

Mass Gap of Lattice Yang–Mills Theory up to the Dobrushin Ceiling

Deep Bhattacharjee^{1,2,*} and Ushashi Bhattacharya^{3,†}

¹Formerly, Electro-Gravitational Space Propulsion Laboratory (EGSPL), Bhubaneswar, Odisha 751030, India

²Formerly, University of Calcutta, Senate House, 87/1 College Street, Kolkata 700073, West Bengal, India

³Formerly, National Taiwan University, Taipei 106319, Taiwan

(Dated: 18 October 2026)

We prove that the Hessian of the Wilson action on d -dimensional lattice gauge fields is bounded by $4d$, sharply, so lattice $SU(N)$ Yang–Mills theory has a unique infinite-volume state with a positive mass gap for $|\beta| < 1/(8d)$. Every von Mises–Fisher law on the three-sphere moves by at most a quarter of its parameter in Kantorovich distance; consequently four-dimensional $SU(2)$ theory has a mass gap for $\beta_W < 2/9$, the ceiling of all single-link Dobrushin arguments. A gauge-invariant finite-box criterion for decay and the area law is also given. The continuum mass gap remains open.

2020 Mathematics Subject Classification. Primary 81T25, 81T13; Secondary 82B20, 60K35, 58J65, 53C21, 47D07, 49Q22, 22E30, 60J60.

CONTENTS

I. INTRODUCTION

I. Introduction	1	Wilson’s lattice gauge theory [107] ¹ is where every attempt to construct four-dimensional Yang–Mills theory begins. Its configurations are assignments of a group element Q_e of a compact group G to each link e of a lattice, its weight is $\exp(N\beta \sum_p \text{Re Tr } Q_p)$, a sum over elementary squares (plaquettes), and the inverse coupling β is the only parameter. ² When β is small the theory is understood: the cluster expansion of Osterwalder and Seiler [91, 97] produces a positive gap in the spectrum of the transfer matrix and exponential decay of correlations. Shen, Zhu and Zhu [100] found a second route, through the geometry of the configuration space: they verified the Bakry–Émery criterion [2, 3] on the product manifold G^E and obtained log-Sobolev inequalities uniform in the volume, uniqueness of the infinite-volume limit and decay of correlations. The strength of that route is set by one number, the constant in the bound
II. Setting	3	
III. The sharp Hessian bound	4	
IV. Consequences of the curvature bound	6	
V. $SU(2)$: the single-link law	6	
VI. A gauge-invariant finite-box criterion	13	
VII. Transfer matrices and the mass gap	15	
VIII. The remaining gaps	18	
Data and code availability	20	$ \text{Hess } S_0(v, v) \leq 8(d-1) v ^2$ (1)
Ethical declarations	21	for the Hessian of the Wilson action $S_0 = \sum_p \text{Re Tr } Q_p$ in d dimensions. The same constant reappears in later work on Langevin dynamics, on the area law and on the Yang–Mills–Higgs model [27, 87, 99].
A. The Hessian of a plaquette	21	
B. The single-link law of $SU(2)$	22	
C. The weighted coupling	23	
D. A Kantorovich–Dobrushin comparison	24	
E. Two-dimensional estimates	25	
F. The sharp quarter	25	

* itsdeep@live.com; * Corresponding author. Second address: d.bhattacharjee@erl-forschung.de.

† rai.bhattacharya.in@gmail.com

¹ Wilson introduced the lattice action to explain confinement: at strong coupling the expectation of a large Wilson loop decays like the exponential of its area, which is the statement that a static quark and antiquark are bound by a linear potential. The Hamiltonian version, with continuous time and a lattice in space, is due to Kogut and Susskind [70]; there the electric energy is the Laplacian on each link group and the magnetic energy is the plaquette term, and the mass gap is the gap above the ground state of this Hamiltonian. The two formulations are related by the transfer matrix of Sec. VII. Standard accounts of the lattice theory, its symmetries, its strong-coupling expansion and its numerical simulation are the books of Montvay and Münster [85] and of Gattringer and Lang [59].

² Throughout, Q_p is the ordered product of the four link variables around the plaquette p , so that $\text{Re Tr } Q_p = N$ exactly when the field is flat on p . Large β favours flat fields; this is the weak coupling regime, which is the one relevant for the continuum limit.

This paper has three aims. The first is to find the best constant in (1). The second is to carry that constant, and a sharper analysis of the single-link law of $SU(2)$, through to uniqueness and to a positive lattice mass gap. The third is structural: a criterion, checkable in one finite box, that produces decay at all scales for gauge fields, and an honest account of what still separates all of this from the continuum mass gap.

A. The Hessian

The bound (1) is obtained plaquette by plaquette. The cross terms between the four links of one plaquette, however, are not independent across the lattice: summed over all plaquettes they reassemble into the square of a covariant divergence, which has a sign. Keeping that square gives the sharp constant.

Theorem A (Sharp Hessian bound). *Let $d \geq 2$, and let G be one of the groups $SU(N)$, $SO(N)$ or $U(N)$. For every gauge field Q on $\mathbb{Z}^d$ or on a torus and every square-summable tangent vector $v = (X_e Q_e)_e$,*

$$\pm \text{Hess } S_0(v, v) + \sum_x |D_x X|^2 \leq 4d |v|^2,$$

where $D_x X$ is the covariant divergence (6). In particular $|\text{Hess } S_0(v, v)| \leq 4d |v|^2$, and for every $N \geq 2$ and $d \geq 2$ the constant $4d$ is attained with both signs.

For $d = 2$ the two constants coincide; for $d = 3$ and $d = 4$ they are 12 against 16 and 16 against 24.³ The two extremal fields are drawn in Fig. 3.

B. Strong coupling

The Ricci curvature of $SU(N)$ in the metric of Sec. II is $N/2$, and that of $SO(N)$ is $(N-2)/4$. Subtracting the Hessian bound gives the Bakry–Émery constant directly.

Theorem B (Curvature regime). *Let $G = SU(N)$ and $|\beta| < 1/(8d)$, or $G = SO(N)$ with $N \geq 3$ and $|\beta| < (N-2)/(16dN)$. The finite-volume Wilson measures satisfy the Bakry–Émery condition with a constant $K > 0$ that depends neither on the volume, nor on the boundary condition, nor on the configuration. They satisfy log-Sobolev and Poincaré inequalities uniform in the volume, and every infinite-volume limit has exponentially decaying correlations of local functions.*

Theorem C (The limit of the curvature method). *For $G = SU(N)$ on every torus of even side,*

$$\inf_Q \lambda_{\min}(\text{Ric} - N\beta \text{Hess } S_0(Q)) = \frac{N}{2} - 4dN|\beta|.$$

No Bakry–Émery argument with a lower bound uniform over configurations, in the bi-invariant metric, reaches beyond $|\beta| = 1/(8d)$.

The curvature bound also controls a coupling of two Langevin dynamics. Shen, Zhu and Zhu built such a coupling and needed a second, separate estimate to make it contract in a weighted metric; that estimate cost a factor of two in the range of β . Applying Theorem A to the whole weighted sum, rather than link by link, removes the loss.

Theorem D (Uniqueness on the whole curvature range). *Under the hypotheses of Theorem B the Langevin dynamics of the Wilson measure on $\mathbb{Z}^d$ has exactly one invariant probability measure μ . The Wilson measures on the periodic tori $(\mathbb{Z}/L\mathbb{Z})^d$ converge to μ as $L \rightarrow \infty$, and every initial law converges to μ exponentially fast in a weighted Wasserstein distance.*

C. $SU(2)$ beyond the curvature range

For $SU(2)$ the conditional law of one link given all the others is a von Mises–Fisher law on the three-sphere, and its moments are explicit. A Dobrushin argument in the chordal metric then goes past the curvature range.

Theorem E (Uniqueness for $SU(2)$). *Let $G = SU(2)$ and $|\beta| < 1/(6(d-1))$. The Wilson theory on $\mathbb{Z}^d$ has exactly one infinite-volume Gibbs measure, and its truncated correlations of local Lipschitz functions decay exponentially with every rate $\lambda < -\log(6(d-1)|\beta|)$. At $d = 4$ the range is $\beta_W < 2/9$. No single-link Wasserstein–Dobrushin argument, in the chordal or the geodesic metric, reaches beyond this range.*

Here $\beta_W = N^2\beta$ is the Wilson coupling of the physics literature.⁴ The range is the largest that a single-link argument can give, and it is reached because of a sharp inequality on the three-sphere, Theorem 7: under every von Mises–Fisher law ν_b , the covariance of a chordally 1-Lipschitz function with a coordinate is at most $\frac{1}{4}$, so that $W_1(\nu_b, \nu_{b'}) \leq \frac{1}{4}|b - b'|$. The constant $\frac{1}{4}$ is attained at $b = 0$ and fails in the geodesic metric. The proof extends a Lipschitz function from the sphere to the ball it bounds and integrates by parts against a harmonic function; the resulting negative Sobolev norm is a series in modified Bessel functions, and the inequality becomes the positivity of the Taylor coefficients of one entire function.

D. A lattice mass gap

Reflection positivity turns decay of correlations in the time direction into a gap in the spectrum of the transfer matrix.

Theorem F (Lattice mass gap). *In the range of Theorem B, and for $SU(2)$ in the range of Theorem E, the infinite-volume state is reflection positive and its transfer matrix $\mathbb{T}$ has a spectral gap: the spectrum of $\mathbb{T}$ on the orthogonal complement of the*

³ We searched arXiv and the journal literature up to October 2026 for the constant $4d$ and did not find it. The bound $8(d-1)$ is stated in [100, Lemma 4.1] and used in [27, 87, 99].

⁴ We follow the normalisation of Shen, Zhu and Zhu [100]: the weight is $\exp(N\beta \sum_p \text{Re Tr } Q_p)$. The physics literature writes $\exp(\frac{\beta_W}{N} \sum_p \text{Re Tr } Q_p)$, so $\beta_W = N^2\beta$, and for $SU(2)$, $\beta_W = 4\beta$.

vacuum lies in $[0, e^{-m}]$ with $m > 0$. For $SU(2)$ one may take $m = -\log(6(d-1)|\beta|)$, which at $d = 4$ reads $m = -\log(\frac{9}{2}\beta_W)$ and is positive for $\beta_W < 2/9$.

This is a gap in lattice units at fixed lattice spacing. It is not the mass gap of the continuum theory: our earlier paper [16] shows that no inequality of this kind, uniform in the spacing and stated in lattice units, can produce the continuum gap, because the physical gap in lattice units must vanish like the spacing.

E. A finite-box criterion

For spin systems, Dobrushin and Shlosman showed that a single finite computation implies uniqueness and decay at all scales [43].⁵ Gauge fields need three changes: boundary data are holonomies, gauge-equivalent boundary conditions must be identified, and disagreement is measured after an interior gauge alignment. With these changes:

Theorem G (Gauge-invariant finite-box criterion). *For every compact group G and every finite-range gauge-invariant interaction, the finite-box inequality of Definition 16 implies uniqueness of the infinite-volume Gibbs measure, exponential decay of all truncated correlations, and the same decay on every large torus, uniformly in its size and in every 't Hooft twist sector. Its box-hereditary form implies Wilson's area law.*

Two-dimensional $SU(2)$ theory meets the criterion at every coupling (Proposition 21). Four-dimensional $U(1)$ theory at weak coupling cannot meet it, because that theory is massless [57, 65]; a criterion that did not fail there would be wrong.

F. What is proved and what is not

Figure 1 places the thresholds on one axis, and Table I in Sec. VIII lists the gaps between this paper and the continuum mass gap, with the status of each. Everything stated as a theorem, proposition or lemma is proved here, by hand, without conditions. The mass gap of the continuum four-dimensional theory is not proved here, at intermediate lattice coupling or in the continuum. What is missing is stated precisely as Problem 27.

⁵ Dobrushin's original criterion [41, 42] compares the conditional laws at a single site: if the total influence $\sum_y \kappa(x, y)$ of all other sites on the conditional law at x is below 1, the Gibbs measure is unique and correlations decay. It is local in space but global in the interaction strength, and it fails before the critical point. Dobrushin and Shlosman replaced the single site by a finite box and asked that the influence of the boundary on the box, averaged over the box, be below its volume; this condition holds in the whole high-temperature phase of many models and is equivalent there to several other forms of complete analyticity. Expositions of the single-site theory are in the books of Georgii [60], Simon [101] and Friedli and Velenik [55]; the coupling formulation used below is that of Ref. [60], Ch. 8.

II. SETTING

Let $G \in \{SU(N), SO(N), U(N)\}$ with Lie algebra $\mathfrak{g}$. We use the inner product $\langle X, Y \rangle = -\text{Tr}(XY) = \text{Re Tr}(X^*Y)$ on $\mathfrak{g}$ and the bi-invariant metric it induces on G . In this metric

$$\text{Ric}_{SU(N)} = \frac{N}{2}, \quad \text{Ric}_{SO(N)} = \frac{N-2}{4}. \quad (2)$$

Indeed $\text{Ric}(X, X) = -\frac{1}{4}B(X, X)$ for the Killing form B of a compact group with a bi-invariant metric [84],⁶ and $B(X, X) = 2N\text{Tr}X^2$ on $\mathfrak{su}(N)$, $B(X, X) = (N-2)\text{Tr}X^2$ on $\mathfrak{so}(N)$.⁷ With this metric $SU(2)$ is the round three-sphere of radius $\sqrt{2}$; its Hopf fibration by great circles, and the Villarceau circles that link them, are described in [19].

Let $\Lambda \subset \mathbb{Z}^d$ be a finite box or a torus, E its set of positively oriented links and P its set of plaquettes. The plaquette p with base point x in the plane of the directions $\mu < \nu$ has the links

$$e_1 = (x, \mu), \quad e_2 = (x + \hat{\mu}, \nu), \quad e_3 = (x + \hat{\nu}, \mu), \quad e_4 = (x, \nu),$$

and $Q_p = Q_{e_1}Q_{e_2}Q_{e_3}^*Q_{e_4}^*$. The Wilson measure is

$$d\mu_{\Lambda, \beta}(Q) = Z^{-1}e^{S(Q)} \prod_e dQ_e, \quad (3)$$

$$S = N\beta S_0, \quad S_0 = \sum_{p \in P} \text{Re Tr } Q_p,$$

with normalised Haar measure dQ_e ; on a box the links outside Λ are frozen to a boundary condition.

A tangent vector at Q is $v = (X_e Q_e)_e$ with $X_e \in \mathfrak{g}$, and $|v|^2 = \sum_e |X_e|^2$. Since $t \mapsto e^{tX}Q$ is a geodesic of the bi-invariant metric, $\text{Hess } S_0(v, v) = \frac{d^2}{dt^2} S_0((e^{tX_e}Q_e)_e)|_{t=0}$.⁸ On $\mathbb{Z}^d$ the Hessian at any field is a bounded quadratic form on square-summable vectors, because it is a sum of finite-range blocks of uniformly bounded size.

⁶ For a bi-invariant metric the Levi-Civita connection on left-invariant fields is $\nabla_X Y = \frac{1}{2}[X, Y]$, so $R(X, Y)Z = -\frac{1}{4}[[X, Y], Z]$ and the sectional curvature of the plane spanned by orthonormal X, Y is $\frac{1}{4}|[X, Y]|^2 \geq 0$. Summing over an orthonormal basis (Y_j) gives $\text{Ric}(X, X) = \frac{1}{4} \sum_j |[X, Y_j]|^2 = -\frac{1}{4} \text{Tr}(\text{ad}_X^2) = -\frac{1}{4}B(X, X)$; this is the bi-invariant case of Milnor's formulas for left-invariant metrics [84]. On $\mathfrak{su}(N)$ one has $\text{Tr}(\text{ad}_X^2) = 2N\text{Tr}X^2$, because ad_X extends to $\mathfrak{gl}(N, \mathbb{C}) = \mathfrak{sl}(N, \mathbb{C}) \oplus \mathbb{C}1$, where it vanishes on the centre and acts as $X \otimes 1 - 1 \otimes X^T$, whose square has trace $2N\text{Tr}X^2 - 2(\text{Tr}X)^2$; for traceless X this is $2N\text{Tr}X^2$. Since $-\text{Tr}X^2 = |X|^2$ in our normalisation, $\text{Ric} = N/2$. The same computation on antisymmetric matrices gives $(N-2)\text{Tr}X^2$ and $\text{Ric} = (N-2)/4$.

⁷ For $U(N)$ the centre contributes a flat direction, so $\text{Ric}_{U(N)}$ is not bounded below by a positive constant, and Theorem B is not stated for $U(N)$. Theorem A holds for $U(N)$ as stated.

⁸ On a compact Lie group with a bi-invariant metric the one-parameter subgroups and their translates are exactly the geodesics. The Hessian of a function f at Q in the direction v is therefore the second derivative of f along the geodesic through Q with velocity v , and the polarisation identity recovers the bilinear form.

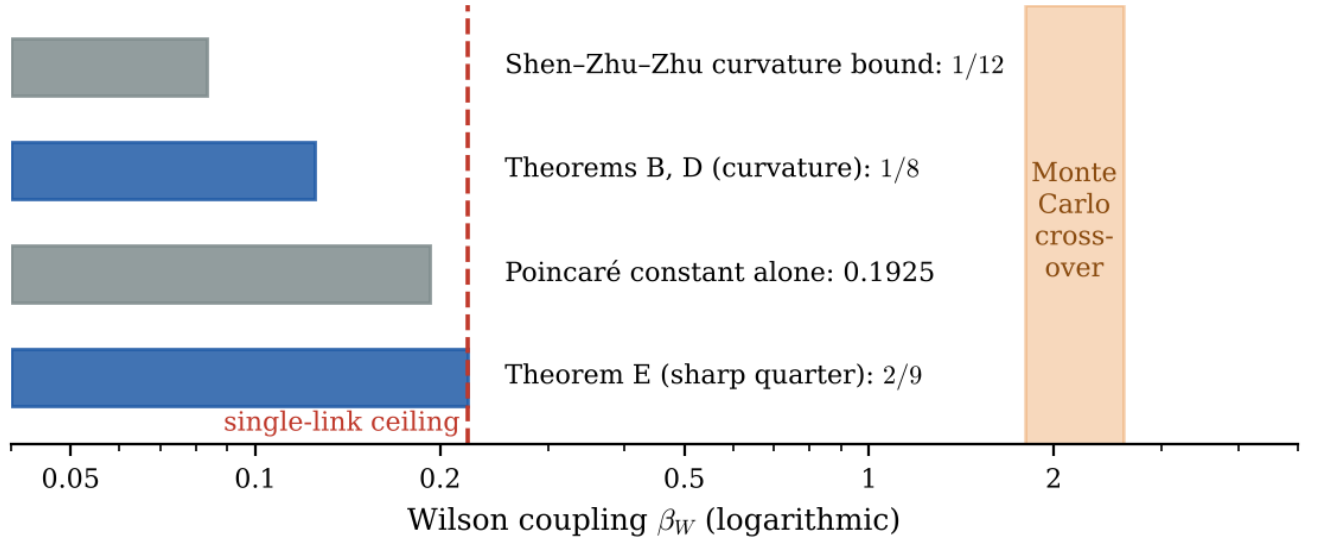


FIG. 1. Thresholds for four-dimensional SU(2) in the Wilson coupling β_W , on a logarithmic scale. Blue: Theorems B and D (curvature, $\beta_W < 1/8$) and Theorem E (chordal Dobrushin, $\beta_W < 2/9$). Grey: the curvature bound $1/12$ that follows from (1), and the range $\beta_W < 1/(3\sqrt{3}) \approx 0.1925$ that the Poincaré constant of Lemma 4 gives without Theorem 7. Red: the ceiling $2/9$ of all single-link Wasserstein–Dobrushin arguments (Proposition 13), which Theorem E reaches. Orange: the region where Monte Carlo simulations place the crossover to weak coupling, which plays no role in any proof.

III. THE SHARP HESSIAN BOUND

A. One plaquette

Fix a plaquette p and a tangent vector $v = (X_e Q_e)$. Transport the four link vectors to the base point of p :

$$Y_1 = X_{e_1}, \quad Y_2 = Q_{e_1} X_{e_2} Q_{e_1}^*, \quad Y_3 = Q_{e_4} X_{e_3} Q_{e_4}^*, \quad Y_4 = X_{e_4}, \quad (4)$$

and write $W = Q_p$. Then the perturbed plaquette variable is $Q_p(t) = e^{tY_1} e^{tY_2} W e^{-tY_3} e^{-tY_4}$ (Appendix A), and

$$\begin{aligned} f_p &:= \frac{d^2}{dt^2} \text{Re Tr } Q_p(t) \Big|_{t=0} \\ &= \sum_{i=1}^4 \text{Re Tr}(Y_i^2 W) + 2 \text{Re Tr}(Y_1 Y_2 W) + 2 \text{Re Tr}(Y_3 Y_4 W) \\ &\quad - 2 \sum_{i \in \{1,2\}, j \in \{3,4\}} \text{Re Tr}(Y_i W Y_j), \end{aligned} \quad (5)$$

so that $\text{Hess } S_0(v, v) = \sum_p f_p$.

B. The covariant divergence

The covariant divergence of X at a vertex x is

$$D_x X = \sum_{\mu=1}^d \left[X_{x,\mu} - Q_{x-\hat{\mu},\mu}^* X_{x-\hat{\mu},\mu} Q_{x-\hat{\mu},\mu} \right] \in \mathfrak{g}. \quad (6)$$

Each summand is the vector of a link with an end at x , transported to x , with sign $+$ for an outgoing and $-$ for an incoming

link.⁹ The corner form of p is

$$c_p = 2\langle Y_1, Y_4 \rangle - 2\langle Y_1, Y_2 \rangle + 2\langle Y_2, W Y_3 W^* \rangle - 2\langle Y_3, Y_4 \rangle. \quad (7)$$

Lemma 1. For every field Q and every square-summable X ,

$$\sum_x |D_x X|^2 = 2|X|^2 + \sum_p c_p + \sum_{x,\mu} \kappa_{x,\mu},$$

where

$$\kappa_{x,\mu} = -2\langle X_{x,\mu}, Q_{x-\hat{\mu},\mu}^* X_{x-\hat{\mu},\mu} Q_{x-\hat{\mu},\mu} \rangle,$$

and $\sum_{x,\mu} |\kappa_{x,\mu}| \leq 2|X|^2$.

Proof. The vector $D_x X$ is a signed sum of $2d$ vectors, one for each end of a link at x . Expanding $|D_x X|^2$ and summing over x gives three kinds of term. The squares contribute $2|X|^2$, since every link has two ends. The cross term between the outgoing and the incoming link in the same direction μ is $\kappa_{x,\mu}$. The cross terms between ends in two perpendicular directions $\mu \neq \nu$ are in bijection with pairs consisting of a plaquette of the plane $\mu\nu$ and one of its corners at x : through x pass four plaquettes of that plane, one for each choice of two signs. Appendix A identifies the four corner terms of each plaquette with the four terms of (7). Finally $|\kappa_{x,\mu}| \leq |X_{x,\mu}|^2 + |X_{x-\hat{\mu},\mu}|^2$ by the Cauchy–Schwarz inequality, and the sum over (x, μ) counts every link twice. $\square$

⁹ At the identity field, $D_x X$ is the lattice divergence of the Lie-algebra valued one-form X , and $\sum_x |D_x X|^2$ is the gauge-fixing term of the Lorenz gauge. The point of Theorem A is that this positive term is hidden inside the Hessian of the Wilson action at every field, not only near the identity.

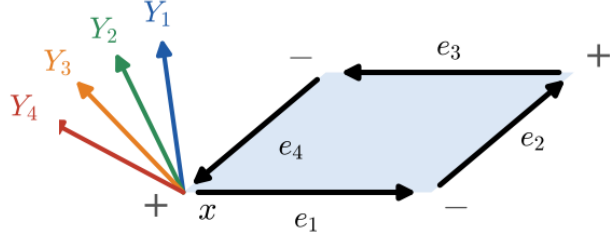
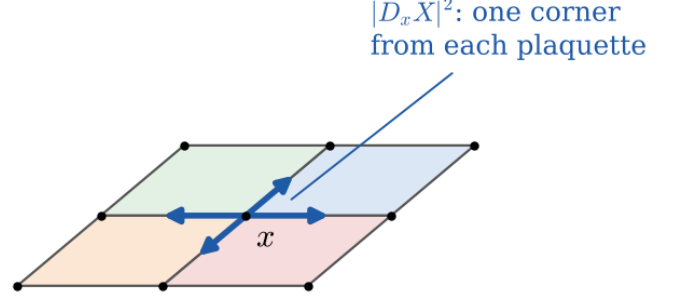
link vectors at x and corner signsfour plaquettes of one plane at x 

FIG. 2. Left: a plaquette with its links $e_1, \dots, e_4$, the link vectors $Y_1, \dots, Y_4$ transported to the base point x , and the sign with which each corner enters the corner form (7). Right: the four plaquettes of one plane through a vertex x ; each contributes one corner to $|D_x X|^2$, which is how the cross terms of the Hessian reassemble into the divergence.

C. A local inequality

Lemma 2. For every N , every $W \in \text{U}(N)$ and all $Y_1, \dots, Y_4 \in \text{u}(N)$,

$$\pm f_p + c_p \leq 2 \sum_{i=1}^4 |Y_i|^2.$$

Proof. Replacing W by $-W$ changes the sign of f_p and leaves c_p unchanged, so it suffices to prove the inequality with the sign $+$. Write $W = VDV^*$ with V unitary and $D = \text{diag}(w_1, \dots, w_N)$, $|w_a| = 1$. Replacing every Y_i by $V^* Y_i V$ changes neither side, so we may assume $W = D$. Every term of f_p and of c_p is then a sum over positions (a, b) of products of the entries $(Y_i)_{ab}$, $(Y_j)_{ba}$ or $(Y_i)_{ab}$, $(Y_j)_{ab}$, with phases from D . Since $(Y_i)_{ba} = -\overline{(Y_i)_{ab}}$, each term couples only the positions (a, b) and (b, a) . Hence the form $F = 2 \sum |Y_i|^2 - f_p - c_p$ is an orthogonal sum of blocks: one for each diagonal position a , in the variables $(Y_i)_{aa} = iy_i$ with $y \in \mathbb{R}^4$, and one for each unordered pair $a \neq b$, in the variables $z_i = (Y_i)_{ab} \in \mathbb{C}$. Appendix A computes these blocks:

$$F_{aa} = (y_1 + y_3)^2 + (y_2 + y_4)^2 + (1 + \cos \varphi)(y_1 + y_2 - y_3 - y_4)^2$$

with $w_a = e^{i\varphi}$, and $F_{ab} = z^* M(m, \theta) z$ with $w_{a,b} = e^{i(\chi \pm \theta)}$, $m = \cos \chi$, and a Hermitian matrix M affine in m that satisfies $M(-1, \theta) = M(1, \theta + \pi)$. Therefore $M(m, \theta) = \frac{1+m}{2} M(1, \theta) + \frac{1-m}{2} M(1, \theta + \pi)$, and after a diagonal unitary change of variables $z = \Delta u$ and with $c = \cos(\theta/2)$,

$$\begin{aligned} z^* M(1, \theta) z &= 2c^2(|u_1 + u_3|^2 + |u_2 + u_4|^2) \\ &+ (1+c)^2|u_1 + u_2 - u_3 - u_4|^2 + (1-c)^2|u_1 - u_2 - u_3 + u_4|^2. \end{aligned}$$

Every block is a sum of squares, so $F \geq 0$. $\square$

The pairing of (a, b) with (b, a) is essential.¹⁰

D. Proof of Theorem A

Proof of the upper bound. By density it suffices to take X finitely supported. Every link lies in $2(d-1)$ plaquettes and $|Y_i| = |X_{e_i}|$, so summing Lemma 2 over p gives $\sum_p (\pm f_p + c_p) \leq 4(d-1)|X|^2$. By Lemma 1, $\sum_p c_p = \sum_x |D_x X|^2 - 2|X|^2 - \sum \kappa \geq \sum_x |D_x X|^2 - 4|X|^2$. Hence

$$\pm \text{Hess } S_0(v, v) \leq 4(d-1)|X|^2 + 4|X|^2 - \sum_x |D_x X|^2. \quad \square$$

Proof of sharpness. Let the torus have even side and let $X_{x,\mu} = (-1)^{|x|} \varepsilon_\mu T$ with $|x| = \sum_v x_v$, a vector $\varepsilon \in \mathbb{R}^d$ with $\sum_\mu \varepsilon_\mu = 0$, and $0 \neq T \in \mathfrak{g}$.

The sign $-$. At the identity field $W = 1$ and $Y_i = X_{e_i}$, and (5) becomes $f_p = \text{Tr}((Y_1 + Y_2 - Y_3 - Y_4)^2) = -|Y_1 + Y_2 - Y_3 - Y_4|^2$. For the plaquette at x in the plane $\mu\nu$, $Y_1 + Y_2 - Y_3 - Y_4 = 2(-1)^{|x|}(\varepsilon_\mu - \varepsilon_\nu)T$. Each vertex therefore contributes

$$\begin{aligned} -4|T|^2 \sum_{\mu < \nu} (\varepsilon_\mu - \varepsilon_\nu)^2 &= -4|T|^2 (d|\varepsilon|^2 - (\sum_\mu \varepsilon_\mu)^2) \\ &= -4d|\varepsilon|^2 |T|^2, \end{aligned}$$

while it carries $|\varepsilon|^2 |T|^2$ of $|X|^2$. So $\text{Hess } S_0(v, v) = -4d|X|^2$. Moreover $D_x X = 2(-1)^{|x|}(\sum_\mu \varepsilon_\mu)T = 0$, so the inequality of Theorem A is an equality.

The sign $+$. Let $z = \text{diag}(-1, -1, 1, \dots, 1)$, which lies in $\text{SU}(N)$, $\text{SO}(N)$ and $\text{U}(N)$, and $Q_{x,\mu} = z^{\sum_{\lambda < \mu} x_\lambda}$, which is well defined on an even torus since $z^2 = 1$. For the plaquette at x in the plane $\mu < \nu$ one finds $Q_{e_1} = Q_{e_3} = z^{\sum_{\lambda < \mu} x_\lambda}$, $Q_{e_4} = z^{\sum_{\lambda < \nu} x_\lambda}$ and $Q_{e_2} = z Q_{e_4}$, so every plaquette variable equals z . Take the same X with T supported in the upper left 2×2 block, so that T commutes with z and $zT = -T$. All transports act trivially on T , every term of (5) changes sign relative to the identity field, and so $\text{Hess } S_0(v, v) = +4d|X|^2$ and again $D_x X = 0$. $\square$

not positive semidefinite: at $w_a = -1$, $w_b = 1$ each of them has a negative eigenvalue. Only after the identification $(Y)_{ba} = -\overline{(Y)_{ab}}$, which uses that the Y_i are anti-Hermitian, does the form become a sum of squares.

¹⁰ The blocks of the ordered positions (a, b) and (b, a) , taken separately, are

Figure 4 compares the bound with the spectrum of the Hessian computed numerically for $SU(2)$ on the torus $(\mathbb{Z}/4\mathbb{Z})^3$. At random fields the spectrum fills a part of $[-12, 12]$; at the two extremal fields it reaches the end points.

IV. CONSEQUENCES OF THE CURVATURE BOUND

Proof of Theorem B. By Theorem A,

$$|\text{Hess} S(v, v)| = N|\beta| |\text{Hess} S_0(v, v)| \leq 4dN|\beta| |v|^2.$$

The Ricci curvature of G^E , or of $G^{E_{\text{int}}}$ with the boundary links frozen, equals Ric_G in every direction. By (2),

$$\text{Ric} - \text{Hess} S \geq K := \frac{N}{2} - 4dN|\beta| \quad (SU(N)),$$

and $K := \frac{N-2}{4} - 4dN|\beta|$ for $SO(N)$, uniformly in the volume, the boundary condition and the field, with $K > 0$ in the stated ranges. The Bakry–Émery criterion on a compact manifold [3, Cor. 5.7.2] gives $\text{Ent}_\mu(f^2) \leq \frac{2}{K}\mu|\nabla f|^2$ and the Poincaré inequality with constant $1/K$.¹¹

For the decay of correlations we follow [100, Sec. 4]. The argument there uses the Langevin dynamics and the gradient bound $|\nabla P_t f| \leq e^{-Kt} P_t |\nabla f|$, which is equivalent to the Bakry–Émery condition with constant K ; the Hessian enters only through the value of K . With our K in place of $K_S = \frac{N}{2} - 8N|\beta|(d-1)$, the proofs of [100, Cor. 4.11 and Rem. 4.12] apply without change. $\square$

Proof of Theorem C. Theorem A gives the lower bound $\frac{N}{2} - 4dN|\beta|$ for every field and every v . For the reverse inequality take $\beta \neq 0$ and the vector of the proof of sharpness: the frustrated field if $\beta > 0$, the identity if $\beta < 0$. Then $N\beta \text{Hess} S_0(v, v) = 4dN|\beta| |v|^2$, and since the bi-invariant metric of $SU(N)$ is Einstein, $(\text{Ric} - N\beta \text{Hess} S_0)(v, v) = (\frac{N}{2} - 4dN|\beta|) |v|^2$. $\square$

A constant rescaling of the metric on all links multiplies Ric and $\text{Hess} S$ by the same factor, and a link-dependent constant rescaling does not change the infimum on a torus, by translation invariance of the extremal fields. Theorem C is a statement about the method; it does not say that anything happens in the theory at $|\beta| = 1/(8d)$.

A. The weighted coupling and Theorem D

We recall the setting of [100, Sec. 5]. Let $|e|$ be the graph distance from the origin to the nearer endpoint of the link e of $\mathbb{Z}^d$. For $a > 1$ the distance

$$\rho_{\infty, a}(Q, Q')^2 = \sum_e a^{-|e|} \rho(Q_e, Q'_e)^2 \quad (8)$$

¹¹ Here the measure is $e^S dQ$, so the relevant curvature-dimension tensor is $\text{Ric} - \text{Hess} S$; the sign convention is that of a measure e^{-V} with $V = -S$.

on $G^{E(\mathbb{Z}^d)}$ is bounded and induces the product topology. Two Langevin dynamics on a torus, extended periodically to $\mathbb{Z}^d$, are coupled so that their noises cancel along the minimal geodesic between them, and the link distances $\rho_e(t) = \rho(Q_e(t), Q'_e(t))$ then satisfy, away from the cut locus and the diagonal,

$$\partial_t \rho_e^2 \leq -2C_{\text{Ric}} \rho_e^2 + 2\rho_e (\langle \nabla S, \dot{\gamma}_e \rangle(Q') - \langle \nabla S, \dot{\gamma}_e \rangle(Q)), \quad (9)$$

where $\dot{\gamma}_e$ is the unit tangent of the geodesic from Q_e to Q'_e and $C_{\text{Ric}} = N/2$ for $SU(N)$, $(N-2)/4$ for $SO(N)$.¹² Shen, Zhu and Zhu estimate the second term of (9) link by link. We estimate the weighted sum at once.

Lemma 3. Let $w_e = a^{-|e|}$ and put

$$K_a = C_{\text{Ric}} - 4dN|\beta| - 3(d-1)N|\beta| \frac{(\sqrt{a}-1)^2}{\sqrt{a}}.$$

Then

$$\sum_e w_e \rho_e (\langle \nabla S, \dot{\gamma}_e \rangle(Q') - \langle \nabla S, \dot{\gamma}_e \rangle(Q)) \leq (C_{\text{Ric}} - K_a) \sum_e w_e \rho_e^2.$$

The proof is in Appendix C; it uses Theorem A for the vector $W^{1/2}V$, where V is the velocity of the product geodesic, and bounds only the commutator of the weights. Figure 5 shows the weights and the sizes of the ratios that enter.

Proof of Theorem D. Insert Lemma 3 into (9). Then

$$\partial_t \sum_e w_e \rho_e^2 \leq -2K_a \sum_e w_e \rho_e^2,$$

which is the conclusion of [100, Lemma 5.1] with the rate K_a in place of

$$\tilde{K}_S = C_{\text{Ric}} - (4 + 4\sqrt{a})N|\beta|(d-1).$$

As $a \downarrow 1$, K_a increases to $C_{\text{Ric}} - 4dN|\beta|$, which is positive in the ranges of Theorem B, so there is $a > 1$ with $K_a > 0$. The rest of the proof of [100, Thm. 1.2] uses only that conclusion and applies verbatim. $\square$

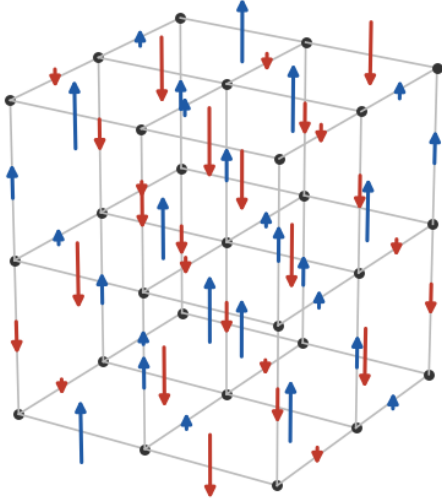
V. $SU(2)$: THE SINGLE-LINK LAW

A. The von Mises–Fisher law

Identify $SU(2)$ with the unit quaternions, that is with the unit sphere $S^3 \subset \mathbb{R}^4$, so that $\text{Re Tr } U = 2x_0(U)$ and $\text{Re Tr}(UV) = 2\langle u, \bar{v} \rangle$ for the Euclidean inner product of $\mathbb{R}^4$ and quaternion

¹² This is [100, Eqs. (5.5), (5.6), (5.10), (5.11)]: the index form of the coupled Brownian motions gives $-C_{\text{Ric}} \rho_e$ by the index lemma, the martingale part vanishes because the noises are parallel transports of each other, and the cut-off near the cut locus and the diagonal is removed by dominated convergence. None of this uses the Hessian bound. In [100] the constant is written $C_{\text{Ric}, N} = \alpha \frac{N+2}{4} - 1$ with $\alpha = 2$ for $SU(N)$, $\alpha = 1$ for $SO(N)$, which is the same.

identity field: $X_{x,\mu} = (-1)^{|x|} \varepsilon_\mu T$



frustrated field: red links carry z

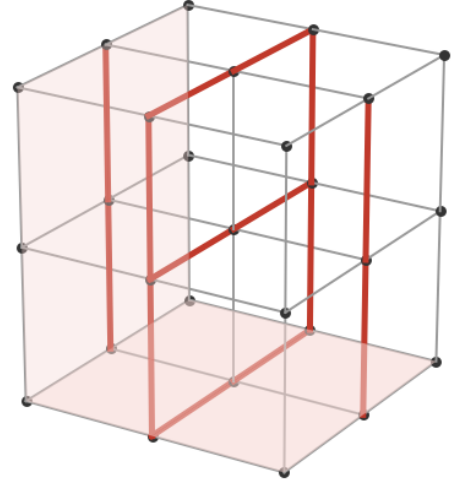


FIG. 3. The two fields at which $\pm \text{Hess} S_0 = 4d$. Left: the identity field with the checkerboard tangent vector $X_{x,\mu} = (-1)^{|x|} \varepsilon_\mu T$, $\sum_\mu \varepsilon_\mu = 0$, drawn on a $3 \times 3 \times 3$ block (arrows of alternating orientation, lengths proportional to ε_μ). Right: the frustrated field $Q_{x,\mu} = z^{\sum_{\lambda < \mu} x_\lambda}$ with $z = \text{diag}(-1, -1, 1, \dots)$; red links carry z , and every plaquette has holonomy z (two faces of the block are shaded).

conjugation $v \mapsto \bar{v}$. Fix a link e and the field off e . Every plaquette through e can be rotated cyclically so that its trace reads $\text{Re Tr}(Q_e A_p)$ with $A_p \in \text{SU}(2)$, the staple of e in p . The action therefore depends on Q_e only through

$$N\beta \sum_{p \ni e} \text{Re Tr}(Q_e A_p) = \langle x, b \rangle, \quad b = 4\beta \sum_{p \ni e} \bar{A}_p \in \mathbb{R}^4,$$

with $x = Q_e \in S^3$. Since there are $2(d-1)$ staples, each of norm one,

$$\kappa := |b| \leq y := 8(d-1)|\beta|. \quad (10)$$

The conditional law of Q_e is the von Mises–Fisher law¹³

$$\nu_b(dx) = Z(\kappa)^{-1} e^{\langle x, b \rangle} \sigma(dx) \quad (11)$$

on the unit sphere, with σ the normalised surface measure. We use coordinates in which $b = \kappa e_1$. The density of $t = x_1$ is proportional to $(1-t^2)^{1/2} e^{\kappa t}$ on $[-1, 1]$, so $Z(\kappa) = 2I_1(\kappa)/\kappa$ with the modified Bessel function I_1 , normalised so that $Z(0) = 1$.

¹³ On the sphere $S^{p-1} \subset \mathbb{R}^p$ the von Mises–Fisher law with mean direction μ and concentration κ has density proportional to $e^{\kappa \langle x, \mu \rangle}$ with respect to the surface measure, and its normalising constant is $(2\pi)^{p/2} I_{p/2-1}(\kappa) / \kappa^{p/2-1}$; it is the maximum entropy law with prescribed mean resultant, and it plays on the sphere the role of the Gaussian in $\mathbb{R}^p$. For $p = 4$ the Bessel index is 1. The mean resultant length $\mathbb{E} \langle x, \mu \rangle = I_{p/2}(\kappa) / I_{p/2-1}(\kappa)$ is the function A of Lemma 5. See Mardia and Jupp [80], Ch. 9, for these facts and for the tangent-normal decomposition used in Appendix B.

Lemma 4. For every $\kappa \geq 0$ the law ν_b on the unit sphere S^3 satisfies the Poincaré inequality $\text{Var}_{\nu_b}(g) \leq \frac{1}{3} \int |\nabla g|^2 d\nu_b$, and the constant $\frac{1}{3}$ is attained at $\kappa = 0$.

Proof. In the polar angle $\theta \in (0, \pi)$ from e_1 the generator of the reversible diffusion of ν_b is $L = \partial_\theta^2 + b(\theta) \partial_\theta + \sin^{-2} \theta \Delta_{S^2}$ with $b(\theta) = 2 \cot \theta - \kappa \sin \theta$. On spherical harmonics of degree ℓ on S^2 it acts as a Sturm–Liouville operator L_ℓ with potential $\ell(\ell+1) \sin^{-2} \theta$. A ground state comparison of Barta’s type [13] with the trial function $\psi = \sin \theta e^{a \kappa \cos \theta}$ gives $-L_1 \psi \geq 3\psi$ for $a = -\frac{1}{5}$, and the same trial function with $a = -\frac{2}{5}$ gives the bound 3 for the derivatives of zonal eigenfunctions; Appendix B carries out both computations. Hence every nonzero eigenvalue of $-L$ is at least 3. At $\kappa = 0$ the first eigenvalue of the Laplacian of the unit S^3 is 3. $\square$

B. Moments

Let $A(\kappa) = \mathbb{E}_{\nu_b} x_1 = Z'(\kappa)/Z(\kappa) = I_2(\kappa)/I_1(\kappa)$.

Lemma 5. For every $\kappa > 0$, in coordinates with $b = \kappa e_1$,

(i) $\mathbb{E} x_2^2 = \mathbb{E} x_3^2 = \mathbb{E} x_4^2 = A/\kappa$, $\mathbb{E}(1 - x_1^2) = 3A/\kappa$, the covariance matrix of x is $\text{diag}(A', A/\kappa, A/\kappa, A/\kappa)$, and $A' = 1 - A^2 - 3A/\kappa$;

(ii) $\frac{\kappa}{2 + \sqrt{\kappa^2 + 4}} \leq A(\kappa) \leq \frac{\kappa}{1 + \sqrt{\kappa^2 + 9}}$;

(iii) $A' \leq A/\kappa \leq (1 + \sqrt{\kappa^2 + 9})^{-1} \leq \frac{1}{4}$;

(iv) $A(\kappa) \geq \kappa / (\frac{3}{2} + \sqrt{\kappa^2 + 25/4})$.

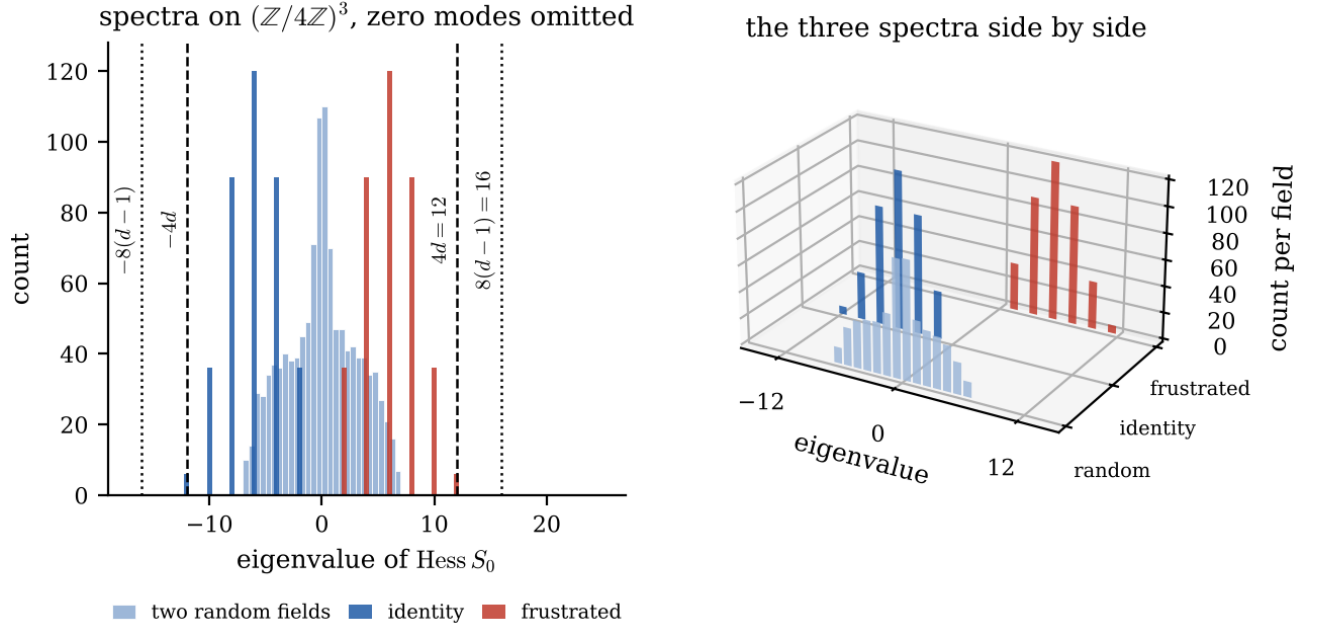


FIG. 4. Eigenvalues of $\text{Hess } S_0$ for $\text{SU}(2)$ on $(\mathbb{Z}/4\mathbb{Z})^3$ ($d = 3$, 576 eigenvalues per field). Histogram: two random Haar fields. Solid bars: the identity field (left) and the frustrated field (right). The dashed lines are Theorem A ($\pm 4d = \pm 12$); the dotted lines are the earlier bound (1) ($\pm 8(d-1) = \pm 16$). The identity and frustrated spectra touch -12 and $+12$. Right: the same three spectra in depth, the random histogram divided by two so that every row counts one field; the spectrum of the frustrated field is the reflection of that of the identity field on this torus.

Proof. (i) Integrating by parts against $\frac{d}{dt}(1-t^2)^{3/2} = -3t(1-t^2)^{1/2}$ gives $\kappa \mathbb{E}(1-x_1^2) = 3\mathbb{E}x_1$, and the three coordinates x_2, x_3, x_4 are exchangeable under v_b , with $\sum_i x_i^2 = 1$. The off-diagonal covariances vanish by the reflections $x_i \mapsto -x_i$, $i \geq 2$, and $A' = \text{Var} x_1$ because $A = (\log Z)'$. (ii) By (i), A solves the Riccati equation $A' = 1 - A^2 - 3A/\kappa$ with $A(\kappa) = \kappa/4 + O(\kappa^3)$. Appendix B shows that the left side of (ii) is a subsolution and the right side a supersolution of this equation, with the same behaviour at 0, and concludes by comparison. (iii) The middle inequality is (ii). For the first, put $s = \sqrt{\kappa^2 + 4}$. By (ii), $4A/\kappa \geq 4/(2+s)$ and $1 - A^2 \leq 1 - \kappa^2/(2+s)^2$, and

$$1 - \frac{\kappa^2}{(2+s)^2} \leq \frac{4}{2+s} \iff (2+s)^2 - \kappa^2 \leq 4(2+s) \\ \iff s^2 - \kappa^2 \leq 4,$$

which holds with equality. So $A' = 1 - A^2 - 3A/\kappa \leq A/\kappa$. (iv) is the comparison of (ii) with a third barrier, which improves the lower bound of (ii); it is carried out in the same appendix. $\square$

C. The Lipschitz rate of the family

Let δ be the chordal metric $\delta(x, x') = |x - x'|$ on S^3 and W_1 the Kantorovich distance for δ . For $c \in \mathbb{R}^4$ and a unit vector $h \in \mathbb{R}^4$ put

$$L(c, h) = \sup_g \text{Cov}_{v_c}(g, \langle h, x \rangle), \quad (12)$$

the supremum over 1-Lipschitz functions g for δ . Since $\frac{d}{dt} v_{c+th}(g) = \text{Cov}_{v_{c+th}}(g, \langle h, x \rangle)$, the Kantorovich–Rubinstein

duality gives, for all b, b' ,

$$W_1(v_b, v_{b'}) \leq |b - b'| \sup_{c \in [b, b']} L\left(c, \frac{b' - b}{|b' - b|}\right). \quad (13)$$

By rotation invariance $L(c, h)$ depends only on $\kappa = |c|$ and on the angle φ between h and $c^\perp$, and we write $L(\kappa, \varphi)$.

Lemma 6. For all c and h , $L(c, h) \leq 1/(2\sqrt{3})$. Moreover:

- (i) at $c = 0$, $L(0, h) = \frac{1}{4}$ for every h ;
- (ii) if $h \perp c$, then $L(c, h) = \mathbb{E}_{v_c} x_2^2 = A(\kappa)/\kappa \leq \frac{1}{4}$;
- (iii) if $\kappa \geq \sqrt{88}/3 \approx 3.127$, then $L(c, h) \leq \frac{1}{4}$ for every h .

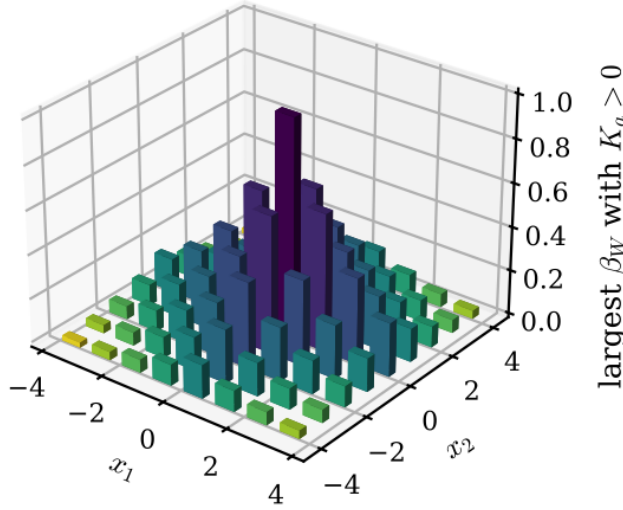
Proof. A 1-Lipschitz function for δ is 1-Lipschitz for the geodesic distance, which is larger, so $|\nabla g| \leq 1$ almost everywhere, and Lemma 4 gives $\text{Var } g \leq \frac{1}{3}$. By Lemma 5(i), (iii) the largest eigenvalue of the covariance of x is $A/\kappa \leq \frac{1}{4}$, so $\text{Var} \langle h, x \rangle \leq A/\kappa$. The Cauchy–Schwarz inequality gives

$$L(c, h) \leq \left(\frac{1}{3}A/\kappa\right)^{1/2} \leq \left(3(1 + \sqrt{\kappa^2 + 9})\right)^{-1/2} \leq \frac{1}{2\sqrt{3}}. \quad (14)$$

(i) and (ii) are reflection arguments. Let R be the reflection in the hyperplane $h^\perp$, and suppose v_c is R -invariant, which is the case when $c = 0$ or $h \perp c$. Then $\mathbb{E} \langle h, x \rangle = 0$ and

$$\begin{aligned} \text{Cov}(g, \langle h, x \rangle) &= \mathbb{E}[(g(x) - g(Rx))\langle h, x \rangle; \langle h, x \rangle > 0] \\ &\leq \mathbb{E}[2\langle h, x \rangle^2; \langle h, x \rangle > 0] = \mathbb{E} \langle h, x \rangle^2, \end{aligned}$$

$w_e = a^{-|e|}$ on horizontal links, $a = 1.6$



Theorem D for $SU(2)$, $d = 4$

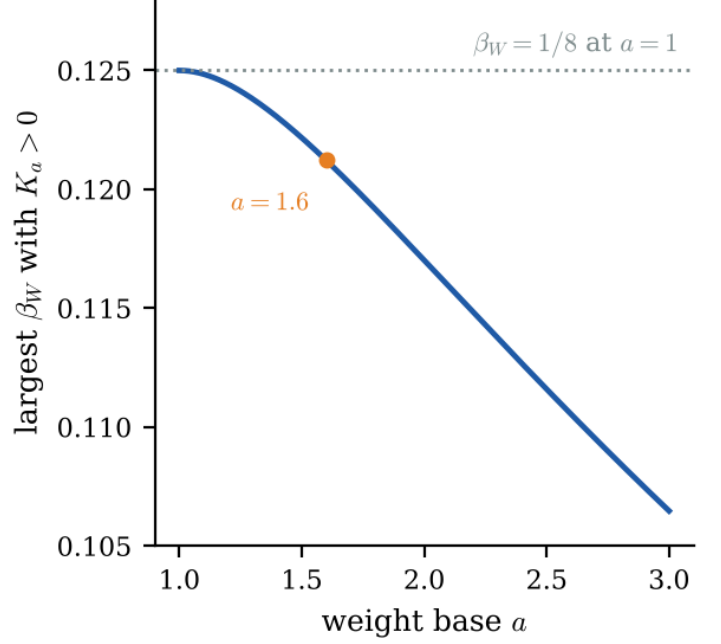


FIG. 5. The weights $w_e = a^{-|e|}$ of the distance (8) on the horizontal links of a 9×9 patch of $\mathbb{Z}^2$ (bar heights, $a = 1.6$). Links in a common plaquette have $||e| - |f|| \leq 1$, so the ratio w_e/w_f lies in $[a^{-1}, a]$; the proof of Lemma 3 pays only $\frac{1}{2}(r^{1/2} + r^{-1/2}) - 1 \leq (\sqrt{a} - 1)^2/(2\sqrt{a})$ for each such ratio r , which vanishes as $a \rightarrow 1$. Right: the largest β_W for which the constant K_a of Theorem D is positive, for four-dimensional $SU(2)$; it decreases from $1/8$ at $a = 1$, and at $a = 1.6$ it is 0.1212 .

since $\delta(x, Rx) = 2|\langle h, x \rangle|$. Equality holds for $g = \langle h, x \rangle$, which is 1-Lipschitz. At $c = 0$, $\mathbb{E}\langle h, x \rangle^2 = \frac{1}{4}$; for $h \perp c$, $\mathbb{E}\langle h, x \rangle^2 = A/\kappa$. (iii) By (14), $L \leq \frac{1}{4}$ as soon as $1 + \sqrt{\kappa^2 + 9} \geq \frac{16}{3}$, that is $\kappa^2 \geq \frac{88}{9}$. $\square$

The reflection is drawn in Fig. 7. It settles the directions $h \perp c$ and the point $c = 0$; for an oblique h the reflection no longer preserves v_c , and the Cauchy–Schwarz bound (14) is too weak for $\kappa < \sqrt{88}/3$. The next theorem closes this case.

Theorem 7 (The sharp quarter). *For every $c \in \mathbb{R}^4$ and every unit vector $h \in \mathbb{R}^4$,*

$$L(c, h) \leq \frac{1}{4},$$

with equality at $c = 0$. Equivalently, $W_1(v_b, v_{b'}) \leq \frac{1}{4}|b - b'|$ for all $b, b' \in \mathbb{R}^4$, and the constant $\frac{1}{4}$ is the best possible.

The equivalence is (13) in one direction and the derivative formula above it in the other; the sharpness is Lemma 6(i). The metric matters.¹⁴ The proof has four steps. A Lipschitz

function on S^3 extends to the unit ball $B \subset \mathbb{R}^4$, and Green’s formula in B bounds the covariance by a negative Sobolev norm of the density (Lemma 8). This norm splits into a zonal and a transverse part, both explicit series in modified Bessel functions (Lemma 9). The transverse part is at most $\frac{1}{4}$ because a certain power series has nonnegative coefficients (Proposition 11), and the zonal part is at most $\frac{1}{4}$ by a Riccati comparison (Proposition 12).

D. Harmonic extension

Let σ be the normalised surface measure of S^3 and H_k the space of spherical harmonics of degree k , the restrictions to S^3 of harmonic homogeneous polynomials of degree k on $\mathbb{R}^4$. The spaces H_k are orthogonal in $L^2(\sigma)$ and span it; we write u_k for the component of u in H_k , and for u with $\int u d\sigma = 0$ we put

$$\mathcal{Q}(u) = \sum_{k \geq 1} \frac{1}{k} \|u_k\|_{L^2(\sigma)}^2. \quad (15)$$

This is $\langle u, \Lambda^{-1}u \rangle$ for the Dirichlet-to-Neumann operator Λ of B , which acts on H_k as multiplication by k .

Lemma 8. *Let u be a real-analytic function on S^3 with $\int u d\sigma = 0$ and let g be 1-Lipschitz for the chordal metric. Then*

$$\int_{S^3} g u d\sigma \leq \frac{1}{2} \mathcal{Q}(u)^{1/2}.$$

¹⁴ In the geodesic metric of S^3 the statement is false already at $c = 0$: the function $\arcsin x_2$ is geodesically 1-Lipschitz and its covariance with x_2 under the uniform law is $\frac{8}{9\pi} \approx 0.283 > \frac{1}{4}$. A proof therefore has to use that the chordal distance between two points is shorter than their geodesic distance, that is, it has to see the sphere from the ball it bounds. The proof below does exactly this: it extends a chordally Lipschitz function to the ball and integrates by parts there.

Proof. The chordal metric is the Euclidean metric of $\mathbb{R}^4$ restricted to S^3 , so McShane's extension $G(y) = \inf_{x \in S^3} (g(x) + |x - y|)$ is 1-Lipschitz on $\mathbb{R}^4$ and equals g on S^3 [82]; by Rademacher's theorem $|\nabla G| \leq 1$ almost everywhere. Let $\psi = \sum_{k \geq 1} k^{-1} r^k u_k(\omega)$ in polar coordinates $y = r\omega$. Each term is a harmonic polynomial, the series converges with all derivatives on the closed ball because u is real-analytic,¹⁵ and $\partial_r \psi = u$ on S^3 . Green's formula for the Lipschitz function G and the smooth harmonic function ψ reads $\int_{S^3} G \partial_r \psi dS = \int_B \nabla G \cdot \nabla \psi dy$. With $|S^3| = 2\pi^2$ and $|B| = \pi^2/2$ the Cauchy–Schwarz inequality gives

$$\int g u d\sigma = \frac{1}{2\pi^2} \int_B \nabla G \cdot \nabla \psi dy \leq \frac{|B|^{1/2}}{2\pi^2} \left(\int_B |\nabla \psi|^2 dy \right)^{1/2}.$$

Green's formula once more gives $\int_B |\nabla \psi|^2 = \int_{S^3} \psi \partial_r \psi dS = 2\pi^2 \sum_k k^{-1} \|u_k\|^2 = 2\pi^2 \mathcal{Q}(u)$, and the right side becomes $\frac{1}{2} \mathcal{Q}(u)^{1/2}$. $\square$

The lemma is a form of Thomson's principle. By Kantorovich–Rubinstein duality, $\sup_g \int g u d\sigma$ is the least cost of a flow in $\mathbb{R}^4$ that carries the positive part of $u d\sigma$ to its negative part, the cost being the total length travelled by the mass.¹⁶ The harmonic field $\nabla \psi$ is one such flow, through the interior of the ball, and the lemma computes its cost. At $c = 0$ the flow is optimal: for $u = \langle h, x \rangle$ one has $\psi = \langle h, y \rangle$, the field $\nabla \psi = h$ is constant, the Cauchy–Schwarz inequality is an equality, and $\mathcal{Q}(u) = \|u\|^2 = \frac{1}{4}$.

E. Spectral decomposition

Fix $c = \kappa e_1$ with $\kappa > 0$ and write $h = \cos \varphi e_1 + \sin \varphi e_2$, which is no loss of generality by rotation invariance. Let $\rho = dv_c/d\sigma = e^{\kappa x_1}/Z(\kappa)$, $Z(\kappa) = \int e^{\kappa x_1} d\sigma = 2I_1(\kappa)/\kappa$, and

$$u = (\langle h, x \rangle - \mathbb{E}_{v_c} \langle h, x \rangle) \rho,$$

so that $\text{Cov}_{v_c}(g, \langle h, x \rangle) = \int g u d\sigma$ and $\int u d\sigma = 0$. Let U_n be the Chebyshev polynomials of the second kind and put

$$d_n(\kappa) = \frac{(n+1)I_{n+1}(\kappa)}{I_1(\kappa)}, \quad n \geq 0. \quad (16)$$

Lemma 9. *With this notation*

$$\mathcal{Q}(u) = \cos^2 \varphi \mathcal{Q}_z(\kappa) + \sin^2 \varphi \mathcal{Q}_x(\kappa),$$

¹⁵ For real-analytic u the norms $\|u_k\|$ decay geometrically in k , and the supremum of a function in H_k is at most $(\dim H_k)^{1/2} = k+1$ times its $L^2(\sigma)$ norm, so the series for ψ and for each of its derivatives converge uniformly on $\bar{B}$.

¹⁶ This is the Beckmann formulation of the transport problem with cost $|x - y|$: the optimal cost equals the least $\int |V|$ over vector measures V on $\mathbb{R}^4$ with divergence $u d\sigma$. Restricting the flow to the closed ball does not change the infimum, since projecting onto $\bar{B}$ shortens paths. The gradient $\nabla \psi$ on B is one admissible flow. It minimises the energy $\int |V|^2$ rather than the cost $\int |V|$, and the Cauchy–Schwarz inequality converts one into the other at the price of the factor $|B|^{1/2}$.

where

$$\mathcal{Q}_z = \sum_{n \geq 1} \frac{d'_n(\kappa)^2}{n}, \quad \mathcal{Q}_x = \frac{1}{3\kappa^2} \sum_{n \geq 1} (n+2) d_n(\kappa)^2.$$

Proof. The functions $U_n(x_1)$, $n \geq 0$, are the zonal harmonics: $U_n(x_1) \in H_n$, and they are orthonormal in $L^2(\sigma)$ (Appendix F). The expansion $e^{\kappa t} = \sum_n \frac{2}{\kappa} (n+1) I_{n+1}(\kappa) U_n(t)$ of the same appendix gives $\rho = \sum_n d_n(\kappa) U_n(x_1)$, with $d_0 = 1$.

Write $u = \cos \varphi u_z + \sin \varphi u_x$ with $u_z = (x_1 - A)\rho$ and $u_x = x_2 \rho$; here $\mathbb{E}_{v_c} x_1 = A$ and $\mathbb{E}_{v_c} x_2 = 0$. The reflection $x_2 \mapsto -x_2$ maps each H_k to itself, fixes u_z and reverses u_x , so the components of u_z and u_x in each H_k are orthogonal and the cross term in $\mathcal{Q}(u)$ vanishes.

Since $A = Z'/Z$, we have $u_z = \partial_\kappa \rho = \sum_n d'_n(\kappa) U_n(x_1)$, and the component of u_z in H_n is $d'_n U_n(x_1)$, of norm $|d'_n|$. This gives $\mathcal{Q}_z$.

For u_x , let $y(t) = \cos t e_1 + \sin t e_2$. The law $v_{\kappa y(t)}$ has density $\rho_t = e^{\kappa \langle y(t), x \rangle} / Z(\kappa) = \sum_n d_n(\kappa) U_n(\langle y(t), x \rangle)$, and $\partial_t \rho_t|_{t=0} = \kappa x_2 \rho$. The function $U_n(\langle y(t), x \rangle)$ lies in H_n for every t , hence so does its derivative $x_2 U'_n(x_1)$ at $t = 0$, and the component of u_x in H_n is $\kappa^{-1} d_n x_2 U'_n(x_1)$. Its norm follows from the eigenvalue $n(n+2)$ of the Laplacian of S^3 on H_n . For $f = U_n(x_1)$,

$$\begin{aligned} n(n+2) &= n(n+2) \|f\|^2 = \int |\nabla_{S^3} f|^2 d\sigma \\ &= \int U'_n(x_1)^2 (1 - x_1^2) d\sigma, \end{aligned}$$

since the tangential gradient of x_1 has squared length $1 - x_1^2$. Given x_1 , the point (x_2, x_3, x_4) is uniform on the sphere of radius $(1 - x_1^2)^{1/2}$, so $\mathbb{E}[x_2^2 | x_1] = \frac{1}{3}(1 - x_1^2)$ and $\|x_2 U'_n(x_1)\|^2 = \frac{1}{3} n(n+2)$. Summing $n^{-1} \kappa^{-2} d_n^2 \cdot \frac{1}{3} n(n+2)$ over $n \geq 1$ gives $\mathcal{Q}_x$. $\square$

By Lemmas 8 and 9,

$$L(\kappa, \varphi) \leq \frac{1}{2} \max(\mathcal{Q}_z(\kappa), \mathcal{Q}_x(\kappa))^{1/2}, \quad (17)$$

and Theorem 7 follows once both parts are at most $\frac{1}{4}$. Both are $\frac{1}{4}$ at $\kappa = 0$, by continuity from the values $d_1 \sim \kappa/2$, $d'_1 \rightarrow \frac{1}{2}$ and $d_n = O(\kappa^n)$.

F. Three Bessel sums

The series of Lemma 9 reduce to closed forms. Write I_j for $I_j(\kappa)$.

Lemma 10. *For every $\kappa > 0$,*

$$\begin{aligned} \sum_{j \geq 1} j^2 I_j^2 &= \frac{\kappa}{4} I_1(2\kappa), & \sum_{j \geq 1} j^3 I_j^2 &= \frac{\kappa^2}{4} (I_0^2 + I_1^2), \\ I_1(2\kappa) &= 2 \sum_{j \geq 1} I_{j-1} I_j. \end{aligned}$$

The first and third identities follow from Parseval's formula for $e^{\kappa \cos \theta} = \sum_{j \in \mathbb{Z}} I_j e^{ij\theta}$, and the second from the recurrences of I_j by telescoping; Appendix F gives the proofs.

G. The transverse part

Proposition 11. $\mathcal{Q}_x(\kappa) \leq \frac{1}{4}$ for every $\kappa \geq 0$.

Proof. By (16), $\mathcal{Q}_x = (3\kappa^2 I_1^2)^{-1} \sum_{j \geq 2} j^2(j+1)I_j^2$, and the sum equals $\sum_{j \geq 1} (j^3 + j^2)I_j^2 - 2I_1^2$. By Lemma 10, $\mathcal{Q}_x \leq \frac{1}{4}$ is equivalent to

$$\mathcal{D}(\kappa) := (8 + 2\kappa^2)I_1^2 - \kappa^2 I_0^2 - \kappa I_1(2\kappa) \geq 0. \quad (18)$$

Each term is an entire function of κ that is even after the factor κ^2 is removed, so $\mathcal{D}(\kappa) = \sum_{n \geq 0} \mathcal{D}_n \kappa^{2n}$. Lemma 32 of Appendix F computes the coefficients: $\mathcal{D}_0 = \mathcal{D}_1 = 0$, and for $n = m + 1 \geq 2$

$$\mathcal{D}_n = \frac{N_m}{4^m m! (m+1)! (m+1)(m+2)},$$

with

$$N_m = \binom{2m}{m} (4(2m+1) + (m-1)(m+1)(m+2)) - 4^m (m+1)(m+2).$$

The numerator N_m vanishes for $m = 0, 1, 2$ and is positive for $m \geq 3$ (Lemma 32). So every coefficient of $\mathcal{D}$ is nonnegative and $\mathcal{D} \geq 0$. $\square$

The first nonzero coefficient is $\mathcal{D}_4 = \frac{1}{2304}$, so $\frac{1}{4} - \mathcal{Q}_x = \mathcal{D}/(12\kappa^2 I_1^2)$ is of order κ^4 at 0: the transverse part leaves the value $\frac{1}{4}$ very slowly, which is why no estimate that loses a constant can prove the proposition.

H. The zonal part

Let $A = I_2/I_1$ as before, so that $d_1 = 2A$ and, by Lemma 5(i), $A' = 1 - A^2 - 3A/\kappa$.

Proposition 12. For $0 < \kappa \leq \sqrt{88}/3$,

$$\mathcal{Q}_z(\kappa) \leq \frac{1}{4} + 4A'^2 + \frac{A^2}{4} - \frac{A}{2\kappa} - \frac{2A^2}{\kappa^2} < \frac{1}{4}.$$

Proof. The higher modes. With $r_v = I_{v+1}/I_v$ the recurrence $I'_v = I_{v+1} + \frac{v}{\kappa} I_v$ gives

$$\frac{d'_n}{d_n} = \frac{I'_{n+1}}{I_{n+1}} - \frac{I'_1}{I_1} = \frac{n}{\kappa} + r_{n+1} - r_1.$$

By Lemma 33 of Appendix F, $r_{n+1} \leq r_1$, and $r_1 = A \leq \kappa/4$ by Lemma 5(iii). Hence $-n/\kappa \leq n/\kappa - \kappa/4 \leq d'_n/d_n \leq n/\kappa$ for $n \geq 2$ and $\kappa^2 \leq 16$, so $d_n'^2 \leq n^2 d_n^2 / \kappa^2$. With $d'_1 = 2A'$,

$$\mathcal{Q}_z \leq 4A'^2 + \frac{1}{\kappa^2} \sum_{n \geq 2} n d_n^2 = 4A'^2 + \frac{S_3 - S_2 - 4I_1^2}{\kappa^2 I_1^2},$$

where $S_2 = \sum_{j \geq 1} j^2 I_j^2$ and $S_3 = \sum_{j \geq 1} j^3 I_j^2$; the second form uses $n d_n^2 I_1^2 = (j-1)j^2 I_j^2$ with $j = n+1$.

The closed form. The recurrences $I_0 - I_2 = \frac{2}{\kappa} I_1$ and $I_1 - I_3 = \frac{4}{\kappa} I_2$ give $I_0/I_1 = 2/\kappa + A$ and $I_2 I_3 / I_1^2 = A - 4A^2/\kappa$. By Lemma 10, $S_3/(\kappa^2 I_1^2) = \frac{1}{4}(2/\kappa + A)^2 + \frac{1}{4}$, and, keeping three terms of the third identity,

$$\frac{S_2}{\kappa^2 I_1^2} \geq \frac{I_0 I_1 + I_1 I_2 + I_2 I_3}{2\kappa I_1^2} = \frac{1}{2\kappa} \left(\frac{2}{\kappa} + 3A - \frac{4A^2}{\kappa} \right).$$

Collecting terms, $\mathcal{Q}_z \leq \frac{1}{4} + 4A'^2 + T(A)$ with $T(a) = \frac{a^2}{4} - \frac{a}{2\kappa} - \frac{2a^2}{\kappa^2}$, which is the first inequality.

The barrier. Let $s = \sqrt{\kappa^2 + 25/4}$ and $A_b = \kappa/(\frac{3}{2} + s)$. By Lemma 5(iv), $A_b \leq A \leq \kappa/4$. The function $a \mapsto 1 - a^2 - 3a/\kappa$ decreases on $a \geq 0$, and $A' \geq 0$ because $A' = \text{Var} x_1$, so $0 \leq A' \leq \alpha := 1 - A_b^2 - 3A_b/\kappa$. For $\kappa^2 \leq 12$ the quadratic T decreases on $[0, \kappa/4]$, since

$$T(a) - T(a') = (a' - a) \frac{2\kappa - (a + a')(\kappa^2 - 8)}{4\kappa^2}$$

and $(a + a')(\kappa^2 - 8) \leq \frac{\kappa}{2}(\kappa^2 - 8) \leq 2\kappa$ there. So $\mathcal{Q}_z \leq \frac{1}{4} + 4\alpha^2 + T(A_b)$. With $\kappa^2 = s^2 - \frac{25}{4}$ and $A_b/\kappa = 2/(2s + 3)$ both terms are rational in s : $\alpha = 16/(2s + 3)^2$, and

$$4\alpha^2 + T(A_b) = \frac{(2s-5)p(s)}{4(2s+3)^4}, \quad (19)$$

$$p(s) = 8s^3 + 28s^2 - 98s - 695.$$

For $0 < \kappa \leq \sqrt{88}/3$ we have $\frac{5}{2} < s \leq \sqrt{577}/6 < 4.004$. On $s \geq \frac{5}{2}$ the cubic p increases, since $p'(s) = 24s^2 + 56s - 98 > 0$, and $p(4.004) < -120$. So the right side of (19) is negative. $\square$

I. Proof of Theorem 7

At $c = 0$ the theorem is Lemma 6(i), and for $|c| \geq \sqrt{88}/3$ it is Lemma 6(iii). For $0 < |c| < \sqrt{88}/3$, Propositions 11 and 12 give $\mathcal{Q}_x, \mathcal{Q}_z \leq \frac{1}{4}$, and (17) gives $L \leq \frac{1}{4}$. $\square$

The two parts of the norm behave differently (Fig. 6). The transverse part stays within 10^{-2} of $\frac{1}{4}$ for $\kappa \leq 3$, and there the theorem is an identity between power series in which no constant can be lost. The zonal part drops below $\frac{1}{4}$ at once, and a cruder argument suffices; its majorant in Proposition 12 ceases to be below $\frac{1}{4}$ only beyond the point where the Cauchy–Schwarz bound of Lemma 6(iii) takes over.¹⁷

J. Proof of Theorem E

Proof. For links $e \neq e'$ let

$$C_{ee'} = \sup \frac{W_1(\gamma_e(\cdot | \omega), \gamma_{e'}(\cdot | \omega'))}{\delta(\omega_{e'}, \omega'_{e'})},$$

¹⁷ The two arguments overlap: the majorant of Proposition 12 is below $\frac{1}{4}$ on all of $(0, \sqrt{88}/3]$, and (14) is at most $\frac{1}{4}$ from $\sqrt{88}/3 \approx 3.127$ on. For Theorem E at $d = 4$ only $\kappa \leq 8(d-1)|\beta| < \frac{4}{3}$ occurs, because the staple sum of a link has at most $2(d-1)$ terms of norm $4|\beta|$ each.

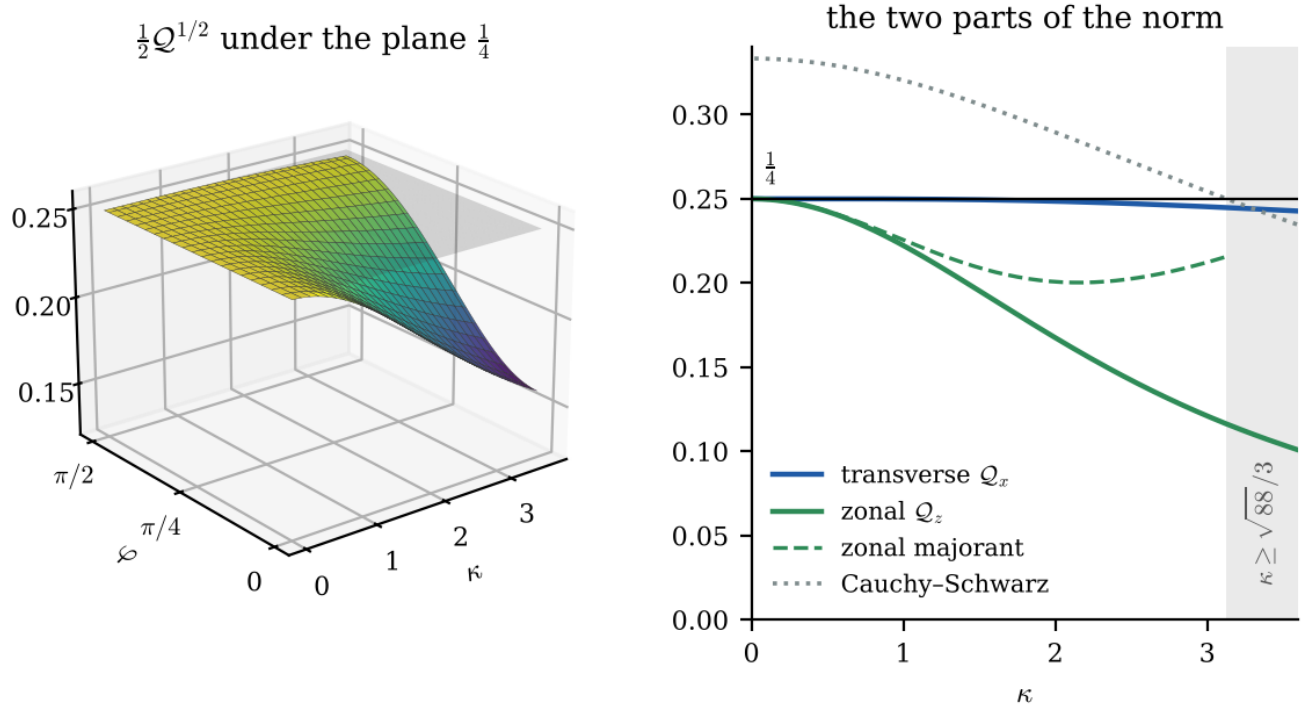


FIG. 6. The proof of Theorem 7. Left: the bound $\frac{1}{2}\mathcal{Q}^{1/2}$ of Lemma 8 on the rate $L(\kappa, \varphi)$ as a function of the field strength $\kappa = |c|$ and the angle φ between h and c , below the plane $\frac{1}{4}$ (grey), which it touches along $\kappa = 0$. Right: the transverse part $\mathcal{Q}_x$ (blue), the zonal part $\mathcal{Q}_z$ (green), the majorant of Proposition 12 (green, dashed) and the square of twice the Cauchy-Schwarz bound (14) (grey, dotted), which reaches $\frac{1}{4}$ at $\kappa = \sqrt{88}/3$ (shaded region to its right).

the supremum over pairs of fields that differ only at e' , where $\gamma_e(\cdot | \omega)$ is the conditional law of Q_e . If e and e' do not share a plaquette, $C_{ee'} = 0$. Otherwise they share exactly one plaquette, and moving $Q_{e'}$ from $\omega_{e'}$ to ω'_e changes one staple of e by an isometric image of the displacement, because left and right multiplication by unit quaternions and conjugation are isometries of $\mathbb{R}^4$. So $|b - b'| = 4|\beta| \delta(\omega_{e'}, \omega'_e)$, and by (13) and Theorem 7,

$$C_{ee'} \leq 4|\beta| \cdot \frac{1}{4} = |\beta|.$$

A link lies in $2(d-1)$ plaquettes, each with three other links, so every row and every column of C has at most $6(d-1)$ nonzero entries and

$$\alpha := \max \left(\sup_e \sum_{e'} C_{ee'}, \sup_{e'} \sum_e C_{ee'} \right) \leq 6(d-1)|\beta|.$$

For $\alpha < 1$ the comparison lemma of Appendix D (Lemma 28) gives uniqueness of the Gibbs measure and decay of truncated correlations of Lipschitz local functions at every rate $\lambda < -\log \alpha$, measured in the distance between link midpoints. At $d = 4$, $\alpha < 1$ reads $|\beta| < 1/18$, that is $\beta_W < 2/9$. $\square$

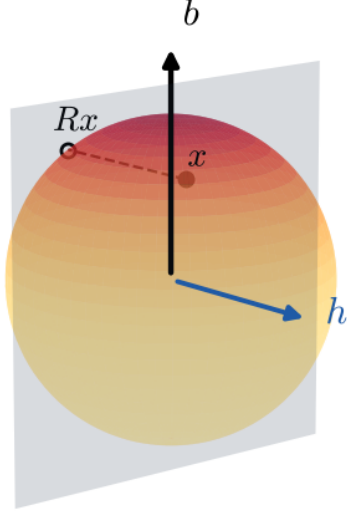
Proposition 13 (The ceiling). *Let δ be the chordal or the geodesic metric of S^3 , used both at e' and for the Kantorovich distance at e . For all links e, e' in a common plaquette, $C_{ee'} \geq |\beta|$. Hence no argument through single-link Dobrushin coefficients proves uniqueness beyond $|\beta| = 1/(6(d-1))$.*

Proof. The $2(d-1)$ staples of e involve pairwise disjoint sets of links other than e' , so they can be chosen with sum zero while the staple through e' is arbitrary. Then $b = 0$ and γ_e is uniform. By Lemma 6(i) the derivative of v_b at $b = 0$ in the direction h has Kantorovich norm $\frac{1}{4}|h|$ in the chordal metric, and at least that in the geodesic one, tested against $\langle x, \hat{h} \rangle$. With $|b - b'| = 4|\beta|\delta$ to first order this gives $C_{ee'} \geq |\beta|$, and summing over the $6(d-1)$ neighbours gives $\alpha \geq 6(d-1)|\beta|$. $\square$

Proposition 13 and the proof of Theorem E together give $\sup C_{ee'} = |\beta|$ exactly: Theorem 7 is the statement that the single-link coupling is optimal, and Theorem E is the best result a single-link argument can give.¹⁸ For $SU(N)$ with $N \geq 3$ the staple sum is not a multiple of a unitary matrix, the single-link law is not of von Mises-Fisher type, and the same argument with the single-link Bakry-Émery bound reproduces only $|\beta| < 1/(16(d-1))$; Theorem D is stronger there.

¹⁸ A 2026 preprint attributed to J. Lockwood, which we have seen only as an abstract, announces uniqueness for four-dimensional $SU(2)$ at $\beta_W < 2/9$ by a Wasserstein-Dobrushin argument; we have not been able to check its proof. By Proposition 13 that range is the ceiling of the method, and Theorem 7 is what the method needs to reach it.

$h \perp b$: the reflection preserves ν_b



h oblique to b : harmonic extension

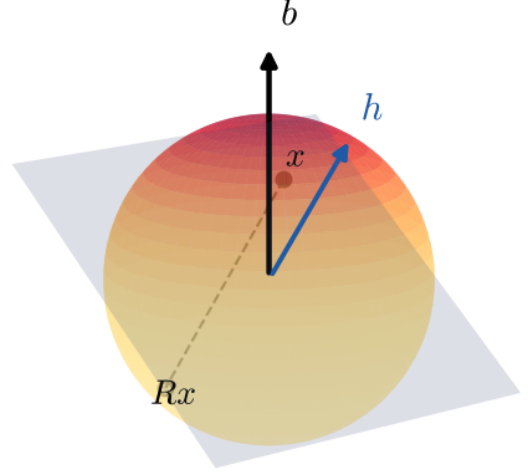


FIG. 7. The single-link law on S^3 , drawn in the three coordinates (x_1, x_2, x_3) . Left: the von Mises–Fisher density for $\kappa = 2$, peaked in the staple direction b ; the plane $h^\perp$ with $h \perp b$ and the reflection $x \mapsto Rx$ that proves Lemma 6(ii), which moves each point by $2|\langle h, x \rangle|$. Right: for h not perpendicular to b the reflection no longer preserves the law, and Theorem 7 is proved instead by harmonic extension into the ball.

VI. A GAUGE-INVARIANT FINITE-BOX CRITERION

A. Interactions and boundary data

Let G be a compact metrisable group with normalised Haar measure λ , and E the set of positively oriented links of $\mathbb{Z}^d$ or of a torus. An interaction is a family $\Phi = (\Phi_X)$ indexed by finite sets $X \subset E$ of diameter at most R , with $\Phi_X: G^X \rightarrow \mathbb{R}$ continuous, invariant under translations, and gauge invariant: $\Phi_X(gU) = \Phi_X(U)$ for every $g: \mathbb{Z}^d \rightarrow G$, where $(gU)_{xy} = g_x U_{xy} g_y^{-1}$. The Wilson action is $\Phi_p = -N\beta \text{Re Tr } U_p$; a renormalised effective action adds further terms.¹⁹

For finite $V \subset E$ let $\partial V = \bigcup \{X \setminus V : X \cap V \neq \emptyset, \Phi_X \neq 0\}$. The specification is

$$\gamma(dU_V \mid \omega) = Z_V(\omega)^{-1} e^{-\sum_{X \cap V \neq \emptyset} \Phi_X(U_V \omega)} \lambda^V(dU_V),$$

which depends on ω only through $\omega_{\partial V}$. Let $I(V)$ be the set of vertices all of whose incident links lie in V , and $\mathcal{G}_V = G^{I(V)}$. A gauge transformation at $x \in I(V)$ moves only links of V , so

$$g_* \gamma(\cdot \mid \omega) = \gamma(\cdot \mid \omega) \quad (g \in \mathcal{G}_V). \quad (20)$$

Lemma 14. *Let Γ_V be the graph with edge set ∂V . If $\omega'_{\partial V} = (h\omega)_{\partial V}$ for a map h from the vertices of Γ_V to G , then*

¹⁹ The definitions are written for the energy $\sum \Phi_X$ with the Gibbs factor $e^{-\sum \Phi_X}$, which is the convention of the probability literature; for the Wilson action this is the same measure as in (3).

$\gamma(\cdot \mid \omega') = (\tilde{h}|_V)_* \gamma(\cdot \mid \omega)$ for every extension $\tilde{h}$ of h . Consequently gauge-invariant expectations $\gamma(F \mid \omega)$ depend only on the class of $\omega_{\partial V}$ in $G^{\partial V} / G^{\text{vert}(\Gamma_V)}$, which for each connected component of Γ_V is the space of holonomy representations of its fundamental group modulo conjugation.

Proof. Change variables $U_V \mapsto \tilde{h}U_V$: Haar measure and the energy are invariant. For the second statement, gauge the links of a spanning tree of each component to the identity; the remaining links are free generators of the fundamental group, and the residual freedom is one constant per component, acting by conjugation.²⁰ $\square$

Plaquette holonomies alone do not determine this class when G is non-abelian; holonomies of larger loops of the shell enter.²¹ The criterion below therefore compares boundary conditions that differ at one link.

Lemma 15 (Gauge lift). *Let Π be a coupling of $\gamma(\cdot \mid \omega)$ and $\gamma(\cdot \mid \omega')$, let $g: G^V \times G^V \rightarrow \mathcal{G}_V$ be measurable, and let h be Haar distributed on $\mathcal{G}_V$, independent of $(U, U') \sim \Pi$. Then $(hU, hg(U, U')U')$ is again a coupling of the two measures, and $\mathbb{P}[(hU)_b \neq (hg(U, U')U')_b] = \Pi[U_b \neq (g(U, U')U')_b]$ for every link b .*

²⁰ This is the holonomy of a gauge field on a graph; for holonomy in the Riemannian sense, in a related $SU(n)$ setting, see [18]; for holonomies of connections used as the gates of a quantum computation, see [20].

²¹ Take two plaquettes of the shell sharing a vertex, with based holonomies (X, Y) and (X, Y') , where $X = Y = i\sigma_3$ and $Y' = i\sigma_1$ in $SU(2)$. Each holonomy is conjugate to its partner, but $\text{Tr}(XY) = -2 \neq 0 = \text{Tr}(XY')$, and XY is the holonomy of the boundary of the union of the two plaquettes.

Proof. By (20), $hU \sim \mathcal{W}(\cdot | \omega)$. Given (U, U') the element hg is Haar distributed, so hgU' is uniform on the $\mathcal{G}_V$ -orbit of U' , and by (20) again $hgU' \sim \mathcal{W}(\cdot | \omega')$. Each h_x acts on a link by a bijection of G , which preserves equality of the two link values. $\square$

B. The criterion

Write V_μ for the links of V in direction μ , $n_{\min} = \min_\mu |V_\mu|$, and $\omega \sim_y \omega'$ when ω, ω' differ only at the link $y \in \partial V$.

Definition 16. The interaction satisfies the gauge Dobrushin–Shlosman condition $\text{GDS}(V, \varepsilon)$ if there are a Borel family of couplings $\Pi_{\omega, \omega'}$ of $\mathcal{W}(\cdot | \omega)$ and $\mathcal{W}(\cdot | \omega')$, for all pairs $\omega \sim_y \omega'$, and numbers $\kappa_V(y, b) \geq \sup_{\omega \sim_y \omega'} \Pi_{\omega, \omega'}[U_b \neq U'_b]$ for $y \in \partial V$, $b \in V$, such that for every direction ν

$$K_V(\nu) := \sum_{y \in \partial V, \text{dir } y = \nu} \sum_{b \in V} \kappa_V(y, b) \leq (1 - \varepsilon) n_{\min}. \quad (21)$$

By Lemmas 14 and 15 the couplings may be built on the gauge quotient and then aligned by any measurable choice of interior gauge, and the supremum is a supremum over classes of shell holonomies (Fig. 8). For a box symmetric under the hypercubic group, (21) reads $\sum_{y, b} \kappa_V(y, b) \leq (1 - \varepsilon)|V|$, the form of Dobrushin and Shlosman [43], except that the supremum over boundary conditions is taken for each b separately; this is what makes the proof below work for an arbitrary compact group.

Theorem 17. Assume $\text{GDS}(V, \varepsilon)$ for a finite-range gauge-invariant interaction with compact G . Let $D = \max\{|b - y| : b \in V, y \in \partial V\}$, with distances between link midpoints, and $0 < \lambda < |\log(1 - \varepsilon)|/D$, so that $\theta := (1 - \varepsilon)e^{\lambda D} < 1$. Then:

- (i) There is exactly one Gibbs measure μ on G^E ; it is invariant under translations and gauge transformations.
- (ii) Let $S \subset E$ be finite and μ_1, μ_2 probability measures invariant under $\mathcal{W}_{V+a}$ for every translate with $(V + a) \cap S = \emptyset$. Let N_S be S together with the links of the translates of V that meet S ; then $|N_S| \leq c_V |S|$. There is a coupling of μ_1 and μ_2 under which, for every link b at distance greater than $\text{diam } V + R$ from S ,

$$\mathbb{P}[\sigma_b \neq \sigma'_b] \leq \frac{\theta}{1 - \theta} \sum_{z \in N_S} e^{-\lambda|b - z|}.$$

- (iii) For bounded functions F and H with finite supports S_F and S_H , the covariance $|\text{Cov}_\mu(F, H)|$ is at most

$$2\|F\|_\infty \|H\|_\infty \frac{\theta c_V}{1 - \theta} |S_F| |S_H| e^{-\lambda(\text{dist}(S_F, S_H) - \text{diam } V - R)}.$$

- (iv) Items (i)–(iii) hold on every torus of side greater than $2(\text{diam } V + R)$ with the same constants, for the untwisted and for every 't Hooft twisted Wilson action.

Proof. A minimal coupling. Let W be E or $E \setminus S$, let $\mathcal{A} = \{a : V + a \subseteq W\}$ and $V_a = V + a$. Fix a link b_0 and put $w_b = e^{-\lambda|b - b_0|}$, so that $\sum_b w_b < \infty$. On the set $\mathcal{C}$ of couplings of μ_1, μ_2 let $\Psi(\pi) = \sum_b w_b \pi[\sigma_b \neq \sigma'_b]$. The set $\mathcal{C}$ is weakly compact because G^E is compact, and each map $\pi \mapsto \pi[\sigma_b \neq \sigma'_b]$ is lower semicontinuous, being the measure of an open set. So Ψ attains its minimum at some π^* . Put $\rho(b) = \pi^*[\sigma_b \neq \sigma'_b]$.

Block resampling. For $a \in \mathcal{A}$ define a new coupling $T_a \pi^*$: sample $(\sigma, \sigma') \sim \pi^*$, keep both fields off V_a , and inside V_a sample from the following path coupling of $\mathcal{W}_a(\cdot | \sigma)$ and $\mathcal{W}_a(\cdot | \sigma')$. List $\partial V_a = \{z_1, \dots, z_k\}$ and let ω^j agree with σ' on $z_1, \dots, z_j$ and with σ elsewhere. Consecutive ω^{j-1}, ω^j are equal or differ only at z_j ; couple their block laws by the identity in the first case and by the translate of $\Pi_{\omega^{j-1}, \omega^j}$ in the second, and chain the couplings through their disintegrations, which depend measurably on the parameters [69, Ch. 3]. This gives $\tau^0, \dots, \tau^k$ with $\tau^0 \sim \mathcal{W}_a(\cdot | \sigma)$, $\tau^k \sim \mathcal{W}_a(\cdot | \sigma')$ and $\mathbf{1}[\tau_b^0 \neq \tau_b^k] \leq \sum_j \mathbf{1}[\tau_b^{j-1} \neq \tau_b^j]$. Since each μ_i is $\mathcal{W}_a$ -invariant, $T_a \pi^* \in \mathcal{C}$, and for $b \in V_a$

$$T_a \pi^*[\tau_b \neq \tau'_b] \leq \sum_{z \in \partial V_a} \kappa_V(z - a, b - a) \rho(z),$$

while nothing changes off V_a . Minimality, $\Psi(T_a \pi^*) \geq \Psi(\pi^*)$, gives

$$\sum_{b \in V_a} w_b \rho(b) \leq \sum_{b \in V_a} w_b \sum_{z \in \partial V_a} \kappa_V(z - a, b - a) \rho(z). \quad (22)$$

Summation over blocks. Sum (22) over $a \in \mathcal{A}$; all series converge, the left side being at most $|V| \sum_b w_b$. Let W° be the set of links b such that every translate of V containing b lies in W ; then $E \setminus W^\circ \subseteq N_S$. A link b of direction μ lies in exactly $|V_\mu| \geq n_{\min}$ translates. On the right, $|z - b| \leq D$ gives $w_b \leq e^{\lambda D} w_z$, and for fixed z of direction ν the sum over all a with $z \in \partial V_a$ and all $b \in V_a$ of $\kappa_V(z - a, b - a)$ equals $K_V(\nu)$. By (21),

$$n_{\min} \sum_{b \in W^\circ} w_b \rho(b) \leq \theta n_{\min} \left[\sum_{z \in W^\circ} w_z \rho(z) + \sum_{z \in N_S} w_z \right],$$

so $\sum_{W^\circ} w \rho \leq \frac{\theta}{1 - \theta} \sum_{z \in N_S} w_z$. Taking $b_0 \in W^\circ$ and keeping only the term $b = b_0$ on the left proves (ii).

Uniqueness. With $S = \emptyset$, $N_S = \emptyset$ and $\rho \equiv 0$, so any two Gibbs measures coincide. Gibbs measures exist by compactness, and translates and gauge transforms of a Gibbs measure are Gibbs measures; since there is only one, it is invariant. This is (i).

Decay. Let $\mu^u = \mu(\cdot | U_{S_H} = u)$ be a regular conditional probability; for μ -almost every u it is invariant under every $\mathcal{W}_a$ with $V_a \cap S_H = \emptyset$. Since $\text{Cov}(F, H) = \int H(u) [\mu^u(F) - \int \mu^{u'}(F) \mu(du')] \mu(du)$, we get $|\text{Cov}(F, H)| \leq \|H\|_\infty \text{ess sup}_{u, u'} |\mu^u(F) - \mu^{u'}(F)|$, and $|\mu^u(F) - \mu^{u'}(F)| \leq 2\|F\|_\infty \sum_{b \in S_F} \rho(b)$. Item (ii) with $S = S_H$ gives (iii).

Tori and twists. On a torus of side greater than $2(\text{diam } V + R)$ every translate is a genuine block and the argument is unchanged. Let $z_p \in Z(G)$ be a twist that is closed, so that on every ball of radius at least $\text{diam } V + R$ it equals $(dn)_p = \prod_{e \in \partial p} n_e^{\pm 1}$ for some $n : E \rightarrow Z(G)$; every 't Hooft twist has this property, because a closed $Z(G)$ -valued 2-cocycle is exact

on a contractible ball. Multiplication M_n of each link by the central element n_e satisfies $\gamma_a^w(\cdot | \omega) = (M_n)_* \gamma_a(\cdot | M_n \omega)$ on such a ball, since central factors commute through, and M_n is a bijection on each link. So the twisted specification satisfies $\text{GDS}(V, \varepsilon)$ with the same κ_V . This is (iv). $\square$

C. Confinement and stability

Let $\mathcal{V}$ be the family of sets $V \cap W'$, where W' runs over the interior link sets of all cubes in all positions. The box-hereditary condition $\text{HGDS}(V, \varepsilon)$ is (21) with $\sum_b \kappa_V(y, b)$ replaced by $\sup_{V' \in \mathcal{V}, y \in \partial V'} \sum_{b \in V'} \kappa_{V'}(y, b)$, the boundary condition of V' including the links of $V \setminus V'$.

Proposition 18. *Assume $\text{HGDS}(V, \varepsilon)$ for the Wilson action of a compact group $G \subset \text{U}(n)$. Then covariances of local functions under the finite-volume measure of every cube with every boundary condition decay exponentially, with constants independent of the cube and the boundary condition. By Chatterjee's theorem [34, Thm. 2.4] the centre symmetry is unbroken and, if the representation acts nontrivially on the centre, Wilson loops obey the area law.*

Proof. Fix a cube B with boundary condition δ , interior link set W_B , and local functions F, H . Condition $\mu_{B, \delta}$ on $U_{S_H} = u$ and on $U_{S_H} = u'$, and run the proof of Theorem 17 in $W = W_B \setminus S_H$ with all blocks $V_a \cap W_B$ that meet W . A link of W of direction μ lies in $|V_\mu| \geq n_{\min}$ blocks, so no boundary layer is lost on the left of the summed inequality. A link $z \in W$ in $\partial(V_a \cap W_B)$ lies in ∂V_a , and its total weight on the right is bounded by the hereditary condition. The other links of $\partial(V_a \cap W_B)$ lie outside B , where both copies carry δ , or in S_H . So the only sources of disagreement are the links of S_H , each with weight at most $e^{\lambda D}(|V| + |\partial V|)|V|/n_{\min}$, and the bound follows as in Theorem 17(iii), uniformly in B and δ . This is the decay property of [34, Def. 2.3]. $\square$

Lemma 19. *Suppose Φ satisfies $\text{GDS}(V, \varepsilon)$ and $\Phi' = \Phi + \mathcal{E}$ with $\mathcal{E}$ gauge invariant of range at most R . Put $t = \tanh(\frac{1}{2} \sum_{X \cap V \neq \emptyset} \|\mathcal{E}_X\|_\infty)$. Then Φ' satisfies $\text{GDS}(V, \varepsilon')$ with $\varepsilon' = \varepsilon - 2t|V| \max_v |\partial V|_v / n_{\min}$, where $|\partial V|_v$ is the number of links of direction v in ∂V .*

Proof. The two energies on V differ by a function of oscillation at most $s = 2 \sum_{X \cap V \neq \emptyset} \|\mathcal{E}_X\|_\infty$, and two probability measures whose log-densities differ by a function of oscillation s are within total variation $\tanh(s/4)$ (Lemma 30). Glue an optimal coupling of $\gamma_V(\cdot | \omega)$ with $\gamma_V(\cdot | \omega)$, then $\Pi_{\omega, \omega'}$, then an optimal coupling of $\gamma_V(\cdot | \omega')$ with $\gamma_V(\cdot | \omega')$. Then $\kappa'(y, b) \leq \kappa(y, b) + 2t$, and summing over $b \in V$ and over y of direction v gives the stated loss. $\square$

The smallness required in Lemma 19 shrinks with the box, since t must be of order $\varepsilon/(|V||\partial V|)$. This matters in Sec. VIII.

D. Two dimensions

In two dimensions the criterion holds at every coupling. Take $G = \text{SU}(2)$ and the plaquette weight $w(U) = e^{\frac{\beta}{2} \text{Tr} U}$, write $c_0 = \int w$, $f = w/c_0$ and $a_j = I_{2j+1}(\beta)/I_1(\beta)$ for $j = \frac{1}{2}, 1, \dots$. The character expansion of Appendix E gives $f^{*m} = \sum_j (2j+1) a_j^m \chi_j$ and

$$\varepsilon_m := \sup_U |f^{*m}(U) - 1| = \sum_{j \geq 1/2} (2j+1)^2 a_j^m,$$

which tends to 0 exponentially as $m \rightarrow \infty$.

Lemma 20 (Disk integration [74, 83]). *For a rectangle D made of k plaquettes, $\int \prod_{p \in D} w(U_p) dU_{\text{int} D} = c_0^k f^{*k}(\text{hol}_{\partial D})$.*

The proof is in Appendix E.

Proposition 21. *For every $\beta > 0$ there is L such that two-dimensional $\text{SU}(2)$ Wilson theory satisfies $\text{GDS}(V, \varepsilon)$ for the set V of the $2L(L-1)$ interior links of an $L \times L$ square and some $\varepsilon > 0$.*

Proof. For $y \in \partial V$ choose an $s \times s$ sub-square D with y on its outer side; let L_D be its $2s(s-1)$ interior links and $L_R = V \setminus L_D$. By Lemma 20, under $\gamma_V(\cdot | \omega)$ the marginal on L_R has density proportional to $\prod_{p \notin D} w(U_p) \cdot f^{*s^2}(\text{hol}_{\partial D})$, and only the last factor depends on ω_y . Two densities proportional to ρg_1 and ρg_2 with $g_i \in [1 - \varepsilon, 1 + \varepsilon]$ are within total variation $\eta = 2\varepsilon/(1 - \varepsilon^2)$ (Appendix E). Couple the two marginals on L_R maximally and sample L_D independently; then $\kappa(y, b) \leq 1$ on L_D and $\kappa(y, b) \leq \eta_{s^2}$ on L_R , with $\eta_m = 2\varepsilon_m/(1 - \varepsilon_m^2)$. Since $|\partial V|_v = 2L$ for each direction and $n_{\min} = L(L-1)$, the condition (21) holds when $2L[2s(s-1) + 2L(L-1)\eta_{s^2}] < L(L-1)$, that is

$$\frac{4s(s-1)}{L-1} + 4L\eta_{s^2} < 1.$$

Choose s with $\eta_{s^2} < 1/(80s^2)$, which is possible because η_{s^2} decays exponentially in s^2 , and then $L = 8s^2$. Both terms are then at most $\frac{1}{2} - \frac{1}{80}$. $\square$

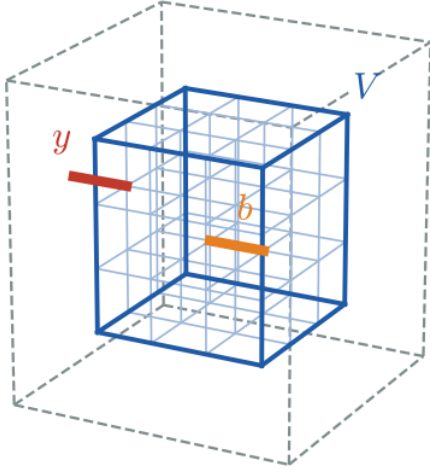
In four-dimensional $\text{U}(1)$ theory at large β correlations decay only as a power [57, 65], so by Theorem 17 no box satisfies GDS there.²²

VII. TRANSFER MATRICES AND THE MASS GAP

In this section $d = 4$ and $G \subset \text{U}(N)$ is compact. Let $\Lambda_L = (\mathbb{Z}/L\mathbb{Z})^3$ be the spatial torus, with $3L^3$ links and $3L^3$ plaquettes,

²² A weaker hypothesis, bounding only the influence of one boundary link on one observable at the centre of the box, would be met by the massless spin-wave part of that theory for large boxes, since that influence decays like the fourth power of the box size. Such a hypothesis cannot imply decay. The criterion has to measure joint couplings of the whole box, as Definition 16 does.

$\kappa_V(y, b)$: disagreement at b caused by y



frozen shell

translates of V containing b

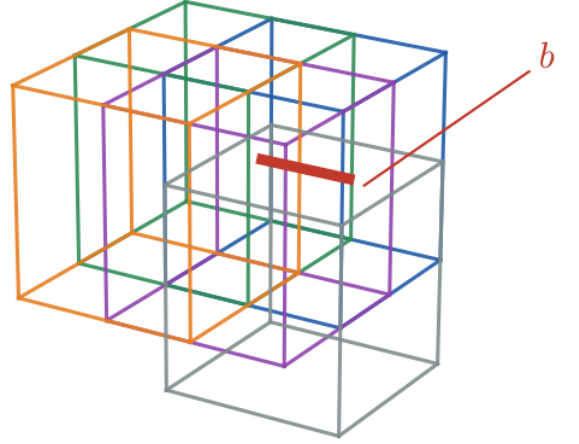


FIG. 8. The finite-box criterion. Left: a box V inside its frozen shell and a boundary link y ; two copies of the box are coupled with boundary conditions that differ only at y , and $\kappa_V(y, b)$ bounds the chance that they disagree at b after the best interior gauge alignment. Right: the translates of V covering a link b ; the proof of Theorem 17 resamples one translate at a time and sums the resulting inequalities.

μ the Haar measure on G^{EL} and $\mathcal{H}_L = L^2(G^{EL}, \mu)$. In temporal gauge the Wilson measure (3) on $\Lambda_L \times \mathbb{Z}/T\mathbb{Z}$ equals $\text{Tr}(\mathbb{T}_L^T)/Z$ for the integral operator with kernel

$$k(U, V) = e^{\frac{N\beta}{2} \sum_p (\text{Re Tr } U_p + \text{Re Tr } V_p)} \times \int_{G^{\Lambda_L}} \prod_{e=(x,y)} e^{N\beta \text{Re Tr}(U_e h_y V_e^{-1} h_x^{-1})} \prod_x dh_x,$$

the integral over h being the integral over the timelike links. This is the transfer matrix of Osterwalder and Seiler [91].²³

Proposition 22 (Finite-volume gap). *For $\beta \geq 0$ the operator $\mathbb{T}_L$ is self-adjoint, nonnegative, Hilbert–Schmidt and positivity improving. Its largest eigenvalue λ_0 is simple with a strictly positive eigenfunction, and every other spectral value λ satisfies $|\lambda| \leq (1 - e^{-12N^2\beta L^3})\lambda_0$.*

Proof. The kernel is continuous on a compact space, hence Hilbert–Schmidt, and symmetric (substitute $h \mapsto h^{-1}$). Write

$\mathbb{T}_L = A\mathcal{K}\Pi A$, with A multiplication by $e^{\frac{N\beta}{2} \sum_p \text{Re Tr } U_p}$, $\mathcal{K}$ the convolution on each link by $\kappa(g) = e^{N\beta \text{Re Tr } g}$, and Π the average over gauge transformations. Since $\text{Re Tr } g = \frac{1}{2}(\chi_F(g) + \chi_F(g))$ and products of characters decompose with nonnegative multiplicities, κ has nonnegative character coefficients, so convolution by it is a nonnegative operator; Π is an orthogonal projection commuting with $\mathcal{K}$, since κ is a class function, so $\mathcal{K}\Pi = \Pi\mathcal{K}\Pi \geq 0$ and $\mathbb{T}_L \geq 0$. From $|\text{Re Tr } g| \leq N$,

$$m_L := e^{-6N^2\beta L^3} \leq k(U, V) \leq e^{6N^2\beta L^3} =: M_L,$$

so $\mathbb{T}_L$ is positivity improving. By Jentzsch’s theorem [68] $\lambda_0 = \|\mathbb{T}_L\|$ is simple with an almost everywhere positive eigenfunction ψ_0 , and $m_L \|\psi_0\|_1 \leq \lambda_0 \|\psi_0\|_1 \leq M_L \|\psi_0\|_1$. The kernel $p(U, V) = k(U, V)\psi_0(V)/(\lambda_0\psi_0(U))$ is Markov, reversible for $\pi = \psi_0^2\mu/\|\psi_0\|_2^2$, and unitarily equivalent to $\mathbb{T}_L/\lambda_0$. It satisfies $p(U, V) \geq \varepsilon\psi_0(V)/\|\psi_0\|_1$ with $\varepsilon = m_L/M_L$, so by Doeblin’s argument $\|P^n f\|_\infty \leq 2(1 - \varepsilon)^n \|f\|_\infty$ for bounded f with $\pi(f) = 0$. For the spectral measure σ_f of P at such f this gives $\int \lambda^{2n} d\sigma_f \leq 4(1 - \varepsilon)^{2n} \|f\|_\infty^2$ for all n , so σ_f vanishes on $\{|\lambda| > 1 - \varepsilon\}$. Bounded mean-zero functions are dense in the mean-zero subspace of $L^2(\pi)$. $\square$

This holds for $U(1)$ as well and carries no information about infinite volume.²⁴

²³ The kernel is a positive operator because the Wilson action is reflection positive for reflections in planes between lattice hyperplanes: the weight splits as $F \cdot \Theta F$ times a factor $e^{N\beta \text{Re Tr}(U \Theta U)}$ on the crossing plaquettes, and Re Tr of a product of a matrix and its reflection expands into a positive combination of products of a function and its reflection, by the character expansion with nonnegative coefficients for $\beta \geq 0$. Reflection positivity in the time direction gives a Hilbert space by the Osterwalder–Schrader construction [89, 90], and the transfer matrix is the operator implementing one step of Euclidean time. Brydges, Fröhlich and Seiler [24] developed this framework for gauge fields, including gauge fixing and the identification of physical states, as the starting point of a construction of the continuum theory; the renormalisation group methods needed to continue from there are surveyed by Rivasseau [95].

²⁴ The bound $1 - e^{-12N^2\beta L^3}$ tends to 1 as $L \rightarrow \infty$, so the finite-volume gap it gives tends to 0; the content of a mass gap is the uniformity in L , Proposition 24.

Let ω be a state on the gauge-invariant local functions of $\mathbb{Z}^4$ that is invariant under translations and the hypercubic group and reflection positive for reflections in planes through sites and in planes bisecting links. Every limit of Wilson states on periodic tori has these properties [91]. The reconstruction gives a Hilbert space with a vector Ω , a map $F \mapsto \Psi_F$ with dense range, and a self-adjoint $\mathbb{T}$ with $0 \leq \mathbb{T} \leq 1$, $\mathbb{T}\Omega = \Omega$ and $\langle \Psi_F, \mathbb{T}' \Psi_H \rangle = \omega(\vartheta(\bar{F})\tau_t H)$, where ϑ is the time reflection and τ_t the time translation.²⁵

Proposition 23. *For $m > 0$ the following are equivalent:*

- (a) *the spectrum of $\mathbb{T}$ on $\Omega^\perp$ lies in $[0, e^{-m}]$;*
- (b) *for every F in a class of gauge-invariant local functions supported at times in a bounded interval and dense in $L^2(\omega)$ on each such support, there is $C_F < \infty$ with $0 \leq \omega(\vartheta(\bar{F})\tau_t F) - |\omega(F)|^2 \leq C_F e^{-mt}$ for all $t \geq 0$.*

Under (a) one may take $C_F = \|F\|_\infty^2$ for every bounded F .

Proof. If (a) holds, then

$$\omega(\vartheta(\bar{F})\tau_t F) - |\omega(F)|^2 = \langle \Psi'_F, \mathbb{T}' \Psi'_F \rangle$$

with $\Psi'_F = \Psi_F - \langle \Omega, \Psi_F \rangle \Omega \in \Omega^\perp$; this number lies in $[0, e^{-mt} \|\Psi_F\|^2]$, and $\|\Psi_F\|^2 \leq \|F\|_\infty^2$. Conversely, let σ be the spectral measure of $\mathbb{T}$ at Ψ'_F and let $\delta > 0$. Then

$$\sigma((e^{-m+\delta}, 1]) e^{(-m+\delta)t} \leq C_F e^{-mt}$$

for all t , so $\sigma((e^{-m+\delta}, 1]) = 0$. The spectral projection of $(e^{-m}, 1]$ restricted to $\Omega^\perp$ therefore annihilates the vectors Ψ'_F of the class, which are dense by the footnote above, and it vanishes. $\square$

Let $\Delta_L(\beta) = \log(\lambda_0/\lambda_1)$ be the gap of $\mathbb{T}_L$ and ω_L its vacuum state.

Proposition 24. *Let $m = \inf_L \Delta_L(\beta)$ and assume $m > 0$. Then every weak limit point ω of the sequence (ω_L) satisfies*

$$|\omega(F\tau_t H) - \omega(F)\omega(H)| \leq \|F\|_\infty \|H\|_\infty e^{-mt}$$

for gauge-invariant F and H on single time slices, and hence has transfer-matrix gap at least m .

Proof. The truncated correlation $\omega_L(F\tau_t H) - \omega_L(F)\omega_L(H)$ equals

$$\langle \bar{F}\Omega_L, (\mathbb{T}_L/\lambda_0)^t (1 - |\Omega_L\rangle\langle\Omega_L|) H\Omega_L \rangle,$$

which is bounded by $\|F\|_\infty \|H\|_\infty e^{-\Delta_L t}$. The bound is uniform in L and passes to weak limits, which exist by compactness. Proposition 23 completes the proof. $\square$

²⁵ Ψ_F is the class of F in the completion of the positive-time local functions for the form $\langle F, H \rangle = \omega(\vartheta(\bar{F})H)$, which is positive semidefinite by reflection positivity. Since $|\omega(\vartheta(\bar{F})F)| \leq \omega(|F|^2)$ by the Cauchy-Schwarz inequality and the invariance of ω under ϑ , functions that are close in $L^2(\omega)$ give close vectors.

A. Proof of Theorem F

Proof. The curvature range. By Theorem D the torus Wilson measures converge to a single measure μ , which is therefore translation and hypercubic invariant and reflection positive, reflection positivity of each torus measure [91] being preserved under weak limits of local expectations. By Theorem B, smooth local functions have exponentially decaying truncated correlations under μ : for F, H smooth and local, $|\text{Cov}_\mu(F, H)| \leq C_{F,H} e^{-c \text{dist}(\text{supp } F, \text{supp } H)}$ with $c > 0$ [100, Cor. 4.11]. Apply this to F supported at times in $[0, s]$ and to $\vartheta(\bar{F})$ and $\tau_t F$, whose supports are at distance at least $t - 1$. Smooth gauge-invariant local functions with support in a fixed set are dense in $L^2(\mu)$ on that set (average a smooth approximant over the compact gauge group of the support). Proposition 23 gives a gap $m \geq c$.

SU(2). In the range of Theorem E, every limit point of the periodic torus measures is a Gibbs measure for the infinite-volume specification [60, Ch. 4], so it equals the unique Gibbs measure μ , which is again reflection positive. Lemma 28 gives, for local functions F, H that are Lipschitz in each link variable, $|\text{Cov}_\mu(F, H)| \leq C_{F,H} e^{-\lambda \text{dist}(S_F, S_H)}$ for every $\lambda < -\log \alpha$, with $\alpha = 6(d-1)|\beta|$. Lipschitz gauge-invariant local functions are dense as before, and Proposition 23 gives a gap at least λ for each such λ , hence at least $-\log \alpha$. At $d = 4$, $\alpha = 18|\beta| = \frac{9}{2}\beta_W$. $\square$

B. Two structural facts

For four-dimensional U(1) at large β the uniformity in L fails, since the power-law decay of [57] and Proposition 23 give $\inf_L \Delta_L = 0$. Two facts separate SU(2) from U(1).

Proposition 25. *Call a map $U_e \mapsto z_e U_e$ with $z: E \rightarrow Z(G)$ and $\prod_{e \in \partial p} z_e^{\pm 1} = 1$ for every plaquette a centre transformation of the torus $(\mathbb{Z}/L\mathbb{Z})^4$. It preserves the Wilson weight, and modulo gauge transformations the centre transformations form $H^1((\mathbb{Z}/L\mathbb{Z})^4, Z(G)) \cong Z(G)^4$: the group $(\mathbb{Z}/2)^4$ for SU(2) and the torus $U(1)^4$ for U(1).*

Proof. Central factors commute, so U_p is multiplied by $\prod_{e \in \partial p} z_e^{\pm 1} = 1$. Closed $Z(G)$ -valued cochains modulo exact ones on the cubical complex of the torus form H^1 of the four-torus with coefficients in $Z(G)$, which is $\text{Hom}(\mathbb{Z}^4, Z(G)) = Z(G)^4$. $\square$

Proposition 26. *Let G be semisimple and F a gauge-invariant C^1 function of finitely many links of $\mathbb{Z}^d$. At every flat field the differential of F in the directions of $\mathfrak{g}$ vanishes.*

Proof. On $\mathbb{Z}^d$ every flat field is a gauge transform of $U \equiv 1$, which is fixed by the constant gauge transformations $U_e \mapsto h U_e h^{-1}$. The differential $dF_1 \in (\mathfrak{g}^E)^*$ is invariant under the diagonal adjoint action, so its restriction to each factor is an Ad-invariant linear form on $\mathfrak{g}$, and for semisimple $\mathfrak{g}$ the only such form is zero. $\square$

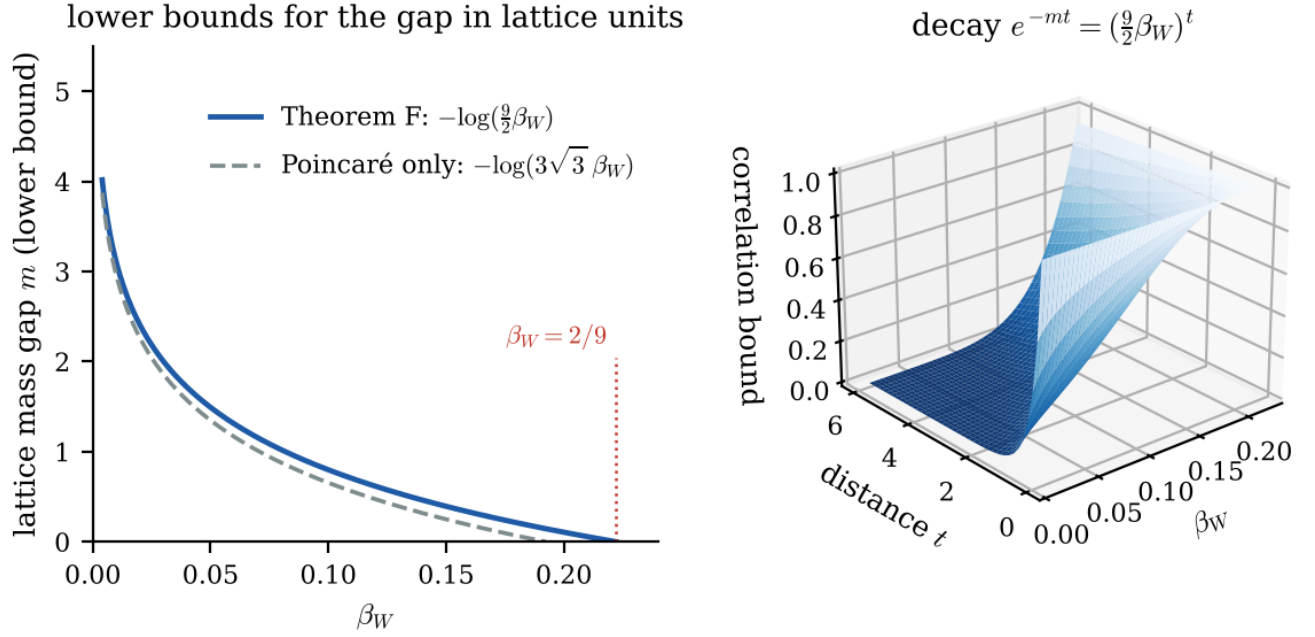


FIG. 9. Left: the lower bound $m \geq -\log(\frac{9}{2}\beta_W)$ of Theorem F for four-dimensional SU(2), in lattice units (blue), with the bound $-\log(3\sqrt{3}\beta_W)$ that the Poincaré constant of Lemma 4 alone gives (grey, dashed). The gap vanishes at the ceiling $\beta_W = 2/9$. Neither bound says anything about the continuum gap, which in lattice units must vanish with the spacing [16]. Right: the resulting bound $(\frac{9}{2}\beta_W)^t$ on truncated correlations at distance t .

For U(1) the observable $\text{Im}U_p$ has nonzero differential at $U \equiv 1$, and its correlations carry the massless photon, the Goldstone mode of a continuous symmetry whose finite-volume shadow is the group $U(1)^4$ of Proposition 25. For SU(2) that group is finite, and every local gauge-invariant observable is at least quadratic in the gauge field near flat fields. Neither fact is an estimate, and neither excludes a phase transition.

VIII. THE REMAINING GAPS

A. The problem as posed

The Yang–Mills problem of the Clay Mathematics Institute, in the formulation of Jaffe and Witten [67], asks for a quantum Yang–Mills theory on $\mathbb{R}^4$ for every compact simple gauge group, satisfying axioms at least as strong as those of Wightman or of Osterwalder and Schrader [89, 90], together with a mass gap: the spectrum of the Hamiltonian on the orthogonal complement of the vacuum is contained in $[\Delta, \infty)$ for some $\Delta > 0$.²⁶ On the lattice route there are four steps between the Wilson measure and that statement. They are (1) control

of the Wilson theory at every fixed coupling, (2) control of the renormalisation group flow at weak coupling, uniformly in the spacing, (3) a crossover between the two, and (4) the continuum limit of the gauge-invariant correlation functions with the axioms. The mass gap in physical units then amounts to a transfer-matrix gap of size ma at spacing a , with m independent of a ; our earlier paper [16] shows that this scaling cannot come from any estimate that is uniform in the spacing and stated in lattice units.

Step (2) is Balaban’s renormalisation group programme [4–12], which proves ultraviolet stability²⁷ of the partition function of four-dimensional lattice Yang–Mills theory on a fixed physical torus, uniformly in the spacing, as long as the running coupling is small; the exposition of Dimock [38–40] treats the scalar case in full detail. Magnen, Rivasseau and Sénéor constructed the four-dimensional theory with an infrared cutoff [79], and Federbush developed a phase-cell approach [50, 51]. None of these controls the theory at the scale where the effective coupling becomes of order one. Step (4) is open in four dimensions even for the partition func-

²⁶ The Osterwalder–Schrader axioms are conditions on the Euclidean Schwinger functions: regularity, Euclidean invariance, reflection positivity, symmetry and clustering. Their reconstruction theorem produces a Hilbert space, a vacuum vector, a positive Hamiltonian H and a unitary representation of the Poincaré group, and the mass gap is the statement $\text{spec}(H) \cap (0, \Delta) = \emptyset$. For a gauge theory the Schwinger functions are those of gauge-invariant observables, such as smeared Wilson loops or smeared

products of curvature components; which family of observables is enough is part of the problem. On the lattice, reflection positivity and the transfer matrix are available for every coupling [77, 91], which is why lattice gauge theory is the natural starting point [61, 97].

²⁷ Ultraviolet stability means bounds on the partition function that are uniform in the cutoff. For gravity, which is not renormalisable by power counting, the corresponding question has a different form: a brane-cluster proposal for it is developed in [21], and the scale dependence of the spacetime dimension under renormalisation is studied in [96]. Neither bears on the gauge theory treated here.

tion's Schwinger functions; in two dimensions the continuum Yang–Mills measure has been constructed and studied in detail [47, 74, 98], and in two and three dimensions the stochastic quantisation programme has constructed the Langevin dynamics and, in two dimensions, its invariant measure [29, 30, 36]. The abelian theory converges to a free field in three and four dimensions [46, 63].

This paper contributes to step (1). The four-dimensional Clay problem is not solved here.

B. The status of each gap

Table I lists the gaps between the results of Sec. III–VII and the continuum mass gap, in the order in which they have to be crossed.

The first eight rows are closed. The last four are the Clay problem.

C. The crossover

The results above, and the cluster expansions they extend [48, 71, 91, 97], control the Wilson theory below a fixed coupling. At the other end Bafaban's renormalisation group controls the block-spin flow (for the geometry of such flows in a gauge–Higgs setting see [17]), with bounds uniform in the lattice spacing, as long as the running coupling stays small. Between the two lies a window of order-one coupling crossed by a number of block-spin steps that does not depend on the spacing (Fig. 10).²⁸

Problem 27 (Crossover). Show that at a block scale $a_* = c_*/\Lambda$ the effective density of the four-dimensional $SU(N)$ Wilson theory is a Wilson weight with coupling in the regime of Theorem B (or of a convergent cluster expansion) times $\exp \sum_X \mathcal{E}_a(X)$, with $\mathcal{E}_a$ gauge invariant, local, and small in the supremum norm over all block configurations, uniformly in the spacing a and in the volume.

A solution of Problem 27, followed by the strong-coupling analysis on the block lattice and reflection positivity of the fine theory, would give a transfer-matrix gap of at least ma at spacing a with m independent of a , which is the lattice form of the mass gap [16]; it would also give a gap at every fixed coupling, by Proposition 24. Problem 27 is open.

²⁸ At one loop the running coupling of $SU(N)$ theory satisfies $g^{-2}(\ell+1) = g^{-2}(\ell) - 2b_0 \log M$ under a block-spin step of scale factor M , with $b_0 = 11N/(48\pi^2)$. Starting from $g^2(0) = g_0^2$ at the spacing a , the coupling reaches a fixed value g_*^2 of order one after $\ell_* = (g_0^{-2} - g_*^{-2})/(2b_0 \log M)$ steps, that is at the scale aM^{ℓ_*} , which tends to the fixed physical length $\Lambda^{-1} e^{-1/(2b_0 g_*^2)}$ as $a \rightarrow 0$ with g_0 tuned along the renormalisation group trajectory. The window between g_*^2 and the strong coupling regime is crossed in a number of further steps that depends on g_* and on the strong-coupling threshold but not on a . The difficulty is not the number of steps but that no estimate is known that controls a single step at order-one coupling.

D. Where the available methods stop

Each of the methods of this paper, and each of the routes proposed in the literature, stops at a point that can be named.

Curvature. Theorem C shows that a lower bound on $\text{Ric} - \text{Hess } S$ uniform over configurations cannot exceed $\frac{N}{2} - 4dN|\beta|$, because the extremal fields of Theorem A are allowed configurations. Any curvature argument that goes further must exploit that those fields are rare, which is an entropy statement, not a curvature statement.²⁹

Single-link couplings. Theorem E reaches $\beta_W = 2/9$, and Proposition 13 shows that no single-link Wasserstein–Dobrushin argument goes further, by the configurations whose staples cancel. For single-link methods the problem is therefore settled. Couplings of larger blocks, as in Theorem G, avoid this ceiling, but then the finite computation has to be done at the coupling in question.³⁰

Finite-box criteria with corrections. Feeding the renormalised action of the crossover into Theorem G requires, by Lemma 19, a smallness of the corrections $\mathcal{E}_a$ of order $\varepsilon/(|V||\partial V|)$. At order-one coupling the box that satisfies GDS has to be large, and the required smallness shrinks with it, so a solution of Problem 27 has to produce corrections that are small on the scale of the box, not only of the block.

Expansions in $1/N$ and the 't Hooft regime. The large- N analyses of lattice Yang–Mills theory [14, 64, 87] and the string-trajectory methods for Wilson loops [1, 26–28, 31, 32] control the theory for β below an N -independent threshold or for finite gauge groups and Higgs models [25, 33, 53]; they do not reach order-one coupling at fixed N .

Abelian and two-dimensional models. In four-dimensional $U(1)$ theory the weak-coupling phase is massless [57, 65], and in three dimensions Polyakov's mechanism gives confinement at all couplings [56, 58, 93]. In two dimensions the $SU(2)$ theory is massive at every coupling (Proposition 21). The closest analogue of the four-dimensional crossover, the two-dimensional $O(3)$ sigma model at weak coupling, is asymptotically free [94], and no proof of its mass gap at weak coupling is published in refereed form.³¹ Three-dimensional $SU(2)$ theory at large β is open as well.

²⁹ Localisation schemes and spectral conditions for Glauber dynamics [35, 49], and the Polchinski-flow proof of log-Sobolev inequalities [15], replace a uniform bound on the Hessian by a bound on the covariance of the measure along a flow. For lattice gauge theory such a bound would itself require control of correlations at the coupling in question, so these methods move the difficulty rather than remove it. The classical relation between log-Sobolev inequalities, spectral gaps and the decay of correlations is in [62, 66, 73, 81, 92, 103, 109].

³⁰ For spin systems with compact spin space Dobrushin's condition [41, 42], its Wasserstein form and its consequences for decay and for transport and Poincaré inequalities [45, 52, 72, 88, 108] are classical, and the Dobrushin–Shlosman finite-box conditions [43, 44] are the block version. Our gauge version, Definition 16, differs from the spin version only through the identification of gauge-equivalent boundary data and the interior gauge alignment of Lemma 15.

³¹ The two-dimensional $O(N)$ models with $N \geq 3$ share with four-dimensional Yang–Mills theory a dimensionless coupling, asymptotic freedom and a dynamically generated mass. For them as well the strong-coupling regime is controlled by convergent expansions, the weak-coupling flow by perturbation theory.

TABLE I. The gaps on the lattice route to the mass gap of four-dimensional $SU(N)$ Yang–Mills theory, and their status after this paper. “Closed” means proved here, by hand and without conditions.

Gap	Status	Where, and what remains
Best constant in the Hessian bound	Closed	Theorem A: $4d$, attained with both signs
Uniqueness on the whole curvature range	Closed	Theorem D: removes the factor 2 of [100], $ \beta < 1/(8d)$
Limit of curvature methods	Closed	Theorem C: no uniform Bakry–Émery bound beyond $1/(8d)$
$SU(2)$ beyond the curvature range	Closed	Theorem E: $\beta_W < 2/9$, by the sharp quarter, Theorem 7
Limit of single-link methods	Closed	Proposition 13: $\beta_W = 2/9$
Lattice mass gap at strong coupling	Closed	Theorem F: $m \geq -\log(\frac{9}{2}\beta_W)$ for $SU(2)$
Finite-box criterion, twists, area law	Closed	Theorem G, Proposition 18
Two dimensions at every coupling	Closed	Proposition 21
Intermediate fixed coupling, $2/9 \leq \beta_W \lesssim 2.2$	Open	No proof of uniqueness or of a gap; Monte Carlo evidence only [37, 75, 76, 86]
Crossover to Balaban’s regime	Open	Problem 27
Continuum limit with the axioms	Open	Balaban controls partition functions, not Schwinger functions [12]
Gap in physical units	Open	Equivalent on the lattice to a gap ma , which no spacing-uniform estimate gives [16]

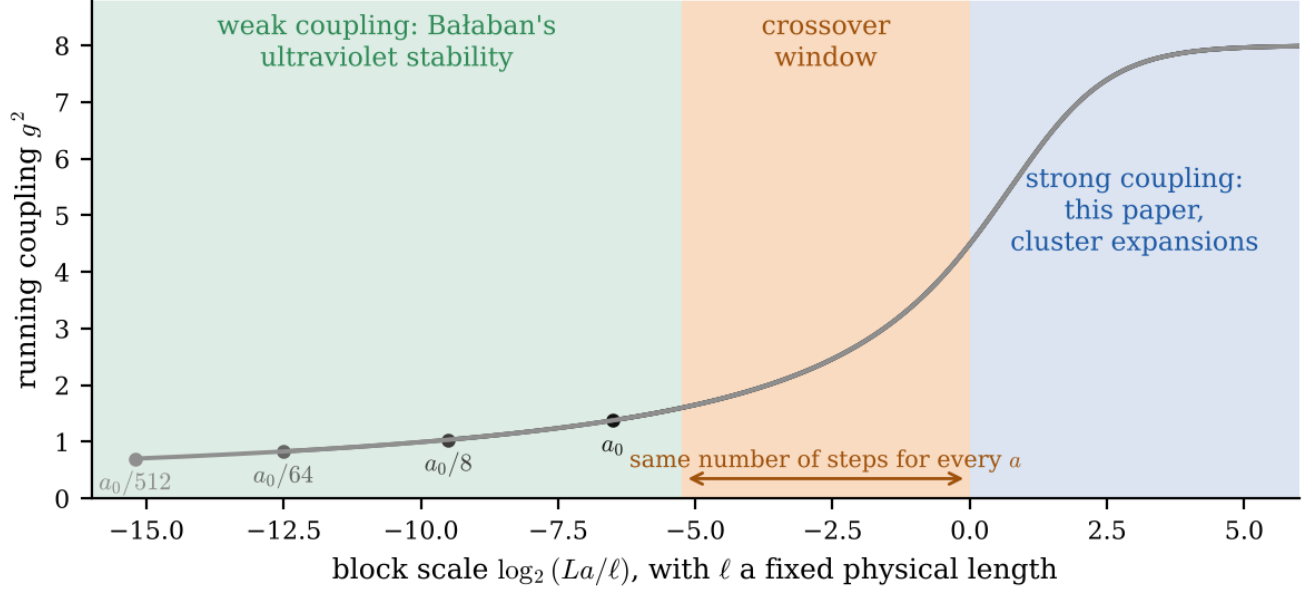


FIG. 10. The three regimes of the block-spin flow, with the running coupling on the vertical axis and the block scale on the horizontal axis. The strong-coupling end (left, blue) is controlled by the results of this paper and by cluster expansions, the weak-coupling end (right, green) by Balaban’s ultraviolet stability. No method controls the window between them (orange), which every trajectory crosses in a number of steps independent of the spacing. Trajectories started at the spacings $a_0, a_0/8, a_0/64, a_0/512$ lie on one curve after the first steps, by asymptotic freedom; the drawing is schematic, with the one-loop shape at weak coupling.

Order parameters. The centre symmetry and the order parameters of Fredenhagen and Marcu and of Bricmont and Fröhlich [23, 54, 78, 104] distinguish phases but give no estimate. Propositions 25 and 26 identify the structural differences

between $SU(2)$ and $U(1)$ that a proof would have to use; neither is an estimate, and neither excludes a phase transition at intermediate coupling.

DATA AND CODE AVAILABILITY

bative renormalisation, and the crossover between them is open. A proof for $O(3)$ would be the natural test of any method proposed for Problem 27.

All proofs in this paper are by hand and are complete without any computation. The repository `creelie/4Y-M` contains

the L^AT_EX source of this paper, independent checks in exact rational arithmetic and in floating point of the identities of Lemma 2, Appendix A, the Barta and Riccati computations of Appendix B, the extremal values of Theorem A, the spectral decomposition of Lemma 9, the Bessel sums of Lemma 10 and the coefficients of Lemma 32, and a Lean 4 certificate (with Mathlib) of the finite and algebraic steps: the coefficient formula and the positivity of Lemma 32, the binomial inequality used for it, the polynomial identities and sign of Proposition 12, the derivative of the third Riccati barrier, and the thresholds of Theorems E and F. The Lean files compile with no sorry and depend only on the standard axioms propext, Classical.choice and Quot.sound. The archive is on Zenodo, doi:10.5281/zenodo.23444338, where later versions are listed, and the PDF of this paper is at <https://github.com/creelie/4Y-M/blob/main/preprintYangMillsDobrushinCeiling/YangMillsDobrushinCeiling.pdf>. Deep Bhattacharjee carried out the computations and the Lean, Python and shell checks with AI assistance from Claude (Anthropic); both authors approved the final text.

ETHICAL DECLARATIONS

Funding. None. *Competing interests.* None.

Appendix A: The Hessian of a plaquette

1. The perturbed plaquette

Perturb every link variable along its geodesic, $Q_e(t) = e^{tX_e} Q_e$. The plaquette variable becomes

$$Q_p(t) = e^{tX_{e_1}} Q_{e_1} e^{tX_{e_2}} Q_{e_2} Q_{e_3}^* e^{-tX_{e_3}} Q_{e_4}^* e^{-tX_{e_4}}.$$

Move each exponential to the base point. Since $Q_{e_1} e^{tX} = e^{tQ_{e_1} X Q_{e_1}^*} Q_{e_1}$ and $Q_{e_3}^* e^{-tX_{e_3}} Q_{e_4}^* = Q_{e_3}^* Q_{e_4}^* e^{-tQ_{e_4} X_{e_3} Q_{e_4}^*}$, the transported vectors of (4) give

$$Q_p(t) = a(t) W b(t), \quad a(t) = e^{tY_1} e^{tY_2}, \quad b(t) = e^{-tY_3} e^{-tY_4},$$

with $W = Q_p$. Now $a(0) = b(0) = 1$, $a'(0) = Y_1 + Y_2$, $b'(0) = -(Y_3 + Y_4)$, $a''(0) = Y_1^2 + 2Y_1 Y_2 + Y_2^2$ and $b''(0) = Y_3^2 + 2Y_3 Y_4 + Y_4^2$, so

$$\left. \frac{d^2}{dt^2} Q_p(t) \right|_{t=0} = a''(0)W + 2a'(0)Wb'(0) + Wb''(0).$$

Taking Re Tr and using the cyclicity of the trace, $\text{Re Tr}(WY_i^2) = \text{Re Tr}(Y_i^2 W)$ and $\text{Re Tr}(WY_3 Y_4) = \text{Re Tr}(Y_3 Y_4 W)$, gives (5).

2. The corner terms

Fix the plaquette p with base point x in the plane $\mu < \nu$. Its four corners are the vertices x , $x + \hat{\mu}$, $x + \hat{\nu}$, $x + \hat{\mu} + \hat{\nu}$, and at each corner two of its links meet, one of direction μ and one of

direction ν . In $|D_y X|^2$ the end of a link at y enters with the sign $+$ if the link leaves y and with the sign $-$ if it arrives at y , an arriving link $(y - \hat{\lambda}, \lambda)$ being transported by $Q_{y-\hat{\lambda}, \lambda}^* \cdot Q_{y-\hat{\lambda}, \lambda}$. The cross term of the two ends at each corner is:

- at x , e_1 and e_4 both leave: $2\langle X_{e_1}, X_{e_4} \rangle = 2\langle Y_1, Y_4 \rangle$;
- at $x + \hat{\mu}$, e_1 arrives and e_2 leaves: $-2\langle X_{e_2}, Q_{e_1}^* X_{e_1} Q_{e_1} \rangle = -2\langle Y_2, Y_1 \rangle$, by invariance of the inner product under conjugation;
- at $x + \hat{\nu}$, e_4 arrives and e_3 leaves: $-2\langle X_{e_3}, Q_{e_4}^* X_{e_4} Q_{e_4} \rangle = -2\langle Y_3, Y_4 \rangle$;
- at $x + \hat{\mu} + \hat{\nu}$, e_2 and e_3 both arrive: $2\langle Q_{e_2}^* X_{e_2} Q_{e_2}, Q_{e_3}^* X_{e_3} Q_{e_3} \rangle$. Since $X_{e_2} = Q_{e_1}^* Y_2 Q_{e_1}$ and $X_{e_3} = Q_{e_4}^* Y_3 Q_{e_4}$, this equals $2\langle Y_2, W Y_3 W^* \rangle$.

The sum is the corner form (7). Conversely every cross term of $\sum_y |D_y X|^2$ between ends of two perpendicular directions $\mu \neq \nu$ at a vertex y is one of these: the two ends determine a quadrant of the plane $\mu\nu$ at y , hence a unique plaquette with a corner at y .

3. The diagonal blocks

Assume $W = D = \text{diag}(w_1, \dots, w_N)$, as in the proof of Lemma 2, and expand every trace in matrix entries:

$$\text{Re Tr}(Y_i Y_j W) = \text{Re} \sum_{a,b} (Y_i)_{ab} (Y_j)_{ba} w_a,$$

$$\text{Re Tr}(Y_i W Y_j) = \text{Re} \sum_{a,b} (Y_i)_{ab} w_b (Y_j)_{ba}.$$

The terms with $a = b$ involve only the diagonal entries $(Y_i)_{aa} = iy_i$, $y_i \in \mathbb{R}$. With $w_a = e^{i\varphi}$ they contribute $-y_i y_j \cos \varphi$ to both traces, so the diagonal part of f_p is

$$-\cos \varphi \left[\sum_i y_i^2 + 2y_1 y_2 + 2y_3 y_4 - 2(y_1 + y_2)(y_3 + y_4) \right] \\ = -\cos \varphi s^2,$$

with $s = y_1 + y_2 - y_3 - y_4$. The inner products $\langle Y_i, Y_j \rangle$ and $\langle Y_2, W Y_3 W^* \rangle$ contribute $y_i y_j$ and $y_2 y_3$ (the phases cancel), so the diagonal part of c_p is $2y_1 y_4 - 2y_1 y_2 + 2y_2 y_3 - 2y_3 y_4$. Expanding the squares,

$$(y_1 + y_3)^2 + (y_2 + y_4)^2 + s^2 \\ = 2 \sum_i y_i^2 - (2y_1 y_4 - 2y_1 y_2 + 2y_2 y_3 - 2y_3 y_4),$$

and therefore the diagonal block of $F = 2 \sum |Y_i|^2 - f_p - c_p$ is $(y_1 + y_3)^2 + (y_2 + y_4)^2 + (1 + \cos \varphi) s^2$.

4. The pair blocks

Fix $a \neq b$ and put $z_i = (Y_i)_{ab}$, so that $(Y_i)_{ba} = -\bar{z}_i$. The terms of the traces at the positions (a, b) and (b, a) give

$$\begin{aligned} \operatorname{Re} \operatorname{Tr}(Y_i Y_j W) &\rightarrow -\operatorname{Re}(z_i \bar{z}_j w_a + \bar{z}_i z_j w_b), \\ \operatorname{Re} \operatorname{Tr}(Y_i W Y_j) &\rightarrow -\operatorname{Re}(z_i \bar{z}_j w_b + \bar{z}_i z_j w_a), \\ \langle Y_i, Y_j \rangle &\rightarrow 2\operatorname{Re}(\bar{z}_i z_j), \\ \langle Y_2, W Y_3 W^* \rangle &\rightarrow 2\operatorname{Re}(\bar{z}_2 z_3 w_a \bar{w}_b), \end{aligned}$$

and $|Y_i|^2$ contains $2|z_i|^2$. Write $w_a = e^{i(\chi+\theta)}$, $w_b = e^{i(\chi-\theta)}$ and $m = \cos \chi$. Collecting terms, the pair block of F is $z^* M(m, \theta) z$ for the Hermitian matrix with diagonal entries $4 + 2m \cos \theta$ and upper off-diagonal entries

$$\begin{aligned} M_{12} = M_{34} &= 2 + 2me^{-i\theta}, & M_{13} = M_{24} &= -2me^{i\theta}, \\ M_{14} &= -2 - 2me^{i\theta}, & M_{23} &= -2e^{2i\theta} - 2me^{i\theta}. \end{aligned}$$

Only $m = \cos \chi$ appears, not $\sin \chi$: in each term the two phases enter through $w_a + w_b = 2e^{i\chi} \cos \theta$ or through the real parts of the combinations displayed.³² The matrix is affine in m , and the substitution $\theta \mapsto \theta + \pi$ changes $e^{i\theta}$ and $\cos \theta$ into their negatives while leaving $e^{2i\theta}$ unchanged, so $M(-1, \theta) = M(1, \theta + \pi)$. Since $m \in [-1, 1]$,

$$M(m, \theta) = \frac{1+m}{2} M(1, \theta) + \frac{1-m}{2} M(1, \theta + \pi),$$

and it suffices to show $M(1, \theta) \geq 0$ for every θ .

Let $\Delta = \operatorname{diag}(1, e^{i\theta/2}, e^{-i\theta}, e^{-i\theta/2})$ and $c = \cos(\theta/2)$. A direct multiplication gives the real symmetric matrix

$$\Delta^* M(1, \theta) \Delta = \begin{pmatrix} 2+4c^2 & 4c & -2 & -4c \\ 4c & 2+4c^2 & -4c & -2 \\ -2 & -4c & 2+4c^2 & 4c \\ -4c & -2 & 4c & 2+4c^2 \end{pmatrix},$$

using $4 + 2\cos \theta = 2 + 4c^2$ and $2 + 2e^{-i\theta} = 4ce^{-i\theta/2}$. Its quadratic form in $u \in \mathbb{C}^4$ is

$$\begin{aligned} 2c^2(|u_1 + u_3|^2 + |u_2 + u_4|^2) &+ (1+c)^2|u_1 + u_2 - u_3 - u_4|^2 \\ &+ (1-c)^2|u_1 - u_2 - u_3 + u_4|^2, \end{aligned}$$

as one checks by comparing coefficients: the diagonal coefficient is $2c^2 + (1+c)^2 + (1-c)^2 = 2 + 4c^2$, the coefficient of $\operatorname{Re}(\bar{u}_1 u_2)$ is $2[(1+c)^2 - (1-c)^2] = 8c$, that of $\operatorname{Re}(\bar{u}_1 u_3)$ is $2[2c^2 - (1+c)^2 - (1-c)^2] = -4$, that of $\operatorname{Re}(\bar{u}_1 u_4)$ is $2[-(1+c)^2 + (1-c)^2] = -8c$, and the others follow by the symmetry $(u_1, u_2, u_3, u_4) \mapsto (u_3, u_4, u_1, u_2)$ with a sign. Every pair block is therefore positive semidefinite.

³² One way to see that $\sin \chi$ drops out is to note that replacing W by $\bar{W}$ and every Y_i by $\bar{Y}_i$ leaves F unchanged, since all the terms are real parts of traces, and maps $\chi \mapsto -\chi$, $\theta \mapsto -\theta$ together with $z \mapsto \bar{z}$. The explicit collection of terms confirms that the coefficients depend on χ only through $\cos \chi$.

Appendix B: The single-link law of SU(2)

1. Barta's inequality

Let $\rho(\theta) = \sin^2 \theta e^{\kappa \cos \theta}$ on $(0, \pi)$, let V be a continuous potential, and let

$$T = -\rho^{-1} \partial_\theta \rho \partial_\theta + V.$$

If $\psi > 0$ is smooth on $(0, \pi)$ with $\rho \psi^2 = O(\sin^4 \theta)$ at the end points and $T\psi \geq \lambda_0 \psi$, then

$$\int (|h'|^2 + Vh^2) \rho d\theta \geq \lambda_0 \int h^2 \rho d\theta$$

for every smooth h vanishing to first order at 0 and π . Indeed, writing $h = \psi u$ and integrating by parts, the boundary terms $\rho \psi \psi' u^2$ vanish and

$$\int (|h'|^2 + Vh^2) \rho = \int \rho \psi^2 |u'|^2 + \int \frac{T\psi}{\psi} h^2 \rho \geq \lambda_0 \int h^2 \rho.$$

This is the inequality of Barta [13] in the form of a ground state transformation.

2. The computations of Lemma 4

The generator of the reversible diffusion of v_b on S^3 is $L = \Delta_{S^3} + \kappa \nabla_{X_1} \cdot \nabla$. In the polar angle θ from e_1 the round metric is $d\theta^2 + \sin^2 \theta g_{S^2}$, so $\Delta_{S^3} = \partial_\theta^2 + 2 \cot \theta \partial_\theta + \sin^{-2} \theta \Delta_{S^2}$, and $\nabla_{X_1} \cdot \nabla = -\sin \theta \partial_\theta$. On $h(\theta) Y_\ell$ with a spherical harmonic Y_ℓ of degree ℓ on S^2 the operator acts as $L_\ell = \partial_\theta^2 + b \partial_\theta - \ell(\ell+1) \sin^{-2} \theta$, $b = 2 \cot \theta - \kappa \sin \theta = \rho'/\rho$, so $-L_\ell = T$ with $V = \ell(\ell+1) \sin^{-2} \theta$.

Degree $\ell \geq 1$. Take $\psi = \sin \theta e^{a \kappa \cos \theta}$. Then $\psi'/\psi = \cot \theta - a \kappa \sin \theta$ and $\psi''/\psi = (\psi'/\psi)^2 + (\psi'/\psi)'$, and a short computation gives

$$\frac{-L_1 \psi}{\psi} = 3 + (1+5a) \kappa \cos \theta - a(1+a) \kappa^2 \sin^2 \theta.$$

For $a = -\frac{1}{5}$ this is $3 + \frac{4}{25} \kappa^2 \sin^2 \theta \geq 3$. Since $V = \ell(\ell+1) \sin^{-2} \theta \geq 2 \sin^{-2} \theta$ for $\ell \geq 1$, Barta's inequality gives $-L_\ell \geq 3$ on every sector with $\ell \geq 1$.

Degree $\ell = 0$. Let $g(\theta)$ be a zonal eigenfunction, $-L_0 g = \lambda g$, g not constant, so that $g'(0) = g'(\pi) = 0$. Differentiating $g'' + b g' = -\lambda g$ gives, for $h = g'$,

$$-(h'' + b h') - b' h = \lambda h, \quad -b' = \frac{2}{\sin^2 \theta} + \kappa \cos \theta,$$

so h is an eigenfunction of T with $V = 2 \sin^{-2} \theta + \kappa \cos \theta$, with eigenvalue λ , and $h \not\equiv 0$ vanishes at both ends. With the same ψ ,

$$\frac{T\psi}{\psi} = 3 + (2+5a) \kappa \cos \theta - a(1+a) \kappa^2 \sin^2 \theta,$$

which for $a = -\frac{2}{5}$ is $3 + \frac{6}{25} \kappa^2 \sin^2 \theta \geq 3$. Barta's inequality applied to h gives $\lambda \geq 3$. Together with the case $\ell \geq 1$ every nonzero eigenvalue of $-L$ is at least 3, which is Lemma 4; the spectrum is discrete because S^3 is compact and the drift is smooth.

3. Moments

The density of $t = x_1$ under v_b is proportional to $(1 - t^2)^{1/2} e^{\kappa t}$. For every smooth ϕ on $[-1, 1]$,

$$\frac{d}{dt} \left[(1 - t^2)^{3/2} e^{\kappa t} \right] = (1 - t^2)^{1/2} e^{\kappa t} (\kappa(1 - t^2) - 3t),$$

and integration by parts gives $\mathbb{E}[\phi'(t)(1 - t^2)] = \mathbb{E}[\phi(t)(3t - \kappa(1 - t^2))]$. With $\phi = 1$ this is $\kappa \mathbb{E}(1 - x_1^2) = 3 \mathbb{E}x_1$, the identity of Lemma 5(i). With $\phi = t^k$ it gives the recursion

$$m_{k+2} = m_k - \frac{k+3}{\kappa} m_{k+1} + \frac{k}{\kappa} m_{k-1}, \quad m_k = \mathbb{E}x_1^k,$$

from $m_0 = 1$ and $m_1 = A$; all polynomial moments of v_b are rational in κ and $A(\kappa)$.³³

4. The Riccati barriers

By Lemma 5(i), A solves $A' = \mathcal{F}(\kappa, A) := 1 - A^2 - 3A/\kappa$ on $(0, \infty)$, and $A(\kappa) = \frac{\kappa}{4} - \frac{\kappa^3}{96} + O(\kappa^5)$ from the series of I_1 and I_2 . Let $s = \sqrt{\kappa^2 + 4}$, $u = \sqrt{\kappa^2 + 9}$, $\phi_- = \kappa/(2 + s)$ and $\phi_+ = \kappa/(1 + u)$. Direct differentiation gives

$$\phi'_- - \mathcal{F}(\kappa, \phi_-) = -\frac{\kappa^2}{s(s+2)^2} \leq 0,$$

$$\phi'_+ - \mathcal{F}(\kappa, \phi_+) = \frac{(u-3)^2}{u(1+u)^2} \geq 0,$$

and $\phi_{\pm} = \frac{\kappa}{4} + O(\kappa^3)$.³⁴ The comparison is singular at 0 because of the term $3A/\kappa$, and we carry it out with an integrating factor. Let $D = A - \phi_-$. Then

$$\begin{aligned} D' &= \mathcal{F}(\kappa, A) - \mathcal{F}(\kappa, \phi_-) - (\phi'_- - \mathcal{F}(\kappa, \phi_-)) \\ &= -\left(A + \phi_- + \frac{3}{\kappa}\right)D + R, \end{aligned}$$

with $R = \kappa^2/(s(s+2)^2) \geq 0$. Put $\mu(\kappa) = \kappa^3 \exp \int_0^\kappa (A + \phi_-)$. Then $(\mu D)' = \mu R \geq 0$, and $\mu D = O(\kappa^6) \rightarrow 0$ as $\kappa \rightarrow 0$, so $\mu D \geq 0$ and $A \geq \phi_-$. The same argument with $D = \phi_+ - A$ and $R = (u-3)^2/(u(1+u)^2)$ gives $A \leq \phi_+$. This is Lemma 5(ii).

For (iv) let $s = \sqrt{\kappa^2 + 25/4}$ and $\phi_b = \kappa/(\frac{3}{2} + s)$. Since $s^2 - \frac{9}{4} = \kappa^2 + 4$, the same differentiation gives

$$\phi'_b - \mathcal{F}(\kappa, \phi_b) = -\frac{5(s - \frac{5}{2})}{2s(\frac{3}{2} + s)^2} \leq 0,$$

³³ For instance $m_2 = 1 - 3A/\kappa$, $m_3 = A - 3/\kappa + 12A/\kappa^2$ and $m_4 = m_2 - 5m_3/\kappa + 2A/\kappa$. The joint moments with x_2, x_3, x_4 follow from exchangeability: $\mathbb{E}[\phi(x_1)x_2^2] = \frac{1}{3}\mathbb{E}[\phi(x_1)(1 - x_1^2)]$.

³⁴ For the first identity write $\phi_- = (s-2)/\kappa$, which follows from $s^2 - 4 = \kappa^2$; then $\phi'_- = (\kappa^2/s - s + 2)/\kappa^2$ and $1 - \phi_-^2 - 3\phi_-/\kappa = (\kappa^2 - (s-2)^2 - 3(s-2))/\kappa^2$, and the difference simplifies with $s^2 = \kappa^2 + 4$. The second is the same computation with $\phi_+ = \kappa(u-1)/(u^2-1) = \kappa(u-1)/(\kappa^2+8)$ in place of ϕ_- and $u^2 = \kappa^2 + 9$ in place of $s^2 = \kappa^2 + 4$. After clearing the denominator $\kappa^2(1+u)^2$ the numerator becomes $\kappa^2(u-3)^2/u$, which is the stated residual.

so ϕ_b is a subsolution, and $\phi_b = \frac{\kappa}{4} - \frac{\kappa^3}{80} + O(\kappa^5)$.³⁵ Hence $A - \phi_b = \frac{\kappa^3}{480} + O(\kappa^5)$, and the integrating-factor argument above, with $D = A - \phi_b$, $R = \frac{5(s-5/2)}{2s(3/2+s)^2} \geq 0$ and $\mu = \kappa^3 \exp \int_0^\kappa (A + \phi_b)$, gives $\mu D = O(\kappa^6)$ at 0, $(\mu D)' = \mu R \geq 0$, and so $A \geq \phi_b$. Since $\frac{3}{2} + \sqrt{\kappa^2 + 25/4} < 2 + \sqrt{\kappa^2 + 4}$ for $\kappa > 0$, this improves the lower bound of (ii).

Appendix C: The weighted coupling

We prove Lemma 3. Let Q, Q' be two fields such that for every link Q'_e is not in the cut locus of Q_e , which is the situation in (9), and let $\gamma_e: [0, 1] \rightarrow G$ be the minimal geodesic from Q_e to Q'_e , of constant speed ρ_e . The product $\gamma(\tau) = (\gamma_e(\tau))_e$ is a geodesic of the product manifold, with velocity $V(\tau) = (\dot{\gamma}_e(\tau))_e$ and $|V_e| = \rho_e$.

Step 1: an integral identity. Let $V^{(e)}$ be the vector field along γ that equals $\dot{\gamma}_e$ at the link e and vanishes elsewhere. It is parallel along γ , because each $\dot{\gamma}_e$ is parallel along the geodesic γ_e and the Levi-Civita connection of a product is the product connection. Hence

$$\frac{d}{d\tau} \langle \nabla S(\gamma(\tau)), V^{(e)}(\tau) \rangle = \text{Hess} S(V(\tau), V^{(e)}(\tau)),$$

and integrating from 0 to 1, with $\langle \nabla S, V^{(e)} \rangle = \rho_e \langle \nabla S, \dot{\gamma}_e / \rho_e \rangle$,

$$\rho_e (\langle \nabla S, \dot{\gamma}_e \rangle(Q') - \langle \nabla S, \dot{\gamma}_e \rangle(Q)) = \int_0^1 \text{Hess} S(V, V^{(e)}) d\tau, \quad (\text{C1})$$

where $\hat{\gamma}_e = \dot{\gamma}_e / \rho_e$ is the unit tangent written $\dot{\gamma}_e$ in (9).

Step 2: symmetrisation. Multiply (C1) by w_e and sum over e . The sum converges absolutely, because $\rho_e \leq \pi\sqrt{2}$ is bounded, $\sum_e w_e < \infty$, and the Hessian is a sum of finite-range blocks. Write H_{ef} for the block of $\text{Hess} S$ between the links e and f . Then

$$\begin{aligned} \sum_e w_e \text{Hess} S(V, V^{(e)}) &= \sum_{e,f} w_f H_{ef}(V_e, V_f) \\ &= \sum_{e,f} \frac{w_e + w_f}{2} H_{ef}(V_e, V_f), \end{aligned}$$

the second equality by the symmetry $H_{ef}(V_e, V_f) = H_{fe}(V_f, V_e)$.

Put $Y_e = w_e^{1/2} V_e$, so that $|Y|^2 = \sum_e w_e \rho_e^2 < \infty$. With $r = w_e / w_f$,

$$\frac{w_e + w_f}{2\sqrt{w_e w_f}} = \frac{r^{1/2} + r^{-1/2}}{2} = 1 + \frac{1}{2}(r^{1/4} - r^{-1/4})^2,$$

and therefore

$$\sum_e w_e \text{Hess} S(V, V^{(e)}) = \text{Hess} S(Y, Y) + B,$$

³⁵ With $\phi_b = \kappa(s - \frac{3}{2})/(\kappa^2 + 4)$ one has $\phi'_b = (\frac{3}{2}s + \frac{25}{4})/(s(\frac{3}{2} + s)^2)$, using $ds/d\kappa = \kappa/s$; and $\mathcal{F}(\kappa, \phi_b) = 1 - \phi_b^2 - 3\phi_b/\kappa$. Clearing the denominator $s(\frac{3}{2} + s)^2$ and replacing κ^2 by $s^2 - \frac{25}{4}$ leaves $-\frac{5}{2}s + \frac{25}{4}$ in the numerator, which is the stated residual. The expansion follows from $s = \frac{5}{2} + \frac{\kappa^2}{5} + O(\kappa^4)$.

with $B = \sum_{e \neq f} H_{ef}(Y_e, Y_f) \cdot \frac{1}{2}(r^{1/4} - r^{-1/4})^2$, the diagonal terms $e = f$ having $r = 1$.

Step 3: the two estimates. Theorem A applies to the square-summable vector Y and gives $|\text{Hess} S(Y, Y)| = N|\beta| |\text{Hess} S_0(Y, Y)| \leq 4dN|\beta||Y|^2$. In B only links $e \neq f$ in a common plaquette contribute, and two distinct links lie in at most one common plaquette. Their mixed term in f_p is one of $2\text{Re Tr}(Y_i Y_j W)$, $-2\text{Re Tr}(Y_i W Y_j)$, and since $|\text{Tr}(XYW)| \leq |X||Y|$ for unitary W , the bilinear form satisfies $|H_{ef}(Y_e, Y_f)| \leq N|\beta||Y_e||Y_f|$ for each of the two orders. The end points of such links have graph distances to the origin that differ by at most one, so $r \in [a^{-1}, a]$ and $\frac{1}{2}(r^{1/4} - r^{-1/4})^2 \leq (\sqrt{a} - 1)^2 / (2\sqrt{a})$. Every link has $6(d-1)$ neighbours in common plaquettes, so

$$\begin{aligned} |B| &\leq N|\beta| \frac{(\sqrt{a}-1)^2}{2\sqrt{a}} \sum_{e \neq f} |Y_e||Y_f| \\ &\leq 3(d-1)N|\beta| \frac{(\sqrt{a}-1)^2}{\sqrt{a}} |Y|^2, \end{aligned}$$

by $|Y_e||Y_f| \leq \frac{1}{2}(|Y_e|^2 + |Y_f|^2)$. Integrating over τ , and using that $|Y|^2 = \sum_e w_e \rho_e^2$ does not depend on τ , gives

$$\sum_e w_e \rho_e (\langle \nabla S, \hat{\gamma}_e \rangle(Q') - \langle \nabla S, \hat{\gamma}_e \rangle(Q)) \leq (C_{\text{Ric}} - K_a) \sum_e w_e \rho_e^2,$$

which is Lemma 3. The constant of Shen, Zhu and Zhu [100, Sec. 5] arises if the weighted sum is estimated link by link with (1); the gain comes from estimating the whole quadratic form $\text{Hess} S(Y, Y)$ with Theorem A and paying only for the commutator of the weights.

Appendix D: A Kantorovich–Dobrushin comparison

Let $\Omega = (S^3)^E$ with the chordal metric δ on each factor, of diameter 2, and let $\gamma_e(\cdot | \omega)$ be the single-link conditional laws of the $\text{SU}(2)$ Wilson theory. Let $C_{ee'}$ be the coefficients of the proof of Theorem E, $\alpha < 1$ their maximal row and column sum, and $|e - e'|$ the Euclidean distance of link midpoints; links in a common plaquette have $|e - e'| \leq 1$. Fix $\lambda > 0$ with $\theta := \alpha e^\lambda < 1$.

Lemma 28. *Let $S \subset E$ be finite, possibly empty, and let μ_1, μ_2 be probability measures on Ω that are invariant under the resampling γ_e of every link $e \notin S$. Let $T \subset E \setminus S$ be finite and $w_b = e^{-\lambda \text{dist}(b, T)}$. There is a coupling π of μ_1, μ_2 such that $\rho(b) = \mathbb{E}_\pi \delta(\sigma_b, \sigma'_b)$ satisfies*

$$\sum_{b \notin S} w_b \rho(b) \leq \frac{2\theta}{1-\theta} \sum_{z \in S} w_z.$$

In particular $\sum_{b \in T} \rho(b) \leq \frac{2\theta}{1-\theta} |S| e^{-\lambda \text{dist}(S, T)}$, and if $S = \emptyset$ then $\mu_1 = \mu_2$.

Proof. A minimal coupling. The set of couplings of μ_1 and μ_2 is weakly compact, and $\Psi(\pi) = \sum_b w_b \mathbb{E}_\pi \delta(\sigma_b, \sigma'_b)$ is weakly continuous, since δ is continuous and bounded and $\sum_b w_b < \infty$. Let π^* be a minimiser and ρ its link distances.

Resampling one link. Fix $e \notin S$. By [105, Cor. 5.22] there is a measurable map assigning to each pair (ω, ω') an optimal coupling of $\gamma_e(\cdot | \omega)$ and $\gamma_e(\cdot | \omega')$ for δ . Sample $(\sigma, \sigma') \sim \pi^*$ and replace the pair (σ_e, σ'_e) by a sample of that coupling; since both μ_i are γ_e -invariant, the result is again a coupling of μ_1, μ_2 , and only the link e changes. Changing the finitely many links that enter γ_e one at a time and using the triangle inequality for W_1 ,

$$W_1(\gamma_e(\cdot | \sigma), \gamma_e(\cdot | \sigma')) \leq \sum_{e'} C_{ee'} \delta(\sigma_{e'}, \sigma'_{e'}).$$

Minimality of π^* therefore gives $\rho(e) \leq \sum_{e'} C_{ee'} \rho(e')$ for every $e \notin S$.

Summation. Multiply by w_e and sum over $e \notin S$. Where $C_{ee'} \neq 0$ the links are in a common plaquette, so $\text{dist}(e, T) \geq \text{dist}(e', T) - 1$ and $w_e \leq e^\lambda w_{e'}$. With the column sums bounded by α ,

$$\begin{aligned} \sum_{e \notin S} w_e \rho(e) &\leq e^\lambda \sum_{e'} w_{e'} \rho(e') \sum_e C_{ee'} \\ &\leq \theta \left[\sum_{e' \notin S} w_{e'} \rho(e') + \sum_{z \in S} w_z \rho(z) \right]. \end{aligned}$$

All sums are finite, and $\rho(z) \leq 2$ on S ; rearranging gives the first claim. On T the weights equal 1, and $w_z \leq e^{-\lambda \text{dist}(S, T)}$ for $z \in S$. If $S = \emptyset$ then $\rho \equiv 0$, so π^* is supported on the diagonal and $\mu_1 = \mu_2$. $\square$

Corollary 29. *In the range of Theorem E there is exactly one Gibbs measure μ , and for a local function F with support S_F that is L_F -Lipschitz in each link variable, and a bounded measurable H with finite support S_H disjoint from S_F ,*

$$|\text{Cov}_\mu(F, H)| \leq \|H\|_{\infty} L_F \frac{2\theta}{1-\theta} |S_H| e^{-\lambda \text{dist}(S_F, S_H)}.$$

Proof. Gibbs measures are invariant under every γ_e , so the case $S = \emptyset$ of Lemma 28 gives uniqueness, and Gibbs measures exist by compactness. Let $\mu^u = \mu(\cdot | \sigma_{S_H} = u)$ be a regular conditional probability [69, Ch. 3]; for μ -almost every u it is invariant under γ_e for every $e \notin S_H$, because the conditional law of σ_e given everything else is γ_e and does not depend on how σ_{S_H} was fixed. Writing $\text{Cov}_\mu(F, H) = \int H(u) [\mu^u(F) - \int \mu^{u'}(F) \mu(du')] \mu(du)$,

$$|\text{Cov}_\mu(F, H)| \leq \|H\|_{\infty} \text{ess sup}_{u, u'} |\mu^u(F) - \mu^{u'}(F)|.$$

Apply Lemma 28 to $\mu^u, \mu^{u'}$ with $S = S_H$ and $T = S_F$: $|\mu^u(F) - \mu^{u'}(F)| \leq L_F \sum_{b \in S_F} \rho(b)$, and the second claim of the lemma completes the proof. $\square$

The decay of Theorem E is Corollary 29; the rate λ can be any number below $-\log \alpha$.

Appendix E: Two-dimensional estimates

1. Total variation under a bounded change of density

Lemma 30. *Let P, P' be probability measures with $dP'/dP = e^\phi/\mathbb{E}_P e^\phi$, where ϕ has oscillation at most s . Then $\|P - P'\|_{TV} \leq \tanh(s/4)$, and the bound is attained.*

Proof. We may assume $0 \leq \phi \leq s$. For an event A with $P(A) = x$ put $p = \mathbb{E}_P[e^\phi; A] \in [x, e^s x]$ and $q = \mathbb{E}_P[e^\phi; A^c] \in [1 - x, e^s(1 - x)]$. Then $P'(A) = p/(p + q)$ increases in p and decreases in q , so

$$P'(A) - P(A) \leq \frac{e^s x}{e^s x + 1 - x} - x =: g(x).$$

The derivative $g'(x) = e^s/(1 + x(e^s - 1))^2 - 1$ vanishes at $1 + x(e^s - 1) = e^{s/2}$, that is $x = 1/(e^{s/2} + 1)$, where $g = x(e^{s/2} - 1) = \tanh(s/4)$. The value is attained for $\phi = s\mathbf{1}_A$ with this $P(A)$. $\square$

If P_i has density proportional to ρg_i with $g_i \in [1 - \varepsilon, 1 + \varepsilon]$, then $\log(g_1/g_2)$ has oscillation at most $2\log \frac{1+\varepsilon}{1-\varepsilon}$, and Lemma 30 gives $\|P_1 - P_2\|_{TV} \leq \varepsilon$. The weaker bound $2\varepsilon/(1 - \varepsilon^2)$ used in the proof of Proposition 21 follows directly: with $c_i = \int \rho g_i \in [(1 - \varepsilon)c, (1 + \varepsilon)c]$, $c = \int \rho$, one has $|g_1/c_1 - g_2/c_2| \leq \frac{1}{c} \left(\frac{1+\varepsilon}{1-\varepsilon} - \frac{1-\varepsilon}{1+\varepsilon} \right) = \frac{4\varepsilon}{c(1-\varepsilon^2)}$, and the total variation is half the L^1 norm.

2. The character expansion

On $SU(2)$ write U with eigenvalues $e^{\pm i\vartheta}$, so that $\text{Tr } U = 2\cos \vartheta$, the Haar measure of class functions is $\frac{2}{\pi} \sin^2 \vartheta d\vartheta$, and the character of spin j is $\chi_j = \sin((2j+1)\vartheta)/\sin \vartheta$. The coefficient of χ_j in $w(U) = e^{\beta \cos \vartheta}$ is

$$\begin{aligned} & \frac{2}{\pi} \int_0^\pi e^{\beta \cos \vartheta} \sin((2j+1)\vartheta) \sin \vartheta d\vartheta \\ &= I_{2j}(\beta) - I_{2j+2}(\beta) = \frac{2(2j+1)}{\beta} I_{2j+1}(\beta), \end{aligned}$$

by $2\sin A \sin B = \cos(A - B) - \cos(A + B)$, the integral $\frac{1}{\pi} \int_0^\pi e^{\beta \cos \vartheta} \cos(n\vartheta) d\vartheta = I_n(\beta)$ and the recurrence $I_{n-1} - I_{n+1} = \frac{2n}{x} I_n$ [106, Sec. 3.71]. So $c_0 = \int w = 2I_1(\beta)/\beta$ and $f = w/c_0 = \sum_j (2j+1) a_j \chi_j$ with $a_j = I_{2j+1}(\beta)/I_1(\beta) \in (0, 1)$ for $j \geq \frac{1}{2}$, since $I_n(\beta)$ decreases in n for $\beta > 0$. Schur orthogonality gives $\chi_j * \chi_k = \delta_{jk} \chi_j/(2j+1)$, hence $f^{*m} = \sum_j (2j+1) a_j^m \chi_j$. Since $|\chi_j| \leq 2j+1$ with equality at $U = 1$,

$$\begin{aligned} \varepsilon_m &= \sup_U |f^{*m}(U) - 1| = \sum_{j \geq 1/2} (2j+1)^2 a_j^m \\ &\leq a_{1/2}^{m-1} \sum_{j \geq 1/2} (2j+1)^2 a_j, \end{aligned}$$

which is finite because $I_n(\beta) \leq (\beta/2)^n e^\beta/n!$, and tends to zero geometrically in m .

3. Disk integration

Proof of Lemma 20. We use the lemma only for rectangles, and prove it for them. Order the plaquettes of the rectangle D row by row, from left to right, and let D_i be the union of the first i . Each D_i is a staircase, hence a topological disk, and p_{i+1} meets D_i along a path γ of one or two consecutive links (its left side, its lower side, or both, which meet at a corner). We show by induction that integrating $\prod_{p \in D_i} w(U_p)$ over the interior links of D_i gives $c_0^i f^{*i}(\text{hol}_{\partial D_i})$; f is a class function, so the base point of the holonomy does not matter. For $i = 1$ there is nothing to integrate. In the step, write $\partial D_i = \gamma \cdot \beta$ and $\partial p_{i+1} = \gamma^{-1} \cdot \eta$, with holonomies ΓB and $\Gamma^{-1} H$, where Γ is the product of the link variables of γ . The new interior links are those of γ , and the integrand depends on them only through Γ . If γ has two links, $\Gamma = U_{e_1} U_{e_2}$ is Haar distributed for each fixed U_{e_1} . In both cases the integral over the links of γ is

$$c_0^{i+1} \int f^{*i}(\Gamma B) f(\Gamma^{-1} H) d\Gamma = c_0^{i+1} f^{*(i+1)}(BH),$$

and BH is the holonomy of $\partial D_{i+1} = \beta \cdot \eta$. $\square$

Appendix F: The sharp quarter

This appendix proves the auxiliary statements used in Sec. V for Theorem 7. Throughout, $I_j = I_j(\kappa)$ with $\kappa > 0$, and $I_{-j} = I_j$ for integers j . We use the series

$$I_\nu(\kappa) = \sum_{i \geq 0} \frac{(\kappa/2)^{2i+\nu}}{i!(i+\nu)!},$$

the integral $I_j(\kappa) = \pi^{-1} \int_0^\pi e^{\kappa \cos \theta} \cos j\theta d\theta$, the Fourier series $e^{\kappa \cos \theta} = \sum_{j \in \mathbb{Z}} I_j e^{ij\theta}$ and the recurrences

$$I_{j-1} - I_{j+1} = \frac{2j}{\kappa} I_j, \quad I'_j = I_{j+1} + \frac{j}{\kappa} I_j, \quad (\text{F1})$$

all of which are in Watson's treatise [106], §§3.7, 3.71, 2.22. All series below converge absolutely, because $0 < I_j \leq (\kappa/2)^j e^\kappa/j!$.

1. Zonal harmonics

Under σ the coordinate x_1 has the density $\frac{2}{\pi}(1-t^2)^{1/2}$ on $[-1, 1]$, the area of the two-sphere $\{x_1 = t\}$ being proportional to $1-t^2$ and its normal speed to $(1-t^2)^{-1/2}$. The Chebyshev polynomials $U_n(\cos \theta) = \sin((n+1)\theta)/\sin \theta$ satisfy

$$\int U_n U_m d\sigma = \frac{2}{\pi} \int_0^\pi \sin((n+1)\theta) \sin((m+1)\theta) d\theta = \delta_{nm},$$

and $U_n = C_n^{(1)}$ is the Gegenbauer polynomial of index $\frac{4-2}{2} = 1$, so $U_n(\langle y, x \rangle)$ is the zonal harmonic of degree n on S^3 with pole y : $|x|^n U_n(\langle y, x \rangle/|x|)$ is a harmonic polynomial on $\mathbb{R}^4$ [102], Ch. IV, §2.

Lemma 31. $e^{\kappa t} = \frac{2}{\kappa} \sum_{n \geq 0} (n+1) I_{n+1}(\kappa) U_n(t)$ in $L^2(\frac{2}{\pi}(1-t^2)^{1/2} dt)$, and uniformly on $[-1, 1]$.

Proof. With $t = \cos \theta$ and $2 \sin \theta \sin((n+1)\theta) = \cos n\theta - \cos(n+2)\theta$, the n -th coefficient is

$$\begin{aligned} \frac{2}{\pi} \int_0^\pi e^{\kappa \cos \theta} \sin((n+1)\theta) \sin \theta d\theta &= I_n - I_{n+2} \\ &= \frac{2(n+1)}{\kappa} I_{n+1}, \end{aligned}$$

by (F1). Since $|U_n| \leq n+1$ and I_{n+1} decays factorially, the series converges uniformly. $\square$

2. Proof of Lemma 10

Let $F(\theta) = e^{\kappa \cos \theta} = \sum_j I_j e^{ij\theta}$. Then

$$F'(\theta) = -\kappa \sin \theta F(\theta) = \sum_j ij I_j e^{ij\theta},$$

and Parseval's formula gives

$$\begin{aligned} \sum_{j \in \mathbb{Z}} j^2 I_j^2 &= \frac{\kappa^2}{2\pi} \int_0^{2\pi} \sin^2 \theta e^{2\kappa \cos \theta} d\theta \\ &= \frac{\kappa^2}{2} (I_0(2\kappa) - I_2(2\kappa)) = \frac{\kappa}{2} I_1(2\kappa), \end{aligned}$$

the last step by (F1) at the argument 2κ . Halving gives the first identity. The coefficient of $e^{i\theta}$ in $F^2 = e^{2\kappa \cos \theta}$ is $I_1(2\kappa) = \sum_{j \in \mathbb{Z}} I_j I_{1-j}$, and the terms with $j \leq 0$ repeat those with $j \geq 1$; this is the third identity.

For the second, $jI_j = \frac{\kappa}{2}(I_{j-1} - I_{j+1})$ by (F1), so

$$\sum_{j \geq 1} j^3 I_j^2 = \frac{\kappa^2}{4} \sum_{j \geq 1} j(I_{j-1}^2 + I_{j+1}^2 - 2I_{j-1}I_{j+1}).$$

Shifting indices, $\sum_{j \geq 1} j(I_{j-1}^2 + I_{j+1}^2) = I_0^2 + \sum_{m \geq 1} 2mI_m^2$, and $\sum_{m \geq 1} 2mI_m^2 = \kappa \sum_{m \geq 1} (I_{m-1}I_m - I_mI_{m+1}) = \kappa I_0I_1$ by telescoping. For the mixed term write $j = \frac{1}{2}((j-1) + (j+1))$ and apply (F1) to $(j-1)I_{j-1}$ and to $(j+1)I_{j+1}$, which is legitimate at $j = 1$ because $I_{-1} = I_1$:

$$\begin{aligned} jI_{j-1}I_{j+1} &= \frac{\kappa}{4} [(I_{j-1}I_j - I_jI_{j+1}) \\ &\quad + (I_{j-2}I_{j+1} - I_{j-1}I_{j+2})]. \end{aligned}$$

Both brackets telescope, to I_0I_1 and to $I_{-1}I_2 = I_1I_2$, so $\sum_{j \geq 1} jI_{j-1}I_{j+1} = \frac{\kappa}{4}(I_0I_1 + I_1I_2)$. Altogether

$$\sum_{j \geq 1} j^3 I_j^2 = \frac{\kappa^2}{4} \left(I_0^2 + \frac{\kappa}{2} I_1(I_0 - I_2) \right) = \frac{\kappa^2}{4} (I_0^2 + I_1^2). \quad \square$$

3. The coefficients of $\mathcal{D}$

Lemma 32. Write $\mathcal{D}(\kappa) = \sum_{n \geq 0} \mathcal{D}_n \kappa^{2n}$ for the function of (18). Then $\mathcal{D}_0 = \mathcal{D}_1 = 0$, and for $m \geq 1$

$$\mathcal{D}_{m+1} = \frac{N_m}{4^m m! (m+1)! (m+1)(m+2)},$$

with $N_m = \binom{2m}{m} (4(2m+1) + (m-1)(m+1)(m+2)) - 4^m(m+1)(m+2)$. Moreover $N_m = 0$ for $m \leq 2$ and $N_m > 0$ for $m \geq 3$.

Proof. The coefficients. Multiplying the series of I_0 and I_1 and collecting powers,

$$\begin{aligned} I_0^2 &= \sum_{m \geq 0} \alpha_m \kappa^{2m}, \quad I_1^2 = \sum_{m \geq 0} \beta_m \kappa^{2m+2}, \\ \kappa I_1(2\kappa) &= \sum_{m \geq 0} \frac{\kappa^{2m+2}}{m! (m+1)!}, \end{aligned}$$

where

$$\begin{aligned} \alpha_m &= \frac{1}{4^m} \sum_{i=0}^m \frac{1}{i!^2 (m-i)!^2}, \\ \beta_m &= \frac{1}{4^{m+1}} \sum_{i=0}^m \frac{1}{i! (i+1)! (m-i)! (m-i+1)!}. \end{aligned}$$

By Vandermonde's identity, $\sum_i \binom{m}{i}^2 = \binom{2m}{m}$ and $\sum_i \binom{m}{i} \binom{m+2}{i+1} = \binom{2m+2}{m+1}$, so

$$\alpha_m = \frac{\binom{2m}{m}}{4^m m!^2}, \quad \beta_m = \frac{\binom{2m+2}{m+1}}{4^{m+1} m! (m+2)!}.$$

The coefficient of κ^{2m+2} in $\mathcal{D}$ is $\mathcal{D}_{m+1} = 8\beta_m + 2\beta_{m-1} - \alpha_m - 1/(m!(m+1)!)$, with $\beta_{-1} = 0$; for $m = 0$ it is $\frac{8}{4} - 1 - 1 = 0$, and $\mathcal{D}_0 = 0$ because every term of $\mathcal{D}$ is $O(\kappa^2)$. For $m \geq 1$ put $\Delta = 4^m m! (m+1)! (m+1)(m+2)$ and $b = \binom{2m}{m}$. With $\binom{2m+2}{m+1} = \frac{2(2m+1)}{m+1} b$,

$$\begin{aligned} 8\beta_m &= \frac{4(2m+1)b}{\Delta}, \quad \alpha_m = \frac{(m+1)^2(m+2)b}{\Delta}, \\ 2\beta_{m-1} &= \frac{2m(m+1)(m+2)b}{\Delta}, \\ \frac{1}{m!(m+1)!} &= \frac{4^m(m+1)(m+2)}{\Delta}, \end{aligned}$$

and $2m - (m+1) = m-1$ gives $\mathcal{D}_{m+1} = N_m/\Delta$.

The sign. Directly, $N_0 = N_1 = N_2 = 0$ ($2 - 2$, $24 - 24$, $192 - 192$), $N_3 = 80$, $N_4 = 1140$ and $N_5 = 10416$. For $m \geq 6$ we first show $16^m \leq 4m \binom{2m}{m}^2$ for $m \geq 1$ by induction: it is an equality at $m = 1$, and $(m+1) \binom{2m+2}{m+1} = 2(2m+1) \binom{2m}{m}$ gives

$$\begin{aligned} 4(m+1) \binom{2m+2}{m+1}^2 &= \frac{16(2m+1)^2}{m+1} \binom{2m}{m}^2 \\ &\geq \frac{16(2m+1)^2}{4m(m+1)} 16^m \geq 16^{m+1}. \end{aligned}$$

Since $(m-1)^2 \geq 4m$ for $m \geq 6$, it follows that $\binom{2m}{m}^2 (m-1)^2 \geq 16^m$, that is $\binom{2m}{m} (m-1) \geq 4^m$, and

$$\begin{aligned} N_m &= \left(\binom{2m}{m} (m-1) - 4^m \right) (m+1)(m+2) \\ &\quad + 4(2m+1) \binom{2m}{m} > 0. \end{aligned}$$

$\square$

4. Monotonicity of Bessel ratios

Lemma 33. For $n \geq 1$ and $\kappa > 0$, $I_{n+2}(\kappa)/I_{n+1}(\kappa) \leq I_2(\kappa)/I_1(\kappa)$.

Proof. We use a lemma of Biernacki and Krzyż [22]: if $f = \sum a_j x^j$ and $g = \sum b_j x^j$ have positive coefficients and a_j/b_j is nonincreasing in j , then f/g is nonincreasing on $x > 0$. Indeed

$$\begin{aligned} f'g - fg' &= \sum_{i,j} (i-j) a_i b_j x^{i+j-1} \\ &= \frac{1}{2} \sum_{i,j} (i-j) b_i b_j \left(\frac{a_i}{b_i} - \frac{a_j}{b_j} \right) x^{i+j-1} \leq 0. \end{aligned}$$

Take $x = \kappa^2$, $f = I_{n+1}(\kappa)/\kappa^{n+1}$ and $g = I_1(\kappa)/\kappa$. Then $a_j/b_j = 2^{-n}(j+1)!/(j+n+1)!$ decreases in j , so $I_{n+1}/(\kappa^n I_1)$ is nonincreasing in κ . Its logarithmic derivative is, by the second recurrence of (F1),

$$\begin{aligned} \frac{n+1}{\kappa} + \frac{I_{n+2}}{I_{n+1}} - \frac{n}{\kappa} - \frac{1}{\kappa} - \frac{I_2}{I_1} \\ = \frac{I_{n+2}}{I_{n+1}} - \frac{I_2}{I_1} \leq 0. \end{aligned}$$

□

-
- [1] A. Adhikari and S. Cao, Correlation decay for finite lattice gauge theories at weak coupling, *Ann. Probab.* **53** (2025), 140–174, doi:10.1214/24-AOP1702.
 - [2] D. Bakry and M. Émery, Diffusions hypercontractives, in *Séminaire de Probabilités XIX*, Lecture Notes in Math. 1123, Springer, Berlin, 1985, 177–206, doi:10.1007/BFb0075847.
 - [3] D. Bakry, I. Gentil and M. Ledoux, *Analysis and Geometry of Markov Diffusion Operators*, Grundlehren Math. Wiss. 348, Springer, Cham, 2014, doi:10.1007/978-3-319-00227-9.
 - [4] T. Bałaban, Propagators and renormalization transformations for lattice gauge theories. I, *Comm. Math. Phys.* **95** (1984), 17–40, doi:10.1007/BF01215753.
 - [5] T. Bałaban, Propagators and renormalization transformations for lattice gauge theories. II, *Comm. Math. Phys.* **96** (1984), 223–250, doi:10.1007/BF01240221.
 - [6] T. Bałaban, Averaging operations for lattice gauge theories, *Comm. Math. Phys.* **98** (1985), 17–51, doi:10.1007/BF01211042.
 - [7] T. Bałaban, Propagators for lattice gauge theories in a background field, *Comm. Math. Phys.* **99** (1985), 389–434, doi:10.1007/BF01240355.
 - [8] T. Bałaban, Renormalization group approach to lattice gauge field theories. I. Generation of effective actions in a small field approximation and a coupling constant renormalization in four dimensions, *Comm. Math. Phys.* **109** (1987), 249–301, doi:10.1007/BF01215223.
 - [9] T. Bałaban, Renormalization group approach to lattice gauge field theories. II. Cluster expansions, *Comm. Math. Phys.* **116** (1988), 1–22, doi:10.1007/BF01239022.
 - [10] T. Bałaban, Convergent renormalization expansions for lattice gauge theories, *Comm. Math. Phys.* **119** (1988), 243–285, doi:10.1007/BF01217741.
 - [11] T. Bałaban, Large field renormalization. I. The basic step of the **R** operation, *Comm. Math. Phys.* **122** (1989), 175–202, doi:10.1007/BF01257412.
 - [12] T. Bałaban, Large field renormalization. II. Localization, exponentiation, and bounds for the **R** operation, *Comm. Math. Phys.* **122** (1989), 355–392, doi:10.1007/BF01238433.
 - [13] J. Barta, Sur la vibration fondamentale d’une membrane, *C. R. Acad. Sci. Paris* **204** (1937), 472–473.
 - [14] R. Basu and S. Ganguly, $SO(N)$ lattice gauge theory, planar and beyond, *Comm. Pure Appl. Math.* **71** (2018), 2016–2064, doi:10.1002/cpa.21788.
 - [15] R. Bauerschmidt and T. Bodineau, A very simple proof of the LSI for high temperature spin systems, *J. Funct. Anal.* **276** (2019), 2582–2588, doi:10.1016/j.jfa.2019.01.007.
 - [16] D. Bhattacharjee and U. Bhattacharya, Spacing-uniform lattice inequalities cannot produce the Yang–Mills mass gap, *Zenodo*, release v1.0.0, 2026, doi:10.5281/zenodo.23444338.
 - [17] D. Bhattacharjee, P. Samal and S. Bhattacharya, Geometric structure and renormalization group flow in chiral Yang–Mills–Higgs theory, *Preprints* (2026), doi:10.20944/preprints202602.2022.v1.
 - [18] D. Bhattacharjee, S. N. Thakur, S. Bhattacharya and P. Samal, $SU(n)$ holonomy deviations in Calabi–Yau manifolds (CY1–CY4), *Preprints* (2026), doi:10.20944/preprints202602.0023.v1.
 - [19] D. Bhattacharjee, Homotopy groups of spheres, Hopf fibrations and Villarceau circles II, *Preprints* (2026), doi:10.20944/preprints202602.2038.v1.
 - [20] D. Bhattacharjee, Holonomic quantum computing, *SSRN* (2026), doi:10.2139/ssrn.6066428.
 - [21] D. Bhattacharjee, S. Singha Roy and P. Samal, Brane-cluster UV completion of quantum gravity, *Research Square* (2026), doi:10.21203/rs.3.rs-9053205/v1.
 - [22] M. Biernacki and J. Krzyż, On the monotonicity of certain functionals in the theory of analytic functions, *Ann. Univ. Mariae Curie-Skłodowska Sect. A* **9** (1955), 135–147.
 - [23] J. Bricmont and J. Fröhlich, An order parameter distinguishing between different phases of lattice gauge theories with matter fields, *Phys. Lett. B* **122** (1983), 73–77, doi:10.1016/0370-2693(83)91171-1.
 - [24] D. Brydges, J. Fröhlich and E. Seiler, On the construction of quantized gauge fields. I. General results, *Ann. Physics* **121** (1979), 227–284, doi:10.1016/0003-4916(79)90098-8.
 - [25] S. Cao, Wilson loop expectations in lattice gauge theories with finite gauge groups, *Comm. Math. Phys.* **380** (2020), 1439–1505, doi:10.1007/s00220-020-03912-z.
 - [26] S. Cao, R. Nissim and S. Sheffield, Expanded regimes of area law for lattice Yang–Mills theories, *arXiv:2505.16585*.
 - [27] S. Cao, R. Nissim and S. Sheffield, Dynamical approach to area law for lattice Yang–Mills, *arXiv:2509.04688*.
 - [28] S. Cao, M. Park and S. Sheffield, Random surfaces and lattice Yang–Mills, *Commun. Amer. Math. Soc.* **5** (2025), 774–896, doi:10.1090/cams/59.
 - [29] A. Chandra, I. Chevyrev, M. Hairer and H. Shen, Langevin dynamic for the 2D Yang–Mills measure, *Publ. Math. Inst. Hautes Études Sci.* **136** (2022), 1–147, doi:10.1007/s10240-022-00132-0.
 - [30] A. Chandra, I. Chevyrev, M. Hairer and H. Shen, Stochastic quantisation of Yang–Mills–Higgs in 3D, *Invent. Math.* **237** (2024), 541–696, doi:10.1007/s00222-024-01264-2.
 - [31] S. Chatterjee, The leading term of the Yang–Mills free energy, *J. Funct. Anal.* **271** (2016), 2944–3005, doi:10.1016/j.jfa.2016.04.032.
 - [32] S. Chatterjee, Yang–Mills for probabilists, in *Probability and Analysis in Interacting Physical Systems*, Springer Proc. Math. Stat. 283, Springer, Cham, 2019, 1–16, doi:10.1007/978-3-030-15338-0_1.

- [33] S. Chatterjee, Wilson loops in Ising lattice gauge theory, *Comm. Math. Phys.* **377** (2020), 307–340, doi:10.1007/s00220-020-03738-9.
- [34] S. Chatterjee, A probabilistic mechanism for quark confinement, *Comm. Math. Phys.* **385** (2021), 1007–1039, doi:10.1007/s00220-021-04086-y.
- [35] Y. Chen and R. Eldan, Localization schemes: a framework for proving mixing bounds for Markov chains, *Duke Math. J.* **174** (2025), 1431–1510, doi:10.1215/00127094-2024-0063.
- [36] I. Chevyrev and H. Shen, Invariant measure and universality of the 2D Yang–Mills Langevin dynamic, *Comm. Pure Appl. Math.* **79** (2026), 1973–2102, doi:10.1002/cpa.70043.
- [37] M. Creutz, Monte Carlo study of quantized SU(2) gauge theory, *Phys. Rev. D* **21** (1980), 2308–2315, doi:10.1103/PhysRevD.21.2308.
- [38] J. Dimock, The renormalization group according to Balaban. I. Small fields, *Rev. Math. Phys.* **25** (2013), 1330010, doi:10.1142/S0129055X13300100.
- [39] J. Dimock, The renormalization group according to Balaban. II. Large fields, *J. Math. Phys.* **54** (2013), doi:10.1063/1.4821275.
- [40] J. Dimock, The renormalization group according to Balaban III. Convergence, *Ann. Henri Poincaré* **15** (2014), 2133–2175, doi:10.1007/s00023-013-0303-3.
- [41] R. L. Dobrushin, The description of a random field by means of conditional probabilities and conditions of its regularity, *Theory Probab. Appl.* **13** (1968), 197–224, doi:10.1137/1113026.
- [42] R. L. Dobrushin, Prescribing a system of random variables by conditional distributions, *Theory Probab. Appl.* **15** (1970), 458–486, doi:10.1137/1115049.
- [43] R. L. Dobrushin and S. B. Shlosman, Constructive criterion for the uniqueness of Gibbs field, in *Statistical Physics and Dynamical Systems*, Birkhäuser, Boston, 1985, 347–370, doi:10.1007/978-1-4899-6653-7_20.
- [44] R. L. Dobrushin and S. B. Shlosman, Completely analytical interactions: constructive description, *J. Stat. Phys.* **46** (1987), 983–1014, doi:10.1007/BF01011153.
- [45] T. C. Dorlas and B. Savoie, On the Wasserstein distance and the Dobrushin uniqueness theorem, [arXiv:2305.19371](https://arxiv.org/abs/2305.19371).
- [46] B. K. Driver, Convergence of the $U(1)_4$ lattice gauge theory to its continuum limit, *Comm. Math. Phys.* **110** (1987), 479–501, doi:10.1007/BF01212424.
- [47] B. K. Driver, YM_2 : continuum expectations, lattice convergence, and lassos, *Comm. Math. Phys.* **123** (1989), 575–616, doi:10.1007/BF01218586.
- [48] J.-M. Drouffe and J.-B. Zuber, Strong coupling and mean field methods in lattice gauge theories, *Phys. Rep.* **102** (1983), 1–119, doi:10.1016/0370-1573(83)90034-0.
- [49] R. Eldan, F. Koehler and O. Zeitouni, A spectral condition for spectral gap: fast mixing in high-temperature Ising models, *Probab. Theory Related Fields* **182** (2022), 1035–1051, doi:10.1007/s00440-021-01085-x.
- [50] P. Federbush, A phase cell approach to Yang–Mills theory. I. Modes, lattice-continuum duality, *Comm. Math. Phys.* **107** (1986), 319–329, doi:10.1007/BF01209397.
- [51] P. Federbush, A phase cell approach to Yang–Mills theory. III. Local stability, modified renormalization group transformation, *Comm. Math. Phys.* **114** (1988), 317–343, doi:10.1007/BF01225039.
- [52] H. Föllmer, A covariance estimate for Gibbs measures, *J. Funct. Anal.* **46** (1982), 387–395, doi:10.1016/0022-1236(82)90053-2.
- [53] M. P. Forssström, J. Lenells and F. Viklund, Wilson loops in the abelian lattice Higgs model, *Probab. Math. Phys.* **4** (2023), 257–329, doi:10.2140/pmp.2023.4.257.
- [54] K. Fredenhagen and M. Marcu, Confinement criterion for QCD with dynamical quarks, *Phys. Rev. Lett.* **56** (1986), 223–224, doi:10.1103/PhysRevLett.56.223.
- [55] S. Friedli and Y. Velenik, *Statistical Mechanics of Lattice Systems: a Concrete Mathematical Introduction*, Cambridge University Press, Cambridge, 2017, doi:10.1017/9781316882603.
- [56] J. Fröhlich and T. Spencer, The Kosterlitz–Thouless transition in two-dimensional abelian spin systems and the Coulomb gas, *Comm. Math. Phys.* **81** (1981), 527–602, doi:10.1007/BF01208273.
- [57] J. Fröhlich and T. Spencer, Massless phases and symmetry restoration in abelian gauge theories and spin systems, *Comm. Math. Phys.* **83** (1982), 411–454, doi:10.1007/BF01213610.
- [58] C. Garban and A. Sepúlveda, Improved spin-wave estimate for Wilson loops in $U(1)$ lattice gauge theory, *Int. Math. Res. Not. IMRN* **2023** (2023), 18142–18198, doi:10.1093/imrn/rnac356.
- [59] C. Gatteringer and C. B. Lang, *Quantum Chromodynamics on the Lattice*, Lecture Notes in Phys. 788, Springer, Berlin, 2010, doi:10.1007/978-3-642-01850-3.
- [60] H.-O. Georgii, *Gibbs Measures and Phase Transitions*, 2nd ed., De Gruyter, Berlin, 2011, doi:10.1515/9783110250329.
- [61] J. Glimm and A. Jaffe, *Quantum Physics: A Functional Integral Point of View*, 2nd ed., Springer, New York, 1987, doi:10.1007/978-1-4612-4728-9.
- [62] L. Gross, Logarithmic Sobolev inequalities, *Amer. J. Math.* **97** (1975), 1061–1083, doi:10.2307/2373688.
- [63] L. Gross, Convergence of $U(1)_3$ lattice gauge theory to its continuum limit, *Comm. Math. Phys.* **92** (1983), 137–162, doi:10.1007/BF01210842.
- [64] D. J. Gross and E. Witten, Possible third-order phase transition in the large- N lattice gauge theory, *Phys. Rev. D* **21** (1980), 446–453, doi:10.1103/PhysRevD.21.446.
- [65] A. H. Guth, Existence proof of a nonconfining phase in four-dimensional $U(1)$ lattice gauge theory, *Phys. Rev. D* **21** (1980), 2291–2307, doi:10.1103/PhysRevD.21.2291.
- [66] R. Holley and D. Stroock, Logarithmic Sobolev inequalities and stochastic Ising models, *J. Stat. Phys.* **46** (1987), 1159–1194, doi:10.1007/BF01011161.
- [67] A. Jaffe and E. Witten, Quantum Yang–Mills theory, in *The Millennium Prize Problems*, Clay Mathematics Institute and American Mathematical Society, Cambridge, MA and Providence, RI, 2006, 129–152, <https://www.claymath.org/wp-content/uploads/2022/06/yangmills.pdf>.
- [68] R. Jentzsch, Über Integralgleichungen mit positivem Kern, *J. Reine Angew. Math.* **141** (1912), 235–244, doi:10.1515/crll.1912.141.235.
- [69] O. Kallenberg, *Foundations of Modern Probability*, 3rd ed., Springer, Cham, 2021, doi:10.1007/978-3-030-61871-1.
- [70] J. Kogut and L. Susskind, Hamiltonian formulation of Wilson’s lattice gauge theories, *Phys. Rev. D* **11** (1975), 395–408, doi:10.1103/PhysRevD.11.395.
- [71] R. Kotecký and D. Preiss, Cluster expansion for abstract polymer models, *Comm. Math. Phys.* **103** (1986), 491–498, doi:10.1007/BF01211762.
- [72] H. Künsch, Decay of correlations under Dobrushin’s uniqueness condition and its applications, *Comm. Math. Phys.* **84** (1982), 207–222, doi:10.1007/BF01208568.
- [73] M. Ledoux, *The Concentration of Measure Phenomenon*, Math. Surveys Monogr. 89, American Mathematical Society, Providence, RI, 2001, doi:10.1090/surv/089.
- [74] T. Lévy, Yang–Mills measure on compact surfaces, *Mem. Amer. Math. Soc.* **166** (2003), no. 790, doi:10.1090/memo/0790.
- [75] B. Lucini and M. Teper, $SU(N)$ gauge theories in four dimensions: exploring the approach to $N = \infty$, *J. High Energy*

- Phys. **2001**, no. 06, 050, doi:10.1088/1126-6708/2001/06/050.
- [76] B. Lucini, M. Teper and U. Wenger, Glueballs and k -strings in $SU(N)$ gauge theories: calculations with improved operators, J. High Energy Phys. **2004**, no. 06, 012, doi:10.1088/1126-6708/2004/06/012.
- [77] M. Lüscher, Construction of a selfadjoint, strictly positive transfer matrix for Euclidean lattice gauge theories, Comm. Math. Phys. **54** (1977), 283–292, doi:10.1007/BF01614090.
- [78] G. Mack and V. B. Petkova, Comparison of lattice gauge theories with gauge groups $\mathbb{Z}_2$ and $SU(2)$, Ann. Physics **123** (1979), 442–467, doi:10.1016/0003-4916(79)90346-4.
- [79] J. Magnen, V. Rivasseau and R. Sénéor, Construction of YM_4 with an infrared cutoff, Comm. Math. Phys. **155** (1993), 325–383, doi:10.1007/BF02097397.
- [80] K. V. Mardia and P. E. Jupp, *Directional Statistics*, Wiley, Chichester, 2000, doi:10.1002/9780470316979.
- [81] F. Martinelli and E. Olivieri, Approach to equilibrium of Glauber dynamics in the one phase region. I. The attractive case, Comm. Math. Phys. **161** (1994), 447–486, doi:10.1007/BF02101929.
- [82] E. J. McShane, Extension of range of functions, Bull. Amer. Math. Soc. **40** (1934), 837–842, doi:10.1090/S0002-9904-1934-05978-0.
- [83] A. A. Migdal, Recursion equations in gauge field theories, Sov. Phys. JETP **42** (1975), 413–418.
- [84] J. Milnor, Curvatures of left invariant metrics on Lie groups, Adv. Math. **21** (1976), 293–329, doi:10.1016/S0001-8708(76)80002-3.
- [85] I. Montvay and G. Münster, *Quantum Fields on a Lattice*, Cambridge University Press, Cambridge, 1994, doi:10.1017/CBO9780511470783.
- [86] C. J. Morningstar and M. Peardon, Glueball spectrum from an anisotropic lattice study, Phys. Rev. D **60** (1999), 034509, doi:10.1103/PhysRevD.60.034509.
- [87] R. Nissim, $U(N)$ lattice Yang–Mills in the 't Hooft regime, arXiv:2510.22788.
- [88] Y. Ollivier, Ricci curvature of Markov chains on metric spaces, J. Funct. Anal. **256** (2009), 810–864, doi:10.1016/j.jfa.2008.11.001.
- [89] K. Osterwalder and R. Schrader, Axioms for Euclidean Green's functions, Comm. Math. Phys. **31** (1973), 83–112, doi:10.1007/BF01645738.
- [90] K. Osterwalder and R. Schrader, Axioms for Euclidean Green's functions. II, Comm. Math. Phys. **42** (1975), 281–305, doi:10.1007/BF01608978.
- [91] K. Osterwalder and E. Seiler, Gauge field theories on a lattice, Ann. Physics **110** (1978), 440–471, doi:10.1016/0003-4916(78)90039-8.
- [92] F. Otto and C. Villani, Generalization of an inequality by Talagrand and links with the logarithmic Sobolev inequality, J. Funct. Anal. **173** (2000), 361–400, doi:10.1006/jfan.1999.3557.
- [93] A. M. Polyakov, Compact gauge fields and the infrared catastrophe, Phys. Lett. B **59** (1975), 82–84, doi:10.1016/0370-2693(75)90162-8.
- [94] A. M. Polyakov, Interaction of Goldstone particles in two dimensions. Applications to ferromagnets and massive Yang–Mills fields, Phys. Lett. B **59** (1975), 79–81, doi:10.1016/0370-2693(75)90161-6.
- [95] V. Rivasseau, *From Perturbative to Constructive Renormalization*, Princeton University Press, Princeton, NJ, 1991, doi:10.1515/9781400862085.
- [96] P. Samal, D. Bhattacharjee and R. Ghor, Renormalization of spacetime dimension, Preprints (2026), doi:10.20944/preprints202602.0531.v1.
- [97] E. Seiler, *Gauge Theories as a Problem of Constructive Quantum Field Theory and Statistical Mechanics*, Lecture Notes in Phys. 159, Springer, Berlin, 1982, doi:10.1007/3-540-11559-5.
- [98] A. Sengupta, Gauge theory on compact surfaces, Mem. Amer. Math. Soc. **126** (1997), no. 600, doi:10.1090/memo/0600.
- [99] H. Shen, R. Zhu and X. Zhu, Langevin dynamics of lattice Yang–Mills–Higgs and applications, Comm. Math. Phys. **407** (2026), doi:10.1007/s00220-025-05528-7.
- [100] H. Shen, R. Zhu and X. Zhu, A stochastic analysis approach to lattice Yang–Mills at strong coupling, Comm. Math. Phys. **400** (2023), 805–851, doi:10.1007/s00220-022-04609-1.
- [101] B. Simon, *The Statistical Mechanics of Lattice Gases, Volume I*, Princeton University Press, Princeton, NJ, 1993, doi:10.1515/9781400863433.
- [102] E. M. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton University Press, Princeton, NJ, 1971, doi:10.1515/9781400883899.
- [103] D. W. Stroock and B. Zegarlinski, The logarithmic Sobolev inequality for continuous spin systems on a lattice, J. Funct. Anal. **104** (1992), 299–326, doi:10.1016/0022-1236(92)90003-2.
- [104] G. 't Hooft, On the phase transition towards permanent quark confinement, Nucl. Phys. B **138** (1978), 1–25, doi:10.1016/0550-3213(78)90153-0.
- [105] C. Villani, *Optimal Transport: Old and New*, Grundlehren Math. Wiss. 338, Springer, Berlin, 2009, doi:10.1007/978-3-540-71050-9.
- [106] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, 2nd ed., Cambridge University Press, Cambridge, 1944.
- [107] K. G. Wilson, Confinement of quarks, Phys. Rev. D **10** (1974), 2445–2459, doi:10.1103/PhysRevD.10.2445.
- [108] L. Wu, Poincaré and transportation inequalities for Gibbs measures under the Dobrushin uniqueness condition, Ann. Probab. **34** (2006), 1960–1989, doi:10.1214/009117906000000368.
- [109] B. Zegarlinski, The strong decay to equilibrium for the stochastic dynamics of unbounded spin systems on a lattice, Comm. Math. Phys. **175** (1996), 401–432, doi:10.1007/BF02102414.