

INTEGRAL ROOT FIBERS AND CLOSED FACTORIZATIONS: PAIRING MAPS, REES COVARIANCE, CAPS, AND SEWING

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ABSTRACT. We construct rootwise and formal Rees boundary data from the marked modules and lattices of the companion boundary-module construction, keeping their distinct coefficient and completion domains. Using a Newton interpolation basis, we determine the Smith invariants and the integral image of the map induced by the Hopf pairing. Factorial/Rees covariance and compatible finite-stage representatives produce a strict rootwise/Rees module with its theta endpoint. At an admissible odd prime, no continuous homomorphism extends algebraic root specialization to the entire completed Rees source; further obstructions exclude unrestricted Newton padding and a state-natural lift of a contracted scalar identity. Established perturbative, Habiro, and stated-skein/Morita constructions give closed comparisons after the declared cap or sewing map, with their boundary state objects kept separate.

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1. INTRODUCTION

A marked boundary state can be read in a reduced root fiber and through a formal Rees coordinate, but these readings have different domains. We compute the integral image of the map induced by the Hopf pairing and construct a handlebody-covariant Rees bridge from common finite-stage representatives. The common representatives retain both readings without making the completed Rees source root-valued: at an admissible odd prime, no continuous homomorphism on that entire completion extends algebraic root specialization to the reduced coefficient field. The reduced root fiber also remains distinct from the disk reading in the companion paper on marked Heegaard boundary modules [35].

The motivation for comparing these readings comes in part from homological blocks: the fivebrane construction of Gukov–Putrov–Vafa [21, Section 6.2] and the WRT reconstruction proposed by Gukov–Pei–Putrov–Vafa [20, Conjecture 2.1] relate disk series to root-of-unity invariants. The torus-boundary series and gluing formulas of Gukov–Manolescu [19, Sections 5–7] make the boundary comparison particularly concrete. The construction here instead starts with integral Hopf pairings and reduced root fibers. Its Rees coordinate is formal, and its closed cyclotomic caps retain their stated domains; none of these outputs is identified with a disk homological block.

Smith invariants in Newton interpolation coordinates compute the integral pairing image in Theorems 4.2 and 4.4, using attributed Hopf, Newton, and integral-TQFT inputs. Section 4 specifies the core variable, nodes, and coefficient ring, and separates the polynomial basis from its rescaled lattice frames. Factorial denominator-clearing factors and compatible representatives supply the covariance and endpoint construction in the first theorem below; the second states the obstructions.

1.1. Local terminology introduced in this paper. The phrase *rootwise/Rees state* is local shorthand for a state carried by the common finite-stage representatives of the marked boundary-module construction, with its reduced root-of-unity fiber and Rees endpoint kept distinct. A *reduced root-of-unity fiber* is the quotient-first specialization at an admissible odd prime, after the root-level admissibility and negligible-class relations stated in the cited theorem have been imposed. Thus *root specialization* below never means evaluation of the whole Rees completion. Likewise, *finite-prime-support data* means the explicitly finite set of root fibers adjoined after closure; it is not a new field-standard structure.

The one-holed-torus diagram separates the disk reading from the rootwise path.

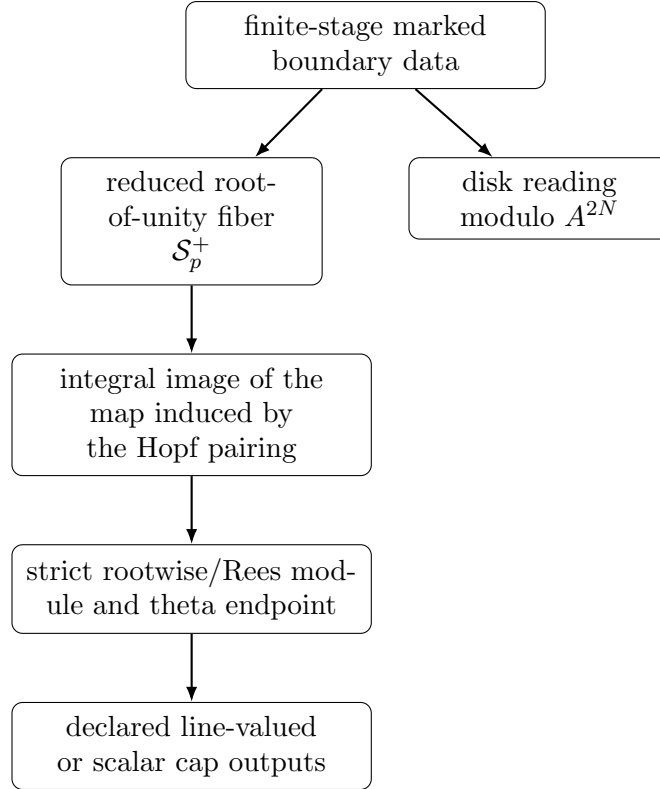


FIGURE 1. The rootwise path in the one-holed-torus running example. The disk reading remains a separate output of the finite-stage boundary construction; only the reduced root fiber enters the pairing-image and rootwise path. The last arrow is defined only on its stated cap domain.

1.2. Main results.

Theorem 1.1 (Rootwise/Rees construction and integral pairing image). *In the root, genus, coefficient, and completion domains stated by the cited results below, Smith invariants in Newton coordinates determine the saturated image of the map induced by the Hopf pairing; the factorially cleared Rees bridge is handlebody-covariant; and common finite-stage representatives form the strict rootwise/Rees module with its theta endpoint.*

Proof. The three clauses are respectively Theorems 4.4, 5.3–5.6, and 6.5–6.7. Each clause retains the domain and target of its cited statement. $\square$

The quantitative bridge uses the polynomial regulators and chart gauges defined in Section 5. Its rapid sector requires every growth rate. A fixed low-height core is chosen before the requested output rate. Ordinary polynomial vector approximation in Theorem 5.7 requires both transported output families to lie in the rapid or exact-intersection sector. Localized finite-core data instead admit Laurent interpolation for prescribed disk precision and finite scalar-root data, with the stated coefficient localization and without an all-root coherence assertion.

Theorem 1.2 (Principal root-specialization and state-lift obstructions). *For each odd prime $p \geq 5$, no continuous homomorphism from all of $\mathcal{R}_\lambda$ to the reduced p -th coefficient field extends $A \mapsto A_p$. Moreover, scalar Galois covariance does not admit the state-natural lift to the strict rootwise/Rees module excluded below. These obstructions do not preclude the stated line-valued or capped scalar outputs.*

Proof. The extension obstruction is Theorem 6.3; the state-lift obstruction is Theorem 7.12. Their positive-domain limitations are part of those statements and yield the final sentence. $\square$

1.3. Boundary inputs and closed comparisons. The finite-stage inputs come from the marked boundary-module construction [35], Theorem ???. The table records their hypotheses and domains.

Imported object	Source result and hypotheses
Filling quotient and generic spine module	Theorems ?? and ??; retain their genus, marking, and coefficient hypotheses.
Finite-stage joint quotient and split completion	Theorems ?? and ??; no root specialization of the whole completion.
Endpoint lattices and transport	Theorems ??, ??, and ??–?; retain endpoint and valuation tags.
Towers and cap domain	Theorems ??, ??, ??, and ??; no disk/cyclotomic identification and no uncapped-state reconstruction.

Established stated-skein/Morita sewing and the generic Habiro cap, followed by the cap-induced universal-construction quotient, give branch-specific closed comparisons on their cited domains. They neither identify uncapped boundary modules nor provide new generic sewing theorems.

2. PRIME-SELECTED THETA STATES AND REDUCED-ROOT PRODUCT CHAMBERS

Can reduced root states be separated while one marked spine is kept fixed? A local spine move changes the channel in which a state is read, and later complementary gluing tests that state. We first determine when each color becomes admissible and build its selected-root coordinates. Only afterward are these coordinates assembled into product chambers and tested along finite recoupling words. The selected-root variable records this construction; it does not replace the marked state or the gluing operation.

The first problem is elementary to state and decisive in practice: construct one generic graph lattice whose specialization knows exactly when a coloring becomes admissible at an odd root. Throughout this section Γ is a fixed finite trivalent spine and $\Gamma_2 = \Theta$ is the canonical closed genus-two theta spine. Lollipop coordinates are used only when the Gilmer–Masbaum form is diagonalized.

The coefficient symbols change with the mathematical operation, so the branch is recorded here before the formulas begin. The joint-completion sections use $q = A^2$. This section instead uses the selected-state variable u . The later rootwise section writes $q = -A = 1 - \lambda$, while the

perturbative/Habiro comparison uses $Q = A^4$. Each convention is local to its stated branch; none is a silent identification of these four parameters.

First identify the exact birth threshold for a coloring and use it to build the minimal cyclotomic selector for the bad admissible reduced roots. The fixed-spine specialization and its kernel then give the local separation statement. Only after that calculation are the rootwise product modules compared, their finite-word operator cores formed, and the separating sewing theorem proved. The final boundary statement records why these product data do not become a scalar dual, a global disk projection, or an unrestricted boundary theory.

Remark 2.1 (Local selected-state root variable). Only in this section, u denotes the polynomial indeterminate specialized at an odd prime to

$$u_p^{\text{sel}} := -q_p^{\text{GM}} = -(A_p^{\text{std}})^2 = -\zeta_p^{(p+1)/2} = e^{\pi i/p}.$$

Thus u_p^{sel} has order $2p$, $(u_p^{\text{sel}})^2 = \zeta_p$, and $\mathbb{Z}[u_p^{\text{sel}}] = \mathbb{Z}[\zeta_p]$. It is not the order- $4p$ bracket root A_p^{std} of the archival convention dictionary. Consequently its evaluation kernel is $(\Phi_{2p}(u))$. In the displayed complex embedding the roots converge to $+1$. In Habiro's adjacency topology, however, they have the limit point -1 , because their orders are $2p$ while -1 has order 2. The $u = 0$ factor below is a raw selected-coordinate disk factor, not an identification with $q_{\widehat{Z}} = A_{\text{bracket}}^4$.

2.1. The exact birth filtration. An even coloring $\alpha : E(\Gamma) \rightarrow 2\mathbb{Z}_{\geq 0}$ is classically admissible when the three colors at each vertex satisfy the triangle inequalities. Put

$$s_v(\alpha) = \sum_{e \ni v} \alpha(e), \quad \mu_\Gamma(\alpha) = \max \left\{ 2 + \max_e \alpha(e), 1 + \frac{1}{2} \max_v s_v(\alpha) \right\},$$

and $b_\Gamma(\alpha) = \mu_\Gamma(\alpha) + 1$.

Lemma 2.2 (Exact root-admissibility birth level). *For every odd prime $p \geq 5$,*

$$\alpha \in \text{Adm}_p(\Gamma) \iff \mu_\Gamma(\alpha) < p \iff b_\Gamma(\alpha) \leq p.$$

For each N , only finitely many colorings have birth level at most N .

Proof. The reduced $SO(3)$ admissibility inequalities are rewritten as one strict birth threshold. Evenness removes the apparent parity ambiguity, and a bounded threshold bounds every edge color, giving the finite-stage support property.

Reduced even $SO(3)$ admissibility is $\max_e \alpha(e) \leq p - 3$ and $s_v(\alpha) < 2p - 2$. Every vertex sum is even, so the second inequality is equivalent to $s_v(\alpha) \leq 2p - 4$. These are precisely the two strict inequalities defining $\mu_\Gamma(\alpha) < p$. A bound on the birth level bounds every edge color, and Γ has finitely many edges. $\square$

For a coloring α , let

$$B_\alpha = \{r \geq 5 : r \text{ prime and } r \leq \mu_\Gamma(\alpha)\}, \quad C_\alpha(u) = \prod_{r \in B_\alpha} \Phi_{2r}(u).$$

After clearing the Jones–Wenzl coupons, choose a polynomial graph vector t_α , and define

$$g_\alpha = u^{2b_\Gamma(\alpha)} C_\alpha(u) t_\alpha.$$

The monomial gives disk decay; C_α kills exactly the excluded reduced fibers.

Lemma 2.3 (Minimal bad-prime selector). *The selector C_α vanishes at every bad admissible reduced root, is a unit at every good admissible reduced root, has constant term one, and divides every monic integer polynomial with the same vanishing requirement.*

Proof. Vanishing and minimality follow from the minimal polynomials Φ_{2r} . For distinct primes p, r , $|\text{Res}(\Phi_{2p}, \Phi_{2r})| = 1$, because exponentiation by r permutes the nontrivial p -th roots. Thus every factor belonging to a bad prime $r \neq p$ evaluates to a unit in the p -th cyclotomic integer ring. Each factor has constant term one. $\square$

2.2. Specialization and state separation. Put $R_u = \mathbb{Z}[u]$. The monomial in the definition of g_α was chosen so that the cleared state has polynomial, rather than merely Laurent-polynomial, raw coordinates. Define the *finite-support selected lattice*

$$\mathcal{L}_\Gamma^{\text{sel}} = \bigoplus_{\alpha \in \text{Adm}_\infty(\Gamma)} R_u g_\alpha.$$

No completion is included in this notation. Let

$$\mathcal{G}_{\Gamma,p}^{\text{std}} = \bigoplus_{\alpha \in \text{Adm}_p(\Gamma)} \mathcal{O}_p^+[\Gamma_\alpha]_p$$

be the standard reduced graph lattice on the physically admissible colorings.

Theorem 2.4 (Prime-selective fixed-spine specialization). *For every odd prime $p \geq 5$, coefficient evaluation followed by the reduced-skein quotient gives a diagonal semilinear surjection*

$$\text{Sp}_p : \mathcal{L}_\Gamma^{\text{sel}} \longrightarrow \mathcal{G}_{\Gamma,p}^{\text{std}}.$$

It sends an inadmissible g_α to zero and every admissible g_α to a unit multiple of $[\Gamma_\alpha]_p$. Its exact kernel on this finite-support domain is

$$\ker \text{Sp}_p = \Phi_{2p}(u) \bigoplus_{\alpha \in \text{Adm}_p(\Gamma)} R_u g_\alpha \oplus \bigoplus_{\alpha \notin \text{Adm}_p(\Gamma)} R_u g_\alpha.$$

Proof. Specialize one selected generator at a time. The selector kills precisely the labels that are not yet root-admissible; for a root-admissible label, the clearing factors are units and the generator reaches one target basis vector. Independence of the target basis then turns this diagonal description into the displayed kernel.

Lemma 2.3 kills precisely the bad fibers. At a good fiber every Jones–Wenzl denominator-clearing factor, selector, and monomial is a unit, so the image is the standard basis vector up to a unit. The target basis is independent. Hence the coefficient kernel on a good summand is (Φ_{2p}) , the entire bad summand is killed, and the displayed kernel and surjectivity follow basiswise. $\square$

Put

$$P_n(u) = (u^2; u^2)_n = \prod_{k=1}^n (1 - u^{2k})$$

and define the distinct *restricted cyclotomic completion*

$$\widehat{\mathcal{L}}_\Gamma^{\text{scal,cyc}} = \varprojlim_n \mathcal{L}_\Gamma^{\text{sel}} / P_n \mathcal{L}_\Gamma^{\text{sel}}.$$

Equivalently, its elements are coefficient families $(x_\alpha)_\alpha$ with x_α in the even Habiro ring such that only finitely many coordinates remain nonzero modulo each P_n . Thus “restricted” describes the support of each element; the stage module itself need not have finite rank.

Theorem 2.5 (Restricted cyclotomic root faithfulness). *The continuously extended reduced-root map is injective:*

$$\prod_{\substack{p \geq 5 \\ p \text{ prime}}} \text{Sp}_p : \widehat{\mathcal{L}}_\Gamma^{\text{scal,cyc}} \hookrightarrow \prod_{\substack{p \geq 5 \\ p \text{ prime}}} \mathcal{G}_{\Gamma,p}^{\text{std}}.$$

Proof. Fixing one coloring reduces the global question to a one-variable Habiro element. It is root-admissible at every sufficiently large prime, so vanishing at those roots forces the coordinate to vanish by root faithfulness; restricted support then lets this coordinatewise conclusion reassemble.

If $x = (x_\alpha)_\alpha$ is in the kernel, fix α . This color is admissible at every sufficiently large admissible odd prime, and the diagonal multiplier in Theorem 2.4 is then a unit. Hence $x_\alpha(u_p^{\text{sel}}) = 0$ for every sufficiently large prime. These roots have Habiro limit point -1 , so Habiro limit-point injectivity forces $x_\alpha = 0$. This holds for every α . $\square$

For the third, again distinct, domain set $J_n = u^{2n}P_n$ and define

$$\widehat{\mathcal{L}}_\Gamma^{\text{scal},J} = \varprojlim_n \mathcal{L}_\Gamma^{\text{sel}} / J_n \mathcal{L}_\Gamma^{\text{sel}}, \quad \widehat{\mathcal{L}}_\Gamma^{\text{scal},u} = \varprojlim_n \mathcal{L}_\Gamma^{\text{sel}} / u^{2n} \mathcal{L}_\Gamma^{\text{sel}}.$$

Since $P_n(0) = 1$, the ideals (u^{2n}) and (P_n) are comaximal. Compatible levelwise Chinese remaindering gives the typed split

$$\widehat{\mathcal{L}}_\Gamma^{\text{scal},J} \cong \widehat{\mathcal{L}}_\Gamma^{\text{scal},u} \times \widehat{\mathcal{L}}_\Gamma^{\text{scal},\text{cyc}}.$$

On this joint domain the root maps mean projection to the second factor followed by the maps of Theorem 2.5; write π_u for projection to the first factor.

Theorem 2.6 (Joint CRT root kernel and disk separation). *Under the preceding split,*

$$\ker \left(\prod_{p \geq 5} \text{Sp}_p \right) = \widehat{\mathcal{L}}_\Gamma^{\text{scal},u} \times \{0\}.$$

Consequently

$$\left(\pi_u, \prod_{p \geq 5} \text{Sp}_p \right)$$

is injective on $\widehat{\mathcal{L}}_\Gamma^{\text{scal},J}$.

Proof. The root product ignores the first CRT factor and is injective on the second factor by Theorem 2.5. The kernel and the joint separation statement follow immediately. $\square$

Here π_u is only a formal raw-coordinate projection; it is not a topological cap, radial limit, or $\widehat{Z}$ comparison. For literal finite-rank height-truncation representatives one uses the separately named height tails

$$\mathcal{L}_{>n} = \bigoplus_{b_\Gamma(\alpha) > n} R_u g_\alpha, \quad \mathfrak{F}_n^{\text{cyc}} = P_n \mathcal{L}_\Gamma^{\text{sel}} + \mathcal{L}_{>n}, \quad \mathfrak{F}_n^J = J_n \mathcal{L}_\Gamma^{\text{sel}} + \mathcal{L}_{>n}.$$

Their quotients are finite rank because only finitely many colors are born by height n ; they must not be conflated with the restricted-support completion above.

Proposition 2.7 (Born-channel pole). *At the 2222 Whitehead wall, the transition from the old channel 0 to the newly born channel 4 has an exact simple Φ_{10}^{-1} pole in the selected coordinates. Hence the same polynomial selected lattice cannot carry the generic recoupling operator.*

Proof. The obstruction is born with one new channel at one admissible odd prime. The selected gauge is trivial on the old channel but contributes exactly the cyclotomic selector on the new channel, so conjugation exposes one uncanceled pole. This is a local failure of a common polynomial lattice, not a failure of the theta coordinates themselves.

At $p = 5$, channel 4 is inadmissible and its selector is $C_4 = \Phi_{10}$, while $C_0 = 1$. The unselected Costantino coefficient is regular and nonzero. Conjugating the transition by the two selected gauges therefore divides the $0 \rightarrow 4$ coefficient by exactly one Φ_{10} . Direct factorization shows that no second numerator factor cancels it. $\square$

This obstruction is not a defect of the theta chart. It is the reason that later chapters enlarge the coefficient category by finite-prime-support extensions and Rees summability instead of claiming a same-lattice generic mapping-class action.

2.3. Gilmer–Masbaum products. At each odd prime let $\mathcal{S}_{p,g}^+$ and $(\mathcal{S}_{p,g}^+)^\sharp$ denote the even Gilmer–Masbaum primal and sharp lattices with their perfect pairing. The ordinary graph lattice embeds in the primal lattice; equality is not assumed. Let $\mathcal{P}_{\text{phys}}$ be the primes $p \geq 5$, and put

$$\mathbf{O} = \prod_{p \in \mathcal{P}_{\text{phys}}} \mathcal{O}_p^+, \quad \mathbf{S}_g = \prod_{p \in \mathcal{P}_{\text{phys}}} \mathcal{S}_{p,g}^+, \quad \mathbf{S}_g^\sharp = \prod_{p \in \mathcal{P}_{\text{phys}}} (\mathcal{S}_{p,g}^+)^\sharp.$$

All three carry the product topology. The dual relevant here is, by definition, the *coordinatewise product-valued dual*

$$\text{Hom}_{\mathbf{O}}^{\text{coord}}(\mathbf{S}_g, \mathbf{O}) := \prod_{p \in \mathcal{P}_{\text{phys}}} \text{Hom}_{\mathcal{O}_p^+}^{\text{anti}}(\mathcal{S}_{p,g}^+, \mathcal{O}_p^+),$$

with “anti” omitted in the bilinear convention. No sum over primes is part of this definition. Compose root specialization with the standard graph inclusion to obtain

$$I_\Gamma : \widehat{\mathcal{L}}_\Gamma^{\text{scal}, \text{cyc}} \longrightarrow \mathbf{S}_g, \quad C_\Gamma = I_\Gamma(\widehat{\mathcal{L}}_\Gamma^{\text{scal}, \text{cyc}}).$$

Theorem 2.5 makes I_Γ injective.

Theorem 2.8 (Primal/dual product comparison). *For fixed finite genus, the product of the standard graph lattices embeds coordinatewise in $\mathbf{S}_g$. The product-valued GM pairing*

$$\mathbf{B}_g((x_p)_p, (y_p)_p) = (B_{p,g}(x_p, y_p))_p \in \mathbf{O}$$

induces an isomorphism

$$\mathbf{S}_g^\sharp \xrightarrow{\sim} \text{Hom}_{\mathbf{O}}^{\text{coord}}(\mathbf{S}_g, \mathbf{O}).$$

Restriction of $\mathbf{S}_g^\sharp$ to C_Γ is injective, and the finite-prime-support sharp cotests separate C_Γ . The quotient by $\mathfrak{F}_N^{\text{cyc}}$ gives exact values at every admissible odd prime $p \leq N$.

In the sourced lollipop notation, let Q' denote the standard graph lattice and let η be the diagonal graph-to-GM gauge. Its relative dual is $\eta^{-1}Q'$; nonzero saturation exponents rule out a same-chamber integral Habiro lift.

Proof. The comparison is primewise before it is global. Rootwise GM perfectness gives the coordinate-dual isomorphism, and finite-support cotests separate a common-image state one admissible odd prime at a time. The P_N -and-color cutoff then isolates the exact finite stages. Finally, the diagonal lollipop gauge identifies the relative dual and its valuation profile; persistent prime-element powers are what obstruct a same-chamber integral Habiro lift.

The finite-level inclusion and perfectness are the GM graph-basis and orthogonal-lollipop theorems [17, 18]. Taking the product of the primewise duality isomorphisms gives precisely the displayed coordinate-dual isomorphism. If a product sharp vector is nonzero at a prime p , the field-spanning standard graph lattice contains a vector pairing nontrivially with it, and surjectivity of Theorem 2.4 supplies a common preimage. Conversely, a nonzero common state survives at some prime by Theorem 2.5 and is detected there by a sharp cotest. The factor P_N vanishes at every admissible odd prime $p \leq N$, while $\mathcal{L}_{>N}$ consists of colors inadmissible at those primes; hence the finite-stage assertion is exact. The diagonal GM norm formula gives the relative dual and its valuation profile. A same-chamber Habiro lift would have to provide the inverse of persistent prime-element powers at infinitely many primes, contradicting Habiro integrality and limit-point injectivity. $\square$

Remark 2.9 (Product-valued does not mean scalar-dual). The preceding perfectness identifies a product sharp vector with a family of primewise, $\mathbf{O}$ -valued coordinate cotests. It does not identify $\mathbf{S}_g^\sharp$ with a scalar-valued continuous dual, does not sum an infinite prime family, and supplies no infinite-product Riesz theorem.

2.4. Product chambers and separating sewing. The common image is dense but proper in the full product: every finite set of coordinates can be interpolated, but arbitrary independent tails need not have one cyclotomic origin. Finite-support product cotests separate the full product rootwise; this separation alone imposes no cross-prime Habiro coherence.

Theorem 2.10 (Maximal product cores and finite-word detection). *Let $\mathfrak{G}$ be a groupoid generated by typed quotient-first recouplings and even extended mapping-class arrows. For every object i , let X_i be its reduced-root product module and let $M_i \subseteq X_i$ be a declared seed module. For each word $w : i \rightarrow j$, let $T_w : X_i \rightarrow X_j$ be the corresponding product isomorphism. Define*

$$P_i = \bigcap_{w:i \rightarrow j} T_w^{-1} M_j, \quad P_i^{(R)} = \bigcap_{\substack{w:i \rightarrow j \\ |w| \leq R}} T_w^{-1} M_j.$$

Then $\{P_i\}$ is the largest family with $P_i \subseteq M_i$ transported isomorphically by every arrow. Moreover

$$M_i = P_i^{(0)} \supseteq P_i^{(1)} \supseteq \cdots, \quad P_i = \bigcap_{R \geq 0} P_i^{(R)},$$

and every failure $x \notin P_i$ is detected at some finite word radius.

Proof. The maximal core is defined by finite-radius tests. The identity word gives the initial containment, closure under a further arrow and its inverse gives transport invariance, and any vector outside the intersection fails at one finite word and hence at one finite radius. Thus the intersection is an arrow-invariant core selected by the declared product chambers, not an arbitrary completion.

The identity word gives $P_i \subseteq M_i$. If $a : i \rightarrow k$ and $x \in P_i$, then for every word $w : k \rightarrow j$, $T_w T_a x = T_{w \circ a} x \in M_j$, so $T_a x \in P_k$. Applying the same argument to a^{-1} gives equality. If $Q_i \subseteq M_i$ is any arrow-invariant family, then $T_w Q_i \subseteq M_j$ for every word, hence $Q_i \subseteq P_i$, proving maximality. The finite-radius sets descend by definition. Finally, $x \notin P_i$ means that one finite word $w : i \rightarrow j$ has $T_w x \notin M_j$; taking $R \geq |w|$ detects the failure. $\square$

For this section write

$$h_p = 1 - (u_p^{\text{sel}})^2 = 1 - \zeta_p.$$

Theorem 2.11 (Separating GM sewing). *Every allowed separating cut has an exact coordinate-wise GM sewing map. In the sourced channel basis its inclusion into the saturated target has diagonal factors $h_p^{\phi(\lambda)}$. The saturation, cokernel, and adjoint are therefore explicit. Whenever $\phi(\lambda) > 0$, a same-chart integral Habiro sewing map is impossible.*

Proof. The sewing map is first constructed rootwise, where the GM modular-functor statement applies. Comparing the source and target lollipop exponents makes the inclusion diagonal, so saturation, cokernel, and adjoint follow from module algebra. The no-go then assumes a generic same-chart coefficient, uses its evaluations at all sufficiently large admissible reduced roots, and invokes limit-point injectivity to obtain an identity whose Taylor order contradicts integrality.

The GM modular-functor sewing theorem gives the rootwise map. Comparing the source and target lollipop normalization exponents diagonalizes the inclusion, so ordinary module algebra gives the saturation and cokernel. If a generic integral same-chart map existed, a channel with $\phi > 0$ would specialize to $h_p^{-\phi}$ at every large prime. If its coefficient were f , then $(1 - u^2)^{\phi} f$ would specialize to (1) at every sufficiently large admissible reduced root. Habiro limit-point injectivity would force the generic identity

$$(1 - u^2)^{\phi} f = 1.$$

But the left side has positive Taylor order at $(u=1)$, whereas the right side has order zero. This contradicts integrality of (f) . $\square$

Remark 2.12 (Boundary of the product theorem). The results above do not assert native generic endpoint equality, nonseparating or multicut sewing, one common generic mapping-class matrix, or a topological vacuum. They are the exact reduced-root product input to the completion theorems below.

3. RECOUPLING OVER FINITE-PRIME-SUPPORT EXTENSIONS, ORBIT POLARS, AND HOMOLOGY-TYPED TARGETS

The selected-root construction leaves two different defects: finitely many born-channel poles and the persistent $u + 1$ phenomenon. This section separates them, builds the targets defined by finite prime support, and tests which of them have topological meaning and which are only algebraic scaffolding.

The born-channel pole of Proposition 2.7 is supported at only finitely many admissible odd primes for each fixed matrix entry. The persistent obstruction is different: powers of $u + 1$ occur at every odd prime. This distinction leads to an intermediate ring controlled by finite prime support and, after checking the topological vacuum sector, to homology-typed targets.

Remark 3.1 (Continuation of the selected-state variable). Through the two topological-correction proofs, u and u_p continue the local convention of Remark 2.1; in particular $u_p = u_p^{\text{sel}}$, not the order- $4p$ bracket root. Although the chosen complex representatives approach $+1$,

$$u_p + 1 = 1 - \zeta_p^{(p+1)/2}$$

is a cyclotomic prime, and -1 is the relevant Habiro-adjacency limit point for the $u + 1$ -adic conductor. The later homology-typed construction uses its separately declared variables q and t_m .

3.1. The eventual localized Habiro ring. Let $\mathcal{O}_p$ be the root-of-unity cyclotomic integer ring and let $\mathcal{H}_u$ be the relevant Habiro completion. Let S_∞ be generated by cyclotomic factors that are units at every sufficiently large admissible prime. Define $\mathcal{E}_{\text{ev}}$ to consist of actual root sequences with an arbitrary finite prefix and one tail represented by $S_\infty^{-1}\mathcal{H}_u$.

Theorem 3.2 (Coefficient extension by finite-support root data). *There is an exact sequence*

$$0 \longrightarrow \bigoplus_p \mathcal{O}_p \longrightarrow \mathcal{E}_{\text{ev}} \longrightarrow S_\infty^{-1}\mathcal{H}_u \longrightarrow 0.$$

The factor Φ_{10}^{-1} is absorbed in this ring, while $\Phi_2^{-1} = (u + 1)^{-1}$ is not.

Proof. An eventual representative determines all but finitely many root values. The difference between two such representatives is therefore a finite-support sequence, while a localized tail can be evaluated away from its finite exceptional set. The two observations give the exact extension sequence and explain why Φ_2 is qualitatively different from Φ_{10} .

Two eventual representatives define the same root sequence precisely when they agree away from finitely many roots. Their difference is therefore a finite-support sequence, giving the kernel. Conversely, a localized Habiro tail has well-defined values outside the finite set on which its denominator vanishes, and arbitrary values may be supplied on that set. The factor Φ_{10} fails only at the fifth reduced root-of-unity fiber; $u + 1$, in contrast, is the prime element in every root-of-unity cyclotomic local ring and is never an eventual unit. $\square$

Activate the fixed-spine basis only at primes where its labels are root-admissible, and let $\mathcal{M}_{\Gamma}^{\text{ev}}$ be the resulting module over $\mathcal{E}_{\text{ev}}$.

Theorem 3.3 (Recoupling over the finite-prime-support extension). *Every finite row-and-column-finite quotient-first recoupling word and every adapted twist acts bicontinuously on the corresponding activated modules. The bigon, square, and pentagon relations hold exactly. The*

global Hopf or modular S matrix has an infinite row and does not preserve this chamber. The ring $\mathcal{E}_{\text{ev}}$ has no canonical disk projection.

Proof. Every finite recoupling word has only finitely many denominator failures, so its tail is allowed in the eventual localization and its exceptional fibers are stored separately. Finite row and column support makes composition meaningful; the Hopf row fails this support test, which is why it lies outside the chamber.

Each fixed recoupling coefficient has only finitely many exceptional cyclotomic denominators, so its tail belongs to $S_{\infty}^{-1}\mathcal{H}_u$; the exceptional fibers are recorded separately. Finite row and column support makes matrix multiplication well-defined, and the rootwise skein identities imply the relations. A polynomial identity holding at all sufficiently large roots holds generically, so no extra relation is introduced. The Hopf row has nonzero entries in unbounded labels and requires persistent inverse powers of the local prime element; it therefore leaves the module. Finally, two root sequences with the same eventual tail can carry independent finite prefixes, and no disk value is defined on that arbitrary data. $\square$

3.2. Orbit polars and the exact Hopf conductor. For a typed groupoid of rootwise operators $T_w : i \rightarrow j$, with declared seed modules $\mathcal{M}_j^0$, put

$$\mathcal{P}_i = \bigcap_{w:i \rightarrow j} T_w^{-1} \mathcal{M}_j^0.$$

Theorem 3.4 (Maximal operator-orbit polar). *The modules $\mathcal{P}_i$ form the maximal invariant carrier satisfying the seed conditions. Intersections over words of radius at most R form an exact descending chamber recursion, and every failure of membership has a finite word and finite cotest certificate.*

Proof. Any invariant carrier contained in every seed must lie in every inverse image, hence in $\mathcal{P}_i$. Conversely, functoriality sends the full intersection at i to the full intersection at the endpoint. If a vector is absent from the infinite intersection, it is absent from one finite subintersection; coordinatewise perfectness then produces a finite cotest detecting the failure. $\square$

For the lowered odd-prime Hopf matrix H_p , write $H_p^2 = \delta_p I$. Its admissible odd prime valuation is $v_{u_p+1}(\delta_p) = p - 3$. Put

$$\mathcal{F} = \bigoplus_p \mathcal{O}_p, \quad \delta = (\delta_p)_p.$$

For the stable even-color index $a \geq 0$, let $\epsilon_{a,p} = 1$ when $a < (p-1)/2$, and zero otherwise, and set

$$\mathcal{M}^{\text{ev}} = \prod_{a \geq 0} \epsilon_a \mathcal{E}_{\text{ev}} e_a, \quad \mathcal{M}^{\text{fd}} = \prod_{a \geq 0} \epsilon_a \mathcal{F} e_a.$$

Take two labeled endpoint copies V_p^+ and V_p^- . The two rootwise lowered coordinate transforms are the typed arrows

$$H_p^+ : V_p^+ \longrightarrow V_p^-, \quad H_p^- : V_p^- \longrightarrow V_p^+,$$

both represented by H_p , so that $H_p^- H_p^+ = \delta_p I_{V_p^+}$ and $H_p^+ H_p^- = \delta_p I_{V_p^-}$. Finally define

$$\mathcal{C}_H = \{x \in \mathcal{M}^{\text{fd}} : Hx \in \mathcal{M}^{\text{fd}}\},$$

and use superscripts $+$ and $-$ for its two endpoint copies.

Theorem 3.5 (Hopf conductor and maximal two-sided coordinate module). *In $\mathcal{E}_{\text{ev}}$,*

$$\{a \in \mathcal{E}_{\text{ev}} : \delta a \in \mathcal{E}_{\text{ev}}\} = \mathcal{F}.$$

The arrows H^+ and H^- exchange $\mathcal{C}_H^+$ and $\mathcal{C}_H^-$. More precisely, if ordinary $\mathcal{E}_{\text{ev}}$ -submodules $C^{\pm} \subset \mathcal{M}^{\text{ev}}$ satisfy

$$H^+ C^+ \subset C^-, \quad H^- C^- \subset C^+,$$

then $C^\pm \subset \mathcal{C}_H^\pm$. Thus these copies are the maximal ordinary two-sided lowered-Hopf coordinate carrier, and they contain neither the stable vacuum nor any state having a fixed-color coefficient with a nonzero localized-Habiro tail.

Proof. The lowered Hopf square multiplies a stable tail by a prime-dependent valuation that grows without bound. A fixed eventual Habiro tail cannot absorb that growth, so only finite-support coefficients survive the conductor test; applying the two opposite arrows then gives the maximal two-sided carrier.

Suppose a has a nonzero eventual Habiro tail. Its Taylor order at $u = -1$ is a fixed integer k . Multiplication by δ_p , whose order is $p - 3$, produces valuation $p - 3 + k$, so δa cannot be represented by a second nonzero eventual localized tail. It cannot have zero tail either, because both factors are nonzero at every sufficiently large prime. This contradiction shows that a has zero tail. The converse holds because finite-support sequences form an ideal and every δ_p is integral.

Now let $x \in C^+$. The two-arrow hypothesis gives $\delta x = H^- H^+ x \in C^+ \subset \mathcal{M}^{\text{ev}}$. Applying the conductor identity to every stable input coefficient shows that $x \in \mathcal{M}^{\text{fd}}$. Applying the same argument at the other endpoint to $H^+ x \in C^-$ shows $H^+ x \in \mathcal{M}^{\text{fd}}$, hence $x \in \mathcal{C}_H^+$; the minus endpoint is identical. Conversely, if $x \in \mathcal{C}_H^+$, then $H^+ x \in \mathcal{M}^{\text{fd}}$ and $H^- H^+ x = \delta x \in \mathcal{M}^{\text{fd}}$, because multiplication by the integral root sequence δ preserves finite prime support. Therefore the two copies are exchanged. A fixed-color nonzero localized-Habiro tail is not in $\mathcal{M}^{\text{fd}}$, which excludes the stated states and, in particular, the stable vacuum. $\square$

3.3. Two topological corrections. This subsection returns from the algebraic carrier to the topological tests that determine whether it can represent the intended states. The first test uses the natural vacuum and its prime-dependent valuation; the second uses lens-space values and their arithmetic variation. Together they exhibit two distinct failures of a single untyped target without changing the finite- prime-support extension constructed above. The conclusions are obstructions to that untyped construction, not replacements for the typed targets already defined.

Candidate statement 3.6 (Natural-vacuum membership no-go). *The natural topological GM cap does not belong to one untyped all-MCG $\mathcal{E}_{\text{ev}}$ orbit polar. In the frozen odd-prime $SO(3)$ normalization,*

$$I_p(S^1 \times S^2) = \mathcal{D}_p, \quad v_{u_p+1}(\mathcal{D}_p) = \frac{p-3}{2}.$$

Independently, for every positive nonsquare m and every admissible odd prime $p \nmid 2m$,

$$I_p(L(m, m-1)) \equiv \left(\frac{m}{p}\right) m^{-1} \pmod{u_p+1}.$$

Neither root sequence is the specialization of one eventual localized Habiro tail.

Proof. The first obstruction is detected by the valuation sequence of the natural vacuum, which cannot be eventually constant in the localized Habiro tail. The second is detected by the quadratic-character residues of lens-space values: one eventual Taylor constant would have to realize both signs on infinitely many admissible odd primes. A fixed central or anomaly unit rescales both classes and cannot remove this incompatibility.

Every nonzero element of $S_\infty^{-1} \mathcal{H}_u$ has eventually constant $(u_p + 1)$ -valuation, namely the order of its first nonzero Taylor term at $u = -1$. The sequence $(\mathcal{D}_p)_p$ is nonzero in every reduced root-of-unity prime and has valuation $(p-3)/2$, so it is not in $\mathcal{E}_{\text{ev}}$. The genus-one word producing $S^1 \times S^2$, paired with the topological complementary cap, is therefore already a finite word/cotest certificate excluding the handlebody vacuum from the all-MCG orbit polar.

For the rational-homology-sphere-only obstruction, the exact even-color Kirby sum gives

$$I_p(L(m, m-1)) = \left(\frac{m}{p}\right) \zeta_p^{(1-m)/2} \frac{\zeta_p^{-2/m} - 1}{\zeta_p^{-2} - 1},$$

where the fractional exponents denote residues modulo p . Reducing the cyclotomic-unit quotient modulo $u_p + 1$ gives the displayed residue. If this sequence had an eventual tail f , then, away from finitely many denominator primes, every such residue would be the reduction of the one rational Taylor constant $c = f(-1)$. For nonsquare m , the character $\left(\frac{m}{p}\right)$ equals $+1$ and -1 on two infinite sets of admissible primes. Hence c would equal both m^{-1} and $-m^{-1}$, a contradiction. A coherent central or anomaly unit only multiplies both residue classes by one fixed nonzero Taylor constant and does not remove either obstruction. $\square$

At an admissible odd prime put

$$q_p = u_p^2, \quad h_p = 1 - q_p = -(u_p - 1)(u_p + 1).$$

The factor $u_p - 1$ is a unit, so h_p -content is the same as $(u_p + 1)$ -content. For the core generator z , use the local notation

$$\{j\}_p = (-u_p)^j - (-u_p)^{-j}, \quad \{n\}_p! = \prod_{j=1}^n \{j\}_p, \quad \{0\}_p! = 1,$$

$$\lambda_{i,p} = -q_p^{i+1} - q_p^{-i-1}, \quad v_p = h_p^{-1}(z + 2).$$

Prime subscripts on $\{n\}!$, λ_i , and v are suppressed below. Let $\mathcal{M}_{\Theta}^{\text{ev}}$ and $\mathcal{M}_{\text{GM}}^{\text{ev}}$ denote the unrestricted activated $\mathcal{E}_{\text{ev}}$ coefficient products in the standard-theta and a fixed invariant-GM graph-like frame, respectively.

Candidate statement 3.7 (Unrestricted standard/GM frame no-go). *There is no bicontinuous $\mathcal{E}_{\text{ev}}$ -linear frame isomorphism*

$$\mathcal{M}_{\Theta}^{\text{ev}} \longrightarrow \mathcal{M}_{\text{GM}}^{\text{ev}}$$

whose root specialization is the canonical identity of the TQFT vector spaces and which maps the declared standard and GM frames to one another. Every fixed nonvacuum GM label has a forced negative h_p -content in standard graph coordinates. After the unique minimal labelwise normalization, the transition still has globally infinite rows, and a varying-maximal-color state maps outside $\mathcal{E}_{\text{ev}}$.

The one-holed-torus obstruction and tail test. The no-go is tested first in the one-holed torus. Before factorial division the frame change is triangular with unit diagonal, while the factorial contributes one unit of local content per color, forcing negative content in the invariant column. Minimal labelwise clearing removes that coefficient defect but not all-color summability: the closed-torus Chebyshev recurrence gives nonzero vacuum output for a state supported at varying maximal colors, and limit-point injectivity rules out the resulting tail. Freezing the other handle transfers the witness to closed genus two.

Proof. In the one-holed torus block of boundary color $2c$, write $L_{c,n}$ for the standard graph basis and

$$Q_n^{(c)} = \frac{1}{\{n\}!} L_{c,0} \prod_{i=c}^{c+n-1} (z - \lambda_i)$$

for the invariant GM basis. Before division by $\{n\}!$, the change matrix is triangular with unit diagonal, while $v_{h_p}(\{n\}!) = n$. Thus the $Q_n^{(c)}$ column has exact h_p -content $-n$, and the inverse

row has content n . Already $n = 1$ has a persistent h_p^{-1} coefficient, which cannot lie in $\mathcal{E}_{\text{ev}}$. In arbitrary genus the same calculation gives the exact GM weight

$$\kappa(\lambda) = \sum_i b_i + \left\lfloor \frac{\sum_i a_i - e}{2} \right\rfloor;$$

in closed genus two this is $a + b_1 + b_2$. Quotient-first integral theta-dumbbell transport preserves content, so the persistent label $(a, b_1, b_2) = (1, 0, 0)$ has theta-coordinate content -1 .

Multiplication by $h_p^{\kappa(\lambda)}$ is the unique minimal labelwise normalization up to units, but it does not repair all-color summability. In the closed torus let $B_j = v^j$, so $z = h_p v - 2$, and write the standard core as $L_n = S_n(z)$. The Chebyshev recurrence gives

$$[B_0]L_n = S_n(-2) = (-1)^n(n+1) \neq 0.$$

Choose an infinite set $\mathcal{P}_+$ of admissible odd primes with infinite complement. At $p \in \mathcal{P}_+$, place coefficient one in the maximal active color $n(p) = (p-3)/2$, and put every other fiber equal to zero. Each fixed input-color sequence has finite prime support and hence belongs to $\mathcal{E}_{\text{ev}}$. Its B_0 output is nonzero on $\mathcal{P}_+$ and zero on its infinite complement. Were this output an eventual localized-Habiro tail, the infinitely many zero evaluations would force that tail to vanish by limit-point injectivity, contradicting the infinitely many nonzero evaluations. Freezing the other handle in the vacuum sector embeds this witness in closed genus two, after which integral Whitehead transport returns it to theta coordinates. Hence the canonical frame change is not even defined on the unrestricted product; in particular it cannot be bicontinuous or invertible there. $\square$

The two preceding statements retain their candidate status and are not used below as proved inputs. Their proofs and independent exact Sage checks are recorded, but the corresponding theorem-matched Wolfram Language checks remain outstanding. Until those checks are completed, the statements remain candidates rather than proved inputs; no passing Wolfram Language check is asserted here.

The two candidate obstructions concern the unrestricted domains. The constructions below instead use homology-typed scalar targets and height/Rees summability for state changes.

3.4. Exponent-typed scalar targets. For a rational homology sphere put $m = \exp(\text{Tor } H_1)$, not generally $|\text{Tor } H_1|$. Define

$$R_m = \mathbb{Z}[1/m], \quad \mathbb{N}_m = \{n \geq 1 : (n, m) = 1\}, \quad t_m = q^{1/m},$$

and, for finitely supported $g : \mathbb{N}_m \rightarrow \mathbb{Z}_{\geq 0}$, put

$$I_g^{(m)} = \left(\prod_{n \in \mathbb{N}_m} \Phi_n(t_m)^{g(n)} \right), \quad \hat{\Lambda}_m = \varprojlim_g R_m[t_m]/I_g^{(m)}.$$

The admissible odd primes are $\mathcal{P}_m = \{p \geq 5 : p \nmid m\}$. If $u_{m,p}m \equiv 1 \pmod{p}$, set

$$t_{m,p} = \zeta_p^{u_{m,p}}, \quad \mathcal{O}_{p,m} = R_m[\zeta_p].$$

This is the canonical branch inside $\langle \zeta_p \rangle$.

Let

$$\Gamma_m = \langle [r] : r \text{ prime and } r \mid m \rangle \subset \mathbb{Q}_{>0}^\times / (\mathbb{Q}_{>0}^\times)^2$$

and define the coherent sheeted tail algebra

$$\mathcal{T}_m = \hat{\Lambda}_m[\Gamma_m].$$

For $F = \sum_{\delta \in \Gamma_m} f_\delta[\delta]$, its root specialization is

$$\text{Ev}_{m,p}(F) = \sum_{\delta \in \Gamma_m} \left(\frac{\delta}{p} \right) f_\delta(t_{m,p}).$$

Write

$$\mathbf{O}_m = \prod_{p \in \mathcal{P}_m} \mathcal{O}_{p,m}, \quad \mathbf{F}_m = \bigoplus_{p \in \mathcal{P}_m} \mathcal{O}_{p,m}, \quad \pi_m : \mathbf{O}_m \longrightarrow \mathbf{O}_m / \mathbf{F}_m,$$

and define the finite-prime-support pullback, rather than merely naming it, by

$$\boxed{\mathcal{E}_{\text{ev}}^{(m)} = \pi_m^{-1}(\pi_m \text{Ev}_m(\mathcal{T}_m))}.$$

Thus a root sequence lies in $\mathcal{E}_{\text{ev}}^{(m)}$ exactly when it agrees with one sheeted Le tail at all sufficiently large admissible primes.

Theorem 3.8 (Homology-typed joint target). *The evaluation*

$$\text{Ev}_m : \mathcal{T}_m \longrightarrow \mathbf{O}_m$$

is injective and remains injective after deleting finitely many admissible primes. Consequently there is a canonically split exact sequence

$$0 \longrightarrow \mathbf{F}_m \longrightarrow \mathcal{E}_{\text{ev}}^{(m)} \xrightarrow{\tau_m} \mathcal{T}_m \longrightarrow 0,$$

where τ_m extracts the unique coherent tail and the section is global evaluation.

The common disk-cyclotomic completion is the explicitly split product

$$\mathcal{J}_m \cong R_m[[t_m]][\Gamma_m] \times \mathcal{T}_m.$$

Every finite disk jet and finite admissible reduced-root battery has a common polynomial representative. No equality of the two factors is asserted.

If $m \mid \ell$, $k = \ell/m$, then

$$j_{m,\ell} : \mathcal{T}_m \hookrightarrow \mathcal{T}_\ell, \quad t_m \mapsto t_\ell^k, \quad [\delta] \mapsto [\delta],$$

is injective and satisfies $\text{Ev}_{\ell,p}(j_{m,\ell}F) = \text{Ev}_{m,p}(F)$ for $p \nmid \ell$. Orientation reversal is

$$\iota_m(t_m) = t_m^{-1}, \quad \iota_m([\delta]) = [\delta].$$

For $\ell = \text{lcm}(m, n)$, connected-sum multiplication is

$$F \boxtimes G = j_{m,\ell}(F)j_{n,\ell}(G) \in \mathcal{T}_\ell.$$

Restriction to primes not dividing ℓ , together with coefficient extension, induces the corresponding finite-prime-support map; unlike the coherent tail map, it need not be injective because it may discard finitely many exceptional fibers.

Proof. The proof separates three kinds of finiteness that the theorem packages together. Character-wise prime specializations detect the coherent sheeted tail, so cofinal injectivity makes that tail unique and leaves only a finite-support ambiguity. Finite-stage CRT then separates disk jets from root-of-unity cyclotomic tests before the inverse limit; it supplies common polynomial approximants without identifying the two factors. Finally, exponent-divisibility substitution transports the marked sheet and its evaluations, while least-common-multiple extension defines connected-sum multiplication and prime restriction accounts for the possible finite-support kernel.

The Taylor map of $\widehat{\Lambda}_m$ at $t_m = 1$ is injective. If an element vanishes at infinitely many prime-order branches, cyclotomic limit-point propagation makes every Taylor coefficient divisible by infinitely many rational primes, hence zero. Every character of the finite elementary 2-group Γ_m occurs as $\delta \mapsto \left(\frac{\delta}{p}\right)$ on infinitely many admissible primes. Restricting a vanishing sheeted tail to each such prime set and inverting the character table proves cofinal injectivity. The pullback definition then gives existence of a coherent tail, cofinal injectivity gives uniqueness, and its kernel is exactly $\mathbf{F}_m$; global evaluation supplies the splitting.

For the joint completion use the ideals

$$K_{N,g}^{(m)} = \left(t_m^{mN} \prod_{n \in \mathbb{N}_m} \Phi_n(t_m)^{g(n)} \right).$$

Since every cyclotomic factor has unit constant term, the disk factor and the cyclotomic factor are comaximal. Finite-stage CRT splits the quotient, and the inverse limit is

$$\varprojlim_{N,g} R_m[t_m][\Gamma_m]/K_{N,g}^{(m)} \cong R_m[[t_m]][\Gamma_m] \times \mathcal{T}_m.$$

The same CRT, with finitely many prime-order cyclotomic factors, proves the common-polynomial assertion.

For $m \mid \ell$, substitution by $t_m = t_\ell^k$ respects the cyclotomic filtrations and has Taylor linear coefficient k , a unit in R_ℓ ; this proves continuity and injectivity of $j_{m,\ell}$. The canonical branch identity $t_{m,p} = t_{\ell,p}^k$ proves evaluation compatibility. The orientation and least-common-multiple formulas then follow directly. $\square$

Remark 3.9 (Algebraic scaffold only). The objects above are coefficient rings and exact transition maps. They do not select a topological state, prove that a general rational-homology-sphere vacuum lies in the pullback, identify the disk and cyclotomic coordinates, or replace the full linking pairing by its exponent.

Homology type decorates a marked word; it is not a quotient of the word groupoid. Distinct Torelli words with the same C -block may still have different invariants.

3.5. Gauss lines and finite linking pairings. This is the first point where the coefficient scaffold is tied back to the torsion linking data of a closed three-manifold. The surgery determinant controls a one-dimensional quadratic-character factor, but that scalar forgets the full linking form. We therefore keep the determinant calculation separate from the finite Fourier and Weil operators built from the nonsingular pairing; the subsequent integral lattice comparison isolates the remaining factor-of-two issue.

Theorem 3.10 (Positive determinant Gauss line). *Let B be a nonsingular integral symmetric surgery matrix, and let p be an odd prime with $p \nmid 2 \det B$. After division by the positive and negative blow-up factors, its odd-prime quadratic Gauss sum is*

$$\left(\frac{|\det B|}{p} \right).$$

There is no residual negative squareclass sheet.

Proof. The Gauss-line calculation diagonalizes the surgery form over the finite field. One-dimensional sums contribute Legendre symbols and a common standard Gauss factor; multiplying collects the determinant symbol, and the negative-signature blow-up normalization cancels the corresponding sign power. The remaining scalar is therefore the absolute-determinant symbol, while the linking-form data must be retained in the separate finite pairing.

Diagonalize the form over $\mathbb{F}_p$. Each one-dimensional Gauss sum is its coefficient's Legendre symbol times the standard Gauss sum. Multiplication gives $(\det B/p)$ and a power of the standard sum. The negative-signature blow-up normalization contributes exactly the compensating $(-1/p)$ -power, leaving $(|\det B|/p)$. $\square$

The scalar determinant line forgets the linking form. For a finite odd-order abelian group T with nonsingular symmetric linking pairing $\ell : T \times T \rightarrow \mathbb{Q}/\mathbb{Z}$, let $m = \exp(T)$, $N = |T|$, and choose $\nu \in \mathbb{Z}$ with $2\nu \equiv 1 \pmod{m}$. The unique homogeneous quadratic refinement is

$$q_0(x) = \nu \ell(x, x);$$

the notation $\frac{1}{2}\ell(x, x)$ always means this inverse of two in the m -torsion subgroup, not division of a rational representative. Set

$$\mathcal{H}_T^{\text{int}} = \mathbb{Z}[\zeta_m][T] = \bigoplus_{x \in T} \mathbb{Z}[\zeta_m] \mathbf{e}_x$$

and define

$$\mathbf{F} \mathbf{e}_y = \sum_{x \in T} e^{-2\pi i \ell(x, y)} \mathbf{e}_x, \quad \mathbf{P} \mathbf{e}_x = e^{2\pi i q_0(x)} \mathbf{e}_x, \quad \mathbf{C} \mathbf{e}_x = \mathbf{e}_{-x}.$$

Choose and retain a square-root line $s_T^2 = N$, and put

$$\begin{aligned} \mathcal{R}_T &= \mathbb{Z}[1/m, \zeta_{8m}, s_T], & \mathcal{H}_T &= \mathcal{R}_T \otimes \mathcal{H}_T^{\text{int}}, & \mathbf{S} &= s_T^{-1} \mathbf{F}, \\ \gamma(T, q_0) &= s_T^{-1} \sum_{x \in T} e^{2\pi i q_0(x)}. \end{aligned}$$

The operators have separate roles: $\mathbf{F}$ is the integral Fourier transform, $\mathbf{P}$ records the quadratic phase, $\mathbf{C}$ is inversion, and $\mathbf{S}$ is the normalized Fourier operator after the square-root line has been adjoined. The scalar $\gamma(T, q_0)$ is the corresponding normalized Gauss sum.

Theorem 3.11 (Odd-torsion Weil module). *The operators above preserve their declared lattices and satisfy*

$$\begin{aligned} \mathbf{F}^2 &= N\mathbf{C}, & \mathbf{S}^2 &= \mathbf{C}, & \mathbf{S}^4 &= 1, & \mathbf{P}\mathbf{C} &= \mathbf{C}\mathbf{P}, \\ (\mathbf{S}\mathbf{P})^3 &= \gamma(T, q_0)\mathbf{C}, & \gamma(T, q_0)^4 &= 1, & \gamma(T, q_0)\gamma(T, -q_0) &= 1. \end{aligned}$$

Isometries act by basis permutation; orientation reversal cyclotomically conjugates $\mathbf{S}, \mathbf{P}, \gamma$; and orthogonal sums tensor these operators after extension to one common coefficient ring and a compatible choice of square-root lines.

The exact 2-primary lattice statement is the following narrower one. For two odd linking modules T_1, T_2 , extend both to

$$\mathcal{R} = \mathbb{Z}[1/\ell, \zeta_{8\ell}, s_1, s_2], \quad \ell = \text{lcm}(\exp T_1, \exp T_2),$$

without inverting 2. Put $\mathcal{H}_i = \mathcal{R}[T_i]$, $\mathcal{H}_i^\pm = \ker(\mathbf{C}_i \mp 1)$, and

$$G = (\mathcal{H}_1 \otimes \mathcal{H}_2)^{\mathbf{C}_1 \otimes \mathbf{C}_2 = +1}, \quad L = (\mathcal{H}_1^+ \otimes \mathcal{H}_2^+) \oplus (\mathcal{H}_1^- \otimes \mathcal{H}_2^-).$$

Then

$$L \subset G, \quad 2G \subset L, \quad G = \text{Sat}_2(L) := (L \otimes_{\mathcal{R}} \mathcal{R}[1/2]) \cap (\mathcal{H}_1 \otimes \mathcal{H}_2),$$

and, with $r_i^- = (|T_i| - 1)/2$,

$$\boxed{G/L \cong (\mathcal{R}/2\mathcal{R})^{r_1^- r_2^-}.$$

Thus G and L agree after, and in general only after, inverting 2.

Finite Fourier identities and lattice comparison. The Weil-module relations are proved in two layers. First, uniqueness of the quadratic refinement and nonsingularity turn character orthogonality and completing the square into the Fourier and projective identities; orthogonal splitting handles the Gauss factors, while changes of summation variables give covariance and tensoring. Second, the integral lattice comparison is an orbit-by-orbit calculation: symmetrizing and antisymmetrizing gives $2G \subset L$, and each pair of nonfixed sign orbits contributes a 2-by-2 Smith block with index 2, while fixed-point blocks contribute none.

Proof. Changing ν by a multiple of m does not change q_0 . Bilinearity gives

$$q_0(x + y) - q_0(x) - q_0(y) = \ell(x, y).$$

If another homogeneous refinement differs from q_0 by a character d , homogeneity at $2x$ gives $2d(x) = 0$; an odd-order group has no nonzero such character. This proves uniqueness.

Nonsingularity identifies T with its character group. Therefore the (x, z) -entry of F^2 is

$$\sum_{y \in T} e^{-2\pi i(\ell(x,y) + \ell(y,z))} = \begin{cases} N, & z = -x, \\ 0, & z \neq -x, \end{cases}$$

which proves the integral Fourier identity. Completing the square with the displayed refinement identity reduces each entry of $(SP)^3$ to the normalized Gauss sum, proving the projective relation. Every odd nonsingular linking form splits orthogonally into cyclic prime-power forms; the classical one-dimensional Gauss sum on each factor is a fourth root of unity. Multiplicativity under orthogonal splitting proves $\gamma^4 = 1$, and reversing the form gives $\gamma(T, -q_0) = \gamma(T, q_0)^{-1}$. The covariance and tensor statements follow by changing summation variables and factoring the pairing and quadratic form.

For the lattice assertion, if $v \in G$, then

$$v + (C_1 \otimes 1)v \in \mathcal{H}_1^+ \otimes \mathcal{H}_2^+, \quad v - (C_1 \otimes 1)v \in \mathcal{H}_1^- \otimes \mathcal{H}_2^-,$$

and their sum is $2v$. Hence $2G \subset L$. Choose one representative in every nonfixed sign orbit of each T_i . For every pair of nonfixed orbits, G has the basis

$$a = \mathbf{e}_x \otimes \mathbf{e}_y + \mathbf{e}_{-x} \otimes \mathbf{e}_{-y}, \quad b = \mathbf{e}_x \otimes \mathbf{e}_{-y} + \mathbf{e}_{-x} \otimes \mathbf{e}_y,$$

whereas L is generated by $a + b$ and $a - b$. Their change matrix $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ has Smith factors 1, 2. Blocks containing the unique fixed point 0 contribute no index. There are $r_1^- r_2^-$ nonfixed-orbit pairs, proving the cokernel and saturation formulas. $\square$

Remark 3.12 (What the 2-cokernel compares). The quotient above compares the global diagonal-sign invariant lattice G with the split $++ \oplus --$ lattice L . It is not an identification with a folded orbit-coordinate lattice, and the finite Weil module is not yet a theta-handlebody state space or a selected topological vacuum.

3.6. Positive first Betti number.

The scalar and comparison class for trefoil zero surgery. The last clause below concerns a particular cyclotomic invariant. Let $N_1 = S_0^3(3_1^r)$ be zero surgery on the right-handed trefoil and set $J_p = \tau_p^{\text{CM}}(N_1)$, for primes $p \geq 5$, using Cochran–Melvin’s Section 4 normalization [11]. To specify its embedding, take

$$A_p = e^{\pi i/p}, \quad \zeta_p = A_p^2, \quad t_p = \zeta_p^{(p+1)/2} = -A_p, \quad s_p = t_p^2 = \zeta_p, \quad q_{\text{CM},p} = t_p^4 = \zeta_p^2.$$

CM’s literal variables are $h_p = t_p - 1 = -\lambda_p$ and $z_{\text{CM},p} = s_p - s_p^{-1}$; $1 - \zeta_p$ is an associate of h_p , not its definition. Their invariant lies in $\mathbb{Q}(\zeta_p)$, and its integral values lie in $\mathcal{O}_p = \mathbb{Z}[A_p]$.

For the comparison class, retain this paper’s total-dimension generator from Section 4:

$$D_p^2 = -\frac{p}{(A_p - A_p^{-1})^2}, \quad D_p > 0 \quad \text{in the fixed complex embedding,}$$

$$F_p^D = \mathbb{Q}(A_p, D_p).$$

All families are compared in $\prod_{p \geq 5} F_p^D$.

Extend the valuation normalized by $v_{\lambda_p}(\lambda_p) = 1$ to this root field. Since $v_{\lambda_p}(p) = p - 1$ and $v_{\lambda_p}(A_p - A_p^{-1}) = 1$,

$$v_{\lambda_p}(D_p) = (p - 3)/2.$$

CM’s unknot bracket $b_0 = -p/(\zeta_p - \zeta_p^{-1})^2$ has a different square: $b_0 = D_p^2/(A_p + A_p^{-1})^2$. No equality of dimension generators or coefficient hulls is inferred.

At exponent one, Γ_1 is trivial and $\mathcal{T}_1 = \hat{\Lambda}_1$, with $t_1 = q$ and $\text{Ev}_{1,p}(f) = f(\zeta_p)$. Here this unsheeted scalar tail defines a comparison class; it does not assign a scalar tail to every

positive- b_1 manifold. Write $D = (D_p)_p$ and define

$$(1) \quad \mathcal{H}_1^D = ((\text{Ev}_1(\mathcal{T}_1) \otimes_{\mathbb{Z}} \mathbb{Q})[D]) \cap \prod_{p \geq 5} \mathcal{O}_p[D_p].$$

Thus a representation is $\sum_{j=0}^J D^j \text{Ev}_1(f_j)$ with finite J , $f_j \in \mathcal{T}_1 \otimes \mathbb{Q}$, and one common rational denominator for its finite coefficient list. The degree and denominator are fixed within a representation, independently of the prime. The intersection imposes root integrality in the displayed order; it does not permit additional coefficients or infinite D -series.

Theorem 3.13 (Split free-homology sector and two no-gos).

For $M = M_0 \# (\#^b S^1 \times S^2)$,

$$I(M) = D^b I(M_0), \quad Z(M) = D^{b-1} I(M_0).$$

Homology alone cannot determine the general positive- b_1 target: T^3 and $\#^3(S^1 \times S^2)$ have the same first homology but different root-growth slopes. For $N_1 = S_0^3(3_1^r)$, the CM-normalized family $J = (\tau_p^{\text{CM}}(N_1))_{p \geq 5}$ satisfies

$$J_p = 0 \iff p \equiv 1 \pmod{3}, \quad J_p \neq 0 \iff p \equiv 2 \pmod{3},$$

and $J \notin \mathcal{H}_1^D$, the unsheeted exponent-one finite D -polynomial hull of (1).

Proof. The connected-sum axiom and the two stated normalizations give the split free-homology formulas immediately. The two rank-three examples then have different exact TQFT growth slopes despite matching first homology, so no homology-only exponent can serve both. For the specified CM family, the Gauss calculation gives two infinite prime classes. A nonzero polynomial in the specified hull instead has a unique least-valuation term at all sufficiently large primes, so is eventually nonzero.

The connected-sum axiom and the two normalizations $Z(S^3) = D^{-1}$, $Z(S^1 \times S^2) = 1$ give the split formulas. The two rank-three examples have exact TQFT dimensions with slopes one and three, so no homology-only exponent can fit both [11].

For the last clause, use CM's Remark 5.3 identification with $(1, 0)$ surgery on the right-handed Whitehead link. Put $n_p = (p-3)/2$ and $a = (p-1)/2$. Its bracket is $b_0 t_a$, and the Section 4 norm is $b_1(b_0/h_p^{n_p})$. Therefore, directly from that surgery normalization,

$$J_p = \frac{h_p^{n_p} t_a}{b_1}, \quad t_a = \sum_{k=1}^{(p-1)/2} s_p^{a(k^2-1)} [k^2]_{\text{CM}}, \quad [m]_{\text{CM}} = \frac{s_p^m - s_p^{-m}}{s_p - s_p^{-1}}.$$

The factors h_p, b_0, b_1 are nonzero by CM's Proposition 3.11; no comparison with an unnamed I or Z is used. Write $G_c = \sum_{k=0}^{p-1} t_p^{ck^2}$. CM's Remark 3.12(b) gives

$$2z_{\text{CM},p} s_p^a t_a = G_{2(a+1)} - G_{2(a-1)} = G_1 - G_{-3} = \left(1 - \left(\frac{-3}{p}\right)\right) G_1.$$

Here $G_1 \neq 0$, and $z_{\text{CM},p} \neq 0$ for $p \geq 5$. Thus the scalar is a nonzero factor times $1 - \left(\frac{-3}{p}\right)$. Quadratic reciprocity gives the stated zero and nonzero classes modulo three. Both classes contain infinitely many primes by Dirichlet's theorem on primes in arithmetic progressions; finite prime tests are not the infinitude argument.

Suppose a nonzero family in $\mathcal{H}_1^D$ is represented by $\sum_{j=0}^J D^j \text{Ev}_1(f_j)$. Taylor injectivity for $\mathcal{T}_1$, used in Theorem 3.8, gives each nonzero f_j a finite first Taylor order k_j at $q = 1$. Clear one common denominator for the coefficient list. Exclude its prime divisors and the prime divisors of the finitely many nonzero leading Taylor numerators. For sufficiently large remaining primes, finite-stage Taylor specialization gives $v_{\lambda_p}(\text{Ev}_{1,p}(f_j)) = k_j$, since $\zeta_p - 1$ is an associate of λ_p . There is no character sheet at $m = 1$ on which a nonzero coefficient can disappear.

Let j_0 be the least degree with nonzero coefficient. For all sufficiently large primes, the finite list of term valuations is

$$v_{\lambda_p}(D_p^j \text{Ev}_{1,p}(f_j)) = j(p-3)/2 + k_j.$$

For every $j > j_0$, the gap $(j-j_0)(p-3)/2 + k_j - k_{j_0}$ is eventually positive. The j_0 term is therefore the unique least-valuation term, and the sum has valuation $j_0(p-3)/2 + k_{j_0}$; in particular it is nonzero. The root-integral intersection does not enlarge the class of finite representations. Every nonzero family in this hull is eventually nonzero at every prime, contradicting the infinite zero set of the nonzero family J . This proves its exclusion from $\mathcal{H}_1^D$. $\square$

The final clause is a statement about the named CM scalar and the displayed $\mathcal{T}_1$ hull. It supplies no corresponding comparison for I/Z , GM, RT or Lê normalizations, an enlarged coefficient base, prime-dependent denominators, or infinite D -series.

There are exact root-side targets using a nonmaximal order, its integral closure, or the intrinsic valuation lattice. Their assembly with a free character sector, the odd Weil module, and an anomaly line remains a candidate architecture rather than a completed non-IHS theorem.

Remark 3.14 (Typed-target boundary). The results in this section do not supply the missing 2-primary linking sector, a full GM-to-Le multiplier, a general positive- b_1 topological assignment, a disk block, or a universal non-IHS Heegaard scalar.

4. GENUS-ONE SMITH INVARIANTS IN NEWTON COORDINATES AND THE INVARIANT GM MIDPOINT

Genus one is the smallest place where the difference between a lowered topological pairing and a state operator can be resolved exactly. The result supplies the local normalization and saturation profiles used in all later theta and handlebody constructions.

In the running one-holed-torus model of Figure 1, this section computes the pairing-induced map and two cap rows on the reduced root-of-unity fiber. That fiber enters the figure's left-hand vertical path; the disk reading remains a separate output of the finite-stage marked boundary construction.

First fix the root and dimension-line conventions and identify the two topological pairings in the genus-one model. The induced Hopf-pairing map is then determined by lowering with the complementary pairing and raising with the identity-double pairing. Next express the Hopf matrix in a Newton basis: its invariant factors describe the integral image, while the factorial basis records the corresponding saturation. The later graph and cap statements use these two coordinate descriptions without turning the matrix into an unqualified state operator.

4.1. Root and line conventions. Fix an odd prime $p \geq 5$, put $d = (p-1)/2$, and use the active even colors $0, 2, \dots, p-3$. In the fixed complex embedding take $A_p = e^{\pi i/p}$, of order $2p$, put $\zeta_p = A_p^2$, and set $\lambda_p = 1 + A_p$ and $\mathcal{O}_p = \mathbb{Z}[A_p] = \mathbb{Z}[\zeta_p]$. All integral lattice comparisons below use this full cyclotomic integer ring, without localization at (λ_p) . Define the lowered Hopf matrix

$$(H_p)_{ab} = [(2a+1)(2b+1)]_p, \quad 0 \leq a, b < d.$$

Let D_p be the chosen positive generator of the total-dimension line, so that

$$D_p^2 = \delta_p = -\frac{p}{(A_p - A_p^{-1})^2}.$$

The line itself is retained in intrinsic formulas; D_p is a chosen trivialization.

Lemma 4.1 (Odd sine orthogonality). *The matrix H_p is symmetric and*

$$H_p^2 = D_p^2 I.$$

Proof. Write the Hopf entries as sine ratios and use orthogonality on the odd residue set. The cosine-product reduction kills the off-diagonal terms and leaves the common norm on the diagonal; the frozen root convention then identifies that norm with the chosen dimension-line square.

Write each quantum integer as a sine ratio. For active odd integers m, n , the sum over odd residues is

$$\sum_{b=0}^{d-1} \sin \frac{m(2b+1)\pi}{p} \sin \frac{n(2b+1)\pi}{p} = \frac{p}{4} \delta_{mn}.$$

This follows by the cosine product identity and the fact that every nontrivial even-frequency odd-residue cosine sum is $-1/2$. Dividing by $\sin^2(\pi/p)$ gives the displayed scalar. The chosen root convention turns that scalar into $-p/(A_p - A_p^{-1})^2$. $\square$

The identity double and complementary gluing are different bilinear pairings:

$$H \cup \overline{H} = S^1 \times S^2, \quad H \cup H' = S^3.$$

In the canonical skein basis their matrices are I and H_p , respectively. The first becomes Hermitian only after applying the anti-linear orientation-reversal map.

The point of the next comparison is that the two pairings have different topological origins even when they are written in the same basis. The complementary handlebody supplies the Hopf matrix, while the identity double supplies the identity-double pairing. The state arrow is obtained by lowering with the first pairing and raising with the second; this is the operation that keeps the matrix, the normalization line, and the topological gluing in their proper roles.

4.2. The map induced by the Hopf pairing.

Theorem 4.2 (State map induced by the identity and complementary pairings). *In the frozen positive-Hopf extended lift and the normalization $\tau_p = D_p Z_p$, the identity-cylinder lowered kernel, complementary handlebody lowered kernel, and raised-index state arrow are*

$$\boxed{G_p = D_p I, \quad P_p = H_p, \quad \mathcal{S}_p = G_p^{-1} P_p = D_p^{-1} H_p.}$$

Intrinsically, $G_p = \mathbf{d}_p \otimes I$ and $\mathcal{S}_p = \mathbf{d}_p^\vee \otimes H_p$. The line-valued square contracts canonically to the identity.

Proof. Compute the identity-double form in the fusion basis, compute the complementary form from the positive Hopf gluing, and relate the two by the mapping cylinder exchanging meridian and longitude. Nondegeneracy then determines the state arrow induced by the two pairings, while the Hopf square checks that the line-valued normalization gives an inverse. The line is part of the construction until the final root-fiber trivialization.

The identity-double Yang–Mills pairing extracts the zero fusion coefficient; multiplicity-free fusion gives δ_{ab} . The complementary cores form a positive zero-framed Hopf link, so Marché’s formula gives H_p with no sign on the even sector. The GM rescaled identity form contributes D_p after the exact root transport $q_{\text{GM}} = -A_p$. If s exchanges meridian and longitude, functorial gluing gives $B_S(x, y) = B_{\text{id}}(x, \mathcal{S}_p y)$, hence $P_p = G_p \mathcal{S}_p$. Nondegeneracy makes the solution unique. Finally Lemma 4.1 gives $\mathcal{S}_p^2 = D_p^{-2} H_p^2 = I$. $\square$

This theorem does not call the raw matrix H_p a state operator. It first lowers an index by the complementary pairing and then raises it by the inverse map induced by the identity-double pairing.

Let $y_p = e_0^T$ and $h_p = y_p H_p$. These are the identity-double and complementary caps in the two orientations.

Corollary 4.3 (Exact cap exchange).

$$h_p \mathcal{S}_p = D_p y_p, \quad D_p y_p \mathcal{S}_p = h_p.$$

Both equations are line-typed before choosing D_p .

Proof. Use $h_p = y_p H_p$, Theorem 4.2, and $H_p^2 = D_p^2 I$. $\square$

4.3. Smith invariants in Newton coordinates: source and image chambers. The matrices are constructed in the even core frame. Let z be the color-one blackboard core and $e_{j+1} = z e_j - e_{j-1}$, with $e_0 = 1$, $e_1 = z$. Define

$$P_0(s) = 1, \quad P_1(s) = s - 1, \quad P_{n+1}(s) = (s - 2)P_n(s) - P_{n-1}(s), \quad e_{2n} = P_n(z^2),$$

and put

$$s_i = (A_p^{2i+1} + A_p^{-2i-1})^2, \quad N_n(s) = \prod_{i < n} (s - s_i), \quad N_n(s) = \sum_{k \leq n} P_k(s) (C_p)_{kn}.$$

Here $N_n(s)$ is the Newton interpolation product in the polynomial variable s , over $\mathcal{O}_p$, with nodes $s_0, s_1, \dots$. Substitution $s = z^2$ gives the even-core columns; the rescaling by λ_p^{-n} in the saturated frame below is a separate normalization. Both polynomial lists are monic in s , so C_p is upper unitriangular and its n -th core column is $c_{n,p} = N_n(z^2)$. In particular $C_p e_0 = e_0$. For $i < a$, set $q_{ai} = (s_a - s_i)/\lambda_p^2$ and define

$$(L_p)_{ab} = \begin{cases} [2a+1]_p \prod_{i < b} q_{ai}, & a \geq b, \\ 0, & a < b. \end{cases}$$

The odd Chebyshev identity $[(2a+1)(2b+1)]_p = [2a+1]_p P_b(s_a)$ evaluates the Newton columns and gives the factorization below. To check it integrally, use

$$s_a - s_i = A_p^{4i+2} (A_p^{4(a-i)} - 1) (1 - A_p^{-4(a+i+1)}).$$

Each parenthesized factor is λ_p times a global cyclotomic unit: the active indices $a - i$ and $a + i + 1$ lie strictly between zero and p . Likewise $[2a+1]_p = A_p^{-2a} (1 - A_p^{4a+2}) / (1 - A_p^2)$ is a global unit. Thus L_p is lower triangular with unit diagonal and $e_0^T L_p = e_0^T$. These explicit choices fix the vacuum and the cap rows as well as the Smith ideals.

There are unimodular matrices $C_p, L_p \in \mathrm{GL}_d(\mathcal{O}_p)$ such that

$$H_p C_p = L_p \Lambda_p, \quad \Lambda_p = \mathrm{diag}(1, \lambda_p^2, \dots, \lambda_p^{p-3}).$$

The columns of C_p are Newton source vectors.

Theorem 4.4 (Saturated image lattice from Smith invariants in Newton coordinates). *The state arrow restricts to an integral isomorphism from the standard active graph lattice to its exact line-valued image:*

$$C_p e_n \mapsto \mathbf{d}_p^\vee \lambda_p^{2n} L_p e_n.$$

The vacuum is $C_p e_0 = e_0$ and has unit Smith weight.

Proof. The Smith factorization is used as a simultaneous source-and-image certificate. Applying the line-valued state arrow to the factorization transports each Newton source column to the corresponding weighted image column; unimodularity makes both lists bases of their declared lattices. The zeroth Newton polynomial and diagonal entry then identify the vacuum without any additional normalization.

Apply $\mathbf{d}_p^\vee H_p$ to the Smith factorization. Since C_p and L_p are unimodular, the displayed source and image columns are bases of their declared lattices. The zeroth Newton polynomial is one and the zeroth diagonal factor is one. $\square$

4.4. Factorial, graph, and invariant GM lattices. For the GM input, transport the entire coefficient convention, including the quotient and pairing, while keeping z fixed. The native data are $q_0 = \zeta_p$ and $A_0 = -q_0^{d+1} = A_p$. The coefficient Galois automorphism and transported data are

$$\sigma_p(q_0) = q = -A_p = \zeta_p^{(p+1)/2}, \quad A = \sigma_p(A_0) = -q^{d+1}, \quad h = 1 - q = \lambda_p,$$

$$\mu_i = -q^{i+1} - q^{-i-1}, \quad F_n = \prod_{k=1}^n ((-A)^k - (-A)^{-k}), \quad F_0 = 1.$$

Gilmer–Masbaum’s Definition 2.1 and Theorem 2.2, with boundary color $c = 0$, give the linear degree- n orthogonal numerator and invariant basis below [18, Section 2, arXiv v3]. This numerator is different from the degree- $2n$ square-node column $c_{n,p} = N_n(z^2)$. If $B = -A = q^{d+1}$, then $B^2 = q$ and $B^k - B^{-k} = B^{-k}(q^k - 1)$. Consequently each factor of F_n/λ_p^n is a root unit times $(1 - q^k)/(1 - q)$, a global cyclotomic unit for $1 \leq k \leq n < d < p$. This proves a unit statement over $\mathcal{O}_p$, not just a valuation at one prime.

Let

$$Q_n = \prod_{i=0}^{n-1} (z - \mu_i), \quad Q'_n = F_n^{-1} Q_n,$$

where F_n is the transported factorial and $(F_n) = (\lambda_p^n)$ in the fixed fiber. The ordinary even graph lattice and invariant GM lattice are

$$\mathcal{G}_p^{\text{ev}} = \bigoplus_{n < d} \mathcal{O}_p Q_n, \quad \mathcal{S}_p^+ = \bigoplus_{n < d} \mathcal{O}_p Q'_n.$$

Here the small cores e_j , $0 \leq j < d$, and active even cores are related by the exact reduced-annulus reflection $e_{p-2-j} = e_j$. It sends a small index j to j when j is even and to $p-2-j$ when j is odd, with no reflection phase. The reflection follows by evaluating $e_m(-u - u^{-1}) = (-1)^m(u^{m+1} - u^{-m-1})/(u - u^{-1})$ at the active nodes $u = q^{i+1}$, where $u^p = 1$. It is an identity in the reduced annular quotient before saturation, independent of any Newton-lattice comparison. Let M_p be this permutation and let R_p be the upper-unitriangular expansion of the monic Q_n in the small monic Chebyshev cores. Then $U_{Q,p} = M_p R_p \in \text{GL}_d(\mathcal{O}_p)$ has columns Q_n in the even frame $\mathbf{e} = (e_0, e_2, \dots, e_{2d-2})$. This is an unscaled graph comparison; saturation still divides the columns by F_n .

Theorem 4.5 (Exact invariant saturation). *The inclusion $\mathcal{G}_p^{\text{ev}} \subsetneq \mathcal{S}_p^+$ is diagonal in the Q'_n -basis and*

$$\mathcal{S}_p^+ / \mathcal{G}_p^{\text{ev}} \cong \bigoplus_{n=0}^{d-1} \mathcal{O}_p / (\lambda_p^n), \quad [\mathcal{S}_p^+ : \mathcal{G}_p^{\text{ev}}] = (\lambda_p^{d(d-1)/2}).$$

The same quotient occurs between the saturated and graph Riesz outputs.

Proof. The saturation statement is a basis comparison rather than a new topological identification. The monic graph vectors and the factorially normalized vectors differ diagonally by the transported factors F_n ; the valuation of those factors records exactly how much the graph lattice misses the invariant midpoint. The pairing-induced map carries the same diagonal comparison to the output lattice.

Both Q_n and the ordinary closed Jones–Wenzl core are monic of degree n , so their change of basis is upper unitriangular. The sourced GM orthogonal-basis theorem gives the Q'_n basis, and $Q_n = F_n Q'_n$ diagonalizes the inclusion. Since $(F_n) = (\lambda_p^n)$, the quotient and index follow without a PID hypothesis. The Riesz composite is injective, so it preserves this quotient on the output side. $\square$

Let B_p be the complementary Hopf pairing and $L^\# = \{x : B_p(x, L) \subset \mathcal{O}_p\}$.

Theorem 4.6 (Self-dual midpoint). *The invariant GM lattice is the self-dual midpoint*

$$\boxed{\mathcal{G}_p^{\text{ev}} \subset \mathcal{S}_p^+ = (\mathcal{S}_p^+)^{\#} \subset (\mathcal{G}_p^{\text{ev}})^{\#}.}$$

The two half-quotients have exponents n , while the full graph-dual quotient has exponents $2n$. Thus the doubled Hopf Smith profile is exactly the sum of the two saturation halves.

Proof. The midpoint is identified by testing each normalized basis vector against the same basis. Unit GM norm makes the Q'_n basis self-dual, while the graph-to-GM factor F_n records the integral cost on each side. One dual comparison contributes n ; composing the two comparisons gives $2n$, which is exactly the Hopf Smith exponent profile.

The GM norm of Q'_n is a unit: its ordinary-brace valuation has $(n+1) - n - 1 = 0$, and every plus-brace is a unit. Hence the Q'_n basis is self-dual. A coefficient $x_n Q'_n$ pairs integrally with $F_n Q'_n$ precisely when $x_n F_n$ is integral. The two diagonal quotients therefore each contribute $v_{\lambda_p}(F_n) = n$; composing them gives $2n$, the Smith exponents of H_p . $\square$

4.5. The saturated Newton frame. Put $v = (z + 2)/\lambda_p$. For every odd m , define

$$r_m(A) = \left(\frac{A^m - A^{-m}}{1 + A} \right)^2.$$

If $m = 2i + 1 < p$, then $r_m(A_p)$ is a unit and the exact identity

$$\frac{z^2 - s_i}{\lambda_p} = -4v + \lambda_p(v^2 - r_{2i+1}(A_p))$$

holds. Let $c_{n,p} = \prod_{i < n} (z^2 - s_i)$ and $w_{n,p} = \lambda_p^{-n} c_{n,p}$.

Theorem 4.7 (Saturated Newton frame and half-Smith image). *For every odd prime,*

$$\boxed{\mathcal{S}_p^+ = \bigoplus_{n=0}^{d-1} \mathcal{O}_p w_{n,p}.}$$

Modulo λ_p , $w_{n,p} = (-4)^n v^n$. Moreover,

$$H_p(C_p \text{diag}(\lambda_p^{-n})) = L_p \text{diag}(\lambda_p^n),$$

and after Riesz raising the right side acquires only the separate factor $\mathbf{d}_p^{\vee}$.

Proof. The Newton-frame proof multiplies the exact one-node identity along the successive roots. This first makes each normalized column integral and identifies its Rees special fiber; the reduced-root degree- d relation then reduces all columns to the declared finite lattice. The unit transition determinant gives equality with the invariant GM basis, and right multiplication of the Smith factorization records the half-Smith weights.

Multiplying the exact one-node identity shows that every $w_{n,p}$ is an integral polynomial in v . The reduced-root relation $Q_d = 0$ becomes a monic degree- d relation in v : its nodes are

$$\beta_i = \frac{2 + \mu_i}{\lambda_p} = \frac{(1 - q^{i+1})(1 - q^{-i-1})}{\lambda_p} \in \lambda_p \mathcal{O}_p.$$

Thus every such polynomial reduces over $\mathcal{O}_p$ in the GM basis $1, v, \dots, v^{d-1}$. Put $D_{\lambda,p} = \text{diag}(\lambda_p^n)$ and $F_p = \text{diag}(F_n)$. The two frame rows are

$$\mathbf{Q}' = \mathbf{e} U_{Q,p} F_p^{-1}, \quad \mathbf{w} = \mathbf{e} C_p D_{\lambda,p}^{-1},$$

so the exact transition is

$$(2) \quad X_p = F_p U_{Q,p}^{-1} C_p D_{\lambda,p}^{-1}.$$

Integral membership of each $w_{n,p}$ gives integral entries of X_p . Moreover

$$\det X_p = \det(U_{Q,p})^{-1} \det(C_p) \prod_{n < d} \frac{F_n}{\lambda_p^n} \in \mathcal{O}_p^\times.$$

The adjugate therefore gives an integral inverse over the full ring $\mathcal{O}_p$. Reduction modulo λ_p is a useful consequence, but is not this global invertibility argument; no inversion of 4 in $\mathcal{O}_p$ is made. Right-multiplying the Smith factorization by $\text{diag}(\lambda_p^{-n})$ proves the half-Smith formula. $\square$

For fixed n , one Laurent polynomial represents the same column at every sufficiently large active prime:

$$W_n(A, v) = \prod_{i=0}^{n-1} [-4v + (1 + A)(v^2 - r_{2i+1}(A))].$$

Its Rees special fiber is $W_n(-1, v) = (-4)^n v^n$. This is an algebraic special fiber, not an analytic disk limit.

4.6. The infinite common matrix and the sharp repair.

Write $W_n = \sum_k t_{kn} v^k$. The coefficient theorem determines its full support and valuation:

$$\text{ord}_{1+A}(t_{kn}) = |k - n|,$$

after the primitive column gauge A^{2n^2} . Thus the forward matrix improves rapidly away from its diagonal, but its inverse need not preserve the earlier rapid carrier.

Theorem 4.8 (Nonsurjectivity of the old rapid module). *The unique inverse Newton expansion of v has infinitely many nonzero coefficients of exact $(1 + A)$ -order $n - 1$ with eventual-unit denominators. It belongs neither to the old rapid tail nor to its finite core. Hence the old common-column bridge is not a two-sided carrier.*

Proof. The obstruction is detected by the inverse matrix, not by the rapidly decaying forward columns. Triangular inversion produces an infinite tail whose valuation stays too small for the old rapid condition and whose support is too large for the finite core. This isolates the failure of the old completion before the localized Neumann repair is introduced.

Triangular inversion gives a recurrence whose leading term at stage n is the product of the inverse diagonal $(-4)^{-n}$ and exactly $n - 1$ off-diagonal prime factors. Induction shows the coefficient is nonzero with order precisely $n - 1$; all remaining denominators are eventual units. Rapid membership would require order growing faster than every prescribed linear rate, while finite-core membership would require only finitely many nonzero terms. Both fail. $\square$

Theorem 4.9 (Localized lambda-Rees automorphism). *Over the $(1 + A)$ -adic completion of $S_{\text{ev}}^{-1}\mathbb{Z}[A, A^{-1}, 1/2]$, normalization by $(-4)^{-n}$ turns the common matrix into $I + \mathbf{N}$ with $\mathbf{N}(P) \subset (1 + A)P$. The Neumann series gives a bicontinuous inverse on the uniform product and on its truncation-dense c_0 submodule. Inverting 2 is sharp.*

Proof. The exact off-diagonal valuation makes the normalized matrix adically close to the identity, so successive powers of its adically small correction move uniformly deeper into the completion. The remaining issue is the diagonal unit: in unbounded degree its reduction is invertible precisely after the declared inversion of 2, which explains the sharp coefficient-ring boundary.

The exact valuation $|k - n|$ makes every off-diagonal entry divisible by $1 + A$, while the diagonal is $(-4)^n$ modulo that element. After normalization, powers $\mathbf{N}^r$ land in the r -th adic neighborhood, so $\sum_{r \geq 0} (-\mathbf{N})^r$ converges and is a two-sided inverse. Without inverting 2, the mod- $(1 + A)$ diagonal $(-4)^n$ is not a unit in unbounded degree, so an automorphism of the stated uniform product is impossible. $\square$

Geometrically, this repaired automorphism is the coefficient-level transport of a marked handlebody state through the declared spine or pairing change. The inverse exists on the repaired product because the normalized off-diagonal terms become adically small; this restores the coordinate transport needed by the subsequent cap constructions, without producing a new reduced-root or disk evaluation.

Remark 4.10 (Rees branch). The localized Rees completion is separately typed. Arbitrary completed elements have no reduced-root, Habiro, or disk evaluation. Only finite entries and finite rectangles retain the proved specializations and CRT approximants. The strict common reduced-root/Rees graph is constructed in Section 6.

5. FACTORIAL CLEARING, JOINT RAPID BRIDGES, AND HANDLEBODY COVARIANCE

A marked-spine move can introduce color-dependent denominators before a handlebody pairing is evaluated. To transport the state, we must clear those denominators and control the loss in the chosen completion. This section compares two proved ways to do so: a legacy all-rate Rees bridge and a factorially selected lattice. Their covariance statements concern these specified coefficient objects.

Candidate 3.7 formulates the proposed obstruction to the unrestricted product topology. It remains a candidate statement, and no argument below uses it as a theorem. The constructions instead use two proved replacements: a legacy all-rate Rees bridge in the recursively cleared gauge, and a new factorial selected lattice with a quadratic joint disk/cyclotomic topology. They are distinct completions; no equality theorem identifies them.

The first theorem records the legacy all-rate Rees bridge and its quantifier order. The factorial Jones–Wenzl denominator-clearing factor then supplies a local denominator removal and its explicit loss estimate. These two repairs are combined in the joint coefficient object, after which the factorial–GM comparison and handlebody covariance transport the chosen lattice. The final stabilization result is stated only for the split zero-handle situation, so the section does not identify the two completions or promote their local covariance to a general boundary equivalence.

5.1. The legacy all-rate bridge. We first specify the regulator shared by the estimates. In this section $A = s^2$ and $q = A^2 = s^4$. This q differs from the transported GM parameter $-A$ of the preceding section. For $N \geq 1$, put

$$e_d(N) = \max\{0, \lfloor N/d \rfloor - 1\}, \quad \overline{K}_N = \prod_{d=1}^N \Phi_d(A^2)^{e_d(N)}, \quad K_N = A^{4N-1} \overline{K}_N,$$

with $\overline{K}_0 = K_0 = 1$, and set $J_N = (K_N)$. The transition maps in both inverse systems are the natural quotient maps:

$$\mathcal{H}_K = \varprojlim_N \mathbb{Z}[A]/(\overline{K}_N), \quad \widehat{\mathcal{J}} = \varprojlim_N \mathbb{Z}[A]/J_N.$$

The exponents increase with N . Comparing each cyclotomic direction and the monomial orders gives

$$\overline{K}_D \overline{K}_N \mid \overline{K}_{D+N}, \quad K_D K_N \mid K_{D+N}, \quad \overline{K}_N \mid (A^2; A^2)_N \mid \overline{K}_{2N}.$$

Indeed $\max(0, a-1) + \max(0, b-1) \leq \max(0, \lfloor (D+N)/d \rfloor - 1)$ for $a = \lfloor D/d \rfloor$, $b = \lfloor N/d \rfloor$. The factorial has exponent $\lfloor N/d \rfloor$, between the two displayed truncated exponents. If $L_N = A^{2N}(A^2; A^2)_N$, then $K_N \mid L_{2N}$ and $L_N \mid K_{2N}$; thus the two joint filtrations are cofinal. These identities also give the loss rule: if a denominator divides K_D , division carries $J_{D+N} \widehat{\mathcal{J}}$ into $J_N \widehat{\mathcal{J}}$ in the common fraction ring. The analogous rule uses $\overline{K}_N$ on the cyclotomic factor.

The legacy selected gauge uses the recursively squared coupon clearer

$$c_0 = c_1 = 1, \quad c_n = \prod_{k=2}^n u_k^{2^{n-k}}, \quad u_k = A^{1-k} \frac{1-q^k}{1-q}.$$

Fix the source and target charts and a word of ℓ Whitehead moves; E bounds their edge counts. In a standard chart use the proper height $h_\Gamma = 1 + \sum_e n_e$, with n_e the doubled colors; in a normalized GM chart use one plus the sum of its nonnegative coordinates. For a product summand, H dominates the endpoint and intermediate heights. All these choices are fixed before a decay rate is requested.

Here is the coefficient topology in the legacy theorem. Use the closed ideals $I_N = \overline{K}_N \mathcal{H}_K$ and $v_K(x) = \sup\{N : x \in I_N\}$, with $v_K(0) = \infty$. Each root coefficient is realized on its active line $e_{\Gamma, \alpha, p}$, with quotient-first evaluation. The rapid sector consists of these realizations of coefficients $x_\alpha \in \mathcal{H}_K$ satisfying, for every integer $s \geq 1$,

$$v_K(x_\alpha) \geq \max\{0, 2^{sh_\Gamma(\alpha)} - c_s\} \quad \text{for one } c_s \text{ and all } \alpha.$$

Its zero-neighborhoods replace the right side by $N + 2^{sh_\Gamma(\alpha)}$, for $N \geq 0$. The algebraic core has finite label support and activated cofinite cyclotomic coefficients, together with finite exceptional-root corrections; it is given the discrete topology. The bridge is its sum with the rapid sector, with the quotient topology from the discrete core plus rapid sector modulo their exact diagonal intersection. One fixed exponential rate cannot replace the condition for every s .

Theorem 5.1 (Uniform legacy Rees bridge). *Fix a finite genus and a finite quotient-first word. In the legacy recursively squared selected gauge, the finite localized core plus the all-exponential-rate K the–Habiro tail is complete, Hausdorff, and has dense finite truncations. The exactly typed selected theta and normalized GM maps extend to mutually inverse bicontinuous maps.*

Proof. The legacy bridge is controlled by one quantifier order. The regulator and Whitehead estimate first determine the loss for a fixed state, so one chooses an input decay rate above that loss and fixes a finite core radius; only then is an arbitrary output seminorm requested. Closed regulator ideals give completeness, proper height gives dense truncations, and the finite-support inverse identities extend to the Hausdorff completions.

The exact regulator multiplication law and the Whitehead factorial estimate bound the loss along a product summand by $\rho_{E, \ell}(H) = (8E + 80\ell + 16)2^H$. For either typed orientation T , affine support gives $h_{\text{out}} \leq ah_{\text{in}} + b$, finite columns, and an input denominator stage at most $\lceil D_T 2^{rh_{\text{in}}} \rceil$, for fixed a, b, D_T, r . Given a rapid state, choose once $t_0 > \max\{r, a\}$ and a cutoff H_0 at which the rate- t_0 bound absorbs this denominator and the output rate one, with the structural activation reserve. Its image $c_T = T(x^{\leq H_0})$ is one finite core, fixed before the output rate. For an arbitrary output rate s , choose $t > \max\{r, as\}$. Above a possibly larger cutoff,

$$2^{th} - c_t \geq D_T 2^{rh} + 2^{s(ah+b)} - c'_s.$$

The loss rule puts this tail in the output rate- s ideals. The middle band above H_0 is already cyclotomic-integral and has finite output support; enlarging c'_s absorbs it without changing c_T . Thus $T(x) - c_T$ satisfies every rate with that same core. Proper height and the remaining regulator order make every row converge; closed ideals preserve its estimates. A sufficiently deep input neighborhood satisfies the same inequality with any requested N , giving continuity. Quotient-first activation avoids specializing an inactive pole. Closed regulator ideals give completeness, and the rate- $s + 1$ estimate makes finite truncations dense at rate s . The finite-support inverse products are the activation identities, so continuity and Hausdorffness extend them to the completed bridges. $\square$

This theorem is cyclotomic only and belongs to the legacy selected gauge.

5.2. The factorial Jones–Wenzl denominator-clearing factor. Use an auxiliary parameter s with $A = s^2$, and put

$$d_n(A) = \prod_{k=1}^n \frac{s^{2k} - s^{-2k}}{s^2 - s^{-2}}, \quad N_n = \frac{n(n-1)}{2}.$$

In the frozen Hecke convention $(T_i - s)(T_i + s^{-3}) = 0$, define

$$X_n = \sum_{w \in S_n} s^{3\ell(w)} T_w, \quad f_n = \frac{X_n}{s^{2N_n} d_n(A)}.$$

The symmetrizer enters for one local geometric reason: after a marked-spine move or handle-body pairing, the Jones–Wenzl coupon can carry a color-dependent denominator. Multiplying by $d_n(A)$ clears that local denominator before the transported coupons are assembled into a word; the subsequent factorial estimate measures the remaining loss in the joint topology.

Lemma 5.2 (All-rank Hecke symmetrizer). *For every rank n ,*

$$X_n T_i = T_i X_n = s X_n, \quad X_n^2 = s^{2N_n} d_n(A) X_n.$$

Thus f_n is idempotent and is annihilated on both sides by every return generator $e_i = s(T_i - s)$.

Proof. Pair w with ws_i whenever $\ell(ws_i) = \ell(w) + 1$. If the coefficient of T_w is c , the pair is $cT_w + cs^3 T_{ws_i}$. The quadratic Hecke relation sends its right product with T_i to s times the pair. Left descents give the left identity. Therefore

$$X_n^2 = \left(\sum_{w \in S_n} s^{4\ell(w)} \right) X_n.$$

The type- A Poincaré product is $s^{2N_n} d_n(A)$, proving the square and idempotence. The return-generator statement follows from the eigenvalue equation. No finite-rank extrapolation is used. $\square$

Theorem 5.3 (Factorial selected image). *On a fixed trivalent spine and frozen crossingless coupon/pants chart, multiplication of every color- n coupon by $d_n(A)$ clears it exactly. If X_α is the cleared graph vector, after equal crossingless coordinates have been combined, and*

$$m_\alpha = -\min \exp_A(X_\alpha), \quad t_\alpha = A^{m_\alpha} X_\alpha,$$

then t_α is A -primitive and $|m_\alpha| = O_\Gamma(h(\alpha)^2)$. The rebuilt factorial selected image has the exact root specialization kernel and root faithfulness. The simple Φ_{10}^{-1} born pole survives, but the old exponential Φ_2 loss becomes linear; a word of length ℓ has cyclotomic stage at most

$$(2E + 36\ell + 16)h.$$

Proof. The factorial repair acts locally on each color- n coupon, where the Hecke symmetrizer supplies the exact denominator-clearing factor. A fixed crossingless assembly contributes only the controlled word loss, so the primitive shift and cyclotomic valuation can be read off before passing to the global selected image. The surviving Φ_{10} pole records the remaining local obstruction rather than being hidden by the estimate.

Lemma 5.2 gives exact clearing. Every cleared coupon has an explicit quadratic Laurent exponent interval; a fixed crossingless assembly adds only linearly many loop factors. The signed primitive shift therefore has two-sided quadratic size and records the exact raw disk order in this chart. Cyclotomic factorization of d_n gives the admissible-root unit range and exact valuations. Rebuilding the bad-prime selector then repeats the basiswise proof of Theorem 2.4. The exact 2222 coefficient has one uncanceled Φ_{10}^{-1} . Summing the linear factorial losses over the fixed word gives the displayed stage. $\square$

For clarity, the chart constants in the joint estimate are fixed as follows. If the crossingless assembly in chart Γ removes at most $c_\Gamma S(\alpha)$ additional loops, put

$$S(\alpha) = \sum_e n_e, \quad N(\alpha) = \sum_e n_e(n_e - 1)/2,$$

$$M_\Gamma = |E(\Gamma)|(c_\Gamma + 1), \quad C_\Gamma = M_\Gamma + |E(\Gamma)| + 8.$$

A cleared color- n coupon has Laurent exponents in $[-n(n-1)/2, n(n-1)]$; assembly places surviving exponents in $[-N(\alpha) - c_\Gamma S(\alpha), 2N(\alpha) + c_\Gamma S(\alpha)]$. Negating the actual least exponent gives the signed shift $m_{\Gamma, \alpha}$, possibly negative, and $|m_{\Gamma, \alpha}| \leq M_\Gamma h_\Gamma(\alpha)^2$. For the selected diagonal scalar

$$s_{\Gamma, \alpha} = A^{2b_\Gamma(\alpha) + m_{\Gamma, \alpha}} C_\alpha \prod_e d_{n_e}(A),$$

b_Γ is the physical birth level and C_α the fixed bad-prime selector, with $C_\alpha(0) = 1$. The whole primitive vector has disk order $2b_\Gamma$, while its scalar has order $2b_\Gamma + m_{\Gamma, \alpha} - N(\alpha)$, whose absolute value is at most $C_\Gamma h_\Gamma(\alpha)^2$. In the fixed normalized GM product chart, polynomial unitriangularity gives the same bound with its declared chart constant. The symbols C_Θ, C_L below mean these two endpoint constants; they are not universal constants over all charts.

Throughout this argument, the Hecke convention and corrected Costantino coefficient are the ones displayed above; the incompatible symmetrizer convention printed in the cited source is not used.

5.3. The joint coefficient object. Use the regulator and its quotient transitions defined above. Since $\bar{K}_N(0) = \pm 1$, CRT and cofinality give

$$\hat{\mathcal{J}} \cong \mathbb{Z}[[A]] \times \mathcal{H}_K.$$

Let S_{ev} be generated by A and finite products of cyclotomic factors that are units at every sufficiently large admissible odd prime, excluding the persistent factor $A + 1$. Then

$$S_{\text{ev}}^{-1} \hat{\mathcal{J}} \cong \mathbb{Z}((A)) \times S_{\text{ev}}^{-1} \mathcal{H}_K.$$

This localization has a Laurent disk coordinate and only a cofinite root germ; exceptional root values are separate finite data. The reduced typed denominators belong to this fixed monoid. A disk-normalized quantum factorial contains only $\Phi_d(q)$ with $d \geq 2$, since its $\Phi_1(q)$ factor has canceled. Under $q = A^2$ these split into nonpersistent cyclotomic factors in A . The endpoint selectors contribute only finitely exceptional $\Phi_{2p}(A)$ factors, the monic changes P, P^{-1} are polynomial, and the remaining denominators are A -monomials. Thus no persistent $A + 1$ denominator is hidden in a typed matrix. The CRT split follows because $\bar{K}_N(0) = \pm 1$, so the disk and cyclotomic ideals are comaximal at each stage. It supplies no map identifying the two inverse-limit factors.

Define the discrete localized core $\mathcal{C}^{\text{loc}}$ by finite support in stable labels. A coefficient is an actual pair: the Laurent disk coordinate of one localized joint fraction and its activated cofinite root tail, augmented by an arbitrary finite root correction. Different fraction presentations change only that finite correction. This paired core does not assign a disk coordinate to an unpaired eventual root sequence. Put $v_J(x) = \sup\{N : x \in J_N \hat{\mathcal{J}}\}$, with $v_J(0) = \infty$. Define the all-slope quadratic rapid group $\mathcal{S}^{\text{quad}}$ by the condition that for every $s \geq 1$ there is c_s with

$$v_J(x_\lambda) \geq \max\{0, sh(\lambda)^2 - c_s\}.$$

Here $x_\lambda \in \hat{\mathcal{J}}$ and the bound holds for all labels with one c_s for each integer $s \geq 1$. Its zero-neighborhoods $V_{\Gamma; s, N}$ require $v_J(x_\lambda) \geq N + sh_\Gamma(\lambda)^2$ for every label. Finally set

$$\mathcal{B}^{\text{joint}} = \frac{\mathcal{C}_{\text{disc}}^{\text{loc}} \oplus \mathcal{S}^{\text{quad}}}{\{(i, -i) : i \in \mathcal{C}^{\text{loc}} \cap \mathcal{S}^{\text{quad}}\}}.$$

Lemma 5.4 (Topology of the joint bridge). *The rapid group is complete and Hausdorff with dense finite truncations. The diagonal intersection is closed, so $\mathcal{B}^{\text{joint}}$ is complete and Hausdorff.*

Proof. The joint object is formed by gluing a discrete finite-support core to the rapid sector along their common finite-support part. Coefficientwise limits preserve every slope estimate, finite truncations control the tail, and the closed diagonal intersection makes the quotient retain both completeness and separation.

Every regulator ideal is closed. A coefficientwise Cauchy limit retains each uniform slope estimate, hence lies in their intersection. Proper height makes a truncation finite and raises its omitted tail into any prescribed neighborhood. Separation follows from $\bigcap_N J_N \widehat{\mathcal{J}} = 0$. The rapid group is an open kernel in the bridge, whose quotient is the discrete core modulo its exact intersection. A Cauchy net is eventually in one open fiber, a translate of the complete rapid group, and converges there. Separation follows in that fiber or in the discrete quotient. Thus one discrete core is used for every slope and neighborhood. $\square$

5.4. The factorial theta–GM bridge. If U_{std} is the standard chart transition, P the normalized-GM polynomial unitriangular change, and S_{Θ}, S_L the selected endpoint diagonals just specified, the two orientations are

$$M_{\text{fac}} = S_L^{-1} P^{-1} U_{\text{std}}^{-1} S_{\Theta}, \quad N_{\text{fac}} = S_{\Theta}^{-1} U_{\text{std}} P S_L.$$

Their physical realizations are quotient-first, with the inverse selected scalar used only on an active coefficient line.

For a word of length ℓ , put

$$R_{\text{joint}}(H) = \max \left\{ (2E + 36\ell + 16)H, \left\lceil \frac{(80\ell + C_{\Theta} + C_L)H^2 + 1}{4} \right\rceil \right\}.$$

Theorem 5.5 (Joint factorial rapid bridge). *For fixed finite theta and normalized-GM charts and a fixed finite quotient-first word, the exactly typed factorial transition matrices have denominators dividing the regulator at $R_{\text{joint}}(H)$. They preserve the localized cores and extend to mutually inverse bicontinuous maps between the corresponding $\mathcal{B}^{\text{joint}}$ spaces. The rapid and exact intersection sectors admit coherent common disk/cyclotomic polynomial approximants.*

Proof. For a fixed word, the factorial estimate controls the cyclotomic loss and the two endpoint gauges control the disk loss. Their maximum is the joint regulator. Finite columns are handled in the localized core; rapid states are handled by choosing an input slope above that regulator and then using a stronger slope for the requested output. Finite-stage CRT supplies the common polynomial approximants.

The linear cyclotomic loss is Theorem 5.3; the two signed endpoint gauges give the quadratic disk loss. Their maximum is the displayed regulator: the monomial order $4R - 1$ clears $(80\ell + C_{\Theta} + C_L)H^2$, while the cyclotomic factor clears the linear stage, with one structural activation factor retained. This also holds for $\ell = 0$. Affine support for either orientation gives $h_{\text{out}} \leq ah_{\text{in}} + b$ and an input denominator stage $\lceil \delta_T h_{\text{in}}^2 \rceil$, with fixed a, b, δ_T . Fixed generic columns are finite and their denominators lie in S_{ev} , so the core is preserved; quotient-first disagreement is confined to finitely many root fibers. For a rapid state, choose once $t_0 > \delta_T$ and separate the fixed low-height core at which the denominator is absorbed. For a requested output slope s , take $t > \delta_T + s(a+b)^2$. The loss rule gives

$$v_J(T_{\mu\lambda}x_{\lambda}) \geq (t - \delta_T)h_{\text{in}}^2 - c_t \geq sh_{\text{out}}^2 - c_t.$$

Finite middle terms enlarge the output constant, not the core. These orders tend to infinity along each row, giving convergence. For continuity choose $t \geq \delta_T + s(a+b)^2 + 1$; then $T(V_{\text{in};t,N}) \subseteq V_{\text{out};s,N}$, with the activation reserve preventing a new finite-core correction. The core and rapid maps agree on their exact intersection. The inverse identities hold on dense finite support

and extend to the completions. Modulo a fixed regulator only finitely many labels remain; CRT supplies one polynomial with the required disk and root residues, compatibly through the inverse system. $\square$

The finite localized core has only finite-battery interpolation unless additional coherence is supplied. This is why finite representatives are stated on the rapid and intersection sectors only. At a rapid stage, choose $u_N, v_N \in \mathbb{Z}[A]$ with $u_N A^{4N-1} + v_N \overline{K}_N = 1$. For disk residue d_N and cyclotomic residue c_N , the single polynomial

$$f_N = c_N u_N A^{4N-1} + d_N v_N \overline{K}_N$$

has both residues; compatibility comes from the given inverse-limit element. For a localized fraction, a prescribed disk order and finite root battery instead give a Laurent-polynomial interpolant after clearing its monomial denominator and inverting finitely many integer resultants. For this interpolation the active root coefficients are in the reduced scalar image $\mathbb{Z}[A_p] = \mathbb{Z}[\zeta_p]$, with $A_p = e^{\pi i/p}$, or in its specified finite integer localization. Finite root corrections have the same finite-battery meaning when their coefficients have this type. The precise localization is given in the proof of Theorem 5.7; it does not enlarge the joint carrier or identify a separate dimension line with scalar coefficients. These interpolants are not asserted to be J -Cauchy or coherent at all roots.

5.5. Arbitrary-genus handlebody covariance. The handlebody group acts by banded spine moves, not by a chosen spatial factorization [24, 16, 30]. The factorial fixed-word estimates and ribbon/framing translation bound every such action in the joint rapid topology.

Theorem 5.6 (Handlebody rapid covariance). *For every fixed finite genus, the natural handlebody action extends to the localized joint rapid bridge, obeys the exact group law, and fixes the vacuum. The construction is independent of the chosen banded-spine factorization, and no hidden Maslov degree occurs on the handlebody side.*

Proof. A handlebody move is reduced to finitely many coupon, recoupling, and framing operations. The factorial bounds control the algebraic moves, the ribbon eigenvalue controls the framing height, and agreement on finite skeins passes the result to the joint completion. This is covariance of the selected handlebody action, not a new scalar stabilization theorem.

The banded handlebody-spine theorem expresses every handlebody map as a finite sequence of coupon, recoupling, and framing operations. The factorial bound controls the first two and the exact ribbon eigenvalue gives a quadratic height bound for the third. The all-slope extension argument from Theorem 5.5 gives a continuous action. Two factorizations represent isotopic handlebody cobordisms, so the generic skein maps agree on dense finite support and hence on the completion. The same reasoning proves the group law. The empty skein is fixed, and a handlebody mapping cylinder has zero relative Maslov contribution. $\square$

Theorem 5.7 (Handlebody chamber lattice and common orbit approximants). *The paired orbit lattices of the positive and negative handlebody actions form an arbitrary-genus chamber lattice.*

For the approximation assertion, fix a finite genus, finite output charts, and a finite collection of transported state/cotest pairs. Suppose that both coefficient families of each pair, in those output charts, belong to the embedded S^{quad} , with coefficients in the unlocalized $\hat{\mathcal{J}}$. For a finite-core output this means membership of its full paired data in $\mathcal{C}^{\text{loc}} \cap S^{\text{quad}}$, not merely regularity of a scalar contraction. Then the transported vectors have simultaneous finite-support $\mathbb{Z}[A]$ -vector approximants, compatible in their joint regulator residue classes and converging in every rapid seminorm. The transported vectors, their residues, and their defined pairings are word-representative independent; the polynomial lifts need not be unique.

For finite localized-core outputs, a separate conclusion holds at a prescribed disk precision $M \geq 0$ and a finite set $\mathcal{P}$ of admissible roots. Require one nonzero integer d , fixed for this

finite data set, to carry every prescribed active root coefficient into $\mathbb{Z}[A_p]$. With the finite coefficient localization specified below, there are simultaneous Laurent-polynomial interpolants matching the disk data modulo A^M and the prescribed root values. No stage coherence, J -Cauchy property, or all-root convergence is claimed for these interpolants.

Proof. Take the orbit of the declared joint carrier under the honest handlebody action and pair it with the contragredient cotest orbit, as before. Theorem 5.6 supplies the exact group law on these full carriers. Two words representing the same map therefore give the same transported state and cotest. Their disk/root residues agree, and naturality of the declared pairing makes its value independent of the orbit chart. This construction imposes no rapid-sector restriction.

Ordinary polynomial approximation uses the additional hypothesis on the outputs. The rapid subgroup embeds in $\mathcal{B}^{\text{joint}}$: the diagonal relation identifies a finite core with a rapid element only on their exact intersection. Thus the stated hypothesis supplies actual coefficient families $x_\lambda \in \hat{\mathcal{J}}$, including both disk and cyclotomic data, for every transported state and cotest. It is not an invariance assertion about initially rapid states. Theorem 5.5 preserves the enlarged joint bridge; its fixed-core argument alone does not put every output in the rapid subgroup.

For the regulator used here, the kernel of projection to stage L is $K_L \hat{\mathcal{J}}$. Indeed, divide compatible kernel residues at stages $T \geq L$ by K_L . Their quotient moduli K_T/K_L are cofinal with the regulator moduli, since $K_L K_m \mid K_{L+m}$; cancellation gives a compatible inverse-limit element. Moreover $K_L \hat{\mathcal{J}} \cap \mathbb{Z}[A] = K_L \mathbb{Z}[A]$, by projection to the polynomial quotient. Thus a coefficient lift with the right stage residues has the required J -valuation, and polynomial differences have the asserted regulator divisibility.

We spell out the simultaneous all-slope approximation. For $n \geq 1$, use the slope $n+1$ bounds of the finitely many output families to choose a common increasing height cutoff H_n , so that outside that cutoff

$$v_J(x_\lambda) \geq n + nh(\lambda)^2.$$

Proper height makes the retained support finite. Choose a common integer $L_n \geq n + nH_n^2$. For each retained coefficient take its disk residue modulo A^{4L_n-1} and its cyclotomic residue modulo $\bar{K}_{L_n}$. Since $\bar{K}_{L_n}(0) = \pm 1$, the integral CRT formula preceding this theorem gives one coefficient in $\mathbb{Z}[A]$ with both residues. Put zero on the omitted labels and call the resulting polynomial vector F_n . Then

$$x - F_n \in V_{\Gamma;n,n}.$$

Hence F_n converges in every $V_{\Gamma;s,N}$. At stage n its residue is the given residue of x ; these classes are compatible under the natural quotient transitions because x is one inverse-limit element. In particular the coefficients of $F_{n+1} - F_n$ lie in $K_n \mathbb{Z}[A]$. The same cutoff and stage work for the entire finite collection. At any root whose cyclotomic factor has entered that stage, quotient-first evaluation gives the prescribed active root residues. Different polynomial lifts can differ by the regulator ideal; the residue classes, rather than a canonical choice of lifts, are determined by the transported data.

For the separate finite-battery assertion, let z_λ^{disk} be the finitely many Laurent disk coordinates. Choose $b \geq 0$ with $A^b z_\lambda^{\text{disk}} \in \mathbb{Z}[[A]]$ for all of them. In the reduced scalar branch put

$$\pi_p(A) = \Phi_{2p}(A), \quad \Delta = \left| d \prod_{\substack{p,r \in \mathcal{P} \\ p < r}} \text{Res}(\pi_p, \pi_r) \right|, \quad R_{\mathcal{P}} = \mathbb{Z}[1/\Delta].$$

Here d clears the denominators of the prescribed finite root coefficients; it can be 1 for integral scalar data. Distinct admissible primes have resultant ± 1 , and $\pi_p(0) = 1$, so in this prime battery $\Delta = |d|$; no further integer inversion is necessary. For the resultant assertion, use

$\pi_p(A) = \Phi_p(-A)$: the quotient of the products of $\zeta_p^{rj} - 1$ and $\zeta_p^j - 1$, for $1 \leq j < p$, is 1, since multiplication by the distinct prime r permutes the nonzero residues modulo p . The ideals (A^{M+b}) and (π_p) are pairwise comaximal over $R_{\mathcal{P}}[A]$. Omit the disk ideal if $M + b = 0$. Choose the disk residue of $A^b z_{\lambda}^{\text{disk}}$ modulo A^{M+b} , and the root residue $A_p^b z_{\lambda,p}$ modulo π_p . Finite CRT gives $P_{\lambda} \in R_{\mathcal{P}}[A]$ with all these residues. Then

$$A^{-b} P_{\lambda} - z_{\lambda}^{\text{disk}} \in A^M R_{\mathcal{P}}[[A]], \quad (A^{-b} P_{\lambda})(A_p) = z_{\lambda,p}.$$

Inactive lines remain absent under quotient-first activation. This construction matches only the specified finite scalar data. It supplies neither an integral joint inverse-limit lift of arbitrary finite root corrections nor compatibility between independently chosen batteries. $\square$

Why the approximation domain is necessary. Let e_0 be the unscaled empty skein. The identity word and $A^{-1}e_0$, paired with the unit empty cotest, are allowed in the localized core because $A \in S_{\text{ev}}$; their root tail is A_p^{-1} . If the selected vacuum generator is $A^{2b_0}e_0$, the state has coefficient $A^{-(2b_0+1)}$ in that chart. For every $f \in \mathbb{Z}[A]$,

$$\text{ord}_A(A^{-1} - f) = -1.$$

The regular disk coordinate of an exact-intersection element cannot remove this pole. Thus finite support or a finite word does not imply ordinary polynomial vector approximation on the full localized carrier. Clearing the monomial and dividing it back gives the Laurent finite-battery statement, not the ordinary polynomial conclusion.

5.6. Split zero-handle stabilization. Let I_g add a zero-colored handle and let P_g remove it. They are defined generically, at every reduced root-of-unity fiber, and on finite-core corrections.

Theorem 5.8 (Split zero-handle state-space stabilization). *The maps I_g and P_g extend continuously to the joint handlebody chambers and satisfy*

$$P_g I_g = \text{id}.$$

The selected multiplier has only linear clearing loss. Hence genus raising is a split inclusion of state spaces in every finite genus.

Proof. The split zero-handle maps are checked first on finite graph states, where the tensor-unit and bubble identities give the retraction. The zero-edge factorial ratio has only linear clearing loss, so the maps are continuous in each joint slope; density and Hausdorffness then extend the retraction to the completed chambers, including the declared finite exceptional-root data.

The tensor-unit and bubble identities give $P_g I_g = \text{id}$ on finite graph states. The factorial ratio for a zero edge has an explicit linear bound, so, for each output slope, an input slope absorbing that linear loss makes both maps continuous. Dense finite support and Hausdorffness extend the identity to the completions, including finite exceptional-root corrections. $\square$

Remark 5.9 (State-space versus scalar stabilization). Theorem 5.8 is not by itself normalized Heegaard stabilization. It supplies the split state-space factor. The scalar theorem also requires both handlebody caps, the finite-root anomaly line, and closed descent. The closed-IHS cap interfaces are Propositions 9.1 and 9.2; normalized arbitrary-genus Heegaard scalars and their Habiro descent are treated in [34, Theorems 5.8 and 5.13].

6. THE ROOTWISE GRAPH AND THE CANONICAL THETA ENDPOINT

The strict rootwise/Rees module records a formal Rees coordinate together with separately typed reduced-root coordinates. This section constructs their compatible finite-stage representatives and the theta endpoint, then identifies the boundary of the construction. The formal Rees reading here is distinct from the disk coordinate of the marked boundary-module joint completion.

Keep a fixed marked handlebody and its finite boundary states in view. A finite-stage representative is one finite state before the reduced-root and formal Rees readings are taken. Rootwise saturation supplies the integral fibers, the generic fixed-label trunk supplies the formal coordinate, and strict closure selects pairs with these common finite-stage representatives. Their compatibility records a common origin; it does not identify the completed targets or evaluate the entire Rees completion.

The section first builds the finite-prime theta lattice and the generic fixed-label trunk, then identifies the continuity obstruction that prevents the full Rees completion from being root-valued. It next characterizes the strict graph of compatible finite-stage representatives and adjoins finite-support root data, before transporting the result to the canonical theta endpoint. The final Newton pullback records the maximal two-sided domain on which the rootwise and Rees readings can be combined; it does not enlarge either completion beyond that typed domain.

6.1. Rootwise saturation in the theta chart. Put

$$R = \mathbb{Z}[A, A^{-1}], \quad \lambda = 1 + A, \quad q = -A = 1 - \lambda, \quad R^{\text{ev}} = S_{\text{ev}}^{-1} R,$$

where S_{ev} contains precisely the previously declared eventually invertible cyclotomic factors and does not contain λ . Let

$$\mathcal{R}_\lambda = \varprojlim_N R^{\text{ev}} / (\lambda^N).$$

The factor $\Phi_4(A) = A^2 + 1$ is inverted and is congruent to 2 modulo λ ; hence 2 is a unit in $\mathcal{R}_\lambda$.

Fix a finite genus g , a lollipop spine L_g , and a theta/ladder spine Γ_g in the same handlebody. At an odd prime $p \geq 5$, put

$$d_p = \frac{p-1}{2}, \quad \mathcal{O}_p = \mathbb{Z}[\zeta_p], \quad h_p = 1 + A_p.$$

For a stable small-lollipop label $\alpha = (a, b, c)$, with trunk half-color e and $s_\alpha = \sum_i a_i$, define

$$\kappa_\alpha = \sum_i b_i + \left\lfloor \frac{s_\alpha - e}{2} \right\rfloor, \quad \kappa_\alpha^\# = \sum_i b_i + \left\lceil \frac{s_\alpha + e}{2} \right\rceil, \quad \nu_\alpha = \kappa_\alpha^\# - \kappa_\alpha.$$

If $w_{\alpha,p}^L$ is the saturated lollipop frame and $U_{\Gamma L,p} \in \text{GL}(\mathcal{O}_p)$ is the actual quotient-first endpoint matrix, put $w_{\alpha,p}^\Gamma = U_{\Gamma L,p} w_{\alpha,p}^L$.

The indices κ_α and $\kappa_\alpha^\#$ measure the primal and dual denominator clearings, while ν_α is the remaining gap. The matrix $U_{\Gamma L,p}$ transports those clearings from the lollipop frame to the theta chart.

Theorem 6.1 (All-prime theta saturation). *For every fixed finite genus and every odd prime $p \geq 5$, the transported numerator frame $Q_{\alpha,p}^\Gamma = h_p^{\kappa_\alpha} w_{\alpha,p}^\Gamma$ is an $\mathcal{O}_p$ -basis of the native graph lattice $\mathcal{G}_{\Gamma,p}$, while $\{w_{\alpha,p}^\Gamma\}$ is an orthogonal basis of the invariant lattice $\mathcal{S}_{p,g}^+$. There is an exact typed chain*

$$\mathcal{G}_{\Gamma,p} \subset \mathcal{S}_{p,g}^+ \subset (\mathcal{S}_{p,g}^+)^\# \subset \mathcal{G}_{\Gamma,p}^\#,$$

whose three successive diagonal profiles are respectively $\kappa_\alpha, \nu_\alpha, \kappa_\alpha$. The full graph-pairing profile is

$$2\kappa_\alpha + \nu_\alpha = \kappa_\alpha + \kappa_\alpha^\# = 2 \sum_i b_i + s_\alpha.$$

For the closed uncolored genus-two theta spine, $\nu_\alpha = 0$ and $\kappa_\alpha = \kappa_\alpha^\# = b_1 + b_2 + a$; hence the invariant lattice is the canonical self-dual midpoint.

Proof. The argument first moves the rootwise lollipop frame into the theta frame by a quotient-first unit matrix. It then tracks the powers of h_p in the numerator frame and in its pairing norm. The resulting diagonal calculation is what produces the three inclusions and the self-dual midpoint; no prime-independent frame is being constructed.

The small-to-even reflection in the reduced root-of-unity quotient is a monomial unit matrix. A quotient-first Whitehead move between all-even frames has only quantum integers of index below p in its retained denominators, so it and its inverse lie in $\mathrm{GL}(\mathcal{O}_p)$. The lollipop numerators are unitriangular in the graph frame and satisfy $Q_{\alpha,p} = h_p^{\kappa_\alpha} w_{\alpha,p}$. Their orthogonal norms are units times $h_p^{\nu_\alpha}$. Transport preserves the intrinsic states and pairing, so all four lattices become diagonal in the displayed frame. Taking diagonal quotients gives the three profiles. The identity $\kappa_\alpha + \kappa_\alpha^\sharp = 2 \sum_i b_i + s_\alpha$ gives the full profile. In closed genus two, $e = 0$, $a_1 = a_2 = a$, and the middle profile vanishes. $\square$

The theorem is rootwise. The active label set and the small/even reflection depend on p , so it does not manufacture one prime-independent generic theta matrix.

6.2. The cofinite generic trunk. Write $z_i = hv_i - 2$, where $h = 1 - q = \lambda$, and put

$$\beta_m(q) = \frac{2 - q^m - q^{-m}}{1 - q}.$$

Theorem 6.2 (Fixed-label generic saturated trunk). *For every fixed stable label $\alpha = (a, b, c)$, the frozen planar lollipop banding has a vector in the unspecialized coefficient parameter*

$$T_\alpha(A) = h^{-\lfloor (s_\alpha - e)/2 \rfloor} g(a, 0, c) \quad \text{over } R^{\mathrm{ev}},$$

and a finite common column

$$W_\alpha^L(A) = T_\alpha(A) \prod_{i=1}^g \prod_{j=a_i}^{a_i+b_i-1} (v_i - \beta_{j+1}(q)).$$

At every sufficiently large active prime this column specializes exactly to $w_{\alpha,p}^L$. Exceptional active primes retain their separately typed integral reductions.

Proof. For one fixed label, the generic calculation is local: braid the bands, expand the eyeglass and tripod projectors, and count the factors of $h = 1 - q$. The finite set of denominators is harmless on the eventual prime tail, and the last displayed factor records the remaining color dependence. This explains both the common column and the exceptional-prime qualification.

The pairing-change braid on $2a$ strands has permutation $[1, 2a, 2, 2a-1, \dots, a, a+1]$, whose inversion number $a(a-1)$ is even. Jones–Wenzl absorption therefore contributes an even Laurent monomial in the bracket square root, hence a Laurent monomial in q . Expanding the color-two eyeglass and tripod projectors gives literal generic formulas divisible by h ; the tripod identity is

$$z_{123} + [2]^{-1}(z_1 z_{23} + z_2 z_{13} + z_3 z_{12}) + 2[2]^{-2} z_1 z_2 z_3.$$

After $z_I = hv_I - 2$, it is divisible by h . The three-ball reduction produces at least $\lfloor (s_\alpha - e)/2 \rfloor$ such pieces, proving the declared trunk divisibility. Only finitely many quantum integers and the factor $\Phi_{12}(q)$ occur for fixed α ; they are units at every sufficiently large active prime. Finally, $(z_i + q^{j+1} + q^{-j-1})/h = v_i - \beta_{j+1}(q)$, which gives the column. $\square$

6.3. Why the Rees completion is not a root-valued object.

Theorem 6.3 (Obstruction to root specialization on the completed Rees source). *For each odd prime $p \geq 5$, no continuous homomorphism from all of $\mathcal{R}_\lambda$ to the reduced p -th coefficient field extends $A \mapsto A_p$. This is false even with the h_p -adic topology on the target. Consequently neither the uniform Rees product nor its c_0 submodule has root specialization on every element.*

Proof. The proposed evaluation would have to send an element that is invertible in the localized Rees ring to zero at the chosen admissible reduced root. The explicit series gives the independent topological obstruction: after substitution, its terms fail to tend to zero in the target adic topology. The failure is therefore continuity of the ambient completion, not a missing formula.

The factor $\Phi_{2p}(A)$ is in S_{ev} , because it fails at only the p -th reduced root-of-unity fiber. It is therefore a unit in $\mathcal{R}_\lambda$, but any extension of $A \mapsto A_p$ would send it to zero. There is also a topological obstruction. Since $\Phi_{2p}(-1) = p$, the element p is a unit in the localized completion, so

$$\sum_{m \geq 0} p^{-m^2} \lambda^m \in \mathcal{R}_\lambda.$$

Under the formal substitution $\lambda = h_p$, its m -th term has h_p -valuation $m - (p-1)m^2 \rightarrow -\infty$. The terms do not tend to zero. $\square$

This theorem forces a graph of compatible finite-stage representatives rather than an evaluation map.

6.4. The strict common rootwise/Rees module. Let Λ_g^{st} be the stable label set and let $\mathfrak{P}(\alpha)$ be the cofinite set of primes at which α is active. Let $R_\alpha^{\text{act}} \subset R^{\text{ev}}$ consist of fractions having a reduced denominator that is a unit at every incidence (α, p) with $p \in \mathfrak{P}(\alpha)$, and put

$$\mathcal{A}_g^{\text{fin}} = \bigoplus_{\alpha \in \Lambda_g^{\text{st}}} R_\alpha^{\text{act}} e_\alpha.$$

Define

$$\mathcal{P}_{\lambda, g}^0 = c_0(\Lambda_g^{\text{st}}, \mathcal{R}_\lambda), \quad \mathcal{R}_g^+ = \prod_{p \geq 5} \mathcal{S}_{p, g}^+,$$

where the first factor has the uniform λ -adic topology and the second is a product of discrete finite modules. For finite $x = (x_\alpha)$, set

$$j_L(x) = \left((\text{Taylor}_\lambda x_\alpha)_\alpha, \left(\sum_{\alpha \in \Lambda_{g, p}} \text{Ev}_p(x_\alpha) w_{\alpha, p}^L \right)_{p \geq 5} \right).$$

Definition 6.4 (Strict saturated rootwise/Rees module). The strict lollipop module is

$$\mathcal{C}_{L, g}^{\text{sat}} = \overline{j_L(\mathcal{A}_g^{\text{fin}})} \subset \mathcal{P}_{\lambda, g}^0 \times \mathcal{R}_g^+.$$

Theorem 6.5 (Characterization by compatible finite-stage representatives). *The module is complete and Hausdorff, and its finite common states are dense. Enumerate the odd primes $p_1, p_2, \dots$. A pair $(x, (X_p)_p)$ belongs to the module exactly when, for every N , one finite $a_N \in \mathcal{A}_g^{\text{fin}}$ satisfies*

$$\text{Taylor}_\lambda(a_N) - x \in \lambda^N \mathcal{P}_{\lambda, g}^0, \quad (j_L(a_N))_{p_j} = X_{p_j} \quad (1 \leq j \leq N).$$

The Rees projection

$$\text{pr}_\lambda : \mathcal{C}_{L, g}^{\text{sat}} \longrightarrow \mathcal{P}_{\lambda, g}^0, \quad (x, (X_p)_p) \longmapsto x,$$

is injective. Thus the module is the graph of a uniquely determined root family over the proper submodule $\text{pr}_\lambda(\mathcal{C}_{L, g}^{\text{sat}})$; it is a proper submodule of the ambient product and contains no independent root-product summand.

Proof. The module is defined as the closure of finite states, so membership is tested by a finite-stage representative that matches a prescribed Rees precision and a finite list of root coordinates. If the Rees coordinate vanishes, the activation-safe denominator condition forces every root coordinate into arbitrarily high h_p -power. Krull intersection then removes the apparently independent root component.

Both ambient factors are complete and Hausdorff, and the c_0 condition is closed. Their basic neighborhoods are precisely a uniform Rees congruence and finitely many exact discrete-root conditions, proving the approximant criterion. For injectivity, suppose the Rees coordinate is zero. Fix p and N . A compatible finite-stage representative has coefficients in $\lambda^N \mathcal{R}_\lambda \cap R^{\text{ev}}$. Activation-safe denominators are prime to λ , so Gauss's lemma gives

$$\lambda^N \mathcal{R}_\lambda \cap R^{\text{ev}} = \lambda^N R^{\text{ev}}.$$

Its root-fiber coefficient is therefore divisible by h_p^N . Since the $w_{\alpha,p}^L$ are an $\mathcal{O}_p$ -basis, $X_p \in h_p^N \mathcal{S}_{p,g}^+$ for every N . Krull intersection in the finite free module forces $X_p = 0$, for every p . Hence pr_λ is injective. The ambient root product is nonzero, so a pair $(0, (Y_p)_p)$ with some $Y_p \neq 0$ belongs to the ambient product but not to the module. This proves properness and the absence of an independent root-product summand. $\square$

Theorem 6.6 (Finite-prime-support data must be adjoined after closure). *Let $\mathcal{D}_g^+ = \bigoplus_{p \geq 5} \mathcal{S}_{p,g}^+$ with the discrete topology. The correct extension is the external direct sum*

$$\mathcal{B}_{L,g}^{\text{sat}} = \mathcal{C}_{L,g}^{\text{sat}} \oplus (\mathcal{D}_g^+)^{\text{disc}}.$$

It is complete and Hausdorff and has an injective algebraic realization in the Rees/root ambient object,

$$\rho((x, (X_p)_p), (d_p)_p) = (x, (X_p + d_p)_p).$$

This realization is algebraic, not a topological embedding: the defect summand is declared discrete, while its induced root-product topology is nondiscrete. If the finite-root corrections are inserted before taking closure, their closure contains the whole independent root product and destroys the common-origin condition.

Proof. The closure module already controls the common-origin condition. The finite-root corrections are therefore adjoined as a separately declared discrete direct sum, and the injective realization simply adds them to the root coordinates. Taking that correction before closure would erase the distinction by filling the whole independent product.

If $\rho((x, X), d) = 0$, then $x = 0$. Injectivity of the strict Rees projection gives $X = 0$, after which $d = 0$; hence ρ is injective. A discrete module and the strict rootwise/Rees module are complete Hausdorff, so their external topological direct sum is complete Hausdorff. The algebraic direct sum of the discrete reduced root-of-unity fibers is dense in their product. Placing it in the dense core before closure therefore produces $\{0\} \times \prod_p \mathcal{S}_{p,g}^+$. $\square$

6.5. Canonical theta transport. Define j_Γ by applying $U_{\Gamma L,p}$ only in the p -th reduced-root coordinate, while retaining the same saturated-frame coefficient chart in the Rees coordinate, and put $\mathcal{C}_{\Gamma,g}^{\text{sat}} = j_\Gamma(\mathcal{A}_g^{\text{fin}})$.

Theorem 6.7 (Arbitrary-genus canonical theta endpoint). *The map*

$$\left(\text{id}, \prod_{p \geq 5} U_{\Gamma L,p} \right) : \mathcal{C}_{L,g}^{\text{sat}} \xrightarrow{\sim} \mathcal{C}_{\Gamma,g}^{\text{sat}}$$

is inverse bicontinuous, extends to the discrete defect summands, and is independent of every quotient-first Whitehead factorization. Sharp cotests transport contragrediently. In closed genus two the primal and sharp theta endpoints have the self-dual midpoint identification.

Proof. The canonical endpoint is obtained by transporting the saturated lollipop frame through the quotient-first theta matrix and then taking the prescribed closed uncolored specialization. The rootwise diagonal profiles collapse at the theta endpoint, while the earlier evaluation obstruction prevents the same conclusion from being read as a global Rees evaluation.

Every root-specialized endpoint matrix is an isomorphism of a finite discrete module, so their product is a homeomorphism. It sends the finite lollipop core exactly to the finite theta core and

therefore sends one closure onto the other. Rootwise bigon, square, and pentagon identities give path independence. The inverse transpose gives the sharp statement; the genus-two assertion is the case $\nu_\alpha = 0$ of Theorem 6.1. $\square$

Remark 6.8 (Theta is canonical but not prime-independent). The reduced-root state graph is the theta/ladder graph, with theta canonical in genus two. The Rees coordinate remains the transported saturated-frame coefficient chart. The theorem does not assert one generic theta transition matrix specializing at every prime.

6.6. The maximal genus-one Newton pullback. For genus one define

$$r_i(A) = \left(\frac{A^{2i+1} - A^{-2i-1}}{\lambda} \right)^2, \quad W_n(A, v) = \prod_{i=0}^{n-1} (\lambda v^2 - 4v - \lambda r_i) = \sum_k t_{kn}(A) v^k.$$

Theorem 6.9 (Exact Newton matrix). *One has $t_{kn} \neq 0$ exactly for $0 \leq k \leq 2n$, and throughout that support*

$$\text{ord}_\lambda(t_{kn}) = |k - n|, \quad t_{nn} \equiv (-4)^n \pmod{\lambda^2}.$$

After normalizing the diagonal by $(-4)^n$, the matrix is $I + N$ with N raising uniform λ -order. It therefore has a Neumann inverse on the localized uniform product and on its λ -adic c_0 submodule. The older all-slope rapid bridge is not surjective: the unique inverse Newton expansion of v has coefficients

$$u_n = \lambda^{-n} g[r_0, \dots, r_n], \quad \text{ord}_\lambda(u_n) = n - 1,$$

where

$$g(y) = 2\lambda^{-1} \left(1 - \sqrt{1 + \lambda^2 y/4} \right),$$

with nonintegral leading constants at infinitely many labels.

Proof. The product expansion is organized by how far a coefficient lies from the middle power of v : each off-middle choice contributes one visible λ -order, while the middle coefficient has no order-one correction. After diagonal normalization this makes the Newton operator a uniformly topologically nilpotent perturbation of the identity, so the inverse is a Neumann series; interpolation of the displayed branch then identifies the older rapid-tail obstruction.

In expanding W_n , choose constant, linear, and quadratic terms. The unique term of smallest λ -order gives $|k - n|$, and the diagonal has no order-one correction. Thus the normalized off-diagonal entries are in $\lambda\mathcal{R}_\lambda$, while diagonal corrections are in $\lambda^2\mathcal{R}_\lambda$, so the Neumann series converges uniformly. Newton interpolation of the displayed branch $g(y)$ gives the inverse. Its leading coefficient is $(-1)^n(2n - 3)!!/(2^{3n-1}n!)$, which excludes the integral old rapid tail; its disk order also has only one fixed quadratic slope rather than every slope required there. $\square$

At an admissible odd prime define the quotient-first monomial module

$$V_p = \mathcal{O}_p[v]/(Q_{d_p,p}(v)), \quad Q_{d_p,p}(v) = \prod_{i=0}^{d_p-1} (v - \rho_{i,p}), \quad \rho_{i,p} = \frac{2 - q_p^{i+1} - q_p^{-i-1}}{h_p}.$$

Let $\mathcal{C}_v^{\text{base}}$ be its strict graph of compatible finite-stage representatives, let $\mathcal{C}_W = \mathcal{C}_{L,1}^{\text{sat}}$, and let T_{pair} denote the Rees Newton transform together with the finite quotient-first matrices.

Proposition 6.10 (The full strict rootwise/Rees module is not Newton invariant).

$$T_{\text{pair}}(\mathcal{C}_W) \not\subset \mathcal{C}_v^{\text{base}}.$$

The obstruction occurs at $p = 5$, source label 2: the activation-safe coefficient $\Phi_{10}(A)^{-1}$ produces the target vacuum coefficient

$$\frac{t_{0,2}}{\Phi_{10}(A)} = \frac{144}{5} \lambda^2 + O(\lambda^3),$$

which cannot be the Taylor germ of an all-incidence-safe target coefficient.

Proof. The no-go is detected at one inactive prime rather than by comparing whole completed modules. An activation-safe source denominator is allowed to produce the indicated 5-adic pole at the inactive label, whereas a target coefficient safe at every active incidence must be 5-integral. The first nonintegral Taylor coefficient therefore blocks the criterion for compatible finite-stage representatives.

Label 2 is inactive at $p = 5$, so $\Phi_{10}^{-1}e_2$ is allowed in the source core. Every target denominator must be a unit at $p = 5$, and therefore every Taylor coefficient of an all-safe target fraction is 5-integral. The coefficient $144/5$ violates the target criterion for compatible finite-stage representatives modulo λ^3 . $\square$

Theorem 6.11 (Maximal genus-one two-sided transform). *Put*

$$\mathcal{C}_W^T = \mathcal{C}_W \cap T_{\text{pair}}^{-1}(\mathcal{C}_v^{\text{base}}).$$

Then $\mathcal{C}_W^T$ is closed, complete, and Hausdorff, and

$$T_{\text{pair}} : \mathcal{C}_W^T \xrightarrow{\sim} \mathcal{C}_v^{\text{base}}$$

is inverse bicontinuous. This is maximal among submodules of $\mathcal{C}_W$ whose paired image lands in the monomial module. Inverse images of the finite target core are dense; finite-support source vectors need not be dense in the pullback.

Proof. The admissible two-sided object is cut out before inversion by the pullback condition. On the dense finite target core, divided-difference formulas give finite Newton rows whose active-incidence denominators are units; truncations then converge in the Rees c_0 topology and stabilize in each finite quotient. Density and closedness extend this finite-stage inverse to the maximal module.

Closedness and maximality follow from the defining pullback. For a target monomial cv^k , the inverse row- n coefficient is $\lambda^{-n}(g^k)[r_0, \dots, r_n]$. Divided-difference denominators are products of node differences that are units at every active incidence, and the numerator contains the required λ^n before reduction. Finite truncations therefore belong to the source finite core, converge uniformly in Rees c_0 , and become exact in the p -th quotient once all its active rows have appeared. The compatible finite-stage representative criterion puts the full inverse of the dense finite target core in $\mathcal{C}_W$. Continuity and closedness extend this to the whole target module. $\square$

Proposition 6.12 (Completed padding no-go). *For a finite monomial vector, quotient-first reduction differs from active-zero padding at only finitely many primes and is repaired by the discrete defect summand. There exists, however, a c_0 common vector whose discrepancy is nonzero at every prime in an infinite sequence. Hence finite-support root data do not produce a completed padded isomorphism.*

Proof. Finite monomials have only finitely many primes at which their quotient degree reaches the root-admissibility cutoff, so their padding discrepancy has finite prime support. The completed witness chooses a new cutoff at each prime and uses earlier cyclotomic factors to freeze earlier coordinates; this turns the same finite-support root discrepancy into an infinite-support sequence, beyond the discrete finite-support root repair.

For v^k , a discrepancy can occur only when $k \geq d_p$, hence at finitely many primes. For the completed witness enumerate p_j , set $k_j = d_{p_j}$, and take coefficients $\lambda^j \prod_{i < j} \Phi_{2p_i}(A)$. Later terms vanish exactly at each earlier root, so the quotient-first coordinates stabilize, but at p_j the j -th inactive monomial has a nonzero reduction modulo $Q_{d_{p_j}, p_j}$. The discrepancies therefore have infinite support. $\square$

Remark 6.13 (Boundary of the strict rootwise/Rees module). The results above do not give root evaluation on $\mathcal{R}_\lambda$, a disk or Habiro value on the completed module, a prime-independent generic theta matrix, arbitrary nonhandlebody mapping-class operators, an anomaly descent,

or a closed Heegaard scalar. The next section constructs the exact genus-one image modules of the Hopf-pairing maps and primitive caps; the following section treats the perturbative and operational closed comparisons. For the arbitrary-genus normalized Heegaard scalar obtained by contracting finite-root data before Habiro descent, see [34, Theorems 5.8 and 5.13].

7. HOPF-PAIRING IMAGES, PRIMITIVE CAPS, AND THE SINGLE- S BOUNDARY

This section constructs the permitted pairing-image and cap outputs, then tests which scalar agreements admit a state-level comparison.

Figure 1 is the genus-one instance to keep in view: the maps in this section act on the reduced root-of-unity fiber, and the primitive caps contract that fiber only after the pairing-induced map has been typed. Neither operation reverses the finite-stage arrows in the figure.

This section separates four objects which have similar finite-root formulas but different mathematical types:

- a strict common rootwise/Rees state,
- an honest line-valued root-product state,
- a scalar obtained only after link-functional contraction,
- a generic boundary state over a fraction field.

Only the first three occur here, and they are never identified. The source constructions are organized below by the operation they support: transport, closed contraction, or a boundary-level single- S test.

Half-Smith transport preserves the coefficient vector while keeping the normalization line separate. The maximal Newton-compatible pullback makes these typed maps available on closed images. The two primitive caps then select coefficient zero and identify the common scalar graph, into which the Habiro ring supplies finite-stage representatives with compatible formal and root/cap readings. The normalized single- S color is tested after contraction, while the honest line-valued root state is kept separate. The final no-go results and the exact AS0 radical mark which state-level comparisons fail and which finite quotient statement remains available.

7.1. Coefficient-preserving Hopf-pairing frames. Fix an odd prime $p \geq 5$, put $d_p = (p-1)/2$, and write

$$\mathcal{O}_p = \mathbb{Z}[A_p], \quad \lambda_p = 1 + A_p, \quad \Lambda_p = \text{diag}(1, \lambda_p, \dots, \lambda_p^{d_p-1}).$$

The exact Smith factorization in Newton coordinates and Hopf square are

$$H_p C_p = L_p \Lambda_p^2, \quad C_p, L_p \in \text{GL}_{d_p}(\mathcal{O}_p), \quad H_p^2 = \delta_p I.$$

Let $\mathbf{d}_p$ generate the normalization line with $\mathbf{d}_p^2 = \delta_p$, and put $\mathbf{d}_p^\vee = \mathbf{d}_p^{-1}$. Define

$$w_{n,p} = \lambda_p^{-n} C_p e_n, \quad g_{n,p} = \lambda_p^n L_p e_n, \quad f_{n,p} = \mathbf{d}_p^\vee g_{n,p}.$$

The first frame spans the saturated source $\mathcal{S}_p^+$; the other two span the lowered-Hopf and raised, line-valued output lattices. Here C_p, L_p are the explicit polynomial matrices of Section 4, and their GM comparison is (2). The division by λ_p^n belongs to the invariant saturation. The factor $\mathbf{d}_p^\vee$ belongs to the inverse dimension line and is retained separately.

Lemma 7.1 (Half-Smith transport).

$$H_p w_{n,p} = g_{n,p}, \quad \mathbf{S}_p^+ w_{n,p} = f_{n,p}, \quad \mathbf{S}_p^+ = \mathbf{d}_p^\vee H_p.$$

Thus both arrows preserve the coefficient vector in their displayed typed frames. The oppositely oriented cylinder $\mathbf{S}_p^- : f_{n,p} \mapsto w_{n,p}$ is the inverse after line contraction.

Proof. The Smith factorization is used as a frame change: the inverse diagonal factor clears the source frame, the Hopf matrix produces the integral target frame, and the normalization line supplies the oppositely typed inverse. Thus coefficient preservation is a statement in the displayed frames, not an untyped identification of the two cylinders.

Multiply $H_p C_p = L_p \Lambda_p^2$ on the right by Λ_p^{-1} . Its n -th column is the first identity; the second is obtained by the separately typed line generator. Finally, $(\mathbf{d}_p^\vee)^2 H_p^2 = 1$. $\square$

Let $\mathcal{C}_W^T$ be the maximal Newton-compatible pullback of Theorem 6.11. On its formal coefficient copy the following arrows preserve coefficients; on its p -th root coordinate they apply H_p and $\mathbf{S}_p^+$. Denote the paired maps by $\mathcal{H}_{\text{pair}}$ and $\mathcal{S}_{\text{pair}}^+$, and define

$$\mathcal{C}_H^T = \mathcal{H}_{\text{pair}}(\mathcal{C}_W^T), \quad \mathcal{C}_S^T = \mathcal{S}_{\text{pair}}^+(\mathcal{C}_W^T).$$

Theorem 7.2 (Strict images of the Hopf-pairing maps). *The carriers $\mathcal{C}_H^T$ and $\mathcal{C}_S^T$ are closed, complete, and Hausdorff, and*

$$\mathcal{H}_{\text{pair}} : \mathcal{C}_W^T \xrightarrow{\sim} \mathcal{C}_H^T, \quad \mathcal{S}_{\text{pair}}^+ : \mathcal{C}_W^T \xrightarrow{\sim} \mathcal{C}_S^T$$

are inverse bicontinuous isomorphisms onto their images. The second inverse is the oppositely oriented typed cylinder, not an untyped reuse of the same matrix.

Moreover, with T_{pair} from the preceding section,

$$\mathcal{S}_{\text{mix}} = \mathcal{S}_{\text{pair}}^+ T_{\text{pair}}^{-1} : \mathcal{C}_v^{\text{base}} \xrightarrow{\sim} \mathcal{C}_S^T$$

is bicontinuous. The compatible triple graph

$$\{(u, T_{\text{pair}} u, \mathcal{S}_{\text{pair}}^+ u) : u \in \mathcal{C}_W^T\}$$

is maximal among graphs with source in the full strict Newton carrier and monomial coordinate in $\mathcal{C}_v^{\text{base}}$.

Proof. The two Hopf-pairing maps are first checked on the displayed integral frames, where the half-Smith factors cancel coefficientwise. Their ambient product maps then transport the closed Newton-compatible pullback to closed images. The mixed map is the same comparison composed with the previously fixed Newton identification; no new rootwise choice is made at this stage.

The root maps send an integral basis to an integral basis by Lemma 7.1; the formal map is the identity between typed coefficient copies. Their ambient product maps are therefore homeomorphisms. They transport the closed, complete pullback $\mathcal{C}_W^T$ to closed, complete images. The mixed arrow is the composition with the already proved Newton isomorphism. A source point has an admissible monomial coordinate exactly when it belongs to $T_{\text{pair}}^{-1}(\mathcal{C}_v^{\text{base}}) = \mathcal{C}_W^T$, which proves maximality. $\square$

7.2. Primitive caps and the vacuum pair. Let $y_p = e_0^T$, define the complementary cap $h_p = y_p H_p$, and define the identity-double cap by the line-valued row $\hat{y}_p = \mathbf{d}_p y_p$. Let $R^{\text{all}} \subset R^{\text{ev}}$ be the subring of fractions whose reduced denominators are units at every declared admissible reduced root. Let

$$\mathcal{C}_{\text{sc}} = \overline{\{(\text{Taylor}_\lambda c, (\text{Ev}_p c)_p) : c \in R^{\text{all}}\}} \subset \mathcal{R}_\lambda \times \prod_{p \geq 5} \mathcal{O}_p$$

be the rank-one graph of compatible formal and root/cap readings. It is not the evaluation graph of all $\mathcal{R}_\lambda$.

Theorem 7.3 (Primitive two-cap vacuum). *For all active n ,*

$$h_p(w_{n,p}) = \delta_{n0}, \quad \hat{y}_p(f_{n,p}) = \delta_{n0}.$$

The paired caps extend continuously to

$$h_W : \mathcal{C}_W^T \rightarrow \mathcal{C}_{\text{sc}}, \quad \hat{y}_S : \mathcal{C}_S^T \rightarrow \mathcal{C}_{\text{sc}},$$

and obey

$$\hat{y}_S \mathcal{S}_{\text{pair}}^+ = h_W, \quad h_W \mathcal{S}_{\text{pair}}^- = \hat{y}_S.$$

The vacua

$$\Omega_p^+ = w_{0,p} = e_0, \quad \Omega_p^- = \mathbf{d}_p^\vee H_p e_0$$

are exchanged by the typed S -cylinder and both declared caps have value one. After positive root-fiber trivialization,

$$S_{00,p} = D_p^{-1}, \quad (D_p y_p)(\Omega_p^+) = D_p.$$

The second displayed value remains normalization-line-valued.

Proof. The two caps both select the zeroth vector in their respective typed frames. The graph of compatible formal and root/cap readings makes that coefficient compatible between the formal and finite-root coordinates, and continuity extends the finite calculation to the closed carriers. The scalar S_{00} value appears only after the positive root-fiber trivialization of the normalization line.

The constructed zeroth column $C_p e_0 = e_0$ and row $e_0^T L_p = e_0^T$ fix $y_p L_p \Lambda_p = e_0^T$. Hence

$$y_p H_p C_p \Lambda_p^{-1} = e_0^T, \quad \mathbf{d}_p y_p (\mathbf{d}_p^\vee L_p \Lambda_p) = e_0^T.$$

On formal coefficients the caps select label zero. On finite-stage representatives with compatible formal and root/cap readings that coefficient lies in the all-incidence-safe ring, so the values land in $\mathcal{C}_{\text{sc}}$; continuity extends the caps to the closures. Since $H_{p,00} = 1$, the S_{00} formulas follow. $\square$

In the monomial chart the cap has a particularly useful closed formula.

Theorem 7.4 (Common monomial cap).

For a finite monomial vector $Y = \sum_k c_k v^k$,

$$\varepsilon_v(Y) = \sum_k c_k \left(\frac{\lambda}{A} \right)^k$$

is the zeroth Newton coefficient of $T_{\text{pair}}^{-1} Y$. It extends uniquely to a continuous map $\mathcal{C}_v^{\text{base}} \rightarrow \mathcal{C}_{\text{sc}}$. Formally it is the convergent series above; physically it evaluates the already formed finite quotient class at

$$v = \rho_{0,p} = 1 - q_p^{-1} = \frac{\lambda_p}{A_p}.$$

No completed formal series is evaluated termwise at a root.

Proof. The row-zero inverse divided difference supplies the powers of λ/A , and their positive λ -order controls convergence on the truncation-dense coefficient module. At an admissible reduced root, evaluation is performed only on the finite quotient. Agreement on the dense finite common core then extends the formal and root-specialized cap maps to the common closure.

The row-zero inverse divided difference is η_0^k , where $\eta_0 = (A + A^{-1} + 2)/(1 + A) = \lambda/A$. Its positive λ -order gives uniform convergence and continuity on c_0 . At an admissible reduced root, $\rho_{0,p}$ is a root of the quotient polynomial, so evaluation is well defined on the finite quotient. The two maps agree on the dense finite common core and therefore extend uniquely to its closure. $\square$

7.3. The Habiro ring inside the scalar graph. Let

$$\widehat{\mathbb{Z}[q]}_{\text{H}} = \varprojlim_N \mathbb{Z}[q]/((q)_N), \quad (q)_N = \prod_{j=1}^N (1 - q^j),$$

with $q = -A$ and $\lambda = 1 - q$.

Theorem 7.5 (Habiro-to-common scalar embedding). *The map*

$$j_{\text{H}}(f) = (\iota_{\lambda}(f), (\text{Ev}_{q_p} f)_{p \geq 5})$$

is a continuous injective ring homomorphism $\widehat{\mathbb{Z}[q]}_{\text{H}} \rightarrow \mathcal{C}_{\text{sc}}$. For every requested Taylor order and finite set of primes, one polynomial representative supplies both approximations.

Proof. Modulo $(q)_m$, choose a polynomial representative $a_m(q)$. Since $(q)_m = \lambda^m u_m(\lambda)$, its Taylor error has order at least m . If a root has order at most m , one factor of $(q)_m$ vanishes there, so the same representative is exact at that root. Taking m larger than the requested Taylor order and all selected root orders gives the required finite-stage representative. Habiro's injective Taylor theorem makes j_{H} injective. $\square$

For $N \geq 1$, define the shell polynomial

$$g_N(q) = \frac{\prod_{j=N+1}^{2N+1} (q^j - 1)}{q - 1}.$$

It has exact λ -order N , is divisible by $(q)_N$, and vanishes at a primitive p -th root once $N \geq (p - 1)/2$. The exact leading term is

$$g_N(1 - \lambda) = (-1)^N \frac{(2N + 1)!}{N!} \lambda^N + O(\lambda^{N+1}).$$

The literal finite approximants used below arise from Habiro's uniform positive twist color on a surgery link. We therefore isolate the class of integral homology spheres admitting an algebraically split all- (-1) -framed presentation before stating the approximation theorem.

Definition 7.6 (All-negative Hoste sector). An all-negative admissible presentation is an algebraically split link $L^- \subset S^3$ with every framing -1 . Its linking matrix is $-I$, and its surgery manifold is an integral homology sphere. This sector consists only of manifolds admitting such a presentation.

Let $\omega_{+, < N}$ be Habiro's positive twist color truncated before label N , and set

$$\mathcal{B}_N(L^-) = J_{L^0}(\omega_{+, < N}, \dots, \omega_{+, < N}).$$

Theorem 7.7 (Negative Hoste scalar and literal approximants). *For an all-negative presentation of M , the polynomials $\mathcal{B}_N(L^-)$ converge in the Habiro ring to J_M , and*

$$\mathcal{C}_-(M) = j_{\text{H}}(J_M) \in \mathcal{C}_{\text{sc}}$$

is independent of that presentation. The same $\mathcal{B}_N$ obey

$$\iota_{\lambda}(J_M - \mathcal{B}_N) \in \lambda^N \mathbb{Z}[[\lambda]]$$

and is exactly equal to the p -th root value once $N \geq (p - 1)/2$. Its two coordinates are the Ohtsuki series and the odd WRT values.

Proof. The Hoste truncation is controlled by its successive shell difference. Link divisibility places each new term in a shell that has both the required Taylor order and the required finite-root vanishing; the twist formula then identifies the limit with the presentation-independent Habiro scalar.

Multilinearity writes $\mathcal{B}_{N+1} - \mathcal{B}_N$ as a sum of colored-link terms whose largest filtration label is N . Habiro's colored-link divisibility

puts every term in $g_N(q)\mathbb{Z}[q, q^{-1}]$. The shell facts above make the sequence cyclotomic Cauchy, give exact Taylor order N , and kill every omitted shell at the p -th root after $N \geq (p-1)/2$. Habiro's twist formula identifies the limit with his surgery definition of J_M ; presentation independence and the Ohtsuki/WRT coordinate statements are his unified-invariant theorems. $\square$

7.4. Normalized single- S colors and scalar contraction. Keep the bracket variable $\tilde{A}$, its square a , and Habiro's variable q_H distinct:

$$v = a = \tilde{A}^2, \quad q_H = \tilde{A}^4 = v^2.$$

Write $\{s\}_v = v^s - v^{-s}$ and $\{n\}_v! = \prod_{s=1}^n \{s\}_v$. At order $\kappa \geq 3$, freeze

$$\begin{aligned} \tilde{A}_\kappa &= -e^{\pi i/(2\kappa)}, & \alpha_\kappa &= e^{\pi i/(2\kappa)} = -\tilde{A}_\kappa, \\ q_{H,\kappa} &= e^{2\pi i/\kappa}, & \xi_\kappa &= e^{3\pi i(\kappa-2)/(4\kappa)}. \end{aligned}$$

The sign ensures that Habiro's twist specializes to the frozen T_κ . Let $S_\kappa = D_\kappa^{-1}\Pi_\kappa$, let h_κ be the signed zero-framed cap, and let

$$F_{\epsilon,\kappa} = \sum_j \Delta_{j,\kappa}^2 \theta_{j,\kappa}^\epsilon = D_\kappa \xi_\kappa^\epsilon.$$

Proposition 7.8 (Anomaly-corrected single- S color). *For $\epsilon = \pm 1$,*

$$\Omega_{\kappa,\epsilon} = \frac{T_\kappa^{-\epsilon} \Omega_\kappa}{F_{-\epsilon,\kappa}} = \xi_\kappa^\epsilon T_\kappa^{-\epsilon} S_\kappa e_0, \quad h_\kappa \Omega_{\kappa,\epsilon} = 1.$$

The anomaly factor ξ_κ^ϵ remains root-specific; only the total dimension has canceled.

Proof. The signed dimension column identifies the unnormalized modular color as $D_\kappa S_\kappa e_0$. The Gauss-sum ratio supplies the root-specific anomaly factor, while the signed cap identity fixes the normalization so that the resulting color contracts to one.

The first column of Π_κ is the signed dimension vector, so $\Omega_\kappa = D_\kappa S_\kappa e_0$. The Gauss identities give $D_\kappa/F_{-\epsilon,\kappa} = \xi_\kappa^\epsilon$. The cap identity is the definition of the normalizing Gauss denominator. $\square$

Put

$$E_\kappa = \mathbb{Q}(\alpha_\kappa, \zeta_8), \quad W_\kappa = \bigoplus_{j=0}^{\kappa-2} E_\kappa f_j,$$

and let Ev_κ denote Habiro's regular root evaluation $q_H^{1/4} \mapsto \alpha_\kappa$, followed by scalar extension to E_κ . A *regular test color* is a finite E_κ -linear combination of representation-ring colors whose lifted scalar denominators do not vanish at α_κ . On one distinguished component of an algebraically split, zero-framed link, intersect the kernels of the functionals obtained by putting arbitrary regular test colors on all remaining components. Denote the resulting E_κ -linear radical and quotient by

$$\mathcal{N}_\kappa^{\text{AS}}, \quad \mathcal{Q}_\kappa^{\text{AS}} = W_\kappa / \mathcal{N}_\kappa^{\text{AS}}.$$

Theorem 7.9 (Functional single- S equality). *Habiro's truncated twist color $\overline{\omega}_{\kappa,\epsilon}$ and the normalized modular color have the same class:*

$$[\overline{\omega}_{\kappa,\epsilon}]_{\text{AS}} = [\xi_\kappa^\epsilon T_\kappa^{-\epsilon} S_\kappa e_0]_{\text{AS}}.$$

They are not equal as vectors for any $\kappa \geq 3$.

Proof. The equality is tested only after applying all declared algebraically split link functionals, so it is a quotient statement. Habiro's comparison lemma puts the difference in the resulting radical, while the last supported basis coefficient separates the two vectors before quotienting. This is the local model for the later distinction between scalar contraction and state equality.

Habiro's algebraically split comparison lemma says that every declared link functional has the same value on the two colors, which is exactly membership of their difference in $\mathcal{N}_\kappa^{\text{AS}}$. Vector equality fails by support: the truncated color is supported through $\lfloor (\kappa - 2)/2 \rfloor$, whereas the normalized full color has a nonzero coefficient at $f_{\kappa-2}$. $\square$

For an odd prime p , let $\sigma_p(\zeta_p) = \zeta_p^{(p+1)/2} = q_p^{\text{com}}$.

Theorem 7.10 (Contracted anomaly-corrected single- S formula). *If $M = S_{L-}^3$ is in the all-negative Hoste sector, then*

$$\sigma_p^{-1}(\text{pr}_p \mathcal{C}_-(M)) = J_{L^0}^{(p)}(\xi_p T_p^{-1} S_p e_0, \dots, \xi_p T_p^{-1} S_p e_0) = \tau_{\zeta_p}(M).$$

Equivalently, σ_p may be applied after the complete scalar contraction to recover the root-specialized common-root coordinate. It is not applied to an uncontracted state.

Proof. First contract the integral Habiro scalar, where Galois naturality is available. At an admissible reduced root, Habiro divisibility truncates each twist color; the functional single- S equality can then be applied component by component, and the finite-root surgery formula identifies the final scalar. The Galois operation is consequently post-contraction, not a transport of an uncontracted boundary state.

The common scalar coordinate is $\text{Ev}_{q_p^{\text{com}}} J_M$. Integral coefficients give Galois naturality of the already contracted Habiro evaluation. At ζ_p , Habiro divisibility truncates every twist color. Replace the components successively using Theorem 7.9; after every replacement the remaining colors are still valid test colors. Habiro's finite-root surgery formula identifies the final contraction with $\tau_{\zeta_p}(M)$. $\square$

7.5. The honest line-valued root state. For an odd prime $p \geq 5$, put

$$\tilde{A}_p = -e^{\pi i/(2p)}, \quad E_p^{\text{sk}} = \mathbb{Q}(\tilde{A}_p) = \mathbb{Q}(\zeta_{4p}), \quad E_p^{\text{an}} = E_p^{\text{sk}}(\zeta_8), \quad \xi_p = \zeta_8^{-1} \tilde{A}_p^{-3}.$$

Let

$$W_p^{\text{F}} = \bigoplus_{j=0}^{p-2} E_p^{\text{sk}} f_j, \quad V_p^{\text{E}} = \bigoplus_{a=0}^{(p-3)/2} E_p^{\text{sk}} e_a, \quad e_a = f_{2a}.$$

Let

$$\mathcal{N}_p^{\text{F}} = E_p^{\text{sk}} d_p^{\text{F}}, \quad \mathcal{N}_p^{\text{E}} = E_p^{\text{sk}} d_p^{\text{E}}$$

be the full and even total-dimension lines. Their chosen quadratic trivializations satisfy $(D_p^{\text{F}})^2 = 2(D_p^{\text{E}})^2$. Introduce the square-root line $\mathcal{R}_p = E_p^{\text{sk}} r_p$, with $r_p^{\otimes 2} \mapsto 2$, and the anomaly line $\mathcal{A}_p = E_p^{\text{sk}} \eta_p$, whose base change to E_p^{an} sends $\eta_p \mapsto \xi_p$. The total-dimension identity is retained as the line isomorphism

$$\phi_{N,p} : \mathcal{N}_p^{\text{F}} \xrightarrow{\sim} \mathcal{R}_p \otimes \mathcal{N}_p^{\text{E}}, \quad d_p^{\text{F}} \mapsto r_p \otimes d_p^{\text{E}}.$$

Define the split parity map

$$\iota_p : V_p^{\text{E}} \longrightarrow W_p^{\text{F}}, \quad \iota_p(e_a) = f_{2a} + i^{-p} f_{p-2-2a}.$$

Even-coordinate projection is its left inverse. Dualizing $\phi_{N,p}$, define the typed split injection

$$\mathcal{J}_p = \text{id}_{\mathcal{A}_p} \otimes \phi_{N,p}^{\vee} \otimes \iota_p : \mathcal{A}_p \otimes \mathcal{R}_p^{\vee} \otimes (\mathcal{N}_p^{\text{E}})^{\vee} \otimes V_p^{\text{E}} \longrightarrow \mathcal{A}_p \otimes (\mathcal{N}_p^{\text{F}})^{\vee} \otimes W_p^{\text{F}}.$$

Theorem 7.11 (Exact rootwise parity bridge). *Let*

$$u_p^{E,+} = \eta_p \otimes r_p^\vee \otimes (d_p^E)^\vee \otimes (T_p^E)^{-1} H_p e_0$$

and

$$u_p^{F,+} = \eta_p \otimes (d_p^F)^\vee \otimes (T_p^F)^{-1} \Pi_p e_0.$$

Then

$$\mathcal{J}_p(u_p^{E,+}) = u_p^{F,+}.$$

After line trivialization, the full state is $\xi_p(T_p^F)^{-1} S_p^F e_0$. The product $\prod_{p \geq 5} u_p^{F,+}$ is an honest point of the complete Hausdorff product of finite line-valued root spaces.

Proof. The parity injection is checked on the simple-current pair of labels, whose sign and phase give the second coefficient in the split vector. The normalization-line isomorphism supplies the missing square-root factor, and the continuous left inverses make the product of finite root states a genuine point of the product space.

For the simple current $J(j) = p - 2 - j$,

$$\Delta_{J(j)} = -\Delta_j, \quad \frac{\theta_{J(j)}^{-1} \Delta_{J(j)}}{\theta_j^{-1} \Delta_j} = (-1)^j i^{-p}.$$

For $j = 2a$ this is exactly the second coefficient in $\iota_p(e_a)$. The normalization-line factors agree because the full total dimension is $\sqrt{2}$ times the even total dimension. Products of the split maps have continuous left inverses, so the product state is well defined and has closed image. $\square$

The primitive $SO(3)$ cap of the even state has residual value $\xi_p/\sqrt{2}$ in its anomaly/square-root line. The scalar-primitive full link-functional cap uses the blow-up Gauss contraction. They are different typed caps.

7.6. Three exact state-level no-gos.

Theorem 7.12 (No state-natural scalar Galois transport). *There is no all-prime family of invertible σ_p -semilinear intertwiners between the standard and frozen common-root even twists, even up to a central scalar.*

Proof. It is enough to compare the smallest declared prime: scalar Galois transport changes the unordered eigenvalue ratios, while a central factor changes all eigenvalues together and cannot change those ratios. The first-prime spectral contradiction therefore rules out an all-prime state intertwiner.

At $p = 5$, the spectrum after scalar Galois transport is $\{1, \zeta_5\}$, whereas the frozen twist spectrum is $\{1, \zeta_5^2\}$. Their characteristic polynomials differ. Even after central rescaling, the unordered eigenvalue-ratio sets $\{\zeta_5^{\pm 1}\}$ and $\{\zeta_5^{\pm 2}\}$ are disjoint. $\square$

Theorem 7.13 (No strict common longitude-twist vacuum). *For either longitude-twist sign and every family of central root units c_p , the root family $c_p \omega_{p,\pm}$ is not the root component of a point of the strict genus-one rootwise/Rees module. Consequently the anomaly-corrected even single- S state has no lift to the maximal image module of the Hopf-pairing map.*

Proof. The strict rootwise/Rees module requires eventual divisibility in the formal Newton coordinate, but the last active root-admissible label has a rootwise unit coefficient. Comparing those two statements at the maximal active label gives the contradiction, and central root units cannot change divisibility.

In the saturated Newton frame, the coefficient of the last active label $n = (p - 3)/2$ is a λ_p -adic unit for either twist sign: modulo λ_p , the GM and Newton bases differ by diagonal units, and every twist coefficient has denominator congruent to -2 . If a strict common lift existed, its formal c_0 coordinate would be divisible by λ at all sufficiently large labels. The

exact compatible finite-stage representative criterion would then make the last active root-fiber coefficient divisible by λ_p , contradicting its unit value. Central root units cannot alter this contradiction. $\square$

Corollary 7.14 (Hereditary arbitrary-genus obstruction). *Fix $g \geq 2$. Any proposed common carrier admitting a continuous tensor-local primitive-cap reduction to the existing genus-one target cannot contain*

$$u_p^{E,+} \otimes \mathbf{1}_p^{\otimes(g-1)}$$

as an all-prime common state.

Proof. Contract the last $g - 1$ primitive vacua. Their caps are one, so a hypothetical lift would give the forbidden genus-one lift of Theorem 7.13. $\square$

This is a no-go for the present strict c_0 architecture, not for arbitrary genus itself.

7.7. The exact AS0 radical. Let $p_n = \text{Ev}_\kappa(P'_n)$, $k = \kappa - 2$, and $d_\kappa = \lfloor k/2 \rfloor$. For $0 \leq n \leq k$, the factorial denominator in P'_n is nonzero at α_κ , so $p_n \in W_\kappa$ is well defined.

Lemma 7.15 (Specialized P' -basis). *The vectors $p_0, \dots, p_k$ form an E_κ -basis of W_κ .*

Proof. Both P_n and V_n are monic of degree n in V_1 , so their transition matrix is triangular with diagonal one. Dividing P_n by the nonzero specialized factorial preserves triangularity and leaves a nonzero diagonal. $\square$

Theorem 7.16 (Exact AS0 radical and failure of MCG descent). *For every $\kappa \geq 3$,*

$$\mathcal{N}_\kappa^{\text{AS}} = \text{Span}_{E_\kappa} \{p_{d_\kappa+1}, \dots, p_k\}, \quad \dim \mathcal{Q}_\kappa^{\text{AS}} = \left\lfloor \frac{\kappa}{2} \right\rfloor.$$

Equivalently, in the representation gauge,

$$\mathcal{N}_\kappa^{\text{AS}} = \text{Span}_{E_\kappa} \{V_{k-j} - V_j : 0 \leq j < k - j\}.$$

This radical is preserved by neither the full modular S matrix nor the full twist T . Hence $\mathcal{Q}_\kappa^{\text{AS}}$ is a contracted link-functional quotient, not a torus state quotient.

Divisibility, Borromean detection, and gauge symmetry. The radical computation has two directions. Habiro divisibility kills the high specialized P' -basis vectors under every regular test color, while Borromean tests detect the low coordinates. Symmetry under $j \mapsto \kappa - j$ gives the representation-gauge generators, and explicit S - and T -images supply witnesses that the contracted radical is not a torus-state subspace.

Proof. High indices vanish. Fix $n > d_\kappa$. Expand each regular test color generically as a finite linear combination of V_m 's, then use Habiro's generic identity

$$V_m = \sum_{a=0}^m \left[\begin{matrix} m+a+1 \\ 2a+1 \end{matrix} \right]_v \{a\}_v! P'_a.$$

Habiro's colored-link divisibility applies to the resulting integral generic colored-Jones expression: up to a Laurent unit it is divisible by

$$G_h(q) = \frac{\prod_{s=h+1}^{2h+1} (q^s - 1)}{q - 1},$$

where $h \geq n$ is the maximum P' -index. If $h < \kappa$, then $h + 1 \leq \kappa \leq 2h + 1$; if $h \geq \kappa$, the interval of $h + 1$ consecutive integers contains a multiple of κ . Thus $\text{Ev}_\kappa(G_h) = 0$. The original test denominators are root units, so every declared functional kills p_n . This argument specializes the divisible integral expression, not any individual high P'_a , whose factorial denominator may vanish.

Low coordinates are detected. Conversely, the Borromean rings with the regular colors p_i, p_i give

$$J_{\text{Bor}}(P'_i, P'_i, P'_n) = \delta_{in} \lambda_i, \quad \lambda_i = (-1)^i \frac{\prod_{s=i+1}^{2i+1} (v^s - v^{-s})}{v - v^{-1}},$$

and $\text{Ev}_\kappa(\lambda_i) \neq 0$ for $i \leq d_\kappa$. Expanding a radical vector in the basis of Lemma 7.15, these tests force all its coefficients through d_κ to vanish. This proves the first equality and the dimension formula.

Representation-gauge generators. The specialized coefficients of P'_i in V_j are invariant under $j \mapsto k - j$ for $i \leq d_\kappa$, since $\{\kappa - r\} = \{r\}$. Hence every $V_{k-j} - V_j$ belongs to the radical; one relation per nontrivial orbit is independent, and their number equals the radical dimension, proving the second equality.

Failure of S - and T -stability. For failure of S -stability, put $r_0 = V_k - V_0$. Up to a nonzero common scalar,

$$(Sr_0)_i = ((-1)^i - 1) \sin\left(\frac{(i+1)\pi}{\kappa}\right).$$

The orbit containing V_1 gives a nonzero quotient coordinate for odd κ , for even $\kappa \geq 6$, and at $\kappa = 4$; the same formula gives a nonzero quotient class at $\kappa = 3$. Thus Sr_0 is not radical. For T ,

$$[T(V_{k-j} - V_j)] = (\theta_{k-j} - \theta_j)[V_j], \quad \frac{\theta_{k-j}}{\theta_j} = \exp\left(\frac{\pi i(\kappa - 2 - 2j)}{2}\right).$$

Take $j = 0$, except when $\kappa \equiv 2 \pmod{4}$, where $j = 1$ is an allowed nontrivial orbit for $\kappa \geq 6$ and the ratio is -1 . The quotient class is nonzero in every case, so the radical is not T -stable. $\square$

The resulting quotient records exactly the closed link functionals used to test the colors. Its failure to be preserved by S and T is therefore a failure of torus-state transport, not a failure of the contracted scalar functional; the latter remains a separate closed observable with its own cap.

Remark 7.17 (Regular-color specialization). The inclusion proof above does not specialize arbitrary high P'_a coefficientwise: its factorial denominator can vanish at a κ -th root. It first forms Habiro's integral generic colored-Jones expression, uses its $G_h(q)$ -divisibility, and only then specializes. This is why the other link components are restricted to regular test colors.

Remark 7.18 (State, scalar, and generic targets). The line-valued root-product single- S state exists. Its contracted all-negative Hoste scalar exists and lies in the common Habiro graph. The same state does not lie in the strict rootwise/Rees c_0 module, and scalar Galois transport is not state-natural. None of these statements constructs or obstructs the later fraction-field generic stated-skein state: that is a different carrier with a cap defined only after geometric closure.

8. PERTURBATIVE, TAYLOR, AND OPERATIONAL BOUNDARY LAYERS

Fix a marked cobordism and cut it at the boundary where its state is read. Recapping produces a closed composite and a scalar coefficient. Such coefficients can agree across boundary theories even when their state objects differ. We therefore construct the perturbative/Taylor and operational Habiro branches separately and compare them on the geometric closed composites for which the stated theorem supplies a map.

This section assembles three boundary theories that have the same closed integer-homology-sphere coefficients but genuinely different state objects. The perturbative layer is the Cheptea–Habiro–Massuyeau (CHM) LMO functor with chamber markings. Its scalar quotient gives a rational Taylor-germ theory at the point $Q = 1$. Independently, geometric cups and caps generate an integral Habiro-valued operational envelope. The comparison theorem is coefficientwise on geometric composites; it is not an identification of the two state modules.

The earlier lollipop coordinates, rootwise chambers, and quotient towers do not supply maps between these state spaces. The common closed-output test therefore provides the comparison used here, while each boundary state and cotest space retains its own definition.

Throughout this section we write Q for the quantum variable used in the Habiro ring and put

$$u = Q - 1, \quad Q = e^h.$$

The letter q in CHM's phrase " q -cobordism" denotes a parenthesization and is unrelated to Q .

The comparison fixes the same geometric composite in both theories. Chamber markings cancel in its LMO closure, and the closed scalar bridge identifies its perturbative coefficients with the Habiro Taylor series. Geometric sewing in the operational construction reaches the same Habiro element. The boundary quotients retain their separate tests: all diagrammatic cotests on the Taylor side, and geometric cotests on the operational side. Coefficient agreement on the closed composite therefore supplies the stated comparison without identifying those radical quotients.

8.1. The chambered LMO local system. Let F_g be the oriented genus- g surface with one boundary component, $H_g = H_1(F_g; \mathbb{Z})$, and let $L_g \subset H_g$ be the standard primitive integral Lagrangian. Write

$$\Gamma_g = \pi_0 \text{Homeo}^+(F_g, \partial F_g), \quad P_g = \{s \in \Gamma_g : s_* L_g = L_g\}.$$

The homological action of Γ_g is onto $\text{Sp}(H_g)$ [4], and every primitive integral Lagrangian is the image of L_g . For such a Lagrangian A , set

$$R_A = \{r \in \Gamma_g : r_* L_g = A\};$$

this is a nonempty right P_g -torsor.

The normalized CHM functor is

$$\tilde{Z} : \mathbf{LCob}_q \longrightarrow {}^{ts}\mathcal{A},$$

where the target is the degree-completed category of top-substantial Jacobi diagrams and composition is contraction of matching colors. In the CHM composition convention, $kh = k \circ h$ and $C(kh) = C(k) \circ C(h)$ for mapping cylinders. The skeletal operator $\tilde{Z}(C(s))$ is defined precisely when $s_* L_g = L_g$.

Choose markings $r_A \in R_A$, with $r_{L_g} = 1$. If $h : A \rightarrow hA$ is an arrow in the action groupoid $\text{Lag}_{\mathbb{Z}}(H_g) \rtimes \Gamma_g$, define

$$(3) \quad K_r(h; A) = \tilde{Z}(C(r_{hA}^{-1} h r_A)).$$

Theorem 8.1 (Chambered LMO transport). *For every genus, the operators (3) obey*

$$K_r(kh; A) = K_r(k; hA) \circ K_r(h; A).$$

If $r'_A = r_A s_A$ with $s_A \in P_g$ and $G_A = \tilde{Z}(C(s_A))$, then

$$(4) \quad K_{r'}(h; A) = G_A^{-1} K_r(h; A) G_A.$$

Consequently the intrinsic object is a mapping-class-equivariant local system, not a preferred family of matrices in one fixed diagram space.

Proof. The marking r_A moves each groupoid arrow back to the fixed Lagrangian where the skeletal CHM operator is defined. For a composite arrow, the middle markings cancel, so functoriality gives the stated composition order. Changing the markings inserts a stabilizer operator at each endpoint; these operators appear contragrediently and produce the displayed gauge law.

The conjugated word $r_{hA}^{-1} h r_A$ preserves L_g , so the CHM domain criterion makes (3) legal. For composable arrows the two middle markings cancel:

$$r_{khA}^{-1} k h r_A = (r_{khA}^{-1} k r_{hA}) (r_{hA}^{-1} h r_A).$$

The mapping-cylinder homomorphism and CHM functoriality preserve this order. Replacing r_A by $r_A s_A$ gives $s_{hA}^{-1}(r_{hA}^{-1} h r_A) s_A$, proving (4). Successive changes multiply in the same order, so the gauge maps satisfy the cocycle law. $\square$

Put

$$V_g = {}^{ts}\mathcal{A}(0, g), \quad W_g = {}^{ts}\mathcal{A}(g, 0).$$

The P_g -actions are $x \mapsto \tilde{Z}(C(s)) \circ x$ and $y \mapsto y \circ \tilde{Z}(C(s))^{-1}$. Hence

$$\mathcal{V}_A = R_A \times_{P_g} V_g, \quad \mathcal{W}_A = R_A \times_{P_g} W_g$$

carry canonical transports $h_*[r, z] = [hr, z]$. Their contraction

$$\langle [r, y], [r, x] \rangle_A = y \circ x$$

is well defined because state and coteat gauges cancel contragrediently. No assertion is made that W_g is the full algebraic dual of V_g .

8.2. Geometric caps and the closed scalar triangle. Let

$$X : 0 \rightarrow (F_g, A), \quad M : (F_g, A) \rightarrow (F_f, B), \quad Y : (F_f, B) \rightarrow 0$$

be fixed geometric Lagrangian cobordisms. In chamber markings, their coordinates are obtained by inserting the marking cylinder and its inverse at each internal boundary.

Theorem 8.2 (Fixed geometric closure). *The chamber coordinates satisfy*

$$(5) \quad y_t \circ m_{t,r} \circ x_r = \tilde{Z}(Y \circ M \circ X) = \Omega\left(Y \circ \widehat{M} \circ X\right),$$

where the hat denotes oriented recapping and Ω is the normalized closed LMO invariant. The contraction is independent of r, t . With the recursive parenthesization $w_{g+1} = (w_g \bullet)$, tensoring the triple by the CHM cup-identity-cap pair leaves the closed value unchanged.

Proof. The inserted marking cylinders cancel inside the composite, leaving the closed geometric cobordism. Recapping identifies its CHM value with the closed invariant, while the three endpoint gauge factors cancel in the contraction. The cup-identity-cap statement is the same cancellation after stabilization.

Functoriality cancels the two inserted marking-cylinder pairs before applying $\tilde{Z}$, proving the first equality. The composite is a $0 \rightarrow 0$ integral Lagrangian cobordism, hence an integral homology cube; the CHM recapping theorem gives the second equality. Under a marking change the three coordinates acquire respectively G_A^{-1} , $G_B^{-1}(-)G_A$, and $(-)G_B$, so every factor cancels. Finally CHM tensor functoriality and $\tilde{Z}(\eta) = \tilde{Z}(\epsilon) = \emptyset$ prove the explicit canceling-pair statement. $\square$

A homological Lagrangian does not determine a disk system or a cap. Theorem 8.2 is therefore a theorem for fixed geometric morphisms, not for a bare triple of Lagrangian subgroups.

Let $\mathcal{D} = \mathcal{A}(\emptyset)$ be the completed algebra of closed Jacobi diagrams. Freeze the trace form $\kappa(x, y) = \text{tr}_{\mathbb{C}^2}(xy)$ on $\mathfrak{sl}_2$ and the completed weight system

$$\widehat{W}_{\mathfrak{sl}_2}^{\text{tr}} : \mathcal{D} \rightarrow \mathbb{Q}[[h]], \quad \widehat{W}_{\mathfrak{sl}_2}^{\text{tr}}(\theta) = 12h.$$

Let

$$\text{Exp}_1 : \mathbb{Q}[[u]] \xrightarrow{\sim} \mathbb{Q}[[h]], \quad f(u) \mapsto f(e^h - 1).$$

Theorem 8.3 (Closed LMO–Ohtsuki–Habiro bridge). *For every oriented integer homology sphere N ,*

$$(6) \quad \widehat{W}_{\mathfrak{sl}_2}^{\text{tr}}(\Omega(N)) = \tau^{\mathfrak{sl}_2}(N) = \text{Exp}_1(\tau^O(N)) = \text{Exp}_1(\iota_1(J_N)) \quad \text{in } \mathbb{Q}[[h]].$$

In particular, $\text{Exp}_1^{-1} \widehat{W}(\Omega(N)) = \iota_1(J_N)$ in $\mathbb{Z}[[u]]$.

Proof. The weight system first turns the closed Jacobi-diagram value into the perturbative scalar series. The Kuriya–Lê–Ohtsuki and Habiro comparison results then identify that series in exponential coordinates. Every step is performed after closing the cobordism, which is why the conclusion is a scalar equality and not an inverse statement about boundary diagrams.

The trace form has root length squared 2 and sends the theta graph to $12h$, matching the Kuriya–Lê–Ohtsuki normalization.

Their theorem identifies the weighted LMO invariant with the arithmetic projective $\mathfrak{sl}_2$ perturbative series. On integer homology spheres, at odd levels, the Kirby–Melvin sector and Habiro’s normalized WRT invariant give the same scalar. Degreewise uniqueness of the arithmetic expansion then identifies the perturbative series with the exponential-coordinate Ohtsuki series. Habiro’s Taylor theorem gives $\tau^O(N) = \iota_1(J_N)$, and his Taylor map is injective. These are equalities of closed scalars; no inverse diagram weight system is used. $\square$

The first-order checksum is

$$\widehat{W}(\Omega(N)) = 1 + 6\lambda(N)h + O(h^2),$$

so changing $Q = e^h$ to e^{-h} , reversing the theta cyclic order, or changing the trace normalization would already violate (6).

8.3. The rational Taylor boundary functor. Set

$$\phi_T = \text{Exp}_1^{-1} \circ \widehat{W}_{\mathfrak{sl}_2}^{\text{tr}} : \mathcal{D} \longrightarrow \mathbb{Q}[[u]].$$

The map is onto: a continuous linear section sends $\sum a_d h^d$ to $\sum a_d 12^{-d} \theta^{\sqcup d}$. Thus the scalar quotient is rational, not integral.

For the standard genus- g object define

$$\beta_g(y, x) = \phi_T(y \circ x), \quad y \in W_g, \quad x \in V_g.$$

Let $\mathcal{R}_R^T(g)$ and $\mathcal{R}_L^T(g)$ be the right and left radicals, and put

$$(7) \quad V_g^T = V_g / \mathcal{R}_R^T(g), \quad W_g^T = W_g / \mathcal{R}_L^T(g).$$

Theorem 8.4 (Taylor radical boundary theory). *The radicals in (7) are closed. The quotient spaces are complete Hausdorff topological $\mathbb{Q}[[u]]$ -modules in their quotient diagram-degree topologies, are u -adically separated, and carry a pairing separating in both variables. Every Lagrangian cobordism $C : g \rightarrow f$ induces continuous covariant and contravariant maps*

$$T_C[x] = [\tilde{Z}(C) \circ x], \quad T_C^\vee[y] = [y \circ \tilde{Z}(C)]$$

with exact adjoint sewing

$$\overline{\beta}_f(y, T_C x) = \overline{\beta}_g(T_C^\vee y, x).$$

These quotients descend to the primitive-Lagrangian chamber local system and obey the gauge law (4).

Proof. The radicals are kernels of continuous tests, so they survive passage to the quotient. Diagram contraction preserves them on both sides, and the intersection of all powers of u removes any residual divisible class. The resulting maps therefore sew adjointly while retaining the chamber gauge action.

Each radical is an intersection of kernels of continuous coefficient functionals, hence closed. Diagram spaces are products of finite dimensional homogeneous pieces; quotienting by a closed subspace preserves completeness and Hausdorffness. The central closed-diagram action factors through $\mathcal{D} / \ker \phi_T \cong \mathbb{Q}[[u]]$. If a quotient state is divisible by every power of u , pairing it against every cotest gives a series in $\bigcap_N u^N \mathbb{Q}[[u]] = 0$; separation then kills the state. Associativity of diagram contraction proves adjoint sewing and preservation of both radicals. The P_g -invariance of the radicals allows the same associated-torsor construction as in Theorem 8.1. $\square$

No perfectness, finite generation, projectivity, or equality of the diagram-degree and u -adic topologies is implied.

8.4. The operational Habiro envelope. Let

$$\mathcal{H} = \varprojlim_n \mathbb{Z}[Q]/((Q; Q)_n), \quad (Q; Q)_n = \prod_{j=1}^n (1 - Q^j),$$

and let $\mathfrak{I}_n$ be the kernel of its n th quotient map. For an arbitrary-Lagrangian object S , form the restricted completions

$$(8) \quad \begin{aligned} \widehat{\mathcal{P}}(S) &= \varprojlim_n \bigoplus_{[X] \in \text{Hom}(0, S)} (\mathcal{H}/\mathfrak{I}_n)[X], \\ \widehat{\mathcal{P}}^{\text{ct}}(S) &= \varprojlim_n \bigoplus_{[Y] \in \text{Hom}(S, 0)} (\mathcal{H}/\mathfrak{I}_n)[Y]. \end{aligned}$$

Thus a coefficient family may have growing support, but at every cyclotomic stage its support is finite. These are not unrestricted products over geometric generators.

For geometric $X : 0 \rightarrow S$ and $Y : S \rightarrow 0$, define

$$(9) \quad \mathcal{B}_S([Y], [X]) = J_{\widehat{Y \circ X}} \in \mathcal{H}.$$

Stagewise finite sums extend this uniquely to a continuous pairing on (8). Let $\mathcal{R}_R(S)$ and $\mathcal{R}_L(S)$ be its right and left radicals. Quotient by these closed radicals and denote the resulting modules by $\overline{\mathcal{P}}(S)$ and $\overline{\mathcal{P}}^{\text{ct}}(S)$.

Theorem 8.5 (Operational Habiro sewing and universality). *For every $M : S \rightarrow T$, geometric composition induces continuous maps*

$$M_*[X] = [M \circ X], \quad M^*[Y] = [Y \circ M]$$

on the restricted completions and on their radical quotients. They obey

$$(10) \quad \mathcal{B}_T(Y, M_*X) = \mathcal{B}_S(M^*Y, X).$$

The quotient pairing is separating; both modules are nonzero, complete Hausdorff cyclotomic modules with faithful $\mathcal{H}$ -action.

Moreover, let

$$q_R : \widehat{\mathcal{P}}(S) \twoheadrightarrow Q_R, \quad q_L : \widehat{\mathcal{P}}^{\text{ct}}(S) \twoheadrightarrow Q_L$$

be continuous quotient maps to complete Hausdorff $\mathcal{H}$ -modules. If a continuous pairing

$$\gamma : Q_L \times Q_R \longrightarrow \mathcal{H}$$

is separating and factors the completed operational pairing for every $x \in \widehat{\mathcal{P}}(S)$ and $y \in \widehat{\mathcal{P}}^{\text{ct}}(S)$,

$$(11) \quad \mathcal{B}_S(y, x) = \gamma(q_L y, q_R x),$$

then

$$\ker q_R = \mathcal{R}_R(S), \quad \ker q_L = \mathcal{R}_L(S).$$

Hence the operational construction is universal among continuous separated quotient realizations of the already prescribed completed closed pairing.

Proof. Modulo $\mathfrak{I}_n$ every input is a finite sum, so composition and the pairing are finite and compatible with transition maps. On generators both sides of (10) are literally $J_{\widehat{Y \circ M \circ X}}$; continuity extends the identity. It also shows that both radicals are functorially preserved. Complementary standard cup and cap generators pair to $J_{S^3} = 1$, proving nontriviality and faithfulness. For the universal property, if $x \in \ker q_R$, then (11) makes x pair to zero against every cotest, so $x \in \mathcal{R}_R(S)$. Conversely, if $x \in \mathcal{R}_R(S)$, surjectivity of q_L makes $q_R x$ pair to zero against every element of Q_L ; separation forces $q_R x = 0$. The left-hand argument is symmetric, and the first isomorphism theorem gives uniqueness. $\square$

8.5. The exact coefficient comparison.

Theorem 8.6 (Geometric Taylor–Habiro comparison). *For every fixed geometric triple X, M, Y , let $N = Y \circ \widehat{M} \circ X$. In any chamber markings,*

$$(12) \quad \widehat{W}(y_t \circ m_{t,r} \circ x_r) = \text{Exp}_1(\iota_1(J_N)),$$

$$(13) \quad \overline{\beta}_f([y_t], [m_{t,r} \circ x_r]) = \iota_1(J_N) \in \mathbb{Z}[[u]] \subset \mathbb{Q}[[u]].$$

At every root of unity ξ ,

$$\text{ev}_\xi \mathcal{B}_T([Y], M_*[X]) = \tau_\xi(N).$$

The explicit tensor canceling pair preserves J_N .

Proof. Evaluate the perturbative and operational constructions on the same fixed geometric composite. The closed-scalar triangle identifies their coefficient at every degree, and the finite-stage sewing maps are compatible with that identification. The comparison is consequently coefficientwise on closed composites; it does not identify the two boundary quotients.

The marking cylinders cancel, so the diagrammatic contraction is $\Omega(N)$ by Theorem 8.2. Apply the closed scalar bridge, then Exp_1^{-1} , to obtain the first two identities. Habiro’s all-root theorem gives the evaluation identity. For stabilization, the LMO functor sends the added cup and cap to the empty diagram, so the two closed LMO invariants agree. Injectivity of both Exp_1 and Habiro’s Taylor map forces equality of the two Habiro elements. $\square$

Remark 8.7 (The two radical quotients are not identified). The Taylor radical is tested by every diagrammatic cotest, whereas the operational radical is tested only by geometric cotests. No density theorem identifies those test families. Therefore Theorem 8.6 does not descend to a map in either direction between the completed operational quotient and the Taylor radical quotient. It is an exact agreement on their common geometric coefficient domain.

Remark 8.8 (Mandatory perturbative and operational nonclaims). Nothing in this section constructs a generic stated-skein state module, a root evaluation of an arbitrary Taylor series, an inverse $\mathfrak{sl}_2$ weight system, a map from arbitrary Jacobi diagrams to the Habiro ring, or an identification with finite RT/GM surface modules. The operational radicals have no intrinsic computable presentation here. No cap is determined by a homological Lagrangian alone. The results do not place the single- S state in the excluded strict c_0 carrier, produce a state-level Galois intertwiner, extend to non-IHS coefficients, or define a global Habiro-to-disk map.

8.6. Primary sources and theorem boundaries. The construction uses [arXiv:math/0701277](#) for the CHM category, composition order, LMO functor, cups, caps, and recapping; [arXiv:2008.10142](#) for surjectivity of the bordered symplectic representation; [arXiv:1005.3895](#) and [arXiv:math/0004099](#) for the trace-normalized perturbative comparison; and [arXiv:math/0605314](#) for the Habiro ring, Taylor injection, all-root evaluation, and Ohtsuki equality. The five resulting theorem axes remain separate: Theorems 8.1, 8.3, 8.4, 8.5, and 8.6.

9. ATTRIBUTED CAP AND SEWING INPUTS

The remaining closed comparisons use three established constructions rather than introducing a second boundary theory. First, stated-skein excision and Morita sewing give the generic Heegaard balanced tensor product [13]. Second, on the integral-homology-sphere locus, the ordinary skein module admits the Habiro-type closed cap of [15, Theorem 1.1 and Section 3.1, arXiv v2]. Third, the universal-construction quotient by the gluing-pairing radical is the standard BHMV mechanism [6].

For the fixed variance, marking, and coefficient conventions of this paper, these inputs yield a generic boundary bimodule, its declared closed cap, and the cap-induced radical quotient. The only comparison asserted here is after that cap or sewing map has been applied. In particular:

- (1) the stated-skein/Morita construction is not extended to the integral root fibers of the rootwise construction;
- (2) the Habiro cap is used only on the stated integral-homology-sphere domain and gives no evaluation map on the whole generic localization; and
- (3) the cap-induced radical quotient is not identified with the perturbative or operational boundary radicals.

The cited primary sources contain the imported excision, cap, and universal-construction theorems. The nontrivial perturbative coefficient comparison proved in the preceding section remains part of this paper.

9.1. Closed-cap comparisons for Heegaard scalars. The following two propositions give the closed-cap comparisons used by the companion Heegaard-certificate construction [34]. Their proofs are included here. Let $A = Q^{1/4}$, let

$$\mathcal{H}_A = \mathcal{H} \otimes_{\mathbb{Z}[Q^{\pm 1}]} \mathbb{Z}[A^{\pm 1}],$$

and, for an oriented integer homology sphere M , let

$$\varphi_M : \mathcal{S}_{\mathbb{Z}[A^{\pm 1}]}(M) \longrightarrow \mathcal{H}_A$$

be the ordinary-skein cap of Garoufalidis–Lê [15, Theorem 1.1, Theorem 2.4 and Section 3.1, arXiv v2]. Their variable is $q_{\text{GL}} = Q = A^4$; their $q_{\text{GL}}^{1/4}$ is our Kauffman parameter A . For the stated-skein comparison, the Costantino–Lê parameter is A , the Korinman–Murakami parameter is A^2 , and the stated ground ring and carrier field are respectively $\mathbb{Z}[A^{\pm 1/2}]$ and $\mathbb{Q}(A^{1/2})$. On a framed unoriented link $L = L_1 \cup \cdots \cup L_r$, the cap is

$$\varphi_M([L]) = (-1)^{\sum_i (\text{fr}(L_i) + 1)} I_{(M, L)}(Q),$$

where every component has the fundamental color and $\text{fr}(L_i) = \text{lk}(L_i, L_i)$. This fixes the vacuum sign and the loop value $\varphi_{S^3}(\bigcirc) = -(A^2 + A^{-2})$.

All framed links generate the ordinary skein module. Theorem 2.4 of the cited source constructs the pair invariant on this full IHS domain; it is not restricted to algebraically split zero-framed retained links. Lemma 2.3 and Section 3.1 prove that each signed colored-Jones functional in the convergent surgery sum satisfies the Kauffman crossing relation with coefficients A, A^{-1} and the loop relation above. Linear extension and passage to the Habiro-convergent sum therefore give the displayed module map through both skein relations.

Write $\mathcal{S}^{\text{ev}}(M)$ for the $\mathbb{Z}[Q^{\pm 1}]$ -span of root-lattice-colored links, equivalently two-parallel. This pure sector is a $\mathbb{Z}[Q^{\pm 1}]$ -module; multiplication by A is not part of its definition.

Proposition 9.1 (Pure Habiro sector and vacuum). *The even restriction lands in the ordinary Habiro ring*

$$\varphi_M^{\text{ev}} : \mathcal{S}^{\text{ev}}(M) \longrightarrow \mathcal{H}.$$

The empty skein belongs to this sector and

$$(14) \quad \varphi_M(\emptyset) = J_M.$$

At every root ξ , $J_M(\xi)$ is Habiro’s normalized $SU(2)$ WRT invariant. When the chosen fourth root $A = \xi^{1/4}$ is itself primitive of odd order, this agrees with the normalized $SO(3)$ specialization of the ordinary-skein cap. With $Q = e^h$,

$$\text{Exp}_1(\iota_1(J_M)) = \widehat{W}_{\text{sl}_2}^{\text{tr}}(\Omega(M)).$$

Proof. For even highest weights the exponent $g/4$ in Theorem 2.4 of [15] is integral. Its representation-ring extension therefore lands in the ordinary Habiro ring on the even sector. For the empty link the sign is one and the pair invariant is J_M . Habiro’s all-root theorem [22] gives the root specializations, while Theorem 8.3 gives the Taylor/LMO comparison. These are precisely the asserted restrictions and scalar identities. $\square$

Put

$$\mathcal{K} = \mathcal{H}_A \otimes_{\mathbb{Z}[A^{\pm 1}]} \mathbb{Q}(A^{1/2}).$$

This is a commutative coefficient extension, not a declared field. The generic closed cap is obtained by scalar extension, as follows. The endpoint-free ordinary-to-punctured-stated map becomes an isomorphism over $\mathbb{Q}(A^{1/2})$, using the spherical comparison of [25] and the filled/endpoint-free sphere lemma of [13]. Its integral kernel is torsion. The Habiro target $\mathcal{H}_A$ is torsion-free over $\mathbb{Z}[A^{\pm 1}]$: its Taylor embedding at $Q = 1$, together with the free fourth-root coefficient extension, detects multiplication by a nonzero Laurent polynomial. Thus φ_M annihilates that kernel and descends to the image lattice in the generic carrier. This lattice spans the carrier, so scalar extension gives its $\mathcal{K}$ -valued cap. This argument uses neither an integral spherical isomorphism nor a root evaluation on all of $\mathcal{K}$. Root values in the first proposition are taken on the integral Habiro sector before this generic extension.

On the raw empty-skein geometric domain, the operational coefficient is the Habiro cap of the oriented closed composite, while the generic GNS coefficient is the scalar obtained by applying the same cap after Costantino–Lê sewing. The latter category is the same-oriented opposite of the CHM convention used by the operational construction.

Proposition 9.2 (Variance-reversed operational agreement). *On the raw empty-skein geometric domain, after the coefficient inclusion $\mathcal{H} \hookrightarrow \mathcal{K}$, the operational pairing and the GNS pairing are the same J_M . The identification reverses state/cotest variance by the same-oriented opposite equivalence. It uses neither orientation reversal of the three-manifold nor $Q \mapsto Q^{-1}$.*

Proof. The comparison is made before either radical quotient. A CHM state $0 \rightarrow S$ corresponds to a Costantino–Lê cotest $S \rightarrow 0$, and a CHM cotest to a state, in the same-oriented opposite category. Both constructions glue the same oriented geometric cup, middle cobordism, and cotest, and both then apply the closed cap to that composite. On elementary balanced tensors this is the same sewn tangle followed by the same linear cap; the balancing relations give the same scalar contraction. Empty skeins sew to the empty skein, whose value is J_M by Proposition 9.1. Passing to the same-oriented opposite category interchanges the state and cotest roles but does not reverse the glued manifold and does not change Q . After the displayed coefficient inclusion, the two scalar pairings are therefore identical. $\square$

The second proposition is deliberately restricted to the raw geometric domain. It does not identify the operational and generic GNS radicals.

10. BOUNDARY OUTPUTS AND LIMITS

The constructions give maps with specified domains and closed scalars, not one universal boundary state space. They comprise the integral pairing image, factorial/Rees covariance, the strict rootwise/Rees object and theta endpoint, the permitted line-valued and scalar contractions, and the exact obstructions to extending those constructions. The perturbative, Habiro, operational, and stated-skein/Morita branches retain their separate source types and are compared only after a declared cap or sewing map.

The companion paper on Heegaard realizations and radial comparisons with $\widehat{Z}$ [34] uses the transported integral lattice and Habiro cap in the stated integer-homology-sphere domain, together with its root and Taylor coordinates. This construction does not identify the factor lattices, evaluate a generic boundary element at every root, or reconstruct a boundary state from the resulting closed scalar.

Remark 10.1 (Conditions on applications). Each application retains the target type, genus and root range, chamber or variance convention, cap or scalarization step, and any IHS or primitive-cap hypothesis of the result used. Omitting those qualifiers changes the claim.

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