614, with $\kappa = 1$, a calm completion would give, from $\mathscr{A}$, $(d\varphi)_{p_0} \geq R_{\mathscr{A}} - 2 = \frac{1}{5}$ and, from $\mathscr{B}$, $(d\varphi)_{p_0} \leq 2 + R_{\mathscr{B}} = -\frac{1}{5}$. These strict inequalities, and $|d\varphi| < 1$ off $\Pi(y)$, hold on a neighbourhood of φ_2, on which therefore the blocks off y are calm, $S_X^{\mathrm{lin}} = 0$, and every completion near the inner part of φ_2 has y rough, so $K_X^{\mathrm{lin}} > 0$.

SU(2). Let $U = \exp(\kappa\varphi_2\tau_3)$ on every active link. It is abelian, its plaquettes are $\exp(\kappa(d\varphi_2)_p\tau_3)$ with $\delta \leq \kappa|(d\varphi_2)_p|$, and its sites, products of three links with $|\varphi_2| \leq \sigma$, lie within $3\sigma\kappa < 2\kappa$ of I, so U is regular, the cut-offs equal one, and $V^\sharp(U) = \exp(\kappa V(\varphi_2)\tau_3) = I$. The construction in the proof of Theorem 598(ii), with $\rho = \kappa/10^4$ in place of the ρ chosen there, gives a set $\Gamma \subset F_I$ with $\nu_I(\Gamma) > 0$ on which every plaquette satisfies $\delta(U'_p, U_p) \leq 90\rho$, by bi-invariance and $\delta \leq d$. On Γ every plaquette not owned by y is at most $(m + 0.009)\kappa < \kappa$.

Fix $U' \in \Gamma$ and suppose that its outer variables admit a calm completion U''. Then every plaquette of U'' is at most κ, and by Lemma 360(i) and (iv), with $c = I$, every inside link is within 3κ of I, every path link v satisfies $v = \exp(-\frac{1}{16}\sum_x \log(v^{-1}P_x))$ with $\delta(v^{-1}P_x, I) \leq 24\kappa$, so $\delta(v, I) \leq 8\pi\kappa$ by Lemma 352, and every other link leaving a block is within $(21 + 8\pi)\kappa =: \Delta$ of I. The linear parts of the six faces of a cube cancel as in the linear model, so by Lemma 620, for the third coordinates b_3 of the vector parts,

$$\left|b_3(U''_{p_0}) - b_3(U''_{p_1}) + b_3(U''_{p_2}) - R^b_{\mathscr{A}}\right| \leq 6\varepsilon_4,$$

where $\varepsilon_4 = 3\Delta^2 + 3.54\Delta^3$, with p_1, p_2 the other faces of $\mathscr{A}$ at x and $R^b_{\mathscr{A}}$ the signed sum of b_3 over its outer faces, which are those of U'; likewise for $\mathscr{B}$. Since $|b_3| \leq \kappa/\sqrt{2}$ on calm faces,

$$R^b_{\mathscr{A}} - \sqrt{2}\kappa - 6\varepsilon_4 \leq b_3(U''_{p_0}) \leq R^b_{\mathscr{B}} + \sqrt{2}\kappa + 6\varepsilon_4.$$

At U the outer faces have $b_3 = \sin(\kappa F/\sqrt{2})$ with $F = \pm\frac{4}{3}\sigma = \pm\frac{11}{15}$ as in the table, so $R^b_{\mathscr{A}} = -R^b_{\mathscr{B}} = 3\sin\frac{11\kappa}{15\sqrt{2}}$, and as $\sin\theta \geq \theta(1 - \theta^2/6)$,

$$R^b_{\mathscr{A}} - R^b_{\mathscr{B}} \geq \frac{22}{5}\frac{\kappa}{\sqrt{2}}\left(1 - \frac{\kappa^2}{6}\right) > 3.1111\,\kappa.$$

On Γ each b_3 moves by at most $90\rho/\sqrt{2}$, since $\|g - g'\| \geq \sqrt{2}|b(g) - b(g')|$, so $R^b_{\mathscr{A}} - R^b_{\mathscr{B}}$ moves by at most $540\rho/\sqrt{2} < 0.0382\,\kappa$. As $\pi < 3.1416$, $\Delta < 46.14\,\kappa$, so $\Delta^2 < 2128.9\,\kappa^2$, $\Delta^3 < 98228\,\kappa^3$ and, for $\kappa \leq 10^{-8}$, $\varepsilon_4 < 6386.7\,\kappa^2 + 0.0035\,\kappa^2 < 6387\,\kappa^2$ and $12\varepsilon_4 < 76644\,\kappa^2 < 0.00077\,\kappa$. With $2\sqrt{2} < 2.8285$,

$$\begin{aligned} R^b_{\mathscr{A}} - R^b_{\mathscr{B}} - 2\sqrt{2}\kappa - 12\varepsilon_4 &> (3.1111 - 0.0382)\kappa \\ &\quad - (2.8285 + 0.00077)\kappa > 0.24\,\kappa \end{aligned}$$

at the outer variables of U', which contradicts the previous display. So $S_X = 0$ at the outer variables of every $U' \in \Gamma$; since U' is a completion of them whose blocks off y are calm, y is rough at U'. By the factorisation of Theorem 600(i) and Fubini, the part of $Z_\perp(I)$ from $X = \{y\}$ is at least $\int_\Gamma e^{-\lambda Q}\,d\nu_I > 0$. □

So $Z_\perp$ is a genuine part of the weight, made of boundary values that force a rough block, and it is not a function of the calm law. It is suppressed like a rough block, in the mean by Proposition 619(ii) and pointwise in the outer variables by the bound of Theorem 600(ii) on K_X; a bound on $Z_\perp(c)/Z_\emptyset(c)$ uniform in c is not proved.

The small-field part therefore stands as follows. The representation of the calm weight as $e^{-\lambda q^{\mathrm{lin}}(\log c)}$ times local factors and a polymer sum fails at every $\lambda > 0.5472$ (Theorem 610), and, if $(M_W)_{AB} \neq 0$, as a product of local factors at large λ (Theorem 607); the leading term must be gauge invariant and, under the same condition, nonlocal, and its linear propagator decays at the explicit rate of Theorem 611. In the linear model the relative activities are unbounded under the calm law at every λ (Theorem 614), and for SU(2) at every λ and $\kappa \leq 10^{-8}$ (Theorem 617), so the almost sure hypothesis of Proposition 604(ii) fails in both, while in the linear model their mean is at most $e^{-9.2|X|}$ for $\lambda\kappa^2 \geq 16$, and the Gaussian layer of the step reproduces itself at every scale with the same constants (Theorem 625, Corollaries 626 and 627 below); and $Z_\perp(I) > 0$ for SU(2) (Theorem 621). Not proved are a local expansion around a gauge-invariant leading term, a bound in the mean on the relative activities in the shell for SU(2), and a bound on $Z_\perp/Z_\emptyset$ uniform in c.

The Gaussian layer of the step. The linear model of Theorem 614 is the Gaussian layer of the step. This paragraph expands it around its Gaussian form at every scale. The thresholds are placed on the fluctuation about the minimiser of the form at given averages, the background, and not on the field. With this choice the large-field remainder does not depend on the coarse field, its relative weights are bounded by $e^{-9.2}$ per block for $\lambda\kappa_{\mathrm{f}}^2 \geq 16$, uniformly in the coarse field, the volume and the scale, and the only hypothesis on the form, that it dominates the plaquette form, is reproduced by the step. The conditioning on boundary values, which defeats the almost sure bound in Theorem 614, is replaced by Royen's Gaussian correlation inequality, used in the mean over the boundary values. The SU(2) step is not treated, and the nonlocality of the Gaussian form is not decided here; Remark 630 states what the SU(2) step needs beyond this paragraph.

Let Λ be a torus of even side at least four, with the blocks $B(y)$ of Definition 328, coarse torus Λ', tree links and active links as there, and $\Pi(y)$ the 96 plaquettes owned by the block y (Lemma 595). Let $\mathscr{A}(\Lambda)$ be the space of real fields on the active links, $Q_\Lambda(\varphi) = \sum_p (d\varphi)_p^2$ over the plaquettes of Λ, and $V : \mathscr{A}(\Lambda) \to \mathbb{R}^{\Lambda' \times \{0,\dots,3\}}$ the block averages. For a real field γ on the coarse links put $\mathscr{V}_\gamma = \{\varphi \in \mathscr{A}(\Lambda) : V(\varphi) = \gamma\}$, with Lebesgue measure m_γ in the coordinates given by the values on the links other than the path links, as in the linear model. Since the path link of a pair enters its average alone, with coefficient $\frac{1}{8}$, and enters no other average, $\mathscr{V}_\gamma$ is not empty and m_γ is the image of m_0 under $\psi \mapsto \varphi_0 + \psi$ for every $\varphi_0 \in \mathscr{V}_\gamma$. The proofs of Theorem 330, of Lemma 329 and of Proposition 335(i), (ii) use only the plaquettes and links of one block and of two neighbouring blocks, and their counts are the same on such a torus; so they hold on Λ, and on Λ' when its side is even and at least four.

Definition 622. $\mathscr{G}(\Lambda)$ is the set of positive semidefinite quadratic forms q on $\mathscr{A}(\Lambda)$ with $q \geq Q_\Lambda$. For $q \in \mathscr{G}(\Lambda)$

and $\lambda > 0$, μ_q is the probability measure on $\mathscr{V}_0$ with density $Z_q^{-1} e^{-\lambda q}$ with respect to m_0, where $Z_q = \int_{\mathscr{V}_0} e^{-\lambda q}\, dm_0$.

The restriction of Q_Λ to $\mathscr{A}(\Lambda)$ belongs to $\mathscr{G}(\Lambda)$.

Lemma 623. *Let $q \in \mathscr{G}(\Lambda)$ and $\lambda > 0$.*

(i) *$Z_q < \infty$. For every γ the form q has a unique minimiser $\varphi_q^*(\gamma)$ on $\mathscr{V}_\gamma$, which is linear in γ, and with $q'(\gamma) = q(\varphi_q^*(\gamma))$,*

$$q\big(\varphi_q^*(\gamma) + \psi\big) = q'(\gamma) + q(\psi) \qquad (\psi \in \mathscr{V}_0).$$

(ii) *$q'(\gamma) \geq Q_{\Lambda'}(\gamma)$ for every real field γ on the coarse links. If the side of Λ' is at least four, the restriction of q' to $\mathscr{A}(\Lambda')$ belongs to $\mathscr{G}(\Lambda')$. If $q(\varphi + d\tilde{h}) = q(\varphi)$ for every function $\tilde{h}$ constant on the blocks, then $q'(\gamma + d'h) = q'(\gamma)$ for every function h on the coarse sites, where $(d'h)_{y,\mu} = h_y - h_{y+e_\mu}$.*

(iii) *For distinct plaquettes $p_1, \dots, p_n$ of Λ the covariance matrix Σ of $((d\psi)_{p_1}, \dots, (d\psi)_{p_n})$ under μ_q satisfies $\Sigma \leq (2\lambda)^{-1} I$.*

Proof. (i) By Theorem 330 with W the set of all blocks, $q(\psi) \geq Q_\Lambda(\psi) \geq \frac{1}{52} \sum_e \psi_e^2$ on $\mathscr{V}_0$, so q is positive definite there and $Z_q < \infty$. Fix $\varphi_0 \in \mathscr{V}_\gamma$ depending linearly on γ, for instance the field of Proposition 335(ii). The function $\psi \mapsto q(\varphi_0 + \psi)$ on $\mathscr{V}_0$ is a quadratic polynomial with positive definite leading part, so it has a unique minimiser, a linear function of φ_0. At the minimiser φ^* the derivative vanishes: $q(\varphi^*, \psi) = 0$ for the bilinear form of q and every $\psi \in \mathscr{V}_0$, which gives the identity.

(ii) By Proposition 335(i), $q'(\gamma) = q(\varphi_q^*(\gamma)) \geq Q_\Lambda(\varphi_q^*(\gamma)) \geq Q_{\Lambda'}(V(\varphi_q^*(\gamma))) = Q_{\Lambda'}(\gamma)$, and q' is positive semidefinite as a minimum of a positive semidefinite form over an affine family depending linearly on γ. For h on the coarse sites let $\tilde{h}$ equal h_y on $B(y)$. Then $d\tilde{h}$ vanishes on the links inside the blocks, so $d\tilde{h} \in \mathscr{A}(\Lambda)$, and $V(\varphi + d\tilde{h}) = V(\varphi) + d'h$ by Lemma 329(ii). So $\varphi \mapsto \varphi + d\tilde{h}$ maps $\mathscr{V}_\gamma$ onto $\mathscr{V}_{\gamma + d'h}$ and preserves q.

(iii) Identify $\mathscr{V}_0$ with $\mathbb{R}^N$ by the coordinates of m_0 and write $q(\psi) = \psi^{\mathrm{T}} K \psi$ with K positive definite. Then μ_q is the centred Gaussian law with covariance $(2\lambda K)^{-1}$, and for a linear function $\ell(\psi) = l^{\mathrm{T}} \psi$, by the Cauchy–Schwarz inequality for the inner product of K,

$$\mathrm{Var}_{\mu_q}(\ell) = \frac{l^{\mathrm{T}} K^{-1} l}{2\lambda} = \frac{1}{2\lambda} \sup_{\psi \neq 0} \frac{\ell(\psi)^2}{q(\psi)}.$$

For $a \in \mathbb{R}^n$ and $\ell(\psi) = \sum_i a_i (d\psi)_{p_i}$, $\ell(\psi)^2 \leq |a|^2 \sum_i (d\psi)_{p_i}^2 \leq |a|^2 Q_\Lambda(\psi) \leq |a|^2 q(\psi)$, as the plaquettes are distinct. So $a^{\mathrm{T}} \Sigma a \leq |a|^2/(2\lambda)$. □

Lemma 624. (i) *If Z is a centred real Gaussian variable with $\mathrm{Var} Z \leq \sigma^2$, then $\Pr(|Z| > s) \leq e^{-s^2/(2\sigma^2)}$ for every $s \geq 0$.*

(ii) *If $(Z_1, \dots, Z_n)$ is a centred Gaussian vector with covariance at most $\sigma^2 I$ and $s^2 \geq \sigma^2$, then*

$$\Pr\Big(\sum_i Z_i^2 > n s^2\Big) \leq \Big(\frac{e s^2}{\sigma^2}\Big)^{n/2} e^{-n s^2/(2\sigma^2)}.$$

Proof. (i) It suffices to take $\mathrm{Var} Z = \sigma^2 > 0$ and $\sigma = 1$, since $\Pr(|Z| > s)$ increases with the variance. Let $f(u) = e^{-u^2/2} - \sqrt{2/\pi} \int_u^\infty e^{-v^2/2} dv$. Then $f(0) = 0$, $f(u) \to 0$ as $u \to \infty$, and $f'(u) = e^{-u^2/2}(\sqrt{2/\pi} - u)$, so f increases and then decreases, and $f \geq 0$ on $[0, \infty)$. (ii) Write the covariance as ODO^{T} with O orthogonal and D diagonal with entries $d_i \leq \sigma^2$. Then $\sum_i Z_i^2$ has the law of $\sum_i d_i g_i^2$ with independent standard Gaussian g_i, and for $0 \leq \theta < 1/(2\sigma^2)$, $\mathbb{E} e^{\theta \sum_i d_i g_i^2} = \prod_i (1 - 2\theta d_i)^{-1/2} \leq (1 - 2\theta\sigma^2)^{-n/2}$. By Markov's inequality the probability is at most $(1 - 2\theta\sigma^2)^{-n/2} e^{-\theta n s^2}$, and $\theta = (1 - \sigma^2/s^2)/(2\sigma^2)$ gives the bound. □

Royen's Gaussian correlation inequality, in the form proved in [80], states that a centred Gaussian measure μ on $\mathbb{R}^N$ has $\mu(K \cap K') \geq \mu(K)\mu(K')$ for any two closed convex sets K, K' symmetric about the origin. Taking complements, $\mu(K' \setminus K) \leq \mu(K')\big(1 - \mu(K)\big)$. For a slab K and K' an intersection of slabs this is the inequality of Šidák.

Thresholds on the fluctuation. Let $q \in \mathscr{G}(\Lambda)$, $\lambda > 0$ and $\kappa_{\mathrm{f}} > 0$. For $\varphi \in \mathscr{V}_\gamma$ let $\psi = \varphi - \varphi_q^*(\gamma) \in \mathscr{V}_0$ be its fluctuation. A block y of Λ' is *calm relative to the background* if $|(d\psi)_p| \leq \kappa_{\mathrm{f}}$ for every $p \in \Pi(y)$, and *rough relative to the background* otherwise; $R^{\mathrm{f}}(\varphi)$ is the set of these rough blocks, and for $\psi \in \mathscr{V}_0$, $R^{\mathrm{f}}(\psi)$ is taken at $\gamma = 0$. For $X \subset \Lambda'$ put

$$Z_X(\gamma) = \int_{\mathscr{V}_\gamma} e^{-\lambda q}\, \mathbf{1}\{R^{\mathrm{f}} = X\}\, dm_\gamma, \qquad \pi_X = \mu_q\{R^{\mathrm{f}} = X\},$$

and, with $t = \lambda \kappa_{\mathrm{f}}^2$,

$$\varepsilon_1(t) = 96\sqrt{2et}\, e^{-t}, \qquad \varepsilon(t) = \frac{\varepsilon_1(t)}{(1 - e^{-t})^{96}}.$$

Theorem 625. *Let $q \in \mathscr{G}(\Lambda)$, $\lambda > 0$, $\kappa_{\mathrm{f}} > 0$ and $t = \lambda \kappa_{\mathrm{f}}^2 \geq \frac{1}{2}$.*

(i) *For every coarse field γ and every $X \subset \Lambda'$,*

$$Z_X(\gamma) = Z_q\, \pi_X\, e^{-\lambda q'(\gamma)},$$

and $\sum_X \pi_X = 1$, so that $\int_{\mathscr{V}_\gamma} e^{-\lambda q}\, dm_\gamma = \sum_X Z_X(\gamma) = Z_q e^{-\lambda q'(\gamma)}$.

(ii) *For every set Y of blocks, $\mu_q\{Y \cap R^{\mathrm{f}} = \emptyset\} \geq (1 - e^{-t})^{96|Y|}$; in particular $\pi_\emptyset \geq (1 - e^{-t})^{96|\Lambda'|}$.*

(iii) *For disjoint sets X, Y of blocks,*

$$\mu_q\{X \subset R^{\mathrm{f}},\, Y \cap R^{\mathrm{f}} = \emptyset\} \leq \varepsilon_1(t)^{|X|}\, \mu_q\{Y \cap R^{\mathrm{f}} = \emptyset\}.$$

(iv) *For every $X \subset \Lambda'$, $\pi_X \leq \varepsilon(t)^{|X|} \pi_\emptyset$.*

(v) *ε decreases on $[\frac{1}{2}, \infty)$, and $\varepsilon(t) < e^{-9.2}$ for $t \geq 16$.*

The bounds do not depend on γ, on the side of Λ or on $q \in \mathscr{G}(\Lambda)$.

Proof. (i) By Lemma 623(i) and the translation $\varphi = \varphi_q^*(\gamma) + \psi$, which maps m_0 to m_γ,

$$Z_X(\gamma) = e^{-\lambda q'(\gamma)} \int_{\mathscr{V}_0} e^{-\lambda q(\psi)} \mathbf{1}\{R^{\mathrm{f}}(\psi) = X\}\, dm_0(\psi),$$

since R^{f} depends on φ only through ψ. The events $\{R^{\mathrm{f}} = X\}$ partition $\mathscr{V}_0$.

(ii) The event is the intersection of the slabs $\{|(d\psi)_p| \le \kappa_{\mathrm{f}}\}$, $p \in \Pi(Y)$, which are closed, convex and symmetric, and $|\Pi(Y)| = 96|Y|$ by Lemma 595(i). Removing one slab at a time with the correlation inequality gives $\mu_q \ge \prod_p \mu_q\{|(d\psi)_p| \le \kappa_{\mathrm{f}}\}$. By Lemma 623(iii), $\mathrm{Var}(d\psi)_p \le (2\lambda)^{-1}$, so by Lemma 624(i) each factor is at least $1 - e^{-\lambda\kappa_{\mathrm{f}}^2}$.

(iii) If $X \subset R^{\mathrm{f}}$, every $y \in X$ owns a plaquette p_y with $|(d\psi)_{p_y}| > \kappa_{\mathrm{f}}$; the p_y are distinct, and $\sum_{y\in X}(d\psi)_{p_y}^2 > |X|\kappa_{\mathrm{f}}^2$. So $\{X \subset R^{\mathrm{f}}\}$ is contained in the union, over the at most $96^{|X|}$ choices of $(p_y)_{y\in X}$ with $p_y \in \Pi(y)$, of the events $E = \{\sum_y (d\psi)_{p_y}^2 > |X|\kappa_{\mathrm{f}}^2\}$. The complement of E and $C_Y = \{Y \cap R^{\mathrm{f}} = \emptyset\}$ are closed, convex and symmetric, so $\mu_q(E \cap C_Y) \le \mu_q(E)\mu_q(C_Y)$ by the correlation inequality. By Lemma 623(iii) the covariance of $((d\psi)_{p_y})_{y\in X}$ is at most $(2\lambda)^{-1}I$, and Lemma 624(ii) with $\sigma^2 = (2\lambda)^{-1}$, $s = \kappa_{\mathrm{f}}$, $s^2/\sigma^2 = 2t \ge 1$ and $n = |X|$ gives $\mu_q(E) \le (2et)^{|X|/2} e^{-t|X|}$.

(iv) Let $Y = \Lambda' \setminus X$. By (iii), $\pi_X \le \mu_q\{X \subset R^{\mathrm{f}},\ Y \cap R^{\mathrm{f}} = \emptyset\} \le \varepsilon_1^{|X|}\mu_q(C_Y)$. By the correlation inequality and (ii), $\pi_\emptyset = \mu_q(C_X \cap C_Y) \ge \mu_q(C_X)\mu_q(C_Y) \ge (1 - e^{-t})^{96|X|}\mu_q(C_Y)$, and $\mu_q(C_Y) > 0$.

(v) The derivative of $\log \varepsilon(t)$ is $\frac{1}{2t} - 1 - \frac{96e^{-t}}{1-e^{-t}} < 0$ for $t > \frac{1}{2}$. With $\log 96 = 5\log 2 + \log 3$ and the bounds on $\log 2$, $\log 3$ of the proof of Theorem 601(ii),

$$\log \varepsilon_1(16) = \tfrac{15}{2}\log 2 + \log 3 - 15.5 < -9.20278.$$

As $e > 2.7$, $e^{16} > 2.7^{16} > 7.9 \cdot 10^6$, and $-\log(1-x) \le x/(1-x)$ gives $-96\log(1 - e^{-16}) < 10^{-4}$. So $\log \varepsilon(16) < -9.2$. □

The large-field remainder of the Gaussian layer is therefore exact: the weight of a fibre is the Gaussian factor $e^{-\lambda q'(\gamma)}$ times a constant, the calm part carries the factor $\pi_\emptyset$, and the parts with a rough region X carry $\pi_X \le \varepsilon(t)^{|X|}\pi_\emptyset$, none of which depends on γ. The weights π_X need not factorise over the components of X, and no polymer representation of them is needed, since they do not depend on the coarse field. For $t \ge 16$ they satisfy the bound $e^{-9|X|}$ that Theorem 601 asks of activities, and $-\log \pi_\emptyset \le 96|\Lambda'|e^{-t}/(1 - e^{-t})$. Part (iii) is the analogue of Theorem 596(ii) without its hypothesis on the exterior: the exterior is not fixed but integrated, under the condition that it is calm.

Corollary 626. *Let Λ_0 be a torus of side 2^m, $m \ge 2$, let Λ_k be its k-th coarse torus, $q_0 = Q_{\Lambda_0}$, and for $k \le m-3$ let q_{k+1} be the restriction of q'_k to $\mathscr{A}(\Lambda_{k+1})$. Then for $0 \le k \le m-2$, $q_k \in \mathscr{G}(\Lambda_k)$, q_k is invariant under the gauge transformations of Λ_k that are constant on its blocks, and Theorem 625 holds at the scale k with the same λ, the same κ_{f} and the same $\varepsilon(t)$.*

Proof. Q_{Λ_0} is invariant under all gauge transformations. For $k \le m-3$ the side of Λ_{k+1} is even and at least four, and Lemma 623(ii) passes both properties from q_k to q_{k+1}: the gauge transformations of Λ_{k+1} constant on its blocks are of the form $d'h$. □

So the hypotheses of the Gaussian layer reproduce themselves exactly: the form at the next scale again dominates the plaquette form of its lattice, the coupling λ does not change, and every constant of Theorem 625 is the same at every scale. The restriction to $\mathscr{A}(\Lambda_{k+1})$ is the tree gauge of the coarse field. By the gauge invariance in Lemma 623(ii), and since every coarse field is $\gamma_0 + d'h$ with $\gamma_0 \in \mathscr{A}(\Lambda_{k+1})$ by Lemma 329(i) on Λ_{k+1}, the weight $e^{-\lambda q'_k}$ is determined by its values on $\mathscr{A}(\Lambda_{k+1})$, as the weight of SU(2) is by Lemma 609.

The shell in the mean. At $\gamma = 0$ the background vanishes and the thresholds relative to the background are the thresholds of the linear model. For $X \subset T$ with components X_i let $K_{X_i}^{\mathrm{lin}}$ and $S_{X_i}^{\mathrm{lin}}$ be the integrals of Theorem 600(i) and of the calm weight S_{X_i}, formed in the linear model at $\gamma = 0$ as K_X^{lin} and S_X^{lin} are for $X = \{y\}$, and $\rho_{X_i}^{\mathrm{lin}} = K_{X_i}^{\mathrm{lin}}/S_{X_i}^{\mathrm{lin}}$, defined almost surely under the calm law μ° of the linear model.

Corollary 627. *In the linear model on the torus Λ_L with $\gamma = 0$, let $\kappa > 0$ and $\lambda\kappa^2 \ge \frac{1}{2}$. For every nonempty $X \subset T$ with components X_i,*

$$\mathbb{E}_{\mu^\circ}\Big[\prod_i \rho_{X_i}^{\mathrm{lin}}\Big] \le \varepsilon(\lambda\kappa^2)^{|X|},$$

and $\mu^\circ\{\prod_i \rho_{X_i}^{\mathrm{lin}} > R\} \le \varepsilon(\lambda\kappa^2)^{|X|}/R$ for $R > 0$. For $\lambda\kappa^2 \ge 16$ the mean is at most $e^{-9.2|X|}$.

Proof. Lemma 599 and the factorisation of Theorem 600(i) concern only which links a factor involves, so they hold in the linear model, and as in the proof of Proposition 604(i) the law of the outer variables of X under μ° has density $Z_\emptyset^{-1}\Phi_X \prod_i S_{X_i}^{\mathrm{lin}}$ with respect to Lebesgue measure on them, where $Z_\emptyset = \int e^{-\lambda Q}\prod_y s_y\, dm_0$. Hence

$$\mathbb{E}_{\mu^\circ}\Big[\prod_i \rho_{X_i}^{\mathrm{lin}}\Big] \le Z_\emptyset^{-1}\int \Phi_X \prod_i K_{X_i}^{\mathrm{lin}},$$

and

$$Z_\emptyset^{-1}\int \Phi_X \prod_i K_{X_i}^{\mathrm{lin}} = \frac{\int e^{-\lambda Q}\mathbf{1}\{R = X\}\, dm_0}{\int e^{-\lambda Q}\mathbf{1}\{R = \emptyset\}\, dm_0} = \frac{\pi_X}{\pi_\emptyset},$$

the last for $q = Q_{\Lambda_L}$, $\gamma = 0$ and $\kappa_{\mathrm{f}} = \kappa$, where the rough set R of the linear model is R^{f}. Apply Theorem 625(iv), (v) and Markov's inequality. □

So in the linear model the relative activities are not essentially bounded under the calm law (Theorem 614) but are bounded in the mean, with the explicit bound $\varepsilon(\lambda\kappa^2)^{|X|}$, uniformly in the volume and for every X; the boundary values of Theorem 614, at which the ratio is large, have calm-law probability at most $\varepsilon^{|X|}/R$. This is the mean form asked for in Remark 615. In the mean the regress of Remark 605 does not

arise, because the outer variables are integrated and not fixed in the shell.

The sharp threshold away from $\gamma = 0$. For $\gamma \neq 0$ the threshold of the linear model is placed on $\varphi = \varphi_q^*(\gamma) + \psi$. When the background is calm with a margin, the sharp calm weight lies within the factor $\pi_\emptyset$ of the Gaussian form.

Proposition 628. *Let $L \geq 32$ and $q = Q_{\Lambda_L}$, and for a coarse field γ let $\kappa_c(\gamma)$ be the largest $|(d'\gamma)_P|$ over the coarse plaquettes P. Then for every plaquette p of Λ_L,*

$$\big|(d\varphi_q^*(\gamma))_p\big| \leq K_* \, \kappa_c(\gamma), \qquad K_* < 1.64 \cdot 10^{19}.$$

Proof. In the notation of Theorem 611, $\varphi_q^*(\gamma) = L_0\gamma - PH^{-1}B^{\mathrm{T}}\gamma$. By Proposition 335(ii) every plaquette of $L_0\gamma$ is 0 or a curvature $(d'\gamma)_P$, and since $\gamma^{\mathrm{T}}Bv = \sum_p (dL_0\gamma)_p (dPv)_p$,

$$(B^{\mathrm{T}}\gamma)_i = \sum_p (dL_0\gamma)_p (dPe_i)_p, \qquad |(B^{\mathrm{T}}\gamma)_i| \leq 12\sqrt{2}\,\kappa_c(\gamma),$$

by the bound $\sum_p |(dPe_i)_p| \leq 6 \cdot 2\sqrt{2}$ of the proof of Theorem 611(i). With $\sum_j |(dPe_j)_p| \leq 4p_1$ from the same proof and $S_H = \max_i \sum_j |(H^{-1})_{ij}|$,

$$\big|(d\varphi_q^*(\gamma))_p\big| \leq \big(1 + 48\sqrt{2}\,p_1 S_H\big)\,\kappa_c(\gamma) = \big(1 + \bar{S} S_H\big)\,\kappa_c(\gamma).$$

By Theorem 611(ii) at $\alpha = \frac{1}{500}$, $|(H^{-1})_{ij}| \leq 66.3\, e^{-\alpha d(i,j)}$. A block carries $17 + 28 = 45$ coordinates, and at most $(2d+1)^4 - (2d-1)^4 = 64d^3 + 16d$ blocks lie at distance $d \geq 1$ from a given one. With $x = e^{-\alpha}$, $\sum_{d \geq 1} dx^d \leq (1-x)^{-2}$ and $\sum_{d \geq 1} d^3 x^d = x(1+4x+x^2)(1-x)^{-4} \leq 6(1-x)^{-4}$, and $1 - x \geq \alpha - \alpha^2/2$ gives $(1-x)^{-1} < 500.51 =: r$, so $r^2 < 250511$ and $r^4 < 6.2756 \cdot 10^{10}$. Hence

$$\begin{aligned} S_H &\leq 66.3 \cdot 45 \big(1 + 384 r^4 + 16 r^2\big) \\ &< 66.3 \cdot 45 \cdot 2.4100 \cdot 10^{13} < 7.20 \cdot 10^{16}, \end{aligned}$$

and $K_* = 1 + \bar{S} S_H < 1 + 227.40 \cdot 7.20 \cdot 10^{16} < 1.64 \cdot 10^{19}$. □

The constant is far from the truth. It comes from bounding H^{-1} by the slow rate $\frac{1}{500}$ of Theorem 611, and it is used only to state the margin condition below in terms of the coarse field.

Corollary 629. *Let $q \in \mathscr{G}(\Lambda)$, $0 < \kappa_{\mathrm{f}} < \kappa$, $t = \lambda\,\kappa_{\mathrm{f}}^2 \geq \frac{1}{2}$, and let γ be a coarse field whose background has $|(d\varphi_q^*(\gamma))_p| \leq \kappa - \kappa_{\mathrm{f}}$ for every plaquette p; for $q = Q_{\Lambda_L}$ and $L \geq 32$ this holds when $\kappa_c(\gamma) \leq (\kappa - \kappa_{\mathrm{f}})/K_*$. Let $Z_\emptyset^{\mathrm{lin}}(\gamma) = \int_{\mathscr{V}_\gamma} e^{-\lambda q} \prod_y s_y \, dm_\gamma$, where s_y is the indicator that every plaquette of $\Pi(y)$ has $|(d\varphi)_p| \leq \kappa$. Then*

$$0 \leq -\log Z_\emptyset^{\mathrm{lin}}(\gamma) - \lambda q'(\gamma) + \log Z_q \leq -\log \pi_\emptyset \leq \frac{96|\Lambda'|e^{-t}}{1 - e^{-t}}.$$

Proof. Since $\prod_y s_y \leq 1$, Theorem 625(i) gives the upper bound $Z_\emptyset^{\mathrm{lin}}(\gamma) \leq Z_q e^{-\lambda q'(\gamma)}$. If every block is calm relative to the background, then $|(d\varphi)_p| \leq |(d\varphi_q^*(\gamma))_p| + |(d\psi)_p| \leq \kappa$ for every p, so $\prod_y s_y \geq \mathbf{1}\{R^{\mathrm{f}} = \emptyset\}$ and $Z_\emptyset^{\mathrm{lin}}(\gamma) \geq Z_\emptyset(\gamma) = Z_q \pi_\emptyset e^{-\lambda q'(\gamma)}$. Finally Theorem 625(ii) and $-\log(1-x) \leq x/(1-x)$. □

So on the coarse fields whose background is calm with the margin κ_{f}, the sharp calm weight of the linear model is the Gaussian form up to a factor between $\pi_\emptyset$ and one, and this factor costs at most $96e^{-t}/(1-e^{-t})$ per block. This is a bound on the total, not a local representation of $\gamma \mapsto -\log Z_\emptyset^{\mathrm{lin}}(\gamma) - \lambda q'(\gamma)$. With the sharp threshold the calm events at $\gamma \neq 0$ are not symmetric, the correlation inequality does not apply to them, and the comparison of the proof of Theorem 625(iv) leaves the factor $(1-e^{-t})^{-96|\Lambda'|}$, which is not uniform in the volume. This is why the thresholds are placed on the fluctuation.

Remark 630 (what the SU(2) step needs beyond the Gaussian layer). Theorem 625 and its corollaries concern the linear model only. A one-step expansion of the calm weight $Z_\emptyset(c)$ of SU(2) as a Gaussian factor times a polymer remainder with the same properties is not proved here, and the following parts of it are missing.

(a) *The background.* The minimiser of Q on the fibre F_c of the cut-off average, its existence and uniqueness near the linear minimiser, and a bound of its plaquettes by the coarse curvature, the nonlinear analogue of Proposition 628. By Theorem 610 the leading term must be invariant under coarse gauge transformations, so the background is needed for every c in the class of Corollary 603, not only for c near I.

(b) *The non-Gaussian density.* In the chart of Theorem 363(iii) around the background, the constrained law has the density $e^{-\lambda Q(\Phi(\eta)) - \psi(\eta)}$, a log-concave perturbation of a Gaussian law for large λ; the mass outside the chart is not bounded (Remark 365(a)).

(c) *The conditioning bound.* Royen's inequality, used in Theorem 625(ii)–(iv), is a property of Gaussian measures. A strongly log-concave law conditioned on a convex set is again strongly log-concave, with the same constant, and concentrates about its conditional mean; so the missing input for SU(2) is a bound on the conditional means of the plaquette fluctuations, uniform in the convex conditioning event.

(d) *Locality.* The Gaussian form q' is not of finite range. Its kernel decays at the first step by Theorem 611(iv), and its decay at the later scales of Corollary 626 is not proved. Whether it couples blocks three apart is the question whether $(M_W)_{AB}$ of Theorem 607 vanishes, and $(M_W)_{AB} \neq 0$ is not proved by hand (Remark 608).

(e) *Thresholds at all scales.* With thresholds on the fluctuation at every scale, a fine plaquette is a sum of the lifts of the fluctuations of all scales and of the last background. A bound of these lifts, which would connect Corollary 626 with the single-scale thresholds of Theorem 600, is not proved.

The next step. To repeat the step at the coarse scale with constants (λ_1, κ_1), the proofs above need three properties of the weight $e^{-A_1(c)}$ in place of $e^{-\lambda Q}$: (H1) it is a product of

nonnegative factors attached to the coarse blocks, each depending on c within distance one, up to a polymer expansion of the kind just described; (H2) the factor of a block that owns a coarse plaquette with $\delta(c_P,I)>\kappa_1$ is at most $e^{-\lambda_1\kappa_1^2}$ times the calm weight, with $\lambda_1\kappa_1^2-8\log M-c_1(\lambda_1,1)\geq 9$; (H3) the averages of calm coarse fields are calm. Item (H3) holds at every step for $\kappa_1<\kappa_c$, since Proposition 470(iv) is a statement about one field on any lattice; the threshold then grows by 16.003 per step, or by $4\cdot 1.023$ along the composed average of Theorem 388. The suppression actually available for (H2) is inherited from the fine scale, as follows.

Proposition 631. *Let $G=\mathrm{SU}(2)$, $\kappa<\kappa_c$ and $j\geq 1$.*

(i) *If $\delta(c_P,I)>16.003\kappa$, then $N(P)$ meets $\Pi(R(U))$. Each rough block meets at most 2310 witness regions, so at most $2310|R(U)|$ coarse plaquettes exceed 16.003κ. On them the kernels carry $e^{-\lambda Q^{(i)}/C_*}$, that is, the coupling λ/C_*.*

(ii) *Let $\rho\leq 1.023\cdot 10^{-4}$, $\rho_0=\rho/(1.01\cdot 1.023\cdot 4^j)$, and call a fine block rough at level ρ_0. By Lemma 466 and Theorem 471(i), every bad block of scale j has a rough fine block among the $m_j=(17\cdot 2^{j-1}-2)^4$ blocks that meet its box. A fine block meets the boxes of at most H_j blocks of scale j, with H_j as in that lemma. Hence for a set Y of blocks of scale j and $a_0>0$, $\sum_X e^{-a_0|X|}\leq e^{-A_j|Y|}$, where the sum runs over the sets X of fine blocks inside the union of the boxes of Y that meet every box, and*

$$A_j=\max_{s>0}\Big(\frac{a_0-s}{H_j}-m_j\log(1+e^{-s})\Big).$$

(iii) *With $a_0=a(\beta_W/4,\rho_0)$ and $\rho=10^{-4}$, $A_j\geq 9$ holds for $\beta_W\geq\beta_j^{\mathrm{pw}}$, where $\beta_1^{\mathrm{pw}}=2.733\cdot 10^{14}$, $\beta_2^{\mathrm{pw}}=4.393\cdot 10^{15}$ and $\beta_3^{\mathrm{pw}}=1.095\cdot 10^{17}$, and for $\beta_W\geq 2.90\cdot 10^{13}\cdot 16^j$ whenever $1\leq j\leq 30$; $A_j<9$ for $\beta_W<1.57\cdot 10^{13}\cdot 16^j$. For every $\beta_W\in[10^{22},10^{42}]$, the largest $j\leq 30$ with $\beta_W\geq 2.90\cdot 10^{13}\cdot 16^j$ satisfies $k_*-j\in\{1,2\}$, with k_* as in Corollary 389.*

Proof. (i) is Lemma 597 and Theorem 600(ii), with $N_{\mathrm{ov}}=1441$. (ii) The box of Lemma 466 has $17\cdot 2^j-4$ points per coordinate, from an even to an odd coordinate, so it meets $\frac12(17\cdot 2^j-4)$ fine blocks per coordinate. A witness plaquette with corner in the box is owned by one of these. The number of boxes met by a fine block is the count of that lemma for the two cubes of the block per coordinate, which is H_j. A set X meeting all the boxes of Y has $|X|\geq|Y|/H_j$ and lies in a set of at most $m_j|Y|$ blocks. So the sum is at most $\sum_{k\geq|Y|/H_j}\binom{m_j|Y|}{k}e^{-a_0k}\leq e^{-(a_0-s)|Y|/H_j}(1+e^{-s})^{m_j|Y|}$.

(iii) Let $\lambda=\beta_W/4$, $b=16^{-j}\beta_W$ and $c=1.01\cdot 1.023$, so that $\lambda\rho_0^2=b/(4c^2\cdot 10^8)$ with $4.27025<4c^2<4.27026$, and, with $L_p=\log p$ as in the proof of Theorem 601(ii), $\log\lambda=\log b+(4j-2)L_2$. By that proof $a_0\geq\lambda\rho_0^2-67.5\log\lambda-K_0$ with $K_0<953.06$. Taking $e^s=m_jH_j$ in the maximum, and using $m_j\log(1+1/(m_jH_j))\leq 1/H_j$,

$$A_j\geq\frac{a_0-1-\log(m_jH_j)}{H_j}.$$

So $A_j\geq 9$ holds when $E(b)\geq R_j$, where

$$E(b)=\frac{b}{4c^2\cdot 10^8}-67.5\log b,$$
$$R_j=954.06+9H_j+\log(m_jH_j)+67.5(4j-2)L_2.$$

E increases for $b>3\cdot 10^{10}$. Since $m_j<17^4\,2^{4j-4}$ and $\log 17<4L_2+\frac{1}{16}$, $\log(m_jH_j)<\frac14+(4j+24)L_2$ for $j\leq 2$ and $<\frac14+(4j+12)L_2+8L_3$ for $j\geq 3$; so $R_1<37931.3$, $R_2<38121.3$, $R_3<60496.7$, and $R_j\leq R_{30}<65624.6$ for $1\leq j\leq 30$. With $\log(1.728\cdot 10^{13})=6L_2+3L_3+10\log 10$, $\log(2.7\cdot 10^{13})=3L_3+12\log 10$ and $\log(3\cdot 10^{13})=L_3+13\log 10$ as upper bounds for $\log b$, $E(b)$ exceeds 37938 at $b=2.733\cdot 10^{14}/16$, 38127 at $b=4.393\cdot 10^{15}/16^2$, 60505 at $b=1.095\cdot 10^{17}/16^3$ and 65816 at $b=2.90\cdot 10^{13}$, which gives the thresholds. Conversely $A_j<a_0/H_j<\lambda\rho_0^2/H_j$, since $\log M\geq 0$ and $c_1(\lambda,1)>0$, so $A_j\geq 9$ needs $b>9H_j\cdot 4c^2\cdot 10^8\geq 36864\cdot 4.27025\cdot 10^8>1.57\cdot 10^{13}$.

For the last assertion let j be that largest integer, so that $16^j\leq\beta_W/(2.90\cdot 10^{13})<16^{j+1}$, and $1\leq j\leq 23$ because $2.90\cdot 10^{13}\cdot 16<10^{22}$ and $2.90\cdot 10^{13}\cdot 16^{24}>10^{42}$. By Corollary 389, $X=10^{-4}/(1.01\rho(\beta_W))$ satisfies $16^{k_*}\leq X^2<16^{k_*+1}$ with $X^2=10^{-8}\beta_W/(1.0201\cdot 96(\log\beta_W+C_1))$, and $57<\log\beta_W+C_1<104$, since $C_1=C_0+\frac16(9L_2+2L_5)$ with $C_0\in[4.90816,4.90817]$ (proof of Theorem 465). Hence

$$16^{k_*-j}\leq\frac{X^2}{16^j}<\frac{16\cdot 2.90\cdot 10^{13}X^2}{\beta_W}<\frac{4.64\cdot 10^6}{97.92\cdot 57}<16^3,$$
$$16^{k_*-j}>\frac{X^2}{16^{j+1}}\geq\frac{2.90\cdot 10^{13}X^2}{16\beta_W}>\frac{2.90\cdot 10^5}{1567\cdot 104}>1,$$

and $k_*-j\in\{1,2\}$. □

So the next step stands as follows. Theorem 471 gives (H3) up to $a_{k_*}/8$ as a statement about probabilities under the Wilson measure: the labels of the composed average are rare. It gives neither (H1) nor (H2), which concern the weight e^{-A_1} for every coarse field. Proposition 631(iii) reaches $a_{k_*}/4$, but it bounds sums of fine kernels with entropy of ℓ^∞ adjacency rather than of the joining distance of the islands, and only for activities of the form $e^{-a_0|X|}$ relative to the calm weight, the hypothesis that is not proved. In both cases the suppression at scale j is the fine one at the radius $\rho_0\propto 4^{-j}$, that is, the coupling $\lambda/16^j$ in units of the scale-j threshold. By Proposition 468, fine witnesses carry it no further than a_{k_*}. Iterating the step instead loses the factor C_* of Theorem 598(i), or $H_j\geq 4096$ in Proposition 631(ii), at every scale. What the iteration needs is (H2) with $\lambda_1\geq(1-\varepsilon_1)\lambda$ and summable ε_k, a lower bound $A_1\geq(1-\varepsilon)\lambda S_1$ on large fields, local and relative to the calm weight. That requires a coarse energy for SU(2) at every scale, of the kind Proposition 402 gives for torus-valued fields. It is not proved here, and this subsection gives no bound on the mass gap.

Theorem 494 is a union bound over the window, and its conclusion concerns every island of W, so it is void in windows of more than 10^{28166} blocks. The data of Theorem 491 do not need islands as their units: the sets I_u are any pairwise separated sets of blocks off which every block of Λ is good. If the islands

are grouped into clusters whose separation grows with their size, the island sums and the weighted Dobrushin condition hold in every configuration of the labels, on every torus and in every window. The rarity of the bad blocks then enters only through the size of the clusters, and it bounds that size block by block, with no union bound over a window.

Definition 632 (clusters of bad blocks)**.** Let $R \geq 2s+8$ be an integer and $\kappa > 0$, and put $g(m) = (3/\kappa)\log m$ and $\ell_m = R + 2g(m)$ for $m \geq 1$. A partition $\mathscr{P}$ of a finite set B of blocks of the torus is *separated* if

$$\operatorname{dist}(U,V) > R + g(|U|) + g(|V|) \tag{52}$$

for all distinct $U, V \in \mathscr{P}$.

Lemma 633 (clusters)**.** *Let B be a finite set of blocks of the torus.*

(i) *Among the separated partitions of B there is a finest one; its parts are the* clusters *of B. Starting from the partition into single blocks and merging, in any order, two parts that violate (52), one reaches it after finitely many merges.*

(ii) *Two blocks of B at distance at most R lie in one cluster; in particular every island formed with the joining distance $2s+8$ lies in one cluster, and distinct clusters are at distance more than R.*

(iii) *A cluster of m blocks is connected in the graph on B in which two blocks are adjacent when their distance is at most ℓ_m; its blocks lie within distance $(m-1)\ell_m$ of each other.*

(iv) *If some cluster has more than M_1 blocks, then B contains a set of between M_1+1 and $2M_1$ blocks that is connected in the graph of distance at most ℓ_{M_1}.*

Proof. (i) Let $\mathscr{Q}$ be finer than a separated partition $\mathscr{P}$, and let $U, V \in \mathscr{Q}$ violate (52). If $U \subset U'$ and $V \subset V'$ with distinct $U', V' \in \mathscr{P}$, then, since g increases,

$$\operatorname{dist}(U',V') \leq \operatorname{dist}(U,V) \leq R + g(|U'|) + g(|V'|),$$

which contradicts the separation of $\mathscr{P}$; so $U \cup V$ lies in one part of $\mathscr{P}$, and the partition obtained by merging U and V is again finer than $\mathscr{P}$. The partition $\{B\}$ is separated, so the merging starts below every separated partition and stays below it at each step. Each merge lowers the number of parts, so the process stops, at a separated partition finer than every separated partition, which is therefore the finest one and does not depend on the order of the merges. (ii) Two single blocks at distance at most R violate (52), since $g(1) = 0$, so by the first step of (i) they lie in one part of every separated partition. An island is connected through steps of length at most $2s+8 \leq R$. The last assertion is (52) with $g \geq 0$. (iii) By induction over the merges, every part of m blocks of an intermediate partition is connected at distance ℓ_m: a single block is, and if U and V are merged, with $|U|+|V| = m$, then $\operatorname{dist}(U,V) \leq R + g(|U|) + g(|V|) \leq \ell_m$, while U and V are connected at distances $\ell_{|U|}, \ell_{|V|} \leq \ell_m$. Two blocks of a connected set of m blocks are joined by a path of at most $m-1$ steps. (iv) Consider the first merge of the process of (i) that produces a part of more than M_1 blocks. It merges two parts of at most M_1 blocks each, at distance at most $R + 2g(M_1) = \ell_{M_1}$, and each of them is connected at distance ℓ_{M_1} by (iii). □

Theorem 634 (the island sums for clusters)**.** *Let $s \geq 1$, $r \leq 8 \cdot 10^{-5}$, $0 \leq \vartheta < 1$, $\kappa = 2(1-\vartheta)\alpha_G$, $R \geq 2s+8$, and form the clusters of the set of bad blocks of the torus with these R and κ. Let the data be those of Theorem 491 on a set Λ of blocks, under its hypotheses, with $I_u = U_u \cap \Lambda$ for distinct clusters U_u, $u \in \mathscr{I}$, every bad block of Λ lying in some I_u. Put*

$$c_1 = \frac{1920}{\beta_W}\bar{\Lambda}_2^2 r^2 \cdot 45(2s+7)^4 e^{\kappa(2s+9)},$$
$$\tau = c_1 e^{-\kappa R/2}\sigma(\kappa/2),$$

with $\bar{\Lambda}_2$ as before Theorem 494 and $\sigma(\cdot)$ as in Proposition 493, and for $u \in \mathscr{I}$ let $\hat{T}_u = \sum_{v \in \mathscr{I}\setminus\{u\}} \Delta_{uv} e^{2\vartheta\alpha_G \ell(u,v)}$, with ℓ the pseudometric of Corollary 492. Then, for every configuration of the labels:

(i) *$\hat{T}_u \leq \tau$ for every $u \in \mathscr{I}$; hence $S_a, S_b \leq \tau$, the sums T_a, T_b of Corollary 492 are at most τ, and*

$$\Delta_{ab} \leq \min\bigl\{\tau,\ c_1 e^{-\kappa R/2} e^{-\kappa \operatorname{dist}(I_a,I_b)/2}\bigr\};$$

(ii) *$\Delta_* \leq \Delta_{ab} + \frac{1}{4}\tau^2$, and if $\tau < 4$ the weighted Dobrushin condition of Corollary 492 holds with $\theta \leq \tau/4$ and*

$$\Delta_* \leq \Delta_{ab} + \frac{e^{-2\vartheta\alpha_G \ell(a,b)}\tau^2}{4-\tau};$$

(iii) *for $\beta_W \geq \beta_0 > 0$ at fixed R, $\tau(\beta_W) \leq (\beta_W/\beta_0)\,\tau(\beta_0)$.*

At $\rho = 2.17 \cdot 10^{-16}$, $r = 43\rho$ and $\beta_W = \beta^*$ of Theorem 494: for $s = 1$, $\log_{10} c_1 \leq 21.74$ and $\tau \leq 10^{-1047}$ at $\vartheta = 0$, $R = 10^7$, and $\tau \leq 10^{-1045}$ at $\vartheta = \frac{1}{2}$, $R = 2 \cdot 10^7$; for $s = 5$, $\tau \leq 10^{-1045}$ and $\tau \leq 10^{-1044}$ at the same ϑ and R. By (iii) the same bounds hold at $\beta_W \geq \beta^*$ with R replaced by $R + \lceil (2/\kappa)\log(\beta_W/\beta^*)\rceil$.

Proof. The data. The sets I_u are pairwise disjoint and, by Lemma 633(ii), at distance more than $R \geq 2s+8$, so $w_{uv} \geq 2s+8$, and every block of Λ outside $\bigcup_u I_u$ is good. The proof of Lemma 487 bounds the norm of $\partial_{x_u}\nabla_z \mathscr{U}$ by the largest sum, over one x_u-link or one z-link, of the norms of the blocks of that matrix, and these are determined by the plaquettes at one link; as in the paragraph before Theorem 494, it applies to each I_u and gives $\Lambda_2 \leq \bar{\Lambda}_2$. Every x_u-link is based in a block at distance at most $s+3$ from I_u, so $n_u \leq 45(2s+7)^4 |I_u| \leq 45(2s+7)^4 m_u$, where $m_u = |U_u|$.

One pair. Let $u \neq v$ and $d = \operatorname{dist}(I_u, I_v)$, so that $w_{uv} - 2s - 8 = d - 2s - 9 \geq 0$ and $\ell(u,v) \leq d - 2s - 9$. Since $2\alpha_G - 2\vartheta\alpha_G = \kappa$,

$$\Delta_{uv} e^{2\vartheta\alpha_G \ell(u,v)} \leq c_1 \sqrt{m_u m_v}\, e^{-\kappa d}.$$

By (52), $d \geq \operatorname{dist}(U_u, U_v) > R + g(m_u) + g(m_v)$, so that

$$e^{-\kappa d} \leq e^{-\kappa d/2} e^{-\kappa R/2} (m_u m_v)^{-3/2},$$

since $e^{-\kappa g(m)/2} = m^{-3/2}$. Hence

$$\Delta_{uv} e^{2\vartheta \alpha_G \ell(u,v)} \le c_1 e^{-\kappa R/2} \frac{e^{-\kappa d/2}}{m_u m_v} \le \frac{c_1 e^{-\kappa R/2}}{m_u} e^{-\kappa d/2}.$$

The sum. For $v \neq u$ let y_v be a block of I_v nearest to I_u; distinct v give distinct y_v, and $e^{-\kappa \operatorname{dist}(I_u,I_v)/2} \le \sum_{x \in I_u} e^{-\kappa \operatorname{dist}(x,y_v)/2}$. Summing over v and using the preimages of the proof of Proposition 493(ii),

$$\sum_{v \neq u} e^{-\kappa \operatorname{dist}(I_u,I_v)/2} \le |I_u|\, \sigma(\kappa/2) \le m_u\, \sigma(\kappa/2),$$

so $\hat{T}_u \le \tau$. The sums S_a, T_a and the row sums of the corollary are sums of terms of $\hat{T}_a$ or $\hat{T}_q$, with factors at most one in the case of S_a, and Δ_{ab} is one term of $\hat{T}_a$; the second bound on Δ_{ab} is the bound of one pair with $m_a m_b \ge 1$ and $e^{-2\vartheta \alpha_G \ell(a,b)} \le 1$. This proves (i).

(ii) The first form of Δ_* in Theorem 491 and (i) give the first bound. By $\tanh t \le t$, the sum of Corollary 492 at q is at most $\frac{1}{4}\hat{T}_q \le \tau/4$, so $\theta \le \tau/4 < 1$, and the corollary with $T_a, T_b \le \tau$ gives the second.

(iii) $\bar{\Lambda}_2(\beta)/\beta$ decreases, so $\bar{\Lambda}_2(\beta_W) \le (\beta_W/\beta_0)\bar{\Lambda}_2(\beta_0)$ and $\bar{\Lambda}_2^2/\beta_W$ grows at most linearly.

The numbers. Let $L_p = \log p$ with the bounds of the proof of Theorem 601(ii); the same series at $x = \frac{1}{21}$ and $\frac{1}{73}$ gives $\log\frac{11}{10}$ and $\log\frac{37}{36}$ to within 10^{-10}, so $2.39789527 < L_{11} < 2.39789528$ and $3.61091791 < L_{37} < 3.61091792$. The function $g(\beta) = \beta\rho^2/16 - 6(\log\beta + C_0) - |\log q^*|$ increases for $\beta > 96/\rho^2$ and β^* is its largest zero, with $C_0 < 4.90817$ (proof of Theorem 465) and $\rho^2 = 4.7089 \cdot 10^{-32}$. Since $N_s < (4s+17)^4$,

$$|\log q^*| < H\big(L_2 + 8\log(4s+17)\big) + 1 + 4L_{11} + 4L_2 + 4L_3,$$

which is below 32481.69 for $s = 1$, with $\log 21 = L_3 + L_7$, and below 38354.08 for $s = 5$. At $\beta_1 = 1.1221 \cdot 10^{37}$, $\beta_1\rho^2/16 > 33024.10$ and, with $\log 1.1221 < \log\frac{9}{8}$, $6(\log\beta_1 + C_0) < 541.33$, so $g(\beta_1) > 1$; at $\beta_5 = 1.3217 \cdot 10^{37}$, $\beta_5\rho^2/16 > 38898.45$ and, with $\log 1.3217 < \log\frac{4}{3}$, $6(\log\beta_5 + C_0) < 542.35$, so $g(\beta_5) > 2$. Hence $\beta^* < \beta_s$ for $s = 1, 5$. As $\bar{\Lambda}_2^2/\beta_W = (99.0001 + 4.65 \cdot 10^6/\beta_W)^2\beta_W$ increases with β_W, and $e^x \le 1 + x + x^2$ for $0 \le x \le 1$, at $\beta_W = \beta^*$

$$c_1 \le 1920 \cdot 99.0002^2 \cdot 43^2 \cdot 45(2s+7)^4\big(1 + x + x^2\big)\beta_s\rho^2,$$

where $x = \kappa(2s+9)$. For $s = 1$ this is below $5.4581 \cdot 10^{21}$ at $\vartheta = 0$ and $5.4431 \cdot 10^{21}$ at $\vartheta = \frac{1}{2}$, and for $s = 5$ below $8.2171 \cdot 10^{22}$ and $8.1779 \cdot 10^{22}$. In particular $\log_{10} c_1 < 21.74$ for $s = 1$, $\vartheta = 0$, as $5.4581 < 3^7/400$ and $7L_3 - 4L_2 - 2L_5 < 0.74\log 10$. By the proof of Proposition 493, $\sigma(\kappa/2) \le 1 + 384(\kappa/2 - \kappa^2/8)^{-4} + 16(\kappa/2 - \kappa^2/8)^{-2}$, which is below $9.8354 \cdot 10^{16}$ at $\vartheta = 0$, $\kappa/2 = 2.5 \cdot 10^{-4}$, and below $1.5733 \cdot 10^{18}$ at $\vartheta = \frac{1}{2}$. In all four cases $\kappa R/2 = 2500$, and $e^{-2500} < 10^{-1085.736}$ since $1085.736\log 10 < 2500$. The product $c_1\sigma(\kappa/2)$ is below $5.37 \cdot 10^{38} < 10^{38.736}$ for $s = 1$, $\vartheta = 0$, as $\log 5.4 = 3L_3 - L_5 < 0.736\log 10$; below 10^{40} for $s = 1$, $\vartheta = \frac{1}{2}$ and for $s = 5$, $\vartheta = 0$; and below $1.3 \cdot 10^{41} < 10^{41.736}$ for $s = 5$, $\vartheta = \frac{1}{2}$. This gives the four bounds on τ. □

The theorem has no exceptional event: the conditions that Theorem 494 obtains outside probability $10^{-28166}|W|$ hold for the clusters in every window of every size. What the rarity has to give is that the clusters are small, and this is a statement about one block at a time. Write $U(y)$ for the cluster of a bad block y and $I(y)$ for its island.

Theorem 635 (the size of the clusters)**.** *Let $G = \mathrm{SU}(2)$, $s \ge 1$, ρ, r, η, β^*, q^*, $M = 11^4$ and $H = 6^4$ be as in Theorem 494, $\beta_W \ge \beta^*$, $p = q_\rho(\beta_W) \le q^*$, $A = eMHp$, and form the clusters of the bad blocks of a torus of L^4 blocks with $R \ge 2s+8$ and $\kappa > 0$. Put $N_5 = 11^4 - 1$, $N_\ell = (2\ell+1)^4 - 1$ for $\ell \ge 5$, and*

$$\nu_\ell = 4\big(N_5 + N_\ell A^{1/2}\big)A^{1/(2H)}.$$

(i) *For every block y and $m \ge 1$, $\mu_L\{y \in B,\ |U(y)| = m\} \le A^{(H+1)/(2H)}\nu_{\ell_m}^{m-1}$.*

(ii) *If $\nu_{\ell_{M_1}} < 1$, then*

$$\mu_L\{\text{a cluster has more than } M_1 \text{ blocks}\} \le \frac{L^4 A^{(H+1)/(2H)}\nu_{\ell_{M_1}}^{M_1}}{1 - \nu_{\ell_{M_1}}}.$$

(iii) *If $1 \le m_0 \le M_1$ and $\nu_{\ell_{M_1}} < 1$, then $\mu_L\{y \in B,\ U(y) \neq I(y)\}$ is at most*

$$(2(m_0-1)\ell_{m_0}+1)^4(2\ell_{m_0}+1)^4(Mp)^2 + \frac{A^{(H+1)/(2H)}\nu_{\ell_{M_1}}^{m_0}}{1 - \nu_{\ell_{M_1}}} + \varepsilon_{M_1},$$

with ε_{M_1} the bound of (ii).

At $\rho = 2.17 \cdot 10^{-16}$, $\beta_W = \beta^*$, $s = 1$, $\vartheta = 0$, $\kappa = 2\alpha_G$, $R = 10^7$ and $M_1 = 10^{10^{1700}}$: $\ell_{M_1} \le 10^{1705}$, $\nu_\ell \le 0.2130$ for every $\ell \le \ell_{M_1}$, the bound of (ii) is at most $L^4\, 10^{-0.67M_1}$, and (iii) with $m_0 = 10^5$ gives

$$\mu_L\{y \in B,\ U(y) \neq I(y)\} \le 10^{-28126} + L^4\, 10^{-0.67M_1}.$$

At $\vartheta = \frac{1}{2}$, $R = 2 \cdot 10^7$ the first term is at most 10^{-28124}. For $s = 5$, $\nu_\ell \le 0.0222$, the bound of (ii) is at most $L^4\, 10^{-1.65M_1}$, and the first term of (iii) is at most 10^{-33227} at $\vartheta = 0$, $R = 10^7$ and 10^{-33224} at $\vartheta = \frac{1}{2}$, $R = 2 \cdot 10^7$. The same numbers hold at $\beta_W \ge \beta^*$ with R replaced by $R + \lceil (2/\kappa)\log(\beta_W/\beta^*)\rceil$.

Proof. (i) By Lemma 633(iii), on the event $U(y)$ is a set $S \ni y$ of m bad blocks, connected at distance $\ell = \ell_m$. Let $S_1, \dots, S_k$ be the components of S in the graph of distance at most 5; they are pairwise at distance at least 6, so Step 1 of the proof of Theorem 494 gives

$$\mu_L(S \subset B) \le \prod_i A^{\lceil |S_i|/H \rceil} \le A^{k/2 + m/(2H)},$$

since $\lceil n/H \rceil \ge \frac{1}{2}(1 + n/H)$ and $A < 1$. Choose a spanning tree of S formed by spanning trees of the S_i at distance 5 and $k-1$ further edges of length at most ℓ, which exist because S

is connected at distance ℓ. A walk of $2(m-1)$ steps around the tree from y recovers S: it is fixed by the positions of the $m-1$ forward steps, at most 4^{m-1} choices, by which $k-1$ of them are long, at most $\binom{m-1}{k-1}$, and by their offsets, N_5 for a short step and N_ℓ for a long one. Summing over k,

$$\begin{aligned}&\mu_L\{y\in B,\ |U(y)|=m\}\\&\quad\le A^{\frac12+\frac{m}{2H}}\sum_{k=1}^{m}4^{m-1}\binom{m-1}{k-1}N_5^{m-k}N_\ell^{k-1}A^{\frac{k-1}{2}}\\&\quad=A^{\frac12+\frac{m}{2H}}\big(4(N_5+N_\ell A^{1/2})\big)^{m-1},\end{aligned}$$

which is (i). (ii) By Lemma 633(iv) some block lies in a set of $m\in(M_1,2M_1]$ bad blocks connected at distance ℓ_{M_1}; the count of (i) with $\ell=\ell_{M_1}$, summed over $m>M_1$ and over the L^4 blocks, gives (ii). (iii) If $|U(y)|>m_0$, either $|U(y)|\le M_1$, which by (i) and $\nu_{\ell_m}\le\nu_{\ell_{M_1}}$ for $m\le M_1$ has probability at most the second term, or some cluster has more than M_1 blocks. If $|U(y)|\le m_0$ and $U(y)$ contains a second island, then $U(y)$, connected at distance ℓ_{m_0}, has an edge between blocks z,z' of different islands: z lies within distance $(m_0-1)\ell_{m_0}$ of y, and $2s+9\le\operatorname{dist}(z,z')\le\ell_{m_0}$. Step 1 of the proof of Theorem 494 with $S_1=\{z\}$, $S_2=\{z'\}$ gives $(Mp)^2$, and a union bound over z and z' gives the first term.

For $\beta_W=t\beta^*$, $t\ge1$, Step 6 of the proof of Theorem 494 gives $\log(p/q^*)\le f(\beta_W)-f(\beta^*)\le 6\log t-32481(t-1)$, since $\beta^*\rho^2/16>|\log q^*|>32481$ for $s=1$ and $s=5$, by the bounds below. With R raised by $\lceil(2/\kappa)\log t\rceil\le1+(2/\kappa)\log t$, the length $\ell_{m_0}\ge R$ is multiplied by at most $1+10^{-7}+4\cdot10^{-4}\log t$, and the first term of (iii) by at most $(1+10^{-7}+4\cdot10^{-4}\log t)^8t^{12}e^{-64962(t-1)}<1+10^{-6}$, since $\log t\le t-1$; the margins below absorb this factor. The factor N_ℓ in ν_ℓ grows by a power of $1+\log t$ that $A^{1/2}$ absorbs, and the other factors decrease.

The numbers. Let $L_p=\log p$ with the bounds of the proofs of Theorems 601(ii) and 634. By the definition of q^*, with $N_s=(4s+17)^4-1$ as in Theorem 494, $A\le eMHq^*=(2N_s^2)^{-H}$ and $Mp\le Mq^*=(2N_s^2)^{-H}/(eH)$. From $4\log(4s+17)-1/N_s\le\log N_s<4\log(4s+17)$, $\log(2N_s^2)$ exceeds 25.049316 at $s=1$ and 29.580489 at $s=5$, while $\log(eH)=1+4L_2+4L_3>8.167037$. Let $s=1$ and $\vartheta=0$, so $\kappa=5\cdot10^{-4}$, $6/\kappa=12000$ and $\ell_{M_1}=10^7+12000\cdot10^{1700}\log10<10^{1705}$; at $\vartheta=\frac12$, $\ell_{M_1}<2\cdot10^7+24000\cdot10^{1700}\log10<10^{1705}$ as well. Since $2N_1^2=75644940800>275000^2$, $A^{1/(2H)}<1/275000$, and $A^{1/2}<10^{-6998}$ as $2N_1^2>10^{11}/1.5>10^{10.8}$, by $1.5^5<10$; with $N_\ell<10^{6822}$ for $\ell\le10^{1705}$,

$$\nu_\ell<\frac{4\cdot14640}{275000}+10^{-176}<0.2130<\frac{16}{75}<10^{-0.67},$$

the last since $\log\frac{16}{75}=4L_2-L_3-2L_5<-0.67\log10$. As $A^{(H+1)/(2H)}<10^{-6998}$ and $1/(1-\nu_\ell)<2$, the bound of (ii) is at most $L^4\,10^{-0.67M_1}$. In (iii), $\ell_{m_0}=10^7+60000\log10<10138156$ and $(2(m_0-1)\ell_{m_0}+1)(2\ell_{m_0}+1)<4.125\cdot10^{19}=\frac{33}{8}\cdot10^{19}$, so the logarithm of the first term is below

$$\begin{aligned}&4\big(L_3+L_{11}-3L_2+19\log10\big)\\&\qquad-2592\cdot25.049316-2\cdot8.167037<-64763.49,\end{aligned}$$

while $28126\log10<64762.51$; the second term is below $10^{-6998}\cdot0.213^{10^5}$, so the sum of the first two terms is below 10^{-28126}. At $\vartheta=\frac12$, $R=2\cdot10^7$, $\ell_{m_0}<20276311$, the product is below $1.65\cdot10^{20}=3\cdot5\cdot11\cdot10^{18}$, and the logarithm of the first term is below $4(L_3+L_5+L_{11}+18\log10)-2592\cdot25.049316-2\cdot8.167037<-64757.95<-28124\log10$. For $s=5$, $2N_s^2>2650000^2$ gives $\nu_\ell<4\cdot14640/2650000+10^{-176}<0.0222<\frac{1}{45}<10^{-1.65}$, since $\log\frac{1}{45}=-2L_3-L_5<-1.65\log10$, hence the bound of (ii); and the logarithm of the first term of (iii) is below $4(L_3+L_{11}-3L_2+19\log10)-2592\cdot29.580489-2\cdot8.167037<-76508.29<-33227\log10$ at $\vartheta=0$, and, with the bracket $4(L_3+L_5+L_{11}+18\log10)$ of $\vartheta=\frac12$, below $-76502.75<-33224\log10$ at $\vartheta=\frac12$. □

On a torus of at most $10^{M_1/2}$ blocks, $M_1=10^{10^{1700}}$, the theorem gives, outside probability $10^{-M_1/10}$, that no cluster has more than M_1 blocks, and the expected number of blocks of a window W that lie in a cluster other than a single island is at most $|W|(10^{-28126}+10^{-M_1/10})$ for $s=1$: a density, not a union bound. Theorems 634 and 635 thus remove the window size from the island comparison of item (ii) of Sec. XIII V, given the labels and with clusters as the units. The size of the torus remains only in the term ε_{M_1} of Theorem 635, at sizes far beyond any other in this paper. The next proposition shows that a separation growing with the size of the units is needed, and where the count of (i) stops.

Proposition 636 (the reach of the clusters)**.**

(i) *Let $R'\ge2s+8$ and $m\ge1$, and on a torus of side at least $2(m+2)(R'+1)$ let the bad blocks be $u_i=iR'e_1$ and $v_i=u_i+(-1)^i(R'+1)e_2$, $1\le i\le m$. In the finest partition of these blocks into parts at mutual distance more than R', the u_i form one part U and each v_i is a part V_i, and with $c_0=\frac{1920}{\beta_W}\bar\Lambda_2^2r^2\cdot45(2s+7)^4e^{-2\alpha_G(R'-2s-8)}$,*

$$\sum_i\bar\Delta_{UV_i}=c_0m^{3/2},\qquad\sum_{i,j}\bar\Delta_{u_jv_i}\ge c_0m.$$

So no partition whose parts are separated by a distance independent of their size bounds the sums $\hat T_u$, through $\bar\Delta$ either for the parts or for their islands, uniformly in the configuration.

(ii) *With $A=eMHq^*$, $s=1$: $\nu_\ell<1$ for $\log_{10}\ell\le1763.2$ and $\nu_\ell>1$ for $\log_{10}\ell\ge1763.3$, and $N_\ell A\ge1$ for $\log_{10}\ell\ge3524.5$. For ℓ_{M_1} beyond the first value the count of Theorem 635(i) is void, and beyond the second the bound $N_\ell A$ on the expected number of bad blocks within distance ℓ of a block exceeds one.*

Proof. (i) Consecutive u_i are at distance R' and so lie in one part; the distance of v_i to u_j is $\max\{|i-j|R',R'+1\}$, and to v_j, $j\ne i$, at least $\min\{2R',2R'+2\}>R'$, so the finest partition is the one stated, with $\operatorname{dist}(U,V_i)=R'+1$ and $w_{UV_i}=R'$. The neighbourhoods of radius $s+3$ of the u_i are disjoint, since $R'>2s+6$, so $\bar n_U=45m(2s+7)^4$ and $\bar n_{V_i}=45(2s+7)^4$, which gives the first identity. The pair (u_i,v_i) alone contributes c_0 to the second sum. Both sums grow without bound in m at fixed R'. (ii) $\nu_\ell<1$ is $N_\ell<A^{-1/2}(A^{-1/(2H)}/4-N_5)$. By

the definition of q^*, $A=(2N_1^2)^{-H}$, so $A^{-1/(2H)}=\sqrt{2}N_1\in(275036,275037)$, as $275036^2<2N_1^2=75644940800<275037^2$, and $A^{-1/(2H)}/4-N_5$ lies between $54000=2^4 3^3 5^3$ and $54675=3^7 5^2$. With the bounds of the proof of Theorem 635, $25.049316<\log(2N_1^2)<25.049327$, so the logarithm of the right side lies between 16242.85 and 16242.88. For $\log_{10}\ell\le 1763.2$, $\log N_\ell<4\log(2\cdot 10^{1763.2}+1)<4(L_2+1763.2\log 10)+10^{-1000}<16242.45$, and for $\log_{10}\ell\ge 1763.3$, $\log N_\ell\ge 4(L_2+1763.3\log 10)>16243.36$. For $\log_{10}\ell\ge 3524.5$, $\log N_\ell>32464.6>H\log(2N_1^2)=-\log A$. The expected number of bad blocks among the N_ℓ blocks within distance ℓ of a block is at most $N_\ell A$. □

The count of Theorem 635 stops at M_1, and the factor L^4 in its term ε_{M_1} cannot be removed: on every large enough torus the clusters of the bad blocks percolate. Two facts give this. Every block is bad with a conditional probability bounded below, by the finite energy of the Wilson measure, and the separation of the clusters grows with their size, which Proposition 636(i) and Proposition 642 below show to be necessary. What replaces the clusters of the torus are the clusters of the bad blocks of a window. Lemma 633(i)–(iii) and their proofs use of g only that it is nondecreasing and nonnegative, and so does the next lemma; for such g the clusters are formed by (52) and $\ell_m=R+2g(m)$.

Lemma 637 (subsets and absorption)**.** *Let $g\ge 0$ be nondecreasing.*

(i) *If $B'\subset B$ are finite sets of blocks, every cluster of B' lies in a cluster of B.*

(ii) *If C is a cluster of B and $z\in B$ has $\operatorname{dist}(z,C)\le R+g(|C|)$, then $z\in C$.*

Proof. (i) Let $\mathscr{P}$ be the partition of B into its clusters and $\mathscr{P}'$ the partition of B' into the nonempty sets $U\cap B'$, $U\in\mathscr{P}$. For distinct $U,V\in\mathscr{P}$ meeting B',

$$\begin{aligned}\operatorname{dist}(U\cap B',V\cap B')&\ge\operatorname{dist}(U,V)>R+g(|U|)+g(|V|)\\&\ge R+g(|U\cap B'|)+g(|V\cap B'|),\end{aligned}$$

since g is nondecreasing, so $\mathscr{P}'$ is a separated partition of B'. The partition of B' into its clusters is finer than $\mathscr{P}'$ by Lemma 633(i), so each cluster of B' lies in some $U\cap B'\subset U$. (ii) If z lay in a cluster $V\ne C$, then $\operatorname{dist}(C,V)\le\operatorname{dist}(z,C)\le R+g(|C|)+g(|V|)$, against (52). □

Lemma 638 (clusters of a window)**.** *Let Λ be a set of blocks of the torus, and form the clusters of $B\cap\Lambda$, with g and the parameters of Theorems 634 and 635. Each of them lies in a cluster of B, and:*

(i) *Theorem 634 holds with $I_u=U_u$ for the distinct clusters U_u of $B\cap\Lambda$;*

(ii) *Theorem 635 holds with $U(y)$ the cluster of y in $B\cap\Lambda$, $I(y)$ the component of y in $B\cap\Lambda$ at the joining distance $2s+8$, and L^4 replaced by $|\Lambda|$, with the same numbers.*

In a window Λ of at most $10^{M_1/2}$ blocks, on a torus of any size, no cluster of $B\cap\Lambda$ has more than M_1 blocks outside probability $10^{-M_1/10}$, and for $s=1$ the expected number of blocks y of Λ with $U(y)\ne I(y)$ is at most $|\Lambda|(10^{-28126}+10^{-M_1/10})$.

Proof. The first assertion is Lemma 637(i). (i) The proof of Theorem 634 uses of the clusters only that the sets U_u are distinct parts of a separated partition of a set of blocks that contains every bad block of Λ, with $I_u=U_u\cap\Lambda$: through Lemma 633(ii), through $|I_u|\le|U_u|$, and through (52). The clusters of $B\cap\Lambda$ are such parts, with $I_u=U_u$. (ii) The proof of Theorem 635(i) uses that $U(y)$ is connected at distance ℓ_m, which Lemma 633(iii) gives for the clusters of $B\cap\Lambda$, and bounds $\mu_L(S\subset B)$ for each set S it counts. In (ii), Lemma 633(iv) applied to $B\cap\Lambda$ gives a set of bad blocks containing a block of Λ, and the union bound runs over the $|\Lambda|$ blocks of Λ. In (iii) an edge of $U(y)$ between blocks of different components of $B\cap\Lambda$ has length between $2s+9$ and ℓ_{m_0}, as before. For the last assertion, (ii) gives at most $|\Lambda|10^{-0.67M_1}\le 10^{-0.17M_1}$, and (iii) with $m_0=10^5$ gives the bound for each block of Λ. □

Lemma 639 (one link makes a block bad)**.** *Let $G=\mathrm{SU}(2)$, let the labels have radius $r\le\frac{1}{10}$, let B be the set of bad blocks of a torus under μ_L, with the coarse field $c=V^\sharp(U)$ of the configuration, and put $p_0=\frac{1}{2}e^{-12\beta_W}$. Let ξ be a random set of blocks that contains each block independently with probability p_0. Then $\mu_L(B\in\mathscr{E})\ge\mathbb{P}(\xi\in\mathscr{E})$ for every increasing event $\mathscr{E}$, that is, every set of sets of blocks that contains each superset of each of its members.*

Proof. In the tree gauge μ_L has density proportional to $\exp(\beta_W\sum_p\frac{1}{2}\operatorname{Re}\operatorname{Tr}U_p)$ with respect to the product of the Haar measures on the active links (Sec. XII B). For a block y let $k=(y,1)$, with path link $\ell_k=U_{2y+e_1,1}$, let f_y be the link $(2y+e_1+e_2,1)$, a non-path private link of k, and put

$$G_y=\{d(U_{f_y},\ell_k)>r+\tfrac{\pi}{3}\}.$$

G_y makes y bad. The link $(2y,1)$ is a tree link, so $P_{2y}=\ell_k$, and by (35) $c_k=\ell_k e^Y$ with $Y=\frac{\chi_k}{16}\sum_x\log(\ell_k^{-1}P_x)$. If $\chi_k=0$, $c_k=\ell_k$. If $\chi_k>0$, then $\delta(\ell_k^{-1}P_x,I)=\delta(P_x,\ell_k)<\rho_G=1$ for every x, so $\|\log(\ell_k^{-1}P_x)\|<\pi/3$ by Lemma 352, and $\|Y\|<\pi/3$; the eigenvalues of Y are $\pm i\theta$ with $\theta<\pi$, so $\log e^Y=Y$ and $d(c_k,\ell_k)<\pi/3$. The reconstruction value of f_y is c_k, and on G_y

$$d(U_{f_y},c_k)\ge d(U_{f_y},\ell_k)-d(\ell_k,c_k)>r,$$

so y is bad.

One link. The link f_y lies in six plaquettes, each contributing $\beta_W\frac{1}{2}\operatorname{Re}\operatorname{Tr}U_p\in[-\beta_W,\beta_W]$ to the exponent, so given all the other active links the law of U_{f_y} has a density with respect to the Haar measure whose values lie within a factor $e^{12\beta_W}$ of each other, hence a density at least $e^{-12\beta_W}$, and G_y is the complement of the ball of radius $r+\pi/3$ about ℓ_k. For SU(2), by Lemma 678 and bi-invariance a ball of radius $\sqrt{2}t$ has Haar measure at most $\frac{2t^3}{3\pi}$. With $r+\pi/3\le 1.15$, $t\le 0.82$ and the measure is at most $2\cdot 0.552/(3\pi)<\frac{1}{2}$. So the conditional probability of G_y given all the other active links is at least p_0.

The coupling. Enumerate the blocks as $y_1,\dots,y_n$ and put $X_j=\mathbf{1}_{G_{y_j}}$. The event G_y depends only on the two private links ℓ_k and U_{f_y} of the pair $(y,1)$, which belong to no other pair, so $X_1,\dots,X_{j-1}$ are functions of the active links other than $U_{f_{y_j}}$, and $\mathbb{P}(X_j=1\mid X_1,\dots,X_{j-1})\ge p_0$. Let $V_1,\dots,V_n$ be independent and uniform on $[0,1]$, and put recursively $X'_j=\mathbf{1}[V_j\le \mathbb{P}(X_j=1\mid X_{<j}=X'_{<j})]$ and $Y_j=\mathbf{1}[V_j\le p_0]$. By induction on j the vector X' has the law of X, while $Y_j\le X'_j$ and the Y_j are independent with mean p_0. With $G=\{y: X_y=1\}\subset B$, for increasing $\mathscr{E}$, $\mu_L(B\in\mathscr{E})\ge\mu_L(G\in\mathscr{E})=\mathbb{P}(X'\in\mathscr{E})\ge\mathbb{P}(Y\in\mathscr{E})$. □

The next theorem is about independent bad blocks of any density and any separation that grows at least logarithmically. Two blocks of a cube of side a, a translate of $\{0,\dots,a-1\}^4$, and of a neighbouring cube of the same grid, one whose grid index differs by at most one in each coordinate, are at distance at most $2a-1$.

Theorem 640 (the clusters percolate)**.** *Let $g\ge 0$ be nondecreasing, $R\ge 1$, $\gamma>0$, $c_0\ge 0$, and suppose $R+g(m)\ge\gamma\log m-c_0$ for every $m\ge 1$.*[47] *Let $0<p<1$ and let ξ be a random set of blocks of a torus of side L that contains each block independently with probability p. Let $a_0\ge 2$ be an integer with $\gamma pa_0^3\ge 24$ and $pa_0^4\ge 23+4c_0/\gamma$, put $M_0=\lceil e^{(2a_0+c_0)/\gamma}\rceil$ and, for $j\ge 1$,*

$$K_j=\lfloor e^{pa_{j-1}^4/8}\rfloor,\qquad a_j=\lfloor\gamma pa_{j-1}^4/8\rfloor,$$
$$\ell_j=\max\{M_0,K_ja_{j-1}\}.$$

Then $a_j\ge 2a_{j-1}$, and K_j and ℓ_j increase to infinity. For $J\ge 1$, $L\ge\ell_J$, a block y, $Q=y+\{0,\dots,\ell_J-1\}^4$ and a set $\Lambda\supset Q$ of blocks, there is an event that depends only on $\xi\cap Q$, of probability at least $\frac12 p^{M_0}$, on which the cluster of y in $\xi\cap\Lambda$ contains the M_0 blocks $y+ie_1$, $0\le i<M_0$, and has at least K_J^4 blocks.

Proof. The scales. Write $a=a_{j-1}$. Since $\lfloor t\rfloor\ge t-1$ and $\gamma pa^3\ge 24$, $a_j\ge\gamma pa^4/8-1\ge 3a-1\ge 2a$, so $pa_j^4\ge 16pa^4$ and $\gamma pa_j^3\ge\gamma pa^3$; by induction $pa_j^4\ge 23\cdot 16^j$, both conditions on a_0 hold for every a_j, and K_j and ℓ_j increase to infinity. As $e^{pa^4/8}\ge 1$ and $\lfloor t\rfloor\ge t/2$ for $t\ge 1$, $\log K_j\ge pa^4/8-\log 2$, so

$$\gamma\log K_j^4-c_0\ge\frac{\gamma pa^4}{2}-4\gamma\log 2-c_0\ge\frac{\gamma pa^4}{4}\ge 2a_j-1, \quad (53)$$

the middle step by $pa^4\ge 16\log 2+4c_0/\gamma$, as $16\log 2<12$. The boxes $y+\{0,\dots,K_ja_{j-1}-1\}^4$ and the segment $S_0=\{y+ie_1: 0\le i<M_0\}$ lie in Q, which $L\ge\ell_J$ embeds in the torus, and the distance of the torus is at most that of $\mathbb{Z}^4$.

The events. Let $E_0=\{S_0\subset\xi\}$. For $1\le j\le J$ divide $y+\{0,\dots,K_ja_{j-1}-1\}^4$ into K_j^4 cubes of side a_{j-1}, the first of which contains y, and let E_j be the event that each of them contains a block of $\xi\setminus S_0$. These events depend on $\xi\cap Q$, E_0 on $\xi\cap S_0$ and the others on $\xi\cap(Q\setminus S_0)$. A cube of side a meets S_0 in at most a blocks, so

$$\mathbb{P}(E_j^c)\le K_j^4(1-p)^{a^4-a}\le e^{pa^4/2-pa^4+pa}\le e^{-pa^4/4},$$

as $a^3\ge 4$. Summing, $\sum_{j\ge1}\mathbb{P}(E_j^c)\le\sum_{i\ge0}e^{-23\cdot16^i/4}<e^{-5.75}(1+e^{-80})<0.004$, since $e^{5.75}>e^5\cdot 2>296$. By independence, $\mathbb{P}(E_0\cap\dots\cap E_J)\ge p^{M_0}(1-0.004)\ge\frac12 p^{M_0}$.

The cluster. On this event let C be the cluster of y in $\xi\cap\Lambda$. Consecutive blocks of S_0 are at distance $1\le R$, so $S_0\subset C$ by Lemma 633(ii), and $R+g(|C|)\ge\gamma\log M_0-c_0\ge 2a_0$. Suppose $R+g(|C|)\ge 2a_{j-1}-1$ for some $1\le j\le J$. Order the cubes of E_j so that each is a neighbour of an earlier one, the first containing y, and in each cube other than the first choose a block of $\xi\setminus S_0$. If the earlier cubes contain blocks of C, the chosen block of the next cube lies within distance $2a_{j-1}-1\le R+g(|C|)$ of C, so it lies in C by Lemma 637(ii). Hence every cube contains a block of C, $|C|\ge K_j^4$, and $R+g(|C|)\ge\gamma\log K_j^4-c_0\ge 2a_j-1$ by (53). Induction on j gives $|C|\ge K_J^4$. □

Fig. 44 draws the construction and the clusters of a window.

Corollary 641 (the bad blocks percolate)**.** *Let $G=\mathrm{SU}(2)$, $r\le\frac{1}{10}$, $\beta_W>0$, and let B be the set of bad blocks of a torus of side L under μ_L. Let g, R, γ and c_0 be as in Theorem 640, which for the clusters of Definition 632 holds with $\gamma=3/\kappa$ and $c_0=0$, and let $p=p_0=\frac12 e^{-12\beta_W}$, with a_0, M_0, K_j, ℓ_j as there. Then for every $J\ge1$ and $L\ge\ell_J$:*

(i) *for every block y and every $\Lambda\supset y+\{0,\dots,\ell_J-1\}^4$, in particular the torus, the cluster of y in $B\cap\Lambda$ has at least $\max\{M_0,K_J^4\}$ blocks with probability at least $\frac12 p_0^{M_0}$;*

(ii) *if $n\ge1$ is an integer with $n\ell_J\le L$, some cluster of B has at least K_J^4 blocks with probability at least $1-(1-\frac12 p_0^{M_0})^{n^4}$.*

Hence, for every K, $\mu_L\{y\in B,\ |U(y)|>K\}\ge\frac12 p_0^{M_0}$ on every large enough torus, and the probability that some cluster has more than K blocks tends to one as $L\to\infty$. With $\gamma=3/\kappa$, $c_0=0$ and $a_0=\lceil\max\{(24/(\gamma p_0))^{1/3},(23/p_0)^{1/4},2\}\rceil$, if $\kappa\ge 2.5\cdot10^{-4}$ and $\beta_W\ge10^{37}$, then $\log M_0>e^{10^{37}}$, so $M_0>M_1=10^{10^{1700}}$.

Proof. For a set Λ and a block y the event that $y\in B$ and the cluster of y in $B\cap\Lambda$ has at least K blocks is increasing, by Lemma 637(i), and so is the event that some cluster of B has at least K blocks. (i) is Lemma 639 and Theorem 640. (ii) Choose n^4 disjoint translates $Q_i=y_i+\{0,\dots,\ell_J-1\}^4$. The events of Theorem 640 for $(y_i,Q_i,\Lambda=Q_i)$ depend on disjoint parts of ξ, so they are independent; on each, the cluster of y_i in $\xi\cap Q_i$ has at least K_J^4 blocks, and it lies in a cluster of ξ by Lemma 637(i). Then apply Lemma 639. The consequence follows since $K_J\to\infty$. For the numbers, $a_0\ge(24/(\gamma p_0))^{1/3}=(48/\gamma)^{1/3}e^{4\beta_W}$ and $\log M_0\ge 2a_0/\gamma$. With $\gamma\le12000$, $2/\gamma\ge1/6000$ and $(48/\gamma)^{1/3}\ge250^{-1/3}>\frac17$,

[47] The proof is a multiscale construction: the occupied boxes of one scale are the sites of the next, and joining distances that grow like the logarithm of the size connect them, as in hierarchical and long-range percolation.

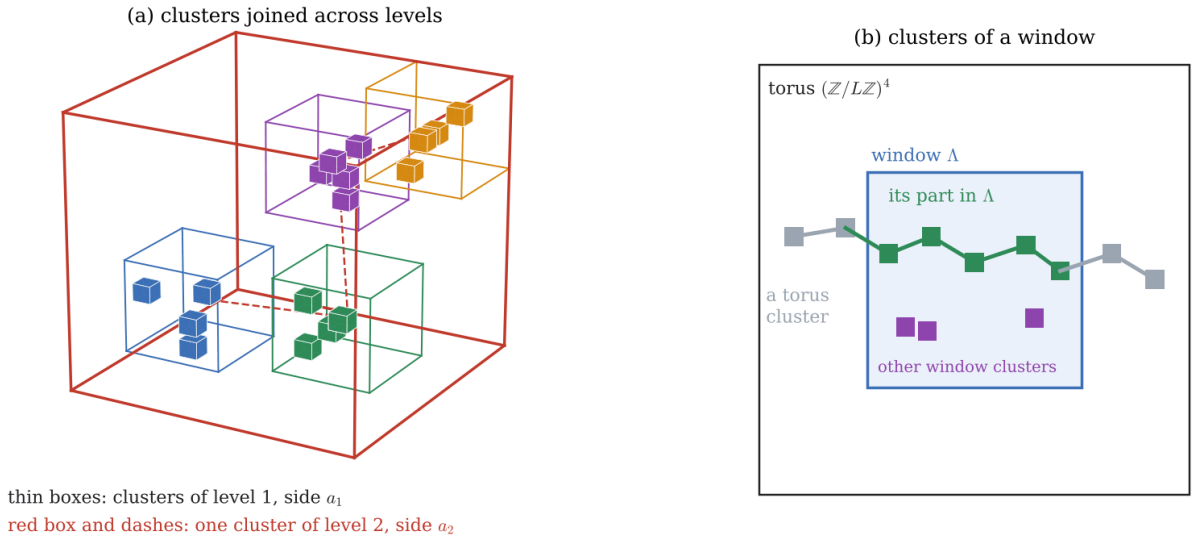


FIG. 44. Clusters of bad blocks, schematic, in three of the four dimensions. (a) The hierarchical construction of Theorem 640: bad blocks grouped into clusters of the first level in boxes of side a_1, and these joined into one cluster of the second level in a box of side a_2; the joining distance grows like the logarithm of the size. (b) A window Λ of the torus: a cluster of the torus that crosses its boundary is cut into clusters of the window, the units of Lemma 638.

so $\log M_0 > e^{4\beta_W}/42000 > e^{4\beta_W - 11} \geq e^{10^{37}}$, while $\log M_1 = 10^{1700}\log 10 < 10^{1701} < e^{10^{37}}$. □

Theorem 635 is not affected: its bound (i) concerns clusters of at most M_1 blocks, while the clusters of Corollary 641 have more than $M_0 > M_1$ blocks, and its bound (ii) grows with L^4. The corollary shows that this growth is not an artefact of the union bound. The next proposition shows that no choice of the separation avoids it while keeping the island sums bounded.

Proposition 642 (the separation the sums need)**.** *Let $g \geq 0$ be nondecreasing, $R \geq 2s+8$ with $R > 2g(1)$, and form the clusters with (52). For $m \geq 1$ put $h = R + \lfloor g(m) + g(1) \rfloor + 1$, and on a torus of side at least $2(m+2)(h+1)$ let the bad blocks be $u_i = iRe_1$ and $v_i = u_i + (-1)^i he_2$, $1 \leq i \leq m$. Then the clusters are $U = \{u_1, \dots, u_m\}$ and $V_i = \{v_i\}$, and with $\bar{\Delta}$ for clusters as in Proposition 636 and $c_2 = b_* \cdot 45(2s+7)^4 e^{-2\alpha_G(R+g(1)-2s-8)}$,*

$$\sum_i \bar{\Delta}_{UV_i} \geq c_2 m^{3/2} e^{-2\alpha_G g(m)}.$$

If these sums are at most τ_g for every m, then $R + g(m) \geq \gamma \log m - c_0$ for every m, with $\gamma = 3/(4\alpha_G)$ and $c_0 = \max\{0, (2\alpha_G)^{-1}\log(\tau_g/c_2)\}$, and the conclusions of Corollary 641 hold for these clusters.

Proof. The distances are $\mathrm{dist}(u_i,u_j) = |i-j|R$, $\mathrm{dist}(v_i,u_j) = \max\{|i-j|R, h\}$, and $\mathrm{dist}(v_i,v_j) \geq \min\{2R, 2h\} = 2R$ for $i \neq j$, since v_i and v_j differ by $2h$ in the direction e_2 when $i-j$ is odd; the side of the torus exceeds $2mR$ and $4h$, so these are the distances on the torus. The partition into U and the V_i is separated: $\mathrm{dist}(U,V_i) = h > R + g(m) + g(1)$ and $\mathrm{dist}(V_i,V_j) \geq 2R > R + 2g(1)$. Consecutive u_i are at distance R, so they lie in one cluster by Lemma 633(ii); hence U is a cluster, and the finest separated partition is this one. The neighbourhoods of radius $s+3$ of the u_i are disjoint, since $R > 2s+6$, so $\bar{n}_U = 45m(2s+7)^4$ and $\bar{n}_{V_i} = 45(2s+7)^4$, and $w_{UV_i} = h - 1 \leq R + g(m) + g(1)$. So $\bar{\Delta}_{UV_i} \geq c_2\sqrt{m}\, e^{-2\alpha_G g(m)}$ for each of the m values of i. If the sums are at most τ_g, then $2\alpha_G g(m) \geq \frac{3}{2}\log m - \log(\tau_g/c_2)$, which with $R \geq 0$ gives the condition of Theorem 640. □

For the clusters of Definition 632, $g(m) = (3/\kappa)\log m$ with $\kappa \leq 2\alpha_G$ exceeds $(3/(4\alpha_G))\log m$, as Theorem 634 needs. So every separation rule of the form (52) that bounds the island sums uniformly in the configuration produces clusters that percolate, by Corollary 641, and no bound on the size of the clusters of the torus is uniform in the torus. The clusters of a window replace them: by Lemma 638 the comparison of Theorem 634 holds with the clusters of $B \cap \Lambda$ as units, and in a window of at most $10^{M_1/2}$ blocks their size is bounded uniformly in the torus. The size of the window enters only through the term $|\Lambda| 10^{-0.67M_1}$, and it cannot be removed: by the proof of Corollary 641(ii), in a window that contains n^4 disjoint boxes of side ℓ_1 some cluster of $B \cap \Lambda$ has at least $M_0 > M_1$ blocks with probability at least $1 - (1 - \frac{1}{2}p_0^{M_0})^{n^4}$. The covariance of the conditional probabilities of the labels, the term of Proposition 455, is not touched by the clusters.

S. The comparison across one step and the conditional iteration

This subsection settles the constants of the comparison between the coarse action A_1 of Sec. XIII R and λ times the coarse curvature Q_1, and states the iteration over scales as a conditional theorem. The factor C_* of Theorem 598(i) is replaced by 2956 for every coarse field and by $1/\theta(\kappa_0) =$

$1+O(\kappa_0)$ for the coarse plaquettes below κ_1, which is about $0.19\,\kappa_0$ for small κ_0. No two-sided comparison of A_1 with λQ_1 has a ratio of constants below $4t_L^2$, which tends to 4: the factor four between the block-spin form and the coarse curvature is attained by a staggered coarse field and is intrinsic. The conditional theorem names one-step hypotheses (H_k), proves that they compose when their relative errors are summable, and locates the couplings at which the large-field estimates of Sec. XIII R stop. Throughout, $G=\mathrm{SU}(2)$, $L=2^m\geq 32$, and the notation is that of Sec. XIII R.

Norms. For $g\in\mathrm{SU}(2)$ with angle $2x$, $\delta(g,I)=2\sqrt{2}\sin x$ and $\|\log g\|=2\sqrt{2}x$ (Sec. XIII K), so $\delta(g,I)\leq\|\log g\|$, and the ratio $x/\sin x$ increases with x. If $\delta(g,I)\leq\kappa_0\leq 10^{-3}$, then $x\leq 1.001\sin x$, $x^2\leq 1.002\,\kappa_0^2/8$ and $(x/\sin x)^2\leq(1-x^2/6)^{-2}\leq 1+x^2/2.98$, so

$$\|\log g\|^2\leq\Big(1+\frac{\kappa_0^2}{20}\Big)\delta(g,I)^2,\qquad \|\log g\|\leq\Big(1+\frac{\kappa_0^2}{40}\Big)\kappa_0. \tag{54}$$

For $0<\kappa_0\leq 9.99\cdot 10^{-4}$ put

$$\theta(\kappa_0)=\Big[\Big(1+\frac{\kappa_0^2}{20}\Big)\big(1+6031\,\kappa_0\big)^2\Big]^{-1},$$
$$\kappa_1=\frac{\kappa_0}{\sqrt{28\,\theta(\kappa_0)}}.$$

Then $\theta(\kappa_0)\geq 1-12063\,\kappa_0$, since $(1+u)^{-2}\geq 1-2u$, and

κ_0	$\theta(\kappa_0)$	κ_1/κ_0
10^{-4}	0.38911…	0.30296…
10^{-6}	0.98804…	0.19012…
10^{-8}	0.99987…	0.18899…

Theorem 643 (comparison constants of one step)**.** *For every coarse field c on the torus:*

(i) $S_1(c)\geq Q_1(c)/2956$, *and* $A_1(c)\geq\lambda Q_1(c)/2956-4|T|\log M$;

(ii) *for* $0<\kappa_0\leq 9.99\cdot 10^{-4}$,

$$S_1(c)\geq\theta(\kappa_0)\sum_P\min\big\{\delta(c_P,I)^2,\kappa_1^2\big\},$$

and $A_1(c)$ is at least λ times the right side minus $4|T|\log M$; in particular, if $\delta(c_P,I)\leq\kappa_1$ for every coarse plaquette, $A_1(c)\geq\theta(\kappa_0)\lambda Q_1(c)-4|T|\log M$;

(iii) *for* $\lambda\geq 100$ *and* $\eta>0$, $A_1(c)\leq 4(1+\eta)\lambda Q_1(c)+c_1(\lambda,\eta)|T|$.

Proof. (i) Every $U\in F_c$ is in the tree gauge and has $V^\sharp(U)=c$, so Theorem 353(ii) gives $Q_1(c)\leq(8+2688\pi^2/9)Q(U)$, and $8+2688\pi^2/9<2956$. Take the infimum over F_c and use the first inequality of Theorem 598(i).

(ii) Let $U\in F_c$ and let $\mathscr{B}$ be the set of fine plaquettes with $\delta(U_p,I)>\kappa_0$. For a coarse plaquette P with sides $\ell\subset P$ let $D(P)$ be the union of the squares $\square_{2y+o}$ and of the sets $\mathscr{U}_\ell$ of the proof of Lemma 358. That proof uses its hypothesis on P only through the plaquettes of $D(P)$: the cut-off of a side ℓ depends on the paths of ℓ, whose surfaces lie in $\mathscr{U}_\ell$, and $K_{P,o}$ is the holonomy of $\square_{2y+o}$. Let $\mathscr{P}_1$ be the coarse plaquettes with $D(P)\cap\mathscr{B}=\emptyset$ and $\mathscr{P}_2$ the others. Let E_{g} and Q_{g} be the sums of $\|\Theta_p\|^2$ and of $\delta(U_p,I)^2$ over the fine plaquettes off $\mathscr{B}$. By (54) these have $\|\Theta_p\|\leq\eta:=(1+\kappa_0^2/40)\kappa_0<10^{-3}$, and $E_{\mathrm{g}}\leq(1+\kappa_0^2/20)Q_{\mathrm{g}}$.

For $P\in\mathscr{P}_1$ the first part of the proof of Lemma 358 applies with this η and gives the two inequalities of (37) and $r_P\leq 28.26\,\eta$. Its summation, restricted to $\mathscr{P}_1$, involves only plaquettes off $\mathscr{B}$, and its multiplicities (a plaquette in four squares $\square_x$, in at most four sets $\mathscr{U}_\ell$, a coarse link in six coarse plaquettes) do not grow under the restriction. The triangle inequality for the Euclidean norm over $\mathscr{P}_1$ therefore gives

$$\Big(\sum_{P\in\mathscr{P}_1}\|\log c_P\|^2\Big)^{1/2}\leq(1+6030\,\eta)\,E_{\mathrm{g}}^{1/2}.$$

Since $6030\,\eta\leq 6031\,\kappa_0$ and $\delta(c_P,I)\leq\|\log c_P\|$,

$$\sum_{P\in\mathscr{P}_1}\delta(c_P,I)^2\leq\theta(\kappa_0)^{-1}Q_{\mathrm{g}}.$$

A fine plaquette lies in $\square_{2y+o}$ for at most four pairs (P,o), since the pairs are in bijection with the squares $\square_x$, and in $\mathscr{U}_\ell$ for at most four coarse links, each a side of six coarse plaquettes. So it lies in $D(P)$ for at most 28 coarse plaquettes, and $|\mathscr{P}_2|\leq 28|\mathscr{B}|$. Hence

$$Q(U)=Q_{\mathrm{g}}+\sum_{p\in\mathscr{B}}\delta(U_p,I)^2$$
$$\geq\theta(\kappa_0)\sum_{P\in\mathscr{P}_1}\delta(c_P,I)^2+\frac{\kappa_0^2}{28}\,|\mathscr{P}_2|.$$

Each term of $\theta\sum_P\min\{\delta(c_P,I)^2,\kappa_1^2\}$ is at most the corresponding term on the right, as $\theta\kappa_1^2=\kappa_0^2/28$. Take the infimum over F_c and apply Theorem 598(i).

(iii) Theorem 598(ii) with $U=R(c)$, which is regular, and $Q(R(c))=4Q_1(c)$. □

The constant 2956 of (i) is less than $C_*/555$. Part (ii) is the small-field form: the coarse plaquettes below $\kappa_1\approx 0.19\,\kappa_0$ are compared with the factor $\theta(\kappa_0)$, which tends to one with κ_0, and every larger coarse plaquette costs at least $\theta\kappa_1^2=\kappa_0^2/28$. The upper constant 4 of (iii) and the lower constant $\theta(\kappa_0)$ of (ii) cannot be brought together, by the next theorem.

For a real field γ on the coarse links of the torus let

$$q_1(\gamma)=\min\Big\{\sum_p(d\varphi)_p^2:\varphi\text{ real on the links of }\Lambda_L,\ V(\varphi)=\gamma\Big\},$$

the linear block-spin form, with V the block average of Definition 328 and the curvature $(d\varphi)_p=\varphi_\mu(x)+\varphi_\nu(x+e_\mu)-\varphi_\mu(x+e_\nu)-\varphi_\nu(x)$. By Proposition 335(i), whose proof applies on every lattice, $q_1\geq Q_1$. Let $s(y)=(-1)^{y_0+y_1}$ and define γ_B by $\gamma_{B,(y,0)}=s(y)$, $\gamma_{B,(y,1)}=-s(y)$ and zero in the directions 2, 3; let $n=L/2$ and γ_A equal $\cos(2\pi y_0/n)$ on the coarse links $(y,1)$ and zero on the others. Put $t_L=\frac{3}{4}+\frac{1}{4}\cos(4\pi/L)$ and $f(u)=4(1-\cos(u/\sqrt{2}))$, so that $\delta(e^{u\tau},I)^2=f(u)$ for $\|\tau\|=1$ and $u^2(1-u^2/24)\leq f(u)\leq u^2$.

Theorem 644 (the factor four is intrinsic)**.** *Let $\tau \in \mathfrak{g}$ with $\|\tau\| = 1$.*

(i) *$q_1(\gamma_B) = 64|T| = 4\,Q_1(\gamma_B)$, attained at the field $\varphi^*(\gamma_B)$ of Proposition 335(ii), and $q_1(\gamma_A) \le t_L^{-2} Q_1(\gamma_A)$.*

(ii) *There is C, depending on L, such that $c^\varepsilon = e^{\varepsilon\tau\gamma_B}$ has $Q_1(c^\varepsilon) = |T| f(4\varepsilon) \le 16\varepsilon^2|T|$ and $S_1(c^\varepsilon) \ge (64 - C\varepsilon)\varepsilon^2|T|$ for $0 < \varepsilon \le C^{-1}$.*

(iii) *For $0 < \varepsilon \le C^{-1}$ the field $c_\varepsilon = e^{\varepsilon\tau\gamma_A}$ has a regular $U_\varepsilon \in F_{c_\varepsilon}$ with $Q(U_\varepsilon) \le t_L^{-2}(1 + C\varepsilon^2)\, Q_1(c_\varepsilon)$.*

(iv) *If $a, b > 0$ and $e(\lambda)$ with $e(\lambda)/\lambda \to 0$ satisfy*

$$a\lambda Q_1(c) - e(\lambda) \le A_1(c) \le b\lambda Q_1(c) + e(\lambda)$$

for every c and every $\lambda \ge 100$, then $b \ge 4$ and $a \le t_L^{-2}$. So $b/a \ge 4t_L^2$, which is above 3.849 for $L \ge 32$ and tends to 4.

Proof. (i) Write $\varphi^* = \varphi^*(\gamma_B)$, equal to $\gamma_{B,(y,\mu)}$ on the links (x,μ), $x \in B(y)$, $o_\mu = 1$, and to zero elsewhere. By Proposition 335(ii), $V(\varphi^*) = \gamma_B$ and $\sum_p (d\varphi^*)_p^2 = 4Q_1(\gamma_B)$, and its curvature is $(d\gamma_B)_P$ on the four plaquettes at $x \in B(y)$ with $o_\mu = o_\nu = 1$ and zero elsewhere. Here $(d\gamma_B)_P = 4s(y)$ in the plane $\{0,1\}$, since s changes sign under e_0 and e_1, and zero in the other planes, so $Q_1(\gamma_B) = 16|T|$. We claim

$$d^{\mathsf T} d\varphi^* = 4\varphi^* = V^{\mathsf T}(32\,\gamma_B). \tag{55}$$

Only plaquettes in the plane $\{0,1\}$ are curved. For the link $(x,0)$ the plaquettes at x and at $x - e_1$ in that plane enter with signs $+$ and $-$; the first is $4s(y)$ when $o_0 = o_1 = 1$, and the second is $4s(y - e_1) = -4s(y)$ when $o_0 = 1$, $o_1 = 0$. So $(d^{\mathsf T} d\varphi^*)_{(x,0)} = 4s(y)$ when $o_0 = 1$ and zero otherwise. The link $(x,1)$ is the same with the roles of the directions exchanged and the sign of the orientation reversed, which gives $-4s(y)$ when $o_1 = 1$. Links in the directions $2,3$ lie in no curved plaquette. For the second equality, a link (x,μ) with $o_\mu = 1$ enters only $V_{y,\mu}$, with coefficient $\frac18$, and a link with $o_\mu = 0$ enters $V_{y,\mu}$ and $V_{y-e_\mu,\mu}$ with $\frac{1}{16}$ each, and $\gamma_{B,(y-e_\mu,\mu)} = -\gamma_{B,(y,\mu)}$. Now for every real φ with $V(\varphi) = \gamma$,

$$\sum_p (d\varphi)_p^2 \ge 2\langle d\varphi, d\varphi^*\rangle - \sum_p (d\varphi^*)_p^2 = 64\langle \gamma, \gamma_B\rangle - 64|T|, \tag{56}$$

which for $\gamma = \gamma_B$, $|\gamma_B|^2 = 2|T|$, is $64|T|$, with equality at φ^*. For γ_A, T of Lemma 337 multiplies γ_A by t_L, since γ_A depends on y_0 alone and $\tau_0 \cos(2\pi y_0/n) = (\frac34 + \frac14 \cos\frac{2\pi}{n}) \cos(2\pi y_0/n)$. Lemma 337 with $k = 1$ on the coarse torus of side n, whose proof uses only the local structure, gives $V(\Pi(\gamma_A/t_L)) = \gamma_A$ and $\sum_p (d\Pi(\gamma_A/t_L))_p^2 \le Q_1(\gamma_A)/t_L^2$.

(ii) The curvature of c^ε is $e^{4\varepsilon s(y)\tau}$ in the plane $\{0,1\}$ and I elsewhere, so $Q_1(c^\varepsilon) = |T| f(4\varepsilon)$. Let $U \in F_{c^\varepsilon}$; we may assume $Q(U) \le 64\varepsilon^2|T|$. Then every plaquette has $\delta(U_p, I) \le 8\varepsilon|T|^{1/2}$, and for ε small Lemma 360(i) and (iv) with $W = T$ put every link within $C\varepsilon$ of I, as in the proof of Theorem 607(ii), using $\delta(c_k^\varepsilon, I) \le \varepsilon$. By Lemma 352, $U = e^X$ with $|X| = \max_e \|X_e\| \le C\varepsilon$. By Lemma 357, $\|\log U_p - (dX)_p\| \le C\varepsilon^2$, and $\delta(e^Y, I)^2 \ge \|Y\|^2(1 - \|Y\|^2/24)$, so $Q(U) \ge \sum_p \|(dX)_p\|^2 - C\varepsilon^3|T|$. As in that proof, $\log V_k^\sharp(e^X) = V_k(X) + O(|X|^2)$, so the component $X^\tau = \langle X, \tau\rangle$ has $V(X^\tau) = \varepsilon\gamma_B + r$ with $|r_k| \le C\varepsilon^2$. Apply (56) to X^τ/ε:

$$\begin{aligned}\sum_p \|(dX)_p\|^2 \ge \sum_p (dX^\tau)_p^2 &\ge 64\varepsilon^2|T| - 64\varepsilon\langle r, \gamma_B\rangle \\ &\ge (64 - C\varepsilon)\varepsilon^2|T|.\end{aligned}$$

(iii) By Lemma 336(iii), whose proof is local, $\Pi(\gamma_A/t_L)$ plus a gradient that vanishes under V is a field φ_A in the tree gauge with $V(\varphi_A) = \gamma_A$ and $\sum_p (d\varphi_A)_p^2 \le Q_1(\gamma_A)/t_L^2$. Let $U_\varepsilon = e^{\varepsilon\tau\varphi_A}$. All its links lie in the one-parameter group $e^{\mathbb{R}\tau}$, so the path product P_x of a coarse link is $e^{\varepsilon\tau\sigma_x}$, σ_x the sum of φ_A along the path of length two from x. For ε small, $\delta(P_x, P_{2y}) \le \varepsilon|\sigma_x - \sigma_{2y}|$ is below $\frac14$, so U_ε is regular, and Theorem 353(i) gives $V_{y,\mu}^\sharp(U_\varepsilon) = e^{\varepsilon\tau V_{y,\mu}(\varphi_A)}$, that is $V^\sharp(U_\varepsilon) = c_\varepsilon$. Then $Q(U_\varepsilon) = \sum_p f(\varepsilon(d\varphi_A)_p) \le \varepsilon^2 Q_1(\gamma_A)/t_L^2$ and $Q_1(c_\varepsilon) = \sum_P f(\varepsilon(d\gamma_A)_P) \ge \varepsilon^2 Q_1(\gamma_A)(1 - \varepsilon^2/6)$, as $|(d\gamma_A)_P| \le 2$.

(iv) By Theorem 598(i), $\lambda S_1(c) - 4|T|\log M \le A_1(c) \le b\lambda Q_1(c) + e(\lambda)$; divide by λ and let $\lambda \to \infty$: $S_1(c) \le bQ_1(c)$. At c^ε, (ii) gives $b \ge 4 - C\varepsilon/16$ for every small ε. By Theorem 598(ii) with any $\eta > 0$ and the regular field U_ε, $a\lambda Q_1(c_\varepsilon) - e(\lambda) \le (1 + \eta)\lambda Q(U_\varepsilon) + c_1(\lambda, \eta)|T|$, and $c_1(\lambda,\eta)$ grows like $\log\lambda$; so $aQ_1(c_\varepsilon) \le Q(U_\varepsilon)$, and (iii) gives $a \le t_L^{-2}(1 + C\varepsilon^2)$. The value $4t_{32}^2 = 3.8491\ldots$ and $t_L \to 1$ give the last statement. □

Remark 645. Part (i) makes exact the right end of the Bloch spectrum stated after Proposition 335: γ_B is the mode of momentum $(\pi,\pi,0,0)$ in the plane $\{0,1\}$, and (55) is the Lagrange condition of q_1 at it, with multiplier $64\gamma_B$. The field γ_A is the long-wavelength end. So C_* is a constant of the witness argument of Lemma 597 and not of the step: the best lower constant lies between $\theta(\kappa_0)$ on the small coarse fields and t_L^{-2}, the best upper constant is 4, and a comparison with λQ_1 with constants $1 + o(1)$ on both sides does not exist. Constants close to one on both sides hold for the fibre infimum itself: by Theorem 598, $\lambda S_1(c) - 4|T|\log M \le A_1(c) \le (1 + \eta)\lambda Q(U) + c_1(\lambda,\eta)|T|$ for every regular $U \in F_c$. At the linear level S_1 is the block-spin form q_1, which is not a sum of local terms if $(M_W)_{AB} \ne 0$ (Theorem 607). The reference forms R_k below are built from the block-spin form for this reason.

The conditional iteration. Let $K \ge 1$ and $\lambda_0 > \lambda_1 > \cdots > \lambda_K > 0$. For $0 \le k \le K$ let $\mathscr{X}_k$ be a finite dimensional real inner product space, for $k < K$ let $B_k : \mathscr{X}_k \to \mathscr{X}_{k+1}$ be linear and onto, and let H_k be a positive semidefinite quadratic form on $\mathscr{X}_k$. For a form H on $\mathscr{X}_k$ put $\mathscr{M}_k H(w) = \inf\{H(x) : B_k x = w\}$, and define the reference forms by

$$R_0 = H_0, \qquad R_{k+1} = \frac{\lambda_{k+1}}{\lambda_k}\, \mathscr{M}_k R_k.$$

In the application H_k is the gauge-invariant Hessian $\mathscr{H}A_k$ of the effective action of scale k at a chain of backgrounds, in exponential coordinates, and B_k the derivative of the cut-off average. At the trivial background, with $\mathscr{X}_k$ the $\mathfrak{g}$-valued

fields on the active links of spacing 2^k, $B_k = V$ and $H_0 = 2\lambda_0 \sum_p \|(dX)_p\|^2$, the Hessian of $\lambda_0 Q$ at I, the identity $V^{(k+1)} = V \circ V^{(k)}$ gives $R_k = 2\lambda_k S_k$, where $S_k(w) = \inf\{\sum_p \|(d\varphi)_p\|^2 : V^{(k)}(\varphi) = w\}$ on the torus; by Lemma 336(iii), whose proof is local, R_1 is $2\lambda_1 q_1$ on each component. The hypotheses at step $k < K$ are:

(H1) *relative error:* for some $0 \le \varepsilon_k < 1$,

$$(1-\varepsilon_k)^{\frac{\lambda_{k+1}}{\lambda_k}} \mathscr{M}_k H_k \le H_{k+1} \le (1+\varepsilon_k)^{\frac{\lambda_{k+1}}{\lambda_k}} \mathscr{M}_k H_k;$$

(H2) *locality:* on a subspace $\mathscr{Y}_{k+1} \subset \mathscr{X}_{k+1}$ on which R_{k+1} and H_{k+1} are positive definite, a slice of the coarse gauge orbits, with a metric on its coordinates and $\alpha_{k+1} \ge 0$, for every $W = e^{\alpha_{k+1}\phi}$ with ϕ 1-Lipschitz,

$$\|W R_{k+1}^{-1} W^{-1}\| \le G_{k+1}, \qquad \|W(H_{k+1} - R_{k+1})W^{-1}\| \le \frac{\ell_{k+1}}{G_{k+1}},$$

with $\ell_{k+1} < 1$;

(H3) *large-field cost:* the weight of a coarse field of scale $k+1$ is $Z_{k,\emptyset}(c)\,\Xi_k(c)$, where $\Xi_k(c)$ is the mean, under a probability law, of the sum over families of pairwise compatible nonempty ℓ^∞-connected sets X_i of blocks of scale $k+1$ of products $\prod_i \rho_{X_i}$, with $0 \le \rho_X \le e^{-9|X|}$ almost surely and compatibility as in Theorem 601.

For $k = 0$, the hypothesis of Proposition 604(ii) gives (H3) only when $Z_\perp = 0$. For SU(2), $Z_\perp(I) > 0$ (Theorem 621), so (H3) at $k = 0$ has to keep $Z_\perp$ as part of the weight, and is not available in the form stated.

Theorem 646 (conditional iteration)**.** *Suppose that (H1)–(H3) hold at every step $k < K$, and let $\sigma = \sum_{k<K} \varepsilon_k$, $|T_k| = |T|/16^{k-1}$ the number of blocks of scale k.*

(a) *For $k \le K$, $\prod_{j<k}(1-\varepsilon_j) R_k \le H_k \le \prod_{j<k}(1+\varepsilon_j) R_k$; in particular $(1-\sigma)R_K \le H_K \le e^\sigma R_K$.*

(b) *Let $\lambda_k = \lambda_0 - tk$, $t = 2b_0 \log 2$, the one-loop running of Remark 345(c), with $t = 0.0643779\ldots$ and $1/t < 15.534$ for* SU(2)*. If $\varepsilon_k \le c_\varepsilon \lambda_{k+1}^{-2}$ for every $k < K$, then*

$$\sigma \le c_\varepsilon \Big(\frac{1}{\lambda_K^2} + \frac{1}{t\lambda_K} \Big),$$

uniformly in λ_0 and K; so $\sigma \le \frac{1}{2}$ once $\lambda_K \ge 32,311,3107$ for $c_\varepsilon = 1, 10, 100$, that is $\beta_K \ge 128, 1244, 12428$. If instead $\varepsilon_k \ge c\,\lambda_{k+1}^{-1}$ for every k, then $\sigma \ge (c/t)\log(\lambda_0/\lambda_K)$, which is unbounded as $\lambda_0 \to \infty$ at fixed λ_K.

(c) *For $1 \le k \le K$, $\|W H_k^{-1} W^{-1}\| \le G_k/(1-\ell_k)$ on $\mathscr{Y}_k$ for every W as in (H2), and $|(H_k^{-1})_{ij}| \le G_k (1-\ell_k)^{-1} e^{-\alpha_k \operatorname{dist}(i,j)}$.*

(d) *For $k < K$, $\Xi_k = \mathbb{E}[e^{\Lambda_k}]$, where Λ_k is the absolutely convergent sum of the connected clusters, those incompatible with a given block add up to at most $\frac{1}{10}$, and $0 \le \log \Xi_k \le |T_{k+1}|/10$; the sum over the steps is at most $\frac{8}{75}|T|$.*

Proof. (a) If $H \le H'$ then $\mathscr{M}_k H \le \mathscr{M}_k H'$, and $\mathscr{M}_k(sH) = s\mathscr{M}_k H$ for $s \ge 0$; this is the argument of Proposition 460, which takes infima over the same affine subspaces and does not use definiteness. Induction on k with (H1) gives the products, and $\prod(1-\varepsilon_j) \ge 1-\sigma$, $\prod(1+\varepsilon_j) \le e^\sigma$. (b) $\lambda_{k+1} = \lambda_K + t(K-k-1)$, so $\sum_{k<K} \lambda_{k+1}^{-2} \le \lambda_K^{-2} + \int_0^\infty (\lambda_K + tx)^{-2} dx$. For the lower bound, $t\lambda_{k+1}^{-1} \ge \int_{\lambda_{k+1}}^{\lambda_k} d\lambda/\lambda$. With $1/t < 15.534$ the bound is at most 0.4865, 0.4997 and 0.49999 at $\lambda_K = 32$, 311, 3107 for $c_\varepsilon = 1, 10, 100$, and it decreases in λ_K. (c) The last statement of Proposition 460 with $H_0 = R_k$ and $\varepsilon = \ell_k$; for the entries take $\phi = \operatorname{dist}(\cdot, j)$ as in Lemma 459. (d) Theorem 601(i), whose proof uses only the activity bound and the count of connected sets of blocks, applied on the torus of scale $k+1$ to each realisation of the activities, as in the proof of Proposition 604(ii); and $\sum_{k\ge 0} 16^{-k}/10 = \frac{8}{75}$. □

The theorem composes hypotheses; it does not supply them. The relative form (H1) is the one that survives composition with factor one, while an absolute error grows by at least $4/(1+2724\kappa)$ per step (Proposition 461), and order-g^2 errors are not summable along the running, by (b). The locality (H2) is posed at each scale: Lemma 459 carries a bound on $H_k - R_k$ to the Schur complement only with the factor $(1+b_k g_k)^2$, and a relative bound (H1) carries no locality (Proposition 460).

Proposition 647 (what is proved of (H_k))**.** (i) *(H1) is not proved at any step. At $k = 0$, Proposition 565 gives it on a box from the absolute hypothesis (H) there, with ε' containing $\delta_G C_n/(2c_*\lambda)$, of order $g^2 n^2$, and Corollary 568 under a loop representation of the remainder that is not proved.*

(ii) *(H2): at the trivial background the reference $R_k = 2\lambda_k S_k$ is local as a form uniformly in k on $\mathbb{Z}^4$ (Theorem 340), and Theorem 367 gives decay of covariances at one scale. The bound on $H_{k+1} - R_{k+1}$ is not proved for any k.*

(iii) *(H3) at $k = 0$: the activities satisfy $\rho_X \le e^{-9|X|}$ where the boundary values admit a completion calm at $\kappa/30$, for $\lambda \ge \lambda^{**}(\kappa)$ (Theorem 602, Proposition 604); the shell $(\kappa/30, \kappa]$ and $Z_\perp$, which is positive for* SU(2)*, are not bounded, so (H3) is not proved at $k = 0$, and nothing is proved for $k \ge 1$.*

(iv) *The rates of Theorems 600(iii) and 602 are increasing in κ, and $\kappa < \kappa_c$ is required, where $0.0837613 \le \kappa_c \le 0.0837617$ (Sec. XIII K). For every $\kappa < \kappa_c$, $a(\lambda,\kappa) < 9$ when $100 \le \lambda \le 245000$ and $a_2(\lambda,\kappa) < 9$ when $100 \le \lambda \le 809300$, whatever $M \ge 1$; with $M < 2.3621 \cdot 10^4$, $a(\lambda,\kappa_c) \ge 9$ for $\lambda \ge 257000$ and $a_2(\lambda,\kappa_c) \ge 9$ for $\lambda \ge 833000$. So, with β^* and β^{**} at κ_c defined by these rates,*

$$9.800 \cdot 10^5 < \beta^*(\kappa_c) \le 1.028 \cdot 10^6,$$
$$3.2372 \cdot 10^6 < \beta^{**}(\kappa_c) \le 3.332 \cdot 10^6,$$

and along the one-loop running more than $1.257 \cdot 10^7$ steps lie between $\lambda = 809300$ and $\lambda = 1$.

Proof. (i)–(iii) restate the cited results. (iv) Both rates contain κ only through $\lambda\kappa^2$ with a positive coefficient, and

$\log M \geq 0$, so for $\kappa < \kappa_c$, $a(\lambda,\kappa) < h(\lambda) = \lambda\kappa_c^2 - c_1(\lambda,1)$ and $a_2(\lambda,\kappa) < h_2(\lambda) = 0.52\lambda\kappa_c^2 - c_2(\lambda,1)$. Since $c_1(\lambda,1)$ and $c_2(\lambda,1)$ are affine in $\log\lambda$ with positive coefficients 67.5 and 109.5, h and h_2 are convex, so on an interval they are below the larger of their values at its ends. The logarithms of 1.957, 14, 23040λ, 31958λ and 23621 at the values of λ used follow from $x = 2^k y$ with $1 \leq y < 2$ and $\log y = 2\operatorname{artanh}\frac{y-1}{y+1}$, the series with its tail bounded geometrically. They give $c_1(100,1) > 1183$ and $c_2(100,1) > 1958$, so $h(100) < 0$ and $h_2(100) < 0$, and, with $\kappa_c \leq 0.0837617$, $h(245000) \leq 8.966$ and $h_2(809300) \leq 8.947$. With $\kappa_c \geq 0.0837613$ and $\log M < \log 23621$, $a(\lambda,\kappa_c) \geq \lambda\kappa_c^2 - 8\log 23621 - c_1(\lambda,1)$, which increases for $\lambda > 67.5/\kappa_c^2$, where $67.5/\kappa_c^2 < 9621$, and is at least 9.35 at $\lambda = 257000$; and $a_2(\lambda,\kappa_c) \geq 0.52\lambda\kappa_c^2 - 8\log 23621 - c_2(\lambda,1)$, which increases for $\lambda > 109.5/(0.52\kappa_c^2)$, where $109.5/(0.52\kappa_c^2) < 30015$, and is at least 11.66 at $\lambda = 833000$. With $\beta = 4\lambda$ this gives the display. The count is $(809300-1)/t$ with $t = 11\log 2/(12\pi^2)$, and $\pi \geq 3.14159265$ gives $1/t \geq 15.53326$, so $(809300-1)/t > 1.2571\cdot 10^7$. □

Remark 648. Theorem 643(ii) is not a substitute for (H1), because its loss is linear in the field size. Suppose it is used at a step whose coarse fields are calm at a threshold κ' satisfying the rate condition $\lambda'\kappa'^2 \geq 9 + c_1(\lambda',1)$ of Theorem 600(iii) at the next coupling $\lambda' \geq 100$, so that $\kappa' \geq 34.4\,\lambda'^{-1/2}$, and suppose $\kappa_1 \geq \kappa'$. Then $\log\theta(\kappa_0) \leq -2\log(1+6031\kappa_0) \leq -1716\,\kappa_0$, and either $\theta \leq \frac{1}{2}$ or $\kappa_0 \geq \sqrt{14}\,\kappa'$, so $-\log\theta(\kappa_0) \geq \min\{\log 2, 2.2\cdot 10^5\lambda'^{-1/2}\}$, which is $\log 2$ for $\lambda' \leq 10^{11}$. The product of these factors along the running tends to zero as $\lambda_0 \to \infty$. The loss comes from Lemma 358, whose bound is linear in η; at the linear level there is no loss, since $q_1 \geq Q_1$, and the long-wavelength fields of Theorem 644(iii) show that no lower constant above t_L^{-2} holds.

T. The induction over scales

This subsection asks which hypotheses of Theorem 646 follow from the conclusions of the previous step, at the trivial background. The relative error (H1) and the large-field cost (H3) are inputs of one step each; they are not implied by the earlier steps. The locality hypothesis (H2) is different: it bounds the inverse of the whole Hessian, and Proposition 650 shows that it cannot hold with constants independent of the volume, since the inverse carries the long-wavelength modes. It is replaced below by a bound on the form H_k itself. The induction step is then proved: the relative comparison, the fibre gap and the locality of the fibre covariance pass from step k to step $k+1$ under one condition (C) on the error of the step k alone, relative and local (Proposition 651, Theorem 652), the locality with constants that grow with k. Theorem 652 states the induction, and Proposition 653 counts the scales.

Setting. Throughout, $H_0 = 2\lambda_0\sum_p \|(dX)_p\|^2$ on $\mathscr{X}_0$, $\lambda_k = \lambda_0 - tk$ with $t = 11\log 2/(12\pi^2)$ as in Theorem 646(b), and $R_k = 2\lambda_k S_k$. The torus of scale k has side $n_k = L/2^k$ in units of its spacing. Let $\hat{\mathscr{X}}_k$ be the $\mathfrak{g}$-valued fields on all links of that torus, so that $\mathscr{X}_k \subset \hat{\mathscr{X}}_k$, and let $\hat{R}_k = 2\lambda_k S_k$ on $\hat{\mathscr{X}}_k$; R_k is its restriction to $\mathscr{X}_k$. The map $B_k = V$ sends $\mathscr{X}_k$ onto $\hat{\mathscr{X}}_{k+1}$ (Proposition 335(ii)), $\mathscr{M}_k H$ is a form on $\hat{\mathscr{X}}_{k+1}$, and (H1) compares its restriction to $\mathscr{X}_{k+1}$ with H_{k+1}. All forms are written as symmetric matrices with blocks $H_{ab} \in \operatorname{End}(\mathfrak{g})$ indexed by links, and $\|H_{ab}\|$ is the operator norm. A link of scale k sits at its midpoint, measured in units of the spacing 2^k, and d_k is the ℓ^∞ distance on the torus between midpoints. For a matrix M indexed by a set with a pseudometric d and $\alpha \geq 0$ put

$$\|M\|_{(\alpha)} = \sup_a \sum_b \|M_{ab}\| e^{\alpha d(a,b)}.$$

For symmetric M the Schur test gives $\|M\| \leq \|M\|_{(0)} \leq \|M\|_{(\alpha)}$, and $\|\cdot\|_{(\alpha)}$ does not increase under restriction to a subset of the indices.

Lemma 649. *For $0 < u \leq 1$ and every link a of a torus of any scale, or of $\mathbb{Z}^4$, $\sum_b e^{-u\,d(a,b)} \leq 2532\,u^{-4}$.*

Proof. The midpoints of the links of one direction form a translate of a lattice $\mathbb{Z}^4$, or of a discrete torus, and the number of its points at ℓ^∞ distance at most r from a given point is at most $(2r+1)^4$. Writing $e^{-ud} = \int_d^\infty u e^{-ur}\,dr$ and exchanging sum and integral, the sum over one direction is at most $\int_0^\infty u e^{-ur}(2r+1)^4\,dr = 384u^{-4} + 192u^{-3} + 48u^{-2} + 8u^{-1} + 1$, since $\int_0^\infty u e^{-ur} r^j\,dr = j!\,u^{-j}$. For $u \leq 1$ this is at most $633u^{-4}$, and there are four directions. □

Proposition 650 ((H2) depends on the volume)**.** *Let $k \geq 1$ with $n_k \geq 4$, and let $\mathscr{Y} \subset \mathscr{X}_k$ be a subspace on which R_k is positive definite and with $\mathscr{Y} + (\mathscr{X}_k \cap \ker R_k) = \mathscr{X}_k$. Every slice of the coarse gauge orbits on which R_k is positive definite has this property, since the gauge directions lie in* $\ker R_k$.

(i) *There is $\tilde{w} \in \hat{\mathscr{X}}_k$, orthogonal to $\ker\hat{R}_k$, with $\hat{R}_k(\tilde{w}) \leq 10\pi^2\lambda_k|\tilde{w}|^2/n_k^2$, and*

$$\|(R_k|_{\mathscr{Y}})^{-1}\| \geq \frac{n_k^2}{10\pi^2\lambda_k}.$$

(ii) *If the bounds of (H2) hold on $\mathscr{Y}$ at scale k with constants G_k, ℓ_k, then $G_k \geq n_k^2/(10\pi^2\lambda_k)$ and $\|(H_k - R_k)|_{\mathscr{Y}}\| \leq 10\pi^2\ell_k\lambda_k/n_k^2$; and the norm of H_k^{-1} on $\mathscr{Y}$, bounded by $G_k/(1-\ell_k)$ in Theorem 646(c), is at least $n_k^2/(10\pi^2\lambda_k)(1+\ell_k)^{-1}$.*

(iii) *If $\mathscr{Y}$ is the orthogonal complement of $\mathscr{X}_k \cap \ker R_k$ in $\mathscr{X}_k$, then $H_k = cR_k$ with $c > 0$ satisfies the bounds of (H2) at scale k only if $|c-1| < 5\pi^2/(6n_k^2)$.*

(iv) *On the same $\mathscr{Y}$, the form $E = \varepsilon\lambda_k|x|^2$ satisfies $|E| \leq \varepsilon' R_k$ on $\mathscr{Y}$ only if $\varepsilon' \geq \varepsilon n_k^2/(10\pi^2)$.*

Proof. (i) Write $n = n_k$, $N = 2^k n$, $q = 2\pi/N$, and fix $Z \in \mathfrak{g}$ with $\|Z\| = 1$. Let φ be the fine field with $\varphi_{(x,0)} = \cos(qx_1)Z$ and $\varphi = 0$ on the links of the other directions.[48] Its only nonzero

[48] This is the lowest nonconstant Fourier mode of the torus. Its energy is of order N^{-2} times its norm, the gap of the discrete Laplacian, and this is why the bound depends on the volume.

curvatures are on the plaquettes of the plane $\{0,1\}$, where $(d\varphi)_p = (\cos qx_1 - \cos q(x_1+1))Z = 2\sin\frac{q}{2}\sin q(x_1+\frac12)Z$, so

$$Q(\varphi) = N^3\cdot 4\sin^2\frac{\pi}{N}\sum_{x_1=0}^{N-1}\sin^2 q\Big(x_1+\frac12\Big) = 2N^4\sin^2\frac{\pi}{N}.$$

By Lemma 336(i), an identity of linear maps that holds on the torus as well, $V^{(k)}(\varphi)$ vanishes off the direction 0 and

$$V^{(k)}_{y,0}(\varphi) = \sum_{j=0}^{2^k-1}\cos q(2^k y_1 + j) = \rho\,\tilde w_{y,0},$$

with $\tilde w_{y,0} = \cos(2\pi y_1/n + c_0)Z$, $c_0 = q(2^k-1)/2$, $\tilde w = 0$ on the other directions, and $\rho = \sin(\pi/n)/\sin(\pi/N)$. Since $n\ge 3$, $|\tilde w|^2 = n^4/2$. Hence

$$\frac{S_k(\tilde w)}{|\tilde w|^2} \le \frac{Q(\varphi)}{\rho^2|\tilde w|^2} = \frac{4\cdot 16^k\sin^4(\pi/N)}{\sin^2(\pi/n)} \le \frac{4\pi^4}{n^4\sin^2(\pi/n)} \le \frac{4\pi^2}{0.8049\,n^2} < \frac{5\pi^2}{n^2},$$

using $\sin x\le x$, and $\sin x\ge x(1-x^2/6)$ at $x=\pi/n\le\pi/4$ with $(1-\pi^2/96)^2 > 0.8049$. So $\hat R_k(\tilde w)\le 10\pi^2\lambda_k|\tilde w|^2/n^2$.

$\tilde w$ is orthogonal to $\ker\hat R_k$. Indeed $S_k\ge Q_k$ by Proposition 335(i), whose proof applies on every lattice, so a field w in $\ker\hat R_k$ has zero curvature. The curvature of a plaquette in the plane $\{0,1\}$ at y is $w_{y,0}+w_{y+e_0,1}-w_{y+e_1,0}-w_{y,1}$; summed over y_0 around the torus, the terms in the direction 1 cancel, so $f(y_1,y_2,y_3)=\sum_{y_0}w_{y,0}$ does not depend on y_1. Then $\langle\tilde w,w\rangle = \sum_{y_2,y_3}\langle Z, f(\cdot,y_2,y_3)\rangle\sum_{y_1}\cos(2\pi y_1/n+c_0) = 0$.

The form S_k is invariant under the coarse gauge directions: for h on the sites of scale k, let h_f be the fine function equal to $h(y)$ on $B_k(y)$; then $V^{(k)}(dh_f) = -d_k h$ by Lemma 336(ii), so $\varphi\mapsto\varphi - dh_f$ maps the fields with $V^{(k)}(\varphi)=w$ onto those with $V^{(k)}(\varphi)=w+d_kh$ and preserves Q. As in the proof of Lemma 336(iii), applied to the blocks of side two of the lattice of scale k, there is h with $t=\tilde w+d_kh\in\mathscr{X}_k$, and $t-\tilde w\in\ker\hat R_k$. By hypothesis $t=y+z$ with $y\in\mathscr{Y}$ and $z\in\mathscr{X}_k\cap\ker R_k\subset\ker\hat R_k$. A positive semidefinite form is constant on the cosets of its kernel, so $R_k(y)=\hat R_k(\tilde w)$; and $y\in\tilde w+\ker\hat R_k$ with $\tilde w\perp\ker\hat R_k$, so $|y|\ge|\tilde w|$. Therefore $\|(R_k|_{\mathscr{Y}})^{-1}\|\ge|y|^2/R_k(y)\ge n^2/(10\pi^2\lambda_k)$.

(ii) The function $\phi=0$ is 1-Lipschitz, so (H2) contains $\|(R_k|_{\mathscr{Y}})^{-1}\|\le G_k$ and $\|(H_k-R_k)|_{\mathscr{Y}}\|\le\ell_k/G_k$; apply (i). At the vector y of (i), $H_k(y)\le R_k(y)+\|(H_k-R_k)|_{\mathscr{Y}}\||y|^2\le(1+\ell_k)\,10\pi^2\lambda_k|y|^2/n^2$, which gives the last bound.

(iii) R_k is block diagonal for $\mathscr{X}_k=\mathscr{Y}\oplus(\mathscr{X}_k\cap\ker R_k)$, so its norm on $\mathscr{Y}$ is its largest eigenvalue on $\mathscr{X}_k$, which is at least $R_k(\delta_eZ)$ for the field δ_eZ equal to Z on one active link e and zero elsewhere. By Proposition 335(iii), $R_k(\delta_eZ)\ge 2\lambda_kQ_k(\delta_eZ)=12\lambda_k$, since a link lies in six plaquettes. If $H_k=cR_k$, then $\|(H_k-R_k)|_{\mathscr{Y}}\|=|c-1|\|R_k|_{\mathscr{Y}}\|\ge 12|c-1|\lambda_k$, and (ii) with $\ell_k<1$ gives $12|c-1|<10\pi^2/n^2$.

(iv) At the vector y of (i), $\varepsilon\lambda_k|y|^2$ is at most $\varepsilon' R_k(y)$, which is at most $10\pi^2\varepsilon'\lambda_k|y|^2/n^2$. □

The norm of H_k^{-1} on a slice is therefore of order n_k^2/λ_k for every H_k with $\|H_k-R_k\|<\lambda_k$ on it, and the entry bound of Theorem 646(c) is at least $n_k^2/(10\pi^2\lambda_k)(1+\ell_k)^{-1}$ at distance zero: the covariance of the massless modes does not decay exponentially, uniformly in the volume. By (iii), (H2) excludes a change of the coupling by any factor independent of L, even one that (H1) allows with ε_k fixed. Part (iv) is the torus form of the conversion cost C_n of Remark 585. The exponential decay that holds uniformly is that of the fibre covariance, where Proposition 335(iv) gives a gap, and it follows from a bound on the form H_k rather than on its inverse:

(H2*) *local form:* $\|H_k\|_{(\alpha_k)}\le\Lambda_k\lambda_k$ on $\mathscr{X}_k$ with the distance d_k, for some $0<\alpha_k\le 1$ and $\Lambda_k\ge 1$;

(D) *local one-step error:* the difference $D_k = H_{k+1} - \frac{\lambda_{k+1}}{\lambda_k}\mathscr{M}_kH_k|_{\mathscr{X}_{k+1}}$ has $\|D_k\|_{(\alpha_{k+1})}\le\delta_k\lambda_{k+1}$ with the distance d_{k+1}.

At $k=0$, $H_0=2\lambda_0Q$ has diagonal blocks $12\lambda_0$ and, in each row, eighteen further blocks of norm $2\lambda_0$, at the other links of the six plaquettes of the link, all at distance at most one; so (H2*) holds at $k=0$ for every $\alpha_0\in(0,1]$ with $\Lambda_0=12+36e^{\alpha_0}$. Hypothesis (D) concerns one integration only, while (H2) concerns the composition of all earlier ones.

For $k<K$ let $\mathscr{C}_k$ be the inverse of H_k on $\ker B_k\cap\mathscr{X}_k$, extended by zero on its orthogonal complement in $\mathscr{X}_k$ (the covariance of the fibre, up to the factor of the Gaussian normalisation), and $\mathscr{T}_k:\hat{\mathscr{X}}_{k+1}\to\mathscr{X}_k$ the map that sends w to the minimiser of H_k on $\{B_kx=w\}$. A coarse link (y,μ) is placed, in the units of scale k, at the centre $2y+e_\mu+\frac12(1,1,1,1)$ of $B(y)\cup B(y+e_\mu)$; on the union of the links of scale k and these points d denotes the ℓ^∞ distance in units of scale k, so that $d=2d_{k+1}$ between two coarse links.

Proposition 651 (what passes from step k to step $k+1$)**.** *Let $k<K$ with $n_k\ge 4$, and let H_k be positive semidefinite on $\mathscr{X}_k$.*

(i) *If $k=0$, or if (H1) holds at the steps $j<k$ with $\sigma_k=\sum_{j<k}\varepsilon_j\le\frac12$, then $H_k(x)\ge\lambda_k|x|^2/52$ for $x\in\ker B_k\cap\mathscr{X}_k$.*

(ii) *Suppose the bound of (i) and (H2*) at scale k, and put $\alpha_*=\alpha_k/(11900\Lambda_k)$. Then for all links a,b of scale k and i,j of scale $k+1$,*

$$\|(\mathscr{C}_k)_{ab}\|\le\frac{2186}{\lambda_k}e^{-\alpha_*d(a,b)},$$
$$\|(\mathscr{T}_k)_{ai}\|\le 2186\,\Lambda_k e^{-\alpha_*d(a,i)},$$
$$\|(\mathscr{M}_kH_k)_{ij}\|\le 2186\,\Lambda_k^2\lambda_k\,e^{-2\alpha_*d_{k+1}(i,j)}.$$

(iii) *Under the same hypotheses, with $C_*=2186\cdot2532\cdot11900^4$, which is less than $1.11\cdot10^{23}$,*

$$\Big\|\frac{\lambda_{k+1}}{\lambda_k}\mathscr{M}_kH_k|_{\mathscr{X}_{k+1}}\Big\|_{(\alpha_*)}\le C_*\frac{\Lambda_k^6}{\alpha_k^4}\lambda_{k+1};$$

so if (D) holds with $\alpha_{k+1}=\alpha_*$, then (H2*) holds at scale $k+1$ with $\Lambda_{k+1}=C_*\Lambda_k^6\alpha_k^{-4}+\delta_k$.

(iv) *For every positive semidefinite H on $\mathscr{X}_k$ and $\varepsilon\ge0$, the form $E=\varepsilon\lambda_k|x|^2$ has $\|E\|_{(\alpha)}=\varepsilon\lambda_k$ for every α, while $\mathscr{M}_k(H+E)-\mathscr{M}_kH\ge4\varepsilon\lambda_k|w|^2$ on $\hat{\mathscr{X}}_{k+1}$. So no bound $\|\mathscr{M}_k(H+E)-\mathscr{M}_kH\|\le C\|E\|_{(\alpha)}$ holds for all H and E with $C<4$.*

Proof. The averages. By the formula of the proof of Theorem 330, $V_{y,\mu}$ has the coefficient $\frac{1}{8}$ on the eight links (x,μ), $x\in B(y)$, $o_\mu=1$, which are never tree links, and $\frac{1}{16}$ on the links of direction μ with $o_\mu=0$ in $B(y)$ and in $B(y+e_\mu)$. So a row of $B=B_k$ has absolute sum at most 2, a link has total weight at most $\frac{1}{8}$ in the rows, and every link in the support of a row lies within distance 1 of the point of its coarse link: the transverse coordinates of the midpoints differ from those of the centre by $\frac{1}{2}$, and the coordinate μ by at most 1. The diagonal entries of $BB^\top$ are at least $8/64$, and the row (y,μ) meets only the rows $(y\pm e_\mu,\mu)$, in at most eight links with coefficients $\frac{1}{16}$, so its off-diagonal entries add up to at most $2\cdot\frac{8}{256}$. By Gershgorin, $BB^\top\ge\frac{1}{16}$, and likewise $BB^\top\le\frac{8}{64}+\frac{16}{256}+\frac{2}{32}=\frac{1}{4}$. These bounds act on each component in an orthonormal basis of $\mathfrak{g}$.

(i) At $k=0$, $H_0=R_0$. Otherwise Theorem 646(a), whose proof uses only (H1), gives $H_k\ge(1-\sigma_k)R_k\ge\lambda_k S_k$, and Proposition 335(iv), whose proof uses only Theorem 330 and $S_k\ge Q_k$ and so applies on the torus, gives $S_k(x)\ge|x|^2/52$ on $\ker B_k\cap\mathscr{X}_k$, the space Φ_W of the lattice of scale k with W all its blocks.

(ii) Let $\lambda=\lambda_k$, $\Lambda=\Lambda_k$, $s=\Lambda\lambda$, and $K=\left(\begin{smallmatrix}H_k & sB^\top\\ sB & 0\end{smallmatrix}\right)$ on $\mathscr{X}_k\oplus\widehat{\mathscr{X}}_{k+1}$. Lemma 332 applies with $h=\|H_k\|\le\Lambda\lambda$, $\beta=\frac{1}{16}$ and $a=\lambda/52$: with $\Lambda\ge1$, $|x_1|\le4|g|/(\Lambda\lambda)$,

$$|x|\le\frac{52(|f|+4|g|)+4|g|}{\lambda}=\frac{52|f|+212|g|}{\lambda},$$
$$|y|\le\frac{4|f|}{\Lambda\lambda}+4|x|\le\frac{212|f|+848|g|}{\lambda},$$

so $|x|+|y|\le(264|f|+1060|g|)/\lambda$, and $\|K^{-1}\|\le1093/\lambda$ since $264^2+1060^2<1093^2$. With weights $e^{\alpha_k d}$, a row of H_k in K has sum at most $\Lambda\lambda+\frac{1}{8}se^{\alpha_k}$, and a row of sB at most $2se^{\alpha_k}$; with $\alpha_k\le1$ both are at most $2e\Lambda\lambda\le5.44\Lambda\lambda$. Lemma 333 needs a finite range, and is replaced as follows. For 1-Lipschitz ϕ and $0\le\alpha\le\alpha_k$, $e^{\alpha\phi}Ke^{-\alpha\phi}=K+E$ with $\|E_{ab}\|\le(e^{\alpha d(a,b)}-1)\|K_{ab}\|$, and $e^{\alpha r}-1\le(\alpha/\alpha_k)(e^{\alpha_k r}-1)$ by convexity in α; by the Schur test $\|E\|\le(\alpha/\alpha_k)5.44\Lambda\lambda$. For $\alpha=\alpha_*$, $\|K^{-1}E\|\le5.44\cdot1093/11900<\frac{1}{2}$, and as in Lemma 333 $\|e^{\alpha_*\phi}K^{-1}e^{-\alpha_*\phi}\|\le2186/\lambda$. With $\phi=d(\cdot,b)$ every block of K^{-1} satisfies $\|(K^{-1})_{ab}\|\le(2186/\lambda)e^{-\alpha_* d(a,b)}$. If $K(x,y)=(f,0)$, then $x\in\ker B$ and $H_kx-f\perp\ker B$, so $\mathscr{C}_k$ is the upper left block of K^{-1}. If $K(x,y)=(0,sw)$, then $Bx=w$ and $H_kx+sB^\top y=0$, the Lagrange condition of the minimum, which is unique by (i); so $\mathscr{T}_k=s(K^{-1})_{12}$, and $H_k(x)=-s\langle Bx,y\rangle=-s^2\langle w,(K^{-1})_{22}w\rangle$, that is, $\mathscr{M}_kH_k=-s^2(K^{-1})_{22}$. The three bounds follow, with $d=2d_{k+1}$ between coarse links.

(iii) By (ii) and Lemma 649 with $u=\alpha_*\le1$,

$$\Big\|\frac{\lambda_{k+1}}{\lambda_k}\mathscr{M}_kH_k\Big\|_{(\alpha_*)}\le2186\Lambda_k^2\lambda_{k+1}\sup_i\sum_j e^{-\alpha_* d_{k+1}(i,j)}$$
$$\le2186\cdot2532\,\Lambda_k^2\alpha_*^{-4}\lambda_{k+1},$$

which is the bound, and restriction to $\mathscr{X}_{k+1}$ does not increase it. Adding D_k gives (H2*) at scale $k+1$.

(iv) The infimum of a sum is at least the sum of the infima, and $\inf\{|x|^2:Bx=w\}=\langle w,(BB^\top)^{-1}w\rangle\ge4|w|^2$ because $BB^\top\le\frac{1}{4}$. This is Proposition 461 at $\kappa=0$. □

For $k<K$ write $D_k=H_{k+1}-\frac{\lambda_{k+1}}{\lambda_k}\mathscr{M}_kH_k|_{\mathscr{X}_{k+1}}$ for the error of the step k, and consider the single condition on these errors

(C) *one-step error, relative and local:* for every $k<K$,

$$-\frac{\lambda_{k+1}}{\lambda_k}\frac{\mathscr{M}_kH_k}{\lambda_{k+1}^2}\le D_k\le\frac{\lambda_{k+1}}{\lambda_k}\frac{\mathscr{M}_kH_k}{\lambda_{k+1}^2}\quad\text{on }\mathscr{X}_{k+1},$$

and $\|D_k\|_{(\alpha_{k+1})}\le\delta_k\lambda_{k+1}$ for some $\delta_k\ge0$.

It is (H1) with $\varepsilon_k=\lambda_{k+1}^{-2}$ together with (D); both concern one integration only.

Theorem 652 (the induction over scales under one condition)**.** *Let $L/2^K\ge4$, $\lambda_K\ge32$, $0<\alpha_0\le1$, $\Lambda_0=12+36e^{\alpha_0}$, and define recursively $\alpha_{k+1}=\alpha_k/(11900\Lambda_k)$ and $\Lambda_{k+1}=C_*\Lambda_k^6\alpha_k^{-4}+\delta_k$, with C_* as in Proposition 651(iii). For $k\le K$ let P_k be the conjunction of*

(a) *$(1-\sigma_k)R_k\le H_k\le e^{\sigma_k}R_k$, where $\sigma_k=\sum_{j<k}\lambda_{j+1}^{-2}$;*

(b) *if $k<K$, $H_k(x)\ge\lambda_k|x|^2/52$ on $\ker B_k\cap\mathscr{X}_k$;*

(c) *(H2*) at scale k with (α_k,Λ_k), and, if $k<K$, the bounds of Proposition 651(ii) for the fibre covariance, the minimiser map and the block infimum.*

(i) *P_0 holds, and for every $k<K$, P_k and (C) at the step k imply P_{k+1}. No other hypothesis enters.*

(ii) *Hence under (C), P_k holds for every $k\le K$, with $\sigma_k\le0.4865$, so $\frac{1}{2}R_k\le H_k\le1.65R_k$; the bounds of (a) and (b) are uniform in λ_0, K and L, and those of (c) are uniform in L.*

(iii) *The reference chain $H_k=R_k$ satisfies (C) with $D_k=0$, so P_k holds for it for every k, unconditionally.*

(iv) *The constants of (c) are not bounded in k: $\Lambda_k\ge48^{6^k}$ and $\alpha_{k+1}\le\alpha_k/(5.7\cdot10^5)$.*

Proof. (i) P_0: $H_0=R_0$, Proposition 651(i) at $k=0$, and the computation after (D). Let P_k and (C) at the step k hold. By the first half of (C), (H1) holds at the step k with $\varepsilon_k=\lambda_{k+1}^{-2}<1$; as in the proof of Theorem 646(a), $\mathscr{M}_k$ is monotone and positively homogeneous, so (a) at k gives $(1-\sigma_k)(1-\varepsilon_k)R_{k+1}\le H_{k+1}\le e^{\sigma_k}(1+\varepsilon_k)R_{k+1}$, which is (a) at $k+1$ since $(1-\sigma_k)(1-\varepsilon_k)\ge1-\sigma_{k+1}$. The bound $\sigma_{k+1}\le\sigma_K\le0.4865$ of Theorem 646(b) with $c_\varepsilon=1$ and $\lambda_K\ge32$, and Proposition 651(i), give (b) at $k+1$. The bounds of (c) at k are Proposition 651(ii), from (b) and (H2*) at k, and (H2*) at $k+1$ is Proposition 651(iii) with the second half of (C). (ii) Induction with (i); $1-0.4865\ge\frac{1}{2}$ and $e^{0.4865}\le1.65$. (iii) $R_{k+1}=\frac{\lambda_{k+1}}{\lambda_k}\mathscr{M}_kR_k$ by definition. (iv) $\Lambda_0\ge48$, $C_*\ge1$ and $\alpha_k\le1$ give $\Lambda_{k+1}\ge\Lambda_k^6$, and $11900\cdot48>5.7\cdot10^5$. □

Hypothesis (H2) is not used, and by Proposition 650 it does not hold with G_k and ℓ_k independent of L. The large-field cost (H3) is independent of (C): it concerns the values of the action far from the background, which no bound on Hessians

at the background controls, and the proof of Theorem 646(d) uses (H3) alone, so that part holds at every step at which (H3) holds. The theorem is the induction that the present estimates support: the relative comparison and the fibre gap pass from step k to step $k+1$ with constants independent of k, and the locality of the fibre passes with constants that grow faster than any exponential. The decay length of the fibre covariance at scale k, α_k^{-1} in units of 2^k, is more than $(5.7\cdot 10^5)^k$ in those units, so the bound of (c) says nothing beyond the first steps at any volume. A bound uniform in k cannot come from Proposition 651(ii), (iii) alone, which use only the size of H_k and its gap, and by part (iv) of that proposition a comparison of H_k with the uniformly local reference R_k (Theorem 340) in an absolute weighted norm loses a factor of at least four per step. Condition (C) is not proved for any k (Proposition 647(i), Remark 584).

Proposition 653 (the count of scales)**.** *Along the one-loop running $\lambda_k=\lambda_0-tk$, with $\beta=4\lambda$ and $b=11/(3\pi^2)$, so that $4t=b\log 2$:*

(i) *The hypothesis $\lambda_K\geq 32$ of Theorem 652 allows $K=\lfloor(\lambda_0-32)/t\rfloor$ steps, which is unbounded as $\lambda_0\to\infty$; under (C) the bounds (a) and (b) of Theorem 652 hold uniformly over all of them.*

(ii) *The large-field estimates of Sec. XIII R, in the form proved at the first step (Theorem 602, Proposition 604), give no activity bound at a coupling $\lambda\leq 809300$, so at most the steps with $\lambda_k>809300$ can receive (H3) from them. At least $(809300-\lambda_K)/t-1$ steps $k<K$ have $\lambda_k\leq 809300$, more than $1.257\cdot 10^7$ when $\lambda_K\leq 1$, whatever λ_0. The lattice spacing at the last scale with $\lambda_k>809300$ is less than $e^{-8.713\cdot 10^6}$ times the spacing of the scale at which λ_k reaches one.*

(iii) *The islands of Theorem 467 at the Wilson coupling $\beta_W=4\lambda_0$ are controlled at the scales $j\leq j_{\max}$, and if $j_{\max}\geq 1$ then $j_{\max}\leq 1+\log_{16}(\beta_W/(2.895\cdot 10^{23}))$, while the running from β_W to $\beta=4$ has $(\beta_W-4)/(b\log 2)$ steps.*

(iv) *Under (H1) with $\varepsilon_k\leq\lambda_{k+1}^{-2}$ and $\lambda_K\geq 32$, the ratio of the upper to the lower comparison constant of H_k with R_k is at most 3.3 for every k, and on $\mathbb{Z}^4$, $2\lambda_k Q_k\leq R_k\leq 512\lambda_k Q_k$ for every k. The factor four of Theorem 644 enters R_k once and does not compound.*

Proof. (i) $\lambda_K=\lambda_0-tK\geq 32$ if and only if $K\leq(\lambda_0-32)/t$. (ii) By Proposition 647(iv), for every $\kappa<\kappa_c$ the rates satisfy $a(\lambda,\kappa)<9$ and $a_2(\lambda,\kappa)<9$ for $100\leq\lambda\leq 809300$, and the estimates require $\lambda\geq 100$. The steps $k<K$ with $\lambda_k\leq 809300$ are those with $(\lambda_0-809300)/t\leq k\leq K-1$, and $K=(\lambda_0-\lambda_K)/t$; the bound $(809300-1)/t>1.257\cdot 10^7$ is in Proposition 647(iv). With the same count for the coupling λ_{k+1} of the step, the conclusion is unchanged. One step doubles the spacing, and k steps lower β by $bk\log 2$; so between $\beta=4\cdot 809300=3237200$ and $\beta=4$ the spacing grows by $2^{(3237200-4)/(b\log 2)}=e^{(3237200-4)/b}$, and $b\leq 0.371512$ gives $(3237200-4)/b>8.713\cdot 10^6$. (iii) $\beta^*_{j+1}>16\beta^*_j$ and $\beta^*_1\geq 2.895\cdot 10^{23}$ give $\beta^*_j\geq 16^{j-1}\cdot 2.895\cdot 10^{23}$, and $\beta_W\geq\beta^*_{j_{\max}}$. The count of steps is (i) with $\lambda_K=1$ and $t=b\log 2/4$. (iv) Theorem 652(ii) gives the constants $\frac12$ and 1.65, whose ratio is 3.3. The second statement is Theorem 338 and $R_k=2\lambda_k S_k$. The reference is built from infima, $R_{k+1}=\frac{\lambda_{k+1}}{\lambda_k}\mathscr{M}_k R_k$, and not from a comparison with $\lambda_k Q_k$ at each step. □

Fig. 45 draws the induction and the growth of its constants.

Remark 654 (how the constants evolve)*.* Along the induction of Theorem 652:

(i) the coupling decreases by t per step; the number of steps is unbounded as $\lambda_0\to\infty$, and nothing in P_k requires it to be bounded;

(ii) the summed relative error σ is at most $\lambda_K^{-2}+1/(t\lambda_K)$ for errors $\varepsilon_k\leq\lambda_{k+1}^{-2}$, uniformly in λ_0; errors $\varepsilon_k\geq c\lambda_{k+1}^{-1}$ give $\sigma\geq(c/t)\log(\lambda_0/\lambda_K)$ (Theorem 646(b)), so one-step errors of order g^2 do not close;

(iii) the comparison constants of H_k with R_k stay in $[\frac12,1.65]$ and the fibre gap stays $\lambda_k/52$; the lower constant $1/2956$ and the factors $\theta(\kappa_0)$ of Theorem 643 concern the coarse action A_1, not H_k, and their products over the steps tend to zero (Remark 648);

(iv) the locality constants obey $\alpha_{k+1}=\alpha_k/(11900\Lambda_k)$ and $\Lambda_{k+1}=C_*\Lambda_k^6\alpha_k^{-4}+\delta_k$ and are unbounded; an absolute weighted bound on H_k-R_k grows by a factor of at least four per step (Proposition 651(iv)), as in Remark 585;

(v) the large-field input fails first: with the first-step estimates it stops at $\lambda=809300$, more than $1.257\cdot 10^7$ steps above coupling one (Proposition 653(ii));

(vi) the island thresholds grow by a factor of more than 16 per scale, so the islands are controlled on $O(\log\beta_W)$ of the $(\beta_W-4)/(b\log 2)$ scales;

(vii) at a curved background, outside the setting of this subsection, the one-step factor of a plaquette remainder is $1+130\|\kappa^{(j)}\|_2$ at scale j (Proposition 577(iii)), and the product over the scales is bounded exactly when $\sum_j\|\kappa^{(j)}\|_2<\infty$ (Remark 581); on calm windows at the levels K_j this holds if $\sum_j K_j<\infty$ (Corollary 579). Levels $K_j\leq c\lambda_{j+1}^{-2}$ have $\sum_j K_j\leq c(\lambda_K^{-2}+1/(t\lambda_K))$, as in Theorem 646(b). Levels $K_j\geq c\lambda_{j+1}^{-1/2}$, the order of the calm thresholds that the large-field rate condition requires (Remark 648), have $\sum_{j<K}K_j\geq(2c/t)(\lambda_0^{1/2}-\lambda_K^{1/2})$, since $t\lambda_{j+1}^{-1/2}\geq\int_{\lambda_{j+1}}^{\lambda_j}\lambda^{-1/2}d\lambda$; then the sufficient condition of Remark 581 fails as $\lambda_0\to\infty$.

So the induction step is proved, and the induction closes over all scales under the one condition (C): uniformly in λ_0 for the relative comparison and the fibre gap, and with constants growing in k for locality. The first open input on the way is (C) at $k=0$. The large-field cost is a separate input, and the estimates proved for it at the first step give nothing at $\lambda_k\leq 809300$.

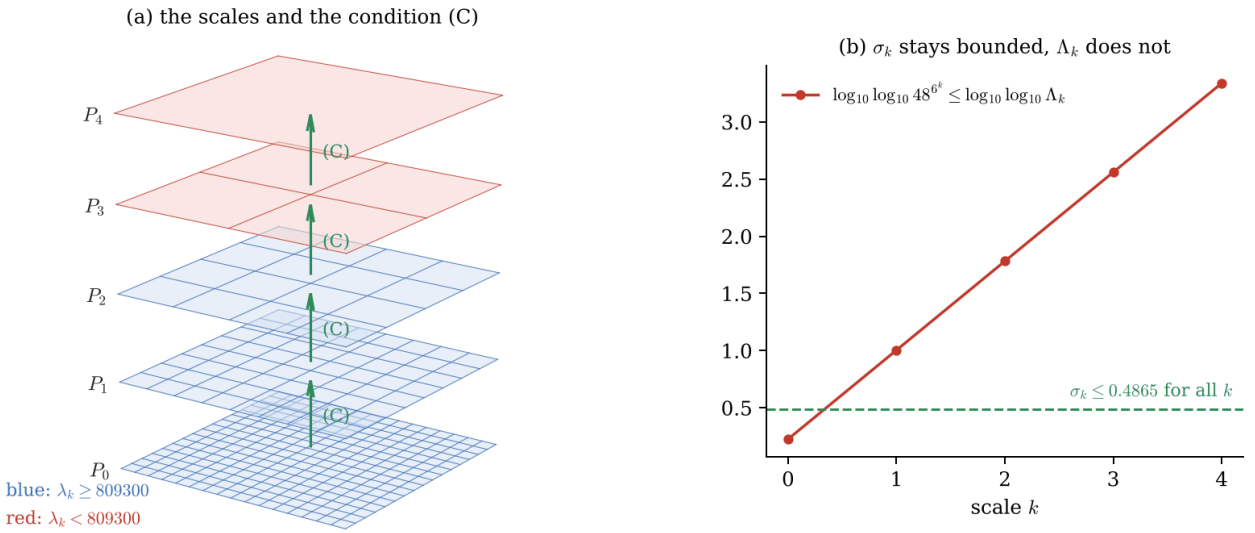


FIG. 45. The induction over scales of Theorem 652, schematic in (a). (a) The tori of the scales $k = 0, \dots, 4$, the statement P_k on each and the condition (C) on each step; the couplings λ_k are at least 809300 on the blue tori and below it on the red ones, and along the one-loop running more than $1.257 \cdot 10^7$ steps lie between 809300 and 1 (Proposition 647). (b) The lower bound $\Lambda_k \geq 48^{6^k}$ on the constants of the induction, as $\log_{10}\log_{10}$, against the bound $\sigma_k \leq 0.4865$, which holds for every k.

U. Gauge-invariant local remainders

The factor C_n of Proposition 565 and the loss κn^2 of Proposition 572 come from the gap of the whole window off its gauge directions. The inverse of the gauge-fixed form of the window does not decay uniformly in n: by Lemma 562(iv) its norm is at least C_n^-, so by the Schur test and Lemma 649 no bound $Ce^{-\alpha d(a,b)}$ on its blocks holds with C and α independent of n, and on the torus Proposition 650 gives the same. What decays uniformly is the covariance of the fibre (Proposition 651(ii)), and it makes the terms of an induced action local. This subsection shows that locality removes C_n when each local term is gauge invariant on a box of its own: the gap of that box replaces the gap of the window. It also shows that the hypothesis (H), stated for the chart density, is incompatible with a local invariant density on large windows, that at curved backgrounds the loss is of order $\|\kappa\|_2$ for every local remainder and not of order κ uniformly in n, and that condition (C) of Theorem 652 follows from such a representation of the one-step error. Gauge-invariant localised effective actions of this kind are what the cluster expansions of Bałaban [10, 11] produce for his averaging operations; for the cut-off averages they are a hypothesis here.

Setting. Let $G \subset \mathrm{U}(N)$ be compact and connected, with a Lie algebra $\mathfrak{g}$ of trivial centre, such as $\mathrm{SU}(N)$, with $\langle X, Y\rangle = -\mathrm{Tr}(XY)$ on $\mathfrak{g}$ and operator norms for matrices in $\mathrm{U}(N)$, as in the curved part of Sec. XIII P. A *box* X of side $r \geq 1$ is the set of sites $\prod_\nu [b_\nu, b_\nu + r]$ of the coarse lattice, which is $\mathbb{Z}^4$ or a torus of side at least $r+2$. Let L_X be its links, those with both ends in X, P_X its plaquettes, those with all four corners in X, and $\ell_X = |L_X| = 4r(r+1)^3$. A *local term* on X is a C^2 function f of the links of X with $f(gV) = f(V)$ for every $g \in G^X$, where $(gV)_{y,\mu} = g_y V_{y,\mu} g_{y+e_\mu}^{-1}$. A *local remainder* is a sum $R = \sum_{X \in \mathscr{X}} f_X$ over a family of boxes, repetitions allowed, with f_X a local term on X, whose series converges with its first two derivatives near the background. For $V \in G^{L_X}$, $\mathrm{Hess}\, f(V)$ is the Hessian at 0 of $\omega \mapsto f(Ve^\omega)$ on $\mathfrak{g}^{L_X}$, and $h_X = \|\mathrm{Hess}\, f_X(I)\|$ is its norm at the trivial field, which depends on f_X alone. For a background V_0 and $w \in \mathfrak{g}^K$, extended by zero off K, put $Q_X^{V_0}(w) = \sum_{p \in P_X} \|Z_p(w)\|^2$, so that $\sum_X Q_X^{V_0}$ counts each plaquette of Q^{V_0} once for every box that contains it. Put

$$\rho_r = \frac{1}{4\sin^2(\pi/(2r+2))} \leq \frac{(r+1)^2}{4}.$$

Lemma 655 (a box with every site gauged)**.** *Let X be a box of side r and $(d_X \eta)_{y,\mu} = \eta_y - \eta_{y+e_\mu}$ for $\eta \in \mathfrak{g}^X$. If $u \in \mathfrak{g}^{L_X}$ is orthogonal to $d_X(\mathfrak{g}^X)$, then*

$$|u|^2 \leq \rho_r \sum_{p \in P_X} \|(du)_p\|^2,$$

where $(du)_p = u_{k_1} + u_{k_2} - u_{k_3} - u_{k_4}$ for p with links $k_1, \dots, k_4$ in the order of Sec. XIII P.

Proof. Componentwise in an orthonormal basis of $\mathfrak{g}$ it suffices to treat real fields. The cochain complex of X in degrees $0, 1, 2$, with the sites, links and plaquettes of X as cells and the coboundaries d_X and d, is the fourfold tensor product of the complex of the path $[0, r]$, in which $C^0 = \mathbb{R}^{r+1}$, $C^1 = \mathbb{R}^r$ and d is the difference operator; the orientations of the cells change only signs, which do not change norms. With the product inner products the Laplacian $\Delta = dd^* + d^*d$ of the product is the sum of the Laplacians of the factors, as in the proof of Lemma 562(ii). On C^0, d^*d is the Laplacian of the path with $r+1$ vertices, with eigenvalues $4\sin^2(\pi j/(2r+2))$ for $0 \leq j \leq r$. Since d maps C^0 onto C^1, dd^* is invertible on C^1,

with the nonzero eigenvalues of d^*d, the least being ρ_r^{-1}. A 1-cochain of the product is a sum of tensors with exactly one factor of degree one, so $\Delta_1 \geq \rho_r^{-1}$. If $u \perp d_X(C^0)$, then $d_X^* u = 0$ and $\sum_p (du)_p^2 = \langle u, \Delta_1 u\rangle \geq \rho_r^{-1}|u|^2$. Finally $\sin x \geq 2x/\pi$ on $[0,\pi/2]$. □

Lemma 656 (a local term at a flat background)**.** *Let f be a local term on a box X of side r, and let $V_0 \in G^{L_X}$ have $V_{0,p} = I$ for every $p \in P_X$. Then:*

(i) *the gradient of $\omega \mapsto f(V_0 e^{\omega})$ vanishes at 0;*

(ii) $\mathrm{Hess}\, f(V_0)[w, d^X_{V_0}\eta] = 0$ *for every $w \in \mathfrak{g}^{L_X}$ and $\eta \in \mathfrak{g}^X$, where $(d^X_{V_0}\eta)_{y,\mu} = \mathrm{Ad}(V_{0,y,\mu}^{-1})\eta_y - \eta_{y+e_\mu}$;*

(iii) $\|\mathrm{Hess}\, f(V_0)\| = \|\mathrm{Hess}\, f(I)\|$, *and for every $w \in \mathfrak{g}^{L_X}$*

$$\big|\mathrm{Hess}\, f(V_0)[w,w]\big| \leq \|\mathrm{Hess}\, f(I)\|\, \rho_r\, Q_X^{V_0}(w).$$

Proof. As in the proof of Lemma 571, the links (x,μ) of X with $x_\nu = b_\nu$ for every $\nu < \mu$ form a spanning tree, and some $g \in G^X$ has $gV_0 = I$ on it. For another link (x,μ) let $\nu < \mu$ be the least direction with $x_\nu > b_\nu$; the plaquette in the plane $\{\nu,\mu\}$ at $x - e_\nu$ lies in P_X and its two links of direction ν lie in the tree, so its holonomy I gives $(gV_0)_{x,\mu} = (gV_0)_{x-e_\nu,\mu}$, and induction gives $gV_0 = I$ on L_X. With $(Tw)_{y,\mu} = \mathrm{Ad}(g_{y+e_\mu})w_{y,\mu}$, an isometry, $(gV_0)e^{Tw} = g(V_0 e^w)$, so $f(gV_0 e^{T\omega}) = f(V_0 e^{\omega})$, $Q_X^{gV_0}(Tw) = Q_X^{V_0}(w)$ and $T d^X_{V_0}\eta = d^X_{gV_0}(\mathrm{Ad}(g)\eta)$, as in that proof. So it suffices to take $V_0 = I$, where $d^X_I = d_X$.

(i) A constant $h \in G$ acts by $e^{\omega} \mapsto e^{\mathrm{Ad}(h)\omega}$, so the gradient $(\nabla_k f(0))_k$ is fixed by every $\mathrm{Ad}(h)$, hence commutes with $\mathfrak{g}$ and lies in the centre of $\mathfrak{g}$, which is 0. (ii) The proof of Lemma 564(i), with Γ_W replaced by G^X and K by L_X, gives $\mathrm{Hess}\, f(0)[w, d_X\eta] = -Df(0)[D\zeta(0)w]$, which vanishes by (i). (iii) The first statement is the transport above. By (ii) the symmetric form $\mathrm{Hess}\, f(0)$ vanishes on $d_X(\mathfrak{g}^X)$ in each argument, so $\mathrm{Hess}\, f(0)[w,w] = \mathrm{Hess}\, f(0)[u,u]$ for the orthogonal projection u of w off $d_X(\mathfrak{g}^X)$, which is at most $\|\mathrm{Hess}\, f(0)\||u|^2$. Lemma 655 and $du = dw$, since the curvature of $d_X\eta$ vanishes, bound $|u|^2$ by $\rho_r \sum_{p\in P_X}\|(dw)_p\|^2$, and at $V_0 = I$, $Z_p(w) = (dw)_p$ by (49). □

Theorem 657 (local remainders at flat backgrounds)**.** *Let $R = \sum_{X\in\mathscr{X}} f_X$ be a local remainder, let V_0 be flat on every plaquette of every box of $\mathscr{X}$, the links off K included, and put*

$$\mathscr{L}_0 = \sup_p \sum_{X:\, p\in P_X} h_X \rho_{r_X}.$$

(i) $|\mathrm{Hess}\, R(V_0)[w,w]| \leq \mathscr{L}_0\, Q^{V_0}(w)$ *for every $w \in \mathfrak{g}^K$, on the window and, with all links variable, on $\mathbb{Z}^4$ or on a torus.*

(ii) *On the window, R is Γ_W-invariant and $|\mathscr{H}R(w,w)| \leq \mathscr{L}_0\, Q^{V_0}(\Pi_{V_0} w)$. At $V_0 = I$ this is at most $\mathscr{L}_0 Q_1(w) \leq (\mathscr{L}_0/c_*)\, S_W(w)$, for every n.*

(iii) *If each box carries at most one term and $h_X \leq \eta e^{-\alpha r_X}$ with $\eta, \alpha > 0$, then $\mathscr{L}_0 \leq 180\, e^{\alpha}\eta\,(1-e^{-\alpha})^{-7}$.*

Proof. (i) The Hessian of the series is the series of the Hessians. By Lemma 656(iii), applied to the restriction of w to L_X,

$$\begin{aligned}|\mathrm{Hess}\, R(V_0)[w,w]| &\leq \sum_X h_X \rho_{r_X} \sum_{p\in P_X} \|Z_p(w)\|^2 \\ &= \sum_p \|Z_p(w)\|^2 \sum_{X:\, p\in P_X} h_X \rho_{r_X},\end{aligned}$$

which is at most $\mathscr{L}_0 Q^{V_0}(w)$.

(ii) An element of Γ_W fixes the links off K and acts on the links of X through its restriction to the sites of X, an element of G^X; so every f_X, and R, is Γ_W-invariant. Lemma 564(ii) gives $\mathscr{H}R(w,w) = \mathrm{Hess}\, R(V_0)[\Pi_{V_0} w, \Pi_{V_0} w]$, and (i) applies. At $V_0 = I$, $Q^I = Q_1$ vanishes on $\mathscr{O}$ and is positive semidefinite, so $Q_1(\Pi w) = Q_1(w)$, and $Q_1 \leq S_W/c_*$ by Lemma 562(iii).

(iii) A plaquette in the plane $\{\mu,\nu\}$ at x lies in the box of side r with corner b only if $b_\mu \in [x_\mu - r + 1, x_\mu]$, $b_\nu \in [x_\nu - r + 1, x_\nu]$ and $b_\lambda \in [x_\lambda - r, x_\lambda]$ for the two other directions, so in at most $r^2(r+1)^2$ boxes of side r. Hence $\mathscr{L}_0 \leq \frac{\eta}{4}\sum_{r\geq 1}(r+1)^6 e^{-\alpha r}$. Since $m^6 \leq 720\binom{m+6}{6}$ and $\sum_{m\geq 0}\binom{m+6}{6}x^m = (1-x)^{-7}$, $\sum_{r\geq 1}(r+1)^6 e^{-\alpha r} \leq e^{\alpha}\sum_{m\geq 0} m^6 e^{-\alpha m} \leq 720\, e^{\alpha}(1-e^{-\alpha})^{-7}$. □

The bound holds for every local term; for the filled loops of Proposition 567 the factor ρ_r is not needed, since their Hessian at a flat background is already a form in the curvatures. At the trivial background it gives the one-step bound of Proposition 565 without C_n, once the hypothesis is put on the gauge-invariant density.

Corollary 658 (one step with a local invariant remainder)**.** *Let $U = I$, so that $V_0 = I$, and let $A^{\mathrm{gi}} = A_0 + R$ near I, where A_0 is Γ_W-invariant with $\mathscr{H}A_0 = 2\lambda S_W$ and R is a local remainder with $\mathscr{L}_0 \leq \lambda\varepsilon$. Then the error $E = \mathscr{H}A^{\mathrm{gi}} - 2\lambda S_W$ vanishes on $\mathfrak{g}^K \times \mathscr{O}$ and*

$$|E(w,w)| \leq \lambda\varepsilon\, Q_1(w) \leq \frac{\varepsilon}{2c_*}\cdot 2\lambda S_W(w) \qquad (w \in \mathfrak{g}^K),$$

so that $(1 - \frac{\varepsilon}{2c_*})2\lambda S_W \leq \mathscr{H}A^{\mathrm{gi}} \leq (1 + \frac{\varepsilon}{2c_*})2\lambda S_W$ for every n, with $\frac{1}{2c_*} < 3.433$.

Proof. By Lemma 564(ii) the covariant Hessian is linear in the function, so $E = \mathscr{H}R$. It vanishes on $\mathfrak{g}^K \times \mathscr{O}$ by Lemma 564(ii), and Theorem 657(ii) bounds it. The constant follows from $2c_* > 0.29131$, as in Corollary 568. □

The factor C_n of Proposition 565 and of Corollary 568 therefore comes from the form of the hypothesis. Hypothesis (H) bounds the Hessian of the chart density A, which differs from the invariant density A^{gi}, the density with respect to Haar measure, by the Haar term. The two cannot both be local on a large window unless the error in (H) is of order g^2.

Proposition 659 (the chart hypothesis against locality)**.** *Let $G = \mathrm{SU}(2)$ and $U = I$. Suppose that $A^{\mathrm{gi}} = A_0 + R$ as in Corollary 658, with constant $\mathscr{L}_0$, and that the chart density A satisfies the upper half of (H):* $\mathrm{Hess} A(0)[\sigma,\sigma] \leq 2\lambda S_W(\sigma) + \lambda\varepsilon|\sigma|^2$ *for $\sigma \in \mathscr{O}^{\perp}$. Then*

$$\lambda\varepsilon \geq \frac{1}{3} - \frac{\mathscr{L}_0}{C_n^-} \geq \frac{1}{3} - \frac{3\pi^2\mathscr{L}_0}{(n+1)^2}.$$

In particular $\lambda\varepsilon \ge \frac{1}{6}$ *on every window with* $(n+1)^2 \ge 18\pi^2\mathscr{L}_0$.

Proof. On the slice $A^{\mathrm{gi}}(\Upsilon(s,\eta)) = A(s) + \sum_k \log j(s_k)$, so, as in the proof of Proposition 565(ii),

$$\mathscr{H}A^{\mathrm{gi}}(w,w) = \mathrm{Hess}A(0)[\Pi w,\Pi w] + \frac{1}{12}\sum_k \mathrm{Tr}\big(\mathrm{ad}^2_{(\Pi w)_k}\big),$$

and the last sum is $-\frac{1}{3}|\Pi w|^2$ for SU(2). Let $w \neq 0$ be the field of Lemma 562(iv), with $w \in \mathscr{O}^\perp$, so $\Pi w = w$, and $|w|^2 = C_n^- Q_1(w)$. The hypothesis gives $\mathscr{H}A^{\mathrm{gi}}(w,w) - 2\lambda S_W(w) \le (\lambda\varepsilon - \frac{1}{3})|w|^2$, and Corollary 658 gives that this difference is at least $-\mathscr{L}_0 Q_1(w) = -\mathscr{L}_0|w|^2/C_n^-$. Divide by $|w|^2$ and use $C_n^- \ge (n+1)^2/(3\pi^2)$. □

An error of order $g^2 = 1/\lambda$ in each step is not summable over the scales (Remark 585). The measure defines the invariant density, and the localised actions of Bałaban are invariant; so the hypothesis belongs on A^{gi}, and in that form the Haar term does not appear.

Curved backgrounds. At a background that is not flat on a box, a local term has a gradient there, and Lemma 656 fails. The term is compared with its value at the trivial field after a gauge transformation of the box, which costs a Euclidean term proportional to the curvature in the box, and the Euclidean term is paid with the ℓ^4 bound of Lemma 576. The following lemma gives the constants from bounds on complex neighbourhoods, the form in which cluster expansions bound their terms.

Lemma 660 (derivatives from analytic bounds)**.** *Let f be a function of the links of a finite set L, $a \ge 0$ and $\rho > 0$; $\mathfrak{g}\otimes\mathbb{C}$ carries the Hermitian extension of the norm, and $e^\xi e^\omega$ is the product link by link.*

(i) *If $\omega \mapsto f(e^\omega)$ extends holomorphically to $\{\omega \in (\mathfrak{g}\otimes\mathbb{C})^L : \|\omega_k\| < \rho \ \forall k\}$ with modulus at most a, then $\|\mathrm{Hess}\, f(I)\| \le 2a/\rho^2$.*

(ii) *If the function $(\xi,\omega) \mapsto f(e^\xi e^\omega)$ extends holomorphically to the set where $\|\xi_k\| < \rho$ and $\|\omega_k\| < \rho$ for every k, with modulus at most a, then for every $v \in \mathfrak{g}^L$ and $V = e^\xi$ with $\xi \in \mathfrak{g}^L$ and $\delta = \max_k\|\xi_k\| \le \rho/2$,*

$$\big|\mathrm{Hess}\, f(V)[v,v] - \mathrm{Hess}\, f(I)[v,v]\big| \le \frac{4a\delta}{\rho^3}|v|^2.$$

(iii) *If $G = \mathrm{SU}(N)$ and $\|I-V\| \le \min(1, 3/N)$, then $V = e^\xi$ with $\xi \in \mathfrak{g}$ and $\|\xi\| \le \frac{\pi}{3}\sqrt{N}\,\|I-V\|$.*

Proof. (i) For $v \in \mathfrak{g}^L$, $v \neq 0$, let $m = \max_k\|v_k\|$. The function $z \mapsto f(e^{zv})$ is holomorphic for $|z| < \rho/m$ and bounded by a, and its second derivative at 0 is $\mathrm{Hess}\, f(I)[v,v]$, since $\omega = zv$ is a line. Cauchy's inequality gives $|\mathrm{Hess}\, f(I)[v,v]| \le 2am^2/\rho^2 \le 2a|v|^2/\rho^2$. (ii) Let $G(s,z) = f(e^{s\xi}e^{zv})$, holomorphic for $|s| < \rho/\delta$ and $|z| < \rho/m$, with $|G| \le a$. Then $\mathrm{Hess}\, f(V)[v,v] = \partial_z^2 G(1,0)$ and $\mathrm{Hess}\, f(I)[v,v] = \partial_z^2 G(0,0)$. For $0 \le s \le 1$ the bidisc about $(s,0)$ with radii $\rho/(2\delta) \le \rho/\delta - 1$ and ρ/m lies in the domain, so Cauchy's inequality gives $|\partial_s\partial_z^2 G(s,0)| \le 2a\cdot 2\delta m^2/\rho^3$; integrate over $s \in [0,1]$. (iii) Let $e^{i\theta_j}$, $|\theta_j| \le \pi$, be the eigenvalues of V. Then $2|\sin(\theta_j/2)| \le \|I-V\| \le 1$, so $|\theta_j| \le \pi/3$, and $\sin x \ge 3x/\pi$ on $[0,\pi/6]$ gives $|\theta_j| \le \frac{\pi}{3}\|I-V\|$. The skew-Hermitian ξ with the eigenvalues $i\theta_j$ on the eigenspaces of V has $e^\xi = V$, and $\sum_j \theta_j \in 2\pi\mathbb{Z}$ with $|\sum_j\theta_j| \le \frac{N\pi}{3}\|I-V\| \le \pi$, so $\mathrm{Tr}\,\xi = 0$ and $\xi \in \mathfrak{g}$. Finally $\|\xi\|^2 = \sum_j\theta_j^2$. □

Theorem 661 (local remainders at curved backgrounds)**.** *Let $R = \sum_{X\in\mathscr{X}} f_X$ be a local remainder near $V_0 \in G^K$, the links off K given, every box of $\mathscr{X}$ containing a link of K. Let $\kappa_p = \|I - V_{0,p}\|$, $\kappa_X = \max_{p\in P_X}\kappa_p$ and $\delta_X = 3r_X\kappa_X$, and suppose:*

(a) $\|\kappa\|_2 \le \frac{1}{9}$, *with $\|\kappa\|_2$ as in Lemma 576;*

(b) *for every X there is $t_X \ge 0$ with*

$$\big|\mathrm{Hess}\, f_X(V)[v,v] - \mathrm{Hess}\, f_X(I)[v,v]\big| \le t_X \max_{k\in L_X}\|I-V_k\|\,|v|^2$$

whenever $\max_{k\in L_X}\|I-V_k\| \le \delta_X$.

Put $\rho_X = \rho_{r_X}$, $b_X = 3r_X\big(h_X\rho_X(\sqrt{432}+432\delta_X)+t_X\big)$,

$$\mathscr{L}_\kappa = \sup_p \sum_{X:\,p\in P_X} h_X\rho_X\big(1+\sqrt{432}\,\delta_X\big),$$

$$B_1 = \sup_{k\in K}\sum_{X:\,k\in L_X} b_X, \qquad B_2 = \sup_p \sum_{X:\,p\in P_X} b_X\ell_X,$$

and $\kappa_* = (\sum_p \kappa_p^2)^{1/2}$, the sum over the plaquettes of the boxes of $\mathscr{X}$. Then for every $w \in \mathfrak{g}^K$

$$|\mathscr{H}R(w,w)| \le \big(\mathscr{L}_\kappa + c_4(B_1B_2)^{1/2}\kappa_*\big)\,Q^{V_0}(\Pi_{V_0}w).$$

None of the constants depends on n. If V_0 is flat on the boxes, the factor is $\mathscr{L}_0$ of Theorem 657. Under the hypotheses of Lemma 660(ii), (iii) *for f_X, with constants a_X, ρ, hypothesis* (b) *holds with $t_X = \frac{4\pi\sqrt{N}}{3}a_X\rho^{-3}$ when $\delta_X \le \min(1, \frac{3}{N}, \frac{3\rho}{2\pi\sqrt{N}})$.*

Proof. Write $u = \Pi_{V_0}w$ and $|u_X|^2 = \sum_{k\in L_X}\|u_k\|^2$, with $u = 0$ off K. As in Theorem 657(ii), R is Γ_W-invariant and $\mathscr{H}R(w,w) = \sum_X \mathrm{Hess}\, f_X(V_0)[u,u]$.

Fix X. The tree argument in the proof of Lemma 571, carried out in X, where $\sum_{\nu<\mu}(x_\nu - b_\nu) \le 3r_X$ for every link (x,μ) of X and every plaquette used lies in P_X, gives $g \in G^X$ with $\|I - (gV_0)_k\| \le \delta_X$ for every $k \in L_X$. With $V' = gV_0$ and $v = Tu$ as in the proof of Lemma 656, $\mathrm{Hess}\, f_X(V_0)[u,u] = \mathrm{Hess}\, f_X(V')[v,v]$, $|v| = |u_X|$ and $Q_X^{V'}(v) = Q_X^{V_0}(u)$. By (b),

$$|\mathrm{Hess}\, f_X(V')[v,v] - \mathrm{Hess}\, f_X(I)[v,v]| \le t_X\delta_X|u_X|^2,$$

and Lemma 656(iii) at I gives

$$|\mathrm{Hess}\, f_X(I)[v,v]| \le h_X\rho_X\sum_{p\in P_X}\|(dv)_p\|^2.$$

As in the proof of Lemma 571, $\|Z_p(v) - (dv)_p\| \le 2\delta_X(3b_1 + 2b_2 + 2b_3 + b_4)$ at V', with b_i the norms of v on the links of p, and a link lies in at most six plaquettes, so

$$\Big(\sum_{p\in P_X}\|(dv)_p\|^2\Big)^{1/2} \le Q_X^{V_0}(u)^{1/2} + \sqrt{432}\,\delta_X|u_X|.$$

With $(s+t)^2 \le (1+\theta)s^2 + (1+\theta^{-1})t^2$ at $\theta = \sqrt{432}\,\delta_X$, and directly if $\delta_X = 0$,

$$|\mathrm{Hess}\, f_X(V_0)[u,u]| \le h_X \rho_X \big(1+\sqrt{432}\,\delta_X\big) Q_X^{V_0}(u) + b_X \kappa_X |u_X|^2,$$

since $\delta_X(h_X\rho_X(\sqrt{432}+432\delta_X)+t_X) = b_X\kappa_X$.

Summing over X and exchanging the sums, the first terms give at most $\mathscr{L}_\kappa Q^{V_0}(u)$, and the second terms give $\sum_{k\in K}\beta_k\|u_k\|^2$ with $\beta_k = \sum_{X:k\in L_X} b_X\kappa_X$. By Cauchy–Schwarz, $\beta_k^2 \le B_1 \sum_{X:k\in L_X} b_X\kappa_X^2$ and $\kappa_X^2 \le \sum_{p\in P_X}\kappa_p^2$, so

$$\sum_k \beta_k^2 \le B_1 \sum_X b_X \ell_X \sum_{p\in P_X} \kappa_p^2 \le B_1 B_2 \kappa_*^2,$$

and $\sum_k \beta_k\|u_k\|^2 \le (\sum_k \beta_k^2)^{1/2}|u|_4^2$. Since $u \in \mathscr{O}_{V_0}^\perp$, Lemma 576 and (a) give $|u|_4^2 \le c_4 Q^{V_0}(u)$. The last statement follows from Lemma 660(ii), (iii): for $\max_k\|I-V_k\| = \delta \le \delta_X$, $V = e^\xi$ with $\max_k\|\xi_k\| \le \frac{\pi}{3}\sqrt{N}\delta \le \rho/2$. □

Theorem 661 extends Proposition 577 from loop sums, whose existence for the induced action is open (Remark 583), to every local remainder, with the same order $\|\kappa\|_2$ of the loss. For the plaquette sum that order is attained (Corollary 578). Measured by the supremum of the curvature the loss is not of order κ uniformly in n, even for the plaquette sum, a local remainder whose terms do not depend on n.

Proposition 662 (no loss of order κ uniformly in n)**.** *Let $G = \mathrm{SU}(2)$, $0 < \kappa \le 0.02$, $n \ge (31/\kappa)^{1/2}$, and let P be the sum of $s(V_p)$ over the plaquettes with their corners in Λ'. There are a background V_0 with $\|I - V_{0,p}\| \le \kappa$ for every such plaquette and $u \in \mathscr{O}_{V_0}^\perp$ with $\mathscr{H}P(u,u) < 0 \le \frac12 Q^{V_0}(u)$. Hence a bound*

$$(1-\phi)\,\tfrac12 Q^{V_0}\circ\Pi_{V_0} \le \mathscr{H}P \le (1+\phi)\,\tfrac12 Q^{V_0}\circ\Pi_{V_0},$$

valid at every background with plaquettes within κ of I on windows of side n, requires $\phi > 1$; for every $C \le 1/\kappa$ none holds with $\phi = C\kappa$ for all n.

Proof. Take $\sin^2\alpha = \kappa/2 \le 10^{-2}$ and $\varsigma = -1$ in Proposition 573: the plaquettes in the plane $\{0,1\}$ have $\|I - V_{0,p}\| = 2\sin^2\alpha = \kappa$, the others are flat, $n \ge (31/0.02)^{1/2} > 39$ and $\kappa(n+1)^2 > 31$, so part (ii) gives $u \in \mathscr{O}_{V_0}^\perp$ with $\mathscr{H}P(u,u) < 0$, and $\Pi_{V_0}u = u$. A lower bound with $\phi \le 1$ would give $\mathscr{H}P(u,u) \ge 0$. □

Within $\kappa(n+1)^2 \le 2$, Proposition 573(i) gives the lower bound $\phi \ge 0.037\,\kappa(n+1)^2$. Since $\|\kappa\|_2 \le \sqrt{6}(n+1)(n+2)\kappa_\infty$, the order κn^2 in the supremum and the order $\|\kappa\|_2$ in the action are the same estimate; the action is the measure of the background that does not depend on n, and on the calm windows of Corollary 579 it is at most $39.5K$.

Condition (C). At the trivial background on the torus of the next scale, Theorem 657(i) gives both halves of condition (C) of Theorem 652 from a local representation of the one-step error.

Corollary 663 (condition (C) from local terms)**.** *In the setting of Sec. XIII T, let $k < K$ and assume P_k of Theorem 652. Suppose that D_k is the restriction to $\mathscr{X}_{k+1}$ of $\mathrm{Hess}\, R(I)$ for a local remainder R on the torus of scale $k+1$, all of whose boxes have side at most $n_{k+1}-2$, with*

$$\mathscr{L}_0 \le \frac{1.027}{\lambda_{k+1}}, \qquad \Delta = \sup_a \sum_{X:a\in L_X} h_X \ell_X e^{\alpha_{k+1} r_X} < \infty.$$

Then condition (C) holds at the step k with $\delta_k = \Delta/\lambda_{k+1}$.

Proof. By Theorem 657(i) at $V_0 = I$, $|D_k(x,x)| \le \mathscr{L}_0 Q_{k+1}(x)$ for $x \in \mathscr{X}_{k+1}$. By Proposition 335(iii), whose bound $S_k \ge Q_k$ holds on every lattice, $Q_{k+1} \le R_{k+1}/(2\lambda_{k+1})$. By P_k(a), $H_k \ge (1-\sigma_k)R_k$, and $\mathscr{M}_k$ is monotone and positively homogeneous, so $(1-\sigma_k)R_{k+1} = (1-\sigma_k)\frac{\lambda_{k+1}}{\lambda_k}\mathscr{M}_k R_k \le \frac{\lambda_{k+1}}{\lambda_k}\mathscr{M}_k H_k$ on $\mathscr{X}_{k+1}$. With $\sigma_k \le \sigma_K \le 0.4865$, as in the proof of Theorem 652,

$$|D_k(x,x)| \le \frac{\mathscr{L}_0}{2\lambda_{k+1}(1-\sigma_k)}\frac{\lambda_{k+1}}{\lambda_k}\mathscr{M}_k H_k(x) \le \frac{1}{\lambda_{k+1}^2}\frac{\lambda_{k+1}}{\lambda_k}\mathscr{M}_k H_k(x),$$

because $2(1-0.4865) = 1.027$; this is the first half of (C). The blocks of D_k are $(D_k)_{ab} = \sum_{X:a,b\in L_X}(\mathrm{Hess}\, f_X(I))_{ab}$, each of norm at most h_X; two links of a box of side r have midpoints at ℓ^∞ distance at most r, and a box has ℓ_X links, so $\sum_b\|(D_k)_{ab}\|e^{\alpha_{k+1}d_{k+1}(a,b)} \le \Delta$, and restriction to $\mathscr{X}_{k+1}$ does not increase it. □

Remark 664 (what the local form leaves open). Four points are not settled by the results above.

(a) The induced action of the cut-off averages is not shown to be of the form $A_0 + R$ with R a local remainder. By Theorem 607 its leading term is nonlocal when $(M_W)_{AB} \ne 0$, and the theorems above concern only the remainder; the bound $\mathscr{L}_0 \le 1.027/\lambda_{k+1}$ of Corollary 663 asks for terms of order g^4 relative to the action, after the order g^2 is absorbed into λ_{k+1}.

(b) The large-field estimates of Sec. XIII R bound the values of polymer activities on real fields. Lemma 660 needs bounds on complex neighbourhoods, and by Remark 569(b) real bounds on values give no bound on Hessians.

(c) Each term of side r costs $\rho_r \le (r+1)^2/4$, the gap of its own box, so the theorems need $\sum_r r^2(r+1)^4 \sup_{r_X=r} h_X < \infty$, which exponential decay of the terms in their size provides (Theorem 657(iii)).

(d) The locality constants Λ_k of Theorem 652 grow with k whatever δ_k is; Corollary 663 does not change that.

V. What remains

The results of this section leave the following statements, which together with Theorem 367 and Sec. XII G are what the cube condition (D) needs for SU(2) past the Dobrushin ceiling.

(i) *The bad side.* For a pair whose non-path links are bad, a bound on the Kantorovich response of the law of its eight private links, given the inside links of its two blocks, uniform in the exterior. Proposition 418 shows that the response of the path-link law alone to its non-path links is infinite, so the non-path links must be integrated over the bad set; Lemma 415 bounds the response of that law to every link that enters only the action, and Proposition 416 settles the good side to radius $1/32$. The inside links move the fibre $\{V_k^\sharp = c_k\}$ while the bad set does not; Proposition 422 shows that the path-link law also jumps in an inside link. For $\mu = 3$ the inside μ-links are tree links and Lemma 415 settles the law of the eight private links; for $\mu \le 2$, Theorem 423 shows that its response to an inside link is not bounded uniformly in the set over which the non-path links are restricted. In the relative chart of Lemma 425 the fibre is one graph over the coordinates $g^{-1}\ell_a$, and Theorem 427 bounds the response for the two sets that the labels produce (Proposition 424) under one condition, the continuity in the mean (J) of the Jacobian of that chart. Theorem 431 proves (J) when the inside μ-links u_a lie within $\frac{1}{32}$ of I, whatever the links w_a, and, in the coordinates $\ell_a g^{-1}$, when the w_a do, whatever the u_a; this includes every configuration of B_r^{bad} that the labels produce, with a constant uniform in the configuration, and for the first set it bounds the response whenever the inside μ-links of one block lie within $\frac{1}{32}$ of I (Corollary 432). The constant contains the largest normalised area of the hypersurfaces where the two paths of one index tie at the top, which is finite by o-minimality (Proposition 430) and not computed; a bound of order t on the Haar measure of the set where these paths tie at the top within t is false (Proposition 433). Condition (J) also holds, and the response to the inside μ-links of one block is bounded, when those of the other block equal I. For the first set the chart exists unless both blocks contain an inside μ-link u with $\|\mathrm{Ad}(u) - 1\| \ge 1/\kappa$, $\kappa < 2.2670$; on an open set of such configurations neither chart exists, the equation $V_k^\sharp = c_k$ having two solutions over an open set of ξ (Proposition 434), and the response there is not proved. The chains of Definition 435 carry the inside links among their variables, partition the bad side, and respond to every other link with the constants of Theorem 421 in which 24 is replaced by their size $n \le 23n_b + 1$ (Proposition 436); they cost no collar for $s \ge 2$, and for $s = 1$ the collar $s = 2$ (Corollary 437). For the projected average $\widehat{V}$ of Sec. XIII C, a different coarse map whose derivatives of order at most two agree with those of $V^\sharp$ at every field whose sixteen paths agree (Proposition 440), item (i) holds: the path-link equation has exactly one solution at every field, with Jacobian in $[\frac{1}{512}, \frac{3375}{512}]$ (Theorem 441); the path-link law responds to the non-path and inside links with the constants 1 and $\frac{1}{2}$, independent of β_W and of the size of the links (Corollary 442); and the law of the eight private links restricted to any fixed set of positive Haar measure of the non-path links responds to an inside link with the constant $24\beta_W + 180.5$, with no hypothesis on the inside links, no lower bound on the measure of the set, and no condition (J) (Theorem 443). The chart decay of Theorem 367 holds for $\widehat{V}$ with the same constants (Proposition 447); the route to (D) with $\widehat{V}$ is Remark 449, and the labels, the islands and the results off the chart are proved for $V^\sharp$ only (Remark 448). For $V^\sharp$ item (i) stays as stated above.

(ii) *The island comparison.* For two islands in a good region, Theorem 483 factorises their weight through one kernel on the good layer between them, Theorem 485 proves the covariance bound (41) for the law of that layer in the coordinates of the labels, and Corollary 486 bounds the cross-ratio by Δ, exponentially small in the width. Lemma 487 evaluates the prefactor, $\Lambda_2 \le 99.0001\beta_W + 4.65 \cdot 10^6$, so that $\Delta \le 1$ at the threshold needs a good layer of width at least 99072. Theorem 491 performs the induction over all the islands of a window given their labels. Theorem 494 gives the smallness of its island sums and its Dobrushin condition in every window of at most 10^{28000} blocks, outside probability 10^{-166}. With the clusters of Definition 632 as units, Theorem 634 gives these conditions in every configuration of the labels and every window, and Theorem 635 bounds the size of the clusters block by block, uniformly in the torus up to $M_1 = 10^{10^{1700}}$ blocks. No bound on the size of the clusters of the torus is uniform in the torus: every block is bad with conditional probability at least $\frac{1}{2}e^{-12\beta_W}$, and for every separation rule that bounds the island sums the clusters percolate on large tori (Corollary 641, Proposition 642). The clusters of the bad blocks of a window replace them: the comparison holds with them as units, and in a window of at most $10^{M_1/2}$ blocks their size is bounded uniformly in the torus (Lemma 638). Sec. XIII O bounds the covariance of the labels with the others averaged, and the covariance of two local observables under μ^c, in the mean over the coarse field and on the whole torus: Theorem 497 improves the trivial bound $2\varepsilon|N_A||N_B|$ by a factor decaying at the rate $2\alpha_G$ down to 10^{-7011}, and Theorem 498 gives decay at the same rate down to 10^{-14062}, or as $e^{-\sqrt{d}/2}$. For a fixed coarse field, Proposition 503 gives the same bounds for every c outside a set of probability at most $2|\Lambda|\varepsilon^{1/2}$, and Corollary 504 gives decay as $e^{-\sqrt{d}/20}$ in the mean and for these c. What is not done is a bound for every coarse field, uniform in the exterior, and decay at the rate $2\alpha_G$ without a floor. By Theorem 505 neither follows from rarity and the group cross-ratios: for laws with only these properties the decay is at best stretched exponential, and across a component that is bad for every labelling they bound nothing. In the SU(2) model, Theorem 526 bounds the constrained law inside such a component by the chart, at the rate α_G and with no cost in its length beyond the mass p of that law off the chart, whenever the fibre has a background with plaquettes below $6.0 \cdot 10^{-13}$; by Proposition 525 this holds for coarse plaquettes up to that size when the exterior agrees with

$R(c)$, by Proposition 531 it holds for some exteriors up to $2.39\cdot 10^{-12}$, and no background exists once a coarse plaquette exceeds $2.418\cdot 10^{-12}$. Theorem 529 shows that p tends to one as the component grows at fixed β_W, so that the bound improves on the trivial one only for $\beta_W > 10^{24}(4.40+\ln|W|)$. Proposition 532 identifies the constrained law, with the links that bind the entering averages moved to the exterior, as the conditional law under μ^c, and Corollary 533 bounds the cross-ratio of the weight of the interior for ends that enter the action only, at the rate α_G up to a term $432\beta_W^2\bar{p}$. Charts local in the component do not remove the dependence on the length: by Theorem 535 a window lies on a chart fixed by its exterior with conditional probability below $0.033^{|W|}$ for $\beta_W \le 10^{24}$, whatever the exterior and the coarse field. For the coarse fields c^θ of Proposition 536 the exterior $R(c)$ carries a background up to $2.39\cdot 10^{-12}$, and the reduced fibre up to $2.09\cdot 10^{-12}$; for opposite fluxes at adjacent coarse plaquettes no background exists above $1.8001\cdot 10^{-12}$, for any field and any exterior, while the dipole $c^{\theta,d}$ has one with exterior $R(c)$ up to $1.79\cdot 10^{-12}$ (Proposition 560), so the threshold is not a function of the size of the coarse plaquette; a general coarse field between $6.0\cdot 10^{-13}$ and $2.418\cdot 10^{-12}$ with exterior $R(c)$ is not decided. An end that enters an average enters one pair factor of the fibre measure (Lemma 537), and that factor jumps under arbitrarily small moves of the end at every value of the coarse link (Proposition 538); after the non-path links are integrated, its mass changes by less than $3.2\cdot 10^{-4}$ under every move (Lemma 540), and along a tube of blocks the cross-ratio of two such ends decays by the factor $1-e^{-D_i}$ per slab, $D_i = |\Sigma_i|(72\beta_W + 3.2\cdot 10^{-4})$, at every β_W and without a chart (Theorem 541, Corollary 542). That factor has no content for $\beta_W \ge 2.3\cdot 10^{17}$. The derivative of the integrated pair factor in an end is bounded in the mean over the coarse link at every configuration (Corollary 546), and uniformly in the coarse link when the links of one of the two blocks of its pair lie within $\frac{1}{32}$ of I (Corollary 548, Lemma 557), with a constant linear in β_W that involves one integer, the crossing number $\mathfrak{b}_{\mathrm{tie}}$, finite by o-minimality and not computed (Theorem 556). Along a tube this gives mixed differences in two ends that decay at the rate of Theorem 541 (Corollary 549), and on the chart of Theorem 526 mixed differences that decay at the rate α_G, independent of β_W, up to a term proportional to the mass off the chart, when the blocks next to the end pairs lie outside W (Theorem 558). The bound uniform in c_k when both blocks of the pair have links far from I, and a decay between two ends at a rate independent of β_W off the chart, are not proved (Remarks 539, 543, 561 and 559). Lemma 506 gives the mixed differences of every order in terms of the diameter; the rate $2\alpha_G$ needs them in terms of the length of the shortest joining tree (Remark 507). That form is not proved, and it follows neither from the second differences in every environment (Proposition 508) nor from the layer bound (41) (Proposition 509); Remark 510 names the two inputs it needs. Theorem 511 and Corollary 512 give the differences of order three of $\log\tilde{K}$ in tree form at the rate α_G from the Hessian bound behind (41) alone, which is optimal for that bound; Theorem 515 and Corollary 516 give them at the rate $2\alpha_G$ of the second differences, in the coarse length of the shortest joining tree, with the walls of D_z included, using in addition the decay along trees of the derivatives of order three of $\mathscr{U}$ that its finite range supplies. For a smooth potential of finite range the tree form holds at every order at the rate of the covariances (Theorem 518); for the layer law the orders $k \ge 4$, for which the Hessian bound alone gives no tree form uniform in the order (Proposition 513), are proved only for its smooth approximations, with constants that grow with the approximation. The passage to $\log Z^c$ reduces to the joint cumulants of the gas of islands (Lemma 520), of which only the covariances are bounded, and no rate uniform in the pair cross-ratios holds (Proposition 521, Remark 522). Proposition 455 shows that no rarity bound replaces this comparison.

(iii) *The island threshold.* Theorem 465 gives the explicit coupling $\beta^*(\rho,s)$ past which the islands of bad blocks at one scale are rare enough to beat their entropy, at the exponent $1/H$ of Proposition 451 and the collar of Proposition 412. It is of order 10^{13} at the witness radius 10^{-4} of Theorem 471, and of order 10^{37} under the calmness of Corollary 486; the tree count of Proposition 474 lowers it by a factor 1.7 to 1.8, and the union bound over connected sets by at most a further factor 1.39. The same statement at the scale of the cube is not given, and Theorem 476 shows that no bound of this kind gives it: at scale j every witness of a coarse label of radius $\eta < \eta_c$ has radius at most $1.00015\cdot 4^{-j-1}\eta$, by the constant curvature of Proposition 468; the normalisation of the chessboard estimate charges at least $\frac{9}{2}(\log\beta_W - c_I)$ per fine site (Lemma 475); and so every bound obtained from Theorem 383 or Proposition 403 is below one only at scales whose blocks are smaller than the cube of $D(n,\alpha)$ by a factor at least $4.08(1-\alpha)^{1/4}$, at every coupling, with the normalisation of Lemma 382 or, for $\beta_W \ge 10^3$, the exact one. For $\alpha \le \frac{1}{2}$ and the labels of Theorem 471 the cube lies at least three dyadic scales higher, and for the calm labels at least twenty (Corollary 477). The fields of constant curvature that carry the ceiling are critical points of the action (Lemma 479), so every first-order estimate vanishes at them, the forces of Theorem 404 and Proposition 407 included. By Theorem 480 the ceiling stays when the chessboard estimate need cover only the defects in an open set that contains the critical points, in particular those left by finitely many first-order estimates, which never cover a neighbourhood of these fields (Corollary 481). What is missing is a bound on the weight of the coarse defects relative to Z_L that does not charge the fluctuation volume of the cube and detects the defects near the critical points (Remarks 478 and 482), and (iv) is needed before any threshold bears on the gap.

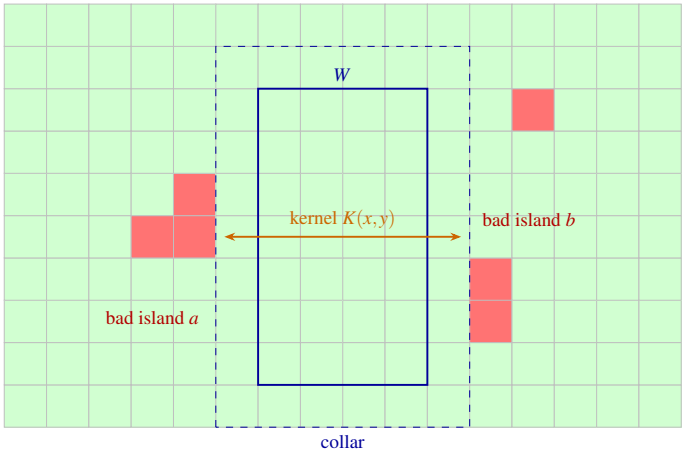
(iv) *The scales.* An effective action at every scale whose

one-step errors are summable in the sense of Proposition 460 and whose locality norms obey Lemma 459, down to the last $O(1)$ scales where the running coupling is of order one; this is the large-field expansion of Bałaban, and it is not carried out here. Theorem 467 and Theorem 471 settle the entropy count of the islands at every scale up to $a_{k_*}/8$, and Proposition 468 shows that fine witnesses carry it no further; the law of the iterates given their labels, which the effective action is for, is untouched. Proposition 461 shows that the one-step errors must be relative bounds by a gauge-invariant form: an error bounded by a multiple of the Euclidean form grows by a factor at least $4/(1+2724\kappa)$ at each step. Proposition 565 gives such a relative bound for one step at the trivial background on a box of side n, from an absolute bound on the Hessian of the induced action off the gauge directions; its factor C_n, of order n^2, is necessary for an absolute bound (Remark 585). For a remainder made of filled Wilson loops the bound is relative at every flat background (Proposition 567), and C_n falls on the Haar term alone (Corollary 568). At a background whose plaquettes lie within κ of I the covariant Hessian of such a remainder is bounded by the covariant plaquette form with a loss of order κn^2, at critical and non-critical points alike (Proposition 572); the loss is attained, and for $\kappa(n+1)^2 \geq 31$ the plaquette action itself has a negative direction off the gauge directions (Proposition 573). In terms of the ℓ^2 norm $\|\kappa\|_2$ of the curvature, the square root of the Wilson action of the background on the window, the loss is of order $\|\kappa\|_2$ for every n (Proposition 577), the unstable mode appears at a background with $\|\kappa\|_2 < 31.8$ (Corollary 578), and on calm windows both forms of the loss are of order K (Corollary 579). That the induced action is a loop sum with summable coefficients is not proved: for SU(2) loop sums are dense among invariant functions (Proposition 582), which controls neither values nor Hessians, and a relative bound on values gives no Hessian bound at curved backgrounds (Remark 569). Sec. XIII R carries out the large-field part of one step for SU(2): each rough block costs $e^{-\lambda\kappa^2}$, the weight of a coarse field is an exact sum over large-field regions, and the large-field sum is the calm weight times the mean of a polymer sum under the calm law (Proposition 604), whose activities are at most e^{-9} per block wherever the boundary values admit a completion calm at $\kappa/30$ (Theorem 602). That bound fails at calm fields with plaquettes in the shell $(\kappa/30, \kappa]$: the Bianchi identity of a cube holds in SU(2) exactly up to conjugation (Lemma 616), and at the trivial coarse field the relative activity of one block is not essentially bounded under the calm law, at every λ and every $\kappa \leq 10^{-8}$ (Theorem 617). So Proposition 604(ii) does not apply, and the shell needs a bound in the mean, which is a bound on the ratios $\mu_c\{R = X\}/\mu_c\{R = \emptyset\}$ of the constrained law and is not proved (Remark 618). Not proved are also a bound on the term $Z_\perp$ and the locality of the calm weight in c; by Theorem 607 the calm weight is not a product of local factors times a polymer sum if the rational number $(M_W)_{AB}$ of that theorem is nonzero, as an exact computation finds (Remark 608), and then a local form has to expand around the Gaussian block-spin form. Nor can the leading term be $\lambda q^{\mathrm{lin}}(\log c)$ for $\lambda > 0.5472$, by gauge invariance (Theorem 610); the linear propagator of such an expansion decays at an explicit rate (Theorem 611); in the linear model the relative activities in the shell are unbounded under the calm law at every λ (Theorem 614); and $Z_\perp(I) > 0$ for SU(2) (Theorem 621). The coarse action is comparable to λ times the coarse curvature only up to a constant factor. Sec. XIII S replaces the factor C_* of the comparison by 2956 for every coarse field and by $1 + O(\kappa_0)$ on the coarse plaquettes below $\kappa_1 \approx 0.19\,\kappa_0$ (Theorem 643), and shows that no such comparison has a ratio of constants below $4t_L^2 > 3.849$ (Theorem 644), so that a next step has to be compared with the block-spin form. Theorem 646 composes the steps under three hypotheses at every scale, a relative error, a locality bound and a large-field cost; none is proved beyond the first step in the required form (Proposition 647), and the large-field estimates of Sec. XIII R give no activity bound for $\beta_W < 3.2372 \cdot 10^6$, more than $1.257 \cdot 10^7$ one-loop steps above coupling one. The locality hypothesis cannot hold uniformly in the volume (Proposition 650). Theorem 652 proves the step from one scale to the next under a single condition on the error of that step, relative and local; under it the relative comparison and the fibre gap pass to every scale uniformly in λ_0, and the locality of the fibre with constants that are not bounded in the scale. Sec. XIII U proves the relative one-step bound without the factor C_n when the remainder of the action is a sum of gauge-invariant local terms, exactly at flat backgrounds (Theorem 657) and with a loss of order $\|\kappa\|_2$ at curved ones (Theorem 661); a loss of order κ uniformly in n is false (Proposition 662). Condition (C) follows from such a representation of the one-step error (Corollary 663); that the induced action has one is not proved (Remark 664).

The covariance of item (ii) for a fixed coarse field and uniform in the exterior, with decay at the rate $2\alpha_G$ without a floor, is the single-scale part of (D) that remains open; its mean over the coarse field, and its value for the coarse fields of Proposition 503, are bounded by Theorems 497, 498 and Corollary 504, and by Theorem 505 the rest needs the mixed differences of $\log Z^c$ in tree form and the response of the law across a bad component, which Theorem 526 gives on the chart up to the mass of the law off the chart, which tends to one as the component grows, for coarse plaquettes below $6.0 \cdot 10^{-13}$, for the coarse fields of Proposition 536 up to $2.39 \cdot 10^{-12}$, and for none above $2.418 \cdot 10^{-12}$ (Proposition 525, Proposition 536, Theorem 535, Theorem 529, Remark 534); item (i) being settled for the chains of the bad blocks by Proposition 436 and item (ii) given the labels of all islands by Theorem 491; item (iii) is settled by Theorem 465 at one scale and by Theorem 471 at every scale up to $a_{k_*}/8$; item (iv) is the multiscale part. Theorem 66 replaces (D) by the averaged condition BP(n, ϖ) of Definition 62, a spectral gap of the cube dynamics on gauge-invariant functions in L^2 of the Wilson measure on tori, with

no gauge and no supremum over exteriors; it follows from its supremum form (Proposition 67), the obstructions of the forest gauge and those uniform in the coarse field do not apply to it, it fails in the Coulomb phase of the Villain theory at every cube size (Corollary 68), it holds on every torus at strong coupling and fails at fixed cube size on small tori at weak coupling (Corollary 74, Theorem 79), the latter already on functions of nonzero centre charge, while the continuum step needs only its local form (Theorem 84, Corollary 85), the failure of the neutral part on small tori reduces for $\mathrm{SU}(2)$ to a lower bound on the law of the Polyakov loop (Theorem 92), and neither form is proved for any non-abelian group along the tuning (Remark 69, Remarks 90 and 93); the condition passes from the fine lattice to the law of the block variables of every local gauge-covariant block-spin map, so that along the tuning it contains the condition at one cube size for those laws at a fixed coarser spacing, the local form passes only in an integrated form that gives no gap (Theorem 94, Corollary 96, Remark 97); in the opposite direction, the fibre and coarse bounds posed jointly as a split inequality give the upward step, and along the tuning the condition at the coarsest scale with the split inequality at every scale, its constants bounded in product, gives $\mathrm{BP}(n_k, \varpi)$ with an explicit ϖ; the split inequality holds with the exact constants for the product of Haar measures and, with the same constants, for the Haar refinement of every gauge-invariant law of the block variables, so that along a tower of refinements the condition at the coarsest scale passes to every finer scale without loss, the two bounds posed separately lose a fixed factor at each scale already for the product measure, the fine Dirichlet form splits into a fibre form and a coarse form for every measure, the Wilson measure at $\beta \neq 0$ is not a Haar refinement, the split inequality follows for every measure from a fibre Poincaré inequality, a bound on the information of the fine exterior of a cube on the block variables and the almost orthogonality of that information for different cubes, it holds for the Wilson measure in the strong-coupling range on every torus, with constants within factors $1 + O(\beta)$ of those of the refinements, and on each torus it holds for every measure with a continuous positive density, with constants that grow with the volume, while the obstructions at weak coupling on small tori bound the condition for the law of the block variables and put no constraint on it; neither the split inequality with constants independent of the torus for the laws of the block variables below the finest scale or for the Wilson measure outside the strong-coupling range nor the condition at the coarsest scale along the tuning is proved (Theorems 100, 101, 110, 114 and 120, Corollaries 102, 104, 111 and 121, Propositions 103, 109, 112, 115 and 122, Remarks 105, 113 and 123); the condition at the coarsest scale follows, for every measure with a continuous positive density and without locality, from a bound of Dobrushin type on the single-link coefficients of the law of the block variables or from a Hessian bound on its induced action, uniform along the tuning, and these bounds are not proved outside the strong-coupling range; given the split inequality the condition is equivalent to the conclusion of the bootstrap up to one unit of cube size, it is independent of the split inequality, and it fails for the coarse law of a measure whose local gauge-invariant covariances vanish beyond the coarsest spacing, and the closeness of each single-link conditional law to the Haar measure does not replace the bound on the sums of the coefficients (Theorem 125, Corollaries 127 and 129, Propositions 130, 131, 132 and 133, Remarks 134 and 113); one step of the straight-path map with blocks of side b contracts a weighted Dobrushin sum of the single-link coefficients by $e^{-(\lambda-2\kappa)\lfloor b/2 \rfloor}$ on every torus, so that along a tower such a bound at one level for every k gives the condition at the coarsest scale, while gauge covariance bounds the Dobrushin sum of every gauge-invariant law below by two minus four times the normalised spread of its single-link conditional laws, which excludes the $\mathrm{SU}(2)$ Wilson measure at $\beta_W \geq \frac{8}{3}$ and every law of the block variables whose plaquette variables are concentrated, forces the mean distance of the coarse holonomies from I to exceed a quarter of the diameter, and does not obstruct the condition at the coarsest scale along the tuning for $\mathrm{SU}(2)$, where the coarse conditional laws are spread out; the Hessian of the induced action is the mean curvature of the fine action minus the covariance of the forces in the fibre law, and no bound on the coefficients along the tuning is proved (Theorems 137 and 140, Corollaries 138, 141 and 142, Propositions 143 and 144, Remark 145); Sec. X P bounds the coarsest coefficients by $D^2/4$ times the mixed derivatives of the induced action, with no condition on its single-link Hessian, shows that for straight-path maps with blocks of side at least three these are the covariances of the forces on distinct marks in the fibre law, so that (H1) follows from a decorrelation (FC) of these forces, which also forces their variances to equal the mean curvatures up to a bounded error, computes them in a Gaussian abelian model as $-\beta$ times a co-closed effective quadratic form with diagonal between $1/b^2$ and $6/b$, row sums at least twice the diagonal and bounded uniformly in the torus by a number γ_b; Sec. X R writes the symbol of this form in closed form and shows that γ_b lies between absolute multiples of $(\log b)/b$ and $(1+\log b)/b$, so that the bound (G) of order $1/b$ fails while the Gaussian form of (FC) holds along the tuning, and that in every representation of the coarse density as a mixture of products over the coarse links the same numbers are the covariances of the scores; (FC) for $\mathrm{SU}(2)$ is not proved along the tuning (Lemmas 146 and 147, Propositions 148, 152, 153 and 156, Corollaries 149, 154 and 174, Theorem 173, Remarks 157, 175 and 176); Sec. X Q shows for $\mathrm{SU}(2)$ that convolution with a one-link density contracts every nontrivial representation, so that the conditional densities of the block variables and their first two derivatives are within $48(1+6\beta_W)^2 q_W^{b-2}$ of those of the Haar measure, the single-link curvature of the induced action tends to zero along every tuning with $\beta_k = O(k)$, and the mixed derivatives are posterior covariances of the scores of the interior path products, each tending to zero; this gives the condition at the coarsest scale on every torus of side at most $e^{2^k/(17k)}$, a column sum at most $\frac{1}{3}$ gives it with no condition on the tails, and a bound on the sums of the posterior covariances uniform in the torus, which is not proved, gives (H1) (Theorems 160, 161 and 164, Corollaries 165 and 166, Propositions 167 and 168, Remark 169). Sec. XIII Q carries the single-scale results of this section to every compact simple group, with explicit constants, except the parts listed in Remark 592. Sec. XI I states the conditional continuum chain for

every compact simple group, with the averaged condition in its local form, the centre sectors of $Z(G)$ and the Schwinger functions at separated points (Theorem 321, Proposition 320), proves the averaged condition and the triviality of the limits for $N\beta \le \kappa_G/48$ (Corollary 322), and extends the obstruction at fixed cube size under a condition (CP) that holds for $\mathrm{SU}(N)$ and fails for $\mathrm{Sp}(n)$ with n even (Proposition 324); its hypotheses are not proved for any group, and an analogue of the sharp quarter for $G \neq \mathrm{SU}(2)$ is not proved (Proposition 323). On the continuum side, Lemmas 462 and 463 reduce the nontriviality of the limit to a lower bound on one ratio of lattice two-point functions at a fixed physical time, uniform along the tuning, which is the open input of Theorem 48; Sec. XI B states it exactly as the condition (N_u), shows that neither the strong-coupling range nor the cube estimate in the form of Theorem 45 supplies it, and reduces it to the trial states of Corollary 194. Sec. XI C bounds the slab energy of every Lipschitz slice function by the plaquette energy, which is at most $(3.7148+0.149/\beta_W)/\beta_W$ for $\mathrm{SU}(2)$, and shows that this route gives the trial condition only where $a_k\beta_k$ stays bounded below, not along the one-loop tuning (Proposition 199). The same subsection shows that no set of upper bounds on correlations at physical separations, with constants not exponentially small in $1/a_k$, implies (N_u) (Proposition 209); that trial states smeared in time over a physical interval, flowed Wilson loops among them, meet the first trial condition exactly when, up to the choice of the times, a ratio of their correlations at physical times is bounded below (Corollary 211), which removes the lattice scale from the condition but leaves a lower bound at physical separations; and that a comparison with the Gaussian theory gives (N_u) from relative bounds at two physical separations, which a scale-by-scale argument along the one-loop tuning supplies only if its losses summed over the scales stay bounded, and not with a loss of first order in the running coupling at each scale (Proposition 212). This is the summability required in item (iv), now for the ratio, and (N_u) is not proved here in infinite volume; in a box of fixed size it holds for packets odd under the centre of $\mathrm{SU}(2)$, with degenerate limits (Corollary 253). Choosing the couplings by the ratio (Sec. XI E) makes (N_u) and the normalisation of Proposition 206(iii) hold by construction; such a choice exists if and only if (N_u) holds along some sequence of couplings (Theorem 244), and it leaves (D), (M), the family $\mathfrak{F}$ and $\beta_k \to \infty$ open. Sec. XI D shows that, given (D) along the tuning, the Osterwalder–Schrader axioms for the spatially flowed fields, with the symmetry $\mathrm{O}(3)\times\mathrm{O}(1)$, follow from three statements, a moment bound (M), one spatial rotation (W) and one limit (U), each of them necessary and none of them proved. Full $\mathrm{O}(4)$ invariance is not among these conclusions: at positive flow time the spatial flow breaks it even for the free field (Proposition 207(c)), so it has to be posed for $\mathrm{O}(4)$-covariant observables, such as fields of the four-dimensional flow, and the four-dimensional flow loses reflection positivity already for the free field (Proposition 217). Of the three statements, (M) is a local moment bound together with the bound $\sup_k \Upsilon_k < \infty$, which contains (N_0) for product packets and fails in the strong-coupling range (Proposition 213), and the flow alone cannot give it (Proposition 214); its spatial comparison $(\mathrm{M}_{\mathrm{sp}})$ follows from spatial reflection positivity (SR_0) of the slice field (Theorem 228), which cannot hold for a field and its Laplacian together in a state that clusters in space (Corollary 234) and is destroyed by the linear heat flow (Proposition 235); (W) is the Ward identity for one spatial generator (Proposition 216); and under the renormalisation condition (RC), (U) is the vanishing of the moment differences between consecutive scales together with a totally disconnected set of limits, and a first-order bound in the running coupling for those differences does not give it (Proposition 219). For unsmoothed local fields at separated points (Sec. XI H) the inputs are of the same kind: the separated moment bound $(\mathrm{M}_{\mathrm{s}})$ for $n \ge 3$ is equivalent to bounds on reflected norms of products of separated packets (Proposition 285), which do not follow from the two-point properties, reflection positivity and the gap (Proposition 310), and its rate $(\mathrm{G}_{\mathrm{s}})$ is a rate of the form $K_1^m (m!)^{\beta_1}(1+L)^{q_1 m}\delta^{-p_1 m}$ in these bounds (Proposition 311), which bounds at each order do not supply (Proposition 312); one inequality of Gaussian domination type, (GD), would give both from the two-point statements (Proposition 313); for $n=2$, $(\mathrm{M}_{\mathrm{s}})$ is equivalent to the comparison (SD), which holds if and only if the largest reflected norm of a unit packet near the reflection plane is carried by energies bounded in physical units, bounds the physical gap from above, fails in the strong-coupling range and is not implied by reflection positivity and a gap bounded above and below in physical units (Propositions 282 and 283); and under (SD) the nonvanishing condition (NE) is the nontriviality of every limit (Proposition 284) and is equivalent to the condition that spatial momenta bounded in physical units carry a fixed fraction of that largest reflected norm (Proposition 317); (NE) follows from (SD) for $\mathrm{U}(1)$ (Corollary 316), is not implied by (SD) together with reflection positivity and a gap bounded above and below in physical units (Proposition 318). The rotations $(\mathrm{W}_{\mathrm{s}})$ and the uniqueness $(\mathrm{U}_{\mathrm{s}})$ follow from rotation invariance of the renormalised lattice moments at large lattice distance with a pointwise bound (Theorem 288), and from a mass condition together with uniqueness of the limit up to dilations (Theorem 292), and the normalisation by one reflected correlation does not fix the scale (Proposition 291). Reflection positivity reduces the pointwise bound for $n=2$ to a bound on a coordinate axis and (RI_2) to a comparison with the axis (Proposition 293), and that bound does not follow from the separated moment bounds, the mass condition and $(\mathrm{U}_{\mathrm{s}})$ (Proposition 294(b)); the gap and $(\mathrm{M}_{\mathrm{s}})$ give the two halves of the mass condition, with two rates that need not coincide (Proposition 295); and $(\mathrm{O}_{\mathrm{s}})$ is equivalent to the convergence of the moments after a change of the spacing by bounded factors and does not follow from the other properties together with the mass condition (Propositions 296 and 294(c)). For $n=2$, (SD) and (NE) follow from the two-sided short-distance law (AF) predicted by asymptotic freedom, which excludes the trivial limits of the strong-coupling range (Theorem 305); the inequality that (AF) requires of the Wilson measure is stated in Remark 308. None of these inputs, (AF) included, is proved for the Wilson measure.



label	witness and result
non-path link off $R(c)$	fine defect, Theorem 353
coarse plaquette $> \eta$	fine defect, Lemma 409
path link	none: determined, Lemma 411
bad non-path links	fold, Proposition 418
bad pair	chain, Proposition 436
inside link	fold, Proposition 422
all links	nonconvex, Proposition 413
good window	graph law, Proposition 412
n bad labels	$A^{n/H}$, Lemma 450
two islands	one kernel, Theorem 483
layer law	chart bound, Theorem 485
two islands	$e^{-2\alpha_G(w-2s-8)}$, Corollary 486
two islands	$\frac{1}{4}\tanh(\Delta/4)$, Corollary 453
many islands	Δ_*, Theorem 491
rarity alone	floor $q^2/16$, Proposition 455
island of n blocks	2^{-n} past β^*, Theorem 465
islands at scale j	2^{-n} past β_j^*, Theorem 467

FIG. 46. Left: a window W of coarse blocks (solid) with its collar (dashed), good blocks in green and bad islands in red. On the good blocks the conditional law is the graph law of Proposition 412; the weight of two islands is compared through a kernel on the good layer between them, whose cross-ratio bounds their covariance (Lemma 452, Corollary 453). Right: the labels considered in Sec. XIII, their fine witnesses, and the estimate or the obstruction that each produces.

XIV. CONCLUSION

For every compact simple gauge group, four-dimensional lattice Yang–Mills theory has a unique infinite-volume state, exponential clustering and a transfer-matrix gap whenever $N|\beta| < \frac{2}{29}\kappa_G$ (Theorem 5), and plaquette blocks extend the range for $\mathrm{SU}(2)$ and $\mathrm{SU}(3)$ (Theorems 3 and 7). One averaged estimate in a cube of fixed physical size, the bound $\mathrm{BP}(n_k, \varpi)$ with $n_k a_k \le \ell$, places the spectrum of every continuum limit in $\{0\} \cup [16\varpi/(625e^2\ell), \infty)$, so every nontrivial limit has a mass gap (Theorem 66). Along the tuning this estimate follows from two statements about block variables: the condition (H1) at the coarsest scale and the split inequality (H2) at every finer scale (Corollary 102).

Both statements are proved in part. The split inequality holds on every torus at strong coupling, with constants within factors $1 + O(\beta)$ of those of the product measure (Theorem 120). For $\mathrm{SU}(2)$ the block variables of the straight-path map approach the Haar measure in every representation, their single-link curvature vanishes along every tuning with $\beta_k = O(k)$, and the condition at the coarsest scale holds on every torus of side up to $e^{2^k/(17k)}$ (Corollary 166). On tori of every size it follows from a bound, uniform in the torus, on the sum of the posterior covariances of the scores of the interior path products (Proposition 168). In the Gaussian abelian model the corresponding row sums are of order $(\log b)/b$, so that model meets the decorrelation along the tuning, although it is not a lattice gauge measure (Theorem 173 and Corollary 174).

Three inputs remain: that covariance bound along the tuning, the split inequality at weak coupling, and a nontrivial continuum limit with the properties of Sec. XI H. Given them, the results above give the continuum theory a mass gap; without them the mass gap is not proved here.

Lattice gauge fields also define four-dimensional gauge theories with chiral fermions in every anomaly-free representation, once the lattice field is interpolated to the continuum and coupled to continuum fermions [117]. The mass gap problem concerns the pure gauge theory, to which every result here refers.

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This paper rests on the work of many earlier scientists. Lattice gauge theory is due to Wilson [116] and its Hamiltonian form to Kogut and Susskind [79]; reflection positivity and the reconstruction of the transfer matrix to Osterwalder and Schrader [95, 96], and their lattice form, with the first proof of a gap at strong coupling, to Osterwalder and Seiler [97, 105]. The uniqueness method is Dobrushin's [42–44], the curvature method is that of Bakry and Émery [4, 5] in the form of Shen, Zhu and Zhu [107], the covariance estimate under Dobrushin's condition is Föllmer's [53], the chessboard estimate is that of Fröhlich, Israel, Lieb and Simon [54] and Fröhlich and Spencer [55], disagreement percolation is due to van den Berg and Maes [112], the Brascamp–Lieb inequality to Brascamp and Lieb [25], the correlation representation to Helffer and Sjöstrand [69] and Naddaf and Spencer [92], the perturbation lemma to Holley and Stroock [68], the derivative of a matrix function to Daleckii and Krein [41], and the strict positivity of the transfer matrix to Lüscher [83], the gauge fixing on a forest is Creutz's [38], and the curvature of Lie groups is Milnor's [90]; the localisation estimate of Sec. XII A

is that of Combes and Thomas [39]. The analysis of the single-link law uses results of Barta [14], Biernacki and Krzyż [27], McShane [89], Stein and Weiss [110] and Watson [114], and the continuum problem is the one formulated by Jaffe and Witten [73] and studied by Bałaban [6, 7, 12] through the renormalisation group, in the account of Dimock [40]; gauge fields with cut-offs were constructed by Brydges, Fröhlich and Seiler [31] and by Magnen, Rivasseau and Sénéor [86]; recent probabilistic work on lattice Yang–Mills theory is due to Chatterjee [36, 37], Basu and Ganguly [15], Adhikari and Cao [1], and Cao, Nissim and Sheffield [33], and the two-dimensional theory was constructed by Driver [46], Gross, King and Sengupta [62] and Lévy [81].

DATA AND CODE AVAILABILITY

The proofs are analytic. Remarks 36 and 39 report evaluations for $\mathrm{SU}(3)$ in interval arithmetic with rigorous error bounds, and Remarks 331 and 695 report computations with certificates checked in exact arithmetic; no result uses them. The value of $(M_W)_{AB}$ in Remark 608 is computed in exact rational arithmetic; no proof uses it, and the statements that the calm weight is not local are conditional on $(M_W)_{AB} \neq 0$. Every number used in a proof is derived in the text. The programs of these steps, in Python with python-flint (FLINT and Arb) and a C kernel for products of Taylor polynomials, their outputs and the list of certified boxes are in the repository `https://github.com/creelie/4Y-M`, archived on Zenodo (doi:10.5281/zenodo.23444338; release v4.37.0, doi:10.5281/zenodo.23992883), together with a Lean 4 certificate of the line field, Lean 4 files for the comparison arithmetic, the coupling lemmas of Sec. XIII F and the finite arithmetic of Theorem 485, Corollary 486, Proposition 461, Proposition 470 and Theorem 471, the counting and trace identities and the constants of Secs. X F–X I and of Theorem 556, the linear algebra of Sec. X S, independent exact or numerical checks of the other finite statements, our checks of the certificate of [102], and the certificate of Remark 331 with its exact check. The code was written by Deep Bhattacharjee, James Montgomery Lockwood and Tristan Romanov.

DECLARATIONS

Author contributions. Deep Bhattacharjee conceived the project and, with the help of AI agents working under his direction, made its constructions and proved its main results: the forest gauge on the period-two line field and the free-boundary state with its transfer-matrix gap at strong coupling; the plaquette blocks of $\mathrm{SU}(2)$ and $\mathrm{SU}(3)$ with their constants proved by hand; the reduction of the mass gap to one averaged estimate in a cube of fixed physical size; the labels of Bałaban's blocks and the islands of bad blocks; the averaged condition, its descent to block variables, the upward step and the split inequality; the single-link coefficients across scales, the forces in the fibre law, the smoothing along the paths for $\mathrm{SU}(2)$ and the row sums of the Gaussian model; and the conditional chain for every compact simple group. He wrote the programs of the certified computations and the paper, and checked every statement. Ushashi Bhattacharya checked the proofs and the references and edited the text. James Montgomery Lockwood contributed the analysis of constrained cubes [82] and its code, and the Lean 4 formalisations of the quaternion model of $\mathrm{SU}(2)$, the open-gluon certificates, the coupling lemmas of Sec. XIII F, the strong-coupling comparison and the limit of couplings, and the composition of metrics across scales of Sec. X S with its Lean 4 formalisation. Tristan Romanov contributed the certified bound on the forest-gauge Hessian [102] and the research continuation on which Sec. XIII is built: the finite Dobrushin arithmetic, the minimality of the coupling cost and its measure-level form, the relative and canonical charts, the matching of labels on independent links, the rough-background and branch obstructions, the rarity witnesses and the island kernel, the analysis of the lattice slice field with the exact Wick coefficients on finite boxes, the block-response diagnostics, the flat fields of Remark 410, four Lean 4 modules, and the programs and certificates behind them. All four authors approved the final text.

Use of artificial intelligence. The computations, the checks and parts of the drafting were done with AI assistance (Claude Opus 5.5, Claude Fable 5.1, GPT-6.1 Sol Ultra and GPT-6 Astra Ultra) under the direction of Deep Bhattacharjee, who checked every statement and is responsible for the text. The combination of the forest gauge with the sharp quarter behind Theorem 1 was first assembled by Anaxia, the reasoning engine of the BLLM system [18, 19]; the line field, the free-boundary limit, the ceiling, the curvature bound, the $\mathrm{SU}(3)$ part, the plaquette blocks and the certified constants are the authors' own.

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Appendix A: The line field

Table VI lists, for each residue class of x modulo two, the direction $\sigma(x)$ and the permutation $\pi_x(\mu) = \sigma(x + e_\mu)$. In every row π_x fixes $\sigma(x)$ and is a three-cycle on the other directions, which is the condition of Lemma 8.

Appendix B: The comparison lemma

Let (G, δ) be a compact group with a bi-invariant metric of diameter D, $\Omega = G^A$, and $C_{ee'}$ the coefficients of Sec. III, with row and column sums at most $\alpha < 1$ and $C_{ee'} = 0$ unless e, e' share a plaquette, so that $|e - e'| \leq 1$ for the distance of link midpoints. Fix $\lambda > 0$ with $\theta = \alpha e^{\lambda} < 1$.

Lemma 665. *Let $S \subset A$, possibly infinite, and let μ_1, μ_2 be probability measures on Ω invariant under γ_e^F for every $e \notin S$. Let $T \subset A \setminus S$ be finite, $w_b = e^{-\lambda \operatorname{dist}(b,T)}$, and suppose $\sum_{z \in S} w_z < \infty$. There is a coupling of μ_1, μ_2 whose link distances*

TABLE VI. The line field: x modulo two (digits $x_0x_1x_2x_3$), the direction $\sigma(x)$ and the values $\sigma(x+e_\mu)$ for $\mu=0,1,2,3$.

x	$\sigma(x)$	$\mu=0$	$\mu=1$	$\mu=2$	$\mu=3$
0000	0	0	2	3	1
1000	0	0	3	1	2
0100	2	3	0	2	1
1100	3	2	0	1	3
0010	3	1	2	0	3
1010	1	3	1	0	2
0110	2	1	3	2	0
1110	1	2	1	3	0
0001	1	2	1	3	0
1001	2	1	3	2	0
0101	1	3	1	0	2
1101	3	1	2	0	3
0011	3	2	0	1	3
1011	2	3	0	2	1
0111	0	0	3	1	2
1111	0	0	2	3	1

$\rho(b)=\mathbb{E}\,\delta(\sigma_b,\sigma'_b)$ *satisfy*

$$\sum_{b\notin S} w_b\rho(b)\le\frac{\theta D}{1-\theta}\sum_{z\in S} w_z.$$

If $S=\emptyset$ *then* $\mu_1=\mu_2$.

Proof. The couplings form a weakly compact set and $\Psi(\pi)=\sum_b w_b\mathbb{E}_\pi\delta(\sigma_b,\sigma'_b)$ is weakly continuous because $\sum_b w_b<\infty$; let π^* be a minimiser. For $e\notin S$ replace the pair (σ_e,σ'_e) by a sample of an optimal coupling of $\gamma_e^F(\cdot\mid\sigma)$ and $\gamma_e^F(\cdot\mid\sigma')$, chosen measurably [113, Cor. 5.22]. Both μ_i are invariant, so the result is again a coupling, and changing the finitely many links that enter γ_e^F one at a time gives $W_1\le\sum_{e'}C_{ee'}\delta(\sigma_{e'},\sigma'_{e'})$. Minimality gives $\rho(e)\le\sum_{e'}C_{ee'}\rho(e')$. Multiply by w_e and sum over $e\notin S$; where $C_{ee'}\neq0$, $w_e\le e^{\lambda}w_{e'}$, so

$$\sum_{e\notin S} w_e\rho(e)\le\theta\Big[\sum_{e'\notin S} w_{e'}\rho(e')+\sum_{z\in S} w_z\rho(z)\Big],$$

all sums are finite, and $\rho\le D$ on S. If $S=\emptyset$, $\rho\equiv0$. □

Corollary 666. *Let μ be the Gibbs measure of γ^F, F a local function with support S_F, L_F-Lipschitz in each link, and H bounded with finite support S_H disjoint from S_F. Then*

$$|\operatorname{Cov}_\mu(F,H)|\le\|H\|_\infty L_F\frac{\theta D}{1-\theta}|S_H|\,e^{-\lambda\operatorname{dist}(S_F,S_H)}.$$

Proof. Condition on the links of S_H [77, Ch. 3]; the conditional measures $\mu^\sigma=\mu(\cdot\mid\sigma_{S_H})$ are invariant under γ_e^F for $e\notin S_H$. Lemma 665 with $S=S_H$ and $T=S_F$, for which $w_b=1$ on S_F and $w_z\le e^{-\lambda\operatorname{dist}(S_F,S_H)}$ on S_H, gives $|\mu^\sigma(F)-\mu^{\sigma'}(F)|\le L_F\sum_{b\in S_F}\rho(b)\le L_F\frac{\theta D}{1-\theta}|S_H|e^{-\lambda\operatorname{dist}(S_F,S_H)}$ for any two boundary values. Since H depends only on σ_{S_H}, $\operatorname{Cov}_\mu(F,H)=\iint H(\sigma)\big(\mu^\sigma(F)-\mu^{\sigma'}(F)\big)\,d\mu(\sigma)\,d\mu(\sigma')$, and the bound follows. □

Lemma 667. *Let the blocks partition A, let C_{Bj} be the block coefficients of Sec. V, with $\sum_B C_{Bj}\le\alpha<1$ for every j and $C_{Bj}=0$ unless every link of B lies within distance 2 of j, and fix $\lambda>0$ with $\theta=\alpha e^{2\lambda}<1$. Let $S\subset A$ be a union of blocks and let μ_1,μ_2 be invariant under $\gamma_B^{\mathscr{B}}$ for every block $B\not\subset S$. With T, w_b as in Lemma 665 and $\sum_{z\in S}w_z<\infty$, there is a coupling with $\sum_{b\notin S}w_b\rho(b)\le\frac{\theta D}{1-\theta}\sum_{z\in S}w_z$, and $\mu_1=\mu_2$ if $S=\emptyset$.*

Proof. Take a minimiser π^* of Ψ as before. For a block $B\not\subset S$ replace (σ_B,σ'_B) by an optimal coupling, for the metric d, of the two block laws; it is again a coupling, and changing the boundary links one at a time bounds the expected new d-distance by $\sum_j C_{Bj}\rho(j)$. Put $\bar w_B=\max_{i\in B}w_i$. Minimality gives

$$\sum_{i\in B} w_i\rho(i)\le\bar w_B\sum_j C_{Bj}\rho(j),$$

and $\bar w_B\le e^{2\lambda}w_j$ whenever $C_{Bj}\neq0$. Summing over the blocks outside S and using the column sums,

$$\sum_{b\notin S} w_b\rho(b)\le\theta\Big[\sum_{j\notin S} w_j\rho(j)+\sum_{z\in S} w_z\rho(z)\Big],$$

which gives the claim as in Lemma 665. □

Lemma 668. *Let $n\ge2$ be even, let V_x, $x\in2\mathbb{Z}^4$, be the translates of Sec. X, and suppose that $D(n,\alpha)$ holds.*

(a) *If a probability measure μ on G^A is invariant under γ_e^F for every e in a finite set V, it is invariant under γ_V.*

(b) *Fix $\lambda>0$ with $\theta=\alpha e^{\lambda\rho_n}<1$. Let $S\subset A$, and let μ_1,μ_2 be invariant under γ_{V_x} for every x with $V_x\cap S=\emptyset$. With T, w_b as in Lemma 665 and $\sum_{z\in S}w_z<\infty$, there is a coupling with*

$$\sum_{b\notin S} w_b\rho(b)\le\frac{D(1+c_ne^{\lambda\rho_n})}{1-\theta}\sum_{z\in S} w_z,$$

and $\mu_1=\mu_2$ if $S=\emptyset$.

Proof. (a) Let ν_η be the conditional law of Q_V given the links η outside V. For bounded f on G^V and bounded g of η, $\gamma_e^F(fg)=g\,\gamma_e^F f$ for $e\in V$, so invariance gives $\int g(\eta)\big(\nu_\eta(\gamma_e^F f)-\nu_\eta(f)\big)\,d\mu=0$ for all g; taking f in a countable dense set of continuous functions, ν_η is invariant under every $\gamma_e^F(\cdot\mid\cdot,\eta)$, $e\in V$, for μ-almost every η. So is $\gamma_V(\cdot\mid\eta)$, whose conditional laws of single links are the γ_e^F. The composition P_η of the kernels γ_e^F, $e\in V$, in a fixed order has a density with respect to Haar measure on G^V bounded below by a positive constant, because each single-link density is at least $e^{-12N^2\beta}$ times its maximum over the group. A Markov kernel with such a lower bound has at most one invariant probability measure (Doeblin's condition), so $\nu_\eta=\gamma_V(\cdot\mid\eta)$.

(b) Let π^* minimise Ψ as in Lemma 665 and call x admissible if $V_x\cap S=\emptyset$. For admissible x replace $(\sigma_{V_x},\sigma'_{V_x})$ by an optimal coupling, for d_{V_x}, of the two block laws; both μ_i are

invariant, so the result is a coupling, and changing the boundary links one at a time bounds the new expected d_{V_x}-distance by $\sum_j C_{V_x,j}\rho(j)$. With $\bar w_x = \max_{i\in V_x} w_i$, minimality gives

$$\sum_{i\in V_x} w_i\rho(i) \le \bar w_x \sum_{j\notin V_x} C_{V_x,j}\rho(j),$$

and $\bar w_x \le e^{\lambda\rho_n} w_j$ whenever $C_{V_x,j}\neq 0$. Let S' be the set of links outside S that lie in a translate meeting S; each is within distance ρ_n of a link of S, so $\sum_{i\in S'} w_i \le c_n e^{\lambda\rho_n}\sum_{z\in S} w_z$. A link outside $S\cup S'$ lies in N_n translates, all admissible. Summing over admissible x and using $\sum_{x:\, j\notin V_x} C_{V_x,j} = K_{c(j)} \le \alpha N_n$,

$$N_n \sum_{i\notin S\cup S'} w_i\rho(i) \le \theta N_n\Big[\sum_{i\notin S\cup S'} w_i\rho(i) + \sum_{i\in S\cup S'} w_i\rho(i)\Big].$$

All sums are finite and $\rho \le D$, so $\sum_{i\notin S\cup S'} w_i\rho(i) \le \frac{\theta D}{1-\theta}(1 + c_n e^{\lambda\rho_n})\sum_{z\in S} w_z$; adding $\sum_{i\in S'} w_i\rho(i) \le D c_n e^{\lambda\rho_n}\sum_{z\in S} w_z$ gives the bound. If $S=\emptyset$ then $S'=\emptyset$ and $\rho\equiv 0$. □

The covariance bound of Corollary 666 follows from part (b) in the same way, with $\theta D/(1-\theta)$ replaced by $D(1 + c_n e^{\lambda\rho_n})/(1-\theta)$, since the conditional measures μ^σ there are invariant under γ_{V_x} for $V_x \cap S_H = \emptyset$ by part (a).

Appendix C: The plaquette blocks

Table VII lists the block plaquettes of Sec. V modulo $2\mathbb{Z}^4$: the plaquette at x in the plane $\{\mu,\nu\}$ is a block plaquette exactly when x modulo two and $\{\mu,\nu\}$ appear in the table. Eight lie in the plane $\{0,1\}$ and eight in $\{2,3\}$; together they contain the 48 active links of a cell of side two, each once.

TABLE VII. The sixteen block plaquettes modulo two: the base point x (digits $x_0x_1x_2x_3$ modulo two) and the plane.

x	plane	x	plane
0000	$\{0,1\}$	0000	$\{2,3\}$
0001	$\{0,1\}$	0011	$\{2,3\}$
0010	$\{0,1\}$	0100	$\{2,3\}$
0011	$\{0,1\}$	0111	$\{2,3\}$
1100	$\{0,1\}$	1000	$\{2,3\}$
1101	$\{0,1\}$	1011	$\{2,3\}$
1110	$\{0,1\}$	1100	$\{2,3\}$
1111	$\{0,1\}$	1111	$\{2,3\}$

In closed form, the plaquette at x in $\{0,1\}$ is a block plaquette when $x_0 \equiv x_1$ and the plaquette at x in $\{2,3\}$ when $x_2 \equiv x_3$; the blocks do not depend on the line field beyond the fact that every plaquette contains one fixed link. The bounds of Lemma 16 and the chain (3) are proved by hand in Sec. V. The archive also evaluates the functions of Lemma 16 in ball arithmetic at 128 bits, from the series of Z_0 and of its first three derivatives in $s = r^2$ summed to forty-eight terms with the tail bounded by twice the first omitted term; these sharper constants are not used by any result.

For SU(3) the constants of Remark 39 are evaluated on the 841 boxes of side $\frac{3}{16}$ at $r=\frac{3}{2}$ and the 784 boxes of side $\frac{19}{100}$ at $r=\frac{38}{25}$ and the 826 boxes of side $\frac{77}{400}$ at $r=\frac{77}{50}$ with the moments of Appendix E, and the constants of (15) and α_3 are evaluated from these bounds and the covers of Remark 36 in ball arithmetic at 120 bits. The covers at $r=\frac{38}{25}$ and $\frac{77}{50}$ are computed as in Appendix E with $r=\frac{38}{25}$ and the threshold 1.03, and with $r=\frac{77}{50}$ and the threshold 1.0275. No result of the paper uses these numbers; Theorem 7 uses the constants of Lemma 31.

Appendix D: The blocks of Lemma 20

Diagonal blocks. Let $w_a = e^{i\phi}$. On the diagonal part, $f_p = -\cos\phi\,(y_1+y_2-y_3-y_4)^2$ and $c_p = 2(y_1y_4 - y_1y_2 + y_2y_3 - y_3y_4)$, so

$$\mathscr{F} = \sum_i \alpha_i y_i^2 + \cos\phi\,(y_1+y_2-y_3-y_4)^2 - \tfrac{9}{5}(y_1y_4 - y_1y_2 + y_2y_3 - y_3y_4)$$

with $y_k = 0$ and weights $\alpha_{k\pm1} = \frac{21}{10}$, $\alpha_{k+2} = \frac{5}{4}$. It is affine in $\cos\phi$, so positivity at $\cos\phi = \pm1$ suffices. For $k=1$ and $\cos\phi = 1$ the matrix in (y_2,y_3,y_4) is

$$\begin{pmatrix} \frac{31}{10} & -\frac{19}{10} & -1 \\ -\frac{19}{10} & \frac{9}{4} & \frac{19}{10} \\ -1 & \frac{19}{10} & \frac{31}{10} \end{pmatrix},$$

and Table VIII lists the leading principal minors for every position; all are positive.

TABLE VIII. Leading principal minors of the diagonal blocks of Lemma 20.

Fixed link	$\cos\phi = 1$	$\cos\phi = -1$
e_1, e_4	$\frac{31}{10}$, $\frac{673}{200}$, $\frac{8421}{2000}$	$\frac{11}{10}$, $\frac{53}{200}$, $\frac{21}{2000}$
e_2	$\frac{31}{10}$, $\frac{861}{100}$, $\frac{8421}{2000}$	$\frac{11}{10}$, $\frac{21}{100}$, $\frac{21}{2000}$
e_3	$\frac{9}{4}$, $\frac{673}{200}$, $\frac{8421}{2000}$	$\frac{1}{4}$, $\frac{53}{200}$, $\frac{21}{2000}$

Pair blocks. Let $w_a = e^{i(\chi+\vartheta)}$ and $w_b = e^{i(\chi-\vartheta)}$. The terms of f_p carry $w_a + \bar w_b = 2\cos\chi\, e^{i\vartheta}$ or its conjugate, and c_p carries $w_a\bar w_b = e^{2i\vartheta}$, so the Hermitian matrix $M(m,\vartheta)$ of $\mathscr{F}$, $m = \cos\chi$, is affine in m with $M(-1,\vartheta) = M(1,\vartheta+\pi)$, and $m=1$ suffices. For the fixed link e_2, in the variables (z_1,z_3,z_4) and with $c = \cos\vartheta$,

$$M = \begin{pmatrix} \frac{21}{5} + 2c & -2e^{i\vartheta} & -\frac{9}{5} - 2e^{i\vartheta} \\ -2e^{-i\vartheta} & \frac{21}{5} + 2c & \frac{9}{5} + 2e^{-i\vartheta} \\ -\frac{9}{5} - 2e^{-i\vartheta} & \frac{9}{5} + 2e^{i\vartheta} & \frac{5}{2} + 2c \end{pmatrix}.$$

With $u = \cos^2(\vartheta/2) \in [0,1]$ its leading principal minors are

$$\frac{20u+11}{5},\quad \frac{(20u+1)(20u+21)}{25},\quad \frac{400u(40u^2-23u+4)+21}{250},$$

and $40u^2-23u+4$ has negative discriminant $529-640$. The other positions give the same determinant and the leading minors $\frac{20u+11}{5}$ or $\frac{8u+1}{2}$ and $\frac{800u^2-180u+53}{50}$, whose discriminant $32400-169600$ is negative. Every block is positive definite.

The inequality is not tight. In floating point the least eigenvalue of the whole form over $W\in \mathrm{U}(N)$, $N\le 4$, is 0.0149, which is also the least eigenvalue of the exact blocks; this slack is what the choice of the weights $\frac{21}{10}$, $\frac{5}{4}$, $\frac{9}{10}$ trades against the constant $\frac{29}{2}$.

Appendix E: The certified computation

This appendix describes the computation of Remark 36, which no result of the paper uses.

Truncation. For a corner b of a box the moments of Lemma 34 are derivatives of Z at $(b,\bar b)$. The truncated series is $\mathbb{E}_H[A_t(UD)B_t(\bar U D')]$ with A_t the part of $\sum_\lambda c_\lambda s_\lambda(UD)$ over partitions with $\lambda_1-\lambda_3\le 34$ and $\lambda_3\le 11$, and B_t likewise. Every omitted partition has $|\lambda|>34$, because $3\cdot 12>34$, and the omitted part is at most $T(\|D\|)=\sum_{n>34}(\frac{3}{2}\|D\|)^n/n!$ in modulus. On the polydisc $|D_{ij}-b_{ij}|\le\rho$, $\|D\|\le\|b\|+3\rho$, and Cauchy's estimate bounds the α-th derivative of the omitted part by $\alpha!\rho^{-|\alpha|}T(\|b\|+3\rho)$. The derivatives of the full exponential are at most $2^{-|\alpha|}e^{v/2}$ with $v=\sum_k|b_k|$, since $|\operatorname{Tr}(Ub)|\le\sum_k|b_k|$ for diagonal b. With $\varepsilon_\alpha=2^{|\alpha|}\alpha!\rho^{-|\alpha|}T(\|b\|+3\rho)$ the truncated moment differs from the true one by at most $\varepsilon_\alpha e^{v/2}+(e^{v/2}+\varepsilon_\alpha)\varepsilon_\beta$, and ρ is chosen to minimise $\rho^{-6}T$. The largest error bound of an unnormalised moment at any corner is 1.7×10^{-11}.

Weight zero. Conjugation by the diagonal torus T preserves Haar measure and $\operatorname{Re}\operatorname{Tr}(Ub)$ for diagonal b, and multiplies u_{ij} by $t_it_j^{-1}$; a moment of a monomial on which T acts by a nontrivial character is therefore 0, and only 8278 of the 134596 monomials of degree at most six have the trivial character. The functions $\operatorname{Tr}(X)$, $e_2(X)$, $\det X$ of $X=DD'^{\top}$ and $\det D$, $\det D'$ are invariant under the same action on the variables, so after these are formed in the full ring of Taylor polynomials of degree six in the 18 variables, the series is computed in the subring of weight-zero monomials.

Rounding. The truncated series is a Taylor polynomial of total degree six in the 36 real directions of $(D-b,D'-\bar b)$, computed in double precision. Every operation is repeated on absolute values and carries an operation count n, so that the computed coefficient differs from the exact one by at most $\gamma_n=nu/(1-nu)$, $u=2^{-53}$, times the absolute-value coefficient [67, Sec. 3.1]. With C in $2^{-30}\mathbb{Z}$, the polynomials $2^{31}q_h$, $X_a(2^{31}q_h)$, $12\cdot2^{31}(\varphi_h+\Delta q_h)$ and $12\cdot2^{31}\langle\nabla\varphi_{F_k},\nabla q_h\rangle$ for the coordinate vectors h have coefficients in $\mathbb{Z}[i]$, and every partial sum in their computation is bounded, through the same operations on absolute values, by less than 2^{50}; they are therefore computed exactly. Normalisation by Z, the products with the corner coordinates, the quadratic forms and their error matrices are formed with the same bookkeeping, and the largest eigenvalues are bounded in ball arithmetic [74].

Separation. The functional c of a discarded box is the dual solution of a linear program for the gauge of the centre of the box with respect to a 72×72 grid of T. Only the final inequality is used: $\max_T\ell$ is bounded by its maximum over a 512×512 grid plus $\frac12 H\delta^2/2$, with $\delta=2\pi/512$ and $H=|c_1|+|c_2|+2|c_3|$ a bound for its Hessian on the torus, and the inequality is checked with relative margin 10^{-12}.

Appendix F: The transfer matrix

This appendix recalls the Osterwalder–Schrader reconstruction for the lattice [95–97] in the form used for the gaps, following our earlier paper [20].

Let ω be a state on the gauge-invariant local functions of $\mathbb{Z}^4$ that is invariant under translations and the hypercubic group and reflection positive for reflections in planes through sites and in planes bisecting links; by Proposition 11(iii)–(iv) and its analogues in the other theorems, the states of this paper are of this kind. The reconstruction gives a Hilbert space with a vector Ω, a map $F\mapsto\Psi_F$ with dense range, and a self-adjoint $\mathbb{T}$ with $0\le\mathbb{T}\le1$, $\mathbb{T}\Omega=\Omega$ and $\langle\Psi_F,\mathbb{T}^t\Psi_H\rangle=\omega(\vartheta(\bar F)\tau_tH)$, where ϑ is the time reflection and τ_t the time translation.[49]

Proposition 669. *For $m>0$ the following are equivalent:*

(a) *the spectrum of $\mathbb{T}$ on $\Omega^\perp$ lies in $[0,e^{-m}]$;*

(b) *for every F in a class of gauge-invariant local functions supported at times in a bounded interval and dense in $L^2(\omega)$ on each such support, there is $C_F<\infty$ with $0\le\omega(\vartheta(\bar F)\tau_tF)-|\omega(F)|^2\le C_Fe^{-mt}$ for all $t\ge0$.*

Under (a) *one may take $C_F=\|F\|_\infty^2$ for every bounded F.*

Proof. If (a) holds, then

$$\omega(\vartheta(\bar F)\tau_tF)-|\omega(F)|^2=\langle\Psi'_F,\mathbb{T}^t\Psi'_F\rangle$$

with $\Psi'_F=\Psi_F-\langle\Omega,\Psi_F\rangle\Omega\in\Omega^\perp$; this number lies in $[0,e^{-mt}\|\Psi_F\|^2]$, and $\|\Psi_F\|^2\le\|F\|_\infty^2$. Conversely, let σ be the spectral measure of $\mathbb{T}$ at Ψ'_F and let $\delta>0$. Then

$$\sigma\big((e^{-m+\delta},1]\big)e^{(-m+\delta)t}\le C_Fe^{-mt}$$

for all t, so $\sigma((e^{-m+\delta},1])=0$. The spectral projection of $(e^{-m},1]$ restricted to $\Omega^\perp$ therefore annihilates the vectors Ψ'_F of the class, which are dense by the footnote above, and it vanishes. □

The class in (b) cannot be replaced by one observable. On $\mathbb{C}^3$ let $\mathbb{T}=\operatorname{diag}(1,e^{-m},e^{-m/2})$, $\Omega=e_1$ and $\Psi'_F=e_2$: the function in (b) is e^{-mt}, while the spectrum on $\Omega^\perp$ reaches $e^{-m/2}$. The decay of one two-point function places only the spectral measure of Ψ'_F in $[0,e^{-m}]$, and the gap follows when such vectors are dense in $\Omega^\perp$, as in (b) and in Theorem 46. Proposition 8.8 of [50] draws the gap on $\Omega^\perp$ from the two-point function of one fixed observable.

[49] Ψ_F is the class of F in the completion of the positive-time local functions for the form $\langle F,H\rangle=\omega(\vartheta(\bar F)H)$, which is positive semidefinite by reflection positivity. Since $|\omega(\vartheta(\bar F)F)|\le\omega(|F|^2)$ by the Cauchy–Schwarz inequality and the invariance of ω under ϑ, functions that are close in $L^2(\omega)$ give close vectors.

Appendix G: Spectral measures and Laplace transforms

This appendix collects the facts on spectral measures that the continuum step uses, with proofs.

Lemma 670. *Let T be a self-adjoint operator on a Hilbert space with $0 \le T \le 1$ and let ψ be a unit vector. There is a unique probability measure ν on $[0,1]$ with $\langle \psi, h(T)\psi\rangle = \int h\,d\nu$ for every bounded Borel function h on $[0,1]$. If $T\psi' = \psi'$ implies $\langle \psi', \psi\rangle = 0$, then $\nu(\{1\}) = 0$. If μ is the image of ν under $\lambda \mapsto -c^{-1}\log\lambda \in [0,\infty]$ for some $c > 0$, then $\langle \psi, T^{\sigma/c}\psi\rangle = \int_{[0,\infty]} e^{-\sigma E} d\mu(E)$ for $\sigma > 0$, with the integrand 0 at $E = \infty$.*

Proof. The spectral theorem [101, Sec. VII.2] gives the measure ν, carried by the spectrum of T, which lies in $[0,1]$; uniqueness holds because the moments $\int \lambda^n d\nu = \langle \psi, T^n\psi\rangle$ determine a measure on a compact interval, polynomials being dense in $C[0,1]$. The spectral projection $P = \mathbf{1}_{\{1\}}(T)$ projects onto the eigenvectors of T with eigenvalue 1, so $\nu(\{1\}) = \|P\psi\|^2 = \langle P\psi, \psi\rangle = 0$ under the hypothesis. The last identity is the change of variables $\lambda^{\sigma/c} = e^{-\sigma E}$ for $\lambda = e^{-cE}$, with $\lambda = 0$ corresponding to $E = \infty$. □

Lemma 671. *Every sequence of probability measures on a compact metric space K has a subsequence that converges weakly, that is, against every continuous function, to a probability measure on K.*

Proof. $C(K)$ is separable; let (h_j) be dense. The numbers $\int h_j\,d\mu_k$ are bounded by $\|h_j\|_\infty$, so a diagonal subsequence makes all of them converge. By density the limit $\Lambda(h) = \lim \int h\,d\mu_k$ exists for every $h \in C(K)$; Λ is linear, positive and $\Lambda(1) = 1$, so by the Riesz representation theorem it is integration against a probability measure. □

Lemma 672. *Let μ be a probability measure on $[0,\infty]$ and $\rho(\sigma) = \int e^{-\sigma E} d\mu(E)$ for $\sigma > 0$.*

(a) *ρ is nonincreasing and log-convex: $\rho(\sigma)^2 \le \rho(\sigma - s)\rho(\sigma + s)$ for $0 < s < \sigma$.*

(b) *$\rho(\sigma) > 0$ for one $\sigma > 0$ if and only if $\mu([0,\infty)) > 0$, and then $\rho(\sigma) > 0$ for every $\sigma > 0$.*

(c) *$\rho(\sigma) \to \mu([0,\infty))$ as $\sigma \downarrow 0$.*

(d) *If μ is carried by $[m,\infty]$, then $\rho(\sigma) \le e^{-m\sigma}$.*

(e) *For $M > 0$, $\mu([0,M]) \ge \rho(\sigma) - e^{-\sigma M}$.*

(f) *If $\mu_k \to \mu$ weakly on $[0,\infty]$, then $\int e^{-\sigma E} d\mu_k \to \rho(\sigma)$ for every $\sigma > 0$.*

Proof. (a) The integrand decreases in σ. By the Cauchy–Schwarz inequality,

$$\left(\int e^{-\sigma E} d\mu\right)^2 \le \int e^{-(\sigma - s)E} d\mu \int e^{-(\sigma + s)E} d\mu.$$

(b) The integrand is positive exactly on $[0,\infty)$. (c) The integrand increases to $\mathbf{1}_{[0,\infty)}$ as $\sigma \downarrow 0$, and monotone convergence applies. (d) The integrand is at most $e^{-m\sigma}$ on $[m,\infty]$. (e) Split the integral at M: it is at most $\mu([0,M]) + e^{-\sigma M}$. (f) For $\sigma > 0$ the function $E \mapsto e^{-\sigma E}$, extended by 0 at ∞, is continuous on $[0,\infty]$. □

By (b) and (e), a lower bound $\rho(\sigma) \ge c > 0$ at one σ places mass at least $c/2$ on the bounded interval $[0, \sigma^{-1}\log(2/c)]$; for the measures μ_k of Theorem 48 a bound uniform in k therefore makes them tight on $[0,\infty)$ with a fixed fraction of their mass.

Part (b) also excludes a positive Laplace representation for a spectral cutoff of compact support. If $\chi(\lambda) = \int_{[0,\infty)} e^{-s\lambda} d\nu(s)$ with ν a positive measure and $\chi(0) = \nu([0,\infty)) > 0$, then (b), applied to $\nu/\chi(0)$, gives $\chi(\lambda) > 0$ for every $\lambda > 0$, so χ vanishes nowhere. The cutoff χ_σ of [50, Sec. 9, Eqs. (9.11), (9.12)] equals 1 on $[0,\sigma]$ and 0 on $[2\sigma,\infty)$ and is asserted to have such a representation.

Appendix H: Spectral gaps on cyclic subspaces

The local form $\mathrm{BP}_{\mathrm{loc}}$ of Definition 83 asks for a spectral gap of the cube chain on the cyclic subspace of each local function. This appendix records the elementary facts about such gaps that Sec. X H uses, for a self-adjoint operator K on a real or complex Hilbert space H with $0 \le K \le I$. For $g \in H$ let ν_g be its spectral measure, so that $\langle g, \phi(K)g\rangle = \int \phi\,d\nu_g$ for bounded Borel ϕ on $[0,1]$, and for a Borel set B let $E(B)$ be the spectral projection.

Lemma 673. *Let $g \in H$ and $0 < \varepsilon < 1$. The following are equivalent.*

(i) *$\|K^t g\| \le (1-\varepsilon)^t \|g\|$ for every integer $t \ge 0$.*

(ii) *$\nu_g((1-\varepsilon, 1]) = 0$.*

(iii) *$E((1-\varepsilon, 1])g = 0$.*

Under these conditions $\langle g', (I-K)g'\rangle \ge \varepsilon\|g'\|^2$ for every g' in the closed cyclic subspace generated by g.

Proof. $\|E(B)g\|^2 = \nu_g(B)$, so (ii) and (iii) are equivalent. Under (ii), $\|K^t g\|^2 = \int \lambda^{2t} d\nu_g \le (1-\varepsilon)^{2t}\nu_g([0,1])$, which is (i). If (ii) fails, there is $\eta > 0$ with $m = \nu_g((1-\varepsilon+\eta, 1]) > 0$, since these sets increase to $(1-\varepsilon, 1]$ as $\eta \downarrow 0$; then $\|K^t g\|^2 \ge m(1-\varepsilon+\eta)^{2t}$, which exceeds $(1-\varepsilon)^{2t}\|g\|^2$ for large t, and (i) fails. The closed cyclic subspace of g is the closure of $\{\phi(K)g\}$ over polynomials ϕ; it lies in the range of $E([0, 1-\varepsilon])$, since $E((1-\varepsilon, 1])$ commutes with K, and on that range $I - K \ge \varepsilon$. □

Lemma 674. *Let $F \subset H$ and $0 < \varepsilon < 1$. Every $g \in F$ satisfies the conditions of Lemma 673 if and only if the range of $E((1-\varepsilon, 1])$ is orthogonal to F, and then every vector of the closed linear span of F satisfies them. In that case, for every g in that span and every $h \in H$,*

$$|\langle g, h\rangle|^2 \le \|g\|^2 \,\frac{\langle h, (I-K)h\rangle}{\varepsilon}.$$

Proof. $E = E((1-\varepsilon,1])$ is an orthogonal projection, so $Eg = 0$ if and only if g is orthogonal to the range of E; the set of such g is a closed subspace. For g in it, g and $h' = (I-E)h$ lie in the range R of $I-E$, which is invariant under K and on which $I-K \geq \varepsilon$. Let T be the restriction of $(I-K)^{1/2}$ to R; it is invertible with $\|T^{-1}\| \leq \varepsilon^{-1/2}$. Then $\langle g,h\rangle = \langle g,h'\rangle = \langle T^{-1}g, Th'\rangle$, and

$$|\langle g,h\rangle|^2 \leq \frac{\|g\|^2}{\varepsilon}\,\langle h',(I-K)h'\rangle \leq \frac{\|g\|^2}{\varepsilon}\,\langle h,(I-K)h\rangle,$$

the last step because E commutes with K and $\langle Eh,(I-K)Eh\rangle \geq 0$. □

So $\mathrm{BP}_{\mathrm{loc}}(n,\varpi)$ on Λ_L says that the spectral projection of the cube chain for $(1-\varpi n^4/L^4,1]$ annihilates the closed span of the local gauge-invariant functions of mean zero, and the inequality of Theorem 84(iv) is Lemma 674 for that span. The span is not an algebra: a product of two local functions in distant boxes is not local, and $\mathrm{BP}_{\mathrm{loc}}$ says nothing about it. A Poincaré inequality for one function bounds the mean of $1-\lambda$ under its spectral measure and does not give a gap on its cyclic subspace.

Lemma 675. *For all $0<\eta<\varepsilon<1$ and $0<\theta<\frac{\pi}{2}$ there are H, K and g with $\|g\|=1$, $\langle g,(I-K)g\rangle \geq \cos^2\theta$, and $\|K^t g\| \geq (1-\eta)^t \sin\theta$ for every integer $t\geq 0$; so g fails the conditions of Lemma 673.*

Proof. Take $H=\mathbb{R}^2$, $K=\mathrm{diag}(0,1-\eta)$ and $g=(\cos\theta,\sin\theta)$. Then $\langle g,(I-K)g\rangle = \cos^2\theta+\eta\sin^2\theta$ and $K^t g = (0,(1-\eta)^t\sin\theta)$ for $t\geq 1$, and $\nu_g(\{1-\eta\}) = \sin^2\theta > 0$ with $1-\eta \in (1-\varepsilon,1]$. □

With θ close to 0 the Rayleigh quotient of g is close to one, the best possible value, while a fraction $\sin^2\theta$ of its spectral mass relaxes at the rate η only. This is why Proposition 87, a Poincaré inequality for each function of a finite set of links, does not give $\mathrm{BP}_{\mathrm{loc}}$.

Appendix I: The character expansion

Let G be a compact connected Lie group with normalised Haar measure dU. For an irreducible unitary representation ρ_λ of dimension d_λ write $\chi_\lambda = \mathrm{Tr}\,\rho_\lambda$. The characters form an orthonormal basis of the square-integrable class functions (Peter–Weyl and Schur orthogonality [32, Chs. 2–4]).

Lemma 676. *Let $w \in L^1(G)$ be a class function with $w(U^{-1}) = w(U)$, and put*

$$\hat{c}_\lambda = d_\lambda^{-1}\int w\,\chi_\lambda\,dU.$$

Then $\int w(U)\rho_\lambda(U)\,dU = \hat{c}_\lambda I$, and $\int w(U)\chi_\lambda(U^{-1}V)\,dU = \hat{c}_\lambda\chi_\lambda(V)$.

Proof. $M = \int w(U)\rho_\lambda(U)dU$ commutes with every $\rho_\lambda(V)$, since $w(VUV^{-1}) = w(U)$ and Haar measure is invariant; by Schur's lemma $M = cI$, and taking traces gives $c = \hat{c}_\lambda$. Then $\int w(U)\chi_\lambda(U^{-1}V)dU = \mathrm{Tr}(\int w(U)\rho_\lambda(U^{-1})dU\,\rho_\lambda(V))$, and the substitution $U\mapsto U^{-1}$ with $w(U^{-1})=w(U)$ turns the integral into M. □

Lemma 677. *Let $G=\mathrm{SU}(N)$ and $\beta>0$, and let*

$$w_\beta(U) = e^{\frac{\beta}{N}\mathrm{Re}\,\mathrm{Tr}U}\Big/\int e^{\frac{\beta}{N}\mathrm{Re}\,\mathrm{Tr}V}dV.$$

Then $\hat{c}_\lambda(\beta) = d_\lambda^{-1}\int w_\beta\chi_\lambda\,dU$ is real, $0\leq\hat{c}_\lambda(\beta)\leq 1$, and $\hat{c}_\lambda(\beta)<1$ for $\lambda\neq 0$. Moreover

$$\hat{c}_\lambda(\beta) = 1 - \frac{NC_2(\lambda)}{\beta} + O(\beta^{-2}) \qquad (\beta\to\infty),$$

where $-\sum_a \rho_\lambda(T_a)^2 = 2C_2(\lambda)I$ for a basis (T_a) of $\mathfrak{su}(N)$ orthonormal for $|X|^2 = -\mathrm{Tr}X^2$, so that $C_2 = (N^2-1)/(2N)$ on the fundamental representation.

Proof. Since $w_\beta(\bar{U}) = w_\beta(U)$ and $\chi_\lambda(\bar{U}) = \overline{\chi_\lambda(U)}$, and complex conjugation preserves Haar measure, $\int w_\beta\chi_\lambda = \int w_\beta\bar{\chi}_\lambda$, which is the complex conjugate of the left side; so $\hat{c}_\lambda$ is real. Next, w_β is a multiple of $\exp(\frac{\beta}{2N}(\chi_{\mathrm{F}}+\bar{\chi}_{\mathrm{F}})) = \sum_m \frac{1}{m!}(\frac{\beta}{2N})^m(\chi_{\mathrm{F}}+\bar{\chi}_{\mathrm{F}})^m$, and each power is the character of a representation, hence a sum of irreducible characters with non-negative integer coefficients. By orthonormality $\int w_\beta\bar{\chi}_\lambda \geq 0$, so $\hat{c}_\lambda\geq 0$. As $|\chi_\lambda(U)|\leq d_\lambda$, with $\mathrm{Re}\,\chi_\lambda(U) = d_\lambda$ only where $\rho_\lambda(U)=I$, a proper closed subgroup when $\lambda\neq 0$ and so a Haar null set, and $w_\beta>0$ everywhere, $\hat{c}_\lambda = d_\lambda^{-1}\int w_\beta\,\mathrm{Re}\,\chi_\lambda < 1$ for $\lambda\neq 0$.

For the expansion, $\mathrm{Re}\,\mathrm{Tr}U \leq N$ with equality only at $U=I$, so outside a neighbourhood $\{U=e^X : |X|<\delta\}$ both integrals $\int e^{\frac{\beta}{N}\mathrm{Re}\,\mathrm{Tr}U}\chi_\lambda$ and $\int e^{\frac{\beta}{N}\mathrm{Re}\,\mathrm{Tr}U}$ are smaller than their values near I by a factor $e^{-c\beta}$. Near I write $U=e^X$; the Haar density in these coordinates is $j(X) = 1+O(|X|^2)$, even in X. For antihermitian X, $\mathrm{Tr}X^3$ is imaginary, so

$$\tfrac{1}{N}\mathrm{Re}\,\mathrm{Tr}e^X = 1 - \tfrac{1}{2N}|X|^2 + r(X), \qquad 0\leq r(X)\leq C|X|^4,$$

and $d_\lambda^{-1}\chi_\lambda(e^X) = 1 + d_\lambda^{-1}\mathrm{Tr}\,\rho_\lambda(X) + \frac{1}{2d_\lambda}\mathrm{Tr}\,\rho_\lambda(X)^2 + c_3(X) + O(|X|^4)$, with c_3 cubic and $\mathrm{Tr}\,\rho_\lambda(X)=0$, since $\mathfrak{su}(N)$ is perfect. Let $\langle\cdot\rangle$ denote the average against the Gaussian weight $e^{-\beta|X|^2/(2N)}$, under which the coordinates x_a of $X=\sum x_a T_a$ are independent with variance N/β. With $h = j\,e^{\beta r}$,

$$\hat{c}_\lambda(\beta) = \frac{\langle d_\lambda^{-1}\chi_\lambda(e^X)\,h\rangle}{\langle h\rangle} + O(e^{-c\beta}) = 1 + \frac{\langle\frac{1}{2d_\lambda}\mathrm{Tr}\,\rho_\lambda(X)^2 h\rangle}{\langle h\rangle} + \frac{\langle(c_3(X)+O(|X|^4))h\rangle}{\langle h\rangle},$$

the averages being taken over $|X|<\delta$. Under $\langle\cdot\rangle$, $|X|^{2m}$ has mean of order β^{-m} and $\beta|r(X)|\leq C\beta|X|^4$ has mean of order β^{-1}, so $h = 1+O(\beta^{-1})$ in every moment. The weight h is even in X, since $\mathrm{Tr}X^k$ is imaginary for odd k and j is even, so the

odd term c_3 contributes nothing; the quartic remainder and the correlation of the quadratic term with $h-1$ contribute $O(\beta^{-2})$. Finally $\langle \operatorname{Tr}\rho_\lambda(X)^2\rangle = \frac{N}{\beta}\sum_a \operatorname{Tr}\rho_\lambda(T_a)^2 = -\frac{2NC_2(\lambda)d_\lambda}{\beta}$, which gives the expansion. For $\mathrm{SU}(2)$ and the representation of spin j the integrals are explicit, $\hat{c}_j(\beta) = I_{2j+1}(\beta)/I_1(\beta)$ with the modified Bessel functions I_n, in agreement with $C_2 = j(j+1)$. □

Appendix J: Haar measure on SU(2)

Write $\mathbf{i},\mathbf{j},\mathbf{k}$ for the matrices $i\sigma_3$, $i\sigma_2$, $i\sigma_1$. The map $(a,b)\mapsto aI + b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$ identifies $\mathbb{R}^4$ with the real span of $\mathrm{SU}(2)$, and $\mathrm{SU}(2)$ with the unit sphere S^3; the Hilbert–Schmidt norm of $aI + b\cdot(\mathbf{i},\mathbf{j},\mathbf{k})$ is $\sqrt{2}\,(a^2+|b|^2)^{1/2}$. Left and right multiplication by elements of $\mathrm{SU}(2)$ preserve this norm, so they act on S^3 by rotations, and the normalised Haar measure is the normalised surface measure of S^3. Every $U\in\mathrm{SU}(2)$ is conjugate to $\operatorname{diag}(e^{i\theta},e^{-i\theta})$ for one $\theta\in[0,\pi]$, its eigenangle, and $\frac{1}{2}\operatorname{Tr}U = \cos\theta$ is the inner product of U and I in $\mathbb{R}^4$; so θ is the angle between U and I on S^3.

Lemma 678. (i) *The eigenangle θ has Haar density $\frac{2}{\pi}\sin^2\theta$ on $[0,\pi]$.*

(ii) $\mathrm{Haar}\{\theta\le t\} = \frac{1}{\pi}(t-\sin t\cos t) \le \frac{2t^3}{3\pi}$ *for $0\le t\le\pi$.*

(iii) $|U-I| = 2\sqrt{2}\sin\frac{\theta}{2}$*, and the distance of the bi-invariant metric induced by the Hilbert–Schmidt inner product is $d(U,I) = \sqrt{2}\,\theta$. With this metric* $\mathrm{SU}(2)$ *is the round sphere of radius $\sqrt{2}$.*

Proof. (i) The unit vectors at angle θ from I form a two-sphere of radius $\sin\theta$ and area $4\pi\sin^2\theta$, and S^3 has area $2\pi^2$.

(ii) $\int_0^t \sin^2\theta\,d\theta = \frac{1}{2}(t-\sin t\cos t)$, and $\sin^2\theta\le\theta^2$.

(iii) The eigenvalues of $U-I$ are $e^{\pm i\theta}-1$, so $|U-I|^2 = 2|e^{i\theta}-1|^2 = 8\sin^2\frac{\theta}{2}$. By the first paragraph the metric is that of the sphere of radius $\sqrt{2}$, on which the points at angle $\theta\le\pi$ are at distance $\sqrt{2}\,\theta$. □

Appendix K: The single-link law of SU(2)

This appendix proves the sharp quarter, $W_1(\nu_b,\nu_{b'})\le\frac{1}{4}|b-b'|$ for the one-link law of $\mathrm{SU}(2)$, which Theorems 1 and 3 use, and the exact value at $b=0$, which Theorem 2 uses. The argument is that of our earlier paper [20]; it is given in full so that the present paper is self-contained.

1. The von Mises–Fisher law

Identify $\mathrm{SU}(2)$ with the unit quaternions, that is with the unit sphere $S^3\subset\mathbb{R}^4$, so that $\operatorname{Re}\operatorname{Tr}U = 2x_0(U)$ and $\operatorname{Re}\operatorname{Tr}(UV) = 2\langle u,\bar v\rangle$ for the Euclidean inner product of $\mathbb{R}^4$ and quaternion conjugation $v\mapsto\bar v$. Fix a link e and the field off e. Every plaquette through e can be rotated cyclically so that its trace reads $\operatorname{Re}\operatorname{Tr}(Q_eA_p)$ with $A_p\in\mathrm{SU}(2)$, the staple of e in p. The action therefore depends on Q_e only through

$$N\beta\sum_{p\ni e}\operatorname{Re}\operatorname{Tr}(Q_eA_p) = \langle x,b\rangle, \qquad b = 4\beta\sum_{p\ni e}\bar A_p\in\mathbb{R}^4,$$

with $x = Q_e\in S^3$. Since there are six staples, each of norm one,

$$\kappa := |b| \le y := 24|\beta|. \tag{K1}$$

The conditional law of Q_e is the von Mises–Fisher law[50]

$$\nu_b(dx) = Z(\kappa)^{-1}e^{\langle x,b\rangle}\,\sigma(dx) \tag{K2}$$

on the unit sphere, with σ the normalised surface measure. We use coordinates in which $b=\kappa e_1$. The density of $t=x_1$ is proportional to $(1-t^2)^{1/2}e^{\kappa t}$ on $[-1,1]$, so $Z(\kappa) = 2I_1(\kappa)/\kappa$ with the modified Bessel function I_1, normalised so that $Z(0)=1$.

Lemma 679. *For every $\kappa\ge0$ the law ν_b on the unit sphere S^3 satisfies the Poincaré inequality* $\operatorname{Var}_{\nu_b}(g)\le\frac{1}{3}\int|\nabla g|^2d\nu_b$*, and the constant $\frac{1}{3}$ is attained at $\kappa=0$.*

Proof. In the polar angle $\theta\in(0,\pi)$ from e_1 the generator of the reversible diffusion of ν_b is $L = \partial_\theta^2 + b(\theta)\partial_\theta + \sin^{-2}\theta\,\Delta_{S^2}$ with $b(\theta) = 2\cot\theta - \kappa\sin\theta$. On spherical harmonics of degree ℓ on S^2 it acts as a Sturm–Liouville operator L_ℓ with potential $\ell(\ell+1)\sin^{-2}\theta$. A ground state comparison of Barta's type [14] with the trial function $\psi = \sin\theta\,e^{a\kappa\cos\theta}$ gives $-L_1\psi\ge3\psi$ for $a=-\frac{1}{5}$, and the same trial function with $a=-\frac{2}{5}$ gives the bound 3 for the derivatives of zonal eigenfunctions; Appendix L carries out both computations. Hence every nonzero eigenvalue of $-L$ is at least 3. At $\kappa=0$ the first eigenvalue of the Laplacian of the unit S^3 is 3. □

2. Moments

Let $A(\kappa) = \mathbb{E}_{\nu_b}x_1 = Z'(\kappa)/Z(\kappa) = I_2(\kappa)/I_1(\kappa)$.

Lemma 680. *For every $\kappa>0$, in coordinates with $b=\kappa e_1$,*

(i) $\mathbb{E}x_2^2 = \mathbb{E}x_3^2 = \mathbb{E}x_4^2 = A/\kappa$, $\mathbb{E}(1-x_1^2) = 3A/\kappa$*, the covariance matrix of x is* $\operatorname{diag}(A',A/\kappa,A/\kappa,A/\kappa)$*, and $A' = 1-A^2-3A/\kappa$;*

(ii) $$\frac{\kappa}{2+\sqrt{\kappa^2+4}} \le A(\kappa) \le \frac{\kappa}{1+\sqrt{\kappa^2+9}};$$

(iii) $A'\le A/\kappa\le\left(1+\sqrt{\kappa^2+9}\right)^{-1}\le\frac{1}{4}$;

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[50] On the sphere $S^{p-1}\subset\mathbb{R}^p$ the von Mises–Fisher law with mean direction μ and concentration κ has density proportional to $e^{\kappa\langle x,\mu\rangle}$ with respect to the surface measure, and its normalising constant is $(2\pi)^{p/2}I_{p/2-1}(\kappa)/\kappa^{p/2-1}$; it is the maximum entropy law with prescribed mean resultant, and it plays on the sphere the role of the Gaussian in $\mathbb{R}^p$. For $p=4$ the Bessel index is 1. The mean resultant length $\mathbb{E}\langle x,\mu\rangle = I_{p/2}(\kappa)/I_{p/2-1}(\kappa)$ is the function A of Lemma 680. See Mardia and Jupp [87], Ch. 9, for these facts and for the tangent-normal decomposition used in Appendix L.

(iv) $A(\kappa) \geq \kappa/\big(\tfrac{3}{2}+\sqrt{\kappa^2+25/4}\big)$.

Proof. (i) Integrating by parts against $\frac{d}{dt}(1-t^2)^{3/2} = -3t(1-t^2)^{1/2}$ gives $\kappa\mathbb{E}(1-x_1^2) = 3\mathbb{E}x_1$, and the three coordinates x_2, x_3, x_4 are exchangeable under ν_b, with $\sum_i x_i^2 = 1$. The off-diagonal covariances vanish by the reflections $x_i \mapsto -x_i$, $i \geq 2$, and $A' = \operatorname{Var} x_1$ because $A = (\log Z)'$. (ii) By (i), A solves the Riccati equation $A' = 1 - A^2 - 3A/\kappa$ with $A(\kappa) = \kappa/4 + O(\kappa^3)$. Appendix L shows that the left side of (ii) is a subsolution and the right side a supersolution of this equation, with the same behaviour at 0, and concludes by comparison. (iii) The middle inequality is (ii). For the first, put $s = \sqrt{\kappa^2+4}$. By (ii), $4A/\kappa \geq 4/(2+s)$ and $1 - A^2 \leq 1 - \kappa^2/(2+s)^2$, and

$$1 - \frac{\kappa^2}{(2+s)^2} \leq \frac{4}{2+s} \iff (2+s)^2 - \kappa^2 \leq 4(2+s) \iff s^2 - \kappa^2 \leq 4,$$

which holds with equality. So $A' = 1 - A^2 - 3A/\kappa \leq A/\kappa$. (iv) is the comparison of (ii) with a third barrier, which improves the lower bound of (ii); it is carried out in the same appendix. □

3. The Lipschitz rate of the family

Let δ be the chordal metric $\delta(x,x') = |x - x'|$ on S^3 and W_1 the Kantorovich distance for δ. For $c \in \mathbb{R}^4$ and a unit vector $h \in \mathbb{R}^4$ put

$$L(c,h) = \sup_g \operatorname{Cov}_{\nu_c}\big(g, \langle h, x\rangle\big), \tag{K3}$$

the supremum over 1-Lipschitz functions g for δ. Since $\frac{d}{dt}\nu_{c+th}(g) = \operatorname{Cov}_{\nu_{c+th}}(g, \langle h, x\rangle)$, the Kantorovich–Rubinstein duality gives, for all b, b',

$$W_1(\nu_b, \nu_{b'}) \leq |b - b'| \sup_{c \in [b,b']} L\Big(c, \frac{b'-b}{|b'-b|}\Big). \tag{K4}$$

By rotation invariance $L(c,h)$ depends only on $\kappa = |c|$ and on the angle φ between h and $c^{\perp}$, and we write $L(\kappa, \varphi)$.

Lemma 681. *For all c and h, $L(c,h) \leq 1/(2\sqrt{3})$. Moreover:*

(i) *at $c = 0$, $L(0,h) = \frac{1}{4}$ for every h;*

(ii) *if $h \perp c$, then $L(c,h) = \mathbb{E}_{\nu_c} x_2^2 = A(\kappa)/\kappa \leq \frac{1}{4}$;*

(iii) *if $\kappa \geq \sqrt{88}/3 \approx 3.127$, then $L(c,h) \leq \frac{1}{4}$ for every h.*

Proof. A 1-Lipschitz function for δ is 1-Lipschitz for the geodesic distance, which is larger, so $|\nabla g| \leq 1$ almost everywhere, and Lemma 679 gives $\operatorname{Var} g \leq \frac{1}{3}$. By Lemma 680(i), (iii) the largest eigenvalue of the covariance of x is $A/\kappa \leq \frac{1}{4}$, so $\operatorname{Var}\langle h, x\rangle \leq A/\kappa$. The Cauchy–Schwarz inequality gives

$$L(c,h) \leq \big(\tfrac{1}{3}A/\kappa\big)^{1/2} \leq \Big(3\big(1+\sqrt{\kappa^2+9}\big)\Big)^{-1/2} \leq \frac{1}{2\sqrt{3}}. \tag{K5}$$

(i) and (ii) are reflection arguments. Let R be the reflection in the hyperplane $h^{\perp}$, and suppose ν_c is R-invariant, which is the case when $c = 0$ or $h \perp c$. Then $\mathbb{E}\langle h, x\rangle = 0$ and

$$\begin{aligned}\operatorname{Cov}(g, \langle h, x\rangle) &= \mathbb{E}\big[(g(x) - g(Rx))\langle h, x\rangle;\ \langle h, x\rangle > 0\big] \\ &\leq \mathbb{E}\big[2\langle h, x\rangle^2;\ \langle h, x\rangle > 0\big] = \mathbb{E}\langle h, x\rangle^2,\end{aligned}$$

since $\delta(x, Rx) = 2|\langle h, x\rangle|$. Equality holds for $g = \langle h, x\rangle$, which is 1-Lipschitz. At $c = 0$, $\mathbb{E}\langle h, x\rangle^2 = \frac{1}{4}$; for $h \perp c$, $\mathbb{E}\langle h, x\rangle^2 = A/\kappa$. (iii) By (K5), $L \leq \frac{1}{4}$ as soon as $1 + \sqrt{\kappa^2+9} \geq \frac{16}{3}$, that is $\kappa^2 \geq \frac{88}{9}$. □

The reflection settles the directions $h \perp c$ and the point $c = 0$; for an oblique h the reflection no longer preserves ν_c, and the Cauchy–Schwarz bound (K5) is too weak for $\kappa < \sqrt{88}/3$. The next theorem closes this case.

Theorem 682 (The sharp quarter)**.** *For every $c \in \mathbb{R}^4$ and every unit vector $h \in \mathbb{R}^4$,*

$$L(c,h) \leq \tfrac{1}{4},$$

with equality at $c = 0$. Equivalently, $W_1(\nu_b, \nu_{b'}) \leq \frac{1}{4}|b - b'|$ for all $b, b' \in \mathbb{R}^4$, and the constant $\frac{1}{4}$ is the best possible.

The equivalence is (K4) in one direction and the derivative formula above it in the other; the sharpness is Lemma 681(i). The metric matters.[51] The proof has four steps. A Lipschitz function on S^3 extends to the unit ball $B \subset \mathbb{R}^4$, and Green's formula in B bounds the covariance by a negative Sobolev norm of the density (Lemma 683). This norm splits into a zonal and a transverse part, both explicit series in modified Bessel functions (Lemma 684). The transverse part is at most $\frac{1}{4}$ because a certain power series has nonnegative coefficients (Proposition 686), and the zonal part is at most $\frac{1}{4}$ by a Riccati comparison (Proposition 687).

4. Harmonic extension

Let σ be the normalised surface measure of S^3 and H_k the space of spherical harmonics of degree k, the restrictions to S^3 of harmonic homogeneous polynomials of degree k on $\mathbb{R}^4$. The spaces H_k are orthogonal in $L^2(\sigma)$ and span it; we write u_k for the component of u in H_k, and for u with $\int u\, d\sigma = 0$ we put

$$\mathscr{Q}(u) = \sum_{k \geq 1} \frac{1}{k} \|u_k\|^2_{L^2(\sigma)}. \tag{K6}$$

This is $\langle u, \Lambda^{-1} u\rangle$ for the Dirichlet-to-Neumann operator Λ of B, which acts on H_k as multiplication by k.

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[51] In the geodesic metric of S^3 the statement is false already at $c = 0$: the function $\arcsin x_2$ is geodesically 1-Lipschitz and its covariance with x_2 under the uniform law is $\frac{8}{9\pi} \approx 0.283 > \frac{1}{4}$. A proof therefore has to use that the chordal distance between two points is shorter than their geodesic distance, that is, it has to see the sphere from the ball it bounds. The proof below does exactly this: it extends a chordally Lipschitz function to the ball and integrates by parts there.

Lemma 683. *Let u be a real-analytic function on S^3 with $\int u\,d\sigma = 0$ and let g be 1-Lipschitz for the chordal metric. Then*

$$\int_{S^3} g\,u\,d\sigma \le \tfrac{1}{2}\,\mathscr{Q}(u)^{1/2}.$$

Proof. The chordal metric is the Euclidean metric of $\mathbb{R}^4$ restricted to S^3, so McShane's extension $G(y) = \inf_{x\in S^3}\big(g(x) + |x-y|\big)$ is 1-Lipschitz on $\mathbb{R}^4$ and equals g on S^3 [89]; by Rademacher's theorem $|\nabla G| \le 1$ almost everywhere. Let $\psi = \sum_{k\ge1} k^{-1} r^k u_k(\omega)$ in polar coordinates $y = r\omega$. Each term is a harmonic polynomial, the series converges with all derivatives on the closed ball because u is real-analytic,[52] and $\partial_r \psi = u$ on S^3. Green's formula for the Lipschitz function G and the smooth harmonic function ψ reads $\int_{S^3} G\,\partial_r\psi\,dS = \int_B \nabla G\cdot\nabla\psi\,dy$. With $|S^3| = 2\pi^2$ and $|B| = \pi^2/2$ the Cauchy–Schwarz inequality gives

$$\int g\,u\,d\sigma = \frac{1}{2\pi^2}\int_B \nabla G\cdot\nabla\psi\,dy \le \frac{|B|^{1/2}}{2\pi^2}\Big(\int_B |\nabla\psi|^2 dy\Big)^{1/2}.$$

Green's formula once more gives $\int_B |\nabla\psi|^2 = \int_{S^3}\psi\,\partial_r\psi\,dS = 2\pi^2\sum_k k^{-1}\|u_k\|^2 = 2\pi^2\,\mathscr{Q}(u)$, and the right side becomes $\tfrac12\mathscr{Q}(u)^{1/2}$. □

The lemma is a form of Thomson's principle. By Kantorovich–Rubinstein duality, $\sup_g \int g\,u\,d\sigma$ is the least cost of a flow in $\mathbb{R}^4$ that carries the positive part of $u\,d\sigma$ to its negative part, the cost being the total length travelled by the mass.[53] The harmonic field $\nabla\psi$ is one such flow, through the interior of the ball, and the lemma computes its cost. At $c = 0$ the flow is optimal: for $u = \langle h, x\rangle$ one has $\psi = \langle h, y\rangle$, the field $\nabla\psi = h$ is constant, the Cauchy–Schwarz inequality is an equality, and $\mathscr{Q}(u) = \|u\|^2 = \tfrac14$.

5. Spectral decomposition

Fix $c = \kappa e_1$ with $\kappa > 0$ and write $h = \cos\varphi\, e_1 + \sin\varphi\, e_2$, which is no loss of generality by rotation invariance. Let $\rho = d\nu_c/d\sigma = e^{\kappa x_1}/Z(\kappa)$, $Z(\kappa) = \int e^{\kappa x_1} d\sigma = 2I_1(\kappa)/\kappa$, and

$$u = \big(\langle h, x\rangle - \mathbb{E}_{\nu_c}\langle h, x\rangle\big)\rho,$$

so that $\mathrm{Cov}_{\nu_c}(g, \langle h, x\rangle) = \int g u\, d\sigma$ and $\int u\, d\sigma = 0$. Let U_n be the Chebyshev polynomials of the second kind and put

$$d_n(\kappa) = \frac{(n+1)\,I_{n+1}(\kappa)}{I_1(\kappa)}, \qquad n \ge 0. \tag{K7}$$

Lemma 684. *With this notation*

$$\mathscr{Q}(u) = \cos^2\varphi\,\mathscr{Q}_z(\kappa) + \sin^2\varphi\,\mathscr{Q}_x(\kappa),$$

where

$$\mathscr{Q}_z = \sum_{n\ge1}\frac{d_n'(\kappa)^2}{n}, \qquad \mathscr{Q}_x = \frac{1}{3\kappa^2}\sum_{n\ge1}(n+2)\,d_n(\kappa)^2.$$

Proof. The functions $U_n(x_1)$, $n \ge 0$, are the zonal harmonics: $U_n(x_1) \in H_n$, and they are orthonormal in $L^2(\sigma)$ (Appendix M). The expansion $e^{\kappa t} = \frac{2}{\kappa}\sum_n (n+1) I_{n+1}(\kappa) U_n(t)$ of the same appendix gives $\rho = \sum_n d_n(\kappa) U_n(x_1)$, with $d_0 = 1$.

Write $u = \cos\varphi\, u_z + \sin\varphi\, u_x$ with $u_z = (x_1 - A)\rho$ and $u_x = x_2\rho$; here $\mathbb{E}_{\nu_c} x_1 = A$ and $\mathbb{E}_{\nu_c} x_2 = 0$. The reflection $x_2 \mapsto -x_2$ maps each H_k to itself, fixes u_z and reverses u_x, so the components of u_z and u_x in each H_k are orthogonal and the cross term in $\mathscr{Q}(u)$ vanishes.

Since $A = Z'/Z$, we have $u_z = \partial_\kappa\rho = \sum_n d_n'(\kappa) U_n(x_1)$, and the component of u_z in H_n is $d_n' U_n(x_1)$, of norm $|d_n'|$. This gives $\mathscr{Q}_z$.

For u_x, let $y(t) = \cos t\, e_1 + \sin t\, e_2$. The law $\nu_{\kappa y(t)}$ has density $\rho_t = e^{\kappa\langle y(t), x\rangle}/Z(\kappa) = \sum_n d_n(\kappa) U_n(\langle y(t), x\rangle)$, and $\partial_t \rho_t|_{t=0} = \kappa x_2 \rho$. The function $U_n(\langle y(t), x\rangle)$ lies in H_n for every t, hence so does its derivative $x_2 U_n'(x_1)$ at $t = 0$, and the component of u_x in H_n is $\kappa^{-1} d_n x_2 U_n'(x_1)$. Its norm follows from the eigenvalue $n(n+2)$ of the Laplacian of S^3 on H_n. For $f = U_n(x_1)$,

$$\begin{aligned} n(n+2) = n(n+2)\|f\|^2 &= \int |\nabla_{S^3} f|^2 d\sigma \\ &= \int U_n'(x_1)^2 (1 - x_1^2)\, d\sigma, \end{aligned}$$

since the tangential gradient of x_1 has squared length $1 - x_1^2$. Given x_1, the point (x_2, x_3, x_4) is uniform on the sphere of radius $(1-x_1^2)^{1/2}$, so $\mathbb{E}[x_2^2 \mid x_1] = \frac13(1 - x_1^2)$ and $\|x_2 U_n'(x_1)\|^2 = \frac13 n(n+2)$. Summing $n^{-1}\kappa^{-2} d_n^2 \cdot \frac13 n(n+2)$ over $n \ge 1$ gives $\mathscr{Q}_x$. □

By Lemmas 683 and 684,

$$L(\kappa, \varphi) \le \tfrac12 \max\big(\mathscr{Q}_z(\kappa), \mathscr{Q}_x(\kappa)\big)^{1/2}, \tag{K8}$$

and Theorem 682 follows once both parts are at most $\frac14$. Both are $\frac14$ at $\kappa = 0$, by continuity from the values $d_1 \sim \kappa/2$, $d_1' \to \frac12$ and $d_n = O(\kappa^n)$.

6. Three Bessel sums

The series of Lemma 684 reduce to closed forms. Write I_j for $I_j(\kappa)$.

Lemma 685. *For every $\kappa > 0$,*

$$\sum_{j\ge1} j^2 I_j^2 = \frac{\kappa}{4} I_1(2\kappa), \qquad \sum_{j\ge1} j^3 I_j^2 = \frac{\kappa^2}{4}\big(I_0^2 + I_1^2\big),$$
$$I_1(2\kappa) = 2\sum_{j\ge1} I_{j-1} I_j.$$

The first and third identities follow from Parseval's formula for $e^{\kappa\cos\theta} = \sum_{j\in\mathbb{Z}} I_j e^{ij\theta}$, and the second from the recurrences of I_j by telescoping; Appendix M gives the proofs.

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[52] For real-analytic u the norms $\|u_k\|$ decay geometrically in k, and the supremum of a function in H_k is at most $(\dim H_k)^{1/2} = k+1$ times its $L^2(\sigma)$ norm, so the series for ψ and for each of its derivatives converge uniformly on $\bar B$.

[53] This is the Beckmann formulation of the transport problem with cost $|x-y|$: the optimal cost equals the least $\int |V|$ over vector measures V on $\mathbb{R}^4$ with divergence $u\,d\sigma$. Restricting the flow to the closed ball does not change the infimum, since projecting onto $\bar B$ shortens paths. The gradient $\nabla\psi$ on B is one admissible flow. It minimises the energy $\int |V|^2$ rather than the cost $\int |V|$, and the Cauchy–Schwarz inequality converts one into the other at the price of the factor $|B|^{1/2}$.

7. The transverse part

Proposition 686. *$\mathscr{Q}_x(\kappa) \le \frac14$ for every $\kappa \ge 0$.*

Proof. By (K7), $\mathscr{Q}_x = (3\kappa^2 I_1^2)^{-1}\sum_{j\ge2} j^2(j+1)I_j^2$, and the sum equals $\sum_{j\ge1}(j^3+j^2)I_j^2 - 2I_1^2$. By Lemma 685, $\mathscr{Q}_x \le \frac14$ is equivalent to

$$\mathscr{D}(\kappa) := (8+2\kappa^2)I_1^2 - \kappa^2 I_0^2 - \kappa I_1(2\kappa) \ge 0. \tag{K9}$$

Each term is an entire function of κ that is even after the factor κ^2 is removed, so $\mathscr{D}(\kappa) = \sum_{n\ge0}\mathscr{D}_n\kappa^{2n}$. Lemma 689 of Appendix M computes the coefficients: $\mathscr{D}_0 = \mathscr{D}_1 = 0$, and for $n = m+1 \ge 2$

$$\mathscr{D}_n = \frac{N_m}{4^m\, m!\,(m+1)!\,(m+1)(m+2)},$$

with

$$N_m = \binom{2m}{m}\big(4(2m+1) + (m-1)(m+1)(m+2)\big) - 4^m(m+1)(m+2).$$

The numerator N_m vanishes for $m = 0,1,2$ and is positive for $m \ge 3$ (Lemma 689). So every coefficient of $\mathscr{D}$ is nonnegative and $\mathscr{D} \ge 0$. □

The first nonzero coefficient is $\mathscr{D}_4 = \frac{1}{2304}$, so $\frac14 - \mathscr{Q}_x = \mathscr{D}/(12\kappa^2 I_1^2)$ is of order κ^4 at 0: the transverse part leaves the value $\frac14$ very slowly, which is why no estimate that loses a constant can prove the proposition.

8. The zonal part

Let $A = I_2/I_1$ as before, so that $d_1 = 2A$ and, by Lemma 680(i), $A' = 1 - A^2 - 3A/\kappa$.

Proposition 687. *For $0 < \kappa \le \sqrt{88}/3$,*

$$\mathscr{Q}_z(\kappa) \le \frac14 + 4A'^2 + \frac{A^2}{4} - \frac{A}{2\kappa} - \frac{2A^2}{\kappa^2} < \frac14.$$

Proof. The higher modes. With $r_\nu = I_{\nu+1}/I_\nu$ the recurrence $I_\nu' = I_{\nu+1} + \frac{\nu}{\kappa}I_\nu$ gives

$$\frac{d_n'}{d_n} = \frac{I_{n+1}'}{I_{n+1}} - \frac{I_1'}{I_1} = \frac{n}{\kappa} + r_{n+1} - r_1.$$

By Lemma 690 of Appendix M, $r_{n+1} \le r_1$, and $r_1 = A \le \kappa/4$ by Lemma 680(iii). Hence $-n/\kappa \le n/\kappa - \kappa/4 \le d_n'/d_n \le n/\kappa$ for $n \ge 2$ and $\kappa^2 \le 16$, so $d_n'^2 \le n^2 d_n^2/\kappa^2$. With $d_1' = 2A'$,

$$\mathscr{Q}_z \le 4A'^2 + \frac{1}{\kappa^2}\sum_{n\ge2} n\, d_n^2 = 4A'^2 + \frac{S_3 - S_2 - 4I_2^2}{\kappa^2 I_1^2},$$

where $S_2 = \sum_{j\ge1} j^2 I_j^2$ and $S_3 = \sum_{j\ge1} j^3 I_j^2$; the second form uses $n\,d_n^2 I_1^2 = (j-1)j^2 I_j^2$ with $j = n+1$.

The closed form. The recurrences $I_0 - I_2 = \frac{2}{\kappa}I_1$ and $I_1 - I_3 = \frac{4}{\kappa}I_2$ give $I_0/I_1 = 2/\kappa + A$ and $I_2 I_3/I_1^2 = A - 4A^2/\kappa$. By Lemma 685, $S_3/(\kappa^2 I_1^2) = \frac14(2/\kappa + A)^2 + \frac14$, and, keeping three terms of the third identity,

$$\frac{S_2}{\kappa^2 I_1^2} \ge \frac{I_0 I_1 + I_1 I_2 + I_2 I_3}{2\kappa I_1^2} = \frac{1}{2\kappa}\Big(\frac{2}{\kappa} + 3A - \frac{4A^2}{\kappa}\Big).$$

Collecting terms, $\mathscr{Q}_z \le \frac14 + 4A'^2 + T(A)$ with $T(a) = \frac{a^2}{4} - \frac{a}{2\kappa} - \frac{2a^2}{\kappa^2}$, which is the first inequality.

The barrier. Let $s = \sqrt{\kappa^2 + 25/4}$ and $A_\flat = \kappa/(\frac32 + s)$. By Lemma 680(iv), $A_\flat \le A \le \kappa/4$. The function $a \mapsto 1 - a^2 - 3a/\kappa$ decreases on $a \ge 0$, and $A' \ge 0$ because $A' = \mathrm{Var}\, x_1$, so $0 \le A' \le \alpha := 1 - A_\flat^2 - 3A_\flat/\kappa$. For $\kappa^2 \le 12$ the quadratic T decreases on $[0, \kappa/4]$, since

$$T(a) - T(a') = (a' - a)\,\frac{2\kappa - (a+a')(\kappa^2 - 8)}{4\kappa^2}$$

and $(a+a')(\kappa^2-8) \le \frac{\kappa}{2}(\kappa^2-8) \le 2\kappa$ there. So $\mathscr{Q}_z \le \frac14 + 4\alpha^2 + T(A_\flat)$. With $\kappa^2 = s^2 - \frac{25}{4}$ and $A_\flat/\kappa = 2/(2s+3)$ both terms are rational in s: $\alpha = 16/(2s+3)^2$, and

$$4\alpha^2 + T(A_\flat) = \frac{(2s-5)\,p(s)}{4(2s+3)^4}, \qquad p(s) = 8s^3 + 28s^2 - 98s - 695. \tag{K10}$$

For $0 < \kappa \le \sqrt{88}/3$ we have $\frac52 < s \le \sqrt{577}/6 < 4.004$. On $s \ge \frac52$ the cubic p increases, since $p'(s) = 24s^2 + 56s - 98 > 0$, and $p(4.004) < -120$. So the right side of (K10) is negative. □

9. Proof of Theorem 682

At $c = 0$ the theorem is Lemma 681(i), and for $|c| \ge \sqrt{88}/3$ it is Lemma 681(iii). For $0 < |c| < \sqrt{88}/3$, Propositions 686 and 687 give $\mathscr{Q}_x, \mathscr{Q}_z \le \frac14$, and (K8) gives $L \le \frac14$. □

Appendix L: Computations for the single-link law

1. Barta's inequality

Let $\rho(\theta) = \sin^2\theta\, e^{\kappa\cos\theta}$ on $(0,\pi)$, let V be a continuous potential, and let

$$T = -\rho^{-1}\partial_\theta\,\rho\,\partial_\theta + V.$$

If $\psi > 0$ is smooth on $(0,\pi)$ with $\rho\psi^2 = O(\sin^4\theta)$ at the end points and $T\psi \ge \lambda_0\psi$, then

$$\int (|h'|^2 + Vh^2)\rho\, d\theta \ge \lambda_0 \int h^2 \rho\, d\theta$$

for every smooth h vanishing to first order at 0 and π. Indeed, writing $h = \psi u$ and integrating by parts, the boundary terms $\rho\psi\psi' u^2$ vanish and

$$\int (|h'|^2 + Vh^2)\rho = \int \rho\psi^2 |u'|^2 + \int \frac{T\psi}{\psi} h^2\rho \;\ge\; \lambda_0 \int h^2\rho.$$

This is the inequality of Barta [14] in the form of a ground state transformation.

2. The computations of Lemma 679

The generator of the reversible diffusion of ν_b on S^3 is $L = \Delta_{S^3} + \kappa \nabla x_1 \cdot \nabla$. In the polar angle θ from e_1 the round metric is $d\theta^2 + \sin^2\theta\, g_{S^2}$, so $\Delta_{S^3} = \partial_\theta^2 + 2\cot\theta\,\partial_\theta + \sin^{-2}\theta\,\Delta_{S^2}$, and $\nabla x_1 \cdot \nabla = -\sin\theta\,\partial_\theta$. On $h(\theta)Y_\ell$ with a spherical harmonic Y_ℓ of degree ℓ on S^2 the operator acts as $L_\ell = \partial_\theta^2 + b\,\partial_\theta - \ell(\ell+1)\sin^{-2}\theta$, $b = 2\cot\theta - \kappa\sin\theta = \rho'/\rho$, so $-L_\ell = T$ with $V = \ell(\ell+1)\sin^{-2}\theta$.

Degree $\ell \geq 1$. Take $\psi = \sin\theta\, e^{a\kappa\cos\theta}$. Then $\psi'/\psi = \cot\theta - a\kappa\sin\theta$ and $\psi''/\psi = (\psi'/\psi)^2 + (\psi'/\psi)'$, and a short computation gives

$$\frac{-L_1\psi}{\psi} = 3 + (1+5a)\kappa\cos\theta - a(1+a)\kappa^2\sin^2\theta.$$

For $a = -\frac{1}{5}$ this is $3 + \frac{4}{25}\kappa^2\sin^2\theta \geq 3$. Since $V = \ell(\ell+1)\sin^{-2}\theta \geq 2\sin^{-2}\theta$ for $\ell \geq 1$, Barta's inequality gives $-L_\ell \geq 3$ on every sector with $\ell \geq 1$.

Degree $\ell = 0$. Let $g(\theta)$ be a zonal eigenfunction, $-L_0 g = \lambda g$, g not constant, so that $g'(0) = g'(\pi) = 0$. Differentiating $g'' + bg' = -\lambda g$ gives, for $h = g'$,

$$-\big(h'' + bh'\big) - b'h = \lambda h, \qquad -b' = \frac{2}{\sin^2\theta} + \kappa\cos\theta,$$

so h is an eigenfunction of T with $V = 2\sin^{-2}\theta + \kappa\cos\theta$, with eigenvalue λ, and $h \not\equiv 0$ vanishes at both ends. With the same ψ,

$$\frac{T\psi}{\psi} = 3 + (2+5a)\kappa\cos\theta - a(1+a)\kappa^2\sin^2\theta,$$

which for $a = -\frac{2}{5}$ is $3 + \frac{6}{25}\kappa^2\sin^2\theta \geq 3$. Barta's inequality applied to h gives $\lambda \geq 3$. Together with the case $\ell \geq 1$ every nonzero eigenvalue of $-L$ is at least 3, which is Lemma 679; the spectrum is discrete because S^3 is compact and the drift is smooth.

3. Moments

The density of $t = x_1$ under ν_b is proportional to $(1-t^2)^{1/2}e^{\kappa t}$. For every smooth ϕ on $[-1,1]$,

$$\frac{d}{dt}\Big[(1-t^2)^{3/2}e^{\kappa t}\Big] = (1-t^2)^{1/2}e^{\kappa t}\big(\kappa(1-t^2) - 3t\big),$$

and integration by parts gives $\mathbb{E}[\phi'(t)(1-t^2)] = \mathbb{E}\big[\phi(t)\big(3t - \kappa(1-t^2)\big)\big]$. With $\phi = 1$ this is $\kappa\mathbb{E}(1-x_1^2) = 3\mathbb{E}x_1$, the identity of Lemma 680(i). With $\phi = t^k$ it gives the recursion

$$m_{k+2} = m_k - \frac{k+3}{\kappa}\,m_{k+1} + \frac{k}{\kappa}\,m_{k-1}, \qquad m_k = \mathbb{E}x_1^k,$$

from $m_0 = 1$ and $m_1 = A$; all polynomial moments of ν_b are rational in κ and $A(\kappa)$.[54]

4. The Riccati barriers

By Lemma 680(i), A solves $A' = \mathscr{F}(\kappa, A) := 1 - A^2 - 3A/\kappa$ on $(0,\infty)$, and $A(\kappa) = \frac{\kappa}{4} - \frac{\kappa^3}{96} + O(\kappa^5)$ from the series of I_1 and I_2. Let $s = \sqrt{\kappa^2+4}$, $u = \sqrt{\kappa^2+9}$, $\phi_- = \kappa/(2+s)$ and $\phi_+ = \kappa/(1+u)$. Direct differentiation gives

$$\phi_-' - \mathscr{F}(\kappa,\phi_-) = -\frac{\kappa^2}{s(s+2)^2} \leq 0,$$
$$\phi_+' - \mathscr{F}(\kappa,\phi_+) = \frac{(u-3)^2}{u(1+u)^2} \geq 0,$$

and $\phi_\pm = \frac{\kappa}{4} + O(\kappa^3)$.[55] The comparison is singular at 0 because of the term $3A/\kappa$, and we carry it out with an integrating factor. Let $D = A - \phi_-$. Then

$$\begin{aligned} D' &= \mathscr{F}(\kappa,A) - \mathscr{F}(\kappa,\phi_-) - \big(\phi_-' - \mathscr{F}(\kappa,\phi_-)\big) \\ &= -\Big(A + \phi_- + \frac{3}{\kappa}\Big)D + R, \end{aligned}$$

with $R = \kappa^2/(s(s+2)^2) \geq 0$. Put $\mu(\kappa) = \kappa^3\exp\int_0^\kappa (A+\phi_-)$. Then $(\mu D)' = \mu R \geq 0$, and $\mu D = O(\kappa^6) \to 0$ as $\kappa \to 0$, so $\mu D \geq 0$ and $A \geq \phi_-$. The same argument with $D = \phi_+ - A$ and $R = (u-3)^2/(u(1+u)^2)$ gives $A \leq \phi_+$. This is Lemma 680(ii).

For (iv) let $s = \sqrt{\kappa^2 + 25/4}$ and $\phi_\flat = \kappa/(\frac{3}{2}+s)$. Since $s^2 - \frac{9}{4} = \kappa^2 + 4$, the same differentiation gives

$$\phi_\flat' - \mathscr{F}(\kappa,\phi_\flat) = -\frac{5\,(s-\frac{5}{2})}{2s\,(\frac{3}{2}+s)^2} \leq 0,$$

so $\phi_\flat$ is a subsolution, and $\phi_\flat = \frac{\kappa}{4} - \frac{\kappa^3}{80} + O(\kappa^5)$.[56] Hence $A - \phi_\flat = \frac{\kappa^3}{480} + O(\kappa^5)$, and the integrating-factor argument above, with $D = A - \phi_\flat$, $R = \frac{5(s-5/2)}{2s(3/2+s)^2} \geq 0$ and $\mu = \kappa^3\exp\int_0^\kappa (A+\phi_\flat)$, gives $\mu D = O(\kappa^6)$ at 0, $(\mu D)' = \mu R \geq 0$, and so $A \geq \phi_\flat$. Since $\frac{3}{2} + \sqrt{\kappa^2 + 25/4} < 2 + \sqrt{\kappa^2+4}$ for $\kappa > 0$, this improves the lower bound of (ii).

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[54] For instance $m_2 = 1 - 3A/\kappa$, $m_3 = A - 3/\kappa + 12A/\kappa^2$ and $m_4 = m_2 - 5m_3/\kappa + 2A/\kappa$. The joint moments with x_2, x_3, x_4 follow from exchangeability: $\mathbb{E}[\phi(x_1)x_2^2] = \frac{1}{3}\mathbb{E}[\phi(x_1)(1-x_1^2)]$.

[55] For the first identity write $\phi_- = (s-2)/\kappa$, which follows from $s^2 - 4 = \kappa^2$; then $\phi_-' = (\kappa^2/s - s + 2)/\kappa^2$ and $1 - \phi_-^2 - 3\phi_-/\kappa = (\kappa^2 - (s-2)^2 - 3(s-2))/\kappa^2$, and the difference simplifies with $s^2 = \kappa^2 + 4$. The second is the same computation with $\phi_+ = \kappa(u-1)/(u^2-1) = \kappa(u-1)/(\kappa^2+8)$ in place of ϕ_- and $u^2 = \kappa^2 + 9$ in place of $s^2 = \kappa^2 + 4$. After clearing the denominator $\kappa^2(1+u)^2$ the numerator becomes $\kappa^2(u-3)^2/u$, which is the stated residual.

[56] With $\phi_\flat = \kappa(s - \frac{3}{2})/(\kappa^2+4)$ one has $\phi_\flat' = (\frac{3}{2}s + \frac{25}{4})/(s(\frac{3}{2}+s)^2)$, using $ds/d\kappa = \kappa/s$; and $\mathscr{F}(\kappa,\phi_\flat) = 1 - \phi_\flat^2 - 3/(\frac{3}{2}+s)$. Clearing the denominator $s(\frac{3}{2}+s)^2$ and replacing κ^2 by $s^2 - \frac{25}{4}$ leaves $-\frac{5}{2}s + \frac{25}{4}$ in the numerator, which is the stated residual. The expansion follows from $s = \frac{5}{2} + \frac{\kappa^2}{5} + O(\kappa^4)$.

Appendix M: Bessel functions

This appendix proves the auxiliary statements used in Appendix K for Theorem 682. Throughout, $I_j = I_j(\kappa)$ with $\kappa > 0$, and $I_{-j} = I_j$ for integers j. We use the series

$$I_\nu(\kappa) = \sum_{i \ge 0} \frac{(\kappa/2)^{2i+\nu}}{i!\,(i+\nu)!},$$

the integral $I_j(\kappa) = \pi^{-1}\int_0^\pi e^{\kappa\cos\theta}\cos j\theta\,d\theta$, the Fourier series $e^{\kappa\cos\theta} = \sum_{j\in\mathbb{Z}} I_j e^{ij\theta}$ and the recurrences

$$I_{j-1} - I_{j+1} = \frac{2j}{\kappa} I_j, \quad I_j' = I_{j+1} + \frac{j}{\kappa} I_j, \tag{M1}$$

all of which are in Watson's treatise [114], §§3.7, 3.71, 2.22. All series below converge absolutely, because $0 < I_j \le (\kappa/2)^j e^\kappa / j!$.

1. Zonal harmonics

Under σ the coordinate x_1 has the density $\frac{2}{\pi}(1-t^2)^{1/2}$ on $[-1,1]$, the area of the two-sphere $\{x_1 = t\}$ being proportional to $1-t^2$ and its normal speed to $(1-t^2)^{-1/2}$. The Chebyshev polynomials $U_n(\cos\theta) = \sin((n+1)\theta)/\sin\theta$ satisfy

$$\int U_n U_m\,d\sigma = \frac{2}{\pi}\int_0^\pi \sin((n+1)\theta)\sin((m+1)\theta)\,d\theta = \delta_{nm},$$

and $U_n = C_n^{(1)}$ is the Gegenbauer polynomial of index $\frac{4-2}{2} = 1$, so $U_n(\langle y,x\rangle)$ is the zonal harmonic of degree n on S^3 with pole y: $|x|^n U_n(\langle y,x\rangle/|x|)$ is a harmonic polynomial on $\mathbb{R}^4$ [110], Ch. IV, §2.

Lemma 688. *$e^{\kappa t} = \frac{2}{\kappa}\sum_{n\ge0}(n+1)I_{n+1}(\kappa)\,U_n(t)$ in $L^2\big(\frac{2}{\pi}(1-t^2)^{1/2}dt\big)$, and uniformly on $[-1,1]$.*

Proof. With $t = \cos\theta$ and $2\sin\theta\sin((n+1)\theta) = \cos n\theta - \cos(n+2)\theta$, the n-th coefficient is

$$\frac{2}{\pi}\int_0^\pi e^{\kappa\cos\theta}\sin((n+1)\theta)\sin\theta\,d\theta = I_n - I_{n+2} = \frac{2(n+1)}{\kappa} I_{n+1},$$

by (M1). Since $|U_n| \le n+1$ and I_{n+1} decays factorially, the series converges uniformly. □

2. Proof of Lemma 685

Let $F(\theta) = e^{\kappa\cos\theta} = \sum_j I_j e^{ij\theta}$. Then

$$F'(\theta) = -\kappa\sin\theta\,F(\theta) = \sum_j ijI_j e^{ij\theta},$$

and Parseval's formula gives

$$\sum_{j\in\mathbb{Z}} j^2 I_j^2 = \frac{\kappa^2}{2\pi}\int_0^{2\pi}\sin^2\theta\,e^{2\kappa\cos\theta}d\theta = \frac{\kappa^2}{2}\big(I_0(2\kappa) - I_2(2\kappa)\big) = \frac{\kappa}{2} I_1(2\kappa),$$

the last step by (M1) at the argument 2κ. Halving gives the first identity. The coefficient of $e^{i\theta}$ in $F^2 = e^{2\kappa\cos\theta}$ is $I_1(2\kappa) = \sum_{j\in\mathbb{Z}} I_j I_{1-j}$, and the terms with $j \le 0$ repeat those with $j \ge 1$; this is the third identity.

For the second, $jI_j = \frac{\kappa}{2}(I_{j-1} - I_{j+1})$ by (M1), so

$$\sum_{j\ge1} j^3 I_j^2 = \frac{\kappa^2}{4}\sum_{j\ge1} j\big(I_{j-1}^2 + I_{j+1}^2 - 2I_{j-1}I_{j+1}\big).$$

Shifting indices, $\sum_{j\ge1} j(I_{j-1}^2 + I_{j+1}^2) = I_0^2 + \sum_{m\ge1} 2mI_m^2$, and $\sum_{m\ge1} 2mI_m^2 = \kappa\sum_{m\ge1}(I_{m-1}I_m - I_mI_{m+1}) = \kappa I_0 I_1$ by telescoping. For the mixed term write $j = \frac{1}{2}((j-1)+(j+1))$ and apply (M1) to $(j-1)I_{j-1}$ and to $(j+1)I_{j+1}$, using $I_{-1} = I_1$ at $j = 1$:

$$jI_{j-1}I_{j+1} = \frac{\kappa}{4}\big[(I_{j-1}I_j - I_jI_{j+1}) + (I_{j-2}I_{j+1} - I_{j-1}I_{j+2})\big].$$

Both brackets telescope, to I_0I_1 and to $I_{-1}I_2 = I_1I_2$, so $\sum_{j\ge1} jI_{j-1}I_{j+1} = \frac{\kappa}{4}(I_0I_1 + I_1I_2)$. Altogether

$$\sum_{j\ge1} j^3 I_j^2 = \frac{\kappa^2}{4}\Big(I_0^2 + \frac{\kappa}{2} I_1(I_0 - I_2)\Big) = \frac{\kappa^2}{4}\big(I_0^2 + I_1^2\big).$$ □

3. The coefficients of $\mathscr{D}$

Lemma 689. *Write $\mathscr{D}(\kappa) = \sum_{n\ge0}\mathscr{D}_n\kappa^{2n}$ for the function of (K9). Then $\mathscr{D}_0 = \mathscr{D}_1 = 0$, and for $m \ge 1$*

$$\mathscr{D}_{m+1} = \frac{N_m}{4^m\,m!\,(m+1)!\,(m+1)(m+2)},$$

with $N_m = \binom{2m}{m}\big(4(2m+1) + (m-1)(m+1)(m+2)\big) - 4^m(m+1)(m+2)$. Moreover $N_m = 0$ for $m \le 2$ and $N_m > 0$ for $m \ge 3$.

Proof. The coefficients. Multiplying the series of I_0 and I_1 and collecting powers,

$$I_0^2 = \sum_{m\ge0}\alpha_m\kappa^{2m}, \qquad I_1^2 = \sum_{m\ge0}\beta_m\kappa^{2m+2},$$

$$\kappa I_1(2\kappa) = \sum_{m\ge0}\frac{\kappa^{2m+2}}{m!\,(m+1)!},$$

where

$$\alpha_m = \frac{1}{4^m}\sum_{i=0}^{m}\frac{1}{i!^2(m-i)!^2},$$

$$\beta_m = \frac{1}{4^{m+1}}\sum_{i=0}^{m}\frac{1}{i!\,(i+1)!\,(m-i)!\,(m-i+1)!}.$$

By Vandermonde's identity, $\sum_i \binom{m}{i}^2 = \binom{2m}{m}$ and $\sum_i \binom{m}{i}\binom{m+2}{i+1} = \binom{2m+2}{m+1}$, so

$$\alpha_m = \frac{\binom{2m}{m}}{4^m\, m!^2}, \qquad \beta_m = \frac{\binom{2m+2}{m+1}}{4^{m+1} m!\,(m+2)!}.$$

The coefficient of κ^{2m+2} in $\mathscr{D}$ is $\mathscr{D}_{m+1} = 8\beta_m + 2\beta_{m-1} - \alpha_m - 1/(m!(m+1)!)$, with $\beta_{-1} = 0$; for $m = 0$ it is $\frac{8}{4} - 1 - 1 = 0$, and $\mathscr{D}_0 = 0$ because every term of $\mathscr{D}$ is $O(\kappa^2)$. For $m \ge 1$ put $\Delta = 4^m\, m!\,(m+1)!\,(m+1)(m+2)$ and $b = \binom{2m}{m}$. With $\binom{2m+2}{m+1} = \frac{2(2m+1)}{m+1} b$,

$$8\beta_m = \frac{4(2m+1)\,b}{\Delta}, \qquad \alpha_m = \frac{(m+1)^2(m+2)\,b}{\Delta},$$
$$2\beta_{m-1} = \frac{2m(m+1)(m+2)\,b}{\Delta},$$
$$\frac{1}{m!(m+1)!} = \frac{4^m(m+1)(m+2)}{\Delta},$$

and $2m - (m+1) = m-1$ gives $\mathscr{D}_{m+1} = N_m/\Delta$.

The sign. Directly, $N_0 = N_1 = N_2 = 0$ ($2-2$, $24-24$, $192-192$), $N_3 = 80$, $N_4 = 1140$ and $N_5 = 10416$. For $m \ge 6$ we first show $16^m \le 4m\binom{2m}{m}^2$ for $m \ge 1$ by induction: it is an equality at $m = 1$, and $(m+1)\binom{2m+2}{m+1} = 2(2m+1)\binom{2m}{m}$ gives

$$4(m+1)\binom{2m+2}{m+1}^2 = \frac{16(2m+1)^2}{m+1}\binom{2m}{m}^2 \ge \frac{16(2m+1)^2}{4m(m+1)}16^m \ge 16^{m+1}.$$

Since $(m-1)^2 \ge 4m$ for $m \ge 6$, it follows that $\binom{2m}{m}^2(m-1)^2 \ge 16^m$, that is $\binom{2m}{m}(m-1) \ge 4^m$, and

$$N_m = \left(\binom{2m}{m}(m-1) - 4^m\right)(m+1)(m+2) + 4(2m+1)\binom{2m}{m} > 0.$$

□

4. Monotonicity of Bessel ratios

Lemma 690. *For $n \ge 1$ and $\kappa > 0$, $I_{n+2}(\kappa)/I_{n+1}(\kappa) \le I_2(\kappa)/I_1(\kappa)$.*

Proof. We use a lemma of Biernacki and Krzyż [27]: if $f = \sum a_j x^j$ and $g = \sum b_j x^j$ have positive coefficients and a_j/b_j is nonincreasing in j, then f/g is nonincreasing on $x > 0$. Indeed

$$f'g - fg' = \sum_{i,j}(i-j)a_i b_j x^{i+j-1} = \frac{1}{2}\sum_{i,j}(i-j)b_i b_j\left(\frac{a_i}{b_i} - \frac{a_j}{b_j}\right)x^{i+j-1} \le 0.$$

Take $x = \kappa^2$, $f = I_{n+1}(\kappa)/\kappa^{n+1}$ and $g = I_1(\kappa)/\kappa$. Then $a_j/b_j = 2^{-n}(j+1)!/(j+n+1)!$ decreases in j, so $I_{n+1}/(\kappa^n I_1)$ is nonincreasing in κ. Its logarithmic derivative is, by the second recurrence of (M1),

$$\frac{n+1}{\kappa} + \frac{I_{n+2}}{I_{n+1}} - \frac{n}{\kappa} - \frac{1}{\kappa} - \frac{I_2}{I_1} = \frac{I_{n+2}}{I_{n+1}} - \frac{I_2}{I_1} \le 0.$$

□

Appendix N: The certificate bound

This appendix reduces a bound on h_F to finitely many positivity statements, by hand, and reports the computation of [102] that verifies them for $h_F \le 14.420001$ (Remark 26); no result of the paper uses that value. The argument is the proof of the bound in Theorem 4 with two changes. The divergence identity of Lemma 19, taken there with the factor $\frac{9}{10}$, is replaced by a positive semidefinite form at each site, and the weights $\frac{21}{10}$ and $\frac{5}{4}$ of Lemma 20 are replaced by weights chosen plaquette by plaquette. The reduction is proved by hand; the positivity of ninety-six matrices of trigonometric polynomials and the arithmetic of finitely many rational numbers are checked by computer.[57]

1. Ends and corners

At a site y the eight *ends* are

$$o_\mu(y) = X_{y,\mu}, \qquad i_\mu(y) = Q^*_{y-e_\mu,\mu} X_{y-e_\mu,\mu} Q_{y-e_\mu,\mu},$$

for $\mu = 0,1,2,3$, as in the proof of Lemma 19; the end of a link in F or outside the box is 0. Each link $e = (y,\mu)$ has the end o_μ at its tail y and the end i_μ at its head $y + e_\mu$, both of norm $|X_e|$.

Lemma 691. *Let p be the plaquette at x in the plane $\mu < \nu$. Then*

$$\langle Y_1, Y_4\rangle = \langle o_\mu, o_\nu\rangle(x),$$
$$\langle Y_1, Y_2\rangle = \langle i_\mu, o_\nu\rangle(x+e_\mu),$$
$$\langle Y_3, Y_4\rangle = \langle o_\mu, i_\nu\rangle(x+e_\nu),$$
$$\langle Y_2, W Y_3 W^*\rangle = \langle i_\nu, i_\mu\rangle(x+e_\mu+e_\nu).$$

Every pair of ends at one site in two different directions occurs in exactly one of these identities, for exactly one plaquette.

Proof. The first identity is the definition of the ends. The inner product is invariant under conjugation by a unitary matrix, so $\langle X_{e_1}, Q_{e_1} X_{e_2} Q^*_{e_1}\rangle = \langle Q^*_{e_1} X_{e_1} Q_{e_1}, X_{e_2}\rangle$, which is the second identity, and the third follows in the same way

[57] The constant 14.420001 is not the best possible for this kind of certificate: by the search reported in Remark 26 it cannot go below 14.384, while the true value of h_F is at least 13.123.

from $\langle Q_{e_4} X_{e_3} Q_{e_4}^*, X_{e_4}\rangle = \langle X_{e_3}, Q_{e_4}^* X_{e_4} Q_{e_4}\rangle$. For the fourth, $WQ_{e_4} = Q_{e_1}Q_{e_2}Q_{e_3}^*$, so that $WY_3W^* = RX_{e_3}R^*$ with $R = Q_{e_1}Q_{e_2}Q_{e_3}^*$, and conjugating both vectors by $(Q_{e_1}Q_{e_2})^*$ gives $\langle Q_{e_2}^* X_{e_2} Q_{e_2}, Q_{e_3}^* X_{e_3} Q_{e_3}\rangle$. The links e_2 and e_3 both have their head at $x+e_\mu+e_\nu$. For the last statement, the pairs (o_μ, o_ν), (i_μ, o_ν), (o_μ, i_ν) and (i_μ, i_ν) at y come from the plaquettes in the plane $\mu\nu$ at y, $y-e_\mu$, $y-e_\nu$ and $y-e_\mu-e_\nu$ respectively. There are $4\binom{4}{2} = 24$ such pairs at each site and $6\cdot 4 = 24$ corners of plaquettes per site. □

For each class of sites modulo 2 let G_y be a real symmetric 8×8 matrix indexed by the ends, with $G_y[o_\mu, i_\mu] = 0$ for every μ. For the plaquette p at x in the plane $\mu < \nu$ put

$$\begin{aligned} c_p^G = {} & 2G_x[o_\mu, o_\nu]\langle Y_1, Y_4\rangle + 2G_{x+e_\mu}[i_\mu, o_\nu]\langle Y_1, Y_2\rangle \\ & + 2G_{x+e_\nu}[o_\mu, i_\nu]\langle Y_3, Y_4\rangle \\ & + 2G_{x+e_\mu+e_\nu}[i_\mu, i_\nu]\langle Y_2, WY_3W^*\rangle. \end{aligned}$$

The matrix G^0 with $G^0[a,a] = 2$, $G^0[o_\mu, o_\nu] = G^0[i_\mu, i_\nu] = 1$ and $G^0[o_\mu, i_\nu] = -1$ for $\mu \neq \nu$ is the matrix of the form $|\sum_\mu (o_\mu - i_\mu)|^2 + \sum_\mu |o_\mu + i_\mu|^2$ of Lemma 19, and $c_p^{G^0}$ is the corner form c_p of (7).

Lemma 692. *On $\mathbb{Z}^4$, or on a box with free boundary, for every field Q and every square-summable X,*

$$\begin{aligned} & \sum_y \sum_{a,b} G_y[a,b]\,\langle \mathrm{end}_a, \mathrm{end}_b\rangle(y) \\ & \quad = \sum_p c_p^G + \sum_e \big(G_{t(e)}[o,o] + G_{h(e)}[i,i]\big)\,|X_e|^2, \end{aligned}$$

where $t(e)$ and $h(e)$ are the tail and the head of e, and o, i are its two ends. If every G_y is positive semidefinite, the left side is nonnegative.

Proof. Split the sum over a,b at each site into the diagonal terms, the pairs (o_μ, i_μ), whose coefficient is 0, and the pairs of ends in different directions, each counted twice. The diagonal terms give the last sum, since each link has one end at its tail and one at its head, and the pairs in different directions give $\sum_p c_p^G$ by Lemma 691. In a box the argument at the end of the proof of Lemma 19 applies. At each site the left side is the trace of G_y times the Gram matrix of the eight ends, and the trace of a product of two positive semidefinite matrices is nonnegative. □

2. From plaquettes to the Hessian

Number the positions of a plaquette as in (5) and let k_p be the position of its link in F; by Lemma 9 there is exactly one.

Lemma 693. *Let every G_y be positive semidefinite with $G_y[o_\mu, i_\mu] = 0$, and let $a_{p,r} \geq 0$ for the three positions $r \neq k_p$ of each plaquette. Suppose that for every plaquette p, every N, every $W \in \mathrm{U}(N)$ and all $Y_1, \dots, Y_4 \in \mathfrak{u}(N)$ with $Y_{k_p} = 0$,*

$$f_p + c_p^G \leq \sum_{r \neq k_p} a_{p,r} |Y_r|^2. \tag{N1}$$

Then $|\operatorname{Hess} S_0(v,v)| \leq h|v|^2$ for every field Q and every $v = (X_e Q_e)$ with $X_e = 0$ for $e \in F$, on $\mathbb{Z}^4$ and on every box, where

$$h = \max_{e \notin F} \Big(\sum_{p \ni e} a_{p,e} + G_{t(e)}[o,o] + G_{h(e)}[i,i] \Big)$$

and $a_{p,e}$ is the weight of the position of e in p.

Proof. Summing (N1) over the plaquettes and adding the identity of Lemma 692, whose left side is nonnegative,

$$\begin{aligned} \operatorname{Hess} S_0(v,v) = \sum_p f_p &\leq \sum_p \Big(\sum_r a_{p,r} |Y_r|^2 - c_p^G \Big) \\ &\leq \sum_{e \notin F} \Big(\sum_{p \ni e} a_{p,e} + G_{t(e)}[o,o] + G_{h(e)}[i,i] \Big) |X_e|^2 \\ &\leq h|v|^2, \end{aligned}$$

since $|Y_r| = |X_e|$ for the link e at position r. In a box the links near the boundary lie in fewer plaquettes, and the weights are nonnegative. For the lower bound, apply (N1) at $-W$. By (6) this changes the sign of f_p, while c_p^G does not change, and the same chain gives $-\operatorname{Hess} S_0(v,v) \leq h|v|^2$. □

The hypothesis is needed on $\mathrm{U}(N)$ and not only on $\mathrm{SU}(N)$, since $-W \notin \mathrm{SU}(N)$ when N is odd.

Lemma 694. *If (N1) holds for $N = 2$ and every $W = \operatorname{diag}(e^{i\varphi_1}, e^{i\varphi_2})$, it holds for every N and every $W \in \mathrm{U}(N)$.*

Proof. Both sides are invariant under conjugation of W and all Y_i by one unitary matrix, so we may take $W = \operatorname{diag}(w_1, \dots, w_N)$. Every term of f_p, of c_p^G and of $|Y_r|^2$ is a multiple of $\operatorname{Re}\operatorname{Tr}(Y_i D_1 Y_j D_2)$ with $D_1, D_2 \in \{I, W, W^*\}$, and

$$\operatorname{Re}\operatorname{Tr}(Y_i D_1 Y_j D_2) = \operatorname{Re} \sum_{a,b} (Y_i)_{ab}\,(D_1)_{bb}\,(Y_j)_{ba}\,(D_2)_{aa}.$$

The difference of the two sides of (N1) is therefore a sum of one term for each a, in the entries $(Y_i)_{aa}$, and one term for each pair $a < b$, in the entries $(Y_i)_{ab}$ and $(Y_i)_{ba} = -\overline{(Y_i)_{ab}}$, as in the proof of Lemma 20. The term of the pair $a < b$ is the same difference for $N = 2$ at $W' = \operatorname{diag}(w_a, w_b)$, evaluated on the matrices $Y_i' \in \mathfrak{u}(2)$ with off-diagonal entries $(Y_i)_{ab}$, $(Y_i)_{ba}$ and zero diagonal. The term of a is the difference at W', for any $b \neq a$, evaluated on $Y_i' = \operatorname{diag}((Y_i)_{aa}, 0)$. In both cases $Y_{k_p}' = 0$, so each term is nonnegative. □

In the orthonormal basis $iI/\sqrt{2}$, $i\sigma_1/\sqrt{2}$, $i\sigma_2/\sqrt{2}$, $i\sigma_3/\sqrt{2}$ of $\mathfrak{u}(2)$, with the Pauli matrices σ_j, the difference of the two sides of (N1) at $N = 2$ is a quadratic form on the three free positions, with a real symmetric 12×12 matrix $M_p(\varphi)$. Its entries are affine combinations of 1, $\cos\varphi_a$, $\sin\varphi_a$, $\cos(\varphi_1 - \varphi_2)$ and $\sin(\varphi_1 - \varphi_2)$: the form f_p is linear in W, and W enters c_p^G only through $\langle Y_2, WY_3W^*\rangle$. Since W commutes with I and σ_3, $M_p(\varphi)$ splits into two blocks of size six, one on the span of $iI, i\sigma_3$ and one on the span of $i\sigma_1, i\sigma_2$, at every position. By Lemmas 693 and 694, a certificate is a choice of G_y and $a_{p,r}$ such that every $M_p(\varphi)$ is positive semidefinite at every φ, and it gives $h_F \leq h$.

3. The certificate

The fixed set F and the field of ends have period two, so it suffices to give G_y for the sixteen classes of sites and $a_{p,r}$ for the ninety-six classes of plaquettes, six planes at each of sixteen sites. The certificate of [102] consists of sixteen positive definite matrices G_y of size eight with $G_y[o_\mu, i_\mu] = 0$, and of nonnegative weights $a_{p,r}$, all rational with denominator 10^6. We checked that the set of fixed links used there is our F up to a symmetry of the lattice. Of the sixty-four links of a period cell, sixteen lie in F, one ending at each site. For each of the other forty-eight the bracket in the definition of h is computed exactly, and the largest is $14420001/10^6$.

Remark 695 (a computation, used by no theorem). The certificate of [102] comes with, for each of the 192 blocks of size six, an integer symmetric matrix P of size thirty, stated to be positive definite and to satisfy, with $z = (1, \cos\varphi_1, \sin\varphi_1, \cos\varphi_2, \sin\varphi_2)$,

$$2^{24} \cdot 4 \cdot 10^6 M(\varphi)_{jk} = \sum_{a,b=0}^{4} z_a z_b P_{6a+j,\,6b+k}$$

for all j,k, as polynomials modulo $\cos^2\varphi_a + \sin^2\varphi_a = 1$. Granted these finite statements, for $x \in \mathbb{R}^6$ and $w = z \otimes x$ one has $2^{24} \cdot 4 \cdot 10^6 x^{\mathsf{T}} M(\varphi) x = w^{\mathsf{T}} P w$, which is positive for $x \neq 0$, since $z_0 = 1$ gives $w \neq 0$; so every block is positive definite at every φ, and Lemmas 693 and 694 give, for every N, every field Q and every $v = (X_e Q_e)$ with $X_e = 0$ for $e \in F$, on $\mathbb{Z}^4$ and on every box,

$$|\operatorname{Hess} S_0(v,v)| \leq 14.420001\,|v|^2.$$

The finite statements are checked by computer, not by hand, so this bound is a computation; no result of the paper uses it.

The bound of Remark 695 rests on computations of three kinds. The positivity of the G_y, the signs of the weights and the forty-eight totals are finite statements about rational numbers. The identities between $M(\varphi)$ and the matrices P, and the positive definiteness of the P, are finite statements about integers. All of these are checked by the Lean 4 kernel in [102], on no axioms beyond `propext` and `Quot.sound`; the program there also checks the positivity at every phase a second time, by interval arithmetic on a subdivision of the torus of phases. Lemmas 691 to 694 and the formula (6) for the Hessian are the written proofs above and are not formalised.[58] We checked the finite data in exact arithmetic and the positivity at every phase by eigenvalues on an adaptive grid with a Lipschitz bound, in floating point, and J. M. Lockwood checked it again in exact rational arithmetic with outward-rounded interval elimination, finding least eigenvalue above $559/716800$ for every plaquette.

With 14.420001 in place of $\frac{29}{2}$ the proofs of Theorems 4 and 5 would go through unchanged and give the ranges stated in Remark 26. Those theorems are stated with $\frac{29}{2}$, whose proof uses no computer.

[58] Formalising them needs matrices over $\mathbb{C}$, the group $\mathrm{U}(N)$ and second derivatives along geodesics, that is, a library such as Mathlib; the Lean files of [102] use only the core library.

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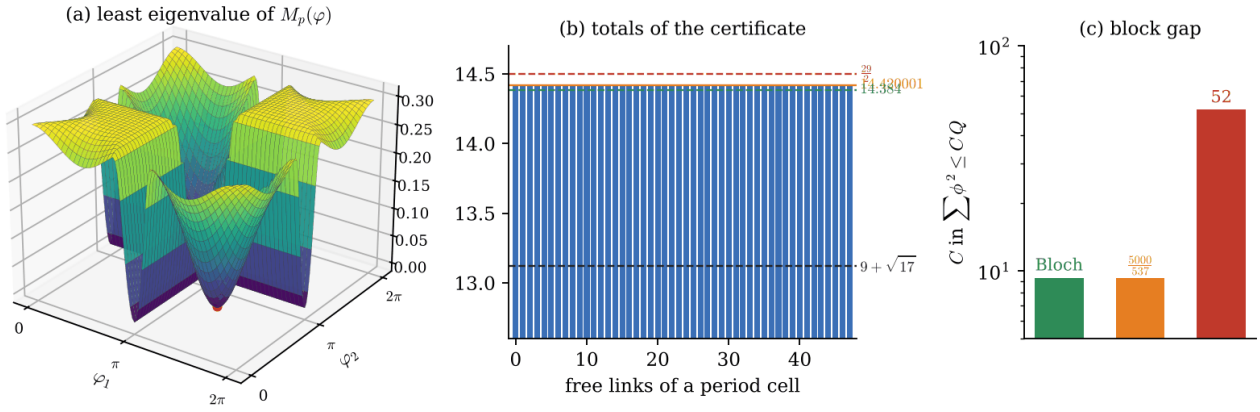


FIG. 47. The certificate of Appendix N. (a) The least eigenvalue of $M_p(\varphi)$ over the torus of phases for the plaquette class with the smallest minimum, about 0.0057 on a grid of 96^2 phases (red dot); the exact computation of Remark 695 finds it positive at every phase. (b) The forty-eight totals in the definition of h, in increasing order, against $\frac{29}{2}$, the value 14.420001 of the largest, the floor 14.384 for certificates on the period-two cell, and the lower bound $9+\sqrt{17}$ for h_F. (c) The constant C in $\sum\phi^2 \le CQ$ for the block gauge: 52 from Theorem 330, $5000/537$ from the computation of Remark 331, and the bulk Bloch value $1/0.10749 = 9.3034$, the best constant when W is the set of all blocks. The eigenvalues in (a) and the Bloch value are in floating point.

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