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A LEAN FORMALIZATION OF ROY'S STRONG SIX EXPONENTIALS THEOREM

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Abstract. We formalize in Lean 4 with Mathlib Roy's strong six exponentials theorem: if x_1, x_2 are linearly independent over $\overline{\mathbb{Q}}$ and y_1, y_2, y_3 are linearly independent over $\overline{\mathbb{Q}}$, then one of the six products $x_i y_j$ lies outside the $\overline{\mathbb{Q}}$-vector space spanned by 1 and the logarithms of algebraic numbers. The proof closure uses only Lean's three standard axioms. Along the way the development proves Waldschmidt's transcendence theorem on $\mathbb{G}_{\mathrm{a}}^{d_0} \times \mathbb{G}_{\mathrm{m}}^{d_1}$ in the form used by Roy, and Philippon's multiplicity estimate with derivatives on $\mathbb{G}_{\mathrm{a}}^{d_0} \times \mathbb{G}_{\mathrm{m}}^{d_1}$ over every algebraically closed field of characteristic zero, following Roy's account in Waldschmidt's book. We reproduce the public Lean statements, summarize the mathematical route, and record how the artifact is verified. The code is available at `https://github.com/telaviv/strong-six-exponentials`.

1. Main theorems

The formalization is public at

`https://github.com/telaviv/strong-six-exponentials`,

and the revision described here is pinned in Section 4. The endpoint is `StrongSix.M2.strongSixExponentials`. Its type is the proposition `StrongSix.StrongSixExponentials`, defined in `StrongSix/Defs.lean` as follows.

```
def logAlg : Set ℂ := {ℓ | IsAlgebraic ℚ (exp ℓ)}

def LTilde : Set ℂ :=
  {z | ∃ (n : ℕ) (β : Fin (n + 1) → ℂ) (ℓ : Fin n → ℂ),
      (∀ i, IsAlgebraic ℚ (β i)) ∧ (∀ i, ℓ i ∈ logAlg) ∧
      z = β 0 + ∑ i, β i.succ * ℓ i}

def QbarIndep {n : ℕ} (v : Fin n → ℂ) : Prop :=
  ∀ c : Fin n → ℂ, (∀ i, IsAlgebraic ℚ (c i)) →
    ∑ i, c i * v i = 0 → ∀ i, c i = 0

def StrongSixExponentials : Prop :=
  ∀ (x : Fin 2 → ℂ) (y : Fin 3 → ℂ), QbarIndep x → QbarIndep y →
```

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```
    ∃ i j, x i * y j ∉ LTilde

theorem StrongSix.M2.strongSixExponentials : StrongSixExponentials
```

(Docstrings are omitted; the source documents `logAlg` as $\mathcal{L}$ and `LTilde` as $\widetilde{\mathcal{L}}$.) The mathematical content is the following.

Theorem 1.1 (Strong six exponentials; Roy [10, Cor. 2])**.** *Let $\mathcal{L} = \{\ell \in \mathbb{C} : e^{\ell} \in \overline{\mathbb{Q}}\}$ and let $\widetilde{\mathcal{L}}$ be the $\overline{\mathbb{Q}}$-vector space spanned by 1 and $\mathcal{L}$. If $x_1, x_2 \in \mathbb{C}$ are linearly independent over $\overline{\mathbb{Q}}$ and $y_1, y_2, y_3 \in \mathbb{C}$ are linearly independent over $\overline{\mathbb{Q}}$, then $x_i y_j \notin \widetilde{\mathcal{L}}$ for some $i \in \{1, 2\}$ and $j \in \{1, 2, 3\}$.*

Here $\overline{\mathbb{Q}}$ is the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$; "algebraic" means algebraic over $\mathbb{Q}$, and $\mathcal{L}$ contains every branch of the logarithm of every nonzero algebraic number. Roy states [10, Cor. 2] for 2×3 matrices: a 2×3 matrix with entries in $\widetilde{\mathcal{L}}$ whose rows and whose columns are linearly independent over $\overline{\mathbb{Q}}$ has rank 2. Theorem 1.1 is the case of the rank-one matrix $(x_i y_j)$, whose rows and columns are $\overline{\mathbb{Q}}$-independent exactly when x and y are. The Lean set `LTilde` is the set of finite sums $\beta_0 + \sum_i \beta_i \ell_i$ with $\beta_i \in \overline{\mathbb{Q}}$ and $\ell_i \in \mathcal{L}$, which is exactly the $\overline{\mathbb{Q}}$-span of $\{1\} \cup \mathcal{L}$. The predicate `QbarIndep` is linear independence over $\overline{\mathbb{Q}}$, phrased with complex coefficients that are required to be algebraic.

Remark 1.2. Independence over $\overline{\mathbb{Q}}$, not over $\mathbb{Q}$, is essential. Since $\widetilde{\mathcal{L}}$ is a $\overline{\mathbb{Q}}$-vector space, the $\mathbb{Q}$-independent families $x = (1, \sqrt{2})$ and $y = (\log 2, \log 3, \log 5)$ put all six products in $\widetilde{\mathcal{L}}$. An independent review of the statements (Section 4) checked that the Lean definitions use $\overline{\mathbb{Q}}$-independence; an earlier version of our README stated the theorem with $\mathbb{Q}$-independence, which is false.

The development exposes four further endpoints, all with the same axiom closure.

Declaration	Content
`StrongSix.M2.waldschmidtTheorem1`	Theorem 1.3
`StrongSix.M2.multEst_of_isAlgClosed`	Theorem 1.4, any $K = \overline{K}$, $\operatorname{char} K = 0$
`StrongSix.M2.multEst_Qb`	Theorem 1.4 for $K = \overline{\mathbb{Q}}$
`StrongSix.SixExp.six_exponentials`	six exponentials theorem [4, Ch. 2], [9]

The six exponentials theorem (with $\mathbb{Q}$-independence and the conclusion $e^{x_i y_j} \notin \overline{\mathbb{Q}}$ for some i, j) is proved separately by the classical auxiliary-function method; it is not used in the proof of Theorem 1.1.

1.1. **Waldschmidt's theorem.** Write $d = d_0 + d_1$, $E = \mathbb{C}^{d_0} \times \mathbb{C}^{d_1}$, and $\Omega = 0 \times 2\pi i \mathbb{Q}^{d_1} \subset E$. A $\mathbb{C}$-subspace of E is *rational over* $\overline{\mathbb{Q}}$ if it is spanned by vectors with algebraic coordinates. The proposition `StrongSix.`

`Roy.WaldschmidtTheorem1` transcribes the following statement [10, Thm. 1], which Roy derives from [12, Thm. 4.1].

Theorem 1.3 (Waldschmidt)**.** *Let $Y \subset \overline{\mathbb{Q}}^{d_0} \times \mathcal{L}^{d_1}$ be a finite-dimensional $\mathbb{Q}$-subspace of E, let $W \subset E$ be a $\mathbb{C}$-subspace rational over $\overline{\mathbb{Q}}$, and let $V \neq E$ be a $\mathbb{C}$-subspace containing Y and W. Then there are d_0', d_1' and a surjective $\mathbb{C}$-linear map $s : E \to E' = \mathbb{C}^{d_0'} \times \mathbb{C}^{d_1'}$ with $s(\overline{\mathbb{Q}}^{d_0} \times 0) \subset \overline{\mathbb{Q}}^{d_0'} \times 0$ and $s(0 \times \mathbb{Q}^{d_1}) \subset 0 \times \mathbb{Q}^{d_1'}$, such that $s(W) \neq E'$ and*

$$\frac{d_1' - \dim_{\mathbb{Q}}\bigl(s(Y) \cap \Omega'\bigr) + \dim_{\mathbb{Q}} s(Y)}{d_0' + d_1' - \dim_{\mathbb{C}} s(W)} \;\leq\; \frac{d_1 - \dim_{\mathbb{Q}}(Y \cap \Omega)}{d_0 + d_1 - \dim_{\mathbb{C}} V}.$$

1.2. **The multiplicity estimate.** The interface between the analytic and the algebraic halves of the proof is the proposition `StrongSix.Thm1.MultEst` K. Points of $G = \mathbb{G}_{\mathrm{a}}^{d_0} \times \mathbb{G}_{\mathrm{m}}^{d_1}$ are pairs $g = (x, y) \in K^{d_0} \times (K^\times)^{d_1}$, polynomials live in $K[X_1, \dots, X_{d_0}, Y_1, \dots, Y_{d_1}]$, and $w = (\xi, \eta) \in K^d$ acts by the invariant derivation $D_w = \sum_k \xi_k\, \partial/\partial X_k + \sum_j \eta_j Y_j\, \partial/\partial Y_j$. A polynomial *$P$ vanishes to order $\geq k$ along W at g* if $D_{w_1} \cdots D_{w_r} P(g) = 0$ for all $r < k$ and $w_i \in W$. Connected algebraic subgroups are $H = h_0 \times T_U$ with $h_0 \subset K^{d_0}$ a subspace, $U \subset \mathbb{Q}^{d_1}$ a subspace and $T_U = \{y : y^\varphi = 1 \text{ for all } \varphi \in U \cap \mathbb{Z}^{d_1}\}$; then $\operatorname{Lie} H = h_0 \times U^\perp$. Two points are *equivalent modulo H* if they differ by an element of H.

Theorem 1.4 (Multiplicity estimate; Philippon [8], in the form of Roy [14, Ch. 8])**.** *Let K be algebraically closed of characteristic 0 and $d = d_0 + d_1$. Put $C = d^d d_0^{d_0} d_1^{d_1}\, d!\, 2^{d_1}$. Let $W \subset K^d$ be a subspace, $S \subset G(K)$ a finite set containing the identity, $T, D_0, D_1 \geq 1$, and $P \neq 0$ a polynomial of degree $\leq D_0$ in X and $\leq D_1$ in Y. Suppose P vanishes to order $\geq dT + 1$ along W at every point $\sigma_1 + \cdots + \sigma_d$ with $\sigma_i \in S$. Then there is a connected algebraic subgroup $H = h_0 \times T_U \neq G$ such that every subset $F \subset S$ of pairwise inequivalent points modulo H satisfies*

$$T^{\ell}\, |F|\, D_0^{\dim h_0} D_1^{d_1 - \dim U} \;\leq\; C\, D_0^{d_0} D_1^{d_1}, \qquad \ell = \dim W - \dim\bigl(W \cap \operatorname{Lie} H\bigr).$$

The Lean proposition `MultEst` K asserts the existence of *some* constant $C(d_0, d_1)$; the proof produces the explicit constant above (`StrongSix.M2.Cst`).

Remark 1.5. Theorem 1.3 is the case $G = \mathbb{G}_{\mathrm{a}}^{d_0} \times \mathbb{G}_{\mathrm{m}}^{d_1}$ of Waldschmidt's theorem [12, Thm. 4.1], often called the linear subgroup theorem. As explained in [13, Ch. 13], special cases of that theorem contain Baker's theorem on linear forms in logarithms, both homogeneous and non-homogeneous. We have not stated or derived Baker's theorem itself from it in Lean; a Lean proof of the qualitative form of Baker's theorem, by a different route, is in [7].

2. Mathematical route

The proof has three parts: the multiplicity estimate (Theorem 1.4), the deduction of Waldschmidt's theorem from it, and Roy's deduction of strong six from Waldschmidt's theorem:

$$\text{Thm. 1.4} \implies \text{Thm. 1.3} \implies \text{Thm. 1.1.}$$

We describe them in the reverse order.

2.1. **From Waldschmidt's theorem to strong six (Roy's reduction).** We follow [10, §§1–4]. Roy works in an *admissible* category in which kernels and cokernels exist and compose, with additive and upper-additive functions on objects; his Section 3 shows that four statements about such functions are equivalent. We encode an admissible category only through the existence of kernels, cokernels and exact triples, which is all the argument uses (`Roy/Abstract.lean`). The concrete category is realized as subquotients T/S of a fixed ambient (E, Y, W, V), with $S \le T$ products of a $\overline{\mathbb{Q}}$-rational and a $\mathbb{Q}$-rational subspace (`Roy/Subquot.lean`). Each subquotient is realized in coordinates by a map preserving the $\overline{\mathbb{Q}}$- and $\mathbb{Q}$-structures (`Roy/Realize.lean`, `Roy/Transport.lean`), so Theorem 1.3 applies to every object; this gives Roy's Theorem 1bis and then his Theorem 2bis. For strong six one writes $x_i y_j = \beta_0^{(ij)} + \sum_r \beta_{r+1}^{(ij)} \ell_r^{(ij)}$, chooses a $\mathbb{Q}$-basis $\omega_1, \dots, \omega_m$ of the number field generated by all β, and lifts the products to $\ell_{kj} \in \overline{\mathbb{Q}}^2 \times \mathcal{L}^{2m}$. With $W = \ker \varphi$, $V = \varphi^{-1}(\mathbb{C}x)$ and $Y = \operatorname{span}_{\mathbb{Q}}\{\ell_{kj}\}$, Theorem 2bis gives $(2m + \dim Y)/2 \le 2m$, whereas $\overline{\mathbb{Q}}$-independence of y gives $\dim Y \ge 3m$ (`Roy/Core.lean`, `Roy/Final.lean`). This layer is about 2,700 lines.

2.2. **From the multiplicity estimate to Waldschmidt's theorem.** This is a transcendence proof by Laurent's interpolation determinants [5], in the setting of [13, Ch. 4, 9, 11]; the choice of parameters is our own derivation. A first reduction (`Thm1/Reduce.lean`) quotients by $\mathbb{C} \cdot (Y \cap \Omega)$ and disposes of the cases $\dim V = 0$ and $\dim V < d_0$. In the main case all parameters are powers of one integer base β (`Thm1/Params.lean`). Rows of the interpolation matrix are pairs (derivation word of length $< T$ over a basis of W, grid point $\sum s_i \eta_i$); columns are monomials $X^a Y^b$ of bidegree $\le (L_0, L_1)$.

- If the columns are dependent, a kernel vector is a nonzero polynomial vanishing along W on the grid. Theorem 1.4 over $\overline{\mathbb{Q}}$ yields an obstruction subgroup $H = h_0 \times T_U$, which is turned into the map s of Theorem 1.3 (`Thm1/Quot.lean`); a count of inequivalent grid points (`Thm1/Count.lean`) gives the inequality.
- Otherwise a maximal minor Δ is nonzero. The analytic side bounds $|\Delta|$ from above through $\Psi(z) = \det\big(D^{t_\nu} f_\lambda(z\zeta_\nu)\big)$, which vanishes to order $M(N - M^n) - \sum t_\nu$ at 0 by a rank count (`Thm1/Rank.lean`,

`Thm1/InterpDet.lean`); analytic derivatives of $P(v_0, e^{v_1})$ are identified with the invariant derivations (`Thm1/Deriv.lean`). The arithmetic side gives Liouville's lower bound from heights of the entries in a number field (`Thm1/Heights.lean`, `Thm1/Core4.lean`). For large β the two bounds are incompatible (`Thm1/Growth.lean`).

This layer is about 3,900 lines.

2.3. **The multiplicity estimate.** Philippon's original proof works with multihomogeneous ideals in $\mathbb{P}^{d_0} \times \mathbb{P}^{d_1}$. We follow instead Roy's affine treatment [14, Ch. 5 and 8], which reduces the deep input to a single theorem of commutative algebra. Let $B = K[X, Y, Z]$ with $d_0 + 2d_1$ variables, G embedded by $(x, y) \mapsto (x, y, y^{-1})$, and $I_G = (Y_j Z_j - 1)_j$.

Filtered Hilbert functions. For an ideal $J \subset B$ and $D = (D_0, D_1)$ let $H(J; t) = \dim F_{tD} - \dim(J \cap F_{tD})$, where F_{tD} is the space of polynomials of block degree $\leq tD$. A Veronese-type bridge to graded Hilbert–Serre theory shows that $H(J; t)$ is eventually polynomial, so J has a Hilbert dimension $\operatorname{hdim} J$ and degree. The key bridge is

$$\operatorname{hdim} L + \operatorname{ht} L = d_0 + 2d_1 \qquad \text{for every proper ideal } L \subset B, \tag{2.1}$$

proved from Macaulay's unmixedness theorem for polynomial rings (an ideal of height r generated by r elements is unmixed) together with the height of maximal ideals. Unmixedness comes from a port of Guan and Hu's Cohen–Macaulay library [2] to our Mathlib version (about 2,200 lines, Section 4); see [6, Thm. 17.6] for the mathematics.

Bézout (Roy's Theorem 8.4). If $I_G \subset A = I_G + (F)$ with F of block degree $\leq D$, then $\deg A \leq \deg I_G$. The proof builds a complete-intersection chain, choosing each new generator outside the associated primes by an avoidance argument over an infinite field.

Algebraic subgroups. An algebraic subgroup of G has the form $V \times T_\Lambda$ (Roy's Proposition 5.6), proved through linear independence of exponential polynomials; we also compute its Hilbert dimension and a lower bound for its degree, separate distinct cosets, and show that the connected component has finite index.

Wüstholz's lemma (Roy's Lemmas 8.14–8.16, Theorem 8.18 [15]). If every element of A vanishes to order $> S_0$ along W on a finite union of cosets $\Gamma + G'$ with ideal J, then $H(A; t + c) \geq (k + 1)^\ell H(J; t)$ whenever $\ell k \leq S_0$. Roy's argument uses the multi-index operators D^α and the commutativity of the D_w. We instead apply derivations along *words* of indices and index the products Q^m by multisets, which removes the need for commutativity; the products $P_a Q^{m_a}$ are then shown to be linearly independent modulo A.

The main argument (Roy §8.4.1). Let $L_r = I_G + (\tau_\sigma D_l P : \sigma \in \Sigma[r],\ |l| \leq rT)$. Since $I(G)$ is prime (the torus is irreducible, proved by mapping B into the domain $K[X, Y][1/\prod Y_j]$), $\operatorname{hdim} L_0 < d$, and a pigeonhole argument gives $r < d$ with $\operatorname{hdim} L_r = \operatorname{hdim} L_{r+1}$. Roy now takes an irreducible component common to the zero sets of L_r and L_{r+1}. We avoid component

theory: we take a prime $p \supseteq L_{r+1}$ with $\operatorname{hdim} p = \operatorname{hdim} L_r$ (which exists by (2.1)) and show that any such prime is a minimal prime of L_r, by strict decrease of hdim along chains of primes. With $V = Z(p)$ and the Nullstellensatz, the stabilizer $G_V = \{h : hV \subseteq V\}$ is an algebraic subgroup, and the zero set of $A = I_G + (P' \circ \tau_v : P' \in L_r,\ v \in V)$ is covered by finitely many cosets of G_V, one for each minimal prime of L_r hit. The elements of A vanish to order $> T$ along W on $S + G_V$; since that set of elements is an ideal (by Leibniz' rule), this is checked on generators. Finally Bézout, Wüstholz's lemma, coset separation and the two degree bounds for G_V^0 and I_G give the inequality of Theorem 1.4. This layer, without the ported library, is about 6,600 lines.

3. AI-assisted formalization

The Lean code, the planning documents and the text of this paper were produced by AI models (Claude, Anthropic; mainly Claude Opus 5.5) working in Claude Code under the direction of the author. Work was organized by written plans (not part of the published repository): an overall blueprint, a plan for deducing Theorem 1.3 from Theorem 1.4, and a plan for proving Theorem 1.4. The deduction of Theorem 1.3 was first proved with Theorem 1.4 as an explicit hypothesis, so that the implication could be checked before the multiplicity estimate was attempted. The plan for the multiplicity estimate was revised once: after a first attempt along Philippon's projective proof, the route was switched to Roy's affine treatment, whose only deep input is the unmixedness theorem; that input was obtained by porting an existing library rather than proving it from scratch.

An independent AI review session, separate from the one that wrote the proofs, checked the statements of Theorems 1.1–1.4 against their sources, the chain of endpoint declarations, and the axiom closure. It found no problem with the Lean statements or the proof. It did find that the README stated the theorem with the wrong (and false) independence hypothesis; this has been corrected (Remark 1.2).

Apart from the check by the Lean kernel, all verification so far has been carried out by AI systems. No human expert has yet reviewed the Lean definitions against the literature or read the proofs; we would be grateful for corrections. The author is responsible for the intended statements and for the claims made here.

4. Formal artifact and verification

The formalization uses Lean 4 [1] and Mathlib [11].

Repository	`https://github.com/telaviv/strong-six-exponentials`
Commit	`1625bdf3c0a7d9222bacd876d738d7540109bea6`
Lean	`leanprover/lean4:v4.35.0`
Mathlib	`v4.35.0`
Mathlib commit	`56a005855ea17e0b0958f73ec801ae140b0a1679`

The toolchain and package revisions are pinned in `lean-toolchain` and `lake-manifest.json`. The library `StrongSix` has 76 files and about 16,600 lines:

Directory	Lines	Content
`Roy/`	2,734	Theorem 1.3 ⇒ Theorem 1.1 (Roy's reduction)
`Thm1/`	3,918	Theorem 1.4 ⇒ Theorem 1.3
`M2/`	6,634	the multiplicity estimate (Theorem 1.4)
`CM/`	2,201	ported Cohen–Macaulay library
other	1,138	`SixExp/` (six exponentials), `Analytic/`, `Arith/`, `Defs.lean`

Third-party code. `StrongSix/CM/` is a port of Guan and Hu's Cohen–Macaulay files [2] from their Mathlib fork (`https://github.com/Thmoas-Guan/mathlib4_fork`, branch `ABS-Criterion-Project-new`, commit `0ff6e01f`; Apache-2.0). The original headers are kept, and local changes are marked `PORT`.

To build the library at the pinned revision:

```
git clone https://github.com/telaviv/strong-six-exponentials
cd strong-six-exponentials
git checkout 1625bdf3c0a7d9222bacd876d738d7540109bea6
lake exe cache get
lake build
```

To inspect the endpoint and its axiom closure, run `lake env lean Axioms.lean`. The file contains, among `#print axioms` queries for the main intermediate results,

```
#check (StrongSix.M2.strongSixExponentials :
  StrongSix.StrongSixExponentials)
#print axioms StrongSix.M2.strongSixExponentials
#print axioms StrongSix.M2.waldschmidtTheorem1
#print axioms StrongSix.M2.multEst_of_isAlgClosed
#print axioms StrongSix.M2.multEst_Qb
```

Every query yields `[propext, Classical.choice, Quot.sound]`. The source contains no `sorry`, no `axiom` declaration and no `native_decide`. These checks concern the named declarations and their dependencies; the mathematical meaning of the result also depends on the definitions in `Defs.lean`, reproduced in Section 1 for review. (The statement `Roy/Theorem1.lean` of Theorem 1.3 matters only for the intermediate endpoint `waldschmidtTheorem1`.)

5. Related work

Karatarakis and Wiedijk formalized the Gelfond–Schneider theorem in Lean [3]. The library [7] proves, among other classical results, the six exponentials theorem and the qualitative form of Baker's theorem (by the Bertrand–Masser route, without a zero estimate), and it carries Roy's strong six exponentials theorem as an explicit hypothesis of several statements; Theorem 1.1 could discharge that hypothesis. We are not aware of an earlier formalization of a zero estimate or a multiplicity estimate in any proof assistant.

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