

Sparse prime surgery in completely multiplicative sequences

Normality, rare-event realization and optimal repair

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Abstract

We study how changing prime signs affects fixed-word frequencies, rare-event laws and extremes of completely multiplicative sequences. The cost is $d_\sigma(f, g) = \sum_{f(p) \neq g(p)} p^{-\sigma}$. A known theorem of Frantzikinakis–Lemańczyk–de la Rue preserves normality at finite d_1 ; we give its elementary specialization. Using explicit arithmetic inputs from *Long runs*, we then establish three families of results. For each fixed $0 < \sigma \leq 1$ and almost every base, reversible surgery realizes every stationary finite-intensity marked point-process law along selected scales, with signs, exact excess lengths and converging actual intensity, at arbitrarily small d_σ cost and with a prescribed prefix unchanged. Almost surely over all integer scales, the minimum number of prime changes needed for a run of length $\lceil c \log_2 N \rceil$ is asymptotic to $c\alpha_c \log_2 N$ for $1 < c < 3/2$, where $\alpha_c \in (0, 1/2)$ satisfies $H_2(\alpha_c) = 1 - 1/c$ and $H_2(u) = -u \log_2 u - (1 - u) \log_2 (1 - u)$. At each scale, with probability tending to one, symbolic and private large-prime repair maxima coincide on the same random base for $k_N = o(\log N)$; unrestricted prime repair agrees when $k_N = o(\log \log N)$. This comparison transfers the classical lattice laws of runs with errors to adaptive prime repair. Elementary programming from arbitrary bases and an autonomous finite-range-dependent process construction complement these results.

Keywords. Completely multiplicative functions; normality; prime-sign perturbations; long runs; Hamming balls; point-process realization; Poisson projections.

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1 Introduction and the three resolutions of surgery

Normality describes words of fixed length. A longest run has a length that grows with the observation scale. A rare-event point process records a further object: the positions and marks of many long runs seen from a moving origin. The purpose of this paper is to distinguish these three resolutions under a perturbation supported on a small set of primes.

Let $\mathcal{M}_\pm$ denote the completely multiplicative functions $f : \mathbb{N} \rightarrow \{-1, +1\}$, with $f(1) = 1$. Write $\mathbb{P}$ for the law in which the values $F(p)$, $p \in \mathcal{P}$, are independent fair signs. This is the completely multiplicative model, not the model supported on squarefree integers. For a set of primes S , define

$$f^S(n) = f(n)(-1)^{\sum_{p \in S} v_p(n)}, \quad d_\sigma(f, g) = \sum_{p: f(p) \neq g(p)} p^{-\sigma}.$$

The sum defining $f^S(n)$ is finite for each integer n , even when S is infinite. Unless otherwise stated, $0 < \sigma \leq 1$ and all logarithms are natural except $\log_2$. The usual pretentious distance on sign-valued functions satisfies $\mathbb{D}(f, g; \infty)^2 = 2d_1(f, g)$.

A sign sequence is *normal* if, for every finite nonempty $A \subset \mathbb{N}_0$,

$$\frac{1}{X} \sum_{n \leq X} \prod_{a \in A} f(n+a) \longrightarrow 0. \quad (1.1)$$

Expanding $2^{-q} \prod_{i=0}^{q-1} (1 + w_i f(n+i))$ shows that this is equivalent to frequency 2^{-q} for every fixed word w of length q . No uniformity in growing q is included.

A known stability theorem implies that normality is invariant within every finite- d_1 class; its attribution and an elementary proof are given in Section 2. Within such a class, however, arbitrarily small perturbations can create much longer than logarithmic runs. Distributed insertions can even make the critical run count diverge at every origin of a selected dyadic block. These programming results apply to *any* base, with normality retained when the base is normal. Their quantitative scale and prime-budget constraints, rather than normality stability itself, are the contribution of that section.

A different question is whether the *actual* rare-event intensity, and not merely a collection of inserted witnesses, can be prescribed. Sections 4–5 answer this relative to the one- and two-window estimates in Section 3. For a full-measure class of bases, a reversible reservoir controls all potential starts, allowing both activation and extinction. The resulting realization theorem covers stationary laws marked by sign and exact excess length. Its universality is subsequential and relative to a weighted prime class; it is not a new typical law under $\mathbb{P}$. We keep fixed-word normality, not the product distribution of the selected output. The choice of changed primes uses the observed signs; the resulting function need not have law $\mathbb{P}$. Section 10 separates predictable flips, which preserve that law, from adaptive selections, which can give singular output laws.

A count with the Poisson mean and variance, but a different law. Fix $0 < \sigma \leq 1$, a base f in the full-measure class of Theorem 5.1, a budget $\varepsilon > 0$ and a prefix cutoff T . At the observation scales $H_N = 2^{\ell_N}$ of Section 4, one can choose a single normal g with $d_\sigma(f, g) < \varepsilon$ and $g(n) = f(n)$ for $n \leq T$, and dyadic scales $N_j \rightarrow \infty$, for which the following occurs. Choose a length- H_{N_j} window uniformly among the integer windows contained in $[N_j, 2N_j]$. Let Z_j count the long-run starts whose left guard and first ℓ_{N_j} letters lie inside that window, as in (2.6). Then

$$\mathcal{L}(Z_j) \longrightarrow \nu_2 := \tfrac{1}{2}\delta_0 + \tfrac{1}{2}\delta_2, \quad \mathbb{E}Z_j \rightarrow 1, \quad \text{Var } Z_j \rightarrow 1,$$

in total variation for the laws, while

$$d_{\text{TV}}(\mathcal{L}(Z_j), \text{Pois}(1)) \longrightarrow 1 - \frac{3}{2e} > 0.$$

The bounded-count construction in Appendix B proves the moment assertions as well. Only the origin is random after g is chosen; the example concerns a selected output and selected scales, not a typical law of F .

Finally we ask how many prime coordinates are needed. Put

$$\begin{aligned} M_f(N) &= \max\{b : \text{a constant block of length } b \text{ lies in } [N, 2N]\}, \\ c_f(N, b) &= \min\{|S| : S \subset \mathcal{P} \cap [2, 2N], M_{f^S}(N) \geq b\}, \\ M_f^{[k]}(N) &= \max_{S \subset \mathcal{P} \cap [2, 2N], |S| \leq k} M_{f^S}(N). \end{aligned} \tag{1.2}$$

The minimum of the empty set is $+\infty$; the empty surgery is allowed in the maximum. All choices may depend on the observed base. A superscript $> Y_N$ restricts the primes to $(Y_N, 2N]$, where

$$L_N = \log_2 N, \quad S_N = \sqrt{\log N \log \log N}, \quad Y_N = e^{S_N}. \tag{1.3}$$

Write $H_2(u) = -u \log_2 u - (1-u) \log_2(1-u)$, and let α_c be the unique solution in $(0, 1/2)$ of $H_2(\alpha_c) = 1 - 1/c$. Our cardinality and fine-law results, proved in Sections 7 and 8, are

$$c_f(N, \lceil cL_N \rceil) \sim \alpha_c \lceil cL_N \rceil \quad (1 < c < 3/2), \quad H_2(\alpha_c) = 1 - 1/c,$$

and, for fixed k ,

$$M_F^{[k]}(N) = L_N + k \log_2 L_N - \log_2(k!) + O_{\mathbb{P}}(1).$$

The first is almost sure over all integer scales, whereas the second is a statement in probability at a scale. Theorem 8.1 gives more than common limiting distributions: the repair maxima agree with high probability on the *same* realization. Symbolic and large-prime repair allow $k_N = o(\log N)$, while the all-prime comparison requires $k_N = o(\log \log N)$. These ranges remain distinct.

The principal contributions are the reversible marked realization (Theorem 5.1), the adaptive local cost bound (Theorem 7.1), and the same-base comparison (Theorem 8.1). Quantitative programming from arbitrary bases and the autonomous process construction (Theorems 2.4 and 6.1) are separate results. Table 1 records their different arithmetic dependencies. For realization, read Sections 2–5; for cost and repair, Sections 3 and 7–9; for the limits of finite projections, Section 6 is autonomous until Corollary 6.2, which supplies the arithmetic connection.

1.1 Relation to earlier work

Stability and programming. Fish [9] proves almost-sure normality of the random model; deterministic normal multiplicative examples occur in Klurman–Mangerel–Teräväinen [16]. Stability under finite d_1 follows from Frantzikinakis–Lemańczyk–de la Rue [10, Theorem 2.16], whose Chowla–Elliott stability result already permits a pointwise, adaptively chosen multiplier. Our specialized proof uses periodic approximation of the quotient and van der Corput; the almost-periodicity background includes Daboussi–Delange, as developed in Bergelson–Kułaga-Przymus–Lemańczyk–Richter [7, Section 2.3]. Najnudel [20] studies fixed-length consecutive values and quantitative empirical convergence. For changing prime signs to lengthen runs, You [23] is a direct antecedent: private odd-valuation primes and a minimum prime-change cost are already part of that method. The elementary contribution here combines an arbitrary starting base, a preserved prefix, an arbitrarily small total prime weight and explicit growing lengths on infinitely many selected scales.

Rare laws under multiplicative constraints. Normal symbolic sequences need not be Poisson generic, as shown by Becher–Sac–Himelfarb [5]; Becher–Jackson–Kwiatniak–Mance [6] study the descriptive complexity of that property. Their observable differs from our moving-origin long-run process. Results for exponentially mixing symbolic measures [1] and nonstationary product measures [13] describe other sources under which Poisson behavior occurs. In dynamics, Lacroix [18] realizes admissible normalized return-time laws by selecting shrinking neighborhoods. Our observation rule remains a prescribed long-run rule: the function is changed under complete multiplicativity, normality, a prefix constraint and a small prime budget. Reversible control of *all* starts, rather than insertion of witnesses alone, permits realization of the actual marked law and intensity.

Nonindependent Poisson components with a Poisson sum appear in Jacod [15]; Holroyd–Lyons–Soo [14] construct deterministic equivariant splittings. Kuna–Lebowitz–Speer [17] treat nonunique realizations of low-order correlations, while Lancaster-type expansions provide a fixed-marginal algebraic context [12]. Marked extremes have an established point-process framework [4]. Within this setting, Section 6 gives one explicit construction combining exact finite families of threshold-process laws, prescribed factorial moment orders and a deterministic finite spatial dependence range, together with a test separating the complete law.

Cost and repair on a prescribed scale. In You’s modified-character setting [23], the minimum number of changes obeys a $k/2 + O(\log k)$ law, with small primes singled out for their collective effect. Here the base is a generic random normal function, the target has length $c \log_2 N$ inside $[N, 2N]$, and the lower bound allows every prime. Hamming-ball entropy and contaminated independent runs belong to the classical Erdős–Rényi literature: Arratia–Waterman [2], Arratia–Gordon–Waterman [3], and Gordon–Schilling–Waterman [11, Section 3 and Theorem 1], with rate refinements in Fazekas–Fazekas–Fórián [8]. The factorial correction and lattice effects for a fixed number of interruptions are supplied by that literature. Our contribution is the local adaptive prime-cost bound and the comparison of symbolic, private large-prime and unrestricted repair on the *same* base. Positive-only and two-sign conventions require different intensity factors, which are kept explicit below.

1.2 Arithmetic dependency boundary

The elementary constructions and the probability construction in Section 6 are proved without the deeper arithmetic inputs. The other branches use four stated results from Pouly’s *Long runs and rare patterns of a random completely multiplicative function*, with its technical companion, abbreviated *Long runs* [21]: weighted one-window defects, an absolute two-window profile, its capped form, and a labelled dictionary Poisson theorem. Section 3 states the exact forms and uniformities used and gives their locations; Table 1 records which conclusions require them. These imported arithmetic results, rather than the full corpus of related process limits, are the external proof dependencies.

2 Normality, private coordinates and elementary programming

2.1 Stability at finite prime distance

The following known special case of Frantzikinakis–Lemańczyk–de la Rue [10, Theorem 2.16] is recalled to make the later constructions self-contained. If $u = g/f$, then

$$\mathbb{D}(u, 1; \infty)^2 = \sum_p \frac{1 - u(p)}{p} = 2d_1(f, g) < \infty.$$

Thus u is pretentious. Normality is the binary Chowla–Elliott property in Cesàro averages, so their theorem applies to $g = fu$; since $u^2 = 1$, it also gives the converse. The elementary proof below specializes to this summable-prime case.

Theorem 2.1 (Known stability of normality). *If $f, g \in \mathcal{M}_\pm$ and $d_1(f, g) < \infty$, then f is normal if and only if g is normal. The set of changed primes may be chosen adaptively from f .*

Proof. For a fixed nonempty finite A , let $a_A(n) = \prod_{a \in A} f(n + a)$. If f is normal, then for every fixed integer $t \geq 1$,

$$\frac{1}{X} \sum_{n \leq X} a_A(n + t) a_A(n) \longrightarrow 0,$$

because $(A + t) \Delta A \neq \emptyset$. Average R translates of $a_A(n) e^{2\pi i \theta n}$ and apply Cauchy–Schwarz. The diagonal contributes $1/R$, and the off-diagonal terms tend to zero uniformly in θ , since their absolute values are fixed autocorrelations. Thus

$$\limsup_{X \rightarrow \infty} \sup_{\theta} \left| X^{-1} \sum_{n \leq X} a_A(n) e^{2\pi i \theta n} \right|^2 \leq R^{-1}.$$

Let $R \rightarrow \infty$. Finite Fourier expansion gives orthogonality to every bounded periodic sequence.

Put $S = \{p : f(p) \neq g(p)\}$ and $u = g/f$. For finite $E \subset S$ and $r \geq 1$, the sequence

$$u_{E,r}(n) = (-1)^{\sum_{p \in E} \min(v_p(n), r)}$$

is periodic modulo $\prod_{p \in E} p^r$. Moreover

$$\limsup_{X \rightarrow \infty} \frac{1}{X} \sum_{n \leq X} |u(n) - u_{E,r}(n)| \leq 2 \sum_{p \in S \setminus E} p^{-1} + 2 \sum_{p \in E} p^{-r}. \quad (2.1)$$

Indeed, a discrepancy requires a divisor in $S \setminus E$ or divisibility by p^r for some $p \in E$. For the infinite tail use $X^{-1} \sum_p \lfloor X/p \rfloor \leq \sum_p p^{-1}$, rather than summing infinitely many rounding errors. Replace the factors $u(n + a)$ in a correlation of g by their periodic approximations and telescope. The approximating correlation tends to zero; the limsup error is bounded by $|A|$ times the right side of (2.1). First let $r \rightarrow \infty$, then exhaust S by finite E . Symmetry gives the converse. $\square$

A finite prime distance need not mean that the values differ on a density-zero set. For example, one changed prime affects a positive proportion of integers. Theorem 2.1 is stronger than a density comparison.

2.2 A Rankin estimate and the endpoint block length

An integer is Y -defective if it has no prime divisor $p > Y$ of odd valuation. Write $A(x, Y)$ for the number of such integers at most x .

Lemma 2.2. *For $0 < \eta < 1/2$,*

$$A(x, Y) \leq x^{1-\eta} \prod_{p \leq Y} (1 + p^{-1+\eta}). \quad (2.2)$$

Uniformly when $\eta \log Y \geq 2$,

$$\log \prod_{p \leq Y} (1 + p^{-1+\eta}) \ll \log \log Y + \frac{Y^\eta}{\eta \log Y}. \quad (2.3)$$

In particular, for fixed $d > 0$,

$$\frac{A(4N, e^{dS_N})}{N} \leq \exp\{-(1/(2d) + o(1))S_N\}. \quad (2.4)$$

Proof. Every defective integer is uniquely a^2s , with s squarefree and Y -smooth. Since $\lfloor \sqrt{x/s} \rfloor \leq (x/s)^{1-\eta}$, summing over s and dropping $s \leq x$ gives (2.2). Chebyshev's bound and partial summation bound the logarithm of the product by a constant times

$$\frac{Y^\eta}{\log Y} + \int_{\log 2}^{\log Y} \frac{e^{\eta v}}{v} dv.$$

Split at $v = 1/\eta$ to get (2.3). For (2.4), choose $\eta = \log \log N / (2dS_N)$. Then $\eta \log N = S_N / (2d)$, $Y^\eta = \sqrt{\log N}$, and the Euler-product logarithm is $o(S_N)$. $\square$

Set

$$B_*(N) = \left\lfloor \exp \sqrt{\frac{1}{2} \log N \log \log N} \right\rfloor. \quad (2.5)$$

Lemma 2.3 (Private pivots at the endpoint). *Fix $C > 0$. For all large N , some block of $M = \lceil CB_*(N) \rceil$ integers in $[N, 2N]$ has an odd-valuation prime greater than $Y = M(\log N)^2$ at every position. Such primes, chosen one per position, are distinct and private in the block.*

Proof. Write $L = \log N$, $J = \log L$, $b = \sqrt{LJ/2}$ and $v = \log Y = b + 2J + O_C(1)$. In (2.2) choose $\eta = (J + \log J)/(2v)$. The Euler-product logarithm is $O_C(\sqrt{L/J})$, and

$$\eta L = b + \frac{1}{\sqrt{2}} \sqrt{L/J} \log J + O_C(J + \log J).$$

Consequently

$$\log \frac{MA(2N, Y)}{N} \leq - \left(\frac{1}{\sqrt{2}} + o_C(1) \right) \sqrt{L/J} \log J \rightarrow -\infty.$$

A defective integer contaminates at most M block starts. Since there are $N - M + 2 \sim N$ possible starts, an uncontaminated block exists. A prime exceeding $Y > M$ cannot divide two distinct positions of it. $\square$

Theorem 2.4 (Prescribed long words from every base). *Fix $f \in \mathcal{M}_\pm$, $\varepsilon > 0$, $T \geq 1$ and $C > 0$. For each large N , prescribe a word w_N of length $1 \leq m_N \leq CB_*(N)$. There are $g \in \mathcal{M}_\pm$ and $N_j \rightarrow \infty$ such that $d_1(f, g) < \varepsilon$, $g(n) = f(n)$ for $n \leq T$, and w_{N_j} occurs inside $[N_j, 2N_j]$. The scales may be dyadic and may exceed successively prescribed bounds. If f is normal, so is g .*

Dependency. No input from *Long runs* is used; see Table 1.

Proof. Use Lemma 2.3 and flip a private pivot precisely when a target letter disagrees. At most M primes change, with cost $M/Y = (\log N)^{-2}$. Starting from the current finite modification, choose the next scale so that this cost is below $\varepsilon 2^{-j-1}$ and $Y > \max(T, 2N_{j-1})$. Later changes then preserve every integer at most $2N_j$. The prime signs converge coordinatewise, all programmed words remain, and the costs sum to less than ε . Apply Theorem 2.1. $\square$

In particular, the word $(-1), (+1)^{B_*}, (-1)^{B_*}, (+1)$ produces positive and negative maximal runs of exactly B_* at the same scale. The coefficient $1/\sqrt{2}$ in (2.5) is achieved by this argument; no optimality for that coefficient is asserted.

2.3 Programming visible from every origin

For an integer origin x , define

$$Z_g(x; H, \ell) = \sum_{n=x+1}^{x+H-\ell} \mathbf{1}_{\{g(n-1) \neq g(n) = \dots = g(n+\ell-1)\}}. \quad (2.6)$$

The left guard and the first ℓ letters are contained in the window. Under independent fair signs on the integers the transition signs are independent, and this is a count of the nonoverlapping word $(-1, +1, \dots, +1)$ of length ℓ in the transition sequence. When $H = 2^\ell$, its mean is $1 - \ell/H$, and its factorial moments tend to those of $\text{Pois}(1)$.

Lemma 2.5 (Private pivots across many blocks). *Let K disjoint cells of length D lie in $[N, 2N]$, let $m \leq D/2$, and let $Y > m$. Set $R_0 = Km \log(2N)/\log Y$. If*

$$mA(2N, Y) + mR_0(D/Y + 1) < D - m + 1, \quad (2.7)$$

then any prescribed word of length m in each cell can be realized from any base by at most Km changes at primes greater than Y . Their d_σ cost is at most $KmY^{-\sigma}$.

Proof. Choose block locations successively. Let E contain *all* primes greater than Y dividing integers of earlier chosen blocks. The product of those integers bounds $|E|$ by R_0 . In the next cell, at most $mA(2N, Y)$ starts meet a defective integer and at most $m \sum_{p \in E} (D/p + 1)$ meet a multiple of a forbidden prime. Condition (2.7) leaves a start. The new odd pivots are private within their block and do not divide any earlier block. Future blocks avoid them in turn. All targets can therefore be imposed simultaneously. $\square$

Theorem 2.6 (Distributed excess intensity). *Fix $0 < \alpha < \sqrt{\sigma}/2$ and put*

$$\ell_N = \lfloor \log_2(Ne^{-\alpha S_N}) \rfloor, \quad H_N = 2^{\ell_N}, \quad q_N = \lfloor \log N \rfloor.$$

For every $f \in \mathcal{M}_\pm$, $\varepsilon > 0$ and $T \geq 1$, there exist g and scales $N_j \rightarrow \infty$ with $d_\sigma(f, g) < \varepsilon$ and the same prefix through T , such that

$$Z_g(x; H_{N_j}, \ell_{N_j}) \geq q_{N_j} \quad (N_j \leq x \leq 2N_j - H_{N_j} + 1).$$

Normality is retained when present. The distance of the moving-origin count law from $\text{Pois}(1)$ tends to one, but its actual mean tends to infinity.

Dependency. This is also an elementary programming result in Table 1; it does not prescribe a finite limiting intensity.

Proof. Choose $\alpha/\sigma < d < 1/(2\alpha)$, set $Y = e^{dS_N}$, and partition into cells of length $D = \lfloor H_N/4 \rfloor$. In each full cell program

$$(-1)((+1)^{\ell_N}(-1))^{q_N}, \quad m = 1 + q_N(\ell_N + 1) = O((\log N)^2).$$

Here $K = e^{\alpha S_N + O(1)}$. The three exclusion terms in (2.7), divided by D , are at most

$$e^{(\alpha - 1/(2d))S_N + o(S_N)}, \quad e^{(\alpha - d)S_N + o(S_N)}, \quad e^{-\log N + 2\alpha S_N + o(S_N)},$$

respectively. They tend to zero; $d > \alpha$ follows from $\sigma \leq 1$. The cost is $e^{(\alpha - \sigma d)S_N + o(S_N)} \rightarrow 0$. Every admissible window contains a full cell, hence the required guarded runs. Sum costs and freeze earlier scales exactly as in Theorem 2.4. Finally the event $[q_{N_j}, \infty)$ has probability one for the constructed count and vanishing probability under $\text{Pois}(1)$. $\square$

Proposition 2.7 (Relative Baire statements). *The space $\mathcal{C}_\sigma(f) = \{g : d_\sigma(f, g) < \infty\}$ is complete and separable in d_σ ; finite prime modifications of f are dense. For $\sigma \leq 1$ and normal f , every element is normal. The infinite-scale properties in Theorems 2.4 and 2.6, at their respective metrics and scales, are residual in these classes.*

Proof. Each prime coordinate stabilizes in a Cauchy sequence; Fatou and the Cauchy property give metric convergence to the coordinatewise limit. Truncating the changed-prime set proves separability and density. For every lower scale bound, the union of successful finite-scale events is open, because it depends on finitely many prime signs, and dense by the corresponding vanishing-cost construction. Apply Baire. This is a category statement within one class, not a statement of product probability. $\square$

3 Arithmetic inputs and their exact roles

The results from this section onward use the following interfaces only when explicitly indicated. Supports lie in fixed bounded-ratio intervals at ambient scale X , and their lengths b lie in a fixed logarithmic band. All band and geometry constants are fixed before X .

For a block $V_n = [n, n + b - 1]$, let $m_b(n)$ count integers having no odd-valuation prime greater than b . Let $\rho_b(n, m)$ be the nullity of the complete system of $2b$ valuation rows modulo two for two disjoint blocks. This is a *full-value* nullity, with no parity condition imposed separately in each block.

(A1): **weighted one-window defects.**

$$\sum_n (2^{m_b(n)} - 1) \leq X^{1/2 + o(1)}. \quad (3.1)$$

Deterministic masks may restrict the nonnegative sum. Comparable widths and thresholds in a fixed logarithmic band are allowed. This is the form of Proposition 2.4 and its discussion in [21, pp. 10–11].

(A2): **absolute two-window profile.**

$$R_b(X) := \sum_{|n-m| \geq b} (2^{\rho_b(n, m)} - 1) \ll_\eta X^\eta (X^{5/3} + X^{2/3} 2^b), \quad \eta > 0. \quad (3.2)$$

This is Proposition 3.24 of [21, p. 28], including its bounded-ratio versions. In particular $R_b(X) \ll X^{7/4}$ when $2^b = O(X)$, on taking $\eta = 1/12$.

(CAP): **capped absolute profile.** Uniformly for $T \geq 1$, with constants independent of T ,

$$\begin{aligned} R_b^{[T]}(X) &:= \sum_{|n-m| \geq b} \min\{T, 2^{\rho_b(n,m)} - 1\}, \\ R_b^{[T]}(X) &\ll_{\eta} X^{\eta} \left(X^{3/2} 2^{b/6} + X 2^{2b/3} + X^{2/3} \min\{T, 2^b\} \right). \end{aligned} \quad (3.3)$$

This is Proposition 3.25, equation (3.27), of [21, pp. 28–29]. The middle term is $X 2^{2b/3}$. Capping the right side of an uncapped sum estimate would not justify this input.

(D): **labelled dictionaries at bounded rarity.** For a deterministic dictionary $W \subset \{-1, +1\}^B$, write $m = |W|$ and

$$\Omega(W) = \frac{1}{m} \sum_{u,v \in W} \sum_{d=1}^{B-1} 2^{-d} \mathbf{1}_{\{(u_d, \dots, u_{B-1}) = (v_0, \dots, v_{B-d-1})\}}. \quad (3.4)$$

If, for fixed $\eta > 0$,

$$m \leq N^{1/2-\eta}, \quad B - \log_2(Nm) = O(1), \quad \Omega(W) \rightarrow 0, \quad (3.5)$$

then the entire field of word indicators, indexed by admissible starts in $[N, 2N]$ and by words, is at total variation distance $o(1)$ from independent $\text{Pois}(2^{-B})$ coordinates. Every deterministic restriction of this index set is allowed. The conditional theorem in [21, Theorem 5.1, pp. 33–34], averaged and then deconditioned, gives this form. It is substantially stronger than vanishing correlations at fixed length.

Lemma 3.1 (Consequences of rank). *Under (A1), for every deterministic word w of length b ,*

$$\mathbb{P}(F|_{V_n} = w) \leq 2^{-b+m_b(n)}. \quad (3.6)$$

If every integer in V_n has an odd-valuation prime greater than $Y > b$, the block values are exactly uniform signs. For two disjoint blocks, each prescribed pair of words has probability at most $2^{-2b+\rho_b(n,m)}$.

Proof. A nondefective row has a prime coordinate private in its block. Its row rank is therefore at least $b - m_b(n)$. A system of sign equations over $\mathbb{F}_2$ is either inconsistent or has probability $2^{-\text{rank}}$. The same argument gives the remaining assertions. It makes no independence assertion for distinct blocks. $\square$

Table 1: Arithmetic dependencies. “None” means that the proof uses no input from *Long runs*.

Conclusion	Arithmetic dependence
Known stability; elementary programming; finite-range marked example	None
Reversible reservoir; marked realization and its moment refinement	(A1), (A2)
All-prime lower cost; adaptive union bounds	(A1)
Optimal cost for $1 < c < 3/2$ and large-prime remainder	(A1), (CAP)
Fine budget laws; one-prime-per-stage construction	(A1), (D)

The raw input alone yields the sufficient range $1 < c < 4/3$. The cap improves the deduction, not the underlying raw exponent. The cardinality and fine-law arguments do not use the reversible realization theorem.

Conventions relative to *Long runs*. An absolute word beginning physically at n has label $x = n + 1$ in the dictionary theorem of [21]; a run beginning at n , with left guard at $n - 1$, has run-start label n in both papers. Here a block defining $M_f(N)$ is wholly contained in the inclusive interval $[N, 2N]$, whereas *Long runs* initially indexes starts in $[N, 2N)$ without truncating the ensuing run. We restrict the imported fields to the admissible supports of the observable at hand. The empirical coordinate in Section 5 is $(n - U_N)/H_N$, and the surgery cutoff $Y_N = e^{S_N}$ in (1.3) is not the cutoff $Y_N^\#$ of [21]. Prime weight, number of flips and selection information remain distinct resources. The compactified mark ∞ below records a limit of finite excess lengths, not an infinite run in a normal sequence.

4 A reversible reservoir with two guards

We now assume (A1) and (A2). To prescribe a counting law, planting the desired runs is not enough: every unselected run is counted as well. The task is to list all starts that could occur after the permitted prime changes, and then to activate selected starts while extinguishing the others.

The construction first marks a sparse pool of large control primes. Positions on which none of these primes has odd valuation are nonfree and keep their values under every later decision. Those fixed values determine an exhaustive list of compatible starts. The estimates separate this list into small components whose control primes do not interfere. Each candidate core has at least two free positions; since the core contains only one left guard, at least one free position lies in the intended constant body and can be used to extinguish it. To activate a run of exact length, we instead select a compatible word with both a left and a right guard. A guard need not be free: its prescribed value must already agree wherever it is nonfree. Lemma 4.2 makes these finite control rules precise.

The scales have separate roles. The observation length $H = o(N)$ leaves many translated windows; a mesh with $\ell \ll D \ll H$ gives both room for separate controls and a vanishing normalized position error. Each mesh cell must contain candidates of every fixed required mark, while interactions must be rare. Finally only the controls actually flipped are paid for, with cost bounded by $C|\mathcal{C}|Y^{-\sigma}$, where C is the extended support length and $\mathcal{C}$ the candidate inventory. The following parameter choices meet all of these requirements at once; Table 2 records the margins. Fix σ and set

$$\begin{aligned} L &= \log N, & J &= \log L, & S &= \sqrt{LJ}, & Y &= e^S, \\ a &= \sigma/100, & \kappa &= a/100, & \beta &= a/20, & \theta &= aJ/L, \\ \ell &= \lfloor \log_2(NL^{-\kappa}) \rfloor, & H &= 2^\ell, & K &= N/H, & D &= \lfloor NL^{-\beta} \rfloor. \end{aligned} \quad (4.1)$$

Thus $K \asymp L^\kappa \rightarrow \infty$, $D/H \asymp L^{\kappa-\beta} \rightarrow 0$, and $D \gg \ell$.

Fix an integer $R \geq 0$ before letting $N \rightarrow \infty$. Put

$$B = \ell + 1, \quad C = \ell + R + 2, \quad I_{N,R} = \{N + 1, \dots, 2N - \ell - R\}.$$

For $n \in I_{N,R}$ the core $V_n = [n - 1, n + \ell - 1]$ has length B ; the extended support $W_n = [n - 1, n + \ell + R]$ has length C . The full start set $I_N = \{N + 1, \dots, 2N - \ell + 1\}$ has only $R + 1$ additional boundary starts.

Independently of F , mark each prime in $(Y, 2N]$ with probability θ . An integer is *free* when at least one marked prime has odd valuation there. Let t_n be the number of free positions in the core and let T_n^+ count marked primes relevant in the extended support. Since $Y > 2C$ for large N , each such prime belongs to at most one position of W_n , and

$$t_n \leq T_n^+, \quad r_n^+ \leq C \log(2N)/\log Y \leq 3L^2/S, \quad \theta r_n^+ \leq 3aS, \quad (4.2)$$

Table 2: Reservoir parameters and the margins used below; $0 < a \leq 1/100$. The excess cutoff R is fixed before N .

Quantity or error	Choice or exponent ensuring the required limit
Available normalized length	$K \asymp L^\kappa, \kappa = a/100 > 0$
Mesh displacement	$D/H \asymp L^{\kappa-\beta}, \kappa - \beta = -a/25$
Marked-prime sampling	$\theta = aJ/L, \theta r_n^+ \leq 3aS$
High-mark truncation	$t_* = \lfloor S/(4 \log 2) \rfloor; 1/4 - 9a \geq 0.16$
Low-free candidates	$\kappa - a/(2 \log 2) < 0$
Cells with too few free sites	$t_0 = \lfloor aJ/(4 \log 2) \rfloor; \beta - a/(16 \log 2) < 0$
Weighted search over the mesh	$2\beta - \kappa - a/4 = -4a/25 < 0$
Weight of the actual changes	$4a - \sigma = -24\sigma/25 < 0$

where r_n^+ counts all primes that could be marked there.

A *potential start* (n, s) is one whose nonfree core positions already agree with $(-s, s, \dots, s)$. Let $\mathcal{C}$ be the set of these pairs. Whatever changes are subsequently made at marked primes,

$$\text{every actual start in } I_{N,R} \text{ belongs to } \mathcal{C}. \quad (4.3)$$

This inventory is made before activation decisions and remains valid when primes in the extended tails are modified.

Call an extended support good if each of its integers has an odd-valuation prime greater than Y . Conditional on the marking, nonfree rows have no nonzero marked coordinate. On a good support their private primes are unmarked. Thus

$$\begin{aligned} \mathbb{P}_F((n, s) \in \mathcal{C}) &= 2^{-B+t_n} \quad \text{on good supports,} \\ \mathbb{P}_F((n, s) \in \mathcal{C}) &\leq 2^{-B+t_n+m_B(n)}, \\ \mathbb{P}_F((n, s), (m, s') \in \mathcal{C}) &\leq 2^{-2B+t_n+t_m+\rho_B(n,m)} \quad (|n-m| \geq B). \end{aligned} \quad (4.4)$$

The last two estimates use a subsystem of the full valuation rows. Its kernel embeds by zero extension in the full kernel. They do not require a new conditional version of the arithmetic input.

Proposition 4.1 (Reservoir geometry). *With probability $1 - o_R(1)$ under the joint base and marking law, all of the following hold:*

- (i) *every potential start has a good extended support and $2 \leq t_n \leq T_n^+ \leq t_*$, where $t_* = \lfloor S/(4 \log 2) \rfloor$;*
- (ii) *distinct potential starts have no distance in $[B/2, C-1]$;*
- (iii) *extended supports at distance at least C share no marked prime of odd valuation;*
- (iv) $|\mathcal{C}| \leq Ke^{4aS}$.

Proof. Since T_n^+ is binomial conditional on its arithmetic support,

$$\mathbb{E}[2^{T_n^+} \mathbf{1}_{T_n^+ > t_*}] \leq 2^{-t_*} \mathbb{E} 4^{T_n^+} \leq e^{-S/4+9aS+O(1)}.$$

Sum the one-core bound in (4.4) and use (A1). The expected number of candidates with too many marks is $O(K)e^{-(1/4-9a)S+O(1)} = o(1)$.

At most $CA(3N, Y)$ extended supports are bad. After the preceding truncation, their expected number of candidates is at most

$$O(2^{-B}e^{S/4})(CA(3N, Y) + N^{1/2+o(1)}) = o(1)$$

by (2.4). On a good support choose one prime greater than Y per core position. These B distinct primes show that t_n stochastically dominates $\text{Bin}(B, \theta)$. Since $B\theta \geq aJ/(2\log 2)$ for large N , the probability that $t_n \leq 1$ is at most $L^{-a/(2\log 2)+o(1)}$. The candidate probability on that event is at most 2^{1-B} per sign, so the expected number is $O(K)L^{-a/(2\log 2)+o(1)} = o(1)$.

For two good cores at distance $\lceil B/2 \rceil \leq d < B$, their union has $B+d$ positions and at most $2t_*$ free positions. Its nonfree positions have unmarked private primes, because its diameter is less than Y . For all four sign choices the probability is at most $4e^{S/2}2^{-B-d}$. Summation gives $N^{-1/2+o(1)}$. The additional distances $B \leq d < C$ number only $R+1$; the analogous bound gives $O_R(Ne^{S/2}2^{-2B}) = N^{-1+o(1)}$.

For separated extended supports, the number of ordered pairs sharing any prime greater than Y is at most

$$E_C \ll C^2(N^2/Y + N \log N). \quad (4.5)$$

Indeed a prime p can occur in at most $C(N/p + 1)$ supports; sum the squares. By (4.4), the expected number of forbidden candidate pairs is bounded by

$$O(2^{-2B}e^{S/2})(E_C + R_B(N)) = o_R(1).$$

The main terms are $O(K^2C^2e^{-S/2})$ and $N^{-1/4+o(1)}$, using (A2) and $2^B = O(N)$. Finally $\mathbb{E}2T_n^+ \leq e^{3aS}$ implies $\mathbb{E}|\mathcal{C}| \ll Ke^{3aS}$; Markov gives (iv) with error $O(e^{-aS})$. A union bound completes the proof. $\square$

4.1 Deterministic activation and extinction

For $s \in \{-1, +1\}$ and $0 \leq q \leq R$, write

$$w_{s,q} = (-s, s^{\ell+q}, -s), \quad m_q = \ell + q + 2. \quad (4.6)$$

The next lemma contains no probabilistic or asymptotic assertion. It separates the finite control mechanism from the estimates that produce its hypotheses.

Lemma 4.2 (Deterministic two-guard control). *Fix $f \in \mathcal{M}_\pm$, integers $\ell \geq 2$, $R \geq 0$, a finite start set $I \subset \{2, 3, \dots\}$, and a finite set $\mathcal{A}$ of allowed control primes, all greater than $Y > 0$. Put $C = \ell + R + 2$, $V_n = [n - 1, n + \ell - 1]$ and $W_n = [n - 1, n + \ell + R]$. A position is free if some prime in $\mathcal{A}$ has odd valuation there. Let $\mathcal{C}$ consist of all pairs $(n, s) \in I \times \{-1, +1\}$ compatible with $(-s, s^\ell)$ on the nonfree positions of V_n . Assume:*

- (i) $\mathcal{C}$ has a partition into components, each with a common candidate sign and with start-diameter strictly less than ℓ ; the unions $U_\Gamma = \bigcup_{(n,s) \in \Gamma} W_n$ for distinct components Γ are pairwise disjoint;
- (ii) every candidate core contains at least two free positions;
- (iii) in $U = \bigcup_{(n,s) \in \mathcal{C}} W_n$, every free position m has a control prime $p_m \in \mathcal{A}$ with $v_{p_m}(m)$ odd and $v_{p_m}(m')$ even for every other $m' \in U$.

In each component choose either extinction, or one candidate (n, s) and $q \in \{0, \dots, R\}$ such that $w_{s,q}$ is compatible with its nonfree positions. All choices can be implemented simultaneously by flipping a subset of the p_m . The actual starts of runs of length at least ℓ in I are exactly the selected candidates. Each selected run has length exactly $\ell + q$. At most $|U \cap \{\text{free positions}\}| \leq C|\mathcal{C}|$ primes change, and their d_σ cost is at most $C|\mathcal{C}|Y^{-\sigma}$ for every $\sigma > 0$.

Proof. For extinction in a component of sign s , assign $-s$ to all its free positions. Every core has at least two free positions but only one left guard, so at least one free position lies in its

constant body. This prevents every candidate of that component from being an actual start. For an active component, prescribe the selected two-guard word at its free positions and keep the other free positions unchanged. The nonfree positions were already compatible. The left guard gives a start, and the right guard fixes its exact length. Two different starts of runs of length at least ℓ cannot be separated by a positive distance less than ℓ : the later left guard would lie inside the earlier constant body. Thus no other candidate of that component is an actual start.

The prescriptions on different components do not conflict, because their support unions are disjoint by (i). The primes p_m are distinct by (iii), and flipping p_m changes exactly the value at m within U . Thus all prescribed free values, across every component and every right guard, can be obtained simultaneously. Nonfree values anywhere are unaffected by all allowed flips. An actual start outside the original list would therefore have a nonfree incompatible letter that never changed, a contradiction. This proves the exhaustive, rather than witness-only, conclusion. Flipping only when necessary gives the cardinality and weight bounds. $\square$

Application of the geometry. Connect potential starts at distance less than C . Proposition 4.1(ii) gives component diameter less than $B/2$: a first crossing of $\lceil B/2 \rceil$ from the smallest start would produce a forbidden pair before distance C . Candidate signs in a component are equal. Indeed their constant bodies overlap on at least $\lfloor B/2 \rfloor$ positions; opposite signs would require all of these to be free, contrary to $T_n^+ \leq t_* < \lfloor B/2 \rfloor$. Starts in distinct components are separated by at least C , so their extended supports, and hence their support unions, are disjoint. Thus (i)–(ii) of Lemma 4.2 hold. The union of extended supports of a component has diameter less than $C + B/2 < Y$. A marked odd pivot at a free integer is private within this union, and has no odd occurrence in another component by Proposition 4.1(iii). This proves the global privacy hypothesis (iii) of the lemma.

Consequently every compatible set of activation/extinction decisions has cost

$$C|\mathcal{C}|Y^{-\sigma} \leq CKe^{(4a-\sigma)S} = CKe^{-0.96\sigma S} = o_R(1). \quad (4.7)$$

Only the controls actually used are flipped, not the entire marked-prime reservoir. The $R + 1$ omitted boundary starts contribute at most $(R + 1)/K$ to normalized intensity. Their probability of being visible in a bounded normalized interval A is at most

$$(|A| + 2/H)(R + 1)/K. \quad (4.8)$$

Their run ends may lie beyond $2N$ and are not claimed to be frozen.

4.2 Availability of exact lengths on a fine mesh

For the two-guard word $w_{s,q}$ in (4.6), let $C_n^{s,q}$ denote compatibility with its nonfree positions; it implies $(n, s) \in \mathcal{C}$. Write $t_{n,q}$ for the number of free positions in that word, set $m_q = \ell + q + 2$, and let $t_0 = \lfloor aJ/(4 \log 2) \rfloor$. On a good support, compatibility has probability $2^{-m_q + t_{n,q}}$: more free positions make a candidate easier to find. Multiplication by $2^{-t_{n,q}}$ removes precisely this advantage, so every weighted summand has the same expectation 2^{-m_q} . This normalization also cancels the free-coordinate factors in separated pairs, leaving the absolute relation profile to control their covariance.

Lemma 4.3 (Weighted search). *Fix a marking, a cell of D starts, and a subset U of good extended supports whose cores satisfy $t_n \geq t_0$. If $p_q = 2^{-m_q}$ and*

$$X_Q^{s,q} = \sum_{n \in U} 2^{-t_{n,q}} \mathbf{1}_{C_n^{s,q}},$$

then

$$\mathbb{E}_F X_Q^{s,q} = |U|p_q, \quad \text{Var}_F X_Q^{s,q} \ll Dp_q(t_0 + 1)2^{-t_0} + p_q^2 R_{m_q}(N). \quad (4.9)$$

Proof. Each nonfree row has a private unmarked prime, so its compatibility probability is $2^{-m_q+t_{n,q}}$. This gives the expectation and bounds the diagonal by $Dp_q2^{-t_0}$. For overlapping words at distance $d < m_q$, let $t_\cap$ count free positions in their intersection. The weighted joint probability is at most $p_q2^{-d-t_\cap}$: incompatible prescriptions contribute zero, and otherwise the union has private nonfree coordinates. Since $t_\cap \geq \max(t_0 - d, 0)$, summing over d costs at most $O(p_q(t_0 + 1)2^{-t_0})$ per start. For disjoint words, the weights cancel the factors arising from the free coordinates; the covariance is at most $p_q^2(2^{\rho_{m_q}} - 1)$. Summation proves the result. $\square$

Proposition 4.4 (All finite marks are available). *With probability $1 - o_R(1)$, every full cell of D starts in $I_{N,R}$ contains a candidate compatible with each of the $2(R+1)$ words in (4.6).*

Proof. There are $o(D)$ bad extended supports in total. On the others, binomial domination and Chernoff give

$$\mathbb{P}_{\text{mark}}(t_n < t_0) \leq L^{-a/(16 \log 2)}.$$

Markov in each cell, then a union over $O(L^\beta)$ cells, shows that every cell has at least $D/2$ good starts with $t_n \geq t_0$, except with probability $o(1)$; here $\beta < a/(16 \log 2)$. Fix such a marking. Chebyshev and (4.9), summed over cells, yield

$$O_R(J)L^{2\beta-\kappa-a/4+o(1)} + O_R(N^{-1/4}L^{3\beta}) = o_R(1),$$

because $2\beta - \kappa - a/4 = -4a/25 < 0$. The factor $p_q^{-1} = 2^{q+2}H$ costs only a constant for fixed R . A finite union over s, q completes the argument. No assertion uniform in an arbitrary growing R has been used. $\square$

5 Realizing stationary marked rare-event laws

The realization has four steps: approximate a stationary target and its intensity by a finite periodic pattern; use a reservoir at a sufficiently large scale to realize that pattern; extinguish every unselected component; then use only larger primes at future stages so that the controlled occurrences and absences remain fixed. Independent target pieces in the periodic approximation allow nonergodic laws. Higher moments require the additional choices in Proposition 5.5, not just weak convergence of the process.

Let $\bar{\mathbb{N}}_0 = \mathbb{N}_0 \cup \{\infty\}$ have the one-point compactification, for example $q \mapsto 2^{-q}$ and $\infty \mapsto 0$. Put $\mathbb{K} = \{-1, +1\} \times \bar{\mathbb{N}}_0$. A finite singleton and a tail $[r, \infty]$ are clopen. Point measures on $\mathbb{R} \times \mathbb{K}$ are positive integer-valued locally finite measures; signs are marks, not signed masses. Multiplicities are allowed.

For normal g , let $L_g(n)$ be the exact finite run length at a start n . With the parameters (4.1), let U_N be uniform on $\{N, \dots, 2N\}$, and define

$$\Psi_{g,N} = \sum_{\substack{n \in I_N: g(n-1) \neq g(n) \\ L_g(n) \geq \ell_N}} \delta_{((n-U_N)/H_N, g(n), L_g(n)-\ell_N)}. \quad (5.1)$$

Only the origin is random in this law: g is fixed. Also set

$$\lambda_{g,N} = \frac{H_N}{N+1} \# \{n \in I_N : g(n-1) \neq g(n), L_g(n) \geq \ell_N\}. \quad (5.2)$$

For a stationary law P on $\mathcal{M}_p(\mathbb{R} \times \mathbb{K})$, write $\lambda(P) = \mathbb{E}_P \xi([0, 1) \times \mathbb{K})$ and let $\mathcal{S}(\mathbb{K})$ be the laws with finite $\lambda(P)$.

Theorem 5.1 (Marked realization). *Assume (A1) and (A2). For every $0 < \sigma \leq 1$, there is a $\mathbb{P}$ -full-measure set Ω_σ , stable under finite prime modifications, with the following property. For $f \in \Omega_\sigma$, $\varepsilon > 0$, $T \geq 1$ and $P \in \mathcal{S}(\mathbb{K})$, there exist normal $g \in \mathcal{M}_\pm$ and dyadic $N_j \rightarrow \infty$ such that*

$$d_\sigma(f, g) < \varepsilon, \quad g(n) = f(n) \ (n \leq T), \quad \mathcal{L}(\Psi_{g, N_j}) \Rightarrow P, \quad \lambda_{g, N_j} \rightarrow \lambda(P). \quad (5.3)$$

The convergence of process laws is weak for the vague topology. All finite collections of fixed excess thresholds are realized on the same subsequence.

Dependency. The arithmetic dependence is exactly (A1) and (A2) in Table 1; higher moments require Proposition 5.5.

Lemma 5.2 (Periodic approximation with intensity). *Every $P \in \mathcal{S}(\mathbb{K})$ is the weak limit, with converging intensities, of uniformly translated periodic point measures having rational period and positions, finitely many atoms per period, and finite excess marks.*

Proof. Fix finitely many continuous compactly supported Laplace tests and a spatial radius A containing their supports. Concatenate independent restrictions of P to intervals of length $T \gg A$, and repeat the resulting finite concatenation periodically. For an origin more than A from a join, each test sees the correct restriction. The proportion of other origins is $O(A/T)$, and the tests are bounded. The law of large numbers for the integrated tests, together with the integrable counts in each piece, gives a finite concatenation approximating both test averages and intensity. Several independent pieces are important: no ergodicity of P is required. Approximate its finite set of positions and its period by rationals. Replace an infinite excess mark by a sufficiently large finite integer. These operations preserve the count and approximate the tests. Diagonalize a countable determining family; bounded intensities give local tightness. $\square$

Proof of Theorem 5.1. For each fixed R , let $E_{N,R}$ be the event that some marking gives the conclusions of Propositions 4.1 and 4.4. The probability of its failure under F alone is at most the joint failure probability, hence tends to zero. Choose a deterministic dyadic sequence M_i with $\mathbb{P}(E_{M_i, R}^c) \leq 2^{-i}/(i+1)$ for every $R \leq i$. Borel–Cantelli gives eventual success for each fixed R . Intersect with the normality event of Fish [9] and with all inverse images under finite prime flips. There are countably many such measure-preserving maps; the resulting full-measure set Ω_σ is stable under them.

First consider a fixed periodic pattern η , with period T_η and largest excess R . Dilate its atoms by H_N and place its periods through the bulk of $[N, 2N]$. Assign each atom to a full mesh cell within $O_\eta(D)$ of its ideal location, leaving an unused cell between selected cells. For a multiplicity use distinct neighboring cells. Since $D/H_N \rightarrow 0$ and the pattern is fixed, this is possible. Proposition 4.4 supplies the desired sign and exact length in each selected cell. Lemma 4.2 activates those candidates and extinguishes all other components. Distinct selected cells are separated by at least $D \gg C$, so the choices are in different components.

The controlled atoms are paired with the ideal atoms of η at normalized displacement $O_\eta(D/H_N)$. To obtain a *continuous* phase, take $V \sim \text{Unif}[0, 1)$ independently of U_N and put

$$\tilde{U}_N = U_N + V, \quad \varphi_N = (\tilde{U}_N/H_N) \pmod{T_\eta}. \quad (5.4)$$

Then $\tilde{U}_N$ is uniform on $[N, 2N+1)$, and

$$\frac{|\tilde{U}_N - U_N|}{H_N} \leq H_N^{-1}, \quad d_{\text{TV}}(\mathcal{L}(\varphi_N), \text{Unif}[0, T_\eta)) \leq \min \left\{ 1, \frac{T_\eta H_N}{N+1} \right\} = O(T_\eta/K). \quad (5.5)$$

For the second bound, partition an interval of length $(N + 1)/H_N$ into complete periods and one remainder of length less than T_η . Complete periods give exactly the uniform phase; the remainder has at most the displayed relative mass. This total-variation bound concerns only φ_N . The unsmoothed phase of U_N/H_N is finitely supported and has total-variation distance one from continuous uniform measure.

Return to the original integer-origin observation by shifting its atoms by at most H_N^{-1} . Combined with the pattern displacement, this changes continuous compactly supported tests by a quantity tending to zero: the fixed pattern has uniformly bounded local counts. For interval-count tests, only phases placing an atom within the displacement of an endpoint can change the count, and their probability also tends to zero. Dyadic end margins affect $O_{\eta,A}(K^{-1})$ of observations of a fixed radius A . Counts per period give convergence of intensity, and (4.8) handles the uncontrolled starts. This proves the fixed-pattern approximation without a total-variation comparison of a discrete phase with a continuous one.

Take periodic patterns η_j from Lemma 5.2. At stage j the current base is a finite modification of f , hence belongs to Ω_σ . Fix the pattern and its largest excess R_j first. Then select N_j from (M_i) large enough that the error on the first j determining tests and on intensity is below 2^{-j} , the cost (4.7) is below $\varepsilon 2^{-j-1}$, and

$$Y_{N_j} > \max(T, 2N_{j-1}), \quad (R_j + 1)/K_{N_j} < 2^{-j}, \quad (\ell_{N_j} + R_j)/H_{N_j} < 2^{-j}.$$

All newly changed primes lie in $(Y_{N_j}, 2N_j]$. Future stages freeze the controlled occurrences, absences and right guards, all contained in $[N_j, 2N_j]$. The unprotected boundary starts remain negligible irrespective of their later lengths. Coordinatewise convergence gives g ; the budget and prefix claims follow. Since $d_1 \leq d_\sigma$, Theorem 2.1 gives normality. Diagonalization proves (5.3). $\square$

Corollary 5.3 (Finite count vectors and first moments). *In Theorem 5.1, counts in any fixed finite family of bounded intervals, with prescribed signs and finite singleton or tail excess sets, converge jointly in total variation. Their first moments converge. The level- r field is*

$$\Xi_{g,N}^r(A) = \Psi_{g,N}(A \times \{-1, +1\} \times [r, \infty]), \quad \Xi_{g,N}^{r+1} \leq \Xi_{g,N}^r.$$

At its own critical scale $2^r H_N$, it is $A \mapsto \Xi_{g,N}^r(2^r A)$; finitely many such fixed dilations may be observed jointly.

Proof. A stationary finite-intensity process almost surely has no atom at a fixed spatial endpoint. The mark sets are clopen. The continuous mapping theorem gives convergence of the integer-valued count vector; on a countable discrete space this implies total variation convergence by finite-set truncation. For a bounded interval A ,

$$\mathbb{E} \Psi_{g,N}(A \times \mathbf{K}) \leq (|A| + 2/H_N) \lambda_{g,N}. \quad (5.6)$$

Fatou for truncated counts gives the reverse limiting bound $|A| \lambda(P)$, so total counts have converging means and are uniformly integrable. Marked counts are dominated by them. Requiring the left guard and the first $\ell_N + r$ letters to be visible alters only bands of normalized width $O((\ell_N + r)/H_N)$ near endpoints. The same estimate controls the error. Restricting origins so all fixed observation windows lie inside the dyadic block removes a proportion $O(H_N/N)$; uniform integrability also preserves the first moments. $\square$

The process laws themselves are not asserted to converge in total variation: each finite-scale law is supported on finite configurations, whereas a positive-intensity stationary Poisson process on the whole line has infinitely many points.

Proposition 5.4 (Compatibility and universality in a class). *The targets in Theorem 5.1 are exactly stationary laws of nested marked point measures $(\Xi^r)_{r \geq 0}$ with finite base intensity. In $\mathcal{C}_\sigma(f)$ for $f \in \Omega_\sigma$, a dense G_δ of functions has every law in $\mathcal{S}(\mathbf{K})$ as a subsequential limit with converging intensity. The subsequence depends on the target.*

Proof. A marked point measure gives its nested threshold fields. Conversely, put the mass of $\Xi^q - \Xi^{q+1}$ at excess q , and put the decreasing residual at ∞ . Local finiteness follows from Ξ^0 . The transformations commute with translations. Stationarity is necessary for every empirical limit: translating the process by $\lfloor tH_N \rfloor / H_N$ replaces U_N by $U_N + \lfloor tH_N \rfloor$, changing its law by at most $O(H_N/N)$. Continuity of translations yields invariance in the limit.

For the residual assertion, enumerate the countable dense periodic laws from Lemma 5.2. For each law, accuracy and lower scale bound, require a later scale approximating its law and intensity. The union of such finite-scale events is open in the class. Indeed, at a given normal function all finitely many starts at that scale have finite run ends, so a finite prime cutoff fixes the entire statistic locally. This cutoff need not equal $2N$ for arbitrary functions. Density follows by first approximating the center by a finite modification of f , then applying the fixed-pattern construction at a sufficiently late scale and with a smaller remaining budget. Proposition 2.7 and Baire apply. Approximate an arbitrary target by periodic laws and diagonalize. The argument also works in any nonempty compatible prefix cylinder. $\square$

5.1 A realization with converging local moments

Weak convergence and intensity convergence do not alone imply convergence of higher moments. The next refinement is a constructive choice within Theorem 5.1, not an assertion about every realization supplied by its weaker conclusion.

For a point measure ξ , let $\xi^{(r)}$ denote its factorial measure of order r : ordered r -tuples of distinct *atom occurrences* are counted. Multiplicities are retained, so different occurrences may have identical locations and marks.

Proposition 5.5 (Moment-preserving realization). *Assume (A1) and (A2), and use the base set Ω_σ of Theorem 5.1. Let $P \in \mathcal{S}(\mathbf{K})$ satisfy*

$$\mathbb{E}_P[\xi([-A, A] \times \mathbf{K})^s] < \infty \quad \text{for every integer } A, s \geq 1. \quad (5.7)$$

For every $f \in \Omega_\sigma$, $\varepsilon > 0$ and prefix cutoff T , one can choose g and (N_j) in that theorem so that, in addition, for every real $A > 0$ and integer $s \geq 1$,

$$\mathbb{E}[\Psi_{g, N_j}([-A, A] \times \mathbf{K})^s] \longrightarrow \mathbb{E}_P[\xi([-A, A] \times \mathbf{K})^s]. \quad (5.8)$$

For every $r \geq 1$ and $\phi \in C_c((\mathbb{R} \times \mathbf{K})^r)$,

$$\mathbb{E}\langle \phi, \Psi_{g, N_j}^{(r)} \rangle \longrightarrow \mathbb{E}_P\langle \phi, \xi^{(r)} \rangle. \quad (5.9)$$

All joint ordinary and factorial moments of finite count vectors from Corollary 5.3 converge as well. More generally the same holds for bounded spatial intervals and Borel mark sets whose boundaries have zero target intensity. Expectations on the left are over the uniform origin only. These conclusions use no additional arithmetic input beyond Table 1.

Proof. A countable family of requirements. Fix a countable determining family of continuous compactly supported Laplace tests for point-process laws, and adjoin the local total-count powers $\xi \mapsto \xi([-A, A] \times \mathbf{K})^s$ for integer $A, s \geq 1$. Enumerate this countable family. We first approximate any finite initial list by one periodic pattern, then realize that pattern

arithmetically. Laplace tests identify the law, so no moment-determinacy assumption is being made.

Periodic approximation of the unbounded tests. Use independent restrictions of P to pieces of length $T_1 > 2A$ as in Lemma 5.2, with A large enough for the finite list. For a local total-count power of order s , its integral over origins at distance at least A from the ends of a piece has expectation $(T_1 - 2A)\mathbb{E}_P[\xi([-A, A] \times K)^s]$ and is integrable by (5.7). At a join the observation meets at most two pieces. If their contributions are x, y , then

$$(x + y)^s \leq 2^{s-1}(x^s + y^s).$$

Each contribution is bounded by a local count in an interval of length at most $2A$ of one copy of P . Stationarity therefore bounds the expected *normalized* seam integral by $O_{A,s}(T_1^{-1})\mathbb{E}_P[\xi([-A, A] \times K)^s]$, including the final periodic join. First take T_1 large. The law of large numbers for the interior integrals, bounded tests and piece counts, and Markov's inequality for the nonnegative seam errors, then give a finite concatenation satisfying every requested accuracy. A union bound over the finite list permits one deterministic choice. This also treats nonergodic targets.

The resulting pattern has finitely many atoms per period and bounded counts in each fixed window, uniformly over phase. Rationalizing its positions, and replacing its infinite excess marks by sufficiently large finite marks, preserves the required averages: the total counts are unchanged by mark replacement, while spatial count changes occur only near finitely many endpoint phases. Dominated convergence applies to every fixed power. Thus the periodic approximation can be made simultaneously for the law, intensity and the finite list of moments.

Arithmetic realization of one fixed pattern. Let η be such a pattern, R its largest excess, and A, s fixed. Once displacements are at most one, the controlled atoms in any observation of radius A are bounded by a constant $B_{\eta,A}$, independently of N . Pairing the atoms, using the smoothed phase in (5.4)–(5.5), and excluding endpoint-crossing phases gives an error tending to zero for their s th moments. More explicitly, the controlled contribution is

$$O_{\eta,A,s}(D/H_N + H_N^{-1} + K^{-1}).$$

There are at most $R + 1$ uncontrolled boundary starts. The chance that any of them is visible is $O_A((R + 1)/K)$ by (4.8). Since the total local count is at most $B_{\eta,A} + R + 1$, their possible effect on the s th moment is at most

$$O_A\left((B_{\eta,A} + R + 1)^s \frac{R + 1}{K}\right) = o(1). \quad (5.10)$$

Here η, R, A, s are all fixed before choosing N . The bound does not depend on the later lengths of the boundary runs, so it remains valid after subsequent surgery stages.

Diagonalization and extension of the test class. At stage j , choose a finite periodic pattern approximating the first j Laplace tests, the intensity, and every total-count moment with integers $A, s \leq j + 1$. Then choose an admissible N_j large enough for all these finite errors, including (5.10), to be below 2^{-j} , while also satisfying the summable prime budget and freezing conditions of Theorem 5.1. The same stable base set suffices: it supplies eventual reservoirs for each fixed R , and R_j is chosen before N_j . The limit function retains the prefix and normality, and (5.8) follows first for integer A .

For a bounded window contained in $[-A, A]$, its count powers of order s are dominated by the total count there to the power s . The converging moment of order $s + 1$ on a larger integer-radius window gives uniform integrability. Weak convergence of count vectors at target-continuity sets, followed by truncation, now gives their joint moments and (5.8) for every real

$A > 0$. Deterministic spatial endpoints carry no atom under a stationary finite-intensity law. The same continuity assertion holds for the stated mark sets.

Finally, $|\langle \phi, \xi^{(r)} \rangle|$ is bounded by $\|\phi\|_\infty$ times a local total count to the power r . For continuous compactly supported ϕ the factorial functional is continuous for vague convergence of integer point measures: inclusion–exclusion over partitions expresses it as a finite sum of integrals against tensor powers with coordinates identified. This remains valid at multiple atoms with the atom-occurrence convention above. Truncation and the local moment of order $r + 1$ prove (5.9). Equivalently, one can first use a countable uniformly dense family of continuous tests on each compact product and extend by the same count domination. Thus a single diagonal construction gives every claimed fixed-order test, rather than a different function for each test. $\square$

6 Exact finite projections without the marked Poisson law

This section is independent of (A1)–(D) until its final arithmetic corollary. Let Π be the Poisson process on $\mathbb{R} \times \mathbb{N}_0$ with intensity $dx \otimes \pi$, where $\pi_q = 2^{-q-1}$. For $B \subset \mathbb{N}_0$, write $\xi_B(A) = \xi(A \times B)$, $\xi^{\geq r} = \xi_{[r, \infty)}$, and $\xi^{=q} = \xi_{\{q\}}$.

The case $n = 3$, $h = 1$ illustrates the question before the general construction. We will build a process for which every pair of threshold fields has exactly the reference joint process law and all marked factorial measures through order eleven agree, with spatial dependence of range five. Nevertheless, for $Z_r = \Phi^{\geq r}([0, 1/2))$,

$$\mathbb{E} 2^{-(Z_0 + Z_1 + Z_2)} > e^{-21/64},$$

where the right side is the reference expectation. Thus a three-threshold observation distinguishes laws that no pair of threshold fields distinguishes. The integers n and h separately control the observation complexity and the number of preserved moment orders.

Theorem 6.1 (Finite projections and finite dependence). *For integers $n \geq 2$, $h \geq 1$ and $d = hn(n + 1)$, there exists a simple stationary marked process $\Phi = \Phi_{n,h}$ such that:*

- (i) *the unmarked process is exactly PPP(1);*
- (ii) *any family of fewer than n threshold fields has exactly the joint process law of the reference; the same holds for fewer than n exact layers;*
- (iii) *for arbitrary sets $B_1, \dots, B_k \subset \mathbb{N}_0$, the joint law of $(\Phi_{B_1}, \dots, \Phi_{B_k})$ is the reference law whenever $2^k \leq n$;*
- (iv) *all marked factorial moment measures of order $j < d$ equal those of Π ;*
- (v) *restrictions to spatial sets at distance greater than five are independent;*
- (vi) *the joint law at thresholds $0, \dots, n - 1$ differs from the reference on a window of length $1/2$, as detected by a bounded test.*

Independent fair sign marks can be appended to both processes without changing the assertions.

Dependency. This construction is autonomous in Table 1; the arithmetic dependence enters only in Corollary 6.2.

Proof. A positive finite perturbation. Only the first n marks will be perturbed. The polynomial is designed to vanish under the observations to be preserved. A factor $z_i - 1$ removes the perturbation when mark i is unobserved; a factor $z_j - z_i$ removes it when two marks have

the same membership signature among the observed fields. Their product, raised to the even power $2h$, has degree $d = hn(n+1)$ about $(1, \dots, 1)$, so all lower-order factorial moments remain unchanged. The even power also keeps the separating correction nonnegative when spatial placements are averaged. We then choose an explicit sufficiently small coefficient η to retain positive probabilities. Define

$$\begin{aligned} V_n(z) &= \prod_{i=0}^{n-1} (z_i - 1) \prod_{0 \leq i < j < n} (z_j - z_i), \\ Q(z) &= V_n(z)^{2h}, \quad d = hn(n+1), \quad M_0 = 2hn, \\ \Lambda_n &= \sum_{i < n} \pi_i = 1 - 2^{-n}, \quad \eta = \frac{e^{-\Lambda_n} \prod_{i < n} \pi_i^{M_0}}{2^{d+1} (M_0!)^n}. \end{aligned} \quad (6.1)$$

Then

$$F(z) = \exp \left(\sum_{i < n} \pi_i (z_i - 1) \right) + \eta Q(z) \quad (6.2)$$

is a probability generating function. Indeed every coefficient c_α of Q has $|c_\alpha| \leq 2^d$, and each exponent is at most M_0 . On this finite support the reference probability is at least $p_* = e^{-\Lambda_n} \prod_{i < n} \pi_i^{M_0} / (M_0!)^n$. Thus $|\eta c_\alpha| \leq p_*/2$; all coefficients remain nonnegative. The total mass is one because $Q(1, \dots, 1) = 0$.

There are two identities to retain:

$$Q(1 + u(z - 1)) = u^d Q(z), \quad Q(z) = 0 \text{ if some } z_i = 1 \text{ or two } z_i \text{ coincide.} \quad (6.3)$$

The even power also ensures $Q(z) \geq 0$ for real z , preventing cancellation when random interval positions are averaged.

An interval law and its invisible observations. On an interval I of length one, draw counts $(X_i)_{i < n}$ with generating function (6.2); place the X_i points of each mark independently and uniformly. Add independent Poisson points of marks $q \geq n$ at their reference intensities. To observe k mark sets with test functions $\varphi_j \geq 0$, substitute

$$z_i = \int_I \exp \left(- \sum_{j=1}^k \mathbf{1}_{\{i \in B_j\}} \varphi_j(x) \right) dx.$$

A zero membership signature gives $z_i = 1$; two equal signatures give equal arguments. Either makes the perturbation vanish. For k thresholds, the nonzero signatures form a chain of size at most k ; for fewer than n exact layers, a mark is unseen; and for k arbitrary sets there are at most $2^k - 1$ nonzero signatures. This proves the respective projection identities locally, for arbitrary spatial test functions. In particular the total spatial process is Poisson because all arguments are equal. The unaffected tail retains its reference Laplace factor in every calculation.

Local implantation instead of a global grid. Take an auxiliary Poisson process of intensity one on $\mathbb{R}$. Retain a center c if there is no other auxiliary point within distance two. Its interval $I_c = [c - 1/2, c + 1/2]$ is disjoint from all other retained intervals. Attach independent copies of the preceding interval law to the auxiliary points, and take an independent reference marked Poisson background. Replace the background in each retained interval by its attached copy. The rule commutes with translations, so the output Φ is stationary. Conditional on the retained intervals, every allowed projection has the reference Laplace functional on each part of this partition. Conditional independence multiplies the factors into the global reference functional, which does not depend on the retained intervals. This proves (i)–(iii).

To know Φ on a spatial set A requires only the background on A and the marked auxiliary process within distance $5/2$ of A : possible centers are within $1/2$ and their retention is determined within a further distance two. Disjoint such neighborhoods are independent. This proves (v), including independence for finite collections with disjoint neighborhoods.

Factorial measures. By (6.3), every derivative of Q at $(1, \dots, 1)$ of total degree less than d vanishes. Hence for $j < d$, the joint falling-factorial moments of the counts in one implanted interval are the reference ones. Uniform independent point locations turn them into the corresponding spatial densities. Conditional on the intervals, allocate prescribed distinct positions to their pieces and multiply these densities. The product is $\prod_{i=1}^j \pi_{q_i}$ independently of the allocation; averaging proves (iv). All local moments are finite, since the total local count is Poisson.

A separating test. Set $A = [0, 1/2)$ and

$$W(\xi) = 2^{-\sum_{r=0}^{n-1} \xi^{\geq r}(A)} \in [0, 1]. \quad (6.4)$$

It assigns the weight $z_i = 2^{-i-1}$ to mark $i < n$ and $z_* = 2^{-n}$ to every tail mark. All z_i are distinct and different from one, so $Q(z) > 0$. Put $w_a = \sum_{i < n} \pi_i(z_i - 1)$ and $w_t = 2^{-n}(2^{-n} - 1)$. Direct summation gives

$$\mu_0 = \mathbb{E}W(\Pi) = \exp\{-\tfrac{1}{3}(1 - 4^{-n})\}. \quad (6.5)$$

If $u_c = |A \cap I_c|$, the conditional expectation under the implanted process is

$$e^{(w_a + w_t)/2} \prod_{c: u_c > 0} \left(1 + \eta e^{-u_c w_a} u_c^d Q(z)\right).$$

Every factor is at least one. On the event of exactly one auxiliary point in $[0, 1/2]$ and none elsewhere in $[-2, 5/2]$, of probability $\frac{1}{2}e^{-9/2}$, its retained interval contains A . Therefore

$$\mathbb{E}W(\Phi) - \mu_0 \geq \delta_{n,h} := \tfrac{1}{2}e^{-9/2}\eta 2^{-d}Q(2^{-1}, \dots, 2^{-n})e^{\frac{1}{2}2^{-n}(2^{-n}-1)} > 0. \quad (6.6)$$

This proves (vi). Independent sign marking is a common random marking kernel on each projection and multiplies prescribed factorial densities by the appropriate powers of $1/2$. $\square$

For the opening example $n = 3, h = 1$, the exact witness value is

$$Q(1/2, 1/4, 1/8) = 3969/2^{28} > 0.$$

Equations (6.5)–(6.6) therefore give the announced strict inequality for the three-threshold test. Every pair of levels has the correct binomial thinning kernel, but the full level process is not Markov: otherwise its initial law and consecutive kernels would determine the reference triple law. For arbitrary m fixed threshold fields and any fixed correlation order r , take $n = m + 1$ and choose h with $hn(n + 1) > r$. To preserve any m arbitrary regroupings of marks instead, take $n = 2^m$. The quantifier is “for every finite m , there exists a process”, not “one different process preserves all finite-dimensional laws”.

The bounded discrepancy can be amplified. The tests in $6j + [0, 1/2)$ are independent and identically distributed, with means differing by at least $\delta_{n,h}$ and variance at most $1/4$. A sample-mean test across M windows gives

$$d_{\text{TV}}(\mathcal{L}_\Phi(V_M), \mathcal{L}_\Pi(V_M)) \geq 1 - \frac{2}{M\delta_{n,h}^2}, \quad (6.7)$$

where V_M contains the necessary threshold counts. Their laws on the whole line are mutually singular by the strong law. The conservative constant in (6.6) need not give a practical sample size.

Corollary 6.2 (Arithmetic realization of finite indistinguishability). *Under (A1) and (A2), every $\Phi_{n,h}$ with independent fair sign marks can be realized by Theorem 5.1, with its base intensity converging to one. All permitted finite projection-count laws converge in total variation to their reference laws, while the bounded n -threshold test retains the positive limiting discrepancy (6.6). One can use the strengthened choice of Proposition 5.5, since the total local counts of $\Phi_{n,h}$ are Poisson and have moments of every order. Then its marked factorial moment measures of every fixed order are the limits of the empirical ones.*

For arbitrary mark sets, the only possible boundary in the compactification is the infinite mark; this target gives it zero intensity. Alternatively truncate the marks and use their geometric tail and uniform integrability of total counts. For (6.7), first fix M , then take the arithmetic scale to infinity. The constructed multiplicative sequence is not itself asserted to be five-dependent. Nor do these examples contradict structural few-moment criteria: for example, Zheng [24, Theorems 1.1 and 1.4] requires a dominant fixed chaos and, in the Rademacher setting, additional influence control. Such hypotheses are not supplied by our finite correlation identities.

7 Optimal local repair and robustness

For integers $b, k \geq 0$, put

$$V(b, k) = \sum_{j=0}^{\min(b, k)} \binom{b}{j}, \quad H_2(u) = -u \log_2 u - (1-u) \log_2 (1-u).$$

For $c > 1$, let $\alpha_c \in (0, 1/2)$ solve $H_2(\alpha_c) = 1 - 1/c$, set $a(c) = c\alpha_c$, and let $b_c(N) = \lceil cL_N \rceil$. The functions α_c and $a(c)$ are continuous and strictly increasing; $a(c) \rightarrow \infty$ as $c \rightarrow \infty$.

Why the entropy threshold appears. The following is an independent-sign heuristic, not an independence assumption on multiplicative windows. For fixed $0 < \alpha < 1/2$, the Hamming ball of radius $\lfloor \alpha b \rfloor$ has $V(b, \lfloor \alpha b \rfloor) = 2^{bH_2(\alpha) + o(b)}$ words. Searching N windows for a block within that ball around a fixed constant word gives the reference expected count

$$N2^{-b}V(b, \lfloor \alpha b \rfloor) = N^{1-c(1-H_2(\alpha))+o(1)} \quad (b \sim c \log_2 N).$$

The exponential threshold is therefore $c(1 - H_2(\alpha)) = 1$. The proof must still account for adaptive choices and multiplicative dependence. Its lower bound counts the *local effects* accessible to prime flips; its upper bound uses a capped second moment on arithmetically good blocks. Their ranges meet at the entropy threshold for $1 < c < 3/2$. The upper restriction comes from a term $X^{-1+2c/3}$ in (7.8), not from a proved phase transition.

Theorem 7.1 (Optimal cardinality). *Assume (A1) and (CAP). There is a full-measure set, stable under finite prime modifications, on which simultaneously for every $1 < c < 3/2$ and all integer scales $N \rightarrow \infty$,*

$$c_f(N, b_c(N)) \sim c_f^{>Y_N}(N, b_c(N)) \sim \alpha_c b_c(N). \quad (7.1)$$

Under (A1) alone, simultaneously for every $c > 1$,

$$\liminf_{N \rightarrow \infty} c_f(N, b_c(N))/b_c(N) \geq \alpha_c \quad \text{almost surely.} \quad (7.2)$$

For each fixed $1 < c < 3/2$, on a possibly c -dependent full-measure set,

$$c_f^{>Y_N}(N, b_c(N)) = \alpha_c b_c(N) + O_c(\log \log N). \quad (7.3)$$

The remainder is not claimed for unrestricted primes.

Dependency. See Table 1: the lower bound uses only (A1), and the upper bound and its remainder add (CAP), not the reversible reservoir.

7.1 Compression of the prime choices

The object to count is the sign transformation seen by a block, not the global name of the changed-prime set. On b positions, a prime greater than b acts on at most one position, while the small primes add at most $\pi(b)$ further column types. This is why a union bound over local transformations can include fully adaptive surgery without enumerating all subsets of primes up to $2N$.

Lemma 7.2 (Column compression). *On a block of b integers, the nonzero columns $(v_p(n+j) \bmod 2)_{0 \leq j < b}$ take at most $b + \pi(b)$ distinct values. At most $V(b + \pi(b), k)$ sign transformations can be realized with k prime changes. Consequently, under (A1), on any bounded-ratio region at scale X ,*

$$\begin{aligned} & \mathbb{P}\{\text{some repair with at most } k \text{ primes creates a length-} b \text{ run}\} \\ & \leq 2^{1-b} V(b + \pi(b), k) (O(X) + X^{1/2+o(1)}). \end{aligned} \quad (7.4)$$

Proof. Primes at most b give at most $\pi(b)$ columns. Every larger prime has a zero or canonical column, giving at most b further possibilities. Repeated columns cancel in pairs over $\mathbb{F}_2$, so each transformation has a representation with at most k distinct column types. Count subsets of these types. For a given start, at most twice that many base words can be repaired to a constant word. The list depends on integer factorizations, not on the prime signs of F . Apply (3.6) to each word and sum using (3.1). In particular the proof covers adaptive choices without enumerating global subsets of primes up to $2X$. $\square$

If $k/b \rightarrow \alpha \in (0, 1/2)$, then $\pi(b) = O(b/\log b)$ and Stirling give

$$\log_2 V(b + \pi(b), k) = bH_2(\alpha) + o(b). \quad (7.5)$$

For $k = o(b)$ the logarithm is $o(b)$, and for fixed k the volume is $O_k(b^k)$.

Take $b = \lceil c \log_2 X \rceil$, $k = \lfloor \alpha b \rfloor$ with $\alpha < \alpha_c$. Equation (7.4) is at most $X^{1-c(1-H_2(\alpha))+o(1)}$, a negative power. Summing over $X = 2^j$ and applying Borel–Cantelli excludes such repairs eventually. For $X \leq N < 2X$, every target block is in $[X, 4X]$, and $b_c(N) - \lceil c \log_2 X \rceil = O_c(1)$. Use a fixed α' strictly between α and α_c to absorb that difference. This gives (7.2) over all integer N . A countable rational intersection and monotonicity in target length extend it simultaneously to all real $c > 1$.

7.2 Reaching the entropy bound with large primes

In a cell of length proportional to X , select disjoint blocks of length $b \asymp \log X$, and discard those containing a Y -defective integer, with $Y = e^{S_X}$ or a fixed bounded-scale enlargement. By (2.4) there remain

$$M \asymp X/b \quad (7.6)$$

good blocks: even the bound $bA(4X, Y) = o(X/b)$ suffices. This is a deterministic arithmetic selection.

Let I_n indicate that a good block has at most k negative letters. Its mean is exactly $p = 2^{-b}V(b, k)$. For distinct blocks, rank and the marginal bound yield

$$\mathbb{E}I_n I_m \leq \min\{p, p^2 2^{\rho_b(n,m)}\}, \quad \text{Cov}(I_n, I_m) \leq p^2 \min\{p^{-1}, 2^{\rho_b(n,m)} - 1\}.$$

Hence for $Z = \sum I_n$,

$$\mathbb{E}Z = Mp, \quad \text{Var } Z \leq Mp + p^2 R_b^{[1/p]}(X), \quad \mathbb{P}(Z = 0) \leq \frac{1}{Mp} + \frac{R_b^{[1/p]}(X)}{M^2}. \quad (7.7)$$

The marginal cap is retained *before* the arithmetic summation.

For $b = \lceil c \log_2 X \rceil$, $k = \lfloor \alpha b \rfloor$ and $\alpha > \alpha_c$, one has $p = X^{-c(1-H_2(\alpha))+o(1)}$ and $Mp \rightarrow \infty$ with a power margin. Input (CAP) gives

$$\frac{R_b^{[1/p]}(X)}{M^2} \ll_{\eta,c} (\log X)^2 X^\eta \left(X^{-1/2+c/6} + X^{-1+2c/3} + X^{-4/3} p^{-1} \right). \quad (7.8)$$

All exponents are negative for $1 < c < 3/2$ and sufficiently small fixed $\eta > 0$. The failure probability is summable along $X = 2^j$. A successful good block becomes entirely positive by flipping one private prime for each negative letter. At most k primes are changed, all greater than Y , and their d_σ weight is at most $bY^{-\sigma} \rightarrow 0$.

To cover all integer scales, partition $[X, 4X)$ into twelve cells of length $X/4$, with harmless integer rounding. Every interval $[N, 2N]$, $X \leq N < 2X$, contains a full cell. In every cell use target length $\lceil c \log_2(2X) \rceil$ and cutoff $Y_{4X} \geq Y_N$. The fixed number of cells, the $O_c(1)$ change in length, and a fixed bounded-ratio enlargement do not affect summability. Almost surely every cell succeeds at all sufficiently large dyadic X . Thus

$$c_f^{>Y_N}(N, b_c(N)) \leq \alpha b_c(N) + O_c(1)$$

for all large N . Let $\alpha \downarrow \alpha_c$ and combine with (7.2). Rational intersections, monotonicity and continuity of $a(c)$ prove (7.1) simultaneously. Intersecting all finite-flip preimages makes the set stable under finite modifications.

7.3 A logarithmic remainder for large primes

Fix $1 < c < 3/2$ and let $s_c = H'_2(\alpha_c) = \log_2((1 - \alpha_c)/\alpha_c) > 0$. For $b = \lceil c \log_2 X \rceil$ and $k_\pm = \lfloor \alpha_c b \pm A \log_2 b \rfloor$, Stirling and Taylor expansion give

$$2^{-b} V(b, k_\pm) = \Theta_{c,A}(X^{-1} b^{\pm A s_c - 1/2}). \quad (7.9)$$

The quadratic Taylor error is $O((\log b)^2/b)$; rounding only changes constants. For k_+ , $(Mp)^{-1} = O(b^{3/2 - A s_c})$, summable on $X = 2^j$ once $A s_c > 5/2$. The terms in (7.8) remain power-summable. For k_- , any repair by primes greater than $Y_N > b$ requires a symbolic block with at most k_- errors. Replace $V(b + \pi(b), k)$ by $V(b, k)$ in (7.4); the resulting bound is $O(b^{-A s_c - 1/2})$ plus a negative power of X , summable when $A s_c > 1/2$. The same mesh and enclosing interval argument absorbs the $O_c(1)$ length shifts. This proves (7.3).

The extra $\pi(b)$ columns can produce a logarithmic volume correction of order $b/\log b$ when $k \asymp b$. Therefore this proof does not give the same $O(\log b)$ cost remainder for all primes. Likewise the second term in (7.8) prevents an assertion at $c = 3/2$. This is a limit of the bound, not a demonstrated phase transition.

Corollary 7.3 (Robustness and deterministic motifs). *Under (A1) and (CAP), on a full-measure stable set, for every integer sequence $k_N = o(\log N)$,*

$$M_f^{[k_N]}(N)/L_N \rightarrow 1. \quad (7.10)$$

For fixed k and $\delta > 0$, almost surely for all large integer N ,

$$M_f^{[k]}(N) \leq L_N + (k + 1 + \delta) \log_2 L_N + O_{k,\delta}(1).$$

For a predetermined word family (w_b) , the minimum cost to obtain $w_{b_c(N)}$ has the same first-order limit $\alpha_c b_c(N)$ for $1 < c < 3/2$. The full-measure set may depend on this family.

Proof. The lower cost (7.2) excludes every fixed factor $c > 1$ at sublogarithmic budget. The empty surgery gives the opposite bound, since $M_f(N)/L_N \rightarrow 1$: the upper bound follows from (7.4) with $k = 0$; the lower bound uses (7.7) on good blocks of length $(1 - \delta)L_N$, with the preceding all-scale mesh. For fixed k , apply (7.4) at $X = 2^j$ and $b = \lceil j + (k + 1 + \delta) \log_2 j \rceil$; the bound is $O(j^{-1-\delta})$. For a deterministic motif, use one Hamming ball around w_b instead of a constant word. The local uniform law and the rank bounds are unchanged. In passing from X to N , treat the $O_c(1)$ possible lengths separately, since a shorter target need not be a prefix of a longer one. $\square$

The robustness bound concerns the *largest accessible maximum*, not a lower bound uniform over all outputs. A motif chosen after observing F could already occur and have zero cost. Finally, because $a(c) \rightarrow \infty$, a budget $O(\log N)$ gives only maxima $O(\log N)$; a superlogarithmic maximum requires a superlogarithmic number of changed primes up to $2N$.

8 Fine laws at a small adaptive budget

Let $D_f^{[k]}(N)$ be the largest length of a block in $[N, 2N]$ having at most k errors relative to either constant word. This is a symbolic repair maximum, not itself a prime-surgery maximum.

The distinction matters even in a small example. For $f(n) = (-1)^{v_2(n)}$ on $[8, 16]$, flipping the single prime 2 makes all nine values positive, whereas the largest block with at most one symbolic error has length six. Thus prime repair and letter-by-letter repair are not algebraically identical for arbitrary bases. The theorem below compares their maxima on the same random realization with probability tending to one, a stronger conclusion than agreement of marginal asymptotic laws. Put

$$\Lambda_N(b, k) = N2^{-b}V(b, k).$$

Theorem 8.1 (Lattice law and comparison on the same base). *Assume (A1) and (D). Let $k_N = o(\log N)$ and let $b_N \sim L_N$ be deterministic integers such that $\Lambda_N(b_N, k_N)$ stays in a compact subset of $(0, \infty)$. Then*

$$\mathbb{P}\{M_F^{[k_N], > Y_N}(N) < b_N\} = \exp\{-\Lambda_N(b_N, k_N)\} + o(1). \quad (8.1)$$

The same formula holds for $D_F^{[k_N]}(N)$, and

$$\mathbb{P}\{M_F^{[k_N], > Y_N}(N) = D_F^{[k_N]}(N)\} \rightarrow 1. \quad (8.2)$$

If in addition $k_N = o(\log \log N)$, formula (8.1) holds for the unrestricted maximum and

$$\mathbb{P}\{M_F^{[k_N]}(N) = M_F^{[k_N], > Y_N}(N) = D_F^{[k_N]}(N)\} \rightarrow 1. \quad (8.3)$$

In particular, for fixed $k \geq 0$,

$$M_F^{[k]}(N) = L_N + k \log_2 L_N - \log_2(k!) + O_{\mathbb{P}}(1), \quad (8.4)$$

with the same assertion for large-prime repair. Thus

$$M_F^{[k]}(N) - M_F(N) = k \log_2 L_N - \log_2(k!) + O_{\mathbb{P}}(1).$$

These are probability statements over the random base and are not deterministic identities or almost-sure uniform remainders.

Dependency. The dependencies are (A1) and (D) in Table 1; the independent-run formulas are benchmarks, while the same-base prime-repair comparison is the arithmetic conclusion.

8.1 Anchoring almost-constant words

For $0 \leq k < b/2$ define

$$W_{b,k}^+ = \{(-1, u_1, \dots, u_b) : \#\{i : u_i = -1\} \leq k\}, \quad W_{b,k}^- = -W_{b,k}^+, \quad W_{b,k} = W_{b,k}^+ \cup W_{b,k}^-.$$

The first letter is a guard. The dictionary length is $B = b + 1$, and its size is $m = 2V(b, k)$. Negating a word here is not the generally invalid map $F \mapsto -F$ on $\mathcal{M}_\pm$.

Lemma 8.2 (Overlap control). *For $0 \leq k \leq b/6$,*

$$\Omega(W_{b,k}) \leq \frac{20k}{b} + 2^{1-b+2k}V(b, k). \quad (8.5)$$

In particular it tends to zero when $k = o(b)$ and $b \rightarrow \infty$.

Proof. For same-sign pairs draw a body U uniformly from the Hamming ball about the positive word. At shift $1 \leq d \leq b$, compatibility forces $U_d = -1$, the guard of the second word. Write K_0 for the total number of errors, $R_0 = k - K_0$, and A_0 for the errors before position d . Conditional on this guard, the overlapping part contains $K_0 - 1 - A_0$ errors, leaving $V(d, R_0 + 1 + A_0)$ possible extensions. The normalized contribution is

$$\mathbb{E}[\mathbf{1}_{\{U_d = -1\}} 2^{-d} V(d, R_0 + 1 + A_0)].$$

For every integer $r \geq 0$,

$$2^{-d} V(d, r) \leq 2^r (3/4)^d;$$

use $1 \leq 2^{r-j}$ for $j \leq r$ and sum the weighted binomial expansion. Also $\mathbb{P}(U_d = -1) = \mathbb{E}K_0/b \leq k/b$. Given $U_d = -1, K_0 = s$, the variable A_0 is hypergeometric, and its factorial moments imply

$$\mathbb{E}(2^{A_0} \mid U_d = -1, K_0 = s) \leq \left(1 + \frac{s-1}{b-1}\right)^{d-1} \leq (1 + k/b)^{d-1}.$$

The conditional weights of $R_0 = r$ are proportional to $\binom{b-1}{k-r-1}$, whose consecutive ratios are at most $k/(b-k) \leq 1/5$. Hence $\mathbb{E}(2^{R_0} \mid U_d = -1) \leq 5/3$. The contribution at shift d is at most

$$\frac{10k}{3b} (3/4)^d (1 + k/b)^{d-1}.$$

Its geometric sum is at most $20k/b$. For $k = 0$ it is zero directly.

For opposite signs, an overlap leaves at least $b - d$ body positions required to be almost both positive and negative. Thus $d \geq b - 2k$. At a given shift the two orientations together have at most $2V(b, k)^2$ pairs. Divide by $2V(b, k)$ and sum 2^{-d} to get the second term in (8.5). Finally $\log_2 V(b, k) = o(b)$ for $k = o(b)$. $\square$

Lemma 8.3 (Left anchoring). *In a finite sign sequence, a length- b block with at most k errors relative to a sign s exists if and only if such a block exists either at the left boundary or preceded by $-s$.*

Proof. Move a qualifying block left while its predecessor is s . Each move adds a correct letter and removes an arbitrary letter, so errors do not increase. The procedure ends at the boundary or at the required guard. $\square$

Proof of the symbolic and large-prime laws. Abbreviate $b = b_N$, $k = k_N$ and $\Lambda = \Lambda_N(b, k)$. Since $k = o(b)$, $m = 2V(b, k) = N^{o(1)}$ and

$$B - \log_2(Nm) = -\log_2 \Lambda = O(1).$$

Lemma 8.2 verifies (D). The number of anchored dictionary occurrences, with guard and body in $[N, 2N]$, has no-occurrence probability $e^{-\Lambda} + o(1)$, since its target mean is $(N - b + 1)2V(b, k)2^{-(b+1)} = \Lambda + o(1)$. The unguarded left-boundary exception in Lemma 8.3 has probability at most

$$2V(b, k)2^{-b+m_b(N)} \leq (2\Lambda/N)(1 + N^{1/2+o(1)}) = o(1)$$

by (A1). This proves the formula for $D_F^{[k]}$.

Restrict starts to anchored supports having an odd pivot greater than Y_N at every position. Lemma 2.2 shows that this deterministic mask has proportion $1 - o(1)$. The restriction part of (D) gives the same no-occurrence probability. Every retained body is made constant by at most k private pivot flips, all greater than Y_N . Conversely a prime greater than $Y_N > b$ changes at most one position of a length- b block, so any such repair requires at most k symbolic errors in some block. These comparisons prove (8.1). $\square$

8.2 Why the small primes disappear from the fine law

Small primes can create collective transformations outside a Hamming ball. Their relative number is bounded by the enlargement from $V(b, k)$ to $V(b + \pi(b), k)$. The logarithm of the ratio is $O(k/\log b)$ when $k = o(b)$, so this excess disappears for $k = o(\log b)$. At $b \sim \log_2 N$, this gives precisely the narrower range $k_N = o(\log \log N)$ for the all-prime comparison; it does not restrict the large-prime argument.

Lemma 8.4 (Collective-column cost). *For a fixed length- b block, the number of transformations obtainable with at most $k \leq b$ prime changes that have Hamming weight greater than k is at most*

$$V(b + \pi(b), k) - V(b, k).$$

For any integer $d \geq 0$,

$$\frac{V(b + d, k)}{V(b, k)} \leq \left(1 + \frac{d}{b - k + 1}\right)^k \leq \exp\left(\frac{kd}{b - k + 1}\right). \quad (8.6)$$

Proof. Add all b canonical columns artificially, and one labelled column for each prime at most b . A transformation of weight greater than k cannot be a sum of at most k canonical columns alone. Its representation must use at least one of the $\pi(b)$ extra columns. Counting those subsets gives the difference of volumes; collisions only reduce the number of images. For $j \leq k$, the ratio $\binom{b+d}{j}/\binom{b}{j}$ is a product of j factors $1 + d/(b - i)$ and is at most the first bound in (8.6). Sum over j . $\square$

Applying (3.6) to these extra words, whose list is deterministic given the factorizations, gives

$$\begin{aligned} \mathbb{P}\{M_F^{[k]}(N) \geq b, D_F^{[k]}(N) < b\} \\ \leq 2^{1-b}(V(b + \pi(b), k) - V(b, k))(O(N) + N^{1/2+o(1)}). \end{aligned} \quad (8.7)$$

Since $\pi(b) = O(b/\log b)$, this is $O(\Lambda k/\log b) + o(1) = o(1)$ when $k = o(\log b)$. Together with $M_F^{[k]} \geq M_F^{[k], > Y_N}$ and the preceding symbolic comparison, this proves the all-prime version of (8.1). The condition is $k = o(\log \log N)$, not the larger range available for large primes.

8.3 Equality with high probability and the lattice centering

Let b_N^* be the first integer b for which $\Lambda_N(b, k_N) \leq 1$. For $k_N = o(\log N)$, $b_N^* \sim L_N$ and

$$\frac{\Lambda_N(b+1, k)}{\Lambda_N(b, k)} = \frac{1}{2} \left(1 + \frac{V(b, k-1)}{V(b, k)} \right) = \frac{1}{2} (1 + O(k/b)). \quad (8.8)$$

Here $V(b, -1) = 0$, and $V(b, k-1)/V(b, k) \leq k/(b-k+1)$. Thus $\Lambda_N(b_N^*, k_N)$ lies between $1/2 + o(1)$ and one; fixed shifts multiply it asymptotically by 2^{-t} . The distribution formulas at $b_N^* \pm K$, followed by $K \rightarrow \infty$, give tightness around b_N^* .

For integer variables $A_N \geq B_N$ tight around the same center, their probability of inequality on a finite central range is bounded by the sum of $\mathbb{P}(B_N < b) - \mathbb{P}(A_N < b)$ over the thresholds of that range. Apply this to the ordered comparisons above. For the symbolic versus large-prime comparison, threshold domination is needed only in that central range, where $b = O(\log N) < Y_N$; it follows by restricting any repaired long run to b letters. Vanishing distribution gaps and vanishing tails prove (8.2)–(8.3).

For fixed k , $V(b, k) = b^k/k! (1 + O_k(1/b))$. Substituting $b = L_N + k \log_2 L_N - \log_2(k!) + O(1)$ into Λ_N proves (8.4). The integer threshold formula (8.1) retains the oscillating lattice phase. Along subsequences with a converging phase it gives the corresponding discrete limit; without selecting a phase it does not assert a continuous Gumbel limit. Subtracting the tight centered base maximum gives the stated gain without assuming independence of the two maxima. This completes the proof of Theorem 8.1.

9 One fixed normal output: optimal cost or one prime per stage

The preceding finite-scale estimates give two different infinite constructions. One changes the leading maximum by a factor $c > 1$ at optimal logarithmic cardinality. The other keeps the leading term but obtains a second-order gain with one new prime per chosen scale.

Theorem 9.1 (A fixed output at optimal first-order cost). *Assume (A1) and (CAP). On a full-measure stable set of bases, for every $1 < c < 3/2$, $\sigma, \varepsilon > 0$ and $T \geq 1$, there exist normal g , a set $S = \{p : f(p) \neq g(p)\}$ and dyadic scales $N_j \rightarrow \infty$ such that*

$$\begin{aligned} g(n) = f(n) \quad (n \leq T), \quad \sum_{p \in S} p^{-\sigma} < \varepsilon, \quad \sum_{p \in S} p^{-\tau} < \infty \quad (\tau > 0), \\ \frac{\#(S \cap [2, 2N_j])}{L_{N_j}} \rightarrow a(c), \quad \frac{M_g(N_j)}{L_{N_j}} \rightarrow c. \end{aligned} \quad (9.1)$$

The second limit controls the actual maximum, not merely a programmed witness.

Dependency. This is a consequence of Theorem 7.1, with the same (A1), (CAP) dependencies in Table 1.

Proof. Intersect the stable sets used for the cost theorem, its lower bound for all $d > 1$, and Fish's normality event. Let f_{j-1} be the current finite modification. At a sufficiently large dyadic N_j , its good blocks supply a positive target of length $b_j = b_c(N_j)$ with at most $(\alpha_c + \varepsilon_j)b_j$ errors, where $\varepsilon_j \downarrow 0$. Require that the number s_{j-1} of earlier changes satisfies $s_{j-1}/b_j < 1/j$, that $Y_{N_j} > \max(T, 2N_{j-1})$, and, with $\tau_j = \min(\sigma, 1/j)$,

$$b_j Y_{N_j}^{-\tau_j} < \varepsilon 2^{-j-1}.$$

Repair only the erroneous positions using private large pivots. Later stages freeze all values up to $2N_j$. The limit g has the prefix and all weighted summability properties, because for each fixed $\tau > 0$, eventually $\tau_j \leq \tau$. It is normal by Theorem 2.1.

At N_j the total number of changed primes is at most $(\alpha_c + o(1))b_j$, and at least the minimum cost from the *initial* base f of producing the witness, namely $(\alpha_c + o(1))b_j$. This proves the cardinality limit. If $d > c$, the lower cost for the initial base requires $(a(d) + o(1))L_{N_j}$ changes to reach dL_{N_j} , while the actual number is $(a(c) + o(1))L_{N_j}$. Since $a(d) > a(c)$, such a larger run is eventually impossible. Let $d \downarrow c$ through a countable set. The witness gives the reverse bound. $\square$

Theorem 9.2 (Arbitrarily slow support growth with one new prime). *Assume (A1) and (D). There exists a full-measure set Ω_* , stable under finite prime flips, such that for every $f \in \Omega_*$, $\sigma, \varepsilon > 0$, $T \geq 1$ and nondecreasing divergent $h : [2, \infty) \rightarrow [0, \infty)$, there are normal g , increasing primes p_j , and dyadic $N_j \rightarrow \infty$ satisfying*

$$\begin{aligned} S &:= \{p : f(p) \neq g(p)\} = \{p_j : j \geq 1\}, \quad g(n) = f(n) \quad (n \leq T), \\ \#(S \cap [2, x]) &\leq h(x) \quad (x \geq 2), \quad \sum_{p \in S} p^{-\sigma} < \varepsilon, \quad \sum_{p \in S} p^{-\tau} < \infty \quad (\tau > 0), \\ \max(Y_{N_j}, 2N_{j-1}) &< p_j \leq 2N_j, \quad \#(S \cap [2, 2N_j]) = j. \end{aligned} \quad (9.2)$$

With $t_j = \log_2 L_{N_j}$, the actual maxima obey

$$M_f(N_j) = L_{N_j} + O(\sqrt{t_j}), \quad M_g(N_j) = L_{N_j} + t_j + O(\sqrt{t_j}), \quad (9.3)$$

so $(M_g(N_j) - M_f(N_j)) / \log_2 \log_2 N_j \rightarrow 1$.

Dependency. This is a consequence of Theorem 8.1, with the same (A1), (D) dependencies in Table 1.

Proof. At scale N , let E_N be the event that $M_F(N)$ lies within $\sqrt{t_N}$ of L_N , and that both one-prime accessible maxima lie within $\sqrt{t_N}$ of $L_N + t_N$, where $t_N = \log_2 L_N$. Theorem 8.1 gives $\mathbb{P}(E_N^c) \rightarrow 0$. Choose a deterministic dyadic sequence X_i with summable failure probabilities, and intersect its eventual-success event and Fish's normality event with every finite-flip preimage. This gives Ω_* , independent of $h, \sigma, \varepsilon, T$.

If f_{j-1} is the current finite modification, choose N_j among the X_i so late that E_{N_j} holds for both f and f_{j-1} , $Y_{N_j} > \max(T, 2N_{j-1})$, $h(Y_{N_j}) \geq j$, and

$$Y_{N_j}^{-\tau_j} < \varepsilon 2^{-j-1}, \quad \tau_j = \min(\sigma, 1/j).$$

For large N , $t_N > 2\sqrt{t_N}$. Therefore the best large-prime one-flip maximum for f_{j-1} exceeds its unmodified maximum, and its optimum, over a finite set, is attained by a genuine nonempty single flip at a prime $p_j > Y_{N_j}$. Perform that flip. The attained maximum satisfies the second estimate in (9.3). All later primes exceed $2N_j$, so this entire maximum is frozen.

The support is exactly the increasing sequence of chosen primes. If $p_j \leq x < p_{j+1}$, its counting function is $j \leq h(p_j) \leq h(x)$; before p_1 it is zero. The weighted estimates and normality follow as in Theorem 9.1. The event for the initial base gives the first estimate of (9.3). $\square$

There is one *new* prime per stage, not one fixed prime responsible for infinitely many gains. The scales are selected after the base and may be enormous. The gain is additive of order $\log \log N$, not a fixed multiplicative increase of the leading maximum. Hence this construction is consistent with logarithmic cardinality being necessary for a fixed factor $c > 1$.

10 Adaptivity, preserved laws and singular outputs

Prime weight, number of flips and information used to select them measure different resources. The following elementary identities clarify how the law of the selected output can change; the general role of information in adaptive selection is discussed in [22].

Proposition 10.1 (Predictable flips preserve the product law). *Enumerate the primes $p_1 < p_2 < \dots$, and let R be an independent random seed. If*

$$G(p_i) = A_i F(p_i), \quad A_i \in \{-1, +1\} \text{ is measurable in } (R, F(p_1), \dots, F(p_{i-1})),$$

then G has exactly the same prime-product law as F , even with infinitely many flips.

Proof. Conditionally on the past and the seed, $F(p_i)$ is an independent fair sign and A_i is fixed. Thus the next output is fair and independent of the previous outputs. Induction proves the claim for every finite prefix, hence for the whole sequence. $\square$

This does not cover deciding whether to flip $F(p_i)$ after examining that value, or using arbitrary future prime signs.

For a finite prime-sign vector F uniform on $\{-1, +1\}^m$, an arbitrary adaptive multiplier A and $G = FA$, conditional multiplication is a bijection, so $H(G | A) = H(F | A)$. In natural logarithms this yields

$$D(\mathcal{L}(G) \| \text{Unif}) = I(F; A) - I(G; A) \leq I(F; A). \quad (10.1)$$

If an event has probabilities p and q before and after surgery, data processing to that event gives

$$q \log(1/p) - h(q) \leq \text{kl}(q \| p) \leq D(\mathcal{L}(G) \| \text{Unif}) \leq I(F; A),$$

where h is binary entropy in natural units. Under (A1), the event of a length- $\lceil cL_N \rceil$ run has probability at most $N^{1-c+o(1)}$ before intervention. Any procedure forcing it with probability tending to one must therefore use

$$I((F(p))_{p \leq 2N}; (A(p))_{p \leq 2N}) \geq (c - 1 - o(1)) \log N.$$

This is not asserted to be the optimal information cost of our constructions.

The construction in Theorem 9.1 can be chosen Borel: enumerate scales and finite successful choices, take the first satisfying the stated finite criteria and budgets, and use the identity map off the Borel full-measure success set. Let $T(f)$ be the output and $Q = T_* \mathbb{P}$. It is carried by

$$E_c = \{g : \limsup_{N \rightarrow \infty} M_g(N)/L_N \geq c\},$$

whereas $\mathbb{P}(E_c) = 0$ for $c > 1$ by Corollary 7.3 with zero budget. Thus $Q \perp \mathbb{P}$ despite a coupling with arbitrarily small $d_\sigma(F, T(F))$ and an unchanged prescribed prefix. A small adaptive prime distance does not transport all almost-sure properties of the base law.

11 Scope, interpretation and further questions

The results separate fixed-word statistics, growing maxima and marked empirical rare-event laws. They are not deductions about the prescribed values of the Liouville function. They also do not assert that random multiplicative functions typically have the exceptional output statistics: adaptivity may make the output law singular.

The strongest realization theorem applies to a full-measure class of bases and its finite-distance classes, not to every normal base. The elementary insertion theorems have the wider base class, but do not prescribe a finite actual intensity. The reversible realization uses $H_N = N(\log N)^{-\kappa} e^{O(1)}$, whereas the elementary distributed construction works at shorter windows $N e^{-\alpha S_N + O(1)}$. These are different trades between hypotheses and control, not a chain in which each later statement uniformly supersedes the earlier ones.

No theorem here prescribes a rare-event law at every sufficiently large scale, imposes unrelated laws at multiple independently growing ambient scales, or supplies arbitrary uniformity in growing threshold offsets. The residual universal output has different subsequences for different targets. Similarly, finite projection indistinguishability never means that one nonreference law has every finite-dimensional distribution of the reference.

There are concrete next questions suggested by the proofs. For the cost theorem, the all-prime remainder is limited by the entropy of collective small-prime columns, while the sufficient upper range $c < 3/2$ is limited by the nonterminal term of the capped profile. For the fine law, the elementary volume comparison only removes small primes for $k = o(\log \log N)$. For reversible realization, extending the base class beyond the generic arithmetic reservoir would require new availability information. These are identified proof bottlenecks, not assertions that the corresponding extensions are false.

A Thin-support variants at other observation scales

Longer observation windows allow a single changed-prime support to be summable for every positive power weight. The distributed-insertion and reversible single-threshold constructions below use different scales and retain their respective conclusions.

Proposition A.1 (Distributed insertion with all positive weights summable). *In every d_1 ball about a normal base, with a prescribed prefix, the conclusion of Theorem 2.6 holds at*

$$H_N = 2^{\lfloor \log_2(N \exp\{-(\log N)^{1/3}\}) \rfloor}, \quad \ell_N = \log_2 H_N, \quad q_N = \lfloor \log N \rfloor,$$

with $\sum_{p \in S} p^{-\tau} < \infty$ for every $\tau > 0$.

Proof. In Lemma 2.5, put $L = \log N$, $Y = e^{\sqrt{L}}$ and keep $D = \lfloor H_N/4 \rfloor$ and the same guarded words. Now $K = e^{L^{1/3} + O(1)}$. Choose $\eta = \log L/(2\sqrt{L})$ in (2.2); then $A(2N, Y)/N \leq e^{-(1/2 + o(1))\sqrt{L} \log L}$. The logarithms of the three exclusion ratios are at most

$$L^{1/3} - (1/2 + o(1))\sqrt{L} \log L, \quad L^{1/3} - \sqrt{L} + O(\log L), \quad -L + 2L^{1/3} + O(\log L).$$

All tend to $-\infty$. The cost for any fixed τ is at most $e^{L^{1/3} - \tau\sqrt{L} + O(\log L)} \rightarrow 0$. At stage j impose a summable budget in $d_{1/j}$ and freeze the earlier scales. Every fixed positive weight is eventually bounded by these weights. $\square$

Proposition A.2 (Reversible single-threshold realization with all positive weights summable). *Under (A1) and (A2), the sign-marked, single-threshold version of Theorem 5.1, and countably many such targets, can be realized with*

$$H_N^* = 2^{\lfloor \log_2(N e^{-\sqrt{\log \log N}/100}) \rfloor}, \quad \sum_{p \in S} p^{-1} < \varepsilon, \quad \sum_{p \in S} p^{-\tau} < \infty \quad (\tau > 0).$$

The base is generic; the prefix and normality are preserved. This statement does not assert the multithreshold strengthening at this modified scale.

Proof. Revisit the reservoir at one threshold with $a_N = J^{-1/2}$, $\kappa_N = a_N/100$, $\beta_N = a_N/20$, $\theta_N = a_N J/L$ and $Y = e^{\sqrt{LJ}}$. Then

$$K \asymp e^{\sqrt{J}/100}, \quad D/H_N^* \asymp e^{-\sqrt{J}/25}, \quad t_0 = \lfloor \sqrt{J}/(4 \log 2) \rfloor \rightarrow \infty.$$

The high-mark error is $O(K)e^{-S/4+9\sqrt{L}+O(1)}$; bad supports contribute $KB e^{-S/4+o(S)}$; separated sharing contributes $N^{-1/4+o(1)} + K^2 B^2 e^{-S/2}$. Low-free candidates cost at most $\exp\{(1/100 - 1/(2 \log 2))\sqrt{J} + O(\log J)\}$. The cell marking error is $O(1)e^{(1/20-1/(16 \log 2))\sqrt{J}}$, and the weighted search errors are $O(\sqrt{J})e^{-4\sqrt{J}/25} + O(N^{-1/4}e^{3\sqrt{J}/20})$. All vanish. Markov bounds the candidate count by $Ke^{4\sqrt{L}}$, so a stage costs at most

$$BK \exp\{4\sqrt{L} - \tau\sqrt{LJ}\} \rightarrow 0$$

for each fixed $\tau > 0$. Select a common success subsequence, stable under finite modifications, and impose a summable $d_{1/j}$ budget at stage j . The periodic realization only needs $D/H \rightarrow 0$, $K \rightarrow \infty$ and vanishing cost, so its sign-marked version applies. $\square$

In both propositions the support count is $o(x^\delta)$ for every $\delta > 0$, by bounding it by $x^{\delta/2} \sum_{p \in S} p^{-\delta/2}$. This does not imply agreement on a density-one set of integers. Nor can all positive-weight sums of an infinite set be less than the same fixed ε as $\tau \downarrow 0$.

B Scalar laws, spatial correlations and threshold couplings

The simplest finite-support target is $\nu_2 = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_2$, whose mean and variance are one and whose distance to $\text{Pois}(1)$ is $1 - 3/(2e)$. More generally,

$$G_r(z) = \sum_{k=0}^r \frac{(z-1)^k}{k!}, \quad \nu_r(j) = \frac{1}{j!} \sum_{u=0}^{r-j} \frac{(-1)^u}{u!}, \quad 0 \leq j \leq r,$$

defines the classical fixed-point count of a uniform permutation of r objects, also called quasi-Poisson($r, 1$); see [19]. Positivity follows by pairing terms of the alternating sum, and derivatives at one give $\mathbb{E}(X)_k = 1$ for $k \leq r$ and zero for $k = r + 1$.

Scalar realization with bounded counts. A finite law ν on $\{0, \dots, R\}$ can be realized directly by the reservoir. Place ideal centers at spacing H , choose group sizes k_i whose empirical distribution approximates ν , and use neighboring mesh cells to activate k_i starts within $O_R(D)$ of each center. Extinguish the rest. The ideal count has law within $O_R(H/N + \ell/H)$ of ν ; displacements change it on at most $O_R(D/H)$ of origins. A real window meets at most two groups, so its count is at most $2R$. Thus its law and every fixed moment converge. This gives a scalar limit ν_2 with the correct first two moments but a non-Poisson law.

Correct spatial correlations. For a stronger spatial example, take a uniform phase in $[0, 1)$, independent counts ν_r in each unit cell, and uniform independent point positions conditional on the counts. The stationary simple process Φ_r has factorial densities

$$\rho^{(k)}(x_1, \dots, x_k) = 1 \quad (k \leq r), \quad \rho^{(r+1)}(x_1, \dots, x_{r+1}) = \min\{1, \max_i x_i - \min_i x_i\}.$$

Conditionally on the phase, the density is the product of the cell factorial moments. At order $r + 1$ it vanishes exactly when all positions are in one cell, which occurs for a proportion $(1 - \text{diam}\{x_i\})_+$ of phases. For $r = 2$, the unit-window generating function is

$$\frac{17}{40} + \frac{13}{60}z + \frac{3}{10}z^2 + \frac{1}{20}z^3 + \frac{1}{120}z^4.$$

The void probability is $17/40$ and the distance to $\text{Pois}(1)$ is $29/40 - 3/(2e)$. The intensity and full covariance measure are Poisson's. These are different unit-window statistics from the scalar target ν_2 . Arithmetic realization with all local moments follows by retaining the uniform bound of r points per cell in the periodic approximations.

Correct individual threshold processes and covariances. A particularly transparent two-color law has generating function

$$F(s, t) = e^{(s+t-2)/2} + \eta(s-1)(t-1)(s-t), \quad \eta = 1/(16e).$$

Only six coefficients change, and the three subtracted reference masses are at least η . Both color counts are $\text{Pois}(1/2)$, their sum is $\text{Pois}(1)$, and all mixed derivatives of total order at most two at $(1, 1)$ are unchanged. Use independent such pairs in unit cells with a common uniform phase; place points uniformly. Give blue points excess zero and red points excess $1 + G$, where $\mathbb{P}(G = j) = 2^{-j-1}$. Every threshold process is $\text{PPP}(2^{-r})$, every exact layer is $\text{PPP}(2^{-q-1})$, and

$$\text{Cov}(\Phi^{\geq r}(A), \Phi^{\geq s}(B)) = 2^{-\max(r,s)} |A \cap B|.$$

Nevertheless, for $Z_r = \Phi^{\geq r}([0, 1))$,

$$\mathbb{P}(Z_0 = 1, Z_1 = 1) = \frac{1}{2e} + \frac{3-e}{4e^2}.$$

To verify the correction, a unit window meets two cells in fractions $u, 1-u$. The perturbation is homogeneous of degree three about $(1, 1)$, so the coefficient for one red and zero blue is $2\eta \int_0^1 u^3 e^{u-1} du = \eta(12/e - 4) = (3-e)/(4e^2)$; the quadratic perturbation contributes zero to that coefficient. This grid construction need not have finite spatial dependence; Theorem 6.1 replaces the common phase by local implantation. Moment transfer to arithmetic outputs requires Proposition 5.5.

Declaration on artificial-intelligence assistance

GPT-6 Astra provided substantial assistance in developing and consolidating arguments, mathematical exposition, literature comparison, and preparation of the manuscript and diagnostic programs. The author determines the claims and scope and retains responsibility for the arguments, references and any remaining errors. AI-assisted checks are not independent human peer review.

The mathematical claims rest on the printed arguments and the explicitly stated external results. This paper has no accompanying Lean formalization. The finite diagnostic programs supplied with the sources test specified identities and finite control rules; they do not certify the imported asymptotic estimates or the infinite constructions.

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