

# MASS GAP OF FOUR-DIMENSIONAL LATTICE YANG–MILLS THEORY IN A FOREST GAUGE

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**ABSTRACT.** *We fix the gauge of four-dimensional lattice Yang–Mills theory on a field of straight lines that meets every plaquette in exactly one link, so that each free link interacts with twelve others instead of eighteen. In this gauge the theory has a unique infinite-volume state with exponential decay of correlations and a positive mass gap at fixed lattice spacing, for  $SU(2)$  beyond the limit of every single-link argument in a gauge of this kind, by blocks made of the three free links of a plaquette, for every  $SU(N)$  beyond the curvature limit of the ungauged theory, and for  $SU(3)$  on a range more than two and a half times as large as was known before, where the same blocks raise the gap at the end of that range by two orders of magnitude and carry the range past it to  $\beta_W = 0.76$ .*

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## 1. INTRODUCTION

Wilson’s lattice gauge theory [34] assigns an element  $Q_e$  of a compact group  $G \subset U(N)$  to every link  $e$  of  $\mathbb{Z}^4$  and weighs a configuration by  $\exp(N\beta \sum_p \operatorname{Re} \operatorname{Tr} Q_p)$ , where  $Q_p$  is the ordered product of the four link variables around the plaquette  $p$ .<sup>1</sup> At small  $|\beta|$  the theory

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<sup>1</sup>This is the normalisation of Shen, Zhu and Zhu [31] and of our earlier paper [9]. The conventional inverse coupling is  $\beta_W = N^2\beta$ , so the weight is  $\exp(\frac{\beta_W}{N} \sum_p \operatorname{Re} \operatorname{Tr} Q_p)$ ; for  $SU(2)$ , identified with the unit quaternions,  $\frac{1}{2} \operatorname{Tr} Q_p = \operatorname{Re} Q_p$  and  $\beta_W = 4\beta$ . Standard accounts of the lattice theory are the books of Montvay and Münster [26] and of Gattringer and Lang [18]; the Hamiltonian version is due to Kogut and Susskind [25].

is under control: it has a unique infinite-volume state, correlations decay exponentially, and the transfer matrix has a gap [29, 30]. The question is how far into the coupling range such statements can be proved with explicit constants, since the problem of the continuum limit [22] begins where strong coupling ends and Balaban's weak-coupling analysis [3, 4, 5] takes over.

Two methods give explicit ranges. The Bakry–Émery method [1, 2] of Shen, Zhu and Zhu [31] bounds the Hessian of the action; in [9] we showed that the sharp bound is  $4d = 16$  and that no curvature argument for the ungauged measure goes beyond  $|\beta| < 1/32$ . The Dobrushin method [14, 15] bounds the influence of each link on the conditional law of another; for  $SU(2)$  the single-link conditional law is a von Mises–Fisher law on the three-sphere, the sharp Kantorovich modulus of that family is one quarter [9, Thm. 7], and every link has eighteen partners, which gives  $\beta_W < 2/9$  and nothing beyond, because the coefficients are attained.

Both limits come from the same count. A link of  $\mathbb{Z}^4$  lies in six plaquettes and shares each with three other links. Gauge invariance makes part of this redundant: in a box, the variables on a forest of links can be set to the identity without changing any gauge-invariant expectation [13].<sup>2</sup> The forest we use is a field of straight lines with one line through every site, chosen so that every plaquette contains exactly one fixed link. Then every free link has exactly twelve free partners, two in each of its plaquettes (Lemma 2.2), and both methods improve.

**Theorem 1.1** ( $SU(2)$ ). *Let  $G = SU(2)$  and  $0 \leq \beta_W < \frac{1}{3}$ . For every gauge-invariant local observable  $O$  the expectations  $W_\Lambda(O)$  in boxes with free boundary converge, along every sequence of boxes that exhausts  $\mathbb{Z}^4$ , to a limit  $\omega(O)$  that does not depend on the sequence. The state  $\omega$  is invariant under translations and the hypercubic group and reflection positive; truncated correlations of Lipschitz observables decay at every rate below  $-\log(3\beta_W)$ ; and the transfer matrix has spectral gap*

$$m \geq -\log(3\beta_W) > 0.$$

For  $-\frac{1}{3} < \beta_W < 0$  all statements except reflection positivity and the gap hold.

The step behind Theorem 1.1 is short once the line field is in place.<sup>3</sup> The number  $\frac{1}{3}$  is also the end of the road for this kind of argument.

**Theorem 1.2** (Ceiling). *Let  $F'$  be any set of links of  $\mathbb{Z}^4$  without cycles, and give two distinct links outside  $F'$  that share a plaquette the coefficient  $|\beta|$ , the largest single-link Kantorovich coefficient of  $SU(2)$ . Then the weighted Dobrushin condition with these coefficients fails for  $\beta_W \geq \frac{1}{3}$ . For the line field the coefficients are attained, so no single-link Kantorovich argument in that gauge reaches  $\beta_W = \frac{1}{3}$ .*

<sup>2</sup>Creutz used a maximal tree to remove the gauge freedom completely; the temporal gauge, with all links in one direction fixed, is the special case of a family of straight lines. In a maximal tree almost every link is fixed in some region and none in others, so the number of partners of a free link is not uniform, and the temporal gauge leaves fourteen partners per link. The point here is a pattern that fixes exactly one link in every plaquette.

<sup>3</sup>The combination of a forest gauge fixing with the sharp quarter of [9] was first assembled during a run of Anaxia, the reasoning engine of the BLLM system [7], release v3.0.5. In that run the engine combines proof steps along typed links (a step that produces a single-link Kantorovich bound may only feed a step that consumes one), and in every run with typed crossing it reached a composite accepting a lattice mass gap for four-dimensional  $SU(2)$  at  $\beta_W < \frac{1}{3}$ , which no untyped or random run reached. The composite was first proposed by the language model inside the engine, and the engine's network reproduces it. The engine did not supply the line field, the free-boundary limit, the ceiling, the curvature bound or anything in the  $SU(3)$  part; the authors wrote and checked every proof in this paper. Release v3.1.0 of the engine [8] records its Yang–Mills runs in the failure log: the narrowing of the target from the continuum gap to explicit couplings in this gauge, the change from analytic one-link bounds to bounds certified box by box, and the change to blocks of several links. These records follow the course of the work and supply no step of the proofs.

The ceiling concerns coefficients between single links. Take instead as a block the three free links of one plaquette. The free links of the line field split into such triples with period two (Lemma 5.1), the coupling inside the block plaquette is no longer paid for, and each link keeps ten outside partners. The price is the sensitivity of a law on three spheres instead of one, which we bound analytically by integrating out the middle link of the block.

**Theorem 1.3** (SU(2) with plaquette blocks). *Let  $G = \text{SU}(2)$  and  $0 \leq \beta_W \leq 0.345$ . The number  $\alpha(\beta_W)$  of (5.3) is increasing,  $\alpha(\frac{1}{3}) \leq 0.9561$  and  $\alpha(0.345) \leq 0.9973$ . The conclusions of Theorem 1.1 hold with decay at every rate below  $-\frac{1}{2} \log \alpha(\beta_W)$  and a transfer-matrix gap*

$$m \geq -\frac{1}{2} \log \alpha(\beta_W) > 0;$$

*at  $\beta_W = \frac{1}{3}$  this gives  $m \geq 0.0224$ . For  $-0.345 \leq \beta_W < 0$  all statements except reflection positivity and the gap hold.*

At small coupling the rate of Theorem 1.1 is the better one, so both bounds stand side by side; Theorem 1.3 matters past  $\beta_W = \frac{1}{3}$ , where the gap of Theorem 1.1 has closed.

The curvature method gains as well, and for every  $N$ . Let  $\text{Hess } S_0$  be the Hessian of  $\sum_p \text{Re Tr } Q_p$  on  $G^E$  and  $F$  the set of fixed links.

**Theorem 1.4** (Curvature in the gauge). *For every  $N$ , every field  $Q$  and every tangent vector  $v = (X_e Q_e)$  with  $X_e = 0$  for  $e \in F$ ,*

$$|\text{Hess } S_0(v, v)| \leq \frac{29}{2} |v|^2,$$

*against the sharp constant 16 without gauge fixing. Consequently, for  $G = \text{SU}(N)$  and  $|\beta| < \frac{1}{29}$ , or  $G = \text{SO}(N)$ ,  $N \geq 3$ , and  $|\beta| < \frac{N-2}{58N}$ , the conclusions of Theorem 1.1 hold with some rate  $c > 0$  in place of  $-\log(3\beta_W)$ . For  $\text{SU}(3)$  this is  $\beta_W < \frac{9}{29} = 0.310$ .*

No Bakry–Émery argument for the ungauged measure can pass  $|\beta| = \frac{1}{32}$  [9, Thm. C]; the gauge fixing changes the measure, not the metric, which is why it can. For  $\text{SU}(2)$  the curvature range  $\beta_W < \frac{4}{29}$  lies inside that of Theorem 1.1. For  $\text{SU}(3)$  the Dobrushin route goes much further.

**Theorem 1.5** (SU(3)). *Let  $G = \text{SU}(3)$  and  $0 \leq \beta_W \leq \frac{3}{4}$ . Then the conclusions of Theorem 1.1 hold, with decay at every rate below  $-\log(1.33319 \beta_W)$  and a transfer-matrix gap*

$$m \geq -\log(1.33319 \beta_W) > 0.$$

The proof of Theorem 1.5 bounds the Kantorovich modulus of the one-link measure  $\nu_b(dU) \propto e^{\text{Re Tr}(U^b)} dU$  on  $\text{SU}(3)$  uniformly over staple sums  $b$ . Its analytic part is a lower bound for the spectral gap of  $\nu_b$  through Bochner’s formula with a cross term (Lemma 7.3), a corrector chosen box by box among polynomials of degree two in the matrix entries (Lemma 7.4), the reduction of  $b$  to diagonal matrices (Lemma 7.5), the character expansion of the moments of  $\nu_b$  (Lemma 7.6) and an interpolation inequality on boxes of staple sums (Lemma 7.7). The remaining step evaluates the resulting bound on 2341 boxes with rigorous error bounds; the theorem rests on that computation, which is described in full in Section 7.7 and Appendix E.

The plaquette blocks of Theorem 1.3 work for  $\text{SU}(3)$  as well. They use the same modulus of the one-link measure, but only through the three links of a block, so they need  $\Lambda(2\beta_W)$  small and not  $2r\Lambda(r) < 1$ : near the end of Theorem 1.5 they give a much larger gap, and they go past it.

**Theorem 1.6** (SU(3) with plaquette blocks). *Let  $G = \text{SU}(3)$  and  $0 \leq \beta_W \leq \frac{19}{25} = 0.76$ . Then  $\alpha_3(\beta_W) < 1$  for the number of (8.2), with  $\alpha_3(0.72) \leq 0.9182$ ,  $\alpha_3(\frac{3}{4}) \leq 0.9653$  and  $\alpha_3(0.76) \leq 0.99950$ . The conclusions of Theorem 1.1 hold with decay at every rate below  $-\frac{1}{2} \log \alpha_3(\beta_W)$  and a transfer-matrix gap*

$$m \geq -\frac{1}{2} \log \alpha_3(\beta_W) > 0;$$

TABLE 1. Explicit ranges of uniqueness, decay and a transfer-matrix gap for four-dimensional lattice Yang–Mills theory. The first three rows hold for the ungauged measure; the last three are the results of this paper.

| Method                         | SU(2), $\beta_W$ | SU(3), $\beta_W$   |
|--------------------------------|------------------|--------------------|
| Curvature, Shen–Zhu–Zhu [31]   | $< \frac{1}{12}$ | $< \frac{3}{16}$   |
| Curvature, sharp [9]           | $< \frac{1}{8}$  | $< \frac{9}{32}$   |
| Dobrushin, sharp quarter [9]   | $< \frac{9}{29}$ | $< \frac{9}{29}$   |
| Curvature, forest gauge        | $< \frac{4}{29}$ | $< \frac{9}{29}$   |
| Dobrushin, forest gauge        | $< \frac{1}{3}$  | $\leq \frac{3}{4}$ |
| Plaquette blocks, forest gauge | $\leq 0.345$     | $\leq 0.76$        |

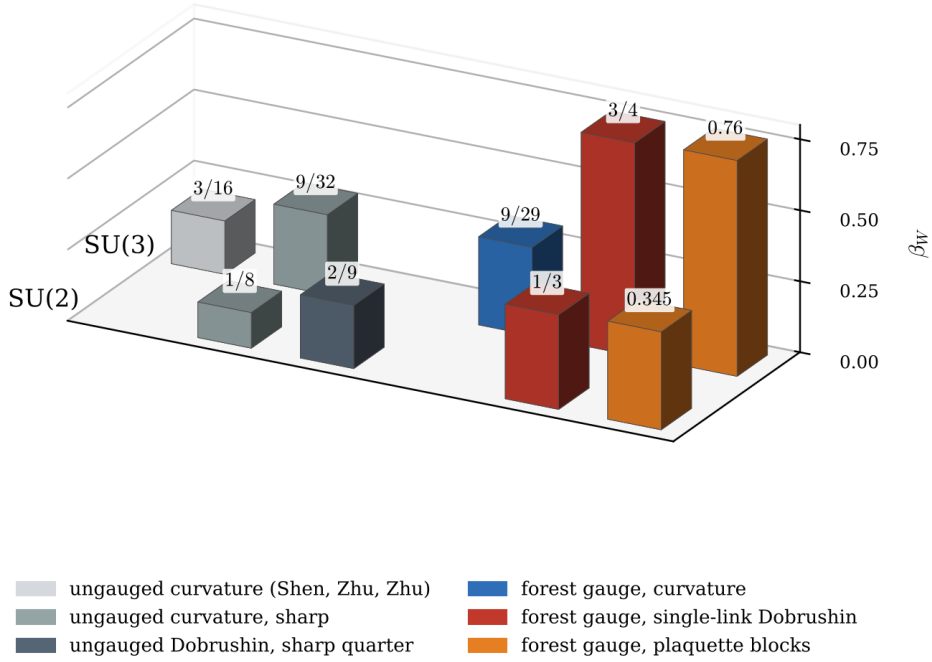


FIGURE 1. Ranges of Table 1. The height of each bar is the largest inverse coupling  $\beta_W$  covered by the method. Grey bars are methods for the ungauged measure; blue, red and orange bars use the forest gauge. For SU(2) the red bar ends at the ceiling of Theorem 1.2 and the orange bar, plaquette blocks, passes it (Theorem 1.3); for SU(3) the red bar is Theorem 1.5 and the orange bar, plaquette blocks, gives a larger gap there and passes it (Theorem 1.6).

at  $\beta_W = \frac{3}{4}$  this gives  $m \geq 0.0177$ , against  $1.1 \cdot 10^{-4}$  from Theorem 1.5, and at  $\beta_W = 0.76$ , where Theorem 1.5 no longer applies,  $m \geq 2.5 \cdot 10^{-4}$ .

All six theorems are statements at fixed lattice spacing. They do not touch the continuum limit, the crossover to Balaban’s regime, or a gap in physical units, which together are the Clay problem; Section 9 says where each method stops and what remains of the four open problems of [9]. The obstruction proved in [10], that a lattice gap uniform in lattice units forces trivial continuum two-point functions, applies to all of them.

Section 2 constructs the line field. Section 3 shows that a Dobrushin bound for the gauge-fixed specification gives the full conclusion of Theorem 1.1. Sections 4, 5, 6 7 and 8 prove Theorems 1.1–1.6, and Section 9 measures what is left. The appendices contain the

residue table of the line field, the comparison lemma and its version for blocks, the blocks of the local Hessian inequality and the details of the certified computation.

## 2. THE LINE FIELD

Write  $e_0, \dots, e_3$  for the unit vectors of  $\mathbb{Z}^4$  and  $(x, \mu)$  for the link from  $x$  to  $x + e_\mu$ . A *direction field* is a map  $\sigma : \mathbb{Z}^4 \rightarrow \{0, 1, 2, 3\}$ ; its links are  $F = \{(x, \sigma(x)) : x \in \mathbb{Z}^4\}$ . At each site put  $\pi_x(\mu) = \sigma(x + e_\mu)$ . The plaquette at  $x$  in the plane  $\{\mu, \nu\}$  has the links  $(x, \mu)$ ,  $(x + e_\mu, \nu)$ ,  $(x + e_\nu, \mu)$ ,  $(x, \nu)$ , so it contains

$$(2.1) \quad [\sigma(x) \in \{\mu, \nu\}] + [\pi_x(\mu) = \nu] + [\pi_x(\nu) = \mu]$$

links of  $F$ .

**Lemma 2.1.** *Every plaquette of  $\mathbb{Z}^4$  contains exactly one link of  $F$  if and only if, at every site  $x$ ,  $\pi_x$  fixes  $\sigma(x)$  and permutes the other three directions cyclically.*

*Proof.* Fix  $x$ , let  $s = \sigma(x)$  and let  $R$  be the set of the other three directions. For a plane  $\{s, \nu\}$  the count (2.1) is  $1 + [\pi_x(s) = \nu] + [\pi_x(\nu) = s]$ ; it equals one for all  $\nu \in R$  exactly when  $\pi_x(s) = s$  and  $\pi_x(\nu) \neq s$  for  $\nu \in R$ , that is, when  $\pi_x$  fixes  $s$  and maps  $R$  into  $R$ . For a plane  $\{\mu, \nu\} \subset R$  the count is  $[\pi_x(\mu) = \nu] + [\pi_x(\nu) = \mu]$ . Draw an arrow  $\mu \rightarrow \pi_x(\mu)$  for every  $\mu \in R$  with  $\pi_x(\mu) \in R \setminus \{\mu\}$ . Each vertex of  $R$  has at most one outgoing arrow, and the condition says that each of the three pairs of  $R$  carries exactly one arrow. Three arrows on three vertices with out-degree at most one give out-degree exactly one everywhere, and a tournament on three vertices with all out-degrees one is a directed three-cycle.  $\square$

From now on  $\sigma$  is the field of period two given in Table 2 of Appendix A, read from  $x$  modulo two. The table lists  $\pi_x$  at each of the sixteen residue classes; in each row  $\pi_x$  fixes  $\sigma(x)$  and is a three-cycle on the rest, so every plaquette contains exactly one link of  $F$  by Lemma 2.1.<sup>4</sup>

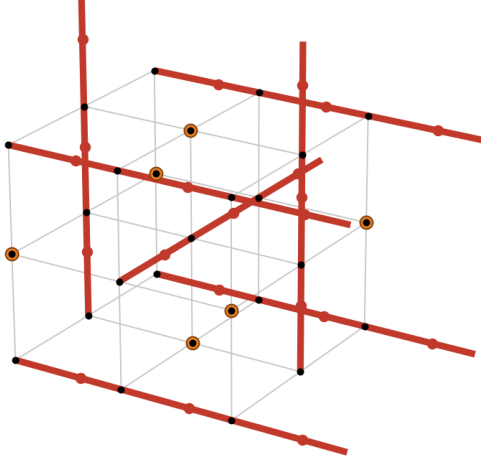
- Lemma 2.2.** (i)  *$F$  is a disjoint union of bi-infinite straight lines, one through every site: every site has exactly one outgoing and one incoming link of  $F$ , and both point in the direction  $\sigma(x)$ .*
- (ii) *For every box  $\Lambda$  (a product of four finite intervals, with the links and plaquettes it contains),  $F \cap E(\Lambda)$  is a disjoint union of segments, hence a forest.*
- (iii) *A link  $e = (x, \mu) \notin F$  lies in six plaquettes. Four of them contain a link of  $F$  that shares an end point with  $e$ , and these lie in the planes  $\{\mu, \sigma(x)\}$  and  $\{\mu, \pi_x(\mu)\}$ . In the other two, which lie in the remaining plane  $\{\mu, \rho\}$ , the link of  $F$  is opposite to  $e$ .*
- (iv) *Every link outside  $F$  shares a plaquette with exactly twelve other links outside  $F$ , two in each of its plaquettes.*

*Proof.* (i) Since  $\pi_x$  fixes  $\sigma(x)$ ,  $\sigma(x + e_{\sigma(x)}) = \sigma(x)$ . Because  $\sigma$  has period two,  $x - e_\mu \equiv x + e_\mu$ , so also  $\sigma(x - e_{\sigma(x)}) = \sigma(x)$ , and the link  $(x - e_{\sigma(x)}, \sigma(x))$  is in  $F$ . Conversely, if  $(x - e_\nu, \nu) \in F$ , then  $\sigma(x - e_\nu) = \nu$ , and straightness at  $x - e_\nu$  gives  $\sigma(x) = \nu$ . So the incoming link is unique and continues the outgoing one.

(ii) A line meets a box in a segment, and distinct lines are disjoint.

(iii) The plaquettes of  $e$  are, for each  $\nu \neq \mu$ , the one at  $x$  and the one at  $x - e_\nu$  in the plane  $\{\mu, \nu\}$ . A link of  $F$  sharing an end point with  $e$  is the outgoing or the incoming link of  $F$  at  $x$  or at  $x + e_\mu$ . At  $x$  both point in the direction  $s = \sigma(x) \neq \mu$ , and they lie in the plaquettes at  $x$  and at  $x - e_s$  in the plane  $\{\mu, s\}$ . At  $x + e_\mu$  both point in the direction  $t = \pi_x(\mu)$ , which lies in  $R \setminus \{\mu\}$  because  $\pi_x$  is a three-cycle on  $R$ , and they lie in the plaquettes at  $x$

<sup>4</sup>There are exactly eight period-two direction fields with this property, all equivalent to  $\sigma$  under the symmetries of the lattice. A field with one fixed link at every site and one in every plaquette exists only in four dimensions: a link lies in  $2(d-1)$  plaquettes and a site carries  $d(d-1)/2$  of them, and  $2(d-1) = d(d-1)/2$  forces  $d = 4$ . The statements of this section are also checked by `decide` in the Lean file `GaugeFix.lean` of the archive.

(a) the line field in the slice  $x_3 = 0$ 

(b) one plaquette in the gauge

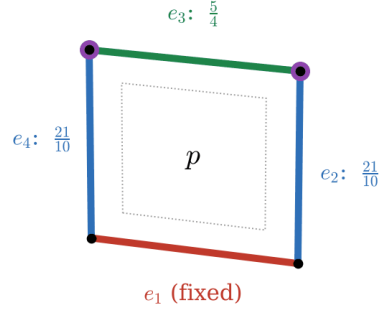


FIGURE 2. (a) The line field in the slice  $x_3 = 0$  of a box of side three. Thick red segments are the links of  $F$  in the directions 0, 1, 2, with a dot marking the direction of travel; an orange ring marks a site whose line leaves the slice in direction 3. Every line is straight (Lemma 2.2(i)), and the segments that leave the box are its boundary stubs. (b) One plaquette with its fixed link  $e_1$ . The weights are those of the local inequality of Lemma 6.2 for the three free links:  $\frac{21}{10}$  on the two links adjacent to  $e_1$  and  $\frac{5}{4}$  on the opposite one. The two purple corners are the ones that enter the corner form (6.3) once  $Y_1 = 0$ .

and at  $x - e_t$  in the plane  $\{\mu, t\}$ . These four plaquettes are distinct because  $s \neq t$ . Each plaquette contains exactly one link of  $F$ , so in the two plaquettes of the third plane  $\{\mu, \rho\}$  the link of  $F$  is a link of direction  $\mu$  or  $\rho$  not touching  $e$ ; the links of direction  $\rho$  in such a plaquette touch  $e$ , so it is the opposite link.

(iv) Each of the six plaquettes contains  $e$ , one link of  $F$  and two other links, and two distinct links share at most one plaquette.  $\square$

### 3. GAUGE FIXING AND THE FREE-BOUNDARY STATE

Let  $G \subset \mathrm{U}(N)$  be compact with Haar measure  $dU$ , and let  $\Lambda$  be a box. The Wilson measure with free boundary is

$$W_\Lambda(dQ) = \frac{1}{Z_\Lambda} \exp\left(N\beta \sum_{p \in P(\Lambda)} \mathrm{Re} \mathrm{Tr} Q_p\right) \prod_{e \in E(\Lambda)} dQ_e.$$

The measure is invariant under the gauge transformations  $Q_{(x,y)} \mapsto g_x Q_{(x,y)} g_y^{-1}$  with  $g \in G^\Lambda$ . Call the links outside  $F$  *active* and write  $A$  for their set. For a function  $O$  of the links,  $O^F$  is  $O$  with every link of  $F$  set to the identity.

**Lemma 3.1** (Gauge fixing). *Let  $\mu_\Lambda^F$  be the probability measure on  $G^{A \cap E(\Lambda)}$  whose density with respect to Haar measure is proportional to  $\exp(N\beta \sum_{p \in P(\Lambda)} \mathrm{Re} \mathrm{Tr} Q_p^F)$ . For every gauge-invariant  $O$  on  $G^{E(\Lambda)}$ ,*

$$W_\Lambda(O) = \mu_\Lambda^F(O^F).$$

*Proof.* Each segment of  $F \cap E(\Lambda)$  is a path  $x_0, x_1, \dots, x_k$  along one direction. Given the variables on  $F$ , put  $g_{x_0} = 1$  and  $g_{x_{i+1}} = g_{x_i} Q_{(x_i, x_{i+1})}$  on every segment. The transformed field has  $Q_e^g = 1$  on  $F \cap E(\Lambda)$ , and for fixed values on  $F$  the map  $Q_A \mapsto Q_A^g$  is a product of left and right translations, which preserves Haar measure. Gauge invariance of the action and of  $O$  and the change of variables give the claim.  $\square$

No leftover holonomy appears because  $F \cap E(\Lambda)$  is a forest. The measures  $\mu_\Lambda^F$  are finite-volume versions of the specification  $\gamma^F$  on  $G^A$ : for  $e \in A$ ,  $\gamma_e^F$  resamples  $Q_e$  from the law with density proportional to  $\exp(N\beta \sum_{p \ni e} \operatorname{Re} \operatorname{Tr} Q_p^F)$  given all other active links. No infinite-volume gauge transformation is performed;  $\gamma^F$  is a specification in its own right. Fix a metric  $\delta$  on  $G$  of diameter  $D$  that is bi-invariant (for  $\operatorname{SU}(2)$  the chordal metric on  $S^3$ ,  $D = 2$ ; for  $\operatorname{SU}(N)$  the Hilbert–Schmidt distance). The Kantorovich coefficient of  $e'$  on  $e$  is

$$C_{ee'} = \sup \frac{W_1(\gamma_e^F(\cdot | Q), \gamma_{e'}^F(\cdot | Q'))}{\delta(Q_{e'}, Q'_{e'})},$$

the supremum over pairs of configurations that differ only at  $e'$ . It vanishes unless  $e$  and  $e'$  share a plaquette.

**Proposition 3.2.** *Suppose that every row and every column of  $(C_{ee'})$  sums to at most  $\alpha < 1$ . Then:*

- (i)  $\gamma^F$  has exactly one Gibbs measure  $\mu_F$ ;
- (ii) for every gauge-invariant local  $O$  the limit  $\omega(O) = \lim W_\Lambda(O)$  exists along every sequence of boxes exhausting  $\mathbb{Z}^4$  and equals  $\mu_F(O^F)$ ;
- (iii)  $\omega$  is invariant under translations and the hypercubic group;
- (iv) if  $\beta \geq 0$ ,  $\omega$  is reflection positive for reflections in hyperplanes through sites and in hyperplanes bisecting links;
- (v) for gauge-invariant local  $O, O'$  with  $O$  Lipschitz in each link,  $|\operatorname{Cov}_\omega(O, O')| \leq C_{O, O'} e^{-\lambda \operatorname{dist}(O, O')}$  for every  $\lambda < -\log \alpha$ ;
- (vi) if  $\beta \geq 0$ , the transfer matrix of  $\omega$  has spectral gap at least  $-\log \alpha$ .

*Proof.* (i) Gibbs measures exist by compactness, and the comparison lemma of Appendix B with  $S = \emptyset$  shows that any two coincide.

(ii) Extend  $\mu_\Lambda^F$  by Haar measure on the active links outside  $\Lambda$ , and let  $S$  be the set of active links with a plaquette outside  $P(\Lambda)$ ; this includes every active link outside  $E(\Lambda)$ . For an active link  $e \notin S$  all six plaquettes of  $e$  lie in  $P(\Lambda)$ , so the conditional law of  $Q_e$  under the extended measure is  $\gamma_e^F$ . Let  $T$  be the support of  $O^F$  and  $\lambda < -\log \alpha$ . The comparison lemma with weights  $e^{-\lambda \operatorname{dist}(\cdot, T)}$  gives

$$|\mu_\Lambda^F(O^F) - \mu_F(O^F)| \leq L \frac{\theta D}{1 - \theta} \sum_{z \in S} e^{-\lambda \operatorname{dist}(z, T)},$$

with  $\theta = \alpha e^\lambda < 1$  and  $L$  a Lipschitz constant of  $O^F$ ; the sum is at most  $C_T R^3 e^{-\lambda R}$  when  $R$  is the distance from  $T$  to the complement of  $\Lambda$ . For  $O$  continuous but not Lipschitz, approximate uniformly. Lemma 3.1 gives  $W_\Lambda(O) = \mu_\Lambda^F(O^F)$ .

(iii)  $W_\Lambda(\tau_v O) = W_{\Lambda-v}(O)$ , and the boxes  $\Lambda - v$  also exhaust  $\mathbb{Z}^4$ ; by (ii) the limit does not depend on the sequence. The line field is a device for computing  $W_\Lambda$  and is never shifted. The same argument applies to the hypercubic group.

(iv) For  $\beta \geq 0$  the measure  $W_\Lambda$  is reflection positive for every box symmetric under the reflection [29], and reflection positivity passes to the limit along such boxes, which is  $\omega$  by (ii).

(v)  $\operatorname{Cov}_\omega(O, O') = \operatorname{Cov}_{\mu_F}(O^F, O'^F)$ , and setting the links of  $F$  to the identity keeps the Lipschitz property. The bound is Corollary B.2.

(vi) Let  $\vartheta$  be a time reflection and  $\tau_t$  a time translation. For gauge-invariant local  $O$  that are Lipschitz in each link, (v) gives  $|\omega(\vartheta(\bar{O})\tau_t O) - |\omega(O)|^2| \leq C_O e^{-\lambda t}$ . On the support of  $O$  these functions are dense in the gauge-invariant part of  $L^2(\omega)$ : gauge-invariant functions are functions on the compact quotient of  $G^E$  by the gauge group, Lipschitz functions for the quotient metric are dense in  $L^2$  of every finite measure on it, and they lift to gauge-invariant Lipschitz functions. By [9, Prop. 23] the spectrum of the transfer matrix on the complement of the vacuum lies in  $[0, e^{-\lambda}]$ , for every  $\lambda < -\log \alpha$ .  $\square$

The proposition replaces the torus limits and the uniqueness of the ungauged Gibbs measure used in [9, Thm. F] by the free-boundary limit, which has all the properties the reconstruction of the transfer matrix needs.<sup>5</sup>

#### 4. SU(2) UP TO ONE THIRD

*Proof of Theorem 1.1.* Identify SU(2) with the unit quaternions and use the chordal metric. For an active link  $e$  and a plaquette  $p \ni e$ , the staple  $s_p$  is the product of the other three link variables in the order that makes  $\text{Re}(Q_e s_p) = \frac{1}{2} \text{Tr } Q_p$ , with links of  $F$  equal to one. Since  $N\beta \text{Re Tr } Q_p = 4\beta \langle Q_e, \bar{s}_p \rangle$  in  $\mathbb{R}^4$ , the conditional law of  $Q_e$  is the von Mises–Fisher law  $\nu_b(dx) \propto e^{\langle x, b \rangle} dx$  on  $S^3$  with  $b = 4\beta \sum_{p \ni e} \bar{s}_p$ . Changing a partner  $e'$  by  $\delta$  in the chordal metric changes one staple by an isometric image of that displacement, so  $b$  moves by  $4|\beta|\delta$ , and the sharp quarter [9, Thm. 7],  $W_1(\nu_b, \nu_{b'}) \leq \frac{1}{4}|b - b'|$ , gives  $C_{ee'} \leq |\beta|$ . By Lemma 2.2(iv) every row and column has twelve nonzero entries, so  $\alpha \leq 12|\beta| = 3|\beta_W|$ . Proposition 3.2 applies for  $|\beta_W| < \frac{1}{3}$ .  $\square$

*Proof of Theorem 1.2.* For the line field, every staple of an active link contains two active links, so the six staples can be chosen with sum zero while the staple through a given partner  $e'$  moves freely. At  $b = 0$  the derivative of  $\nu_b$  in the direction  $h$  has Kantorovich norm  $\frac{1}{4}|h|$  [9, Lemma 6(i)], so  $C_{ee'} = |\beta|$  exactly, and every row sum is  $12|\beta| = 3|\beta_W|$ .

Now let  $F'$  be any forest and  $C'$  the matrix of the statement. Suppose a positive weight  $w$  satisfies  $C'w \leq \alpha w$ . For a box  $\Lambda_n$  of side  $n$  let  $C_n$  be the principal submatrix on the active links of  $E(\Lambda_n)$ ; then  $C_n w \leq \alpha w$  on these links, and the Collatz–Wielandt inequality for nonnegative matrices gives  $\rho(C_n) \leq \alpha$ . Since  $C_n$  is symmetric,  $\rho(C_n)$  is at least the mean row sum. The box has  $4n^4 - O(n^3)$  links and  $n^4$  sites, and a forest has fewer links than sites, so the number  $a$  of active links is at least  $3n^4 - O(n^3)$ . Let  $a_p$  be the number of active links of the plaquette  $p$ ; the box has  $P = 6n^4 - O(n^3)$  plaquettes and  $\sum_p a_p = 6a - O(n^3)$ . Two links share at most one plaquette, so the row sums add up to  $|\beta| \sum_p a_p(a_p - 1)$ , and by convexity

$$\frac{|\beta|}{a} \sum_p a_p(a_p - 1) \geq \frac{|\beta|P}{a} \bar{a}(\bar{a} - 1) = 6|\beta|(\bar{a} - 1) - o(1),$$

with  $\bar{a} = \sum_p a_p / P = 6a/P - o(1) \geq 3 - o(1)$ . Hence  $\alpha \geq 12|\beta| - o(1)$  for every  $n$ , and  $\alpha \geq 12|\beta| = 3|\beta_W|$ .  $\square$

Without gauge fixing the same count gives eighteen partners and the ceiling  $\beta_W = \frac{2}{9}$  of [9, Prop. 13]; the temporal gauge, a family of lines in one direction, leaves fourteen. The line field is optimal among forests: twelve is the mean number of partners forced by the number of free links, and it is attained at every link.

#### 5. SU(2) PAST ONE THIRD: PLAQUETTE BLOCKS

Throughout this section  $G = \text{SU}(2)$  is the group of unit quaternions with the chordal metric  $|x - y|$ ,  $\bar{x}$  is the conjugate,  $\langle x, y \rangle = \text{Re}(x\bar{y})$  is the inner product of  $\mathbb{R}^4$ , and  $dx$  is the normalised Haar measure on  $S^3$ . Left and right multiplication by unit quaternions and conjugation are isometries, and  $\text{Re}(xyz) = \text{Re}(yzx) = \langle y, \bar{z}\bar{x} \rangle$ . Put

$$Z_0(c) = \int_{S^3} e^{\langle c, x \rangle} dx = \sum_{j \geq 0} \frac{(|c|^2/4)^j}{j!(j+1)!}, \quad c \in \mathbb{R}^4,$$

let  $\nu_c(dx) \propto e^{\langle c, x \rangle} dx$  be the von Mises–Fisher law, and write  $g = \log Z_0$ , so that  $\nabla g(c)$  is the mean and  $\text{Hess } g(c)$  the covariance of  $\nu_c$ .

<sup>5</sup>On a torus the line field closes up into cycles, and the holonomies along them cannot be removed by a gauge transformation; the argument never uses a torus. The state  $\omega$  is defined only on gauge-invariant observables, which is all that the Osterwalder–Schrader reconstruction [27, 28, 29] uses.

**5.1. The blocks.** A *block plaquette* is a plaquette of the set  $\mathcal{B}$  described in Table 3 of Appendix C, and its *block* is the set of its three active links.

**Lemma 5.1.** *The set  $\mathcal{B}$  is invariant under translations by  $2\mathbb{Z}^4$  and:*

- (i) *the blocks partition the active links;*
- (ii) *every plaquette  $p \notin \mathcal{B}$  meets three distinct blocks, one in each of its active links;*
- (iii) *every active link lies in exactly five plaquettes outside  $\mathcal{B}$ , so it shares a plaquette outside  $\mathcal{B}$  with exactly ten active links, none of them in its own block;*
- (iv) *if an active link  $j$  and a link of a block  $B$  lie in a common plaquette outside  $\mathcal{B}$ , then every link of  $B$  has its midpoint within distance 2 of the midpoint of  $j$ .*

*Proof.* All four statements are invariant under  $2\mathbb{Z}^4$  and involve plaquettes at distance at most two from one site, so they hold on  $\mathbb{Z}^4$  as soon as they hold on the torus  $(\mathbb{Z}/P\mathbb{Z})^4$  with  $P \geq 6$ , where no such configuration wraps around. They are checked on the tori of side four and six by the program described in Appendix C; the side four already suffices for (i)–(iii). For (iii), an active link lies in six plaquettes, one of which is its block plaquette, and each of the other five contains two further active links, which lie in other blocks by (ii).  $\square$

Fix a block plaquette  $p$  with active links  $e_1, e_2, e_3$ . Reading the plaquette product cyclically and deleting its fixed link,  $\frac{1}{2} \text{Tr } Q_p = \text{Re}(y_1 y_2 y_3)$ , where  $y_i$  is  $Q_{e_i}$  or  $Q_{e_i}^{-1}$ . Every other plaquette containing  $e_i$  meets the block only in  $e_i$  by Lemma 5.1(ii) and contributes  $4\beta \langle y_i, \bar{s} \rangle$  to the exponent, where  $s$  is the staple of  $y_i$  in it. Hence the conditional law of the block given all other active links is

$$(5.1) \quad \mu_b(dy) \propto \exp\left(\sum_{i=1}^3 \langle y_i, b_i \rangle + 4\beta \text{Re}(y_1 y_2 y_3)\right) dy_1 dy_2 dy_3,$$

with  $b_i$  the sum of  $4\beta \bar{s}$  over the five staples, so that  $|b_i| \leq 20|\beta|$ . Inversion is an isometry of  $S^3$  and preserves Haar measure, so we may work with  $y$  instead of  $Q$ . On  $(S^3)^3$  we use the metric  $d(y, y') = \sum_i |y_i - y'_i|$ .

**5.2. The sensitivity of one block.** Let  $\lambda_*$  and  $\tau$  be a lower and an upper bound for the eigenvalues of  $\text{Hess } g(c)$  on the ball  $|c| \leq 24|\beta|$ , and  $M$  an upper bound there for the norm of the third derivative  $D^3 g(c)$  as a trilinear form. Put

$$(5.2) \quad \begin{aligned} w &= 8\beta^2(\tau - \lambda_*), & s &= \frac{1}{4} + (e^w - 1)e^w \sqrt{\tau}, \\ C &= 20|\beta|(4|\beta|\tau s + 16\beta^2 M), & K &= \frac{(1 + |\beta|)s}{1 - C}, \end{aligned}$$

assuming  $C < 1$ .

**Lemma 5.2.** *Let  $\nu$  be a probability measure and  $\lambda = e^R \nu / \nu(e^R)$  with  $\text{osc } R \leq \omega$ . For bounded  $f, h$ ,*

$$\text{Cov}_\lambda(f, h) \leq \text{Cov}_\nu(f, h) + (e^\omega - 1)e^\omega \text{sd}_\nu(f) \text{sd}_\nu(h).$$

*Proof.* We may assume  $0 \leq R \leq \omega$ . The density  $\rho = e^R / \nu(e^R)$  lies in  $[e^{-\omega}, e^\omega]$ , so  $\delta = \|\rho - 1\|_\infty \leq e^\omega - 1$ . With  $\tilde{f} = f - \nu f$  and  $\tilde{h} = h - \nu h$ ,

$$\begin{aligned} \text{Cov}_\lambda(f, h) - \text{Cov}_\nu(f, h) &= \nu((\rho - 1)\tilde{f}\tilde{h}) \\ &\quad - \nu((\rho - 1)\tilde{f}) \nu((\rho - 1)\tilde{h}), \end{aligned}$$

and the right side is at most  $(\delta + \delta^2) \text{sd}_\nu(f) \text{sd}_\nu(h)$  by the Cauchy–Schwarz inequality.  $\square$

**Lemma 5.3.** *Let  $\beta \geq 0$ ,  $a, c_0 \in \mathbb{R}^4$  with  $|a|, |c_0| \leq 20\beta$ , and*

$$\kappa(dx) \propto \exp(\langle x, a \rangle + g(c_0 + 4\beta \bar{x})) dx \quad \text{on } S^3.$$

*For every 1-Lipschitz  $f$  and every  $h \in \mathbb{R}^4$ :*

- (i)  $\text{Cov}_\kappa(f, \langle x, h \rangle) \leq s|h|$ ;
- (ii) *if  $c_0$  moves in the direction  $k$ , then  $|\frac{d}{dt} \kappa(f)| \leq (4\beta\tau s + 16\beta^2 M)|k|$ .*

*Proof.* (i) Taylor's formula at  $c_0$  gives  $g(c_0 + 4\beta u) = g(c_0) + 4\beta \langle \nabla g(c_0), u \rangle + R(u)$  with  $R(u) = 8\beta^2 \text{Hess } g(\xi)(u, u)$  for some  $\xi$  on the segment,  $|\xi| \leq 24\beta$ . For  $u = \bar{x}$  of norm one,  $R(\bar{x}) \in [8\beta^2 \lambda_*, 8\beta^2 \tau]$ , so  $\text{osc } R \leq w$ . Since  $\langle \nabla g(c_0), \bar{x} \rangle = \langle x, \overline{\nabla g(c_0)} \rangle$ ,  $\kappa$  is the tilt  $e^R \nu / \nu(e^R)$  of  $\nu = \nu_c$  with  $c = a + 4\beta \overline{\nabla g(c_0)}$ . The sharp quarter [9, Thm. 7] gives  $\text{Cov}_\nu(f, \langle x, h \rangle) = \frac{d}{dt} \nu_{c+th}(f) \leq \frac{1}{4}|h|$ . A 1-Lipschitz function on a set of diameter two has  $\text{sd} \leq 1$ , and  $\text{sd}_\nu \langle x, h \rangle \leq \sqrt{\tau}|h|$  because  $|c| \leq |a| + 4\beta \leq 24\beta$ . Lemma 5.2 gives (i).

(ii) The derivative is  $\text{Cov}_\kappa(f, \langle \nabla g(c_0 + 4\beta \bar{x}), k \rangle)$ . Write  $\nabla g(c_0 + 4\beta u) = \nabla g(c_0) + 4\beta \text{Hess } g(c_0)u + N(u)$ . The constant term does not contribute. The linear term is  $\langle x, h \rangle$  with  $h = 4\beta \overline{\text{Hess } g(c_0)}k$ ,  $|h| \leq 4\beta \tau |k|$ , and contributes at most  $4\beta \tau s |k|$  by (i). For  $|u|, |u'| \leq 1$ ,  $\|\text{Hess } g(c_0 + 4\beta v) - \text{Hess } g(c_0)\| \leq 4\beta M |v| \leq 4\beta M$  on the segment between them, so  $|N(u) - N(u')| \leq 16\beta^2 M |u - u'|$ ; the function  $\langle N(\bar{x}), k \rangle$  is therefore  $16\beta^2 M |k|$ -Lipschitz, its standard deviation is at most that constant, and its covariance with  $f$  is at most  $16\beta^2 M |k|$ .  $\square$

**Lemma 5.4.** *Let  $C < 1$ . If  $b$  and  $b'$  satisfy  $|b_i|, |b'_i| \leq 20|\beta|$  and differ only in one coordinate  $b_i$ , then*

$$W_1(\mu_b, \mu_{b'}) \leq K |b_i - b'_i|$$

*in the metric  $d$ .*

*Proof.* The map  $y_1 \mapsto -y_1$  turns  $\mu_b$  for  $\beta$  into the law for  $-\beta$  with  $b_1$  replaced by  $-b_1$  and is an isometry, so we may take  $\beta \geq 0$ ; the cyclic symmetry of  $\text{Re}(y_1 y_2 y_3)$  lets us take  $i = 1$ . Let  $f$  be 1-Lipschitz for  $d$  and  $\varepsilon = |b_1 - b'_1|$ .

*The middle link.* Given  $y_1, y_3$ , the law of  $y_2$  is  $\nu_c$  with  $c = b_2 + 4\beta \overline{y_3 y_1}$ , which does not involve  $b_1$ . Changing  $y_1$  by  $\eta$  moves  $c$  by  $4\beta \eta$ , and the sharp quarter moves  $\nu_c$  by at most  $\beta \eta$  in  $W_1$ . Hence  $F(y_1, y_3) = \mathbb{E}[f \mid y_1, y_3]$  is  $(1 + \beta)$ -Lipschitz in each variable, and  $\mu_b(f) = \mu_b^{13}(F)$ , where  $\mu_b^{13}$  is the law of  $(y_1, y_3)$ :

$$\mu_b^{13} \propto e^{\langle y_1, b_1 \rangle + \langle y_3, b_3 \rangle} Z_0(b_2 + 4\beta \overline{y_3 y_1}).$$

Since  $|b_2 + 4\beta \overline{y_3 y_1}| = |b_2 y_3 + 4\beta \bar{y}_1| = |y_1 b_2 + 4\beta \bar{y}_3|$ , its conditional laws are

$$\kappa_1(dy_1 \mid y_3) \propto e^{\langle y_1, b_1 \rangle + g(b_2 y_3 + 4\beta \bar{y}_1)} dy_1,$$

$$\kappa_3(dy_3 \mid y_1) \propto e^{\langle y_3, b_3 \rangle + g(y_1 b_2 + 4\beta \bar{y}_3)} dy_3,$$

both of the form of Lemma 5.3, with  $|c_0| = |b_2|$ .

*Two sites.* Integrating Lemma 5.3(i) along the segment from  $b_1$  to  $b'_1$  gives  $W_1(\kappa_1, \kappa'_1) \leq s\varepsilon$  for the same  $y_3$ . Moving  $y_3$  to  $y'_3$  moves  $c_0 = b_2 y_3$  along a segment of length  $|b_2||y_3 - y'_3| \leq 20\beta |y_3 - y'_3|$  inside the ball  $|c_0| \leq 20\beta$ , so Lemma 5.3(ii) gives  $W_1(\kappa_1(\cdot \mid y_3), \kappa_1(\cdot \mid y'_3)) \leq C|y_3 - y'_3|$ , and the same for  $\kappa_3$ , which does not depend on  $b_1$ . Let  $\pi$  be a coupling of  $\mu_b^{13}$  and  $\mu_{b'}^{13}$  minimising  $d_1 + d_3$ ,  $d_k = \mathbb{E}_\pi |y_k - y'_k|$ . Replacing  $(y_1, y'_1)$  by an optimal coupling of  $\kappa_1(\cdot \mid y_3)$  and  $\kappa'_1(\cdot \mid y'_3)$  keeps a coupling, so minimality gives  $d_1 \leq s\varepsilon + C d_3$ ; in the same way  $d_3 \leq C d_1$ . Hence  $d_1 \leq s\varepsilon / (1 - C^2)$  and

$$|\mu_b(f) - \mu_{b'}(f)| \leq (1 + \beta)(d_1 + d_3) \leq \frac{(1 + \beta)s}{1 - C} \varepsilon = K\varepsilon. \quad \square$$

### 5.3. The constants.

**Lemma 5.5.** *Write  $g(c) = \varphi(|c|^2)$  and  $r = |c|$ . For  $c \neq 0$  the eigenvalues of  $\text{Hess } g(c)$  are  $2\varphi'(r^2)$ , three times, and  $2\varphi'(r^2) + 4r^2\varphi''(r^2)$ , and the norm of  $D^3 g(c)$  is*

$$\max\left(|A + \Gamma|, \frac{2}{3}|A|\right), \quad A = 12r\varphi''(r^2), \quad \Gamma = 8r^3\varphi'''(r^2).$$

*On the ball  $r \leq 2.07$  every eigenvalue of  $\text{Hess } g$  lies between 0.158491 and 0.25000001, and the norm of  $D^3 g$  is at most 0.0615586.*

*Proof.* Differentiating  $\varphi(|c|^2)$  gives  $D^2g(v, v) = 2\varphi'|v|^2 + 4\varphi''\langle c, v \rangle^2$  and  $D^3g(v, v, v) = 12\varphi''\langle c, v \rangle|v|^2 + 8\varphi'''\langle c, v \rangle^3$ . For  $|v| = 1$  and  $t = \langle c/r, v \rangle \in [-1, 1]$  the cubic form is  $At + \Gamma t^3$ , whose largest absolute value is attained at  $t = \pm 1$  or at an interior critical point, where it equals  $\frac{2}{3}|A||t| \leq \frac{2}{3}|A|$ ; by Banach's theorem [6] the supremum of a symmetric trilinear form on the diagonal of the unit sphere is its norm. The numerical bounds are obtained in ball arithmetic [23] from the series of  $Z_0$ , truncated after forty terms with a proved bound on the tail, on  $2 \cdot 10^4$  subintervals of  $[0, 2.07]$ ; the program is described in Appendix C.  $\square$

**5.4. The proof of Theorem 1.3.** Let  $\gamma^B$  be the block specification: for a block  $B$ ,  $\gamma_B^B$  resamples the three links of  $B$  from (5.1). It comes from the same interaction as  $\gamma^F$ , so the two specifications have the same Gibbs measures. Define the block coefficients

$$C_{Bj} = \sup \frac{W_1(\gamma_B^B(\cdot | Q), \gamma_B^B(\cdot | Q'))}{|Q_j - Q'_j|}$$

for an active link  $j \notin B$ , the supremum over pairs of configurations that differ only at  $j$ , with  $W_1$  for the metric  $d$ .

**Proposition 5.6.** *Suppose that  $\sum_B C_{Bj} \leq \alpha < 1$  for every active link  $j$ . Then the conclusions (i)–(vi) of Proposition 3.2 hold with  $-\frac{1}{2} \log \alpha$  in place of  $-\log \alpha$ .*

*Proof.* The proof of Proposition 3.2 goes through with Lemma B.3 of Appendix B in place of Lemma B.1, and with the set  $S$  of part (ii), and the support of  $H$  in Corollary B.2, enlarged to the union of the blocks that meet it. The enlargement multiplies the size of these sets by at most three and moves them by at most one, which changes only the constants.  $\square$

*Proof of Theorem 1.3.* Let  $|\beta| \leq 0.345/4$  and take the constants of Lemma 5.5, whose ball  $r \leq 2.07$  contains  $|c| \leq 24|\beta|$ . With these fixed constants every quantity in (5.2) is increasing in  $|\beta|$ , and so is

$$(5.3) \quad \alpha(\beta_W) = 40|\beta|K.$$

Changing an active link  $j$  by  $\eta$  changes one staple of each of its ten partners outside its block by an isometric image of the displacement, so the field  $b_i$  of each such partner  $i$  moves by at most  $4|\beta|\eta$ , and Lemma 5.4, applied once for each of these ten fields, gives  $\sum_B C_{Bj} \leq 40|\beta|K$ . Evaluating (5.2) in ball arithmetic at  $\beta_W = 0.345$  gives  $w \leq 0.0054459$ ,  $s \leq 0.252746$ ,  $C \leq 0.050243$ ,  $K \leq 0.289069$  and  $\alpha \leq 0.99729$ ; at  $\beta_W = \frac{1}{3}$  it gives  $\alpha \leq 0.95602$ . Proposition 5.6 applies.  $\square$

*Remark 5.7.* Most of the range is kept. A second bound for the same sensitivity, which keeps the law of the middle link instead of linearising it, evaluated in floating point on a quadrature grid and maximised over the fields  $b$  by a random search followed by the Nelder–Mead method, gives 0.2783 at  $\beta_W = 0.34$ , against 0.2881 from Lemma 5.4, and  $40|\beta|$  times it crosses one near  $\beta_W = 0.357$ . That computation is not certified and is not used in the proof. Past this point the gain from integrating out one link is spent, and larger blocks are needed, such as the free links of a cube of the line field, whose sensitivity no longer reduces to two sites.

## 6. CURVATURE IN THE GAUGE

In this section  $G = \text{SU}(N)$ ,  $\text{SO}(N)$  or  $\text{U}(N)$  with the inner product  $\langle X, Y \rangle = -\text{Tr}(XY)$  on the Lie algebra, so that  $\text{Ric}_{\text{SU}(N)} = \frac{N}{2}$  and  $\text{Ric}_{\text{SO}(N)} = \frac{N-2}{4}$ . A tangent vector at a field  $Q$  is  $v = (X_e Q_e)$ ,  $|v|^2 = \sum_e |X_e|^2$ . For the plaquette at  $x$  in the plane  $\mu < \nu$  number its links  $e_1 = (x, \mu)$ ,  $e_2 = (x + e_\mu, \nu)$ ,  $e_3 = (x + e_\nu, \mu)$ ,  $e_4 = (x, \nu)$ , transport their vectors to  $x$ ,

$$(6.1) \quad Y_1 = X_{e_1}, \quad Y_2 = Q_{e_1} X_{e_2} Q_{e_1}^*, \quad Y_3 = Q_{e_4} X_{e_3} Q_{e_4}^*, \quad Y_4 = X_{e_4},$$

and write  $W = Q_p$ . The second derivative of  $\text{Re Tr } Q_p$  along the geodesic  $Q_e(t) = e^{tX_e}Q_e$  is [9, Eq. (5)]

$$(6.2) \quad f_p = \sum_{i=1}^4 \text{Re Tr}(Y_i^2 W) + 2 \text{Re Tr}(Y_1 Y_2 W) + 2 \text{Re Tr}(Y_3 Y_4 W) - 2 \sum_{i \in \{1,2\}, j \in \{3,4\}} \text{Re Tr}(Y_i W Y_j),$$

so that  $\text{Hess } S_0(v, v) = \sum_p f_p$ . The corner form of  $p$  is

$$(6.3) \quad c_p = 2\langle Y_1, Y_4 \rangle - 2\langle Y_1, Y_2 \rangle + 2\langle Y_2, W Y_3 W^* \rangle - 2\langle Y_3, Y_4 \rangle.$$

It collects the inner products of the transported vectors of two links that meet at a right angle at a corner of  $p$ .

**Lemma 6.1.** *On  $\mathbb{Z}^4$ , or on a box with free boundary, for every field  $Q$  and every square-summable  $X$ ,*

$$\begin{aligned} \sum_y \left[ |D_y X|^2 + \sum_{\mu} |X_{y,\mu} + Q_{y-e_{\mu},\mu}^* X_{y-e_{\mu},\mu} Q_{y-e_{\mu},\mu}|^2 \right] \\ = 4|X|^2 + \sum_p c_p, \end{aligned}$$

where  $D_y X = \sum_{\mu} [X_{y,\mu} - Q_{y-e_{\mu},\mu}^* X_{y-e_{\mu},\mu} Q_{y-e_{\mu},\mu}]$  is the covariant divergence and terms with a link outside the box are omitted.

*Proof.* At a site  $y$  write  $o_{\mu}$  for the vector of the outgoing link in direction  $\mu$  and  $i_{\mu}$  for the transported vector of the incoming one, so that  $D_y X = \sum_{\mu} (o_{\mu} - i_{\mu})$ . Every link is outgoing at one end and incoming at the other, so the squares  $|o_{\mu}|^2 + |i_{\mu}|^2$ , which appear once in  $|D_y X|^2$  and once in  $|o_{\mu} + i_{\mu}|^2$ , add up to  $4|X|^2$ . The collinear cross term  $-2\langle o_{\mu}, i_{\mu} \rangle$  of  $|D_y X|^2$  cancels the term  $+2\langle o_{\mu}, i_{\mu} \rangle$  of  $|o_{\mu} + i_{\mu}|^2$ . The remaining cross terms of  $|D_y X|^2$  pair two perpendicular ends at  $y$ , and each such pair is a corner of exactly one plaquette, with the sign and the transport of (6.3) (this is the proof of [9, Lemma 1]). In a box two perpendicular ends at  $y$  span a plaquette of the box, because a box is a product of intervals.  $\square$

**Lemma 6.2.** *Let  $k \in \{1, 2, 3, 4\}$ , let  $k \pm 1 \pmod{4}$  be the positions adjacent to  $k$  and  $k + 2$  the opposite one. For every  $N$ , every  $W \in \text{U}(N)$  and all  $Y_1, \dots, Y_4 \in \mathfrak{u}(N)$  with  $Y_k = 0$ ,*

$$\pm f_p + \frac{9}{10} c_p \leq \frac{21}{10} (|Y_{k-1}|^2 + |Y_{k+1}|^2) + \frac{5}{4} |Y_{k+2}|^2.$$

*Proof.* Replacing  $W$  by  $-W$  changes the sign of  $f_p$  and fixes  $c_p$ , so the sign  $+$  suffices. Conjugation by a unitary matrix preserves both sides, so we may take  $W = \text{diag}(w_a)$ . The form  $\mathcal{F} = \text{RHS} - f_p - \frac{9}{10} c_p$  then splits into a block for each diagonal position  $a$ , in the real variables  $y_i = -i(Y_i)_{aa}$ , and a block for each pair  $a \neq b$ , in the complex variables  $z_i = (Y_i)_{ab}$ , since  $(Y_i)_{ba} = -\bar{z}_i$ . Appendix D shows that every block is positive definite.  $\square$

*Proof of the bound in Theorem 1.4.* Every plaquette has exactly one link of  $F$ , and  $X$  vanishes on it. By Lemma 2.2(iii) each active link is adjacent to the fixed link in four of its plaquettes and opposite to it in two. Summing Lemma 6.2 over the plaquettes,

$$\sum_p (\pm f_p + \frac{9}{10} c_p) \leq (4 \cdot \frac{21}{10} + 2 \cdot \frac{5}{4}) |X|^2 = \frac{109}{10} |X|^2.$$

By Lemma 6.1,  $\frac{9}{10} \sum_p c_p = \frac{9}{10} \Sigma - \frac{36}{10} |X|^2$ , where  $\Sigma \geq 0$  is the left side of the lemma. Hence

$$\pm \text{Hess } S_0(v, v) + \frac{9}{10} \Sigma \leq (\frac{109}{10} + \frac{36}{10}) |X|^2 = \frac{29}{2} |X|^2.$$

In a box, links near the boundary lie in fewer plaquettes and receive less.  $\square$

The factor  $\frac{9}{10}$  is chosen so that the corners are paid by the divergence identity at the price  $\frac{36}{10}$  while the local blocks stay positive; Lemma 6.1 is the divergence identity of [9, Lemma 1] with the collinear term written as a square, so nothing is lost by Cauchy–Schwarz.

**Corollary 6.3.** *Let  $G = \mathrm{SU}(N)$  and  $|\beta| < \frac{1}{29}$ , or  $G = \mathrm{SO}(N)$ ,  $N \geq 3$ , and  $|\beta| < \frac{N-2}{58N}$ . For every box the measure  $\mu_\Lambda^F$  satisfies the Bakry–Émery condition with  $K = \mathrm{Ric}_G - \frac{29}{2}N|\beta| > 0$ , uniformly in the box, and hence a logarithmic Sobolev inequality with constant  $2/K$  and a Poincaré inequality with constant  $1/K$ .*

*Proof.* The set  $\{Q_F = 1\}$  is a product factor, so its Ricci curvature is  $\mathrm{Ric}_G$  on every link, and the Hessian of the restricted action is the restriction of  $N\beta \mathrm{Hess} S_0$  to  $X_F = 0$ . Apply the bound and [2, Cor. 5.7.2].  $\square$

*Proof of the rest of Theorem 1.4. Decay in boxes.*  $\mu_\Lambda^F$  is the Gibbs measure of a nearest-neighbour interaction on a product of copies of  $G$ . The commutator estimate of [31, Section 4] holds with the same constants, because fixing links only removes terms, and the Poincaré constant is  $1/K$ , so the argument of [31, Cor. 4.11 and Rem. 4.12] gives  $|\mathrm{Cov}_{\mu_\Lambda^F}(f, g)| \leq c_1 e^{-c \mathrm{dist}(f, g)} (\|f\| \|g\| + |f|_2 |g|_2)$  for smooth local  $f, g$ , uniformly in the box, in the norms of that paper.

*Limit.* For boxes  $\Lambda \subset \Lambda'$  let  $S_s = S_\Lambda + s(S_{\Lambda'} - S_\Lambda)$  on the active links of  $\Lambda'$ ,  $s \in [0, 1]$ , and  $\mu_s$  the corresponding measure. Its Hessian is a convex combination of two Hessians, each bounded by  $\frac{29}{2}N|\beta|$ , so  $\mu_s$  has the same constant  $K$  and the same commutator constants. Since  $\frac{d}{ds}\mu_s(f) = \sum_{p \in P(\Lambda') \setminus P(\Lambda)} \mathrm{Cov}_{\mu_s}(f, N\beta \mathrm{Re} \mathrm{Tr} Q_p^F)$ ,

$$|\mu_{\Lambda'}^F(f) - \mu_\Lambda^F(f)| \leq C_f R^3 e^{-cR},$$

$R$  the distance from the support of  $f$  to the complement of  $\Lambda$ . The limit therefore exists along every increasing sequence of boxes, and two sequences are compared through their intersections, which are boxes.

*Properties.* Invariance, reflection positivity and decay follow as in Proposition 3.2(iii)–(v), and Proposition 23 of [9], applied to smooth gauge-invariant cylinder functions, gives a gap at least  $c$ .  $\square$

*Remark 6.4.* Let  $h_F$  be the supremum of  $\pm \mathrm{Hess} S_0$  on the gauge-fixed directions, so that uniform curvature reaches  $|\beta| < 1/(2h_F)$  and no further. At the identity field  $-\mathrm{Hess} S_0$  restricted to  $X_F = 0$  has the eigenvalue  $9 + \sqrt{17} = 13.123$  on the torus of side two, so  $9 + \sqrt{17} \leq h_F \leq \frac{29}{2}$ . A gradient search over  $\mathrm{SU}(2)$  fields of periods two, four and six gives  $h_F = 13.97401$ , attained by an abelian field; certificates made of plaquette forms and vertex forms on the period-two cell cannot go below 14.384. With this gauge the curvature method stops between  $|\beta| = \frac{1}{29}$  and  $\frac{1}{26.25}$ .

## 7. $\mathrm{SU}(3)$ UP TO $3/4$

In this section  $G = \mathrm{SU}(3)$ ,  $\beta \geq 0$  and  $\beta_W = 9\beta$ . The distance on  $\mathrm{SU}(3)$  is  $d(U, V) = \|U - V\|_{\mathrm{HS}}$ , which is at most the geodesic distance for  $\langle X, Y \rangle = -\mathrm{Tr}(XY)$ ;  $K = \mathrm{conv} \mathrm{SU}(3)$ ;  $A(M) = \frac{1}{2}(M - M^*)$  and  $P$  removes the trace. With this inner product  $\mathrm{Ric} = \frac{3}{2}$ , the Casimir of the fundamental representation is  $\lambda_F = \frac{8}{3}$  and that of the adjoint representation is 6.

**7.1. Reduction to one link.** The conditional law of an active link is

$$\nu_b(dU) \propto e^{\mathrm{Re} \mathrm{Tr}(U^b)} dU, \quad b = N\beta \sum_{p \ni e} S_p,$$

where  $S_p$  is the staple of  $e$  in  $p$ , a product of  $\mathrm{SU}(3)$  matrices and identities, or its adjoint. Six staples give  $b \in rK$  with  $r = 6N\beta = 2\beta_W$ . For a matrix  $h$  put  $\varphi_h(U) = \mathrm{Re} \mathrm{Tr}(Uh)$  and

$$L_h(b) = \sup\{\mathrm{Cov}_{\nu_b}(g, \varphi_h) : g \text{ 1-Lipschitz for } d\},$$

$$\Lambda(r) = \sup_{b \in rK} \sup_{\|h\|_{\text{HS}}=1} L_h(b).$$

Changing a partner  $e'$  changes one staple by a matrix of norm  $d(Q_{e'}, Q'_{e'})$ . Along the segment  $b_t$  from  $b$  to  $b'$ , which stays in  $rK$ ,  $\frac{d}{dt}\nu_{b_t}(g) = \text{Cov}_{\nu_{b_t}}(g, \text{Re Tr}(U(b' - b)))$ , so  $W_1(\nu_b, \nu_{b'}) \leq N\beta\Lambda(r) d(Q_{e'}, Q'_{e'})$ . With twelve partners per link,

$$(7.1) \quad \alpha \leq 12N\beta\Lambda(r) = 2r\Lambda(r) = 4\beta_W\Lambda(2\beta_W).$$

Since  $K$  is convex and contains 0, the mean of  $I, \omega I, \omega^2 I$  with  $\omega^3 = 1$ ,  $r'K \subset rK$  for  $r' \leq r$  and  $\Lambda$  is nondecreasing.

## 7.2. Gradients and one-forms.

**Lemma 7.1.** *In the left trivialisation,  $\nabla\varphi_c(U) = -PA(cU)$ ; if  $V = \varphi_b$  then  $\text{Hess } V(X, X) = \text{Re Tr}(UX^2b)$ ; and  $\Delta\varphi_c = -\lambda_F\varphi_c$ .*

*Proof.*  $\frac{d}{dt} \text{Re Tr}(Ue^{tX}c)|_{t=0} = \text{Re Tr}(cUX) = -\langle PA(cU), X \rangle$  for  $X \in \mathfrak{su}(3)$ . On a group with a bi-invariant metric  $\nabla_X X = 0$  for left-invariant  $X$ , so the Hessian is the second derivative along  $e^{tX}$ . The function  $\varphi_c$  is a matrix coefficient of the fundamental representation, on which the Casimir acts by  $\lambda_F$ .  $\square$

**Lemma 7.2.** *The bottom of the spectrum of the rough Laplacian  $\nabla^*\nabla$  on  $L^2$  one-forms of  $\text{SU}(3)$  is  $\mu_0 = \frac{7}{6}$ .*

*Proof.* Trivialise one-forms on the left, so that a one-form is a function  $f : \text{SU}(3) \rightarrow \mathfrak{su}(3)$ . For an orthonormal basis  $X_a$  acting as left-invariant vector fields,  $\nabla_{X_a} Y = \frac{1}{2}[X_a, Y]$  and  $\nabla_{X_a} X_a = 0$ , so  $\nabla^*\nabla = -\sum_a (\rho(X_a) + \frac{1}{2}\text{ad } X_a)^2$ . On the isotypic part  $V_\pi \otimes V_\pi^* \otimes \mathfrak{su}(3)$  of  $L^2 \otimes \mathfrak{su}(3)$  put  $A_a = \rho_\pi(X_a)$ ,  $B_a = \text{ad } X_a$  and  $T_a = A_a + B_a$ , the action on the tensor product. Since  $A_a$  and  $B_a$  commute,  $A_a + \frac{1}{2}B_a = \frac{1}{2}(A_a + T_a)$  and  $A_a T_a + T_a A_a = A_a^2 + T_a^2 - B_a^2$ , so on a component  $\sigma \subset \pi \otimes \text{ad}$

$$\nabla^*\nabla = \frac{1}{2}(C_\pi + C_\sigma) - \frac{1}{4}C_{\text{ad}} = \frac{1}{2}(C_\pi + C_\sigma) - \frac{3}{2},$$

with  $C(p, q) = \frac{2}{3}(p^2 + q^2 + pq + 3p + 3q)$  for the representation of highest weight  $(p, q)$  [17]. The components of  $(p, q) \otimes (1, 1)$  are  $(p+1, q+1)$ ,  $(p+2, q-1)$ ,  $(p-1, q+2)$ ,  $(p+1, q-2)$ ,  $(p-2, q+1)$ ,  $(p-1, q-1)$  and  $(p, q)$ , the last with multiplicity two, one or zero as both, one or none of  $p, q$  are positive, and only those with nonnegative labels occur. For  $\pi = (0, 0)$  the only component is  $(1, 1)$ , with value  $\frac{3}{2}$ . For  $\pi = (1, 0)$  the components are  $(2, 1)$ ,  $(0, 2)$ ,  $(1, 0)$ , with values  $\frac{31}{6}$ ,  $\frac{19}{6}$ ,  $\frac{7}{6}$ , and  $(0, 1)$  is conjugate. For  $p+q \geq 2$  the smallest values occur at  $(p-1, q-1)$ ,  $(p, q)$ ,  $(p-2, q+1)$  or  $(p+1, q-2)$ , and each is at least  $\frac{1}{3}(2(p^2 + q^2 + pq) + 3(p+q) - 3) - \frac{3}{2} \geq \frac{3}{2}$ .  $\square$

## 7.3. The gap of the one-link measure.

**Lemma 7.3.** *For  $b \in rK$  the spectral gap of  $L = \Delta + \langle \nabla V, \nabla \cdot \rangle$ ,  $V = \varphi_b$ , on  $L^2(\nu_b)$  is at least*

$$\gamma(r) = \frac{39}{64}\mu_0 + \frac{3}{2} - \frac{r}{2} = \frac{283}{128} - \frac{r}{2}.$$

*Proof.* Let  $-Lf = \lambda f$  with  $\lambda > 0$ ,  $\omega = df$  and  $\eta = e^{V/2}\omega$ ; norms are in  $L^2(dU)$ . Bochner's formula integrated against  $\nu_b$  gives  $\lambda\nu_b|\omega|^2 = \nu_b|\nabla\omega|^2 + \nu_b((\text{Ric} - \text{Hess } V)(\omega, \omega))$ , and since  $e^{V/2}\nabla\omega = \nabla\eta - \frac{1}{2}dV \otimes \eta$ ,

$$(7.2) \quad \lambda\|\eta\|^2 = \|\nabla\eta - \frac{1}{2}dV \otimes \eta\|^2 + \int (\text{Ric} - \text{Hess } V)(\eta, \eta).$$

Put  $\mathcal{A} = \nabla\eta$  and  $\mathcal{B} = \frac{1}{2}dV \otimes \eta$ . Then

$$\langle \mathcal{A}, \mathcal{B} \rangle = \frac{1}{4} \int \langle \nabla V, \nabla |\eta|^2 \rangle = \frac{\lambda_F}{4} \int V |\eta|^2,$$

because  $-\Delta V = \lambda_F V$ . Writing  $\|\mathcal{A} - \mathcal{B}\|^2 = \|\frac{5}{8}\mathcal{A} - \mathcal{B}\|^2 + \frac{39}{64}\|\mathcal{A}\|^2 - \frac{3}{4}\langle\mathcal{A}, \mathcal{B}\rangle$  and dropping the first term,

$$\|\mathcal{A} - \mathcal{B}\|^2 \geq \frac{39}{64}\mu_0\|\eta\|^2 - \frac{1}{2} \int V|\eta|^2$$

by Lemma 7.2. With  $\text{Ric} = \frac{3}{2}$ , (7.2) gives  $\lambda\|\eta\|^2 \geq (\frac{39}{64}\mu_0 + \frac{3}{2})\|\eta\|^2 - \int(\text{Hess } V(\eta, \eta) + \frac{1}{2}V|\eta|^2)$ , and the claim follows from the pointwise bound

$$(7.3) \quad \text{Re Tr}(MX^2) + \frac{1}{2} \text{Re Tr } M \leq \frac{r}{2}|X|^2, \quad M = bU \in rK,$$

since  $\text{Hess } V(X, X) = \text{Re Tr}(MX^2)$  and  $V = \text{Re Tr } M$ . The maximum over  $|X| = 1$  of the left side is convex in  $M$ , so it suffices to take  $M = rW$  with  $W \in \text{SU}(3)$  and, after a conjugation,  $W = \text{diag}(e^{i\theta_k})$  with  $\sum_k \theta_k = 0$ ; put  $c_k = \cos \theta_k$ . Write  $X = iH$  with  $H$  Hermitian, traceless,  $\text{Tr } H^2 = 1$ , with diagonal  $d_k$  and entries  $H_{jk}$ . Then

$$\text{Re Tr}(WX^2) = - \sum_k c_k d_k^2 - \sum_{j < k} (c_j + c_k) |H_{jk}|^2,$$

with  $\sum_k d_k^2 + 2 \sum_{j < k} |H_{jk}|^2 = 1$ , so the maximum is the larger of  $q_{\text{off}} = \max_{j < k} -\frac{1}{2}(c_j + c_k)$  and  $q_{\text{diag}} = \max\{-\sum_k c_k d_k^2 : \sum d_k = 0, \sum d_k^2 = 1\}$ . If  $l$  is the third index,  $q_{\text{off}} + \frac{1}{2} \sum_k c_k = \frac{1}{2}c_l \leq \frac{1}{2}$ . The value  $q_{\text{diag}}$  is minus the smaller eigenvalue of  $\text{diag}(c)$  compressed to the plane  $\sum d_k = 0$ ; the eigenvalues of the compression solve  $\sum_k (c_k - \mu)^{-1} = 0$ , that is  $3\mu^2 - 2e_1\mu + e_2 = 0$  with  $e_1, e_2$  the elementary symmetric functions of the  $c_k$ , so  $q_{\text{diag}} = \frac{1}{3}(\sqrt{e_1^2 - 3e_2} - e_1)$ . The inequality  $q_{\text{diag}} + \frac{1}{2}e_1 \leq \frac{1}{2}$  is  $\sqrt{e_1^2 - 3e_2} \leq \frac{1}{2}(3 - e_1)$ , that is  $e_1^2 - 4e_2 + 2e_1 - 3 \leq 0$ . When  $\sum \theta_k = 0$ ,  $\sum c_k^2 = 1 + 2c_1c_2c_3$ , and with this identity the left side equals  $-2(1 - c_1)(1 - c_2)(1 - c_3) \leq 0$ .  $\square$

The weight  $\frac{5}{8}$  is the one that gives the coefficient  $\frac{1}{2}$  of  $V$ , and (7.3) is an equality at  $W = I$ . At  $r = \frac{3}{2}$ ,  $\gamma = \frac{187}{128}$ .<sup>6</sup>

#### 7.4. A corrector of degree two.

**Lemma 7.4.** *For  $b \in rK$ ,  $\|h\| = 1$  and every smooth real function  $u$  on  $\text{SU}(3)$ ,*

$$L_h(b) \leq \sqrt{\nu_b |\nabla u|^2} + \sqrt{\text{Var}_{\nu_b}(\varphi_h + Lu)/\gamma(r)}.$$

*Proof.* Let  $-L\psi = \varphi_h - \nu_b \varphi_h$ . For a 1-Lipschitz  $g$ ,  $|\nabla g| \leq 1$  almost everywhere, because  $d$  is at most the geodesic distance, so  $\text{Cov}(g, \varphi_h) = \nu_b \langle \nabla g, \nabla \psi \rangle \leq \nu_b |\nabla \psi|$ . Write  $\psi = u + \chi$ . Since  $\nu_b(Lu) = 0$ ,  $\chi$  solves  $-L\chi = X - \nu_b X$  with  $X = \varphi_h + Lu$ , and

$$\nu_b |\nabla \chi|^2 = \text{Cov}(\chi, X) \leq \sqrt{\text{Var } \chi \text{ Var } X} \leq \sqrt{\nu_b |\nabla \chi|^2 \text{ Var } X / \gamma}$$

by Lemma 7.3. Hence  $\nu_b |\nabla \chi|^2 \leq \text{Var } X / \gamma$ , and  $\nu_b |\nabla \psi| \leq \sqrt{\nu_b |\nabla u|^2} + \sqrt{\nu_b |\nabla \chi|^2}$ .  $\square$

The best  $u$  is  $\psi$  itself, and then the second term vanishes. With  $u = s\varphi_h$ , Lemma 7.1 gives  $\varphi_h + Lu = (1 - s\lambda_F)\varphi_h + s\Gamma$  with  $\Gamma = \langle \nabla V, \nabla \varphi_h \rangle$ ; the choice  $s = 1/\lambda_F$  removes  $\varphi_h$  and leaves the drift. This blended corrector reaches  $\beta_W = \frac{45}{64}$  and stops near 0.715 (Section 9).<sup>7</sup> We take  $u$  in a space of polynomials of degree two.

Let  $\mathcal{P}_2$  be the real vector space of real polynomials of degree one or two in the entries  $u_{ij}$  of  $U$  and  $\bar{u}_{ij}$  of  $\bar{U}$ . It has dimension 189: a basis  $(B_k)$  consists of  $\frac{1}{2}(m + \bar{m})$  and  $\frac{1}{2i}(m - \bar{m})$  for each pair  $\{m, \bar{m}\}$  of conjugate monomials with  $m \neq \bar{m}$ , and  $m$  itself when  $m = \bar{m}$ .

<sup>6</sup>The direct Bochner bound, with Minkowski's inequality in place of the cross term, gives  $(\sqrt{\mu_0} - \frac{g}{2}r)^2 + \frac{3}{2} - r$  with  $g^2 = \max_W \|PA(W)\|^2 = 2.0655$ ; it is exact at  $b = 0$ , where the gap is  $\frac{8}{3}$ , but falls below  $\gamma$  for  $r > 0.24$  and vanishes before  $r = 1$ .

<sup>7</sup>An earlier version of this paper proved Theorem 1.5 for  $\beta_W \leq \frac{45}{64}$  with  $u = s\varphi_h$ ,  $s$  chosen box by box, on 865 boxes at  $r = \frac{45}{32}$ ; the largest bound was 0.999657.

Identify  $h \in \mathbb{C}^{3 \times 3}$  with  $\mathbb{R}^{18}$  through the real and imaginary parts of its entries. For a real  $189 \times 18$  matrix  $C$  put

$$q_h = \sum_k \sum_{m=1}^{18} C_{km} h_m B_k \in \mathcal{P}_2,$$

linear in  $h$ . For diagonal  $b = \text{diag}(a_1, a_2, x_3 + iy_3)$  we have  $V = \sum_k b_k \varphi_{F_k}$  with  $(b_k) = (a_1, a_2, x_3, y_3)$  and  $F = (e_{11}, e_{22}, e_{33}, ie_{33})$ , so

$$(7.4) \quad X_h = \varphi_h + Lq_h = \varphi_h + \Delta q_h + \sum_k b_k \langle \nabla \varphi_{F_k}, \nabla q_h \rangle$$

is a polynomial of degree at most three, affine in  $b$ , while  $\nabla q_h$  does not depend on  $b$ . The squares of these functions have degree at most six, so the quadratic forms  $h \mapsto \nu_b |\nabla q_h|^2$  and  $h \mapsto \nu_b [(X_h - \kappa)^2]$  are combinations of moments of  $\nu_b$  of degree at most six.

To compute  $\Delta$  and  $\langle \nabla f, \nabla g \rangle$  exactly we use the basis of  $\mathfrak{su}(3)$

$$e_{jk} - e_{kj}, \quad i(e_{jk} + e_{kj}) \ (j < k), \quad i(e_{11} - e_{22}), \quad i(e_{11} + e_{22} - 2e_{33}),$$

acting as left-invariant fields  $X_a f(U) = \frac{d}{dt} f(Ue^{tX_a})|_{t=0}$ , with weights  $w_a = \frac{1}{2}$  for the first seven and  $\frac{1}{6}$  for the last. Then  $\sum_a w_a X_a \otimes X_a$  is the Casimir tensor of  $\langle X, Y \rangle = -\text{Tr}(XY)$ , so

$$\langle \nabla f, \nabla g \rangle = \sum_a w_a X_a f X_a g, \quad \Delta = \sum_a w_a X_a^2.$$

The fields map monomials to combinations of monomials of the same degree with coefficients in  $\mathbb{Z}[i]$ ; for instance  $\Delta \varphi_h = -\frac{8}{3} \varphi_h$ , as in Lemma 7.1.

### 7.5. Orbits of staple sums.

**Lemma 7.5.** *Let  $V, W \in \text{SU}(3)$ .*

- (i)  $L_h(VbW) = L_{V^{-1}hW^{-1}}(b)$  and  $L_h(\bar{b}) = L_{\bar{h}}(b)$ .
- (ii) Every  $b \in \mathbb{C}^{3 \times 3}$  is in the orbit of some  $\text{diag}(a_1, a_2, z_3)$  with  $a_1 \geq a_2 \geq |z_3|$  and  $\text{Im } z_3 \geq 0$ .
- (iii) A diagonal matrix lies in  $K$  if and only if it lies in the convex hull of the diagonal torus  $T$  of  $\text{SU}(3)$ .

*Proof.* (i) If  $U$  has law  $\nu_{VbW}$  then  $WUV$  has law  $\nu_b$ ; the map preserves  $d$  and Haar measure, and  $\varphi_h(U) = \varphi_{V^{-1}hW^{-1}}(WUV)$ . For  $\bar{b}$  use  $U \mapsto \bar{U}$ . (ii) Take a singular value decomposition  $b = V_1 \Sigma W_1$  and move the determinants of  $V_1, W_1$  into the last entry; a complex conjugation makes  $\text{Im } z_3 \geq 0$ . (iii) Let  $c$  be diagonal with nonzero entries of distinct moduli and let  $U$  maximise  $\text{Re Tr}(cU)$  on  $\text{SU}(3)$ . The derivative along  $\mathfrak{su}(3)$  vanishes, so the anti-Hermitian part of  $cU$  is a multiple  $i\lambda I$  of the identity and  $cU = H + i\lambda I$  with  $H$  Hermitian. Then  $H^2 + \lambda^2 I = cc^*$  is diagonal with distinct entries,  $H$  commutes with it and is diagonal, and  $U \in T$ . Hence  $\max_{\text{SU}(3)} \text{Re Tr}(cU) = \max_T \text{Re Tr}(cU)$  for such  $c$ , and by continuity for every diagonal  $c$ . A diagonal matrix outside  $\text{conv } T$  is separated from it by such a functional, which then separates it from  $K$ .  $\square$

Since every  $b \in rK$  has operator norm at most  $r$ ,

$$\Lambda(r) = \sup \left\{ \sup_{\|h\|=1} L_h(b) : b \in \mathcal{R}_r \right\},$$

where  $\mathcal{R}_r$  is the set of  $\text{diag}(a_1, a_2, x_3 + iy_3) \in r \text{ conv } T$  with  $a_1 \geq a_2 \geq |x_3 + iy_3|$  and  $y_3 \geq 0$ , a subset of the box  $[0, r]^2 \times [-r, r] \times [0, r]$  in the coordinates  $(a_1, a_2, x_3, y_3)$ .

### 7.6. Exact moments and interpolation.

**Lemma 7.6.** *For  $D, D' \in \mathbb{C}^{3 \times 3}$  let  $\mathbb{E}_H$  be the Haar mean on  $\mathrm{SU}(3)$  and*

$$Z(D, D') = \mathbb{E}_H \exp\left(\frac{1}{2} \mathrm{Tr}(UD) + \frac{1}{2} \mathrm{Tr}(\bar{U}D')\right).$$

*Then*

$$Z(D, D') = \sum_{p \geq q \geq 0} \frac{s_{(p,q)}(DD'^\top) G_{p,q}(\det D) G_{p,q}(\det D')}{d_{p,q}},$$

*with  $G_{p,q}(z) = \sum_{k \geq 0} c_{(p+k, q+k, k)} z^k$ ,  $c_\lambda = 2^{-|\lambda|} / \prod \text{hooks}(\lambda)$ ,  $d_{p,q} = \frac{1}{2}(p-q+1)(q+1)(p+2)$  and  $s_{(p,q)}$  the Schur polynomial. The moments of degree  $j$  in  $U$  and  $k$  in  $\bar{U}$  are  $\mathbb{E}_H[u^\alpha \bar{u}^\beta e^{\mathrm{Re} \mathrm{Tr}(Ub)}] = 2^{|\alpha|+|\beta|} \partial_D^\alpha \partial_{D'}^\beta Z(b, \bar{b})$ , the entry  $u_{ij}$  pairing with  $D_{ji}$ .*

*Proof.* We use  $(\mathrm{Tr} X)^n = \sum_{\lambda \vdash n} f^\lambda s_\lambda(X)$ , where  $f^\lambda$  is the number of standard tableaux of shape  $\lambda$ , given by the hook length formula [32, Cor. 7.21.6]. Expanding the exponential gives

$$e^{\frac{1}{2} \mathrm{Tr} X} = \sum_{n \geq 0} \frac{(\mathrm{Tr} X)^n}{2^n n!} = \sum_{\lambda} c_\lambda s_\lambda(X),$$

the sum over partitions with at most three parts. For  $U \in \mathrm{SU}(3)$  one has  $s_{\lambda+(k,k,k)}(UD) = (\det D)^k s_\lambda(UD)$ , so

$$e^{\frac{1}{2} \mathrm{Tr}(UD)} = \sum_{p \geq q} G_{p,q}(\det D) \chi_{p,q}(UD),$$

where  $\chi_{p,q}$  is the irreducible character of highest weight  $(p, q, 0)$  and  $d_{p,q}$  its dimension. Since  $\bar{U} = (U^{-1})^\top$ , we have  $\chi_{p,q}(\bar{U}D') = \chi_{p,q}(U^{-1}D'^\top)$ , and Schur orthogonality [12, Ch. 2],

$$\mathbb{E}_H[\chi_\rho(UA) \chi_{\rho'}(U^{-1}B)] = \delta_{\rho\rho'} \frac{\chi_\rho(AB)}{d_\rho},$$

gives the expansion. The bound below gives absolute convergence. Finally  $\mathrm{Re} \mathrm{Tr}(Ub) = \frac{1}{2} \mathrm{Tr}(Ub) + \frac{1}{2} \mathrm{Tr}(\bar{U}\bar{b})$ .  $\square$

The computation keeps  $p \leq 34$  and  $k \leq 11$ . Every omitted partition has more than 34 boxes, and since  $|s_\lambda(X)| \leq \dim(\lambda) \|X\|^{|\lambda|}$  and  $\sum_{\lambda \vdash n} f^\lambda \dim \lambda = 3^n$  by Schur–Weyl duality, the omitted part of each factor is at most  $\sum_{n > 34} (\frac{3}{2} \|D\|)^n / n!$ ; Cauchy’s estimate on a polydisc around  $b$  turns this into a bound for every derivative (Appendix E).

**Lemma 7.7** (Interpolation on a box). *Let  $Q = \prod_k [l_k, l_k + w_k]$  in the coordinates  $(a_1, a_2, x_3, y_3)$ , with corners  $b_i$  and multilinear weights  $\omega_i(b) \geq 0$ ,  $\sum_i \omega_i(b) = 1$ , and let  $\varsigma = w_1^2 + w_2^2 + \max(w_3^2, w_4^2)$ . For  $b \in Q$ :*

- (i)  $Z(b) \geq e^{-\varsigma/8} \sum_i \omega_i(b) Z(b_i)$ ;
- (ii)  $\nu_b(f) \leq e^{\varsigma/8} \max_i \nu_{b_i}(f)$  for every  $f \geq 0$  that does not depend on  $b$ ;
- (iii) if  $Y_b = (\mathcal{A} + \sum_k (b_k - \beta_k) \mathcal{B}_k)^2$  with functions  $\mathcal{A}, \mathcal{B}_k$  of  $U$  and a fixed point  $\beta$ , then

$$\nu_b(Y_b) \leq e^{\varsigma/8} \left( \max_i \nu_{b_i}(Y_{b_i}) + \sum_k \frac{w_k^2}{4} \frac{\max_i Z(b_i) \nu_{b_i}(\mathcal{B}_k^2)}{\min_i Z(b_i)} \right).$$

*Proof.* Write  $E(b) = e^{\mathrm{Re} \mathrm{Tr}(Ub)} = \prod_k e^{c_k b_k}$ , where

$$c = (\mathrm{Re} u_{11}, \mathrm{Re} u_{22}, \mathrm{Re} u_{33}, -\mathrm{Im} u_{33}),$$

so that  $|c_k| \leq 1$  and  $c_3^2 + c_4^2 \leq 1$ . In each coordinate the chord of  $e^{c_k x}$  over  $[l_k, l_k + w_k]$  lies above the function and, by Hoeffding’s lemma [21] for a two-point variable, below  $e^{c_k x} e^{c_k^2 w_k^2 / 8}$ . Multiplying,

$$(7.5) \quad E(b) \leq \sum_i \omega_i(b) E(b_i) \leq E(b) e^{\sum_k c_k^2 w_k^2 / 8} \leq E(b) e^{\varsigma/8}.$$

Integrating the right inequality gives (i), and (ii) follows from the left one and (i). For (iii) let  $\mathcal{F}(b) = Y_b E(b)$  and  $I_k$  the linear interpolation in the  $k$ -th coordinate, so that  $\mathcal{F} - I_1 I_2 I_3 I_4 \mathcal{F} = \sum_k I_1 \cdots I_{k-1} (\mathcal{F} - I_k \mathcal{F})$ . Along the  $k$ -th coordinate  $\mathcal{F}$  is  $\ell^2 e^{c_k x}$  times a positive factor, with  $\ell$  affine of slope  $\mathcal{B}_k$ , and  $\partial_k^2 \mathcal{F} = ((c_k \ell + 2\mathcal{B}_k)^2 - 2\mathcal{B}_k^2) E \geq -2\mathcal{B}_k^2 E$ . With the Green kernel of the interval, whose integral is at most  $w_k^2/8$ ,

$$\mathcal{F} - I_k \mathcal{F} \leq 2\mathcal{B}_k^2 \int G E \leq \frac{w_k^2}{4} \mathcal{B}_k^2 \sum_i \omega_i^{(k)} E(b_i)$$

by the left inequality of (7.5), with weights  $\omega_i^{(k)} \geq 0$  of sum one that do not depend on  $U$ . The operators  $I_j$  have nonnegative weights of sum one; integrate over  $U$  and divide by  $Z(b)$ , bounded below by (i).  $\square$

**7.7. The cover and the theorem.** Fix a box  $Q$  and a matrix  $C$ . For any constant  $\kappa$ ,  $\text{Var}(X_h) \leq \nu_b[(X_h - \kappa)^2]$ ; take  $\kappa = \langle \kappa(b), h \rangle$  with  $\kappa(b)$  affine in  $b$ , fitted to the means at the corners. By (7.4),  $X_h - \kappa = \mathcal{A} + \sum_k (b_k - \beta_k) \mathcal{B}_k$  with  $\beta$  the centre of  $Q$  and  $\mathcal{B}_k = \langle \nabla \varphi_{F_k}, \nabla q_h \rangle - \langle \partial_k \kappa, h \rangle$ , and Lemmas 7.4 and 7.7 give, for all  $b \in Q \cap \mathcal{R}_r$ ,

$$(7.6) \quad \sup_h L_h(b) \leq \sqrt{e^{c/8} \max_i \lambda_A(b_i)} + \sqrt{\frac{e^{c/8}}{\gamma(r)} \left( \max_i \lambda_V(b_i) + \frac{e_N}{\min_i Z(b_i)} \right)},$$

where  $e_N = \sum_k \frac{w_k^2}{4} \max_i Z(b_i) \lambda_k(b_i)$  and  $\lambda_A, \lambda_V, \lambda_k$  are the largest eigenvalues of the quadratic forms  $h \mapsto \nu_b |\nabla q_h|^2$ ,  $h \mapsto \nu_b [(X_h - \kappa)^2]$  at  $b = b_i$ , and  $h \mapsto \nu_{b_i} [\mathcal{B}_k^2]$  on  $\mathbb{R}^{18}$ . Their entries are moments of  $\nu_{b_i}$  of degree at most six, computed with rigorous error bounds (Appendix E); each largest eigenvalue is bounded above by an  $LDL^\top$  factorisation of  $tI$  minus the computed form with certified positive pivots, plus the norm of its error matrix by Weyl's inequality [19, 23].

The matrix  $C$  is chosen in each box in floating point. At the mean of the corner moments and at  $b = \beta$ , Lemma 7.4 for  $u = q_h$  is a quadratic problem in  $C$ : for  $\theta > 0$  we minimise the trace of  $\nu |\nabla q|^2 + \theta \text{Var}(X)/\gamma$  over the 18 coordinate directions  $h$ , a least-squares problem of size 189, keep the  $\theta$  on a geometric grid of 48 values from 0.05 to 80 that gives the smallest bound, and round  $C$  to multiples of  $2^{-30}$ . Any choice gives a valid bound.

A box is discarded when it contains no point with  $a_1 \geq a_2 \geq |z_3|$ , or when a linear functional  $\ell(b) = \text{Re Tr}(c^* b)$  satisfies  $\min_Q \ell > r \max_T \ell$ , which separates it from  $r \text{conv } T$  and hence, by Lemma 7.5(iii), from  $\mathcal{R}_r$ .

**Proposition 7.8.**  $2r\Lambda(r) \leq 0.999887$  at  $r = \frac{3}{2}$ .

*Proof.* The box  $[0, r]^2 \times [-r, r] \times [0, r]$  is cut into 8192 boxes of side  $r/8$ . Of these, 841 survive the two tests, and 581 satisfy  $2rB_Q < 1$  for the right side  $B_Q$  of (7.6), with  $\gamma(r) = \frac{187}{128}$ . Each of the other 260 is cut into sixteen boxes of side  $r/16$ ; 1760 of these survive the tests, and all satisfy  $2rB_Q < 1$ . The 2341 certified boxes cover  $\mathcal{R}_r$ . The largest bound is 0.999887, on the box  $[1.3125, 1.5] \times [0.9375, 1.125] \times [0.5625, 0.75] \times [0, 0.1875]$ , and the median is 0.983. Figure 3 shows the cover, and Appendix E the error analysis.  $\square$

The same cover at a larger  $r$  bounds  $\Lambda(r)$  where  $2r\Lambda(r)$  is no longer below one; the plaquette blocks of Section 8 use that bound.

**Proposition 7.9.**  $2r\Lambda(r) \leq 1.029967$  at  $r = \frac{38}{25}$ .

*Proof.* As in the proof of Proposition 7.8, with  $\gamma(r) = \frac{4643}{3200}$  and the threshold  $2rB_Q < 1.03$ : of the 8192 boxes of side  $r/8$ , 784 survive the two tests and 690 satisfy the threshold; each of the other 94 is cut into sixteen boxes of side  $r/16$ , of which 373 survive the tests, and all satisfy it. The 1063 certified boxes cover  $\mathcal{R}_r$ , and the largest bound is 1.029967.  $\square$

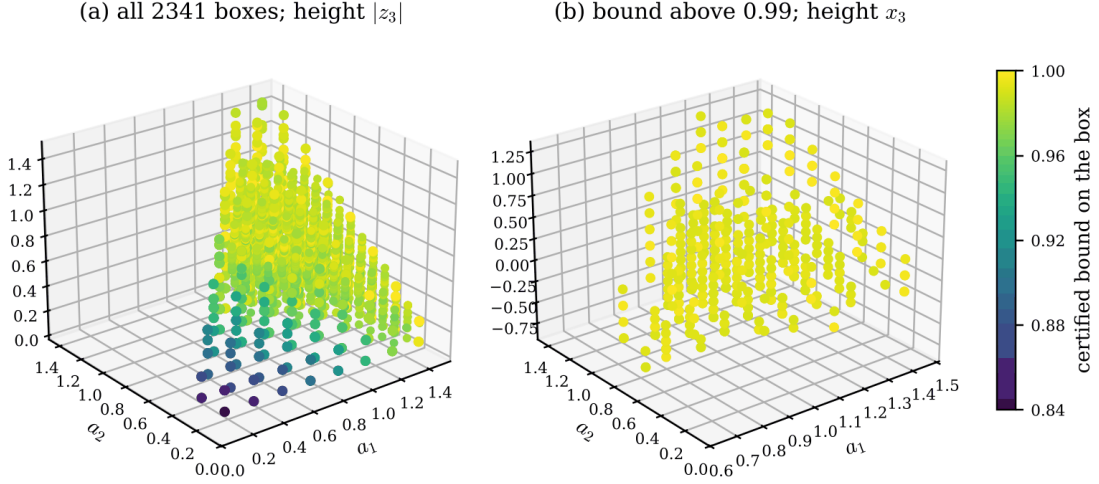


FIGURE 3. The 2341 certified boxes at  $r = \frac{3}{2}$ , plotted at their centres in the coordinates  $(a_1, a_2)$  of the staple sum  $\text{diag}(a_1, a_2, x_3 + iy_3)$ , coloured by the certified bound on  $2r \sup_h L_h$  over the box; the larger points are the boxes of side  $r/8$ . (a) All boxes, with height  $|x_3 + iy_3|$ . (b) The 419 boxes with bound above 0.99, with height  $x_3$ ; the largest bounds sit in the interior of  $\mathcal{R}_r$ , not at its extreme points.

*Proof of Theorem 1.5.* Proposition 7.8 gives

$$\Lambda\left(\frac{3}{2}\right) \leq \frac{0.999887}{3} < 0.333296.$$

For  $\beta_W \leq \frac{3}{4}$  we have  $2\beta_W \leq \frac{3}{2}$ , and (7.1) with the monotonicity of  $\Lambda$  gives

$$\alpha \leq 4\beta_W \Lambda\left(\frac{3}{2}\right) < 1.333183 \beta_W \leq \frac{3}{4} \cdot 1.333183 < 1.$$

Proposition 3.2 applies with the Hilbert–Schmidt metric.  $\square$

## 8. SU(3) WITH PLAQUETTE BLOCKS

In this section  $G = \text{SU}(3)$ ,  $\beta \geq 0$ ,  $\beta_W = 9\beta$ ,  $N\beta = \beta_W/3$ , and the notation is that of Section 7:  $d$  is the Hilbert–Schmidt distance,  $K = \text{conv SU}(3)$ ,  $\nu_c(dU) \propto e^{\text{Re Tr}(Uc)} dU$ ,  $\varphi_h(U) = \text{Re Tr}(Uh)$  and  $\Lambda(r)$  is the modulus of (7.1). Put  $Z(c) = \int e^{\text{Re Tr}(Uc)} dU$  and  $g = \log Z$  on  $\mathbb{C}^{3 \times 3}$ , viewed as  $\mathbb{R}^{18}$  with the inner product  $\langle x, y \rangle = \text{Re Tr}(xy^*)$ . The gradient of  $g$  at  $c$  is the conjugate mean  $m(c)^*$ ,  $m(c) = \nu_c(U) \in K$ , and  $\text{Hess } g(c)(h, h) = \text{Var}_{\nu_c}(\varphi_h)$ . Since  $Z(VcW) = Z(c)$  for  $V, W \in \text{SU}(3)$ , the same holds for  $g$ .

The blocks are those of Lemma 5.1, which concerns the line field only and holds for every group. For a block plaquette with active links  $e_1, e_2, e_3$ ,  $\text{Re Tr } Q_p = \text{Re Tr}(y_1 y_2 y_3)$  with  $y_i = Q_{e_i}$  or  $Q_{e_i}^{-1}$ , and the conditional law of the block is

$$(8.1) \quad \mu_b(dy) \propto \exp\left(\sum_i \text{Re Tr}(y_i b_i) + N\beta \text{Re Tr}(y_1 y_2 y_3)\right) dy,$$

where  $b_i$  is  $N\beta$  times the sum of the five staples of  $y_i$  outside the block plaquette, so  $b_i \in 5N\beta K$  and  $\|b_i\|_{\text{op}} \leq 5N\beta$ . On  $\text{SU}(3)^3$  we use  $d(y, y') = \sum_i \|y_i - y'_i\|_{\text{HS}}$ .

**8.1. The sensitivity of one block.** Let  $r = 2\beta_W = 6N\beta$ . Let  $\lambda_*$  and  $\tau$  be a lower and an upper bound for the eigenvalues of  $\text{Hess } g$  on  $rK$ ,  $M$  an upper bound on  $rK$  for  $|D^3g(c)(u, v, v)|$  over unit vectors  $u, v$ , and  $\gamma = \gamma(r)$  the gap of Lemma 7.3. Put

$$(8.2) \quad \begin{aligned} w &= \frac{3}{2}(N\beta)^2(\tau - \lambda_*), & s &= \Lambda(r) + (e^w - 1)e^w \sqrt{\tau/\gamma}, \\ C &= 5(N\beta)^2 \left( \tau s + \frac{\sqrt{3} N\beta M e^w}{\gamma} \right), \\ K_3 &= \frac{(1 + N\beta\Lambda(r)) s}{1 - C}, & \alpha_3 &= 10N\beta K_3, \end{aligned}$$

assuming  $C < 1$ .

**Lemma 8.1.** *Let  $a, c_0 \in 5N\beta K$  and*

$$\kappa(dx) \propto \exp(\text{Re Tr}(xa) + g(c_0 + N\beta x))dx \quad \text{on } \text{SU}(3).$$

*For every 1-Lipschitz  $f$  and every  $h \in \mathbb{C}^{3 \times 3}$ :*

- (i)  $\text{Cov}_\kappa(f, \varphi_h) \leq s \|h\|_{\text{HS}}$ ;
- (ii) *if  $c_0$  moves in the direction  $k$  inside  $5N\beta K$ , then*

$$\left| \frac{d}{dt} \kappa(f) \right| \leq N\beta(\tau s + \sqrt{3}N\beta M e^w / \gamma) \|k\|_{\text{HS}}.$$

*Proof.* (i) The segment from  $c_0$  to  $c_0 + N\beta x$  lies in  $6N\beta K = rK$ , and Taylor's formula gives

$$g(c_0 + N\beta x) = g(c_0) + N\beta \text{Re Tr}(x m(c_0)) + R(x)$$

with  $R(x) = \frac{1}{2}(N\beta)^2 \text{Hess } g(\xi)(x, x)$ . Since  $\|x\|_{\text{HS}}^2 = 3$ ,  $R \in [\frac{3}{2}(N\beta)^2 \lambda_*, \frac{3}{2}(N\beta)^2 \tau]$  and  $\text{osc } R \leq w$ . Hence  $\kappa$  is the tilt  $e^{R\nu}/\nu(e^R)$  of  $\nu = \nu_{c'}$  with  $c' = a + N\beta m(c_0) \in rK$ . By definition of  $\Lambda$ ,  $\text{Cov}_\nu(f, \varphi_h) \leq \Lambda(r) \|h\|$ . The function  $f$  is 1-Lipschitz for the geodesic distance, which dominates  $d$ , so  $\text{sd}_\nu f \leq \gamma^{-1/2}$  by Lemma 7.3, and  $\text{sd}_\nu \varphi_h \leq \sqrt{\tau} \|h\|$ . Lemma 5.2 gives (i).

(ii) The derivative is  $\text{Cov}_\kappa(f, \langle \nabla g(c_0 + N\beta x), k \rangle)$ . Write  $\nabla g(c_0 + N\beta u) = \nabla g(c_0) + N\beta \text{Hess } g(c_0)u + N(u)$ . The linear term is  $\varphi_h$  with  $\|h\| \leq N\beta \tau \|k\|$  and contributes at most  $N\beta \tau s \|k\|$  by (i). On the segment from  $c_0$  the Hessian moves by at most  $N\beta M \|u\| \leq \sqrt{3}N\beta M$ , so  $x \mapsto \langle N(x), k \rangle$  is  $\sqrt{3}(N\beta)^2 M \|k\|$ -Lipschitz for  $d$ . The law  $\kappa$  is a tilt of  $\nu$  by a function of oscillation at most  $w$ , so its spectral gap is at least  $e^{-w}\gamma$  by the Holley–Stroock lemma [20], and the covariance of two Lipschitz functions under  $\kappa$  is at most  $e^w/\gamma$  times the product of their constants.  $\square$

**Lemma 8.2.** *Let  $C < 1$ . If  $b, b'$  have all fields in  $5N\beta K$  and differ only in  $b_i$ , then  $W_1(\mu_b, \mu_{b'}) \leq K_3 \|b_i - b'_i\|_{\text{HS}}$  for the metric  $d$ .*

*Proof.* By the cyclic symmetry of  $\text{Re Tr}(y_1 y_2 y_3)$  we may take  $i = 1$ . Given  $y_1, y_3$ , the law of  $y_2$  is  $\nu_c$  with  $c = b_2 + N\beta y_3 y_1 \in rK$ . Changing  $y_1$  by  $\eta$  in  $d$  moves  $c$  by  $N\beta \eta$  and  $\nu_c$  by at most  $N\beta \Lambda(r) \eta$  in  $W_1$ , as in (7.1); so  $F(y_1, y_3) = \mathbb{E}[f | y_1, y_3]$  is  $(1 + N\beta \Lambda(r))$ -Lipschitz in each variable. The law of  $(y_1, y_3)$  is proportional to  $e^{\text{Re Tr}(y_1 b_1) + \text{Re Tr}(y_3 b_3)} Z(b_2 + N\beta y_3 y_1)$ , and  $Z(b_2 + N\beta y_3 y_1) = Z(y_3^{-1} b_2 + N\beta y_1) = Z(b_2 y_1^{-1} + N\beta y_3)$ , so its two conditional laws have the form of Lemma 8.1, with  $c_0 = y_3^{-1} b_2$  and  $c_0 = b_2 y_1^{-1}$ . Moving  $y_3$  to  $y'_3$  moves  $y_3^{-1} b_2$  along a segment in  $5N\beta K$  of length at most  $\|b_2\|_{\text{op}} \|y_3 - y'_3\| \leq 5N\beta \|y_3 - y'_3\|$ . The coupling argument of Lemma 5.4 now gives  $d_1 \leq \varepsilon s + C d_3$ ,  $d_3 \leq C d_1$  and  $|\mu_b(f) - \mu_{b'}(f)| \leq (1 + N\beta \Lambda(r))(d_1 + d_3) \leq K_3 \varepsilon$ .  $\square$

## 8.2. The constants and the proof of Theorem 1.6.

**Lemma 8.3.** *At  $r = \frac{3}{2}$  every eigenvalue of  $\text{Hess } g$  on  $rK$  lies in  $[0.056015, 0.247316]$ , and  $|D^3g(c)(u, v, v)| \leq 0.182126$  for  $c \in rK$  and unit vectors  $u, v$ . At  $r = \frac{38}{25}$  the same holds with  $[0.054978, 0.248280]$  and  $0.183546$ .*

*Proof.* For  $V, W \in \text{SU}(3)$ ,  $g(VcW) = g(c)$  and  $x \mapsto VxW$  is an isometry, so the eigenvalues of  $\text{Hess } g$  and the norm of  $D^3g$  are constant on the orbits, and by Lemma 7.5 it suffices to bound them on  $\mathcal{R}_r$ . The derivatives of  $g$  are the cumulants of  $\varphi$  under  $\nu_c$ . Cut the box  $[0, r]^2 \times [-r, r] \times [0, r]$  into boxes of side  $r/8$  and keep the 841 (784 at  $r = \frac{38}{25}$ ) that survive the two tests of Section 7.7. On each box  $Q$  we compute, with rigorous error bounds, the moments of  $\varphi$  of degree at most three at the sixteen corners  $b_i$  and at the centre  $b_0$ , and put  $k = m(b_0)$ , the mean of  $\varphi$  at the centre.

*Upper bound.* For  $b \in Q$  and every  $h$ ,  $\text{Var}_{\nu_b}(\varphi_h) \leq \nu_b[(\varphi_h - \langle k, h \rangle)^2] \leq e^{\varsigma/8} \max_i \nu_{b_i}[(\varphi_h - \langle k, h \rangle)^2]$  by Lemma 7.7(ii).

*Lower bound.* Integrating the right inequality of (7.5) against  $f \geq 0$  and dividing by  $Z(b) \leq \sum_i \omega_i Z(b_i)$ , the left one, gives  $\nu_b(f) \geq e^{-\varsigma/8} \min_i \nu_{b_i}(f)$ . With  $f = (\varphi_h - \nu_b \varphi_h)^2$  this gives  $\text{Var}_{\nu_b}(\varphi_h) \geq e^{-\varsigma/8} \min_i \text{Var}_{\nu_{b_i}}(\varphi_h)$ .

*Third cumulant.* Put  $\delta = m(b) - k$ . The mean value theorem and the upper bound give  $\|\delta\| \leq \tau\rho + e$ , with  $\rho$  the largest distance from  $b_0$  in  $Q$  and  $e$  the error of the computed mean. Expanding the central moment about  $k$ ,

$$\begin{aligned} D^3g(b)(u, v, v) &= T_b(u, v, v) - \delta_u \text{Var}(\varphi_v) \\ &\quad - 2\delta_v \text{Cov}(\varphi_u, \varphi_v) - \delta_u \delta_v^2, \end{aligned}$$

with  $T_b = \nu_b[(\varphi - k)^{\otimes 3}]$ ,  $\delta_u = \langle \delta, u \rangle$  and the moments under  $\nu_b$ . For a function  $G$  of either sign, writing  $G = G^+ - G^-$  and applying the two bounds above to  $G^\pm$  gives  $|\nu_b(G)| \leq (\max_i |\nu_{b_i}(G)| + \varepsilon \max_i \nu_{b_i}|G|)/(1 - \varepsilon)$  with  $\varepsilon = 1 - e^{-\varsigma/8}$ . For  $G = (\varphi_u - k_u)(\varphi_v - k_v)^2$  we use  $|\nu_{b_i}(G)| \leq \|T_{b_i}\|$ , the largest singular value of the  $18 \times 324$  unfolding of  $T_{b_i}$ , and  $\nu_{b_i}|G| \leq 2\sqrt{3} \nu_{b_i}[(\varphi_v - k_v)^2]$ , since  $\|U - k\|_{\text{HS}} \leq 2\sqrt{3}$ . The largest eigenvalues and singular values are bounded as in Section 7.7, and the maxima over these boxes are the numbers stated (Appendix C).  $\square$

*Proof of Theorem 1.6.* Let first  $\beta_W \leq \frac{3}{4}$ , so that  $r = 2\beta_W \leq \frac{3}{2}$ . Every field met in Lemmas 8.1 and 8.2 lies in  $\frac{3}{2}K$ , so the constants of Lemma 8.3 at  $\frac{3}{2}$ ,  $\Lambda(\frac{3}{2}) \leq \frac{0.999887}{3}$  from Proposition 7.8 and  $\gamma = \gamma(\frac{3}{2}) = \frac{187}{128}$  apply; with these fixed constants every quantity in (8.2) is increasing in  $\beta_W$ . For  $\frac{3}{4} < \beta_W \leq \frac{19}{25}$  the fields lie in  $\frac{38}{25}K$ , and the constants of Lemma 8.3 at  $\frac{38}{25}$ ,  $\Lambda(\frac{38}{25}) \leq \frac{1.029967}{3.04}$  from Proposition 7.9 and  $\gamma(\frac{38}{25}) = \frac{4643}{3200}$  apply in the same way. Changing an active link  $j$  by  $\eta$  changes one staple of each of its ten partners outside its block by a matrix of norm  $\eta$ , so the field  $b_i$  of each such partner moves by  $N\beta\eta$ , and Lemma 8.2 gives  $\sum_B C_{Bj} \leq 10N\beta K_3 = \alpha_3(\beta_W)$ . In ball arithmetic, at  $\beta_W = \frac{3}{4}$ ,  $w \leq 0.017935$ ,  $s \leq 0.34088$ ,  $C \leq 0.043520$ ,  $K_3 \leq 0.38609$  and  $\alpha_3 \leq 0.96521$ ; at  $\beta_W = 0.72$ ,  $\alpha_3 \leq 0.91815$ ; at  $\beta_W = 0.76$ , with the second set of constants,  $w \leq 0.018609$ ,  $s \leq 0.34673$ ,  $C \leq 0.045770$ ,  $K_3 \leq 0.39454$  and  $\alpha_3 \leq 0.99950$ . Proposition 5.6, whose proof does not use the group, applies with the Hilbert–Schmidt metric.  $\square$

*Remark 8.4.* Theorem 1.5 ends where  $2r\Lambda(r)$  reaches one. The blocks remove the factor  $\frac{12}{10}$  of the single-link sum and keep most of it after paying for the three links: at  $\beta_W = \frac{3}{4}$  the gap bound rises from  $1.1 \cdot 10^{-4}$  to 0.0177, and the block bound is the better one for  $\beta_W \geq 0.715$ . Before the corrections for the three links their sum is  $\frac{10}{12}$  of the single-link sum  $2r\Lambda(r)$ , so it can stay below one while  $2r\Lambda(r)$  is slightly above one; Proposition 7.9 certifies  $2r\Lambda(r) \leq 1.03$  at  $r = 1.52$ , which is enough at  $\beta_W = 0.76$ , with little room. Evaluated in floating point with the constants of a scan of  $\mathcal{R}_r$ , the block sum stays below one up to  $\beta_W \approx 0.78$ ; that computation is not certified and is not used.

## 9. WHERE THE METHODS STOP

$\text{SU}(2)$ . Theorem 1.2 closes the single-link route in every forest gauge at  $\beta_W = \frac{1}{3}$ . Blocks of several links, where the influence of a boundary link on a whole block is bounded at once, in the spirit of the Dobrushin–Shlosman criterion [16], pass it: plaquette blocks reach

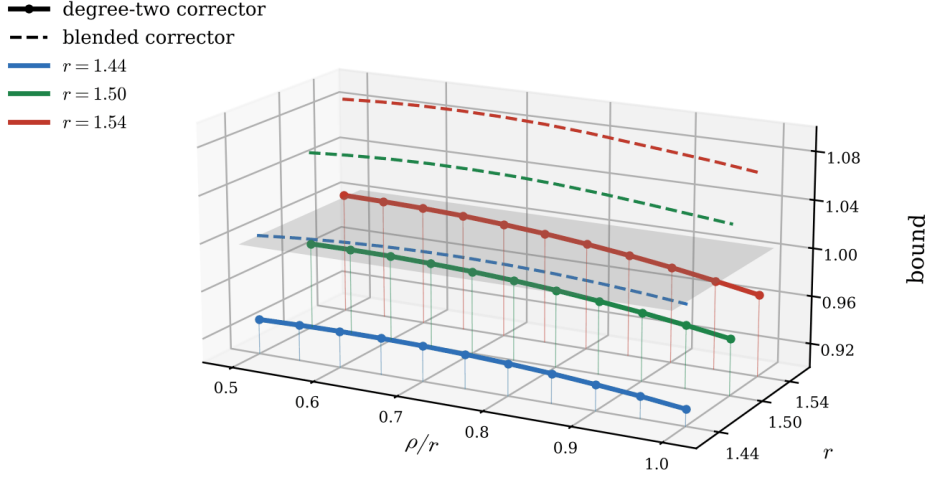


FIGURE 4. The floating-point value of  $2r \sup_h$  of the bound of Lemma 7.4 on the face  $b = \text{diag}(r, \rho, \rho)$ , as a function of  $\rho/r$ , for  $r = 1.44, 1.50$  and  $1.54$ : dashed with the corrector  $u = s\varphi_h$  for the best  $s$ , solid with  $u = q_h$  of degree two for the best  $C$ . The grey plane is the Dobrushin threshold one. The first corrector is above it already at  $r = 1.44$ ; the second stays below it at  $r = 1.50$ , where Proposition 7.8 certifies the bound on all of  $\mathcal{R}_r$ , and crosses it by  $r = 1.54$ .

$\beta_W = 0.345$  (Theorem 1.3), and the sharper floating-point bound of Remark 5.7 stops near 0.357. Larger blocks are the next step.

*Curvature.* Remark 6.4 locates the end of the curvature method in this gauge between  $|\beta| = \frac{1}{29}$  and  $\frac{1}{26.25}$ .

SU(3). The route of Section 7 bounds the  $H^{-1}(\nu_b)$  norm of  $\varphi_h - \nu_b \varphi_h$ , which is at least  $L_h(b)$ , and needs it uniformly on  $rK$ . Its Galerkin values (polynomials of degree two) at matrices of  $rK$  give  $2r$  times the norm equal to 0.79 at  $r = 1.25$ , 0.89 at  $r = 1.4$ , 0.97 at  $r = 1.5$  and 1.009 at  $r = 1.55$ , so the route stops at or below  $\beta_W \approx 0.77$  even with exact constants. With  $u = s\varphi_h$ , Lemma 7.4 stops earlier: in floating point its largest values on  $\mathcal{R}_r$  lie on the face  $\text{diag}(r, \rho, \rho)$  with  $\rho$  between  $0.75r$  and  $0.8r$ , and  $2r$  times the bound is 0.9458 at  $r = 1.36$ , 0.9767 at  $r = 1.40$  and 1.0080 at  $r = 1.44$ , so this corrector ends near  $\beta_W \approx 0.715$ . With  $u = q_h$  of degree two the largest values on the same face are 0.9305 at  $r = 1.44$ , 0.9729 at  $r = 1.50$  and 1.0018 at  $r = 1.54$  (Figure 4), close to the Galerkin values. Lemma 7.3 therefore costs little at these couplings, and most of the distance between the floating-point value at  $r = 1.5$  and the certified 0.999887 of Proposition 7.8 is the interpolation over boxes. Between  $\beta_W = \frac{3}{4}$  and 0.77 there is room only for smaller boxes. The plaquette blocks of Theorem 1.6 use  $\Lambda(2\beta_W)$  as well, with a little more room: they reach 0.76 with the certified cover of Proposition 7.9 and near 0.78 in floating point (Remark 8.4); past that one needs larger blocks, whose sensitivity does not go through the one-link modulus.

*The four open problems.* Our earlier paper [9] listed four problems between the strong-coupling results and the Clay problem [22]: (1) uniqueness and a gap at intermediate fixed coupling; (2) the crossover from strong coupling to Balaban's weak-coupling regime; (3) the continuum limit with the Osterwalder–Schrader axioms; and (4) a gap in physical units. The present paper moves the boundary of (1): for SU(2) from  $\beta_W = \frac{2}{9}$  to 0.345, for SU(3) from  $\frac{9}{32}$  to 0.76, with a gap at  $\frac{3}{4}$  of 0.0177 in lattice units, and for every SU( $N$ ) past the curvature ceiling. It does not close (1); the region between these values and the scaling

region, near  $\beta_W \approx 2.2$  for  $SU(2)$  and  $\beta_W \approx 6$  for  $SU(3)$ , is open, and with it problem (2). Problems (3) and (4) are untouched. Every bound proved here is uniform in lattice units, so by [10] none of them can by itself produce a continuum mass gap; that requires control of the gap in physical units along the renormalisation group flow [3, 11].

#### DECLARATIONS

**Data availability.** The analytic proofs are complete as written, except the evaluations in Propositions 7.8 and 7.9 and Lemma 8.3 and the numerical bounds of Lemma 5.5, which are computations with rigorous error bounds, and the finite check of Lemma 5.1. The archive [creelie/4Y-M](https://github.com/creelie/4y-m/releases/tag/v3.4.0) (release v3.4.0, <https://github.com/creelie/4y-m/releases/tag/v3.4.0>) contains the programs of that computation, in Python with python-flint (FLINT and Arb) and a C kernel for products of Taylor polynomials, with their outputs and the list of certified boxes; the programs and outputs of the earlier certificate at  $\beta_W = \frac{45}{64}$ ; independent checks in exact rational arithmetic of Lemmas 6.2 and 7.2 and of (7.3); numerical checks of Lemma 6.1 on tori; the searches of Remark 6.4 and Section 9; a Lean 4 certificate of the line field (Lemmas 2.1 and 2.2 for the table of Appendix A, by `decide`); the programs of Appendix C for the plaquette blocks, with the floating-point search of Remark 5.7; and the source of the preprint version of this paper. Deep Bhattacharjee carried out the computations and the Lean, Python and shell checks with AI assistance from Claude (Anthropic). The first composite behind Theorem 1.1 was assembled by Anaxia, the reasoning engine of the BLLM system, in release v3.0.5 [7], as described in the footnote to that theorem; release v3.1.0 [8] records the engine’s Yang–Mills runs in its failure log, whose last Yang–Mills entry is the change from single links to blocks of several links; the plaquette blocks of Section 5, the bound of Lemma 5.4, their  $SU(3)$  version in Section 8 and the certified constants were found and checked by the authors. Both authors approved the final text.

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#### APPENDIX A. THE LINE FIELD

Table 2 lists, for each residue class of  $x$  modulo two, the direction  $\sigma(x)$  and the permutation  $\pi_x(\mu) = \sigma(x + e_\mu)$ . In every row  $\pi_x$  fixes  $\sigma(x)$  and is a three-cycle on the other directions, which is the condition of Lemma 2.1.

#### APPENDIX B. THE COMPARISON LEMMA

Let  $(G, \delta)$  be a compact group with a bi-invariant metric of diameter  $D$ ,  $\Omega = G^A$ , and  $C_{ee'}$  the coefficients of Section 3, with row and column sums at most  $\alpha < 1$  and  $C_{ee'} = 0$  unless  $e, e'$  share a plaquette, so that  $|e - e'| \leq 1$  for the distance of link midpoints. Fix  $\lambda > 0$  with  $\theta = \alpha e^\lambda < 1$ .

**Lemma B.1.** *Let  $S \subset A$ , possibly infinite, and let  $\mu_1, \mu_2$  be probability measures on  $\Omega$  invariant under  $\gamma_e^F$  for every  $e \notin S$ . Let  $T \subset A \setminus S$  be finite,  $w_b = e^{-\lambda \text{dist}(b, T)}$ , and suppose  $\sum_{z \in S} w_z < \infty$ . There is a coupling of  $\mu_1, \mu_2$  whose link distances  $\varrho(b) = \mathbb{E} \delta(\sigma_b, \sigma'_b)$  satisfy*

$$\sum_{b \notin S} w_b \varrho(b) \leq \frac{\theta D}{1 - \theta} \sum_{z \in S} w_z.$$

*If  $S = \emptyset$  then  $\mu_1 = \mu_2$ .*

*Proof.* The couplings form a weakly compact set and  $\Psi(\pi) = \sum_b w_b \mathbb{E}_\pi \delta(\sigma_b, \sigma'_b)$  is weakly continuous because  $\sum_b w_b < \infty$ ; let  $\pi^*$  be a minimiser. For  $e \notin S$  replace the pair  $(\sigma_e, \sigma'_e)$  by a sample of an optimal coupling of  $\gamma_e^F(\cdot | \sigma)$  and  $\gamma_e^F(\cdot | \sigma')$ , chosen measurably [33, Cor. 5.22]. Both  $\mu_i$  are invariant, so the result is again a coupling, and changing the finitely

TABLE 2. The line field:  $x$  modulo two (digits  $x_0x_1x_2x_3$ ), the direction  $\sigma(x)$  and the values  $\sigma(x + e_\mu)$  for  $\mu = 0, 1, 2, 3$ .

| $x$  | $\sigma(x)$ | $\mu = 0$ | $\mu = 1$ | $\mu = 2$ | $\mu = 3$ |
|------|-------------|-----------|-----------|-----------|-----------|
| 0000 | 0           | 0         | 2         | 3         | 1         |
| 1000 | 0           | 0         | 3         | 1         | 2         |
| 0100 | 2           | 3         | 0         | 2         | 1         |
| 1100 | 3           | 2         | 0         | 1         | 3         |
| 0010 | 3           | 1         | 2         | 0         | 3         |
| 1010 | 1           | 3         | 1         | 0         | 2         |
| 0110 | 2           | 1         | 3         | 2         | 0         |
| 1110 | 1           | 2         | 1         | 3         | 0         |
| 0001 | 1           | 2         | 1         | 3         | 0         |
| 1001 | 2           | 1         | 3         | 2         | 0         |
| 0101 | 1           | 3         | 1         | 0         | 2         |
| 1101 | 3           | 1         | 2         | 0         | 3         |
| 0011 | 3           | 2         | 0         | 1         | 3         |
| 1011 | 2           | 3         | 0         | 2         | 1         |
| 0111 | 0           | 0         | 3         | 1         | 2         |
| 1111 | 0           | 0         | 2         | 3         | 1         |

many links that enter  $\gamma_e^F$  one at a time gives  $W_1 \leq \sum_{e'} C_{ee'} \delta(\sigma_{e'}, \sigma'_{e'})$ . Minimality gives  $\varrho(e) \leq \sum_{e'} C_{ee'} \varrho(e')$ . Multiply by  $w_e$  and sum over  $e \notin S$ ; where  $C_{ee'} \neq 0$ ,  $w_e \leq e^\lambda w_{e'}$ , so

$$\sum_{e \notin S} w_e \varrho(e) \leq \theta \left[ \sum_{e' \notin S} w_{e'} \varrho(e') + \sum_{z \in S} w_z \varrho(z) \right],$$

all sums are finite, and  $\varrho \leq D$  on  $S$ . If  $S = \emptyset$ ,  $\varrho \equiv 0$ .  $\square$

**Corollary B.2.** *Let  $\mu$  be the Gibbs measure of  $\gamma^F$ ,  $F$  a local function with support  $S_F$ ,  $L_F$ -Lipschitz in each link, and  $H$  bounded with finite support  $S_H$  disjoint from  $S_F$ . Then*

$$|\text{Cov}_\mu(F, H)| \leq \|H\|_\infty L_F \frac{\theta D}{1 - \theta} |S_H| e^{-\lambda \text{dist}(S_F, S_H)}.$$

*Proof.* Condition on the links of  $S_H$  [24, Ch. 3]; the conditional measures are invariant under  $\gamma_e^F$  for  $e \notin S_H$ , and Lemma B.1 with  $S = S_H$ ,  $T = S_F$  bounds the difference of two of them, as in [9, Cor. 29].  $\square$

**Lemma B.3.** *Let the blocks partition  $A$ , let  $C_{Bj}$  be the block coefficients of Section 5, with  $\sum_B C_{Bj} \leq \alpha < 1$  for every  $j$  and  $C_{Bj} = 0$  unless every link of  $B$  lies within distance 2 of  $j$ , and fix  $\lambda > 0$  with  $\theta = \alpha e^{2\lambda} < 1$ . Let  $S \subset A$  be a union of blocks and let  $\mu_1, \mu_2$  be invariant under  $\gamma_B^B$  for every block  $B \not\subset S$ . With  $T$ ,  $w_b$  as in Lemma B.1 and  $\sum_{z \in S} w_z < \infty$ , there is a coupling with  $\sum_{b \notin S} w_b \varrho(b) \leq \frac{\theta D}{1 - \theta} \sum_{z \in S} w_z$ , and  $\mu_1 = \mu_2$  if  $S = \emptyset$ .*

*Proof.* Take a minimiser  $\pi^*$  of  $\Psi$  as before. For a block  $B \not\subset S$  replace  $(\sigma_B, \sigma'_B)$  by an optimal coupling, for the metric  $d$ , of the two block laws; it is again a coupling, and changing the boundary links one at a time bounds the expected new  $d$ -distance by  $\sum_j C_{Bj} \varrho(j)$ . Put  $\bar{w}_B = \max_{i \in B} w_i$ . Minimality gives

$$\sum_{i \in B} w_i \varrho(i) \leq \bar{w}_B \sum_j C_{Bj} \varrho(j),$$

and  $\bar{w}_B \leq e^{2\lambda} w_j$  whenever  $C_{Bj} \neq 0$ . Summing over the blocks outside  $S$  and using the column sums,

$$\sum_{b \notin S} w_b \varrho(b) \leq \theta \left[ \sum_{j \notin S} w_j \varrho(j) + \sum_{z \in S} w_z \varrho(z) \right],$$

which gives the claim as in Lemma B.1.  $\square$

### APPENDIX C. THE PLAQUETTE BLOCKS

Table 3 lists the block plaquettes of Section 5 modulo  $2\mathbb{Z}^4$ : the plaquette at  $x$  in the plane  $\{\mu, \nu\}$  is a block plaquette exactly when  $x$  modulo two and  $\{\mu, \nu\}$  appear in the table. Eight lie in the plane  $\{0, 1\}$  and eight in  $\{2, 3\}$ ; together they contain the 48 active links of a cell of side two, each once.

TABLE 3. The sixteen block plaquettes modulo two: the base point  $x$  (digits  $x_0x_1x_2x_3$  modulo two) and the plane.

| $x$  | plane      | $x$  | plane      |
|------|------------|------|------------|
| 0000 | $\{0, 1\}$ | 0000 | $\{2, 3\}$ |
| 0001 | $\{0, 1\}$ | 0011 | $\{2, 3\}$ |
| 0010 | $\{0, 1\}$ | 0100 | $\{2, 3\}$ |
| 0011 | $\{0, 1\}$ | 0111 | $\{2, 3\}$ |
| 1100 | $\{0, 1\}$ | 1000 | $\{2, 3\}$ |
| 1101 | $\{0, 1\}$ | 1011 | $\{2, 3\}$ |
| 1110 | $\{0, 1\}$ | 1100 | $\{2, 3\}$ |
| 1111 | $\{0, 1\}$ | 1111 | $\{2, 3\}$ |

The program `cover.py` in the folder `strong-coupling/gauge-fixed/blocks/` of the archive finds this set by an exact-cover search on the torus of side two and checks Lemma 5.1 on the tori of side four and six. The program `constants.py` evaluates the bounds of Lemma 5.5 in ball arithmetic with python-flint at 120 bits: the series of  $Z_0$  and of its first three derivatives in  $s = r^2$  is summed to forty terms, and the tail is bounded for  $s \leq 5$  by twice the first omitted term  $(5/4)^{40}40^3/(40!41!)$ , because consecutive terms decrease by a factor below one half. The program `alpha.py` evaluates the constants of (5.2) and  $\alpha$  at a given  $\beta_W$ , read exactly.

For  $SU(3)$  the folder `strong-coupling/gauge-fixed/blocks/su3/` contains `consts.py`, which evaluates the bounds of Lemma 8.3 on the 841 boxes of side  $\frac{3}{16}$  at  $r = \frac{3}{2}$  and the 784 boxes of side  $\frac{19}{100}$  at  $r = \frac{38}{25}$  with the moments of Appendix E, and `alpha3.py`, which reads these bounds and the cover of Proposition 7.8 or Proposition 7.9 and evaluates the constants of (8.2) and  $\alpha_3$  in ball arithmetic at 120 bits; its outputs at  $\beta_W = \frac{3}{4}$  and 0.76 are listed with the bounds of each box. The cover of Proposition 7.9 is computed by the program of Appendix E with `-r 38/25 -thr 1.03`.

### APPENDIX D. THE BLOCKS OF LEMMA 6.2

*Diagonal blocks.* Let  $w_a = e^{i\phi}$ . On the diagonal part,  $f_p = -\cos \phi (y_1 + y_2 - y_3 - y_4)^2$  and  $c_p = 2(y_1y_4 - y_1y_2 + y_2y_3 - y_3y_4)$ , so

$$\mathcal{F} = \sum_i \alpha_i y_i^2 + \cos \phi (y_1 + y_2 - y_3 - y_4)^2 - \frac{9}{5}(y_1y_4 - y_1y_2 + y_2y_3 - y_3y_4)$$

with  $y_k = 0$  and weights  $\alpha_{k\pm 1} = \frac{21}{10}$ ,  $\alpha_{k+2} = \frac{5}{4}$ . It is affine in  $\cos \phi$ , so positivity at  $\cos \phi = \pm 1$  suffices. For  $k = 1$  and  $\cos \phi = 1$  the matrix in  $(y_2, y_3, y_4)$  is

$$\begin{pmatrix} \frac{31}{10} & -\frac{19}{10} & -1 \\ -\frac{19}{10} & \frac{9}{4} & \frac{19}{10} \\ -1 & \frac{19}{10} & \frac{31}{10} \end{pmatrix},$$

and Table 4 lists the leading principal minors for every position; all are positive.

TABLE 4. Leading principal minors of the diagonal blocks of Lemma 6.2.

| Fixed link | $\cos \phi = 1$ |                   |                     | $\cos \phi = -1$ |                  |                   |
|------------|-----------------|-------------------|---------------------|------------------|------------------|-------------------|
| $e_1, e_4$ | $\frac{31}{10}$ | $\frac{673}{200}$ | $\frac{8421}{2000}$ | $\frac{11}{10}$  | $\frac{53}{200}$ | $\frac{21}{2000}$ |
| $e_2$      | $\frac{31}{10}$ | $\frac{861}{100}$ | $\frac{8421}{2000}$ | $\frac{11}{10}$  | $\frac{21}{100}$ | $\frac{21}{2000}$ |
| $e_3$      | $\frac{9}{4}$   | $\frac{673}{200}$ | $\frac{8421}{2000}$ | $\frac{1}{4}$    | $\frac{53}{200}$ | $\frac{21}{2000}$ |

*Pair blocks.* Let  $w_a = e^{i(\chi+\vartheta)}$  and  $w_b = e^{i(\chi-\vartheta)}$ . The terms of  $f_p$  carry  $w_a + \bar{w}_b = 2 \cos \chi e^{i\vartheta}$  or its conjugate, and  $c_p$  carries  $w_a \bar{w}_b = e^{2i\vartheta}$ , so the Hermitian matrix  $M(m, \vartheta)$  of  $\mathcal{F}$ ,  $m = \cos \chi$ , is affine in  $m$  with  $M(-1, \vartheta) = M(1, \vartheta + \pi)$ , and  $m = 1$  suffices. For the fixed link  $e_2$ , in the variables  $(z_1, z_3, z_4)$  and with  $c = \cos \vartheta$ ,

$$M = \begin{pmatrix} \frac{21}{5} + 2c & -2e^{i\vartheta} & -\frac{9}{5} - 2e^{i\vartheta} \\ -2e^{-i\vartheta} & \frac{21}{5} + 2c & \frac{9}{5} + 2e^{-i\vartheta} \\ -\frac{9}{5} - 2e^{-i\vartheta} & \frac{9}{5} + 2e^{i\vartheta} & \frac{5}{2} + 2c \end{pmatrix}.$$

With  $u = \cos^2(\vartheta/2) \in [0, 1]$  its leading principal minors are

$$\frac{20u+11}{5}, \quad \frac{(20u+1)(20u+21)}{25}, \quad \frac{400u(40u^2-23u+4)+21}{250},$$

and  $40u^2 - 23u + 4$  has negative discriminant  $529 - 640$ . The other positions give the same determinant and the leading minors  $\frac{20u+11}{5}$  or  $\frac{8u+1}{2}$  and  $\frac{800u^2-180u+53}{50}$ , whose discriminant  $32400 - 169600$  is negative. Every block is positive definite.

The inequality is not tight. In floating point the least eigenvalue of the whole form over  $W \in \mathbf{U}(N)$ ,  $N \leq 4$ , is 0.0149, which is also the least eigenvalue of the exact blocks; this slack is what the choice of the weights  $\frac{21}{10}, \frac{5}{4}, \frac{9}{10}$  trades against the constant  $\frac{29}{2}$ .

#### APPENDIX E. THE CERTIFIED COMPUTATION

*Truncation.* For a corner  $b$  of a box the moments of Lemma 7.6 are derivatives of  $Z$  at  $(b, \bar{b})$ . The truncated series is  $\mathbb{E}_H[A_t(UD)B_t(\bar{U}D')]$  with  $A_t$  the part of  $\sum_\lambda c_\lambda s_\lambda(UD)$  over partitions with  $\lambda_1 - \lambda_3 \leq 34$  and  $\lambda_3 \leq 11$ , and  $B_t$  likewise. Every omitted partition has  $|\lambda| > 34$ , because  $3 \cdot 12 > 34$ , and the omitted part is at most  $T(\|D\|) = \sum_{n>34} (\frac{3}{2}\|D\|)^n/n!$  in modulus. On the polydisc  $|D_{ij} - b_{ij}| \leq \rho$ ,  $\|D\| \leq \|b\| + 3\rho$ , and Cauchy's estimate bounds the  $\alpha$ -th derivative of the omitted part by  $\alpha! \rho^{-|\alpha|} T(\|b\| + 3\rho)$ . The derivatives of the full exponential are at most  $2^{-|\alpha|} e^{v/2}$  with  $v = \sum_k |b_k|$ , since  $|\text{Tr}(Ub)| \leq \sum_k |b_k|$  for diagonal  $b$ . With  $\varepsilon_\alpha = 2^{|\alpha|} \alpha! \rho^{-|\alpha|} T(\|b\| + 3\rho)$  the truncated moment differs from the true one by at most  $\varepsilon_\alpha e^{v/2} + (e^{v/2} + \varepsilon_\alpha) \varepsilon_\beta$ , and  $\rho$  is chosen to minimise  $\rho^{-6} T$ . The largest error bound of an unnormalised moment at any corner is  $1.7 \times 10^{-11}$ .

*Weight zero.* Conjugation by the diagonal torus  $T$  preserves Haar measure and  $\text{Re Tr}(Ub)$  for diagonal  $b$ , and multiplies  $u_{ij}$  by  $t_i t_j^{-1}$ ; a moment of a monomial on which  $T$  acts by a nontrivial character is therefore 0, and only 8278 of the 134596 monomials of degree at most six have the trivial character. The functions  $\text{Tr}(X)$ ,  $e_2(X)$ ,  $\det X$  of  $X = DD'^\top$  and  $\det D$ ,  $\det D'$  are invariant under the same action on the variables, so after these are formed in the full ring of Taylor polynomials of degree six in the 18 variables, the series is computed in the subring of weight-zero monomials.

*Rounding.* The truncated series is a Taylor polynomial of total degree six in the 36 real directions of  $(D - b, D' - \bar{b})$ , computed in double precision. Every operation is repeated on absolute values and carries an operation count  $n$ , so that the computed coefficient differs from the exact one by at most  $\gamma_n = nu/(1 - nu)$ ,  $u = 2^{-53}$ , times the absolute-value coefficient [19, Sec. 3.1]. With  $C$  in  $2^{-30}\mathbb{Z}$ , the polynomials  $2^{31}q_h$ ,  $X_a(2^{31}q_h)$ ,  $12 \cdot 2^{31}(\varphi_h + \Delta q_h)$  and  $12 \cdot 2^{31} \langle \nabla \varphi_{F_k}, \nabla q_h \rangle$  for the coordinate vectors  $h$  have coefficients in  $\mathbb{Z}[i]$ , and every partial sum in their computation is bounded, through the same operations on absolute values,

by less than  $2^{50}$ ; they are therefore computed exactly. Normalisation by  $Z$ , the products with the corner coordinates, the quadratic forms and their error matrices are formed with the same bookkeeping, and the largest eigenvalues are bounded in ball arithmetic [23].

*Separation.* The functional  $c$  of a discarded box is the dual solution of a linear program for the gauge of the centre of the box with respect to a  $72 \times 72$  grid of  $T$ . Only the final inequality is used:  $\max_T \ell$  is bounded by its maximum over a  $512 \times 512$  grid plus  $\frac{1}{2}H\delta^2/2$ , with  $\delta = 2\pi/512$  and  $H = |c_1| + |c_2| + 2|c_3|$  a bound for its Hessian on the torus, and the inequality is checked with relative margin  $10^{-12}$ .

*Running time.* The moments at one corner take about one and a half seconds and the bound on one box about two and a half seconds on one core. The cover runs in about an hour on four cores and writes every certified box with its  $\theta$  and its bound.

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