

MARKED HEEGAARD BOUNDARY MODULES: FILLING, JOINT COMPLETION, AND INTEGRAL LATTICES

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ABSTRACT. A marked handlebody state must retain its filling relations and coefficient control before complementary capping or root specialization. We construct boundary modules and regulated integral lattices that keep the disk and cyclotomic residues of one finite-stage representative together. Standard handle-attachment and recoupling results supply the geometric inputs; the comparison constructs endpoint lattices, separating transport, raw and saturated quotient towers, and exact cap domains. These objects supply the companion rootwise theory while the two completed topologies remain distinct and root specialization remains restricted to its declared domain.

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1. INTRODUCTION

Start with a marked handlebody state and retain the relations imposed by its chosen filling. Before complementary capping, we want finite representatives that can be read both in the disk topology and cyclotomically. The obstacle is already present in the coefficients: polynomial inclusion has no continuous additive extension from the cyclotomic completion to the disk power-series ring. A common representative therefore cannot identify the two completed targets.

This distinction also matters in the homological-block program. Gukov, Putrov, and Vafa introduced homological blocks through fivebrane constructions [21, Section 6.2]; Gukov, Pei, Putrov, and Vafa formulated their reconstruction of WRT invariants and gave disk-series formulas for negative-definite plumbings [20, Conjecture 2.1 and Appendix A]. Gukov–Manolescu then developed two-variable series for torus-boundary manifolds [19, Sections 5–7]. Here the object is an integral skein module before complementary capping. Its common finite representatives organize disk and cyclotomic readings; they do not identify that module with a homological block or turn a closed cyclotomic invariant into a disk series.

The construction keeps the polynomial lattice in view. At a finite stage, the module Chinese remainder theorem separates the disk and cyclotomic residues; compatible representatives then control the two inverse limits. Regulated endpoint lattices make the pairings and transport meaningful, and raw and saturated slide quotients retain the distinction between integral relations and their generic saturation. The one-holed-torus example follows this construction toward the rootwise pairing and caps of the companion paper on integral root fibers and closed factorizations [34].

The simultaneous disk-adic and cyclotomic product-ideal filtration is $A^{2N}(q; q)_N$.

1.1. Main results. The all-genus filling quotient and generic spine use the attributed input in Theorems 2.1 and 5.3. The module CRT and split completion, Theorems 4.3 and 4.5, keep disk and cyclotomic coordinates of one finite representative together without identifying the completed targets.

The first lollipop norm and regulated lattice are fixed in Theorems 5.2 and 5.5. Recoupling and separating transport use the local-system and common-bridge statements 2.4 and 8.7–8.8, with their marking, valuation, and coefficient hypotheses.

Theorem 10.4 fixes the common-approximant criterion; Theorem 10.5 proves the obstruction above. Theorem 10.6 specifies the cap domain, not a map on the whole completion.

1.2. Local terminology introduced in this paper. A *proper height* is a height function with finite sublevel sets. The adjective *disk–cyclotomic* abbreviates the simultaneous filtration described above. A *lollipop lattice* is the regulated polynomial lattice constructed here from the established lollipop basis. Its Newton interpolation products and regulator are specified in Section 5; they belong to this construction, not to the name of the basis. We use *cotest* as a short name for a transported test functional.

1.3. Inputs to the rootwise theory.

Theorem 1.1 (Boundary constructions for the rootwise theory). *With the genus, marking, coefficient, endpoint, valuation, and cap-domain hypotheses of the cited results, the following data are available to the rootwise construction:*

- (1) the marked filling quotient of Theorem 2.1;
- (2) the finite-stage joint quotient and split completion of Theorems 4.3 and 4.5;
- (3) the regulated endpoint lattices and recoupling transport of Theorems 5.5 and 2.4;
- (4) the separating bridge on its common domain from Theorems 8.7 and 8.8;
- (5) the raw and saturated towers and common-approximant criterion of Theorems 9.7 and 10.4; and
- (6) the no-map boundary and exact cap domain of Theorems 10.5 and 10.6.

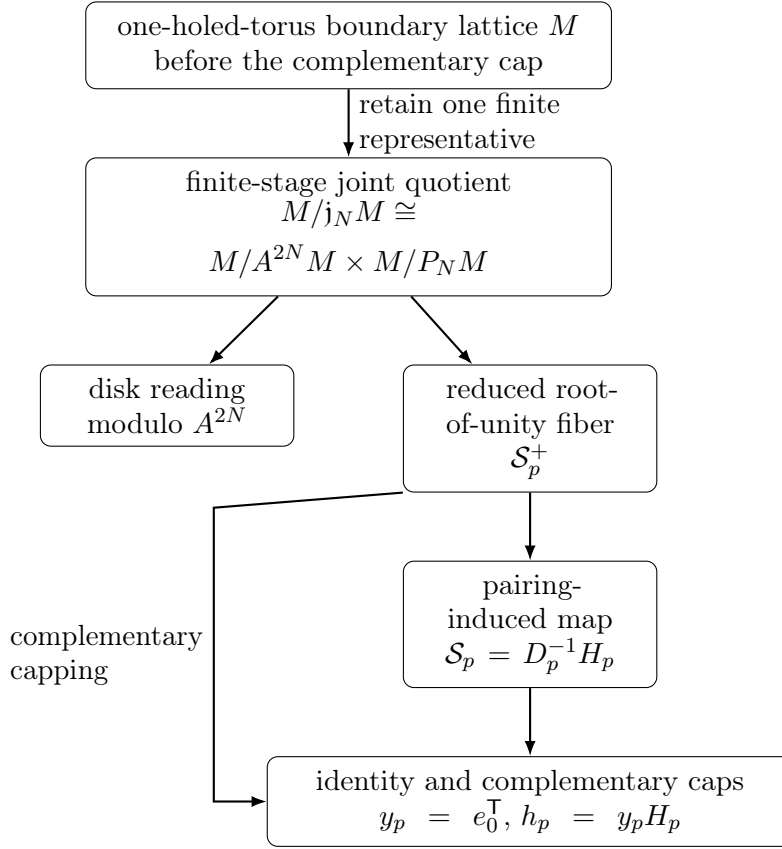


FIGURE 1. The one-holed-torus running example. This paper constructs the upper finite-stage and lattice data; the companion rootwise construction gives the pairing and cap outputs. The reduced root fiber is obtained by its separate quotient-first construction, not by evaluating the whole completion. The scalar caps do not reconstruct an uncapped state.

No clause identifies the disk and cyclotomic completions, evaluates the whole completion at a root, or reconstructs an uncapped state from a cap scalar.

Proof. The six clauses collect the cited theorems without changing their hypotheses or conclusions. The final sentence is exactly their common domain restriction. $\square$

Remark 1.2 (Global non-identifications). This paper does not construct an untyped prime-independent all-root boundary state, a root-specialization map on the whole Rees completion, or a global root evaluation on the generic localized Habiro target. It does not identify Taylor, operational, stated-skein, Rees, or GNS boundary targets. A closed-cap-induced quotient is not promoted to a universal boundary equivalence. The general non-IHS block vector, complete 2-primary sector, positive- b_1 topological assignment, and refined arbitrary-genus theory remain outside the proved boundary program.

2. ATTRIBUTED BOUNDARY INPUTS AND THE COMPLETION INTERFACE

This paper starts from a marked Heegaard splitting $M_{\mathcal{D}} = H_{\alpha} \cup_{\Sigma} H_{\beta}$, a fixed trivalent spine in H_{α} , and the Kauffman-bracket skein module over the coefficient ring specified in each theorem. Handle attachment, graph bases, and normalized recoupling are used in the forms proved in [33, 6, 31, 12]; they are inputs rather than new skein or TQFT constructions.

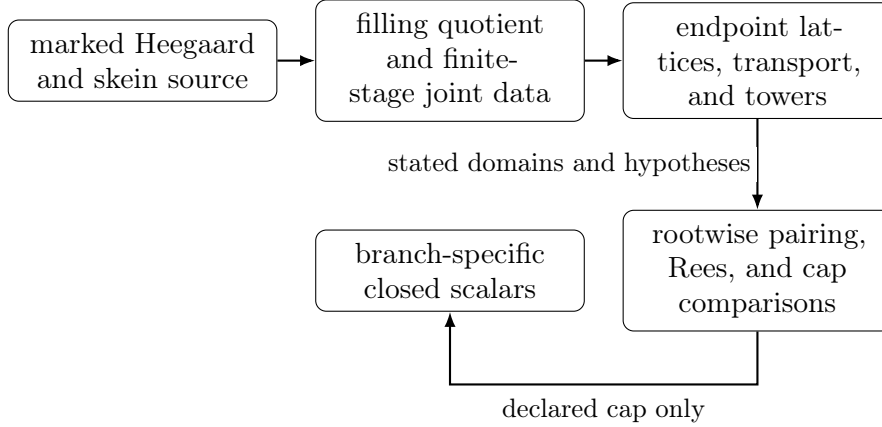


FIGURE 2. From marked boundary modules to rootwise comparisons. This paper constructs the finite-stage data and lattices; the companion rootwise construction gives the pairing and capped comparisons. The connecting arrow retains the exact hypotheses and domains of the cited results; it does not define a universal boundary state space.

2.1. The marked filling quotient. Attach the β -two-handles successively. At each stage take every slide difference through the new handle, pull the resulting kernel back to $\mathcal{S}_A(H_\alpha; R)$, and denote the sum of these pulled-back kernels by $\mathcal{J}_D^R$. Later relations are therefore imposed in the already-filled manifold; a hand-picked list of slides in the initial handlebody is not substituted for this successive kernel.

The one-handle quotient used at each stage is [33, Proposition 2.2(5)].

Theorem 2.1 (Marked filling quotient). *For every $g \geq 0$, inclusion induces a pointed isomorphism*

$$(\mathcal{S}_A(H_\alpha; R)/\mathcal{J}_D^R, [\emptyset]) \cong (\mathcal{S}_A(M_D; R), [\emptyset]).$$

The result is independent of the order of the disjoint β -attachments.

Proof. Przytycki's Proposition 2.2(5) identifies the skein module after one two-handle attachment with the quotient by all slide differences through that handle. Apply this result successively to the ordered disjoint β -curves. The kernel of the composite map is the sum of the stage kernels pulled back to the initial handlebody module, which is exactly $\mathcal{J}_D^R$. After all two-handles are attached, the remaining boundary spheres are capped by three-handles; these attachments induce skein-module isomorphisms by [33, Proposition 2.2(2)(i)]. The composite therefore gives the displayed pointed isomorphism.

Changing the attachment order gives the same filled manifold. For every order, the composite is the inclusion-induced map to its skein module, so the resulting quotient isomorphism is independent of that order. $\square$

2.2. Cyclotomic and disk completions. Put

$$(q; q)_N = \prod_{k=1}^N (1 - q^k), \quad \widehat{\mathbb{Z}[q]}_{\text{cyc}} = \varprojlim_N \mathbb{Z}[q]/((q; q)_N), \quad \widehat{\mathbb{Z}[q]}_{\text{disk}} = \mathbb{Z}[[q]].$$

The first ring is Habiro's cyclotomic completion [22]; the second uses the q -adic topology. They are retained as distinct coordinates.

Theorem 2.2 (No continuous polynomial-identity extension). *There is no continuous additive map*

$$\Psi : \widehat{\mathbb{Z}[q]}_{\text{cyc}} \longrightarrow \mathbb{Z}[[q]]$$

whose restriction to $\mathbb{Z}[q]$ is the ordinary polynomial inclusion. In particular, no continuous ring homomorphism with this property exists.

Proof. The sequence $P_N = (q; q)_N$ tends to zero cyclotomically. If Ψ were continuous and fixed polynomials, then $\Psi(P_N) = P_N$ would tend to zero q -adically, but every P_N has constant coefficient one. $\square$

For a module M , the finite ideals used below are $\mathfrak{j}_N = (A^{2N}P_N)$. Their Chinese-remainder splitting records two residues of one finite representative; it does not select either completed coordinate from the other.

2.3. Conditional closed descent. The later cap discussion uses the following formal implication. An *enhanced Heegaard datum* consists, in every genus, of chambered right and left modules and vacua $\Omega_g^\pm$, a continuous pairing, an exact representation of the declared central extension, coherent lifts of both handlebody subgroups, an anomaly function ω_g , and stabilization data u_g, s_g, ν_g satisfying

$$A_{g+1}(\tilde{\text{st}}_g \tilde{\phi}) = u_g \kappa^{s_g} A_g(\tilde{\phi}), \quad \omega_{g+1} = \omega_g + s_g, \quad \nu_{g+1} u_g = \nu_g,$$

where $A_g(\tilde{\phi}) = \langle \Omega_g^-, \rho_g(\tilde{\phi}) \Omega_g^+ \rangle$. All products and pairings must exist in the stated chambers.

Theorem 2.3 (Conditional enhanced Heegaard descent). *If an enhanced Heegaard datum exists for a class of closed decorated three-manifolds, then*

$$\mathfrak{Z}(M_\phi) = \nu_g \kappa^{-\omega_g(\tilde{\phi})} \langle \Omega_g^-, \rho_g(\tilde{\phi}) \Omega_g^+ \rangle$$

is independent of the central lift, left and right handlebody mapping classes, and Heegaard stabilization. It therefore depends only on the oriented decorated three-manifold, and every declared cyclotomic or chamber projection is separately topological.

Proof. Replacing $\tilde{\phi}$ by $z^n \tilde{\phi}$ multiplies the matrix element by κ^n and increases ω_g by n , so the two factors cancel. Left and right handlebody changes disappear by the two covariance identities, while ω_g is constant on the same double coset. Under stabilization the scalar factor is

$$\nu_{g+1} \kappa^{-\omega_g - s_g} u_g \kappa^{s_g} = \nu_g \kappa^{-\omega_g}.$$

Stable equivalence of Heegaard splittings then proves topological invariance. Continuity makes every declared projection commute with the matrix element. $\square$

This theorem is an implication, not an existence result.

2.4. Recoupling transport. Fix a base marking G_0 , a base chamber α , and its primal and dual lattices. For a verified recoupling degree γ , put

$$L_{\gamma, \alpha} = \rho(\gamma) L_{0, \alpha}, \quad L_{\gamma, \alpha}^\vee = (\rho(\gamma)^{-1})^* L_{0, \alpha}^\vee.$$

Theorem 2.4 (Chamber local system). *For every finite genus, each verified arrow a gives exact Λ -linear identifications*

$$\rho(a) : L_{\gamma, \alpha} \xrightarrow{\sim} L_{a\gamma, \alpha}, \quad (\rho(a)^{-1})^* : L_{\gamma, \alpha}^\vee \xrightarrow{\sim} L_{a\gamma, \alpha}^\vee.$$

They preserve every scalar submodule J_N , extend bicontinuously to the completions, and pure recoupling paths with the same endpoints give identical transported lattices and test functionals.

Proof. Transport the order, meet, and join from the base chamber. The module identity is $\rho(a)\rho(\gamma)L_0 = \rho(a\gamma)L_0$, and the dual identity is its contragredient. Scalar ideals commute with every linear transport, giving the finite-stage preservation and bicontinuous extensions. Finally, normalized recoupling coherence [31, 12] identifies pure recoupling paths with the same endpoints, so the transported lattices and test functionals are path independent. $\square$

No native lattice at a new marking is identified without the separately proved bridge below.

2.5. Homological data of a Heegaard splitting. Choose a symplectic basis

$$(1) \quad (a_1, \dots, a_g, b_1, \dots, b_g)$$

in which the a_i bound disks in the first handlebody, and write

$$(2) \quad \phi_* = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2g, \mathbb{Z}).$$

Let $L_A \subset H_1(\Sigma_g; \mathbb{Z})$ be the meridian Lagrangian of the first handlebody. Mayer–Vietoris gives

$$(3) \quad H_1(M_\phi; \mathbb{Z}) \cong H_1(\Sigma_g; \mathbb{Z}) / (L_A + \phi_* L_A).$$

Modulo L_A , the relations contributed by $\phi_* L_A$ are the columns of C . Thus the direct quotient is the column model $G_{\mathrm{col}} = \mathbb{Z}^g / C\mathbb{Z}^g$; the row model below is its canonical coordinate counterpart.

Theorem 2.5 (Homological data of the Heegaard splitting). *For the block convention of (2),*

$$(4) \quad H_1(M_\phi; \mathbb{Z}) \cong G_{\mathrm{col}} := \mathrm{coker}(C) \cong G_{\mathrm{row}} := \mathrm{coker}(C^T).$$

The canonical isomorphism $G_{\mathrm{row}} \rightarrow G_{\mathrm{col}}$ is $[u] \mapsto [Du]$, with inverse $[x] \mapsto [A^T x]$. If $\det C \neq 0$, then M_ϕ is a rational homology sphere,

$$(5) \quad |H_1(M_\phi; \mathbb{Z})| = |\det C|,$$

and the torsion linking form is represented in the row model by

$$(6) \quad Q_\phi = C^{-1}D \pmod{\mathbb{Z}}.$$

In particular, Q_ϕ is symmetric.

Proof. The first isomorphism is the quotient computation following (3). The map $[u] \mapsto [Du]$ from the row model is well defined because $DC^T = CD^T$; $[x] \mapsto [A^T x]$ is well defined because $A^T C = C^T A$. The identities $A^T D - C^T B = I$ and $DA^T - CB^T = I$ show that the two induced maps are inverse. Nonsingularity of C is equivalent to finiteness, and the group order is $|\det C|$.

In the direct b -column coordinates, the Heegaard linking calculation gives

$$\mathrm{lk}_{\mathrm{col}}([x], [y]) = x^T A C^{-1} y \pmod{\mathbb{Z}}.$$

Indeed, if $Cu = nx$, the corresponding combination of meridian disks has boundary nx after its Au part is capped in the first handlebody; dividing its intersection with y by n gives the displayed matrix in the fixed orientation. It descends because AC^{-1} is symmetric and $(AC^{-1})C = A$ is integral. Pulling the form back by $\overline{D}$ gives

$$u^T D^T A C^{-1} D v.$$

Since $D^T A - B^T C = I$,

$$D^T A C^{-1} D = C^{-1} D + B^T D \equiv C^{-1} D \pmod{\mathrm{Mat}_g(\mathbb{Z})}.$$

Thus (6) is the row-coordinate matrix. It descends directly because $C(C^{-1}D) = D$, and symmetry follows from $CD^T = DC^T$. $\square$

This theorem records homological data only; it does not identify the trivalent-spine operator or any TQFT amplitude.

3. TRACE FUNCTIONALS, INFINITE COLORS, AND CHAMBER OBSTRUCTIONS

At a root of unity, finite-dimensionality turns a surface skein into an endomorphism and turns the Yang–Mills functional into a matrix trace. It is tempting to carry that formula unchanged to generic parameter. This section shows exactly where the temptation fails. The generic identity-double functional does not annihilate an arbitrary filling ideal, the formal infinite color is not a projector in the naive disk topology, and opposite twists cannot preserve one color-split disk lattice. These are obstructions to a particular scalarization architecture, not obstructions to the full-slide quotient of Section 2.1. Mapping-torus trace constructions provide useful comparison data [10]; a Heegaard filling amplitude is nevertheless a mixed vacuum coefficient rather than that trace.

3.1. Finite-root trace architecture. Let V be a finite-dimensional reduced handlebody state space, let $v \in V$ be the normalized empty vector, and let $\lambda \in V^*$ satisfy $\lambda(v) = 1$. Put

$$P = v\lambda,$$

so $P^2 = P$ and P has rank one. Suppose the reduced surface skein algebra $\mathcal{A}$ acts through an isomorphism

$$\Phi : \mathcal{A} \xrightarrow{\sim} \text{End}(V),$$

and let $e_H = \Phi^{-1}(P)$. In a fixed reduced convention, the Yang–Mills functional has the form

$$\text{YM}_r(x) = c_r \text{tr}(\Phi(x)), \quad c_r \neq 0.$$

This is the finite-root trace formula for the skein-theoretic Yang–Mills measure [8].

Theorem 3.1 (Doubled rank-one overlap). *For every $U \in \text{GL}(V)$,*

$$\begin{aligned} \text{YM}_r(e_H \Phi^{-1}(UPU^{-1})) &= c_r \text{tr}(PUPU^{-1}), \\ \text{tr}(PUPU^{-1}) &= (\lambda Uv)(\lambda U^{-1}v). \end{aligned}$$

Consequently

$$\frac{\text{YM}_r(e_H \Phi^{-1}(UPU^{-1}))}{\text{YM}_r(e_H)} = (\lambda Uv)(\lambda U^{-1}v).$$

This ratio is unchanged by $U \mapsto zU$ and by $U \mapsto U^{-1}$. In a unitary convention it is $|\langle v, Uv \rangle|^2$.

Proof. At a finite root, the handlebody vacuum and cotest turn the filling data into the rank-one operator P . Multiplying by the transported rank-one projector and taking the trace produces the two vacuum coefficients; the paired factors explain projective invariance, while a linear lift is needed to retain the chiral phase. The later no-go results identify which parts of this finite calculation fail when the color set or filling ideal becomes generic.

The first equality is the definition of e_H and the trace functional. For $w \in V$,

$$PUPU^{-1}w = v\lambda(Uv)\lambda(U^{-1}w).$$

The trace of the rank-one map $w \mapsto a\mu(w)$ is $\mu(a)$, giving the second equality. Since $\text{tr}(P) = 1$, one has $\text{YM}_r(e_H) = c_r$. Scalar rescaling cancels between U and U^{-1} , and the two factors are exchanged by inversion. In the unitary case they are complex conjugates. $\square$

The doubled expression is projectively safe but loses the oriented phase. A linear lift of the mapping-class action gives the normalized one-sided vacuum matrix coefficient.

Theorem 3.2 (Normalized one-sided vacuum matrix coefficient at a root). *Fix a Maslov or framing central extension and a linear representation $\tilde{\rho}$ on V . For $U_{\tilde{\phi}} = \tilde{\rho}(\tilde{\phi})$ and $u_{\tilde{\phi}} = \Phi^{-1}(U_{\tilde{\phi}})$,*

$$\boxed{\frac{\text{YM}_r(e_H u_{\tilde{\phi}})}{\text{YM}_r(e_H)} = \lambda(U_{\tilde{\phi}} v)}.$$

If the central generator acts by κ , changing the lift by its m -th power multiplies the coefficient by κ^m .

Proof. Since $\Phi(e_H u_{\tilde{\phi}}) = P U_{\tilde{\phi}}$,

$$\mathrm{tr}(P U_{\tilde{\phi}}) = \mathrm{tr}(v(\lambda U_{\tilde{\phi}})) = \lambda(U_{\tilde{\phi}} v).$$

Divide by $\mathrm{YM}_r(e_H) = c_r$. The central statement follows from linearity of the chosen lift. $\square$

For a Heegaard double-coset invariant, conjugation invariance of P is not enough. If h_+ and h_- extend over the right and left handlebodies, one needs the one-sided identities

$$U_{h_+} v = v, \quad \lambda U_{h_-} = \lambda,$$

together with the declared anomaly characters. This distinction between a simultaneously transported trace and a mixed chiral coefficient will remain important at generic parameter.

3.2. The generic Yang–Mills row does not descend. Let $H = S^1 \times D^2$, let $s_0 = \emptyset$, and let $z = s_1$ be the zero-framed core. The generic coefficient functional arising from the identity double $S^1 \times S^2$ is

$$\lambda_H(s_n) = \delta_{n0}.$$

It is the generic handlebody coefficient functional on its declared test domain [8].

Theorem 3.3 (Genus-one no-descent). *Attach a two-handle to H along a longitude with the standard product framing and cap the resulting sphere. The filled manifold is S^3 , and its full slide kernel contains*

$$z + (A^2 + A^{-2})\emptyset.$$

Nevertheless

$$\lambda_H(z + (A^2 + A^{-2})\emptyset) = A^2 + A^{-2} \neq 0$$

at generic A . Hence λ_H does not descend through this filling quotient. The same local obstruction occurs after any number of standard stabilizations.

Proof. After filling, the core z becomes a trivial zero-framed unknot in S^3 . The trivial-circle relation therefore identifies it with $-(A^2 + A^{-2})\emptyset$, proving that the displayed element belongs to the complete kernel. Its value under λ_H is immediate from $\lambda_H(z) = 0$ and $\lambda_H(\emptyset) = 1$. Stabilization can be performed away from the solid-torus summand and does not change the local relation. $\square$

There is no contradiction with the ordinary bracket on S^3 . The bracket is the correct covector for the longitude filling; λ_H is the covector for the identity double. Different fillings generally require different annihilator lines.

3.3. The infinite color is not a naive projector. Write the formal Kirby or Yang–Mills color as

$$\Omega_{\infty} = \sum_{n \geq 0} \Delta_n s_n, \quad \Delta_n = (-1)^n [n+1],$$

and let Ω_N be its cutoff at $n = N$. Fusion contains the vacuum s_0 in $s_i s_j$ precisely when $i = j$, with coefficient one.

Proposition 3.4 (Vacuum-square obstruction). *The vacuum coefficient of the cutoff square is*

$$[s_0](\Omega_N^2) = \sum_{i=0}^N \Delta_i^2.$$

With

$$[n+1] = A^{-2n} R_n(A), \quad R_n(A) = 1 + A^4 + \cdots + A^{4n},$$

the next increment is

$$\Delta_{N+1}^2 = A^{-4(N+1)} R_{N+1}(A)^2.$$

Its A -adic valuation tends to $-\infty$, and its magnitude diverges for $|A| < 1$. Thus Ω_∞^2 is not defined in the naive disk completion. At $A = 1$, the increments are $(N+2)^2$, so the same cutoffs also fail first-stage cyclotomic Cauchy convergence.

Proof. The infinite-color obstruction is read from the vacuum coefficient of a finite cutoff square. Fusion selects equal colors, the balanced quantum-integer factorization gives the exact negative disk valuation of the next increment, and specializing at the cyclotomic point turns the same increment into quadratic growth. The two convergence failures therefore come from the same cutoff viewed in the two topologies.

The fusion rule gives the vacuum coefficient. The displayed factorization of the balanced quantum integer gives the exact valuation $-4(N+1)$ because $R_{N+1}(0) = 1$. Substitution $A = 1$ gives $\Delta_n = (-1)^n(n+1)$. $\square$

The infinite color is rigorously useful as a weak capping distribution on a declared test domain. The proposition only rules out treating a fixed unnormalized color cutoff as an idempotent in the naive coefficient topology.

3.4. Scalar extension erases the simultaneous disk-adic/cyclotomic filtration. Let

$$I_N = q^N(q; q)_N \Lambda$$

in a non-Laurent polynomial ring Λ . The two factors encode different smallness conditions.

Proposition 3.5 (Scalar-extension collapse). *For every N :*

- (1) $q^N(q; q)_N$ is a unit in an algebraic Laurent-series field $\mathbb{C}((q))^{\text{alg}}$;
- (2) in $\mathbb{C}[[q]]$, $(q; q)_N$ is a unit and $I_N \mathbb{C}[[q]] = q^N \mathbb{C}[[q]]$;
- (3) if q is a Laurent monomial in A , then localization at A makes q^N a unit and retains only the cyclotomic factor.

Proof. Scalar extension is tested factor by factor. The cyclotomic factors are units in the disk power-series ring, while the disk factor becomes a unit after passing to a Laurent field; localizing in the surgery variable removes the disk filtration as well. Each allowed extension therefore forgets one of the two smallness conditions.

Every factor $1 - q^k$ has constant coefficient one and is invertible in $\mathbb{C}[[q]]$. The Laurent-series field also contains q^{-N} . If q is a Laurent monomial in A , it is invertible after localizing A . The three claims follow. $\square$

A vector space over a Laurent field can host generic recoupling operators, but it cannot remember simultaneous disk-adic and cyclotomic divisibility. The non-Laurent lattice must be chosen before scalar extension.

3.5. Opposite twists force different disk chambers. The local twist obstruction is already visible on one color line.

Theorem 3.6 (Two-sided color-line obstruction). *Let Λ_y be a discrete valuation ring with uniformizer y and fraction field F . On Fe_n , let*

$$Te_n = (-1)^n y^{n(n+2)} e_n.$$

For $n > 0$, no rank-one fractional Λ_y -lattice spanning Fe_n is preserved by both T and T^{-1} .

Proof. The two-sided twist obstruction reduces every rank-one fractional lattice to a single valuation shift. That shift cancels from preservation by the twist and its inverse, whose valuations have opposite signs for every nonzero color. Hence one fixed diagonal lattice cannot serve both directions; the positive repair is to let arrows move between declared disk chambers.

Every fractional ideal in a discrete valuation ring is principal, so such a lattice is $y^w \Lambda_y e_n$ up to a unit. The basis weight w cancels from the preservation condition. The valuations of T and T^{-1} are $n(n+2)$ and $-n(n+2)$, and both are nonnegative only for $n = 0$. $\square$

The theorem rules out one color-split lattice and every diagonal monomial renormalization. It does not rule out a non-diagonal enlargement. Its positive conclusion is architectural: use a chamber family with arrows

$$U_{\gamma,c} : L_c \longrightarrow L_{\gamma c}$$

rather than demand that every generator preserve one disk lattice.

3.6. Exact disk damping and its sign wall. Set $Q = A^4$ for this calculation. For genus $g > 1$, the empty-channel term of the generic Yang–Mills distribution is

$$Y_{g,n} = [n+1]^{2-2g} = A^{4n(g-1)} R_n(A)^{2-2g} = Q^{n(g-1)} \left(\frac{1-Q}{1-Q^{n+1}} \right)^{2g-2}.$$

It has exact A -valuation $4n(g-1)$, so the untwisted tail is coefficientwise disk-coercive. For

$$\theta_n = (-1)^n A^{n(n+2)},$$

one instead obtains

$$\text{ord}_A(Y_{g,n} \theta_n) = n^2 + (4g-2)n,$$

$$\text{ord}_A(Y_{g,n} \theta_n^{-1}) = -n^2 + (4g-6)n.$$

The first is proper and the second is unbounded below. Replacing A by A^{-1} exchanges the two sectors.

A related local genus-two channel makes the wall especially transparent. Let $c = m + n - 1$ and let $\kappa \in \{0, 1, 2\}$ encode the raw-row, metric-symmetric, and raw-column gauges. For

$$K_{m,n,\kappa}(t) = [t+1]_y^\kappa y^{ct(t+2)},$$

the inside and outside degrees are

$$v_{\text{in}} K = ct(t+2) - 2\kappa t, \quad v_{\text{out}} K = -ct(t+2) - 2\kappa t.$$

Hence the inside chamber is coercive when $m+n > 1$, the outside chamber when $m+n < 1$, and the wall $m+n = 1$ is noncoercive in every displayed gauge. Gauge changes alter the linear term, not the quadratic wall.

Exact computation 3.7 (Escaping ray in the literal Poincaré kernel). For every even $t \geq 0$, one fully resolved path in the marked genus-two Poincaré word has

$$v_{\text{in}}(\text{path}_t) = -3t(7t+16), \quad v_{\text{out}}(\text{path}_t) = 3t(11t+4).$$

The formulas follow by summing the exact twist and recoupling degrees along that admissible ray and were checked symbolically over the integers. The negative first formula disproves termwise coercivity of the literal inside expansion. The positive second formula is evidence only: one ray cannot prove global coercivity of the outside expansion.

3.7. What a chamber trace must contain. A generic chiral construction therefore needs, for each genus and chamber, a completed multiplier algebra $\widehat{\mathcal{A}}_{g,c}$, a completed handlebody module $\widehat{\mathcal{M}}_{g,c}$, a right vacuum $v_{H,c}^+$, a left cotest $\lambda_{H,c}^-$, lifted implementers $u_{\phi,c}^-$, and a trace or pairing on an explicitly declared product domain. The required identities include

$$\begin{aligned} u_{\phi}^- x u_{\phi}^{-1} &= \alpha_{\phi}(x), & u_{\phi}^- u_{\psi} &= u_{\phi}^- u_{\psi}, \\ u_{h+} v_H^+ &= v_H^+, & \lambda_H^- u_{h-} &= \lambda_H^-, \end{aligned}$$

as well as slide annihilation, anomaly normalization, sector-transition cocycles, and stabilization.

The finite-root trace theorem supplies the model for these identities. The no-descent and two-sided-twist theorems prove that the generic objects cannot be obtained by simply reusing one identity-double row and one color lattice for every filling and both twist directions.

4. SIMULTANEOUS DISK-ADIC AND CYCLOTOMIC COMPLETIONS WITH COMPATIBLE FINITE-STAGE REPRESENTATIVES

The chamber analysis dictates a non-Laurent coefficient lattice, but it does not yet couple disk and cyclotomic convergence. Start with a marked boundary state and retain one finite representative for its later disk reading and root-of-unity cap. At every finite stage the two ideals are comaximal, so the Chinese remainder theorem separates the two residues, without freeness or flatness assumptions on the module.

The remaining issue is compatibility through the stages. A single sequence of finite polynomials must have increments in both ideals to represent both limits. This common-origin condition couples the representatives; it does not identify the completed targets. The section first constructs the finite splitting and then states the exact approximation conditions that survive passage to inverse limits.

4.1. The coefficient ring and the joint regulator. Use the standard polynomial, Laurent, and generic rings

$$\Lambda = \mathbb{Z}[A], \quad R = \mathbb{Z}[A, A^{-1}], \quad \mathbb{F}_A = \mathbb{Q}(A),$$

and put $q = A^2$. For $N \geq 0$, define

$$P_N(A) = (q; q)_N = \prod_{k=1}^N (1 - A^{2k}), \quad j_N = A^{2N} P_N(A).$$

Thus $P_0 = j_0 = 1$; this stage-zero value is used for height-zero approximant terms. Unless explicitly stated otherwise, the finite-stage CRT statements below assume $N \geq 1$. We use the same symbols for the generated principal ideals when no confusion is possible. The filtrations are nested because

$$j_{N+1} = A^2(1 - A^{2N+2})j_N \quad (N \geq 0).$$

The factor A^{2N} measures order at the disk origin; P_N measures cyclotomic divisibility. The product ideals therefore define the simultaneous disk-adic and cyclotomic filtration. As in the local terminology introduced above, “disk–cyclotomic” abbreviates this product-ideal construction.

Proposition 4.1 (No naive common skein base change). *For $N \geq 1$, the quotient $\Lambda/(j_N)$ is not a coefficient ring to which the ordinary Kauffman two-handle theorem can be directly base changed: the image of A is not invertible. If A is localized first, then*

$$j_N R = P_N R,$$

so the disk factor disappears. Joint finite stages must therefore be understood as quotients of declared polynomial lattices inside a generic $\mathbb{F}_A$ -skein space, not as Kauffman skein modules over $\Lambda/(j_N)$.

Proof. The base-change failure is detected before any skein calculation: the polynomial quotient contains a nonzero class killed by the image of the parameter, so that parameter is not invertible. Localizing first makes the disk monomial a unit and removes the disk filtration. The joint stage must therefore remain a polynomial-lattice quotient inside the generic skein space.

The nonzero class of $A^{2N-1}P_N$ in $\Lambda/(A^{2N}P_N)$ is killed by A , so A is a zero divisor and cannot be a unit. In R , the factor A^{2N} is a unit, giving the displayed equality of ideals. $\square$

This is the coefficient-ring distinction behind the raw and saturated relations of Section 9: first define the genuine slide submodule over R , intersect its appropriate span with a polynomial lattice, and only then reduce by j_N .

4.2. Complementary CRT idempotents. The two factors of j_N are comaximal because $P_N(0) = 1$. The following explicit complementary idempotents implement the Chinese remainder projection for every module.

Lemma 4.2 (Integral Bézout identity and complementary CRT idempotents). *For $N \geq 1$, let $b_N(A)$ be the truncation through degree $2N - 1$ of the formal power series $P_N(A)^{-1}$. Then $b_N \in \mathbb{Z}[A]$ and there is an $a_N \in \mathbb{Z}[A]$ such that*

$$a_N A^{2N} + b_N P_N = 1.$$

The two elements

$$e_N^{\text{disk}} = b_N P_N, \quad e_N^{\text{cyc}} = a_N A^{2N} = 1 - e_N^{\text{disk}}$$

are complementary orthogonal idempotents modulo j_N , with

$$\begin{aligned} e_N^{\text{disk}} &\equiv 1 \pmod{A^{2N}}, & e_N^{\text{disk}} &\equiv 0 \pmod{P_N}, \\ e_N^{\text{cyc}} &\equiv 0 \pmod{A^{2N}}, & e_N^{\text{cyc}} &\equiv 1 \pmod{P_N}. \end{aligned}$$

Proof. The constant term of P_N makes its truncated inverse integral, and the truncation converts the formal inverse into a finite Bézout separator modulo the disk power. The two complementary residues are then obtained by the selector pair; no freeness of a later module is involved at this stage.

Recursive inversion of a polynomial with constant coefficient one has integral coefficients. Truncation gives $b_N P_N \equiv 1 \pmod{A^{2N}}$, so $(1 - b_N P_N)/A^{2N}$ is the required a_N . The congruences are immediate. They imply idempotence and orthogonality modulo the product ideal. $\square$

The one-holed-torus model. For a marked one-holed torus, keep a boundary state as a vector in the declared polynomial lattice before capping it with the complementary solid torus. The cap supplies the root-of-unity reading, while the same finite polynomial representative can be read at the disk origin. A single Laurent skein module cannot carry both readings: localizing makes the disk factor a unit, whereas retaining the polynomial parameter prevents a direct base change of the ordinary skein relation. The module CRT below is the finite-stage device that keeps these two readings attached to one representative. Figure 1 records this model and the later root-fiber pairing and cap without adding an evaluation map on the completed disk coordinate.

4.3. The module Chinese remainder theorem.

Theorem 4.3 (Module CRT). *For $N \geq 1$ and every Λ -module M , the diagonal map*

$$\Theta_{M,N} : M/j_N M \longrightarrow M/A^{2N} M \times M/P_N M, \quad [x] \longmapsto ([x], [x]),$$

is a natural isomorphism. No flatness, torsion-freeness, finite generation, or freeness hypothesis on M is required. Its inverse is

$$([x], [y]) \longmapsto [e_N^{\text{disk}} x + e_N^{\text{cyc}} y]_{j_N}.$$

Proof. The two selectors show directly that the proposed inverse has the requested disk and cyclotomic residues, proving surjectivity. For injectivity it is enough to prove

$$A^{2N} M \cap P_N M = A^{2N} P_N M.$$

If $x = A^{2N} u = P_N v$ lies in the intersection, the Bézout identity gives

$$x = (a_N A^{2N} + b_N P_N) x = a_N A^{2N} P_N v + b_N P_N A^{2N} u,$$

which lies in the product submodule. The reverse inclusion is immediate. Naturality follows because the selectors are central scalars. $\square$

Corollary 4.4 (CRT after relations). *For every submodule $X \subset M$,*

$$\frac{M}{X + \mathfrak{j}_N M} \cong \frac{M}{X + A^{2N} M} \times \frac{M}{X + P_N M}.$$

Equivalently,

$$(X + A^{2N} M) \cap (X + P_N M) = X + \mathfrak{j}_N M.$$

Proof. The relation submodule is absorbed first by passing from M to M/X . The module CRT on that quotient supplies both the product decomposition and the intersection identity, so adding relations does not require a new flatness or torsion argument.

Apply Theorem 4.3 to the module M/X . $\square$

In particular, the statement applies equally to raw or saturated full-slide quotients. It does not assert that either root fiber is the reduced RT state space; that remains a separate non-flat specialization problem.

4.4. Split inverse limits. Define

$$\widehat{M}_{\text{joint}} = \varprojlim_N M/\mathfrak{j}_N M, \quad \widehat{M}_{\text{disk}} = \varprojlim_N M/A^{2N} M, \quad \widehat{M}_{\text{cyc}} = \varprojlim_N M/P_N M.$$

Theorem 4.5 (Split joint completion). *For every Λ -module M , there is a canonical natural isomorphism*

$$\widehat{M}_{\text{joint}} \cong \widehat{M}_{\text{disk}} \times \widehat{M}_{\text{cyc}}.$$

For $M = \Lambda$, the disk factor is $\mathbb{Z}[[A]]$, while the cyclotomic factor is

$$\varprojlim_N \mathbb{Z}[A]/((A^2; A^2)_N).$$

Proof. For the disk factor we use the ordinary A -adic filtration.

The finite-stage CRT maps commute with every reduction map. Take inverse limits and use that finite products commute with inverse limits. The ideals (A^{2N}) are cofinal with the ordinary A -adic ideals, yielding $\mathbb{Z}[[A]]$. $\square$

The product theorem makes the independence of the two coordinates exact. It also explains why a distinguished compatible disk–cyclotomic lift is extra data: the cyclotomic coordinate alone does not choose the disk coordinate, as proved in Theorem 2.2.

4.5. Strict common-origin approximants.

Definition 4.6 (Finite-stage representative sequence). A sequence $f_N \in M$ is a strict common-origin sequence when

$$f_{N+1} - f_N \in \mathfrak{j}_N M$$

for every N . The same finite representatives then define both the disk and cyclotomic limits.

Theorem 4.7 (Criterion for a finite-stage representative sequence). *For a sequence $f_N \in M$, the following are equivalent:*

- (1) $f_{N+1} - f_N \in \mathfrak{j}_N M$ for every N ;
- (2) simultaneously,

$$f_{N+1} - f_N \in A^{2N} M, \quad f_{N+1} - f_N \in P_N M$$

for every N .

The quotient version is obtained by adding one fixed relation submodule X to all three conditions.

Proof. The common-origin condition is exactly the intersection of the disk and cyclotomic increment conditions. The finite-stage CRT identifies that intersection with the joint regulator ideal, while the quotient formulation simply carries one fixed relation submodule through the same calculation.

Use the intersection identity in Theorem 4.3, or its quotient form in Corollary 4.4. $\square$

The word “same” is essential. One disk-Cauchy sequence and an unrelated cyclotomic-Cauchy sequence determine a pair in the split product, but they do not exhibit a common kernel or a common family of finite polynomials.

Theorem 4.8 (Constructive bicoercive cutoff). *Let $T_x \in M$ for x in a countable set X . Suppose proper functions $h, d : X \rightarrow \mathbb{Z}_{\geq 0}$ satisfy*

$$T_x \in P_{h(x)}M, \quad T_x \in A^{2d(x)}M.$$

Set

$$X_N = \{x : h(x) < N \text{ or } d(x) < N\}, \quad f_N = \sum_{x \in X_N} T_x.$$

Then X_N is finite and $f_{N+1} - f_N \in \mathfrak{j}_N M$.

Proof. The bicoercive cutoff uses two independent properness tests. Proper height makes each finite lower set a finite marked-state sum, while the disk and cyclotomic bounds force every new layer into both branch ideals. Their intersection is the joint regulator ideal, so successive finite sums define one common-origin limit.

Properness makes each of the two sublevel sets finite, hence also their union. Every term in the new layer has $h(x), d(x) \geq N$. The layer lies in both $P_N M$ and $A^{2N} M$, hence in their intersection $\mathfrak{j}_N M$. $\square$

A layerwise variant only asks that the sum of each new finite layer lie in the two branch ideals. It therefore permits exact equal-degree cancellation without assuming termwise positivity.

4.6. Restricted completions and infinite matrices. Let X be countable and let

$$L(X) = \Lambda^{(X)}$$

be the free direct-sum lattice. Its joint completion is naturally the restricted product

$$\widehat{L}_{\text{joint}}(X) = \left\{ (a_x) \in \prod_{x \in X} \widehat{\Lambda}_{\text{joint}} : \begin{array}{l} \text{for every } N, \text{ all but finitely many } a_x \\ \text{vanish modulo } \mathfrak{j}_N \end{array} \right\}.$$

Theorem 4.9 (Restricted-column criterion). *A matrix $U = (u_{yx})$ over $\widehat{\Lambda}_{\text{joint}}$ defines a continuous map*

$$U : \widehat{L}_{\text{joint}}(X) \longrightarrow \widehat{L}_{\text{joint}}(Y)$$

if and only if every column is restricted:

$$\text{for all } x, N, \quad u_{yx} \equiv 0 \pmod{\mathfrak{j}_N} \quad \text{for all but finitely many } y.$$

Restricted matrices are closed under composition.

Proof. At a fixed stage, a restricted-product input has finite support, so continuity is exactly the requirement that each surviving column have finite output support modulo $\mathfrak{j}_N$. The same finite-support observation applied to a composite proves closure under composition. This is a completion-theoretic support test, not a basis-change identity.

Modulo $\mathfrak{j}_N$, an input has finite support. Its image has finite support precisely when each of the finitely many columns it uses has finite support at that stage. Conversely, the image of a basis vector is one column, so continuity forces that column to be restricted. For a composite and one fixed input column, only finitely many intermediate indices survive modulo $\mathfrak{j}_N$; the finite union of the corresponding restricted output supports is finite. $\square$

This criterion detects a basic failure of the ordinary color-basis Hopf kernel. After removing a Laurent monomial, its entry is

$$G_{mn}(q) = 1 + q + \cdots + q^{(m+1)(n+1)-1}.$$

For a fixed n ,

$$G_{mn}(1) = (m+1)(n+1) \neq 0$$

for infinitely many m . Thus the column is not restricted even modulo $1 - q$, and multiplying rows or columns by monomials cannot change the value at $q = 1$. A non-diagonal cyclotomic basis is necessary.

4.7. A diagonal annular seed. In the representation ring, define

$$P_n^H = \prod_{i=0}^{n-1} (V_1 - v^{2i+1} - v^{-2i-1}),$$

and in its even subalgebra define

$$S_n^H = \prod_{i=1}^n (V_1^2 - (v^i + v^{-i})^2).$$

These are bases of two different modules. Their Hopf pairing is diagonal:

$$\langle P_m^H, S_n^H \rangle = \delta_{mn} \{2m+1\}_{2m}.$$

These two integral bases and their diagonal pairing are Habiro's construction [22]. With $q = v^2$, the polynomial part of the diagonal norm, up to sign and a Laurent monomial, is

$$D_n(q) = q^n \prod_{j=2}^{2n+1} (1 - q^j).$$

Theorem 4.10 (Annular joint tail). *For every $n \geq N \geq 0$,*

$$D_n(q) \in q^N (q; q)_N \mathbb{Z}[q].$$

Proof. The annular tail is certified coefficientwise by two independent factors: the leading monomial supplies the disk order, and counting each cyclotomic factor in the product supplies the required P_N -divisibility. This is a pairing tail estimate, not an assertion that the two annular bases define one invertible operator.

The monomial supplies q^N . For $d \geq 2$, the exponent of the cyclotomic polynomial $\Phi_d(q)$ in the product is

$$\left\lfloor \frac{2n+1}{d} \right\rfloor \geq \left\lfloor \frac{N}{d} \right\rfloor,$$

its exponent in $(q; q)_N$. For $d = 1$, the product has $2n$ factors and again supplies at least the required N . $\square$

This is a pairing-tail theorem, not an invertible annular S operator: the two displayed families are not bases of the same module. A relative operator still needs source and target lattices, boundary colors, inverse basis shifts, and the restricted-column property.

4.8. Literal Poincaré finite-stage representatives. The modified Poincaré series has the exact summands

$$T_n(q) = q^n(q^n; q)_n.$$

This is the distinguished joint formula in Habiro's source [22]. For $n \geq 1$,

$$(q^n; q)_n = \frac{(q; q)_{2n-1}}{(q; q)_{n-1}} = (q; q)_n \begin{bmatrix} 2n-1 \\ n \end{bmatrix}_q,$$

and therefore

$$T_n(q) = q^n(q; q)_n \begin{bmatrix} 2n-1 \\ n \end{bmatrix}_q \in q^n(q; q)_n \mathbb{Z}[q].$$

Theorem 4.11 (Poincaré finite-stage representatives). *The literal partial sums*

$$B_N(q) = \sum_{n=0}^{N-1} q^n(q^n; q)_n$$

satisfy

$$B_{N+1} - B_N \in q^N(q; q)_N \mathbb{Z}[q].$$

They therefore define one joint element whose disk and cyclotomic projections come from the same finite polynomials.

Proof. The displayed factorization puts the Poincaré summand at stage n in both required ideals: its leading power supplies the disk order, and its cyclotomic product supplies the divisibility. The literal partial sums therefore represent both limits through the same finite polynomials.

The increment is T_N , and the preceding factorization gives the required divisibility. $\square$

This explicit compatible disk–cyclotomic lift is special data supplied by the formula. It is not obtained from the closed cyclotomic coordinate by a continuous map.

4.9. Proper-height approximants after the slide quotient.

Let $L = \bigoplus_{\lambda \in X} \Lambda u_\lambda$ be a declared polynomial lattice, let $H : X \rightarrow \mathbb{Z}_{\geq 0}$ be proper, and let $\mathcal{X} \subset L$ be either a raw or saturated full-slide relation lattice. Put

$$C_N = \{\lambda : H(\lambda) < N\}.$$

For a polynomial coefficient family $x = (x_\lambda)$, define

$$v_N(x) = \left[\sum_{\lambda \in C_N} j_{H(\lambda)} x_\lambda u_\lambda \right] \in L/(\mathcal{X} + j_N L).$$

Theorem 4.12 (Quotient-valued finite-stage representatives). *The classes $v_N(x)$ are compatible under the stage transition maps and define an element*

$$\varprojlim_N L/(\mathcal{X} + j_N L).$$

More generally, if $C \subset C'$ are finite lower sets and

$$\nu(C) = \min_{\lambda \notin C} H(\lambda),$$

then the corresponding finite sums differ by an element of $j_{\nu(C)} L$.

Proof. Proper height makes each finite stage a finite marked-state sum, while the regulator j_N makes the new height layer invisible at the preceding stage. This gives a common joint limit in the quotient; an exact groupoid arrow transports it only when it transports both the lattice and the declared relation submodule.

The new terms from stage N to stage $N+1$ have height N and hence carry the factor j_N . They vanish at stage N . The same argument with $\nu(C)$ proves the lower-set statement. $\square$

The finite-stage module itself need not have finite rank. Finiteness of the approximant comes from properness of H . If an exact Λ -linear groupoid arrow transports both L and $\mathcal{X}$, it transports every v_N and its joint limit exactly. This comparison is based at one declared source chamber; it does not compare independently chosen height sections or cotests in unrelated chambers.

Finally, a scalar sequence appears only after choosing compatible cotests

$$\epsilon_N : L \longrightarrow \Lambda / (\mathfrak{j}_N)$$

which annihilate $\mathcal{X}$. The universal property of the quotient says that this condition is both necessary and sufficient for descent. The CRT, restricted completion, and proper-height regulator construct the domain of such a cap; they do not manufacture the nonzero root-specialized cap, its one-sided handlebody covariance, or its stabilization normalization.

5. REGULATED LATTICES FROM THE GENERIC LOLLIPOP BASIS

To make the two completion problems calculable, we first need a lattice whose basis records color growth and whose pairing is controlled. This section starts from the established generic lollipop basis and builds the particular Newton-corrected, regulated polynomial lattice used by the later chamber and cap constructions. Thus “lollipop lattice” below names the present lattice construction, not a claim that every lattice associated with a lollipop basis has the same definition.

The construction has three separate ingredients: a generic orthogonal lollipop basis, a proper height on its color set, and a regulator which is simultaneously small in the disk and cyclotomic topologies. Proper height makes each cutoff finite; the regulator places the pairing increments in both filtration ideals, so the finite pairings define one joint inverse-limit class. No mapping-class or filling assertion is needed for this first layer. The relative and higher-genus lollipop normalizations are those of integral TQFT [18]; the cyclotomic regulator follows the Habiro filtration [22].

The intended topological test is to cap a marked handlebody state with its complementary handlebody. The non-Laurent lattice records the states used in this pairing, and the joint inverse-limit class supplies controlled finite approximants for both specializations. The choice of cap remains separate from the basis and regulator.

5.1. The coefficient system and the joint ideals. Let

$$\Lambda = \mathbb{Z}[A], \quad q = A^2, \quad \mathbb{F}_A = \mathbb{Q}(A),$$

and, for $N \geq 0$, put

$$C_N = (q; q)_N = \prod_{j=1}^N (1 - q^j), \quad J_N = q^N C_N = A^{2N} (A^2; A^2)_N.$$

The ideals are nested because

$$\frac{J_h}{J_N} = J_{h-N} \begin{bmatrix} h \\ N \end{bmatrix}_q \in \Lambda \quad (h \geq N).$$

We write

$$\widehat{\Lambda}_J = \varprojlim_N \Lambda / (J_N).$$

The factor A^{2N} makes a J_N -small term small at $A = 0$, while $(q; q)_N$ kills it at every q -root whose order is at most N . A single sequence with increments in (J_N) is therefore a literal common-origin sequence for the two topologies.

5.2. Relative Newton lollipop. Fix a one-holed torus whose boundary has color $2c$. Let $L_{c,n}$ denote the usual generic lollipop basis and let z be the core curve. Set

$$\lambda_i = -q^{i+1} - q^{-i-1}, \quad Q_n^{(c)} = L_{c,0} \prod_{i=c}^{c+n-1} (z - \lambda_i).$$

These are Newton interpolation products in the core variable z , at the successive nodes $\lambda_c, \lambda_{c+1}, \dots$, multiplied by $L_{c,0}$. The basis assertion below is over $\mathbb{F}_A$; polynomial clearing and regulation are separate steps. The relative Hopf pairing will be denoted by B_c .

Theorem 5.1 (Generic relative Newton basis). *For every $c \geq 0$, the vectors $Q_n^{(c)}$, $n \geq 0$, form a monic triangular basis over $\mathbb{F}_A$ and are pairwise orthogonal for B_c .*

Proof. Multiplication by z is monic raising in the lollipop filtration, so the change from $L_{c,n}$ to $Q_n^{(c)}$ is unitriangular. Encircling $L_{c,m}$ by a color-one core has eigenvalue λ_{c+m} . If $k > 0$, the first argument of $B_c(Q_{n+k}^{(c)}, Q_n^{(c)})$ contains $z - \lambda_{c+m}$ for every $0 \leq m \leq n$ appearing in the second argument, hence annihilates each term. Symmetry gives orthogonality in the reverse order. $\square$

Write

$$\eta_{c,n} = B_c(Q_n^{(c)}, Q_n^{(c)}), \quad L = c + n + 1, \quad M = 2c + n + 1,$$

and

$$D(c, n) = (c + n)^2 + 3c + 2n.$$

The generic norm obtained from the lollipop evaluation is

$$\eta_{c,n} = q^{-c(c+1)/2} \{n\}! \{2c + 2n + 1\}!! \{2c + n + 1\}^+! \frac{(\{c\}_q!)^2}{\{1\}_q \{2c\}_q!}.$$

Theorem 5.2 (Exact norm factorization). *One has*

$$\eta_{c,n} = (-1)^c q^{-D(c,n)} \frac{(q; q)_n (q; q^2)_L (-q; q)_M}{(1 - q^2)^{\lceil \frac{2c}{2} \rceil} q^2}.$$

The quotient after the Laurent monomial is removed lies in $\mathbb{Z}[q]$ and is divisible by $(q; q)_n$. Consequently

$$\mathcal{D}_{c,n} := q^{D(c,n)+n} \eta_{c,n} \in q^n (q; q)_n \mathbb{Z}[q], \quad \text{ord}_q \mathcal{D}_{c,n} = n.$$

Proof. The norm factorization is obtained by substituting the three quantum-integer identities, collecting the Laurent powers into the declared exponent, and then comparing cyclotomic exponents by residue class. The residual polynomial has constant coefficient one, so the exponent comparison gives the stated divisibility and sharp disk order; the extra monomial is what supplies the nonzero disk order.

Use

$$\{j\} = -q^{-j/2} (1 - q^j), \quad \{j\}^+ = q^{-j/2} (1 + q^j), \quad \{j\}_q = -q^{-j} (1 - q^{2j}).$$

Collecting Laurent powers gives $-D(c, n)$. For a cyclotomic polynomial $\Phi_d(q)$, the exponent in the numerator is

$$e_d = \left\lfloor \frac{n}{d} \right\rfloor + \mathbf{1}_{d \text{ odd}} \left\lfloor \frac{L + (d-1)/2}{d} \right\rfloor + \mathbf{1}_{d \text{ even}} \left\lfloor \frac{M + d/2}{d} \right\rfloor,$$

whereas the denominator exponent is

$$\delta_d = \mathbf{1}_{d=1 \text{ or } 2} + \left\lfloor \frac{2c \gcd(d, 2)}{d} \right\rfloor - 2 \left\lfloor \frac{c \gcd(d, 2)}{d} \right\rfloor.$$

A residue-class check, separately for odd d , even $d > 2$, and $d = 1, 2$, gives

$$e_d - \delta_d \geq \left\lfloor \frac{n}{d} \right\rfloor.$$

These are precisely the cyclotomic exponents of $(q; q)_n$. The residual polynomial has constant coefficient one, proving both divisibility and the sharp disk order. $\square$

The extra factor q^n is essential: multiplying only by $q^{D(c,n)}$ clears the Laurent pole but leaves disk order zero.

Worked checkpoint. For the first nonvacuum relative lollipop, $c = 0$ and $n = 1$, the normalized norm is

$$\mathcal{D}_{0,1} = q(1-q)^2(1+q^2)(1+q+q^2) = q(1-q)(1-q^3)(1+q^2).$$

Thus

$$\frac{\mathcal{D}_{0,1}}{J_1} = (1-q^3)(1+q^2) \in \mathbb{Z}[q], \quad \text{ord}_q \mathcal{D}_{0,1} = 1.$$

This is the smallest nontrivial instance in which the disk and cyclotomic factors occur simultaneously.

5.3. Orthogonal lollipops in arbitrary genus. Let G be a lollipop spine in a genus- g handlebody. Write a generic admissible coloring as $\lambda = (a, b, c_{\text{tree}})$: the stick colors are $2a_i$, the loop colors are $a_i + b_i$, and c_{tree} records the remaining tree colors. Denote the countable color set by X_G . The generic divided lollipop vectors $\tilde{b}_\lambda$ are obtained by taking the Newton factor in each handle and the ordinary clasped graph on the remaining tree.

Theorem 5.3 (Generic arbitrary-genus orthogonality). *The family $\{\tilde{b}_\lambda\}_{\lambda \in X_G}$ is a $\mathbb{F}_A$ -basis of the generic graph state space V_G . For the higher-genus Hopf pairing,*

$$B_G(\tilde{b}_\lambda, \tilde{b}_\mu) = \delta_{\lambda\mu} \eta_\lambda, \quad \eta_\lambda \in \mathbb{F}_A^\times.$$

Proof. The arbitrary-genus statement is assembled handle by handle. Each Newton change is triangular and invertible over the generic field, while the fixed-tree factors supply the remaining basis. Orthogonality is first known at sufficiently many admissible roots; because each fixed pair gives a rational function of the parameter, those infinitely many equalities promote to generic equality.

In each handle the Newton change of basis is triangular and invertible over $\mathbb{F}_A$, so their tensor product, together with the fixed tree basis, is a basis. Orthogonality and the factorized norm hold at every sufficiently large admissible odd level. For any fixed pair of generic colorings both sides are rational functions of A ; equality at infinitely many roots therefore gives equality in $\mathbb{F}_A$. The diagonal evaluations are nonzero generically. $\square$

This does not identify one generic $\mathbb{Z}[q]$ -lattice whose specializations are all prime-level integral lattices. Jones–Wenzl recursions have color-dependent denominators, and unbounded colors would require infinitely many inversions. The next definition instead chooses a lattice directly inside V_G .

5.4. Proper-height primal lattices and lattices of test functionals. Here a *cotest* is a test functional in the algebraic dual.

Definition 5.4. A height $H : X_G \rightarrow \mathbb{Z}_{\geq 0}$ is *proper* when $H^{-1}([0, N])$ is finite for every N . We require $H(0) = 0$ in the closed vacuum sector and may take the maximum of all dynamic edge colors and derived vertex half-sums. Put

$$\rho_\lambda = J_{H(\lambda)}, \quad P_\lambda = \rho_\lambda \tilde{b}_\lambda, \quad S_\lambda = \eta_\lambda^{-1} \tilde{b}_\lambda,$$

and define

$$L_G^P = \bigoplus_{\lambda} \Lambda P_\lambda, \quad L_G^S = \bigoplus_{\lambda} \Lambda S_\lambda.$$

If $\tilde{b}^\lambda(\tilde{b}_\mu) = \delta_{\lambda\mu}$, let

$$L_G^{S,*} = \bigoplus_{\lambda} \Lambda \tilde{b}^\lambda.$$

The generators may be rational vectors; integrality refers to their declared coefficient lattices.

Theorem 5.5 (Regulated Hopf pairing). *For all λ, μ ,*

$$B_G(P_\lambda, S_\mu) = \tilde{b}^\mu(P_\lambda) = \delta_{\lambda\mu} J_{H(\lambda)}.$$

For arbitrary coefficient families $x_\lambda, y_\lambda \in \Lambda$, the sum

$$\sum_{\lambda \in X_G} x_\lambda y_\lambda J_{H(\lambda)}$$

is well defined in $\hat{\Lambda}_J$.

Proof. The first identity is immediate from orthogonality. Modulo J_N , only the finitely many labels with $H(\lambda) < N$ survive, because $J_{H(\lambda)}$ is a multiple of J_N for every other label. Hence the sum is finite at every stage and determines a unique inverse-limit class. $\square$

The formal coefficient products used here are larger than the J -adic completion of the direct sum. Convergence is supplied by the diagonal regulator, not by silently identifying those two products.

5.5. Literal finite lower-ideal pairing representatives and algebraic word chambers.

Let $\mathcal{C}_N = \{\lambda : H(\lambda) < N\}$. Define

$$K_N(x, y) = \sum_{\lambda \in \mathcal{C}_N} x_\lambda y_\lambda J_{H(\lambda)}.$$

Theorem 5.6 (Common approximants for the disk-adic and cyclotomic filtrations). *Every K_N is a finite polynomial and*

$$K_{N+1} - K_N = \sum_{H(\lambda)=N} x_\lambda y_\lambda J_N \in J_N \Lambda.$$

Thus the same finite formula gives its disk approximation modulo A^{2N} and its cyclotomic approximation at every q -root of order at most N .

Proof. Proper height makes each color cutoff finite, and the difference between successive cutoffs is exactly the new height layer. Every term in that layer carries the same joint regulator J_N , so the finite formula is Cauchy in both the disk and cyclotomic branches. The later word transport preserves this construction only when it carries both sides of the pairing. $\square$

Now let $U_p : V_{G_0} \rightarrow V_G$ be a word in algebraic recoupling maps: both U_p and U_p^{-1} have finite columns on the algebraic direct sums. The full dual is used on cotests, so no row-finiteness assumption is required. Set

$$L_{G,p}^P = U_p L_{G_0}^P, \quad L_{G,p}^{S,*} = (U_p^{-1})^* L_{G_0}^{S,*}.$$

Proposition 5.7 (Word-chamber covariance). *For an algebraic arrow $a : G \rightarrow G'$,*

$$U_a : L_{G,p}^P \xrightarrow{\sim} L_{G',ap}^P, \quad (U_a^{-1})^* : L_{G,p}^{S,*} \xrightarrow{\sim} L_{G',ap}^{S,*},$$

and

$$\langle U_p P_\lambda, (U_p^{-1})^* \tilde{b}^\mu \rangle = \delta_{\lambda\mu} J_{H(\lambda)}.$$

Consequently the finite common approximants are unchanged under simultaneous primal and contragredient transport.

Proof. Word-chamber covariance transports the primal basis and the contragredient cotest together. The arrow composition identities give the new chamber, scalar regulator ideals commute with the generic maps, and inverse-dual pairing preserves the diagonal value. The finite lower-ideal pairing representative is therefore unchanged only under simultaneous two-sided transport.

The transport identities are definitions and $U_{ap} = U_a U_p$. Scalar ideals commute with $\mathbb{F}_A$ -linear maps. Finally $\langle U_p v, (U_p^{-1})^* f \rangle = \langle v, f \rangle$. $\square$

At this stage the word remains part of the chamber label. Pure recoupling path independence, endpoint descent, root-specialized caps, full slide annihilation, stabilization, and comparison with WRT or $\widehat{Z}$ require additional theorems developed below.

6. OPPOSITE TWISTS, POLYNOMIAL ENDPOINTS, AND CHAMBER APPROXIMANTS

The generic longitude twist has infinite columns in the Newton basis, so it is not an endomorphism of the algebraic direct sum. It nevertheless has row-finite triangular coordinates, an exact generic inverse, polynomial source and target endpoint bases, and integral Hopf-contracted kernels after explicit separable clearing. These facts produce an arbitrary-genus chamber system for standard handles without pretending that one non-Laurent primal lattice is invariant under both twist signs. The one-holed-torus basis and twist conventions are matched to [18], with the graphical normalization fixed by [12].

6.1. Positive and negative longitude matrices. Retain

$$\Lambda = \mathbb{Z}[A], \quad q = A^2, \quad \mathbb{F}_A = \mathbb{Q}(A), \quad J_h = q^h(q; q)_h.$$

For boundary color $2c$, set

$$Q_n^{(c)} = \frac{Q_n^{(c)}}{\{n\}!}, \quad \{j\} = (-A)^j - (-A)^{-j}.$$

The positive and negative Kirby multipliers have coefficients

$$\gamma_k^+ = \frac{(-A)^{(k^2+5k)/2}}{\prod_{j=1}^k (A^{2j+1} - 1)}, \quad \gamma_k^- = A^{-3k} \gamma_k^+.$$

With

$$C_{a,b}^l = (-1)^l \frac{\{a\}! \{b\}! \{a+b+1\}!}{\{a-l\}! \{b-l\}! \{a+b+1-l\}! \{l\}!},$$

define, for $n \geq m$ and $k = l + n - m$,

$$b_{n,m,l}^{\pm(c)} = C_{k,m+c}^l \frac{\gamma_k^{\pm} \{n\}!}{\{k\}! \{m\}!}, \quad b_{n,m}^{\pm(c)} = \sum_{l=0}^{m+c} b_{n,m,l}^{\pm(c)}.$$

Then

$$t_{\pm}^* Q_m^{(c)} = \sum_{n=m}^{\infty} b_{n,m}^{\pm(c)} Q_n^{(c)},$$

and the diagonal entries are

$$b_{m,m}^{+(c)} = \mu_{c+m}, \quad b_{m,m}^{-(c)} = \mu_{c+m}^{-1}, \quad \mu_j = (-1)^j A^{j(j+2)}.$$

Each coefficient $b_{n,m}^{\pm(c)}$ is a finite sum, but a fixed column has infinitely many rows. These formulas therefore define formal lower-triangular matrices; the next proposition asserts an entrywise inverse identity, not an endomorphism of the algebraic direct sum.

Proposition 6.1 (Formal inverse identity). *The two infinite lower-triangular matrices B^+ and B^- are formal inverses: every matrix-product entry is a finite sum and equals δ_{nm} in $\mathbb{F}_A$.*

Proof. The inverse statement is checked entry by entry, rather than by multiplying infinite operators on the direct sum. For fixed source and target indices the triangular shape leaves only finitely many intermediate colors. Root-level inverse twists identify that finite rational expression at infinitely many admissible specializations, which forces the generic identity.

For fixed (n, m) the product uses only $m \leq k \leq n$. At every sufficiently large admissible odd level, the chosen lifts are inverse geometric twists, so this finite rational function of A equals δ_{nm} . Equality at infinitely many values of A proves the generic identity. $\square$

6.2. Endpoint-local products. The following general lemma explains why row-finite triangular operators act on product completions even when their columns are infinite.

Theorem 6.2 (Two-sided triangular endpoint products). *Let $V = \bigoplus_{h \geq 0} V_h$, with every V_h finite dimensional, and let $T_{hk} = 0$ for $h < k$ with T_{hh} invertible. There are finite free lattices L_h^- and $L_h^+ = T_{hh} L_h^-$ for which the normalized matrix B of T and its inverse $C = B^{-1}$ are lower unitriangular over Λ .*

On

$$\mathcal{P}_\pm = \prod_{h \geq 0} (\widehat{\Lambda}_J \otimes_\Lambda L_h^\pm)$$

the maps T and T^{-1} are continuous inverse homeomorphisms. For normalized dual-coordinate products the covector push-forward is

$$(T_\# y)_k = \sum_{h \geq k} \frac{J_h}{J_k} C_{hk}^* y_h,$$

and

$$\langle x, y \rangle_\pm = \sum_h J_h y_h(x_h)$$

is convergent and invariant under $(T, T_\#)$.

Proof. The endpoint lattices absorb the diagonal denominators first, and the successive denominator-clearing factors absorb the finitely many lower-triangular blocks at each height. The remaining inverse is integral by finite triangular recursion. The J -filtration then suppresses the distant tail in the dual action, so finite-support pairing invariance extends to the product completion.

Choose reference lattices M_h . Clearing the finitely many blocks $T_{hh}^{-1} T_{hk}$ with $k < h$ by factors Δ_h , and putting $D_h = \prod_{j \leq h} \Delta_j$, gives

$$L_h^- = D_h^{-1} M_h, \quad L_h^+ = T_{hh} L_h^-, \quad B_{hk} = \frac{D_h}{D_k} T_{hh}^{-1} T_{hk} \in \text{Mat}(\Lambda).$$

Finite triangular recursion makes C integral and row finite. For the dual tail use the exact identity

$$\frac{J_h}{J_k} = J_{h-k} \begin{bmatrix} h \\ k \end{bmatrix}_q \in (J_{h-k}).$$

Hence the terms with $h \geq k + N$ vanish modulo J_N . Ordinary finite matrix algebra proves inversion and pairing invariance on finite-support sequences, and continuity and separatedness extend the identities. $\square$

This theorem is formal: arbitrary denominator-clearing factors D_h can have unbounded disk or cyclotomic divisors. The explicit endpoint theorem below removes this defect for a standard longitude coordinate matrix.

6.3. Closed matrices and the universal polynomial endpoint pair. Put

$$d = n - m, \quad M = m + c, \quad \tau_{n,m} = \frac{d(n+m+1)}{2},$$

$$\Xi(c, n, m) = \frac{d^2 + 3d + M^2 + 2Md + M}{2}, \quad D_{M,d}(A) = \prod_{j=1}^d (1 - A^{2(M+j)+1}).$$

Theorem 6.3 (Closed raw matrices). *The divided-Newton matrices satisfy*

$$b_{n,m}^{+(c)} = (-1)^d (-A)^{M-\tau_{n,m}} q^{\Xi(c,n,m)} \frac{\begin{bmatrix} n \\ m \end{bmatrix}_q}{D_{M,d}(A)},$$

$$b_{n,m}^{-(c)}(A) = (-1)^d b_{n,m}^{+(c)}(A^{-1}).$$

Their exact A -orders are

$$v_+ = M(M+2) + d(m+2c) + \frac{d(d+5)}{2}, \quad v_- = -M(M+2) - dm + \frac{d(d-1)}{2}.$$

Proof. The raw coefficient calculation separates three tasks: convert the brace quotients to a Gaussian polynomial, collect the Laurent monomial, and leave the finite denominator $D_{M,d}$ visible. Because the two non-monomial factors have constant coefficient one, the resulting orders are read from the collected monomial without an unproved cancellation claim.

The brace quotient is

$$\frac{\{n\}!}{\{m\}!(q; q)_d} = (-1)^d (-A)^{-\tau_{n,m}} \begin{bmatrix} n \\ m \end{bmatrix}_q.$$

Substitution into the terminating contracted formula proved below gives the positive identity; mirror substitution gives the negative one. Both the Gaussian polynomial and $D_{M,d}$ have constant coefficient one, so the remaining monomial powers give the exact orders. $\square$

Define

$$f(n) = \binom{n}{2}, \quad F_n^{(c)} = (Aq^{c+1}; q)_n,$$

and the endpoint vectors

$$\mathbf{E}_n^- = \frac{A^{-f(n)}}{F_n^{(c)}} Q_n^{(c)}, \quad \mathbf{E}_n^+ = \mu_{c+n} \mathbf{E}_n^-.$$

Theorem 6.4 (Polynomial endpoint pair). *In the source and target endpoint bases, the two twist matrices are*

$$\widehat{\beta}_{n,m}^+ = (-1)^{\tau_{n,m}} \begin{bmatrix} n \\ m \end{bmatrix}_q,$$

$$\widehat{\beta}_{n,m}^- = (-1)^{d+\tau_{n,m}} A^{d(d-1)} \begin{bmatrix} n \\ m \end{bmatrix}_q.$$

They lie in $\mathbb{Z}[A]$, are independent of c , are lower unitriangular, and are formal inverses.

Proof. The endpoint basis is chosen to contain exactly the denominator string that appears in the raw coefficient. Conjugating by those basis factors therefore turns the two twist matrices into signed Gaussian matrices. Finite q -binomial inversion supplies the opposite direction and explains why the result is polynomial and independent of the boundary color.

Because $D_{M,d} = F_n^{(c)} / F_m^{(c)}$, endpoint conjugation cancels the entire denominator. If $G_{n,m} = \begin{bmatrix} n \\ m \end{bmatrix}_q$ and $R_{nn} = (-1)^{n(n+1)/2}$, then $\widehat{\beta}^+ = RGR^{-1}$. The finite q -binomial inversion formula

$$(G^{-1})_{n,m} = (-1)^{n-m} q^{\binom{n-m}{2}} \begin{bmatrix} n \\ m \end{bmatrix}_q$$

gives $\widehat{\beta}^- = RG^{-1}R^{-1}$. $\square$

The factor $F_n^{(c)}$ is an A -adic unit. At the admissible odd-prime branch $A = -q^{(p+1)/2}$ it is also a unit on every admissible color block. This is a cutoff-dependent finite-level statement, not an untruncated branch-independent Habiro operator.

There is a sharp obstruction to using one bounded-below diagonal chart.

Proposition 6.5 (Diagonal two-sided no-go). *If source and target weights $s_m, t_m \in \mathbb{F}_A^\times$ make both raw twist directions entrywise A -regular, and $a_m = \text{ord}_A(s_m)$, $b_m = \text{ord}_A(t_m)$, then*

$$b_m - a_m = (m + c)(m + c + 2), \quad a_m - a_{m+d} \geq dm + \frac{d(d-1)}{2}.$$

In particular $a_n \leq a_0 - n(n-1)/2$. The exponent in Theorem 6.4 is optimal among diagonal coordinate weights.

Proof. The no-go calculation reads the diagonal eigenvalue relation first, then uses the lowest off-diagonal order of the positive twist to force a monotone loss in the source weights. Iterating that inequality gives the quadratic bound, and the explicit endpoint chart attains it; the obstruction is therefore a coordinate-weight obstruction, not a failure of the formal inverse identity.

The diagonal eigenvalues force the first equality. Substitution into the positive off-diagonal order gives the second inequality; setting $m = 0$ gives the bound. The displayed endpoint pair attains equality. $\square$

6.4. Hopf-contracted kernels. Let $\eta_{c,n}$ be the relative norm from Theorem 5.2. Define

$$\mathcal{K}_{n,m,l}^{\pm(c)} = \eta_{c,n} C_{l+n-m,m+c}^l \frac{\gamma_{l+n-m}^{\pm}}{\{l+n-m\}!}, \quad \mathcal{K}_{n,m}^{\pm(c)} = \sum_{l=0}^{m+c} \mathcal{K}_{n,m,l}^{\pm(c)}.$$

Thus $\mathcal{K}_{n,m}^{\pm(c)} = B_c(t_\pm^* Q_m^{(c)}, Q_n^{(c)})$.

Theorem 6.6 (Terminating product and Laurent cancellation). *For $n \geq m$,*

$$\mathcal{K}_{n,m}^{+(c)} = \eta_{c,n} \frac{(-A)^M q^{\Xi(c,n,m)}}{(q; q)_d (Aq^{M+1}; q)_d},$$

and

$$\mathcal{K}_{n,m}^{-(c)}(A) = (-1)^c A^{-2c(c+1)} \mathcal{K}_{n,m}^{+(c)}(A^{-1}).$$

Both kernels lie in $\mathbb{Z}[A, A^{-1}]$ for all indices.

Proof. The Hopf pairing converts the infinite twist column into a finite sum indexed by the contracted color. A terminating q -identity evaluates that sum, and the relative norm supplies precisely the factors needed to cancel the remaining Pochhammer denominator. The positive calculation and its mirrored negative version are performed before the common separable clearing is introduced.

After converting braces to q -products, the positive finite sum is a terminating q -Chu–Vandermonde sum:

$$\sum_{l=0}^M \frac{(q^{-M}; q)_l (q^{n+c+2}; q)_l}{(Aq^{d+1}; q)_l (q; q)_l} q^l.$$

Its finite product evaluation, followed by cancellation of the two Pochhammer strings, gives the first formula. Mirroring each summand gives the second. For Laurent polynomiality, use

$$\frac{\eta_{c,n}}{\eta_{c,m}} = \frac{\{n\}!}{\{m\}!} \prod_{r=1}^d \{2M + 2r + 1\} \prod_{r=1}^d \{2c + m + 1 + r\}^+.$$

The first quotient cancels $(q; q)_d$ up to a Gaussian polynomial, and

$$\frac{\{j\}}{1 - A^j} = -(-A)^{-j} (1 + A^j)$$

cancels each factor of $(Aq^{M+1}; q)_d$. The mirror identity handles the negative kernel. $\square$

The vacuum column is especially simple:

$$\mathcal{K}_{n,0}^{+(0)} = A^{-n(n+1)} \prod_{j=3}^{2n+2} (1 + A^j), \quad \mathcal{K}_{n,0}^{-(0)} = A^{-n(n+4)} \prod_{j=3}^{2n+2} (1 + A^j).$$

Worked checkpoint. For $c = m = 0$ and $n = 1$ these formulas give

$$\mathcal{K}_{1,0}^{+(0)} = A^{-2}(1 + A^3)(1 + A^4), \quad \mathcal{K}_{1,0}^{-(0)} = A^{-5}(1 + A^3)(1 + A^4).$$

Since $R_0(1) + R_0(0) = 8 + 2 = 10$, the common clearing produces

$$A^{10}\mathcal{K}_{1,0}^{+(0)} = A^8(1 + A^3)(1 + A^4), \quad A^{10}\mathcal{K}_{1,0}^{-(0)} = A^5(1 + A^3)(1 + A^4),$$

both in $\mathbb{Z}[A]$.

One separable two-sided polynomial shift is

$$R_c(s) = 2(s + c + 1)^2.$$

Corollary 6.7 (Separable kernel clearing). *For both signs and every $n \geq m$,*

$$A^{R_c(n)+R_c(m)}\mathcal{K}_{n,m}^{\pm(c)} \in \mathbb{Z}[A].$$

Proof. The uniform clearing is obtained by bounding the valuation of each summand, then writing $n = m + d$ so the proposed separable exponent dominates both sign-specific losses coefficientwise. Laurent polynomiality finishes the integrality claim; the sharper cancellation in the positive case is not needed for this uniform statement.

Direct summand valuation gives sufficient exponents

$$E_{\Sigma,+} = n^2 + (4c + 2m + 1)n + 3c^2 + 2cm + 5c + 2m,$$

and $E_{\Sigma,-} = E_{\Sigma,+} + 3(n + c)$. Subtracting these from $R_c(n) + R_c(m)$ gives polynomials with nonnegative coefficients after writing $d = n - m$. Laurent polynomiality then proves the claim. The exact full positive order contains additional cancellation, but is not needed for this uniform separable bound. $\square$

6.5. Arbitrary-genus cutoff and endpoint chambers. For a fixed lollipop marking G , encode each dynamic coloring injectively by a multidegree $d_G(\lambda) \in \mathbb{N}^r$ and order the admissible set componentwise. Let $\mathcal{C}_G$ be its finite lower ideals. Intersection and union make $\mathcal{C}_G$ a directed distributive lattice; every principal lower ideal is finite. The height

$$H_G(\lambda) = \max_j d_G(\lambda)_j$$

is proper and monotone, and $C_N = \{H_G < N\}$ is a cofinal sequence.

For a sign vector $\varepsilon \in \{-, +\}^g$, define

$$\mathbf{e}_\lambda^\varepsilon = \left(\prod_{i=1}^g \alpha_{a_i, b_i}^{\varepsilon_i} \right) \tilde{b}_\lambda,$$

where

$$\alpha_{a,n}^- = \frac{A^{-\binom{n}{2}}}{F_n^{(a)}}, \quad \alpha_{a,n}^+ = \mu_{a+n} \alpha_{a,n}^-.$$

For a finite lower ideal C , let $L_\varepsilon(C) = \bigoplus_{\lambda \in C} \Lambda \mathbf{e}_\lambda^\varepsilon$ and take the inverse limit under coordinate projection.

Theorem 6.8 (Endpoint chamber cube). *If $\varepsilon_s = -$, the positive standard twist on handle s is a continuous isomorphism from the ε product to the product with the s th sign changed to $+$; the negative twist is its inverse. The local matrices are those of Theorem 6.4. Twists on distinct handles tensor and commute, producing a g -dimensional Boolean endpoint cube.*

Proof. The endpoint cube is assembled at finite lower ideals. A fixed output color uses only finitely many lower source colors, so each truncation sees the finite Gaussian inverse pair; coordinate projections commute with those finite matrices. The tensor factors belong to distinct handles, which is what makes the endpoint changes commute without turning the cube into an integral action by repeated same-sign powers.

For any fixed output color only lower source colors occur. A lower ideal therefore contains every source coordinate needed to compute each output. Every finite truncation is an exact Gaussian inverse pair, and coordinate projection is compatible with the matrices. Distinct handles act on distinct tensor factors. $\square$

The cube is square-free, not an integral $\mathbb{Z}^g$ action: repeated same-sign twists introduce a negative off-diagonal disk order. Arbitrary powers require retaining an orbit degree.

6.6. Spectator denominator-clearing factors and scalar chambers. First single out one handle s . Let its stick half-color be $a_s = \kappa$, let $\sigma = (a, b_s, c_{\text{tree}})$ collect the spectator labels, and write

$$r_\sigma = \left\lfloor \frac{-e + \sum_i a_i}{2} \right\rfloor$$

when the trunk color is $2e$. Unprime only the active factor:

$$V_{\sigma,n} = \{n\}! \tilde{b}_{(a,b_s,n,c_{\text{tree}})}.$$

Higher-genus orthogonality factorizes as

$$B_G(V_{\sigma,n}, V_{\tau,m}) = \delta_{\sigma\tau} \delta_{nm} \rho_\sigma(A) \eta_{\kappa,n},$$

where

$$\rho_\sigma = (1-q)^{-2r_\sigma} \langle DG(a, c_{\text{tree}}) \rangle \frac{\{1\}_q}{\{2\kappa+1\}_q} \prod_{i \neq s} \left(\frac{\eta_{a_i, b_i}}{\{b_i\}!^2} \frac{\{1\}_q}{\{2a_i+1\}_q} \right).$$

This factor is independent of n and m .

For $a, b \geq 0$, put

$$E(a, b) = a^2 + 2ab + a + \frac{b^2 + 3b}{2}$$

and

$$S_{a,b}(q) = \frac{(q; q^2)_{a+b+1} (-q; q)_{2a+b+1}}{(q; q)_b \begin{bmatrix} 2a \\ a \end{bmatrix}_{q^2} (1 - q^{4a+2})}.$$

Theorem 6.9 (Polynomial divided-spectator factor). *One has*

$$\frac{\eta_{a,b}}{\{b\}!^2} \frac{\{1\}_q}{\{2a+1\}_q} = (-1)^a q^{-E(a,b)} S_{a,b}(q),$$

with

$$S_{a,b}(q) \in \mathbb{Z}[q], \quad S_{a,b}(0) = 1, \quad \text{ord}_{1-q} S_{a,b} = a.$$

Proof. The factorization is an inventory of cyclotomic exponents after the exact norm substitution. Separating odd and even indices reduces each exponent to a residue calculation, while the $q = 1$ order is read from the remaining linear factors. This proves polynomiality and the stated initial/order data without promoting the factor to a new endpoint operator.

Substitute the exact norm factorization of Theorem 5.2. For a cyclotomic polynomial $\Phi_j(q)$, the exponent in the resulting quotient is

$$\sigma_j = f_j(a, b) - 2 \left\lfloor \frac{b}{j} \right\rfloor + \mathbf{1}_{j|2} - \mathbf{1}_{j|(4a+2)},$$

where $f_j(a, b)$ is the numerator exponent from the norm factorization. If $j > 1$ is odd, write $a = kj + r$ and $b = tj + s$; the remaining residue contribution is

$$k + \left\lfloor \frac{r + s + (j+1)/2}{j} \right\rfloor - \left\lfloor \frac{2r}{j} \right\rfloor - \mathbf{1}_{r=(j-1)/2} \geq 0.$$

If $j = 2h > 2$, write $a = kh + r$ and $b = 2ht + s$; the residue contribution is

$$k + \left\lfloor \frac{2r + s + 1 + h}{2h} \right\rfloor - \left\lfloor \frac{2r}{h} \right\rfloor - \mathbf{1}_{h \text{ odd}, r=(h-1)/2} \geq 0.$$

In each case the numerator floor cancels the only possible subtraction when the residue crosses the midpoint; the remaining k is nonnegative. For $j = 1, 2$ the exponent is exactly a . Thus the quotient is a polynomial, its constant coefficient is one, and its $(1 - q)$ -order is a . $\square$

The theorem cancels every divided local spectator factorial. The remaining possible denominators lie in the trunk factor, the active quantum-dimension factor, and the doubled-tree graph gauge; they are finite in each fixed spectator sector.

Let $S \subset \{1, \dots, g\}$ be nonempty, let θ collect all complementary labels, and unprime the active Newton factors:

$$V_{\theta, \mathbf{n}}^S = \left(\prod_{i \in S} \{n_i\}! \right) \tilde{b}_\lambda.$$

For a signed square-free product $T_{S, \delta} = \prod_{i \in S} t_{\delta_i, i}^*$, higher-genus orthogonality gives

$$B_G(T_{S, \delta} V_{\theta, \mathbf{m}}^S, V_{\theta, \mathbf{n}}^S) = \delta_{\theta \vartheta} \rho_\theta^S \prod_{i \in S} \mathcal{K}_{n_i, m_i}^{\delta_i, (a_i)}.$$

Here $\rho_\theta^S \in \mathbb{F}_A^\times$ is independent of the active indices.

Write $\rho_\theta^S = N_\theta^S / Q_\theta^S$ in lowest terms. A *spectator denominator-clearing factor* is a nonzero $D_\theta^S \in \Lambda$ such that $D_\theta^S \rho_\theta^S \in \Lambda$.

Proposition 6.10 (Lattice of denominator-clearing factors). *Every denominator-clearing factor is a multiple of Q_θ^S , and every nonzero multiple is a denominator-clearing factor. Modulo signs, gcd and lcm give a directed distributive lattice with minimum Q_θ^S . Taking the coordinatewise product over sectors preserves these properties.*

Proof. The denominator-clearing factor statement is a divisibility calculation in the coefficient UFD. Writing the spectator factor in lowest terms isolates the exact denominator; coprimality forces every denominator-clearing factor to contain it, and multiplication by any extra factor preserves clearing. The gcd/lcm operations and the coordinatewise sector product then give the stated directed lattice.

In the UFD $\mathbb{Z}[A]$, $Q \mid DN$ and $(N, Q) = 1$ imply $Q \mid D$. The converse is immediate, and the gcd/lcm claims follow in a UFD. $\square$

Set

$$\widehat{V}_{\theta, \mathbf{n}}^{S, D} = D_\theta^S A^{\sum_{i \in S} R_{a_i}(n_i)} V_{\theta, \mathbf{n}}^S.$$

Theorem 6.11 (Integral square-free scalar chamber). *For every sign choice and $\mathbf{m} \leq \mathbf{n}$,*

$$B_G(T_{S, \delta} \widehat{V}_{\theta, \mathbf{m}}^{S, D}, \widehat{V}_{\theta, \mathbf{n}}^{S, D}) = D_\theta^S (D_\theta^S \rho_\theta^S) \prod_{i \in S} \left[A^{R_{a_i}(n_i) + R_{a_i}(m_i)} \mathcal{K}_{n_i, m_i}^{\delta_i, (a_i)} \right] \in \Lambda.$$

Proof. The scalar chamber proof separates spectator denominators from active twist orders. The sector denominator-clearing factor first makes the spectator factor integral, while the uniform local shift makes each active Hopf-contracted kernel polynomial; the product formula then assembles those independent factors over the selected handles. This is a square-free chamber construction, not a claim that one diagonal chart is stable under arbitrary twist powers.

The first two factors are polynomial by the definition of a denominator-clearing factor, and each bracketed local factor is polynomial by Corollary 6.7. $\square$

For polynomial coefficient families define, on a finite lower ideal C ,

$$\mathbf{G}_C = \sum_{(\theta, \mathbf{n}) \in C} \sum_{\mathbf{m} \leq \mathbf{n}} x_{\theta, \mathbf{n}} y_{\theta, \mathbf{m}} J_{H_G(\theta, \mathbf{n})} B_G(T_{S, \delta} \widehat{V}_{\theta, \mathbf{m}}, \widehat{V}_{\theta, \mathbf{n}}).$$

Theorem 6.12 (Arbitrary-genus common scalar approximants). *If $C \subset C'$ and $\nu(C) = \min_{\lambda \notin C} H_G(\lambda)$, then*

$$\mathbf{G}_{C'} - \mathbf{G}_C \in J_{\nu(C)} \Lambda.$$

In particular, the height sequence has increments in $J_N \Lambda$, so the same finite polynomials converge in the disk and cyclotomic branches.

Proof. The common-net estimate is finite before it is completed. A larger lower ideal adds only finitely many outputs and finitely many lower sources, and the height of every new output supplies the corresponding J -factor. Consequently the same scalar finite sums are Cauchy in the disk and cyclotomic branches; the conclusion belongs to the common chamber because the construction retains both filtrations.

There are finitely many outputs in C' , and each has finitely many lower sources. Every newly added output has height at least $\nu(C)$; the exact quotient $J_h/J_{\nu(C)} \in \Lambda$ supplies the divisibility. $\square$

This scalar chamber and the polynomial endpoint cube are distinct charts. The additional scalar shift satisfies

$$R_c(m) - R_c(n) < 0 \quad (n > m),$$

and destroys the sharp two-sided endpoint inequality.

An endpoint-valuation bridge, not a diagonal identification, is needed to combine them.

7. TWO-SECTOR ENDPOINT-VALUATION GEOMETRY AND THE UNIVERSAL CONTRACTED CORE

The polynomial endpoints of Section 6 are not two bases of one disk-regular module. They are opposite endpoint-valuation sectors. This section computes their exact inside and outside valuations, places one integral scalar core inside every endpoint chamber, proves that the raw twist cannot preserve that core, and constructs the regulated pairing which survives this no-go. The local recoupling and norm inputs use the integral graphical conventions of [12, 18]; all new endpoint-valuation statements are proved below.

The section follows the obstruction that forces the different constructions. We first compare the inside- and outside-disk valuations, which identifies the two endpoint sectors. We then take common denominator-clearing factors and form one universal contracted core. The next step shows that this core lies in every endpoint lattice but is not preserved by the raw twists; that failure is the scalar-primal obstruction. Finally, an output regulator and finite-lower-ideal approximation produce the rigged pairing. Thus the common core is a shared contracted object, while the endpoint lattices and their twist actions remain distinct.

7.1. Exact inside and outside valuations. For $c, n \geq 0$ define

$$\mathcal{U}_{c,n}(X, z) = \frac{\prod_{j=1}^n (1 + X^{2(c+j)} z + X^{4(c+j)})}{\prod_{j=1}^n (1 - X^{2j}) \prod_{j=1}^n (1 - X^{2(c+j)+1})}.$$

It belongs to $\mathbb{Z}[[X]][z]$. Let ord_0 be A -order, and let ord_∞ be the B -order after $A = B^{-1}$.

Lemma 7.1 (Coefficient valuation). *For $0 \leq k \leq n$,*

$$\text{ord}_X[z^k] \mathcal{U}_{c,n}(X, z) = 2ck + k(k+1).$$

Proof. Every denominator is a unit at $X = 0$, so the lowest X -order in the z^k coefficient comes from selecting the z -term in the first k numerator factors and the constant term in the rest. Summing those k exponents gives the displayed valuation.

Every denominator factor has constant coefficient one. The unique lowest term in the z^k coefficient selects the z term from the first k numerator factors and 1 from the rest, with order $\sum_{j=1}^k 2(c+j) = 2ck + k(k+1)$. $\square$

Recall

$$E_n^- = \frac{A^{-\binom{n}{2}}}{(Aq^{c+1}; q)_n} Q_n^{(c)}, \quad E_n^+ = \mu_{c+n} E_n^-.$$

Theorem 7.2 (Exact two-sector polarization). *With $N_n = n(n+1)/2$,*

$$\begin{aligned} E_n^-(A) &= (-1)^{n+N_n} A^{-n(n+2c)} \mathcal{U}_{c,n}(A, z), \\ E_n^+(A) &= (-1)^{c+N_n} A^{c(c+2)+2n} \mathcal{U}_{c,n}(A, z), \\ E_n^-(B^{-1}) &= (-1)^{n+N_n} B^{n(n+1)} \mathcal{U}_{c,n}(B, z). \end{aligned}$$

Consequently

$$\begin{aligned} \text{ord}_0[z^k]E_n^- &= -n(n+2c) + 2ck + k(k+1), \\ \text{ord}_0[z^k]E_n^+ &= c(c+2) + 2n + 2ck + k(k+1), \\ \text{ord}_\infty[z^k]E_n^- &= n(n+1) + 2ck + k(k+1), \\ \text{ord}_\infty[z^k]E_n^+ &= n(n+1) + 2ck + k(k+1) - (c+n)(c+n+2). \end{aligned}$$

Thus E^+ is the inside-disk endpoint and E^- is the outside-disk endpoint; the opposite endpoint has an unbounded pole.

Proof. The endpoint formulas come from one common Newton product, but the monomial factored from that product is different in the inside and outside chambers. The coefficient valuation records the remaining regular factor, and the endpoint exponents then expose the unbounded pole at the opposite end. The same calculation tensorizes handle by handle; it does not identify the two analytic sectors.

Factoring the lowest monomial from each Newton factor gives

$$Q_n^{(c)} = A^{-2cn-n(n+1)} \prod_{j=1}^n (1 + A^{2(c+j)}z + A^{4(c+j)}).$$

Use

$$\{n\}! = (-1)^{n(n+3)/2} A^{-n(n+1)/2} (q; q)_n$$

and the endpoint definitions. The closed forms follow, and Lemma 7.1 gives the coefficient orders. Their minima are at $k = 0$. $\square$

The handlewise tensor product gives the same polarization in every finite genus. A sector-constant graph normalization does not alter it; uniform valuation under a change of marking is a separate recoupling problem.

7.2. The all-subset denominator-clearing data. Fix a genus- g lollipop marking G . Let $\mathcal{C}_G$ be its finite lower-ideal cutoff lattice, and let $\mathcal{D}_{G,S}$ be the denominator-clearing factor lattice of Section 6 for every nonempty $S \subset [g]$. Choose a positive-leading denominator-clearing factor $D_{\theta_S(\lambda)}^S$ in every complementary sector and set

$$D_\star(\lambda) = \text{lcm}_{\emptyset \neq S \subset [g]} D_{\theta_S(\lambda)}^S.$$

The sign cube $\mathbb{B}_g = \{-, +\}^g$, ordered by $- < +$, and the product

$$\mathfrak{A}_G = \mathcal{C}_G \times \mathbb{B}_g \times \prod_{\emptyset \neq S \subset [g]} \mathcal{D}_{G,S}$$

form a directed distributive lattice under coordinatewise intersection or union, minimum or maximum, and gcd or lcm. This statement is for each fixed finite genus; it contains no stabilization assertion.

Define

$$R_a(b) = 2(a+b+1)^2, \quad p_b(A) = A^{b(b+1)/2} \{b\}! = (-1)^{b(b+3)/2} (q; q)_b.$$

For the generic divided orthogonal lollipop vector $\tilde{b}_\lambda$, put

$$(7) \quad \mathbb{U}_\lambda = D_\star(\lambda) A^{\sum_{i=1}^g R_{a_i}(b_i)} \left(\prod_{i=1}^g p_{b_i}(A) \right) \tilde{b}_\lambda, \quad L_\star = \bigoplus_{\lambda} \Lambda \mathbb{U}_\lambda.$$

For a nonempty active set S , the subset-specific scalar vector is

$$\hat{V}_\lambda^S = D_{\theta_S(\lambda)}^S A^{\sum_{i \in S} R_{a_i}(b_i)} \left(\prod_{i \in S} \{b_i\}! \right) \tilde{b}_\lambda.$$

Theorem 7.3 (Universal square-free contracted core). *For every nonempty $S \subset [g]$,*

$$\mathbb{U}_\lambda = E_S(\lambda) \hat{V}_\lambda^S,$$

where

$$E_S(\lambda) = \frac{D_\star(\lambda)}{D_{\theta_S(\lambda)}^S} A^{\sum_{j \notin S} R_{a_j}(b_j) + \sum_{i \in S} b_i(b_i+1)/2} \prod_{j \notin S} p_{b_j}(A) \in \Lambda.$$

If $T_{S,\delta}$ is any square-free signed standard multitwist, then

$$\kappa_{\lambda\mu}^{\star, S, \delta} := B_G(T_{S,\delta} \mathbb{U}_\mu, \mathbb{U}_\lambda) \in \Lambda$$

on the triangular range and is zero otherwise. One lattice therefore supports all $3^g - 1$ nontrivial square-free signed contracted kernels.

Proof. The universal core is assembled by taking the lcm of the finite-sector denominator-clearing factors and adjoining the fixed factorial factors from the complementary handles. Dividing by a selected subset vector leaves the displayed polynomial multiplier; the previously established subset kernel then gives integrality by bilinearity. The construction is therefore a common contracted core, not an invariant primal module.

The selected subset denominator-clearing factor divides $D_\star$. Dividing (7) by $\hat{V}_\lambda^S$ gives the displayed polynomial E_S . Bilinearity expresses the new kernel as $E_S(\lambda)E_S(\mu)$ times the integral subset kernel of Theorem 6.11. $\square$

The theorem concerns contracted pairings; it does not claim $T_{S,\delta} L_\star \subset L_\star$.

7.3. The common core inside every endpoint. Put $F_b^{(a)} = (Aq^{a+1}; q)_b$ and define

$$\begin{aligned} m_{a,b}^- &= (-1)^{b(b+3)/2} A^{R_a(b) + \binom{b}{2}} F_b^{(a)}(q; q)_b, \\ m_{a,b}^+ &= (-1)^{b(b+3)/2 + a+b} A^{R_a(b) + \binom{b}{2} - (a+b)(a+b+2)} F_b^{(a)}(q; q)_b. \end{aligned}$$

The positive exponent simplifies to

$$(a+b+1)^2 + 1 + \frac{b(b-1)}{2} \geq 2b.$$

Theorem 7.4 (Loop-coercive endpoint inclusions). *For every sign vertex $\varepsilon \in \mathbb{B}_g$,*

$$\mathbb{U}_\lambda = M_\lambda^\varepsilon \mathbf{e}_\lambda^\varepsilon, \quad M_\lambda^\varepsilon = D_\star(\lambda) \prod_{i=1}^g m_{a_i, b_i}^{\varepsilon_i} \in \Lambda,$$

and

$$M_\lambda^\varepsilon \in \left(\prod_{i=1}^g J_{b_i} \right) \Lambda.$$

Thus $L_\star$ lies integrally in every endpoint lattice.

Proof. The endpoint inclusion is checked one handle at a time. Dividing the core by an endpoint vector produces the explicit multiplier $m^\pm$; its factorial string and the stated exponent bound give J_b divisibility. Multiplying over handles and by the global denominator-clearing factor places the same core in each endpoint chamber, while the nonunit multipliers explain why this inclusion is not equality.

Divide the core vector by the endpoint scalar handle by handle. The displayed $m^\pm$ result. Each contains $(q; q)_b$ and at least A^{2b} , hence a factor J_b . $\square$

Worked checkpoint. For one handle and unit denominator-clearing factor, the vacuum multipliers are

$$m_{0,0}^- = m_{0,0}^+ = A^2.$$

For the first loop excitation they are

$$m_{0,1}^- = A^8(1 - A^2)(1 - A^3), \quad m_{0,1}^+ = -A^5(1 - A^2)(1 - A^3).$$

Their exact A -orders are 8 and 5, while both contain $J_1 = A^2(1 - A^2)$. The example displays both the opposite-sector asymmetry and the common joint divisibility.

The inclusion is one way: these multipliers are nonunits. They are coercive when loop colors grow, but not when only stick or tree labels grow; the output factor $J_{H_G(\lambda)}$ below supplies uniform decay.

7.4. Closed endpoint-valuation images.

Lemma 7.5 (Cofinal colon estimate). *For every nonzero $z \in \Lambda$ and every k , there is $N = N(z, k)$ such that*

$$(J_N) : z \subset (J_k).$$

Multiplication by z on $\widehat{\Lambda}_J$ is therefore injective, a homeomorphism onto its image, and has closed image.

Proof. The closed-image statement is a valuation-growth argument in the coefficient UFD. A fixed multiplier can remove only finitely much of the irreducible content of J_N , so sufficiently deep regulators force the residual quotient into any prescribed J_k . This gives injectivity and continuity of multiplication and, with completeness, closedness; coordinatewise application then handles the product sector.

In the UFD Λ , $(J_N) : z$ is generated by $J_N / \gcd(J_N, z)$. The fixed polynomial z removes only finitely many powers of finitely many irreducibles, whereas their multiplicities in J_N grow without bound. Choose N so that the residual divisor contains J_k . This also makes division continuous on the image; completeness and separatedness make that image closed. $\square$

Let

$$\mathcal{S}_\star = \prod_\lambda \widehat{\Lambda}_J \mathbb{U}_\lambda, \quad \mathcal{P}_\varepsilon = \prod_\lambda \widehat{\Lambda}_J \mathbf{e}_\lambda^\varepsilon.$$

The ambient identity induces

$$\iota_\varepsilon(x_\lambda)_\lambda = (M_\lambda^\varepsilon x_\lambda)_\lambda.$$

Corollary 7.6 (Closed endpoint-valuation sector). *Each ι_ε is a homeomorphism onto the closed product submodule*

$$\prod_{\lambda} M_{\lambda}^{\varepsilon} \widehat{\Lambda}_J e_{\lambda}^{\varepsilon}.$$

It is not onto the full endpoint product.

Proof. Apply Lemma 7.5 coordinatewise. Product topology requires no uniform bound. Non-surjectivity follows because nonvacuum multipliers are divisible by A . $\square$

7.5. Why the scalar core is not a primal twist lattice. Fix a handle with boundary half-color c and write

$$\widehat{V}_n = DA^{R_c(n)}\{n\}! \widetilde{b}_n = Z_n^- E_n^- = Z_n^+ E_n^+.$$

Direct substitution gives, for $n = m + d$,

$$\begin{aligned} M_{n,m}^+ &= (-1)^{m+c} \frac{A^{(m+c)(m+c+2)-d(2(m+c)+d+1)}}{(q; q)_d (Aq^{m+c+1}; q)_d}, \\ M_{n,m}^- &= (-1)^{m+c} \frac{A^{-(m+c)(m+c+2)-d^2-4d(m+c+1)}}{(q; q)_d (Aq^{m+c+1}; q)_d}. \end{aligned}$$

Proposition 7.7 (Scalar-primal no-go). *Neither signed standard twist preserves the scalar product lattice. In the vacuum column $c = m = 0$, the off-diagonal A -orders are respectively $-d(d+1)$ and $-d^2 - 4d$. No finite free non-Laurent lattice can carry both a finite twist truncation and its inverse unless the determinant is a unit.*

Proof. The scalar-primal obstruction is already visible in the vacuum column: constant-term-one denominators leave the displayed negative monomial orders unchanged. The finite-height determinant packages the same defect invariantly; if a matrix and its inverse preserved one finite free lattice, their determinants would be mutual units, contradicting the nonunit twist eigenvalue.

The denominator products have constant coefficient one, so the displayed monomial exponents are exact. They are negative for every $d > 0$. More invariantly, the determinant through height N is

$$\prod_{n=0}^N \mu_{c+n} = \pm A^{\Delta_{c,N}},$$

with $\Delta_{c,N} > 0$ outside the vacuum. If both a matrix and its inverse were endomorphisms of one finite free Λ -lattice, their determinants would be inverse units, a contradiction. $\square$

The obstruction is the algebraic shadow of the two topological fillings: positive and negative longitude twists require opposite endpoint chambers. The scalar core can therefore be paired with both fillings after the stated regulator is inserted, but it cannot itself be the common primal lattice on which both twist directions act integrally.

7.6. The rigged endpoint-valuation pairing and finite-stage representatives. Define the weighted cotest product

$$S_{\star}^{\dagger} = \prod_{\lambda} J_{H_G(\lambda)} \widehat{\Lambda}_J \mathbb{U}_{\lambda}.$$

For a nonempty S and sign choice δ , set

$$(\mathbb{K}_{S,\delta}^J y)_{\lambda} = J_{H_G(\lambda)} \sum_{\mu \preceq_S \lambda} \kappa_{\lambda\mu}^{\star, S, \delta} y_{\mu}.$$

Theorem 7.8 (Rigged endpoint-valuation bridge). *The displayed formula defines a continuous map $\mathcal{S}_\star \rightarrow \mathcal{S}_\star^\dagger$. For arbitrary coefficient families, the finite-lower-ideal net*

$$\mathfrak{B}_C^{S,\delta}(x, y) = \sum_{\lambda \in C} \sum_{\mu \preceq_S \lambda} x_\lambda y_\mu J_{H_G(\lambda)} \kappa_{\lambda\mu}^{\star, S, \delta}$$

converges in $\widehat{\Lambda}_J$ and defines a continuous bilinear pairing. On finite-support vectors it is precisely the root-specialized Hopf-contracted kernel with the regulator inserted once on the output cotest.

Proof. The rigged bridge uses the universal core kernel only after the output regulator has been inserted. Proper height makes the surviving output rows finite modulo each J_N , and row-finiteness makes each such output depend on finitely many inputs. This stagewise finiteness proves convergence and continuity on the product, while the finite-support calculation identifies the resulting pairing with the root-specialized contracted kernel.

All kernel entries are integral by Theorem 7.3, and every output row is finite. Modulo J_N , only the finitely many outputs with $H_G < N$ remain. Each depends on finitely many input coordinates, proving convergence and continuity. The finite-support identity is bilinearity; continuity extends it to the product domain. $\square$

Collecting the $3^g - 1$ operator choices into one finite vector gives simultaneous finite representatives for the disk truncations and root evaluations: if $C \subset C'$, their difference lies in $J_{\nu(C)} \Lambda^{3^g-1}$. Hence the same finite vector of polynomials gives the disk truncations and all root evaluations. This does not remove scalar denominator-clearing factors or prove unrescaled WRT normalization.

Arbitrary standard-twist powers are represented by orbit lattices

$$L_{\mathbf{k}, C} = T_{1,C}^{k_1} \cdots T_{g,C}^{k_g} L_{\mathbf{0}, C}, \quad \mathbf{k} \in \mathbb{Z}^g,$$

and a restricted orbit–Rees completion. A twist shifts $\mathbf{k}$ and is an exact isomorphism between adjacent degrees. Collapsing the degrees would force both a nonunit eigenvalue and its inverse onto one disk lattice, contradicting Proposition 7.7.

8. SEPARATING-MERIDIAN TWISTS AND BRIDGE CHAMBERS

A separating compressing disk is dual to a bridge in a suitable trivalent handlebody spine. Its Dehn twist is therefore diagonal in the graph basis. This section constructs that missing generic seed, distinguishes scalar denominator-clearing factors from honest polarized endpoints, and adjoins all bridge powers to the recoupling chamber groupoid.

Begin with the disk twist on one graph edge and use bridge parity to identify the integral exponent available on a closed separating bridge. The inverse twist then presents three different tasks: clearing one scalar coefficient, regularizing a bounded symmetric family of pairings, or transporting an honest polarized endpoint lattice. After those tasks are separated, the bridge degrees can be added to the recoupling groupoid and completed with restricted degree support. The final compatibility discussion records the local ribbon relation and its finite-level limitation, without extending the verified generic bridge chamber to an unproved mapping-class presentation.

8.1. Polynomial Gram entries. Fix a closed genus- g lollipop spine G and its universal core $L_\star = \bigoplus_\lambda \Lambda \mathbb{U}_\lambda$. Write

$$\varrho_\lambda^\star = B_G(\mathbb{U}_\lambda, \mathbb{U}_\lambda).$$

Lemma 8.1 (Polynomial Gram diagonal). *For every coloring λ , one has $\varrho_\lambda^\star \in \Lambda$.*

Proof. The Gram diagonal is obtained by using one negative twist eigenvalue as a test rather than dividing by it. The contracted kernel is already polynomial on the universal core, so multiplying back by the monomial eigenvalue proves polynomiality in the coefficient ring without any unstated cancellation.

Choose any standard handle whose loop color is ℓ . The diagonal coefficient of the negative twist is μ_ℓ^{-1} . The corresponding contracted kernel on the universal core is polynomial, hence $\mu_\ell^{-1} \varrho_\lambda^* \in \Lambda$. Multiplication by $\mu_\ell \in \Lambda$ proves the claim; no division in Λ is used. $\square$

8.2. The direct disk twist. Let e be an edge of G with integer Jones–Wenzl color $j_e(\lambda)$, and let D_e be its transverse dual disk. Denote the two disk twists by $T_{e,+}$ and $T_{e,-}$. Costantino’s half twist on spin s is [12]

$$H_s = i^{2s} x^{s(s+1)} \text{id}_s.$$

Use the formal identification $j = 2s$ and $x^{1/2} = A$; this is a formal square-root convention, not an analytic branch simplification.

Theorem 8.2 (Source-matched disk-twist eigenvalue). *On every fixed-color graph line,*

$$T_{e,+} \mathbb{U}_\lambda = \mu_{j_e(\lambda)} \mathbb{U}_\lambda, \quad T_{e,-} \mathbb{U}_\lambda = \mu_{j_e(\lambda)}^{-1} \mathbb{U}_\lambda,$$

where

$$\mu_j = (-1)^j A^{j(j+2)}.$$

Proof. The disk twist is diagonal on the graph line because the transverse disk is dual to one colored edge. Squaring the half twist gives the full-twist eigenvalue, and the bridge parity calculation then identifies the integral exponent available on a separating bridge. This supplies a local chamber seed; it does not make a scalar denominator-clearing factor into an invertible polarized action.

Squaring the half twist gives

$$H_s^2 = i^{4s} x^{2s(s+1)} = (-1)^j A^{j(j+2)}.$$

This agrees with the standard Kauffman full-twist eigenvalue on a colored edge. The universal core vector is a scalar multiple of the same definite edge-color graph vector, so it has the same eigenvalue. $\square$

Lemma 8.3 (Closed bridge parity). *If e is a bridge of a closed trivalent spine, then every admissible coloring has $j_e = 2a_e$ even, and*

$$\mu_{j_e} = A^{4a_e(a_e+1)}, \quad \frac{j_e(j_e+2)}{2} = 2a_e(a_e+1) \in \mathbb{N}.$$

Proof. Remove the bridge and sum the vertex parity equations over one component. Every internal edge is counted twice and the bridge once, forcing its color to be even. The formula follows from Theorem 8.2. $\square$

Conversely, a separating compressing disk can be made dual to a bridge: cut along the disk, choose a spine on each side with one endpoint on the cut disk, and join the two endpoints. Thus every separating meridian has such a seed in an appropriate marking.

If $w(j) = j(j+2)$, then for $n \in \mathbb{Z}$,

$$\text{ord}_{A=0}(\mu_j^n) = nw(j), \quad \text{ord}_{A^{-1}=0}(\mu_j^n) = -nw(j),$$

and every cyclotomic valuation of μ_j^n is zero. Hence all nonmonomial denominators in an F -conjugated disk twist come from the recoupling coordinates, not from the twist eigenvalue.

8.3. Three inequivalent responses to the inverse pole. The raw inverse contracted coefficient is

$$B_G(T_{e,-} \mathbb{U}_\lambda, \mathbb{U}_\lambda) = (-1)^{j_e} A^{-w(j_e)} \varrho_\lambda^*.$$

If $v_\lambda = \text{ord}_A \varrho_\lambda^*$, the least nonnegative one-sided scalar denominator-clearing factor is

$$r_\lambda^- = \max\{0, w(j_e) - v_\lambda\}.$$

It regularizes one scalar coefficient but does not change the inverse operator.

If both pairing arguments are multiplied by A^r , the least symmetric exponent is

$$r_\lambda^{\text{sym}} = \max \left\{ 0, \left\lceil \frac{w(j_e) - v_\lambda}{2} \right\rceil \right\}.$$

For a closed bridge, $w(j_e)/2$ is a canonical norm-independent choice.

For the honest invertible action define polarized endpoint lattices

$$L_{e,k} = \bigoplus_{\lambda} \Lambda \mu_{j_e(\lambda)}^k \mathbb{U}_\lambda, \quad k \in \mathbb{Z}.$$

Then

$$T_{e,+} : L_{e,k} \xrightarrow{\sim} L_{e,k+1}, \quad T_{e,-} : L_{e,k+1} \xrightarrow{\sim} L_{e,k}.$$

Proposition 8.4 (Denominator-clearing factor versus endpoint). *For a nonvacuum color:*

- (1) *a one-sided denominator-clearing factor regularizes one coefficient but is not an inverse twist;*
- (2) *a symmetric half-shift handles a bounded signed family of contracted pairings but not arbitrary negative powers;*
- (3) *only the polarized endpoint sequence carries the exact inverse pair.*

No nonzero finite free Λ -lattice on the eigenline is preserved by both signs.

Proof. The three responses are separated by what they preserve. A one-sided denominator-clearing factor changes a coefficient, a symmetric shift handles only a bounded family of pairings, and the polarized degree sequence is the object on which the twist and its inverse act exactly. The nonunit eigenvalue rules out collapsing those degrees into one finite free lattice.

Preservation by both signs would make μ_j and μ_j^{-1} endomorphism scalars and hence units, impossible for $j > 0$. For a fixed symmetric denominator-clearing factor the k -fold negative coefficient has order $2r + v_\lambda - kw(j) \rightarrow -\infty$. The endpoint identities are immediate from their definition. $\square$

The common intersection of the three degrees $-1, 0, 1$ is the fully shifted line $L_{e,1}$, not the half-shift used for a bilinear contraction.

8.4. Simultaneous bridge scalarization. Let E be a finite set of bridge edges and define

$$w_e(\lambda) = j_e(\lambda)(j_e(\lambda) + 2), \quad W_E(\lambda) = \sum_{e \in E} w_e(\lambda).$$

Every term is even. Put

$$\mathbb{S}_\lambda^E = A^{W_E(\lambda)/2} \mathbb{U}_\lambda,$$

and, for $\delta \in \{-1, 0, 1\}^E$,

$$T_E^\delta = \prod_{e \in E} T_{e,+}^{\delta_e}.$$

The factors commute because their dual disks are disjoint.

Theorem 8.5 (Separating-edge contracted core). *For every square-free signed separating multi-twist,*

$$B_G(T_E^\delta \mathbb{S}_\mu^E, \mathbb{S}_\lambda^E) = \delta_{\lambda\mu} A^{W_E(\lambda) + \sum_e \delta_e w_e(\lambda)} \varrho_\lambda^* \in \Lambda.$$

The all-negative corner has zero additional exponent, so the half-shift is sharp without further Gram divisibility.

Proof. The separating core calculation is diagonal after bridge parity. Orthogonality reduces the contracted pairing to one coloring, the half-shift contributes one copy of the bridge weight, and the signed power contributes the second copy or removes it at the negative corner. The remaining Gram entry is polynomial by the earlier lemma, so every square-free corner is integral.

Bridge parity removes all signs from the eigenvalues. Orthogonality makes the kernel diagonal, and the exponent is $\sum_e (1 + \delta_e) w_e(\lambda) \geq 0$. The remaining Gram factor is polynomial by Lemma 8.1. $\square$

Worked checkpoint. For one closed bridge of color $j = 2$,

$$w(2) = 8, \quad \mu_2 = A^8, \quad \frac{w(2)}{2} = 4.$$

After extracting the polynomial Gram entry $\varrho_\lambda^\star$, the canonically shifted contracted factors at $\delta = -1, 0, 1$ are respectively

$$1, \quad A^8, \quad A^{16}.$$

The all-negative corner has exponent zero exactly, while the other two corners have positive disk order.

A standard-handle triangular twist changes only its active loop excess and fixes all bridge colors. Hence its nonzero contracted entries have the same W_E on source and output.

Corollary 8.6 (Combined standard and separating family). *For every square-free signed standard multitwist $T_{S,\varepsilon}$,*

$$B_G(T_E^\delta T_{S,\varepsilon} S_\mu^E, S_\lambda^E) = A^{W_E(\lambda) + \sum_e \delta_e w_e(\lambda)} \kappa_{\lambda\mu}^{\star, S, \varepsilon} \in \Lambda$$

on the inherited triangular support, and it vanishes elsewhere.

Proof. A standard-handle triangular twist changes only its active loop excess and fixes all bridge colors. The preceding separating-edge theorem therefore applies without changing the bridge weight, and the separating twists contribute exactly $\sum_e \delta_e w_e(\lambda)$. This gives the displayed formula on the inherited triangular support and zero elsewhere. $\square$

For a relative graph, bridge parity can be odd because external colors contribute. Replacing $W_E/2$ by $\lceil W_E/2 \rceil$ leaves residual order zero or one. The closed Heegaard sector needs no ceiling.

8.5. Polarized bridge degrees and literal approximants. For $\mathbf{k} \in \mathbb{Z}^E$, define

$$m_{\mathbf{k}}(\lambda) = \prod_{e \in E} \mu_{j_e(\lambda)}^{k_e},$$

$$\mathbb{U}_{\mathbf{k},\lambda} = m_{\mathbf{k}}(\lambda) \mathbb{U}_\lambda, \quad \mathbb{U}_{\mathbf{k}}^\lambda = m_{\mathbf{k}}(\lambda)^{-1} \mathbb{U}^\lambda.$$

Then

$$\mathbb{U}_{\mathbf{k}+\delta}^\lambda (T_E^\delta \mathbb{U}_{\mathbf{k},\mu}) = \delta_{\lambda\mu}.$$

For a finite lower ideal C , set

$$X_{\mathbf{k},C}(x) = \sum_{\lambda \in C} x_\lambda \mathbb{U}_{\mathbf{k},\lambda},$$

$$F_{\mathbf{k},C}(y) = \sum_{\lambda \in C} J_{H_G(\lambda)} y_\lambda \mathbb{U}_{\mathbf{k}}^\lambda.$$

The index $\mathbf{k}$ records the current bridge-twist degree. The factor $m_{\mathbf{k}}$ changes the primal frame and its inverse changes the cotest, so the pairing remains Kronecker; $X_{\mathbf{k},C}$ and $F_{\mathbf{k},C}$ are the corresponding finite primal and regulated cotest sums.

Theorem 8.7 (Bridge-degree finite representatives). *For every integer power vector δ ,*

$$F_{\mathbf{k}+\delta,C}(y) (T_E^\delta X_{\mathbf{k},C}(x)) = \sum_{\lambda \in C} J_{H_G(\lambda)} x_\lambda y_\lambda.$$

The polynomial is independent of the bridge degree. If $C \subset C'$, its increment is in $J_{\nu(C)}\Lambda$ and hence is simultaneously small in the disk and cyclotomic topologies.

Proof. The bridge-degree identity is labelwise contragredient cancellation. The finite lower-ideal increment is controlled by the height regulator, so the same diagonal polynomial survives both disk and cyclotomic projection. The actual root-specialized kernel still retains its operator coefficient; only the transported coordinate pairing is degree-independent.

The target cotest cancels the source multiplier label by label. Every new label has height at least $\nu(C)$, so its regulator is a multiple of $J_{\nu(C)}$. $\square$

The root-specialized contracted net retains the actual operator coefficient:

$$\sum_{\lambda \in C} \sum_{\mu \preceq_S \lambda} x_\lambda y_\mu J_{H_G(\lambda)} A^{W_E(\lambda) + \sum_e \delta_e w_e(\lambda)} \kappa_{\lambda\mu}^{\star, S, \varepsilon}.$$

It is polynomial and has the same tail estimate, but unlike the coordinate pairing it is not degree-independent.

8.6. The bridge-extended chamber theorem. Adjoin at every marked spine the bridge generators and their inverses to the algebraic recoupling groupoid. Quotient by exact pure recoupling relations, literal twist inverse relations, commutation for disjoint disks, and coordinate naturality

$$T_e^{(G')} = U_{G'G} T_e^{(G)} U_{GG'}$$

whenever both markings describe the same disk twist. Retain all nonunit central degrees. Let D_G^{br} be the resulting degrees from the base marking to G and set

$$\mathfrak{A}_G^{\text{orb, br}} = \bigoplus_{\gamma \in D_G^{\text{br}}}^{\text{lat}} (\mathfrak{A}_0 \sqcup \{\perp\}).$$

Theorem 8.8 (Arbitrary-genus bridge chamber). *For every fixed finite genus:*

- (1) *the bridge orbit chamber is a directed distributive lattice;*
- (2) *every verified recoupling or bridge arrow acts by an exact lattice isomorphism and polarized Λ -linear transport;*
- (3) *the J -completion of the external orbit-Rees sum is the restricted product with finite degree support modulo every J_N ;*
- (4) *the transported finite polynomial of Theorem 8.7 is independent of degree; and*
- (5) *every separating compressing-disk twist and all its powers occur after choosing a bridge marking in the verified recoupling component.*

Proof. The bridge chamber is built by adjoining exact degree shifts to the recoupling orbit. The finite-support and restricted-product arguments transport unchanged, while the bridge marking realizes every separating compressing disk in the verified component. Keeping the nonunit degree visible is what lets all bridge powers coexist with the common completion.

The bridge seed is diagonal and algebraic in both directions over $\mathbb{F}_A$ and acts by an endpoint degree shift. The finite-support lattice and restricted completion arguments of Section 2.4 apply unchanged. Contragredient cancellation proves the common polynomial. The bridge realization following Lemma 8.3 gives the final claim. $\square$

This covers separating meridians, not separating curves which do not bound disks in the chosen handlebody.

8.7. Ribbon consistency and the chain limitation. The phased braiding satisfies the balancing identity

$$\theta_c = \theta_a \theta_b (R_c^{ab})^2, \quad \theta_s = i^{4s} x^{2s(s+1)}.$$

With new channels as rows and old channels as columns, the correct conjugations are

$$D_{\text{old}}^{(\text{new})} = F D_{\text{old}} F^{-1}, \quad D_{\text{new}}^{(\text{old})} = F^{-1} D_{\text{new}} F.$$

Unequal-boundary blocks distinguish this orientation from the reverse one.

For three equal inner colors a and total color d , let $B_1 = R$, $B_2 = F^{-1}RF$, and define

$$A_{12} = B_1^2, \quad A_{13} = B_2 B_1^2 B_2^{-1}, \quad A_{23} = B_2^2.$$

The braid and ribbon identities imply

$$A_{12}A_{13}A_{23} = (B_1 B_2)^3 = \frac{\theta_d}{\theta_a^3} \text{id}.$$

Restoring the two inner-boundary twists in each pair curve gives the local ribbon-lantern identity

$$T_{12}T_{13}T_{23} = \theta_a^3 \theta_d \text{id}.$$

This is a verified local compatibility relation, not a complete arbitrary-label mapping-class presentation. The braid and pair-curve arrows used in this certificate are not adjoined to the bridge-only groupoid above; the identity is an external compatibility certificate, not an additional quotient relation in that groupoid.

At prime level $p \geq 5$, for an admissible integer $0 \leq c \leq (p-3)/2$, the one-holed-torus chain relation [18] gives

$$(tt^*)^6 = q^{-6}T_\delta = A^{-12}T_\delta.$$

This root-level central character does not select a canonical generic disk degree: adding multiples of the root order changes the formal A -valuation without changing the root scalar. The untruncated opposite-torus/Hopf operator also has infinite columns, so no generic chain relation is inferred from this finite-level identity.

9. RAW AND SATURATED SLIDE QUOTIENTS

The chamber data become a closed-manifold construction only after quotienting by every successive attaching-slide relation. This section compares the raw integral quotient with its generic saturation and tracks the torsion that can appear at cyclotomic specialization. There are two natural polynomial relation lattices: the relations inherited from the genuine Laurent skein module and their saturation inside the generic fraction-field slide subspace. They agree generically but can differ by cyclotomic torsion, so the distinction must be made before root specialization.

First form the complete geometric filling quotient by attaching the two-handles and pulling each later slide kernel back through the preceding quotient. Then compare the polynomial relation lattice with its generic saturation inside the localized slide space. The coefficient-ring obstruction explains why localization cannot be performed before the joint disk–cyclotomic regulator is fixed; finite-stage quotients must instead remain inside the embedded polynomial lattice. Finally, transport the relation-before-reduction construction along verified degrees and state the torsion and specialization limits separately.

9.1. The complete generic filling quotient. Fix a finite genus and a Heegaard diagram $\mathcal{D} = (\Sigma_g, \alpha, \beta)$. Put

$$R = \mathbb{Z}[A, A^{-1}], \quad M_R = \text{Sk}_A(H_\alpha; R), \quad V = \mathbb{Q}(A) \otimes_R M_R.$$

Attach the β -two-handles successively. Pull every later slide kernel back to M_R through the preceding quotient maps, and let $\mathcal{J}_\mathcal{D}^R$ be the sum of all these pulled-back kernels. Set

$$\mathcal{I}_\mathcal{D}^{\mathbb{F}_A} = \mathbb{F}_A \mathcal{J}_\mathcal{D}^R \subset V, \quad \mathbb{F}_A = \mathbb{Q}(A).$$

Theorem 9.1 (Complete successive-slide quotient). *In every finite genus,*

$$M_R / \mathcal{J}_\mathcal{D}^R \cong \text{Sk}_A(M_\mathcal{D}; R), \quad V / \mathcal{I}_\mathcal{D}^{\mathbb{F}_A} \cong \text{Sk}_A(M_\mathcal{D}; \mathbb{F}_A).$$

A restricted family of slides need not generate the kernel. This warning is essential even for connected sums of handlebodies [5].

Proof. The quotient is built in the order in which the two-handles are attached. The one-handle theorem supplies each full slide kernel, pullback records the earlier quotient maps, and flat localization gives the generic statement. Only after this complete geometric quotient is fixed do the raw and saturated polynomial relation lattices become comparable.

For one two-handle, the standard handle-attachment theorem [33] identifies the surjection and its full slide kernel. Apply it in order and pull every later kernel back through the preceding surjections. The final three-handle induces an isomorphism. Localization to $\mathbb{F}_A$ is flat and gives the generic vector-space statement. $\square$

The pointed uncompleted quotient is invariant under Heegaard moves because it is the filled skein module. This alone says nothing about the polynomial chamber height, its completion, a cap, or the anomaly line.

9.2. The coefficient-ring obstruction. Set

$$\Lambda = \mathbb{Z}[A], \quad q = A^2, \quad j_N = A^{2N}(q; q)_N \quad (N \geq 0), \quad j_0 = 1.$$

Then

$$j_{N+1} = A^2(1 - q^{N+1})j_N \quad (N \geq 0).$$

Proposition 9.2 (No naive common skein base change). *The finite-stage ring $\Lambda/(j_N)$ is not an admissible coefficient ring for a direct application of the ordinary Kauffman-bracket two-handle theorem, because the image of A is not invertible. If A is localized first, then*

$$j_N \Lambda[A^{-1}] = (q; q)_N \Lambda[A^{-1}],$$

so the disk factor A^{2N} disappears. Thus naive skein base change cannot retain the joint disk-cyclotomic regulator.

Proof. The coefficient-ring obstruction is a noncommuting-operations check. In the polynomial quotient the disk parameter remains a zero divisor, whereas localizing to use the Laurent skein relation makes the disk monomial a unit. The finite stage must therefore be formed inside the already embedded polynomial lattice, keeping disk and cyclotomic control simultaneously.

In $\Lambda/(j_N)$ the nonzero class of A is killed by $A^{2N-1}(q; q)_N$, so it is a zero divisor and cannot be a unit. After localization A^{2N} is a unit. The Kauffman relation uses both A and A^{-1} , while disk order requires A to remain a nonunit; the operations cannot be interchanged. $\square$

Accordingly, the finite stages below are abstract quotients of polynomial lattices already embedded in the generic $\mathbb{F}_A$ -skein space. They are not called Kauffman skein modules over $\Lambda/(j_N)$.

9.3. Diagram-labeled degrees and two relation lattices. Let γ be a verified recoupling/bridge degree with generic operator U_γ and transported lattice $L_{\gamma, \alpha} = U_\gamma L_{0, \alpha}$ for a fixed source chamber α . Carry the filling with the degree:

$$\mathcal{J}_{\mathcal{D}, \gamma}^R = U_\gamma \mathcal{J}_{\mathcal{D}}^R, \quad \mathcal{I}_{\mathcal{D}, \gamma}^{\mathbb{F}_A} = U_\gamma \mathcal{I}_{\mathcal{D}}^{\mathbb{F}_A}.$$

The base label $\mathcal{D}$ remains fixed; the current geometric diagram is $\gamma\mathcal{D}$. Replacing it by an independently chosen diagram would change the filling problem.

Definition 9.3 (Raw and saturated slide lattices). Set

$$\begin{aligned} \mathcal{R}_{\mathcal{D}, \gamma, \alpha} &= L_{\gamma, \alpha} \cap \mathcal{J}_{\mathcal{D}, \gamma}^R, \\ \mathcal{S}_{\mathcal{D}, \gamma, \alpha} &= L_{\gamma, \alpha} \cap \mathcal{I}_{\mathcal{D}, \gamma}^{\mathbb{F}_A}, \end{aligned}$$

and

$$Q_{\mathcal{D}, \gamma, \alpha}^{\text{raw}} = L_{\gamma, \alpha} / \mathcal{R}_{\mathcal{D}, \gamma, \alpha}, \quad Q_{\mathcal{D}, \gamma, \alpha}^{\text{sat}} = L_{\gamma, \alpha} / \mathcal{S}_{\mathcal{D}, \gamma, \alpha}.$$

The raw lattice remembers genuine Laurent slide relations. The saturated lattice is the kernel of the map to the generic filled $\mathbb{F}_A$ -skein space. Both intersections are taken before disk or root reduction.

Proposition 9.4 (Raw torsion and generic saturation). *There are inclusions and an exact sequence*

$$\mathcal{R} \subset \mathcal{S} \subset L, \quad 0 \longrightarrow \mathcal{S}/\mathcal{R} \longrightarrow Q^{\text{raw}} \longrightarrow Q^{\text{sat}} \longrightarrow 0.$$

The two relation lattices have the same $\mathbb{F}_A$ -span. Hence $\mathcal{S}/\mathcal{R}$ is polynomial torsion and may carry cyclotomic information.

Proof. Raw and saturated relations are compared by denominator clearing inside the generic slide subspace. A saturated relation has a finite fraction-field expression in raw generators; one common polynomial multiplier returns it to the raw lattice. This exposes the quotient torsion as the finite specialization defect rather than identifying the two relation systems.

For $s \in \mathcal{S}$, express s as a finite $\mathbb{F}_A$ -linear combination of raw slide elements. Clearing all rational and Laurent denominators gives a nonzero $f \in \Lambda$ with $fs \in \mathcal{J}^R$. Since $s \in L$, also $fs \in L$, so $fs \in \mathcal{R}$. This proves equality of $\mathbb{F}_A$ -spans; the exact sequence is the third isomorphism theorem. $\square$

Lemma 9.5 (Saturation and torsion-freeness). *If $0 \neq f \in \Lambda$, $x \in L$, and $fx \in \mathcal{S}$, then $x \in \mathcal{S}$. Consequently Q^{sat} is torsion-free over Λ .*

Proof. The polynomial f is invertible in $\mathbb{F}_A$, and $\mathcal{I}^{\mathbb{F}_A}$ is a $\mathbb{F}_A$ -subspace. Thus $fx \in \mathcal{I}^{\mathbb{F}_A}$ implies $x = f^{-1}(fx) \in \mathcal{I}^{\mathbb{F}_A}$. Intersecting with L proves the first claim, which is equivalent to torsion-freeness of the quotient. $\square$

Torsion-free is the correct conclusion; $\mathbb{Z}[A]$ is not a PID, so freeness is not inferred.

Theorem 9.6 (Exact quotient transport). *For every verified arrow $a : \gamma \rightarrow a\gamma$, the map U_a carries both $\mathcal{R}$ and $\mathcal{S}$ exactly to their counterparts and induces Λ -linear isomorphisms*

$$Q_{\mathcal{D},\gamma,\alpha}^{\square} \xrightarrow{\sim} Q_{\mathcal{D},a\gamma,\alpha}^{\square}, \quad \square \in \{\text{raw}, \text{sat}\}.$$

Pure recoupling substitutions with the same endpoints induce the same map.

Proof. Transport is exact because the same verified arrow carries the chamber lattice and the pulled-back filling relations. The identity $U_a(L \cap \mathcal{X}) = U_a L \cap U_a \mathcal{X}$ transfers both intersections, and the recoupling groupoid then supplies endpoint path independence.

For either carried relation space $\mathcal{X}$,

$$U_a(L \cap \mathcal{X}) = U_a L \cap U_a \mathcal{X}.$$

This proves exact relation transport and hence the quotient map. Pure recoupling path independence follows from the common underlying skein realization of Section 2.4. $\square$

The directed distributive structure belongs to the chamber index. Images of intersections need not equal intersections of quotient images, so no unproved distributive law for quotient submodules is asserted.

9.4. Relation-before-reduction towers. For $N \geq 1$, define

$$Q_{\mathcal{D},\gamma,\alpha}^{\text{raw},(N)} = \frac{L_{\gamma,\alpha}}{\mathcal{R}_{\mathcal{D},\gamma,\alpha} + j_N L_{\gamma,\alpha}},$$

$$Q_{\mathcal{D},\gamma,\alpha}^{\text{sat},(N)} = \frac{L_{\gamma,\alpha}}{\mathcal{S}_{\mathcal{D},\gamma,\alpha} + j_N L_{\gamma,\alpha}}.$$

The nesting of the j_N gives transition maps and inverse limits $\widehat{Q}^{\square,J}$.

Theorem 9.7 (Two quotient towers). *Every verified arrow preserves j_N , induces exact isomorphisms in both towers at every stage, and extends bicontinuously to both inverse limits. The raw-to-saturated maps commute with all transitions and arrows. Taking the external orbit direct sum before completion yields restricted orbit–Rees quotients with finite degree support modulo every j_N .*

Proof. The two towers are formed by imposing the relation lattice before the j_N reduction. Centrality of the scalar ideal makes transported lattice isomorphisms commute with each stage quotient; the restricted orbit–Rees statement is then the inverse-limit form of finite degree support at every precision.

The scalar ideal is central, so a transported-lattice isomorphism maps $j_N L$ onto $j_N U_a L$. Combine this with Theorem 9.6 and pass to inverse limits. The restricted support statement is the inverse limit of external direct sums. $\square$

The whole stage need not have finite rank. Proper-height approximants are finite because only finitely many output colors survive at a fixed precision.

9.5. Quotient-valued finite-stage representatives. Fix one source chamber α with regulated core $\{U_{\alpha,\lambda}\}_{\lambda \in X_0}$ and proper height H_0 . Put $C_N = \{\lambda : H_0(\lambda) < N\}$. For polynomial coefficients x_λ , define

$$v_{N,\alpha}^{\mathcal{D},\square}(x) = \left[\sum_{\lambda \in C_N} j_{H_0(\lambda)} x_\lambda U_{\alpha,\lambda} \right] \in Q_{\mathcal{D},0,\alpha}^{\square,(N)}.$$

Theorem 9.8 (Universal quotient approximants). *For both raw and saturated towers:*

- (1) *the family v_N is compatible under the stage maps and determines an inverse-limit class;*
- (2) *if $C \subset C'$ are finite lower ideals, the corresponding partial sums differ by an element of $j_{\nu(C)} L$;*
- (3) *in degree γ , define $v_{N,\gamma}$ by the quotient transport of Theorem 9.6; every verified arrow carries it exactly to the next degree; and*
- (4) *modulo A^{2N} the disk projection is determined by height N , while the abstract root projection at q -order ℓ is determined by height ℓ .*

No equality between the raw and saturated root fibers is implied.

Proof. The quotient approximants are built from the same proper-height finite cutoffs in both relation towers. The new height layer is killed by j_N , transport carries the finite sum exactly between degrees, and the two factors of the regulator give the disk and root projections. The construction keeps raw and saturated fibers distinct.

The new height- N layer is multiplied by j_N and vanishes at stage N . The same divisibility proves the lower-ideal estimate. Exact transport follows from Theorem 9.7. Finally j_N contains A^{2N} , and $(q; q)_h$ contains $1 - q^\ell$ for $h \geq \ell$. $\square$

This is a source-gauged regulated section. It does not claim that the height regulator preserves the slide subspace or compare independently chosen source chambers.

9.6. The exact cap criterion. At stage N , a cap is a Λ -linear map

$$\epsilon_{\gamma,N} : L_{\gamma,\alpha} \longrightarrow \Lambda/(j_N).$$

It descends through the raw or saturated stage precisely when it kills the corresponding relation lattice.

Theorem 9.9 (Cap factorization). *Suppose compatible source caps $\epsilon_{0,N}$ annihilate one of the two complete relation lattices. Set*

$$\epsilon_{\gamma,N} = \epsilon_{0,N} \circ U_\gamma^{-1}.$$

Then each transported cap annihilates the complete transported relation lattice, and

$$\epsilon_{\gamma,N}(v_{N,\gamma}) = \epsilon_{0,N}(v_{N,0})$$

is independent of γ . The compatible residue classes define an element of $\widehat{\Lambda}_J$. Conversely, a cotest factors through a stage quotient exactly when it annihilates that stage's relation lattice.

Proof. Cap factorization is a quotient-universal-property argument after transport. Annihilation of the complete source relation lattice survives composition with the inverse arrow, while the transported approximant cancels the same arrow labelwise. Conversely, a row descends exactly when it kills the stage relation submodule.

For s in the source relation lattice,

$$(\epsilon_0 \circ U_\gamma^{-1})(U_\gamma s) = \epsilon_0(s) = 0.$$

The same cancellation proves the scalar equality. Compatibility follows from Theorem 9.8; the converse is the universal property of a quotient. $\square$

Corollary 9.10 (An integral raw cap kills the saturation). *If a Λ -valued row $\epsilon : L \rightarrow \Lambda$ annihilates $\mathcal{R}$, then it also annihilates $\mathcal{S}$ and descends through both towers.*

Proof. For $s \in \mathcal{S}$, Proposition 9.4 gives $0 \neq f \in \Lambda$ with $fs \in \mathcal{R}$. Hence $f\epsilon(s) = 0$. The coefficient ring is a domain, so $\epsilon(s) = 0$. $\square$

The theorem is a criterion, not a construction of a root-specialized cap. Such a cap must additionally be nonzero and normalized, annihilate all successive slides, have coherent left/right covariance and anomaly, stabilize integrally, compare correctly with the reduced root quotient, and yield WRT and the intended disk block.

9.7. Root caps and three obstructions. At a fixed semisimple modular root, the usual complementary-handlebody row

$$\epsilon_{r,\tilde{\phi}}(w) = N_{r,g}(\tilde{\phi}) \langle v_{r,g}, U_{\tilde{\phi}} w \rangle_r$$

uses the modular handlebody vacuum, a coherently lifted gluing operator, and the declared genus and Maslov normalization.

Theorem 9.11 (Reduced-root cap). *In every finite genus the displayed row annihilates the complete reduced attaching-slide kernel, factors through the reduced filled state space, and evaluates the vacuum to the anomaly-corrected WRT invariant. It is invariant under coherently lifted handlebody moves and Heegaard stabilization with the declared normalization.*

The extended-cobordism and Heegaard-vacuum inputs are those of [23, 2].

Proof. The reduced-root cap is obtained from the finite modular TQFT state map, not from polynomial base change. Each successive reduced slide is killed after the corresponding two-handle attachment; complementary gluing supplies the row, and the extended-cobordism law carries the declared anomaly and stabilization factors.

The modular TQFT assigns equal states to reduced skeins which become isotopic after a two-handle attachment. Hence each reduced slide difference lies in the kernel of the corresponding filled-state map. Applying this at every successive attachment annihilates the full successive reduced slide module. Gluing the complementary handlebody produces the displayed row; the extended-cobordism composition law supplies the Maslov factor. Handlebody covariance and the canceling-handle identity give the stated move and stabilization invariance. $\square$

This finite-label theorem is not the specialization of either polynomial tower until a non-flat base-change theorem is proved.

Three elementary obstructions delimit the missing generic cap.

Identity-double row. The normalized identity-double functional is the skein-theoretic Yang–Mills row [8]. In a solid-torus polynomial lattice $L_H = \Lambda 1 \oplus \Lambda(A^2z)$, longitude filling gives the raw slide relation

$$A^2z + (A^4 + 1)1.$$

The coefficient row which is one on the vacuum and zero on nonvacuum core colors evaluates it to $A^4 + 1 \neq 0$, so it does not descend.

Naive infinite-color cutoff. The ordinary truncation of the weak infinite capping color leaves a boundary slide term $(-1)^L[L + 1]$. After any Laurent monomial clearing, its value at $A = 1$ is still $(-1)^L(L + 1) \neq 0$. It is not divisible by $1 - q$, so it fails already at j_1 .

Saturation versus specialization. For the integral two-strand relation

$$s_{\text{TL}} = \begin{pmatrix} A^8 - 1 \\ A^4 - 1 \end{pmatrix} = (A^4 - 1) \begin{pmatrix} A^4 + 1 \\ 1 \end{pmatrix},$$

the raw line has saturation generated by the primitive column and torsion $\Lambda/(A^4 - 1)$. At a root the raw generator may vanish while the primitive saturated generator remains nonzero. Thus saturation and non-flat root specialization need not commute.

The exact row

$$c_{\text{TL}} = (1, -(1 + A^4))$$

annihilates both the raw column and its primitive. In a different projector lattice the same geometric relation can appear as $(A^8 - 1)(0, 1)^T$, with saturation generated by $(0, 1)^T$ and torsion $\Lambda/(A^8 - 1)$. These are different integral lattices, not an integral basis equality. Under an invertible recoupling F and diagonal twist D ,

$$s_{F,k} = D^k F s_0, \quad c_{F,k} = c_0 F^{-1} D^{-k}$$

satisfy $c_{F,k} s_{F,k} = 0$ identically. This verifies simultaneous primal–ideal–cotest transport in a low sector but does not construct a cap for every successive slide.

Worked checkpoint. At $A = 1$, the raw two-strand column and its primitive saturation specialize differently:

$$s_{\text{TL}}(1) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad s_{\text{TL}}^{\text{sat}}(1) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

The cap becomes

$$c_{\text{TL}}(1) = (1, -2), \quad c_{\text{TL}}(1) s_{\text{TL}}^{\text{sat}}(1) = 2 - 2 = 0.$$

Thus specialization erases the raw generator but not its saturation, while the integral cap continues to annihilate both.

For the spin- $\frac{1}{2}$ Costantino block used in the exact certificate, with rows indexed by the new channel and columns by the old channel,

$$F(x) = \frac{1}{1 + x^2} \begin{pmatrix} -x & (1 + x^2 + x^4)/x \\ x & x \end{pmatrix}, \quad x = A^2, \quad \det F = -1.$$

Taking $D = \text{diag}(1, A^8)$ gives the same identity for every integer k for which the expressions are formed. Keeping the cap fixed while moving only the slide is a nonzero negative control.

10. HOPF CAPS AND SPLIT JOINT COMPLETIONS

The joint regulator stores a disk coordinate and a cyclotomic coordinate, but neither coordinate determines the other. This section proves the exact module-level splitting, gives a sharp obstruction to recovering the disk series from Habiro data alone, and constructs the standard complementary-handlebody Hopf cap on a declared endpoint lattice in every finite genus. The cap theorem is intentionally one-sided: it handles the standard filling and its simultaneous transported copies, not arbitrary Heegaard gluings. The section then constructs the rigged genus-one cap pair, proves the contracted stabilization margin, and obtains an all-boundary-color one-way inside-disk endpoint-valuation arrow with a fixed-marking arbitrary-genus tensor

extension. Its exact first-cyclotomic obstruction identifies why this still does not give the universal scalar.

For the cap application, the module Chinese remainder theorem separates the two directions at each finite stage; its inverse limit gives the joint completion and the finite-stage representative criterion. The cap problem then has its own obstruction: one fixed row cannot encode both complementary solid-torus fillings. The rigged genus-one pair supplies the two contractions, and the stabilization margin controls the genus-normalized contracted kernels. The endpoint-valuation arrow is constructed separately on its one-way disk domain. Its fixed-marking arbitrary-genus tensor extension does not remove the first-cyclotomic obstruction: the raw cutoffs still fail to be cyclotomic Cauchy.

10.1. Finite-stage splitting of the disk-adic and cyclotomic factors. Let

$$\Lambda = \mathbb{Z}[A], \quad P_N(A) = \prod_{k=1}^N (1 - A^{2k}),$$

and define the ideals

$$I_N = (A^{2N}), \quad C_N = (P_N), \quad J_N = I_N C_N = (A^{2N} P_N).$$

Lemma 10.1 (Integral Bezout certificate). *The ideals I_N and C_N are comaximal. If $b_N(A)$ is the truncation through degree $2N - 1$ of the formal power series $P_N(A)^{-1}$, then $b_N \in \mathbb{Z}[A]$ and*

$$b_N(A) P_N(A) \equiv 1 \pmod{A^{2N}}.$$

Thus some $a_N \in \mathbb{Z}[A]$ satisfies

$$a_N A^{2N} + b_N P_N = 1.$$

Proof. The constant coefficient of P_N is one, so recursive formal inversion has integral coefficients. Truncating through degree $2N - 1$ gives the congruence, and its difference from one is divisible by A^{2N} . $\square$

Set

$$e_N^{\text{disk}} = b_N P_N, \quad e_N^{\text{cyc}} = 1 - e_N^{\text{disk}}.$$

Modulo J_N , these are complementary orthogonal idempotents: the first is 1 modulo A^{2N} and 0 modulo P_N , while the second has the opposite residues.

Theorem 10.2 (Module Chinese remainder theorem). *For every Λ -module M , without any flatness or finiteness hypothesis,*

$$M/J_N M \xrightarrow{\sim} M/A^{2N} M \times M/P_N M, \quad [x] \mapsto ([x], [x]).$$

An inverse sends $([x], [y])$ to

$$[e_N^{\text{disk}} x + e_N^{\text{cyc}} y]_{J_N}.$$

Proof. The two ideals are separated by a finite Bezout identity: the idempotents produce the two prescribed residues, while the same identity sends an element lying in both submodules into their product. This proves the module statement directly and explains why no flatness or finiteness hypothesis is needed.

The idempotents prove surjectivity. If $x \in A^{2N} M \cap P_N M$, the Bezout identity gives

$$x = (a_N A^{2N} + b_N P_N)x.$$

Because x belongs to both submodules, both summands are in $A^{2N} P_N M$. Hence

$$A^{2N} M \cap P_N M = A^{2N} P_N M = J_N M,$$

which proves injectivity. $\square$

Worked checkpoint. At joint stage $N = 2$,

$$P_2 = 1 - A^2 - A^4 + A^6, \quad J_2 = A^4 - A^6 - A^8 + A^{10}.$$

The Bezout pair $a_2 = 2 - A^4$, $b_2 = 1 + A^2$ satisfies

$$(2 - A^4)A^4 + (1 + A^2)P_2 = 1.$$

Hence

$$e_2^{\text{disk}} = 1 - 2A^4 + A^8, \quad e_2^{\text{cyc}} = 2A^4 - A^8.$$

Their idempotency defects, their product, and $e_2^{\text{disk}} + e_2^{\text{cyc}} - 1$ all vanish modulo J_2 .

Apply this theorem to either raw or saturated quotient module of Section 9. It gives, naturally under every verified diagram-labeled transport,

$$\frac{L}{\mathcal{X} + J_N L} \cong \frac{L}{\mathcal{X} + A^{2N} L} \times \frac{L}{\mathcal{X} + P_N L},$$

where $\mathcal{X}$ is either complete relation lattice.

10.2. The split completion and finite-stage representatives. The finite-stage splittings commute with the transition maps.

Theorem 10.3 (Split joint completion). *For every Λ -module M there is a canonical natural isomorphism*

$$\widehat{M}_{\text{joint}} := \varprojlim_N M/J_N M \cong \widehat{M}_{\text{disk}} \times \widehat{M}_{\text{cyc}},$$

where

$$\widehat{M}_{\text{disk}} = \varprojlim_N M/A^{2N} M, \quad \widehat{M}_{\text{cyc}} = \varprojlim_N M/P_N M.$$

For $M = \Lambda$, the disk factor is $\mathbb{Z}[[A]]$ and the cyclotomic factor is the A^2 -Habiro-type completion $\varprojlim_N \mathbb{Z}[A]/(A^2; A^2)_N$.

Proof. The split completion is obtained in two layers. First the finite-stage Chinese remainder theorem gives compatible disk and cyclotomic coordinates; then inverse limits turn those compatible splittings into the joint completed module. The finite-stage representative criterion is checked against both filtrations, which is why it does not collapse the two completion directions into one.

Take the inverse limit of Theorem 10.2. Finite products commute with inverse limits. The even powers (A^{2N}) are cofinal with the ordinary A -adic filtration. $\square$

Theorem 10.4 (Criterion for finite-stage representatives in the split completion). *For a sequence $f_N \in M$, the following are equivalent:*

- (1) $f_{N+1} - f_N \in J_N M$ for every N ;
- (2) simultaneously

$$f_{N+1} - f_N \in A^{2N} M, \quad f_{N+1} - f_N \in P_N M$$

for every N .

Such a sequence determines one pair in $\widehat{M}_{\text{disk}} \times \widehat{M}_{\text{cyc}}$. Conversely, every pair has compatible joint residues reconstructed by the CRT idempotents.

Proof. The criterion is the module intersection identity behind the finite CRT: the joint regulator is exactly the intersection of the disk and cyclotomic regulators. Passing to inverse limits then records the two compatible coordinates, while the finite idempotents reconstruct a joint residue from any such pair.

The equivalence is exactly $A^{2N} M \cap P_N M = J_N M$ from Theorem 10.2. The inverse-limit statement is Theorem 10.3. $\square$

Thus a proper-height construction is “common” because the same finite polynomials converge in both topologies. The two limits remain independent coordinates.

10.3. No continuous Habiro-to-disk identity map.

Theorem 10.5 (Cyclotomic-to-disk obstruction). *There is no continuous additive map*

$$\Psi : \widehat{\Lambda}_{\text{cyc}} \longrightarrow \mathbb{Z}[[A]]$$

whose restriction to Λ is ordinary polynomial inclusion. In particular, there is no continuous ring homomorphism with that property.

Proof. The obstruction compares the two topologies on the same polynomial sequence. The cyclotomic filtration makes the sequence tend to zero because fixed factors eventually divide it, while its constant coefficient keeps its disk order at zero. A continuous extension of polynomial inclusion would have to preserve both behaviors, giving the contradiction.

The polynomial sequence $P_N = (A^2; A^2)_N$ tends to zero in the cyclotomic topology because every fixed P_m divides P_N for $N \geq m$. If Ψ extended the identity, then $\Psi(P_N) = P_N$. But each P_N has constant coefficient one, hence A -order zero, and does not even tend to zero modulo A . This contradicts continuity at zero. $\square$

The disk factor in J_N is therefore genuine extra data. A closed unified Habiro invariant can supply a cyclotomic coordinate, but not a canonical disk series by continuous polynomial extension. When an identity such as a false-theta formula presents both coordinates, it is a chosen compatible disk–cyclotomic lift, not a consequence of the closed cyclotomic coordinate alone.

10.4. The standard arbitrary-genus Hopf cap. Work in the closed, unpointed handlebody sector. Let $g \geq 1$, let $H_g \subset S^3$ be the standard unknotted genus- g handlebody and H'_g its complement. Define the Laurent skein cap

$$\epsilon_H : \text{Sk}_A(H_g; R) \longrightarrow R, \quad \epsilon_H(x) = \langle x \cup \emptyset \rangle_{S^3}, \quad R = \mathbb{Z}[A, A^{-1}],$$

and extend it $\mathbb{F}_A$ -linearly to the ambient generic space. Let $\mathcal{D}_0$ be an ordered complete disk system of H'_g ; filling along it and adding the final three-handle gives S^3 .

For $g = 0$ the statement is tautological: take $H_0 = B^3$, $L_H = L_\star = \Lambda \mathbf{1}$, and $c_0 = 1$.

Choose any endpoint sign $\varepsilon \in \{-, +\}^g$ and its lattice

$$L_H^\varepsilon = \bigoplus_{\lambda} \Lambda \mathbf{e}_\lambda^\varepsilon.$$

At the all-zero coloring λ_0 , every endpoint scalar equals one, so $\mathbf{e}_{\lambda_0}^\varepsilon = 1$. The universal core basis is

$$\mathbb{U}_\lambda = D_\star(\lambda) A^{\sum_{i=1}^g 2(a_i + b_i + 1)^2} \prod_{i=1}^g p_{b_i}(A) \widetilde{b}_\lambda.$$

Put

$$c_g = D_\star(\lambda_0) A^{2g} \in \Lambda \setminus \{0\}.$$

Theorem 10.6 (Standard Hopf-cap theorem). *For every finite $g \geq 1$:*

(1) *the cap annihilates the complete successive-slide module of the standard complementary filling,*

$$\epsilon_H(\mathcal{J}_{\mathcal{D}_0}^R) = 0;$$

(2) *on every declared endpoint lattice it is the primitive Λ -valued row*

$$\epsilon_H(\mathbf{e}_\lambda^\varepsilon) = \delta_{\lambda, \lambda_0};$$

(3) *on the universal core it is*

$$\epsilon_H(\mathbb{U}_\lambda) = \delta_{\lambda, \lambda_0} c_g;$$

- (4) taking L_H^ε as the source chamber, the cap annihilates both the raw and saturated relation lattices and descends at every joint stage; and
 (5) on a regulated finite core vector $v_N = \sum_\lambda x_{N,\lambda} \mathbb{U}_\lambda$,

$$\epsilon_H(v_N) = c_g x_{N,\lambda_0}.$$

Hence a joint-Cauchy vector net gives a joint-Cauchy scalar net.

Proof. The complementary filling identifies the full successive-slide kernel, while the Hopf pairing isolates the all-zero coloring as the vacuum row. Restricting that row to the declared endpoint lattice makes it integral and allows it to descend through both relation lattices; the universal-core formula then records the same scalar at finite joint stages. The theorem remains tied to this declared filling and lattice.

The successive two-handle theorem [33] identifies the kernel of

$$\mathrm{Sk}_A(H_g; R) \longrightarrow \mathrm{Sk}_A(S^3; R) = R$$

with the full successive-slide module, proving the first claim. The lollipop vectors are orthogonal for the complementary-handlebody Hopf pairing, and the empty complementary vector is the all-zero vector. Pairing with it therefore kills every $\tilde{b}_\lambda$ except λ_0 . Endpoint normalization is one there, proving the primitive row. On the core, all p_{b_i} equal one at the vacuum and the displayed monomial is A^{2g} , giving c_g ; the lollipop pairing normalization is matched to [18].

The endpoint raw relation lattice lies in the full slide module. The $\mathbb{F}_A$ -linear extension of ϵ_H kills its $\mathbb{F}_A$ -span and hence the saturated intersection. Because the endpoint row is Λ -valued, it reduces modulo every J_N . The final scalar formula follows immediately. $\square$

Exact scope. The theorem is an integral cotest theorem on the declared endpoint lattice and its universal-core sublattice. It is not a claim that ϵ_H is Λ -valued on the whole Laurent skein module. For the solid-torus core z ,

$$\epsilon_H(z) = -(A^2 + A^{-2}).$$

It also does not compare two endpoint source chambers chosen independently.

10.5. Transport covariance is not a Heegaard amplitude. For a verified diagram-labeled arrow γ , set

$$L_\gamma^H = U_\gamma L_H^\varepsilon, \quad L_{\gamma,\star} = U_\gamma L_\star, \quad \epsilon_\gamma = \epsilon_H \circ U_\gamma^{-1},$$

and carry the disk system to $\gamma\mathcal{D}_0$.

Theorem 10.7 (Cap local system on the verified orbit). *The row ϵ_γ is Λ -valued on L_γ^H , kills the transported raw and saturated relation lattices, and satisfies*

$$\epsilon_\gamma(U_\gamma \mathbf{e}_\lambda^\varepsilon) = \delta_{\lambda,\lambda_0}, \quad \epsilon_\gamma(U_\gamma \mathbb{U}_\lambda) = \delta_{\lambda,\lambda_0} c_g.$$

Every stage scalar is invariant under the exact relations already imposed in the verified orbit. Unidentified twist degrees remain distinct.

Proof. The cap local system is transported rather than recomputed at each endpoint: conjugate the fixed cap row by the inverse transport, carry the marking and disk system with it, and use exact lattice and relation transport. Evaluation on a transported basis then reduces to the base cap, while the later mixed amplitude keeps one handlebody datum fixed and is therefore a different operation.

Apply ϵ_H after U_γ^{-1} and use exact simultaneous transport of the lattice, diagram, and relation module. $\square$

For every source vector v ,

$$\epsilon_\gamma(U_\gamma v) = \epsilon_H(v).$$

This identity is coordinate covariance, not a root-specialized mixed amplitude. Transporting the state and cap contragrediently cancels the gluing operator. A Heegaard amplitude instead keeps one handlebody datum fixed and takes a mixed coefficient such as

$$\epsilon_H(U_\phi v_0).$$

Proposition 10.8 (Genus-one fixed-cap obstruction). *No nonzero $\mathbb{F}_A$ -linear row on the solid-torus skein module annihilates both the complete longitude-filling and meridian-filling ideals. Thus one normalized row cannot be both the standard S^3 Hopf cap and the $S^1 \times S^2$ identity-double cap.*

Proof. The S^3 cap has

$$\epsilon_H(1) = 1, \quad \epsilon_H(z) = -(A^2 + A^{-2}),$$

whereas the normalized identity-double row [8] has

$$\text{YM}(1) = 1, \quad \text{YM}(z) = 0.$$

Each filled generic skein quotient is one-dimensional, so its annihilator is a one-dimensional row line. The normalized generators differ on z ; the lines are distinct and intersect only in zero. $\square$

This is the sharp separator: re-expressing one cap with its carried diagram is proved, but arbitrary fillings require a coherent family indexed by handlebody double cosets.

10.6. The genus-one rigged two-cap pair. The preceding obstruction forbids one fixed row, not a compatible pair of rows on a rigged primal–cotest system. Habiro’s diagonal annular bases provide such a pair at genus one [22]. Use his variables $v = e^{h/2}$, $q_H = v^2$, and translate to the chamber convention by

$$v = A^2, \quad q_H = A^4 = q^2, \quad q = A^2.$$

The braided dictionary also rephases colors:

$$V_i^H \mapsto (-1)^i f_i^{\mathbb{F}_A}, \quad V_1^H \mapsto -V_1.$$

Indeed, the Kauffman Hopf matrix is $\Pi_{ij} = (-1)^{i+j}[(i+1)(j+1)]_{A^2}$ [30], while Habiro’s is $[(i+1)(j+1)]_v$. The two signs from the rephased colors cancel those in Π . This full dictionary is forced: the scalar shortcut $v = -A^2$ matches the first quantum dimension but gives the wrong odd–odd Hopf entries. Write V_i for the irreducible color basis, so $V_i = U_i(V_1/2)$. Habiro’s Newton and even families are

$$P_n = \prod_{i=0}^{n-1} (V_1 - v^{2i+1} - v^{-2i-1}), \quad S_n = \prod_{i=1}^n (V_1^2 - (v^i + v^{-i})^2).$$

His exact Hopf diagonal is

$$(8) \quad \langle P_m, S_n \rangle_H = \delta_{mn} \prod_{j=2}^{2n+1} (v^j - v^{-j}).$$

The identity-double Yang–Mills row is the coefficient of V_0 , $\text{YM}(V_i) = \delta_{i0}$ [8]. Habiro’s basis-change formula therefore gives

$$(9) \quad h_n(v) := \text{YM}(P_n) = (-1)^n \frac{[2]}{[n+2]} \left[\frac{2n+1}{n+1} \right]_v = (-1)^n v^{-n^2} \mathcal{C}_n(v^2),$$

where

$$(10) \quad \mathcal{C}_n(q) = \frac{1 - q^2}{1 - q^{n+2}} \left[\frac{2n+1}{n+1} \right]_q \in \mathbb{Z}[q].$$

Indeed, if $d \mid n + 2$ and $d \geq 3$, the exponent of $\Phi_d(q)$ in the Gaussian polynomial in (10) is

$$\left\lfloor \frac{2n+1}{d} \right\rfloor - \left\lfloor \frac{n+1}{d} \right\rfloor - \left\lfloor \frac{n}{d} \right\rfloor = 1.$$

For $d = 1, 2$, the numerator $1 - q^2$ supplies the required factor. Thus the quotient is polynomial; it has constant coefficient one, and

$$\mathcal{C}_n(1) = \frac{2}{n+2} \binom{2n+1}{n+1} = \text{Cat}_{n+1}.$$

The raw master regulator does not by itself clear this row:

$$\text{ord}_A(J_n h_n(A^2)) = 2n - 2n^2,$$

which is negative for $n \geq 2$. The required polynomial primal and cotest families are instead

$$\begin{aligned} P_n &= (-1)^n A^{2n^2} P_n|_{v=A^2, V_1^H=-V_1} = \prod_{i=0}^{n-1} (A^{4i+2} V_1 + A^{8i+4} + 1), \\ S_n &= A^{2n^2+8n} S_n|_{v=A^2, V_1^H=-V_1} \\ &= A^{6n} \prod_{i=1}^n (A^{4i} V_1^2 - (A^{4i} + 1)^2). \end{aligned}$$

Put

$$D_n = A^{2n} \prod_{j=2}^{2n+1} (1 - A^{4j}).$$

Then

$$(11) \quad \langle P_m, S_n \rangle_{\mathbb{F}_A} = \delta_{mn} (-1)^n D_n, \quad D_n \in J_n \Lambda,$$

For $n \geq 1$, the divisibility can be seen without localization from the explicit quotient

$$\frac{D_n}{J_n} = \frac{1 - A^{4n+4}}{1 - A^2} \prod_{k=2}^n (1 + A^{2k}) \prod_{j=n+2}^{2n+1} (1 - A^{4j}) \in \Lambda,$$

with empty products understood. Hence $D_n \in J_n \Lambda$, and the chosen power A^{2n^2+8n} is the smallest polynomial clearing of S_n for which the paired diagonal has the master regulator's disk order $2n$. Indeed, raw S_n has the unique lowest order $-2n(n+1)$, while, after the fixed A^{2n^2} primal clearing, an $A^s S_n$ cotest clearing gives paired order $s - 2n^2 - 6n$. Requiring this to be at least $2n$ forces $s \geq 2n^2 + 8n$, with equality above.

Define the fractional cotest generator over $\mathbb{F}_A$ by

$$S_n^\dagger = D_n^{-1} S_n.$$

This is a declaration of a lattice in the generic dual; it does not invert D_n inside $\hat{\Lambda}_J$. Let the ambient restricted primal module be

$$\mathcal{P}^{\text{res}} = \left\{ \begin{array}{c} \sum_{n \geq 0} a_n P_n : a_n \in \hat{\Lambda}_J, \\ \text{for every } N, \text{ all but finitely many } a_n \text{ lie in } J_N \hat{\Lambda}_J \end{array} \right\}.$$

Equivalently, this is the inverse limit of the finite-support coordinate modules modulo J_N . It contains both the algebraic lattice $L_P = \bigoplus_n \Lambda P_n$ and the stricter coercive module

$$\mathcal{P}^{\text{coer}} = \left\{ \sum_{n \geq 0} J_n x_n P_n : x_n \in \hat{\Lambda}_J \right\} \subset \mathcal{P}^{\text{res}}.$$

On the cotest side put

$$\mathcal{S}^\dagger = \prod_{n \geq 0} \widehat{\Lambda}_J S_n^\dagger.$$

Give $\mathcal{P}^{\text{res}}$ its inverse-limit restricted-product topology and $\mathcal{S}^\dagger$ the coefficientwise product topology.

Theorem 10.9 (Rigged genus-one two-cap theorem). *The formula*

$$(12) \quad \left\langle \sum_n a_n P_n, \sum_n b_n S_n^\dagger \right\rangle_{\text{rig}} = \sum_{n \geq 0} (-1)^n a_n b_n$$

defines a continuous $\widehat{\Lambda}_J$ -valued pairing $\mathcal{P}^{\text{res}} \times \mathcal{S}^\dagger \rightarrow \widehat{\Lambda}_J$. The two cotests

$$Y_{S^3} = S_0^\dagger, \quad Y_{\text{id}} = \sum_{n \geq 0} (-1)^n \mathcal{C}_n(A^4) S_n^\dagger$$

represent the standard S^3 and identity-double cap rows. Explicitly,

$$\begin{aligned} \epsilon_{S^3} \left(\sum_n a_n P_n \right) &= a_0, \\ \epsilon_{\text{id}} \left(\sum_n a_n P_n \right) &= \sum_{n \geq 0} \mathcal{C}_n(A^4) a_n. \end{aligned}$$

On $\mathcal{P}^{\text{coer}}$, these become

$$\epsilon_{S^3} = x_0, \quad \epsilon_{\text{id}} = \sum_{n \geq 0} J_n \mathcal{C}_n(A^4) x_n.$$

For polynomial coordinates $x_n \in \Lambda$, their finite partial sums are literal common disk-cyclotomic polynomial approximants. For general $x_n \in \widehat{\Lambda}_J$, the partial sums are joint-Cauchy truncations; polynomial approximants require a compatible choice of representatives at each finite level.

On the algebraic primal lattice $L_P \subset \mathcal{P}^{\text{res}}$, the rows have the correctly typed filling descent

$$\begin{aligned} \epsilon_{S^3}(L_P \cap \mathcal{J}_{\text{longitude}}^R) &= 0, \\ \epsilon_{\text{id}}(L_P \cap \mathcal{J}_{\text{meridian}}^R) &= 0. \end{aligned}$$

They also kill the corresponding saturations inside L_P , and, by continuity, the closures in $\mathcal{P}^{\text{res}}$ of the two displayed intersections.

Proof. The cleared diagonal pairing turns the primal and fractional cotest families into a coefficientwise rigged pairing. Restricted support makes the pairing finite at every joint stage, while the two selected cotests recover the standard S^3 and identity-double rows separately. Filling descent is then inherited from the corresponding relation annihilators; the construction does not identify the two rows.

Equation (11) gives (12). Modulo J_N , the definition of $\mathcal{P}^{\text{res}}$ leaves only finitely many primal coordinates. This proves convergence and continuity; on $\mathcal{P}^{\text{coer}}$, every term with $n \geq N$ vanishes. Pairing with Y_{S^3} selects the zeroth coefficient. Pairing with Y_{id} cancels the two signs in the diagonal and gives $\text{YM}(P_n) = \mathcal{C}_n(A^4)$, proving the displayed rows and their common-tail property. Moreover,

$$P_1 = A^2 V_1 + A^4 + 1 = r_\ell,$$

so the fixed Kauffman S^3 cap kills P_n for every $n \geq 1$, while it sends P_0 to one. This identifies the coefficient selector with the actual cap, rather than merely a formally similar row.

The first row is the standard complementary-handlebody cap, while the second is inclusion into the identity double followed by the Yang–Mills coefficient functional. The successive two-handle theorem therefore annihilates the appropriate full Laurent slide module before restriction to L_P . Corollary 9.10 gives saturated descent, and continuity gives the stated closures inside $\mathcal{P}^{\text{res}}$. $\square$

The rows remain genuinely distinct. For the cleared longitude relation

$$r_\ell = A^2 V_1 + (A^4 + 1)1,$$

one has

$$\epsilon_{S^3}(r_\ell) = 0, \quad \epsilon_{\text{id}}(r_\ell) = A^4 + 1 \neq 0.$$

Thus the theorem constructs two filling-specific row lines; it does not evade Proposition 10.8.

There is also a basis-independent obstruction to turning the generic color Hopf kernel into a bare integral S -isomorphism. Let

$$H_r(v) = ([ab]_v)_{1 \leq a, b \leq r}.$$

Proposition 10.10 (Unbounded integral Hopf index). *One has the exact determinant*

$$(13) \quad \det H_r(v) = \prod_{j=1}^r [j]_v \prod_{1 \leq i < j \leq r} (v^j + v^{-j} - v^i - v^{-i}).$$

Consequently

$$\text{ord}_{v=1} \det H_r = r(r-1),$$

with leading coefficient

$$r! \prod_{i < j} (j^2 - i^2).$$

No source and target lattices both related to the ordinary color lattice by changes invertible over the local ring at $(v-1)$ can make every finite truncation an integral Hopf isomorphism. Equivalently, the $(v-1)$ -adic cokernel length grows without bound with r . Under the braided Kauffman dictionary $v = A^2$, this is the cyclotomic point $A^2 = 1$. The statement obstructs truncation-compatible, filtration-preserving integral Hopf isomorphisms; it does not by itself exclude a conceivable infinite operator that abandons that filtration.

Proof. The obstruction is read from the finite Hopf determinant. Factoring one quantum-integer term from each row leaves a monic polynomial determinant in the variables $v^a + v^{-a}$; expanding their pairwise differences at the cyclotomic point gives the growing valuation. Local-unit changes of basis cannot change that valuation, which proves the stated truncation obstruction while leaving the weaker infinite-operator question open.

Put $x_a = v^a + v^{-a}$. The identity

$$[ab]_v = [a]_v U_{b-1}(x_a/2)$$

factors $\prod_a [a]_v$ from the determinant. Since $U_{b-1}(x/2)$ is monic of degree $b-1$, the remaining determinant is the Vandermonde $\prod_{i < j} (x_j - x_i)$, proving (13). At $v = 1 + t$,

$$x_j - x_i = (j^2 - i^2)t^2 + O(t^3), \quad [j]_v = j + O(t),$$

which proves the order and leading coefficient. Changes of basis invertible over the local ring at $(v-1)$ do not change this determinant valuation. $\square$

This degeneration does not contradict modularity after root reduction: the reduced color set and negligible quotient depend on the level, and the fractional generators $S_n^\dagger$ cannot be specialized term by term where D_n vanishes. The theorem supplies a continuous two-cap pairing, not a state-to-state S , its inverse, an $SL_2(\mathbb{Z})$ relation, or a root base-change theorem.

10.7. The vacuum stabilization gauge.

Assume the vacuum denominator-clearing factors are minimal, so $D_\star(\lambda_0) = 1$. Then

$$\mathbb{U}_{\lambda_0, g} = A^{2g} 1.$$

Proposition 10.11 (Vacuum-line stabilization defect). *Under empty-graph stabilization,*

$$\text{stab}(\mathbb{U}_{\lambda_0, g}) = A^{-2} \mathbb{U}_{\lambda_0, g+1}.$$

Thus the minimally cleared core is not stabilization-integral even on the vacuum line.

Proof. Empty-skein stabilization fixes the underlying vector, while the core normalization changes by a factor of A^2 with the genus. Expressing the old normalized vector in the next vacuum basis therefore requires A^{-2} , which is not integral. This isolates the vacuum-line failure before the later genus-normalized contracted gauge.

Stabilization sends the empty skein to the empty skein, so $A^{2g} 1 = A^{-2} A^{2g+2} 1$. The vacuum line of the higher-genus core is exactly $\Lambda \mathbb{U}_{\lambda_0, g+1}$, and $A^{-2} \notin \Lambda$. $\square$

The vacuum defect suggests the genus-normalized core

$$\bar{\mathbb{U}}_{\lambda, g} = A^{-2g} \mathbb{U}_{\lambda, g}, \quad \bar{L}_{\star, g} = \bigoplus_{\lambda} \Lambda \bar{\mathbb{U}}_{\lambda, g}.$$

The required A^{4g} margin can in fact be proved for every contracted operator constructed in Sections 6–8.

Theorem 10.12 (Contracted stabilization gauge). *Choose the reduced minimum denominator-clearing factors. Then $D_\star(\lambda_0) = 1$, and for every finite genus g :*

(1) *the handlewise endpoint multipliers have exact orders*

$$\begin{aligned} \text{ord}_A m_{a,b}^- &= 2(a+b+1)^2 + \frac{b(b-1)}{2}, \\ \text{ord}_A m_{a,b}^+ &= (a+b+1)^2 + 1 + \frac{b(b-1)}{2}. \end{aligned}$$

Both orders are 2 exactly at $(a, b) = (0, 0)$ and are at least 5 otherwise. Hence $\bar{L}_{\star, g}$ lies in every endpoint chamber, with unit inclusion coefficient on the vacuum and strictly positive A -order on every nonvacuum handle.

- (2) *every square-free signed standard contracted kernel on $L_\star$ is divisible by A^{4g} ;*
- (3) *the Gram diagonal $\varrho_\lambda^\star$ is divisible by A^{4g} , and therefore every canonically half-shifted square-free separating, or combined standard and separating, contracted kernel is divisible by A^{4g} ; and*
- (4) *after replacing both pairing inputs by the genus-normalized core, all these kernels, with the canonical separating half-shift where applicable, remain in Λ . The vacuum entry is exactly one, and empty-graph stabilization fixes the normalized vacuum line literally.*

Vacuum normalization and handlewise divisibility. The vacuum calculation fixes the normalization before the contracted estimates are used: the all-zero local rows have unit denominator, and the endpoint orders then supply the positive margin on every nonvacuum handle. The handlewise margins combine with the exact local pole orders to give the A^{4g} divisibility of the contracted kernels; genus normalization removes the remaining vacuum shift. Thus the conclusion is a lattice statement for the declared contracted operators, not a new universal scalar or an unrestricted stabilization law.

Proof. At the all-zero label, every local contracted kernel and the empty-skein pairing equal one. The spectator factorization therefore gives $\rho_{\text{vac}}^S = 1$ for every nonempty S ; its reduced minimum denominator and hence their least common multiple are one. Thus $D_\star(\lambda_0) = 1$.

The factors $F_b^{(a)}$ and $(q; q)_b$ have constant coefficient one, so the displayed endpoint orders follow directly from the formulas for $m_{a,b}^\pm$. Their difference from 2 is manifestly nonnegative; if $a + b > 0$, the positive endpoint has minimum 5 and the negative endpoint has minimum 8.

For an active standard handle put $c = a$, $n = m + d$, $d \geq 0$. The exact pole orders of the full positive and negative contracted local kernels are

$$\begin{aligned} P_+(c, n, m) &= (n + c)^2 + n + m + 4c, \\ P_-(c, n, m) &= n^2 + (4c + 2m + 1)n + 3c^2 + 2cm + 5c + 2m + 3(n + c). \end{aligned}$$

The first is read from the terminating product in Theorem 6.6; for the second, the mirror formula has a unique lowest term. With $R_c(s) = 2(c + s + 1)^2$, the two shifted margins are

$$\begin{aligned} R_c(n) + R_c(m) - P_+ &= 4 + 4c + 3c^2 + 6m + 6cm + 3m^2 + 3d + 2cd + 2md + d^2, \\ R_c(n) + R_c(m) - P_- &= 4 + (c + m)^2 + 2m + d^2. \end{aligned}$$

They are at least 4, with equality only for $c = n = m = 0$. The passage from the subset-specific scalar vector to the full universal core adds the nonnegative active order $n(n+1)/2 + m(m+1)/2$, so these are conservative lower bounds for the full core kernel. An inactive handle contributes $R_a(b) + R_{a'}(b') \geq 4$ through the two factors E_S . All remaining denominator-clearing and spectator factors in Theorem 7.3 are polynomials, so they cannot lower the A -order. Summing the handlewise bounds proves divisibility by A^{4g} for every nonempty square-free standard family.

For the all-negative standard multitwist, orthogonality and triangularity give on the diagonal

$$\kappa_{\lambda\lambda}^{\star, -[g]} = \left(\prod_{i=1}^g \mu_{a_i+b_i}^{-1} \right) \varrho_\lambda^\star.$$

Multiplying by the polynomial monomial $\prod_i \mu_{a_i+b_i}$ proves $\varrho_\lambda^\star \in A^{4g}\Lambda$. The separating-edge formula adds only

$$\sum_{e \in E} (1 + \delta_e) j_e(j_e + 2) \geq 0$$

to this order, and the combined formula adds the same nonnegative monomial to an already divisible standard kernel. This proves the third claim.

If all handle pairs are $(a_i, b_i) = (0, 0)$, admissibility forces every tree color to vanish: remove the lollipop loops and induct from a cherry of the remaining rooted tree, where a triple $(0, 0, x)$ forces $x = 0$. Hence the global equality case is the unique vacuum.

Regauging both inputs multiplies a contracted kernel by A^{-4g} , so the preceding divisibility proves polynomiality. Finally $\overline{\mathbb{U}}_{\lambda_0, g} = 1$; the empty skein has pairing one and is fixed by every displayed twist and by empty-graph stabilization. $\square$

Scope. The theorem closes the contracted stabilization-margin question for the operator families already constructed: standard square-free twists, separating square-free twists, and their combined contracted kernels. It does not construct the generic Hopf/ S row, prove a complete mixed mapping-class presentation, or supply the Maslov and normalization data required by Theorem 2.3. Nor does it yet prove a genus- g to genus- $(g+1)$ lattice embedding away from the vacuum; that would require compatibility of the nonvacuum denominator-clearing factor systems. Those inputs must be checked on the same normalized core when they are constructed.

10.8. Relative Newton duals and the one-holed-torus disk S arrow. This section isolates a construction which is stronger than the two-cap pairing but weaker than the missing generic mapping-class operator. For every boundary half-color $c \geq 0$, the relative Newton basis admits a polynomial rigging with a jointly restricted Hopf diagonal. In the fixed Gilmer–Masbaum coordinates, the finite word tt^*t then has a coefficientwise A -adic limit after the exact boundary-dependent input shift computed below. The resulting map is a one-way inside-disk valuation

arrow. Its raw cutoffs fail even the first cyclotomic congruence, so it is neither an invertible generic S -operator nor a root base change.

Convention boundary. The relative lollipop formulas use the Gilmer–Masbaum variable $q = A^2$. For the uncolored Habiro pair, the full braided dictionary is

$$v = A^2, \quad q_H = v^2 = A^4, \quad V_i^H \mapsto (-1)^i f_i^{\mathbb{F}_A}.$$

In particular $V_1^H \mapsto -V_1^{\mathbb{F}_A}$. The scalar shortcut $v = -A^2$ happens to reproduce the Newton polynomials but gives the wrong odd–odd Hopf entries; it is not a braided dictionary.

10.8.1. *The all-color relative Newton rigging.* Recall

$$Q_n^{(c)} = L_{c,0} \prod_{i=c}^{c+n-1} (z - \lambda_i), \quad \lambda_i = -q^{i+1} - q^{-i-1},$$

and put

$$D(c, n) = (c + n)^2 + 3c + 2n, \quad J_n = q^n(q; q)_n.$$

Define

$$u_{c,n} = n(n + 2c + 1), \quad v_{c,n} = n^2 + 2cn + 5n + 2c^2 + 6c.$$

Then

$$u_{c,n} + v_{c,n} = 2(D(c, n) + n).$$

Set

$$\begin{aligned} P_{c,n} &= A^{u_{c,n}} Q_n^{(c)} \\ &= L_{c,0} \prod_{r=1}^n (1 + A^{2(c+r)} z + A^{4(c+r)}), \\ R_{c,n} &= A^{v_{c,n}} Q_n^{(c)} = A^{2c^2+6c+4n} P_{c,n}. \end{aligned}$$

The letters P, R avoid confusing this self-dual relative Newton system with Habiro’s distinct P_n, S_n pair. Habiro’s P_n samples the even-index eigenvalue nodes, whereas $Q_n^{(c)}$ uses the consecutive nodes $\lambda_c, \dots, \lambda_{c+n-1}$.

Theorem 10.13 (Relative Newton rigging). *For every $c, n \geq 0$, both displayed families are polynomial relative to the fixed graph $L_{c,0}$, span the same $\mathbb{F}_A$ -space, and satisfy*

$$B_c(P_{c,m}, R_{c,n}) = \delta_{mn} \Delta_{c,n},$$

where

$$\Delta_{c,n} = q^{D(c,n)+n} \eta_{c,n} = (-1)^c q^n \frac{(q; q)_n (q; q^2)_{c+n+1} (-q; q)_{2c+n+1}}{(1 - q^2) \begin{bmatrix} 2c \\ c \end{bmatrix}_{q^2}} \in J_n \mathbb{Z}[q].$$

Consequently

$$R_{c,n}^\dagger = \Delta_{c,n}^{-1} R_{c,n}$$

defines a restricted fractional dual and

$$\left\langle \sum_{n \geq 0} J_n x_n P_{c,n}, \sum_{n \geq 0} b_n R_{c,n}^\dagger \right\rangle = \sum_{n \geq 0} J_n x_n b_n$$

is a continuous joint pairing. At every joint stage only finitely many coordinates survive.

Proof. Relative Newton factors are first cleared into two polynomial families, after which orthogonality supplies the diagonal norm. The exact norm divisibility provides the joint regulator needed for the fractional dual, and the stagewise vanishing of higher colors gives continuity. This is a topology-aware dual construction, not a unimodular state-to-state operator.

Multiplying the r -th Newton factor by $A^{2(c+r)}$ gives the first polynomial product, so $u_{c,n}$ is its exact minimal monomial clearing. The difference

$$v_{c,n} - u_{c,n} = 2c^2 + 6c + 4n$$

is nonnegative, giving the second polynomial family. Orthogonality of $Q_n^{(c)}$, the exponent identity, and the exact relative norm factorization give the displayed diagonal. Its divisibility by J_n is the cyclotomic floor estimate in Theorem 5.2. Modulo J_N , every term with $n \geq N$ vanishes. $\square$

For $c > 0$, the vacuum root norm satisfies $\eta'_{c,0} \sim h^c$. Thus $R_{c,0}^\dagger$ belongs to the restricted fractional dual, not the original integral root lattice. Relative duality is therefore not a unimodular state-to-state S .

10.8.2. *The finite cutoff word.* Use

$$Q_n'^{(c)} = \frac{Q_n^{(c)}}{\{n\}!}, \quad \eta'_{c,n} = B_c(Q_n'^{(c)}, Q_n'^{(c)}).$$

Let $b_{nm}^{(c)}$, $n \geq m$, be the lower-triangular matrix of the positive longitude twist t^* . Hopf adjointness defines the upper-triangular meridian matrix

$$a_{mn}^{(c)} = \frac{\eta'_{c,n}}{\eta'_{c,m}} b_{nm}^{(c)}, \quad m \leq n.$$

Let $A_{c,N}$ and $B_{c,N}$ be their leading blocks through color N , and set

$$S_{c,N} = A_{c,N} B_{c,N} A_{c,N}.$$

At an admissible odd prime p , with $d = (p-1)/2$ and $N = d - c - 1$, this is the finite Gilmer–Masbaum matrix of tt^*t . In the generic calculation N is retained as a cutoff.

The exact orders are

$$\begin{aligned} \text{ord}_A \eta'_{c,n} &= -2c^2 - 6c - 4cn - n^2 - 3n, \\ \text{ord}_A b_{nm}^{(c)} &= \frac{2c^2 + 4c - m + m^2 + 5n + 4cn + n^2}{2}, \\ \text{ord}_A a_{mn}^{(c)} &= \frac{2c^2 + 4c + 5m + 8cm + 3m^2 - n - 4cn - n^2}{2}. \end{aligned}$$

Theorem 10.14 (All-color inside-disk valuation arrow). *For every $c, r, m \geq 0$, the coefficientwise limit*

$$s_{c;rm}^{\text{in}} = \lim_{N \rightarrow \infty} (S_{c,N})_{rm} \in \mathbb{Q}((A))$$

exists in the A -adic topology. Its exact order is

$$(14) \quad \text{ord}_A s_{c;rm}^{\text{in}} = 3c(c+2) + \frac{r(3r+8c+9) - m(m+4c+1)}{2}.$$

For $N > \max\{r, m\}$,

$$(15) \quad (S_{c,N} - S_{c,N-1})_{rm} = a_{rN}^{(c)} \sum_{j=0}^m b_{Nj}^{(c)} a_{jm}^{(c)},$$

and its exact order is

$$(16) \quad 2N + 3c(c+2) + \frac{r(3r+8c+9) - m(m+4c+1)}{2}.$$

Put

$$\chi_{c,m} = \frac{m(m+4c+1)}{2}, \quad \psi_{c,r} = 3c(c+2) + \frac{r(3r+8c+9)}{2}.$$

Then s_c^{in} defines a continuous one-way map

$$\mathcal{S}_c^{\text{in}} : \prod_{m \geq 0} J_m A^{\chi_{c,m}} \mathbb{Q}[[A]] Q_m^{(c)} \longrightarrow \prod_{r \geq 0} A^{\psi_{c,r}} \mathbb{Q}[[A]] Q_r^{(c)}.$$

Its $r = 0$ row is a continuous inside-disk cap functional. Moreover,

$$s_{c;00}^{\text{in}} = (-1)^c A^{3c(c+2)} (1 + O(A)),$$

so the row becomes disk-primitive after the displayed monomial and unit normalization.

Proof. The cutoff difference adds precisely the paths whose new intermediate index is N . A unique lowest-order path controls the coefficient and the increment order rises with N , giving A -adic convergence; the regulated input weights then make the row continuous. The resulting $r = 0$ map is an inside-disk cap on its declared domain, with no reverse transport asserted.

A path contributing to $(S_{c,N})_{rm}$, with intermediate indices i, j , has exact order

$$\frac{1}{2} (6c^2 + 12c + 4i + 4j + 8cj + 4j^2 - m - 4cm - m^2 + 5r + 8cr + 3r^2).$$

Subtracting the value at $i = r, j = 0$ gives

$$2(i - r) + 2j(2c + j + 1).$$

It is nonnegative and vanishes only at $i = r, j = 0$. Thus the lowest path is unique and cannot cancel. Evaluating it proves (14).

Increasing the cutoff from $N - 1$ to N adds exactly the paths with $i = N$, proving (15). Among those paths $j = 0$ is uniquely minimal, and substitution gives (16). Its order increases by two with each cutoff, proving coefficientwise convergence.

Finally,

$$\text{ord}_A (A^{\chi_{c,m}} s_{c;rm}^{\text{in}}) = \psi_{c,r}.$$

The m -th contribution from the regulated input domain has order at least $\psi_{c,r} + 2m$, since $\text{ord}_A J_m = 2m$. Hence each output sum converges continuously. The exact diagonal formula begins

$$b_{00}^{(c)} = (-1)^c A^{c(c+2)} (1 + O(A)).$$

At $r = m = 0$, the uniquely minimal path is $(b_{00}^{(c)})^3$, giving the displayed leading coefficient. $\square$

The term *one-way* is literal. The theorem proves a continuous map between the two displayed, generally different, weighted products. It does not assert surjectivity, a same-domain inverse, or a completed $SL_2(\mathbb{Z})$ action. Mirroring gives an A^{-1} -adic outside-sector candidate, but composing the two sectors requires a separate inside/outside endpoint-connection theorem.

The resulting one-way row is the disk-side reading of a marked boundary state after the inside recapping operation. Its continuity records that the regulated state survives this reading in the declared chamber; it does not provide the reverse boundary transport or identify the disk cap with the cyclotomic cap.

10.8.3. The first-cyclotomic obstruction for every boundary color.

Theorem 10.15 (Raw cutoff word is never cyclotomic Cauchy). *For every $c \geq 0$ and $N \geq 1$,*

$$(17) \quad (S_{c,N} - S_{c,N-1})_{00}|_{A=-1} = \frac{(c + \frac{3}{2})_N}{N!} \neq 0.$$

Consequently, in these fixed coordinates, the raw cutoff sequence is not Cauchy even modulo $1 - A^2$, hence not in the cyclotomic or joint topology. Laurent monomial row and column gauges cannot remove this obstruction.

Proof. The obstruction is detected before any completion argument: the first new intermediate color contributes a nonzero vacuum increment at $A = -1$. Because every multiple of $1 - A^2$ vanishes there, the raw cutoff cannot be Cauchy even at the first cyclotomic level; Laurent monomial gauges only change the value by a sign and therefore cannot remove the defect. This isolates the failure of the raw word from the later question of constructing coherent counterterms or a unified row.

For the vacuum increment,

$$(S_{c,N} - S_{c,N-1})_{00} = a_{0N}^{(c)} b_{N0}^{(c)} a_{00}^{(c)}.$$

At $A = -1$,

$$a_{00}^{(c)} \longrightarrow 1, \quad b_{N0}^{(c)} \longrightarrow (-2)^{-N},$$

and

$$\frac{\eta'_{c,N}}{\eta'_{c,0}} \longrightarrow 2^N \frac{(2c + 2N + 1)!!}{(2c + 1)!!N!}.$$

Since

$$a_{0N}^{(c)} = \frac{\eta'_{c,N}}{\eta'_{c,0}} b_{N0}^{(c)},$$

their product is

$$2^{-N} \frac{(2c + 2N + 1)!!}{(2c + 1)!!N!} = \frac{(c + \frac{3}{2})_N}{N!}.$$

It is nonzero. Every multiple of $1 - A^2$ vanishes at $A = -1$, while a Laurent monomial evaluates to ± 1 , proving both claims. $\square$

Thus a joint construction needs either non-monomial cutoff counterterms with proved mapping-class coherence or an independently unified cyclotomic S -row combined with the disk row through the split joint completion. The theorem concerns every c, N for this raw cutoff family; it is not a no-go theorem for all bases or all non-monomial renormalizations.

10.8.4. *Row, operator, and root base change.* The following statements are distinct.

- (1) The $r = 0$ line of Theorem 10.14 is a continuous row on one declared disk domain.
- (2) A generic mapping-class S must be an invertible arrow with a declared inverse domain and must satisfy the braid and central relations. The one-holed-torus relations include

$$t_\alpha t_\beta t_\alpha = t_\beta t_\alpha t_\beta, \quad (t_\alpha t_\beta)^6 = t_\delta.$$

Those completed relations are not proved here.

- (3) At an admissible odd-prime root, the finite matrix with $N = d - c - 1$ is separately invertible. Gilmer–Masbaum prove for their geometric lifts

$$(tt^*t)^4 = q^{-6+2c(c+1)-p(p+1)/2}.$$

Since $q^p = 1$, the final factor is rootwise one. The cyclotomic obstruction above shows that the finite root matrices are not evaluations of the raw inside-disk limit.

10.8.5. *The fixed-marking arbitrary-genus consequence.* Fix a lollipop marking G , all spectator labels, and boundary half-colors

$$\mathbf{c} = (c_1, \dots, c_g).$$

Tensor the local arrows and put

$$\chi_{\mathbf{c}, \mathbf{m}} = \sum_{i=1}^g \chi_{c_i, m_i}, \quad \psi_{\mathbf{c}, \mathbf{r}} = \sum_{i=1}^g \psi_{c_i, r_i}.$$

Corollary 10.16 (Fixed-marking arbitrary-genus disk arrow). *For any proper height H_G , the tensor matrix defines a continuous map*

$$\widehat{\bigotimes}_{i=1}^g \mathcal{S}_{c_i}^{\text{in}} : \prod_{\mathbf{m}} J_{H_G(\mathbf{m})} A^{\chi_{c,\mathbf{m}}} \mathbb{Q}[[A]] Q'_{\mathbf{m}} \longrightarrow \prod_{\mathbf{r}} A^{\psi_{c,\mathbf{r}}} \mathbb{Q}[[A]] Q'_{\mathbf{r}}.$$

Proof. With the marking and spectator sector fixed, tensoring the local inside-disk arrows makes the output order additive across handles. Properness of the height function turns the resulting lower-order estimate into a finite sum at each output coordinate, which is exactly the continuity statement. The argument stays within one marked decomposition and preserves the one-way inside-disk direction; it does not supply recoupling compatibility or a decomposition-independent handlebody theorem.

Indeed, for fixed output $\mathbf{r}$, an input term has order at least

$$\psi_{c,\mathbf{r}} + 2H_G(\mathbf{m}),$$

and properness makes every bounded-height label set finite. $\square$

This is an arbitrary-genus theorem for one fixed marked decomposition and fixed spectator sector. It is not a decomposition-independent handlebody theorem.

For a recoupling arrow $F : G \rightarrow G'$, the missing typed compatibility is

$$(18) \quad F^{\text{out}} \mathcal{S}_G^{\text{in}} = \mathcal{S}_{G'}^{\text{in}} F^{\text{in}},$$

where F^{in} and F^{out} must be bicontinuous on the distinct boundary-color-dependent input and output endpoint-valuation domains. The existing algebraic F -move theorems do not prove this: an F -move can mix the c -channels entering $\chi_{c,m}$.

A handlebody theorem also requires the S -specific stabilization law

$$\epsilon_{g+1}^{\text{in}}(\text{stab } v) = \epsilon_g^{\text{in}}(v)$$

on a proved lattice embedding. The existing contracted stabilization theorem applies to the previously constructed square-free contracted kernels, not this uncontracted infinite S -arrow. Primary source. Gilmer–Masbaum, *Integral TQFT for a one-holed torus* (arXiv:0908.2796), supplies the relative Newton bases, orthogonality and norm, the mapping-class presentation, Hopf adjointness and the identity $a_{mn}^{(c)} = (\eta'_m/\eta'_n) b_{nm}^{(c)}$, and the root central relation.

The exact generic norm factorization is in Theorem 5.2, and the exact positive-twist orders are in Section 6. These formulas were also checked by exact symbolic computation.

Precise status. Proved: the all- c Newton rigging, exact cutoff difference, coefficientwise inside-disk convergence, continuous weighted domain, fixed-marking tensor extension, and all- c first-cyclotomic obstruction for the raw cutoff.

Not proved: a same-domain inverse, the inside/outside endpoint connection, recoupling compatibility (18), central relations in the completed disk category, stabilization of the new arrow, a cyclotomic relative row, root base change, or WRT/ $\widehat{Z}$ comparison.

10.9. What remains for a universal root-specialized scalar. Closed Habiro theory [22] supplies unified cyclotomic scalars for integral homology spheres and their WRT evaluations. It does not supply the boundary functor or the filling-dependent generic vacuum rows needed here; the homology-handlebody extension is explicitly a further construction. Even a completed closed cyclotomic cap would give only the cyclotomic coordinate, by Theorem 10.5.

For the closed-IHS part, Garoufalidis–Lê already construct a cap on the full ordinary skein module [15, Theorem 1.1, Theorem 2.4 and Section 3.1, arXiv v2]. Their variable is $q_{\text{GL}} = Q = A^4$, so their fourth root is our Kauffman parameter A . On a framed link with fundamental colors the cap is

$$[L] \longmapsto (-1)^{\sum_i (\text{fr}(L_i)+1)} I_{(M,L)}(Q) \quad \text{in} \quad \mathcal{H} \otimes_{\mathbb{Z}[Q^{\pm 1}]} \mathbb{Z}[A^{\pm 1}],$$

where $\mathcal{H}$ is the ordinary Habiro ring. Arbitrary framed links span the source. Their Lemma 2.3 verifies the Kauffman relations term by term, and the convergent surgery sum preserves them; no restriction to algebraically split zero-framed retained links is needed. The even $\mathbb{Z}[Q^{\pm 1}]$ -sector lands in $\mathcal{H}$, with empty-link value J_M . This supplies a closed cap, not the filling-dependent boundary vacuum, mixed Hopf row, or joint completion required by the remaining theorem.

The remaining theorem must construct, for every declared coefficient type, finite genus, handlebody double coset, and block label, a compatible vacuum and a mixed Hopf/ S row which:

- (1) is continuous on a declared primal lattice L_ϕ and annihilates the correctly typed raw relation lattice $L_\phi \cap \mathcal{J}_{\mathcal{D}_\phi}^R$;
- (2) satisfies a proved complete presentation of the recoupling, bridge, twist, one-holed-torus Hopf/ S , and central relations;
- (3) obeys both handlebody covariances and the Maslov anomaly law;
- (4) becomes integral under a stabilization gauge satisfying the exact normalization identities of Theorem 2.3;
- (5) defines one joint disk–cyclotomic inverse-limit element; and
- (6) has normalized WRT and intended $\widehat{Z}$ projections.

The two genus-one filling-cap lines and their common polynomial approximants are now constructed by Theorem 10.9. In fixed Gilmer–Masbaum coordinates, Theorem 10.14 now constructs the all-boundary- color one-way disk arrow and its fixed-marking arbitrary-genus tensor extension. Theorem 10.15 proves that its raw cutoffs cannot supply the cyclotomic coordinate, even at the first ideal.

The smallest unresolved geometric case is therefore a cyclotomic relative row or non-monomial counterterm which is compatible with that disk arrow, together with the inside/outside connection, inverse, and central $SL_2(\mathbb{Z})$ relations. The typed recoupling equation (18) and an S -specific stabilization law must then make the fixed-marking tensor independent of the handlebody decomposition. The determinant obstruction shows why this cannot be a bare integral endomorphism of the ordinary color lattice.

11. BOUNDARY RESULTS AND LIMITATIONS

The constructions developed here provide marked source modules, filling quotients, finite-stage joint quotients, endpoint lattices, exact recoupling and separating transport, raw and saturated quotient towers, and the declared cap domains. They furnish the inputs to the companion paper on integral root fibers, Rees covariance, caps, and sewing, subject to the boundary constructions in Theorem 1.1 and the hypotheses of the individual results.

These results do not define root specialization on an entire completion, identify the disk and cyclotomic completions, or provide an inverse operation that reconstructs a boundary state from a capped scalar. The limitations are structural: the relevant source and target objects lie in different domains.

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