

Moving prime environments and arithmetic cluster limits

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Version 1

This research received substantial assistance from AI systems;
see the declaration at the end of the article.

Abstract

We study shrinking-target occurrences of completely multiplicative functions with arbitrarily moving phases on a fixed finite set of exceptional primes and independent Haar values at the remaining primes. After exact removal of the environment-only points, effective integer-subgroup profiles classify all subsequential marked cluster limits, while target-scale pair overlaps characterize Poisson convergence. The quantitative input is a direct Haar comparison with the moving germ kernel, uniform in the exceptional phases and with stretched-exponential error.

For deterministic profiles and interval targets, the complete spatial covariance measure reconstructs the unmarked law through primitive multiplicative rays and finite interval geometry. A positive forest formula makes this reconstruction valid even with unbounded cluster sizes. For a fixed known bounded Borel target, retaining positions and angular marks yields the corresponding identification from the complete marked covariance, using signed overlap channels.

Restricting the observation to an interior spatial window imposes both angular and logarithmic-spatial constraints. The maximal number of descendants of one germ becomes a rectangle-occupancy number for a subgroup of $\mathbb{R}^2$, finite exactly when that subgroup is discrete. Explicit rank-one laws illustrate what counts forget, while environmental mixtures show why deterministic identification does not extend by averaging covariances.

Keywords. Random multiplicative functions; shrinking targets; Poisson approximation; arithmetic clusters; moving environments; relation lattices; spanning forests.

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1 Introduction

A rare occurrence can ignore most of the dependence in a multiplicative sequence, but it cannot ignore a relation that remains accurate at the resolution of its target. If a target on the circle has mass τ/N , phases separated by order $1/N$ can activate the same regular-prime source. Their integer arguments need not be close: multiplying a common core by 2 and by 3 produces macroscopically separated positions. A limit law must therefore retain both the angular precision of a relation and the spatial geometry of its descendants.

The germ kernel in *Arithmetic clusters* [25, Theorem 4.1] is uniform in the exceptional phases, so its prelimit comparison already permits a moving phase vector. Its Theorem 5.1 fixes the phases before $N \rightarrow \infty$; Remark 5.2 distinguishes that count classification from moving-phase limits. Here we classify the microscopic moving limits and prove a direct comparison for the Haar regular source. The exceptional prime set stays finite and fixed; only its phases may vary arbitrarily.

1.1 The model and the main questions

Let $\mathcal{E}$ be a fixed finite set of primes. At primes $p \in \mathcal{E}$ prescribe phases $F_N(p) = \xi_{N,p}$; at every other prime choose independent Haar values on the circle. Extend F_N completely multiplicatively. For the arc $J_N = \exp(2\pi i(-\tau/2, \tau/2)/N)$, record hits up to N , their positions divided by N , and their phases magnified by N . An integer supported entirely on $\mathcal{E}$ has no remaining regular randomness. We remove this exact subconfiguration, rather than subtracting its expectation. For example, if $F_N(2) = 1$, the powers of 2 alone contribute $\lfloor \log_2 N \rfloor$ hits. The correction is part of the observable, not an asymptotically harmless convention.

A one-prime picture. Take $\mathcal{E} = \{2\}$ and $F_N(2) = e^{2\pi ic/N}$. Writing $n = 2^k d$ with d odd, a germ with enlarged angular coordinate z activates the descendant $2^k d$ exactly when $z \in I - kc$, where $I = (-\tau/2, \tau/2)$. They coincide at $c = 0$; successive intervals overlap for $0 < |c| < \tau$, while all are disjoint up to endpoints for $|c| \geq \tau$. For example, when $\tau/2 \leq |c| \leq \tau$, the corrected count converges to

$$P_1 + 2P_2, \quad P_1 \sim \text{Pois}(|c|), \quad P_2 \sim \text{Pois}((\tau - |c|)/2), \quad P_1 \perp P_2.$$

This is the classical Hermite distribution [13], with parameters supplied here by the arithmetic environment. Section 7 derives the full formula. Figure 1 displays the geometric reason for the double hits: one angular germ can activate two distinct spatial positions.

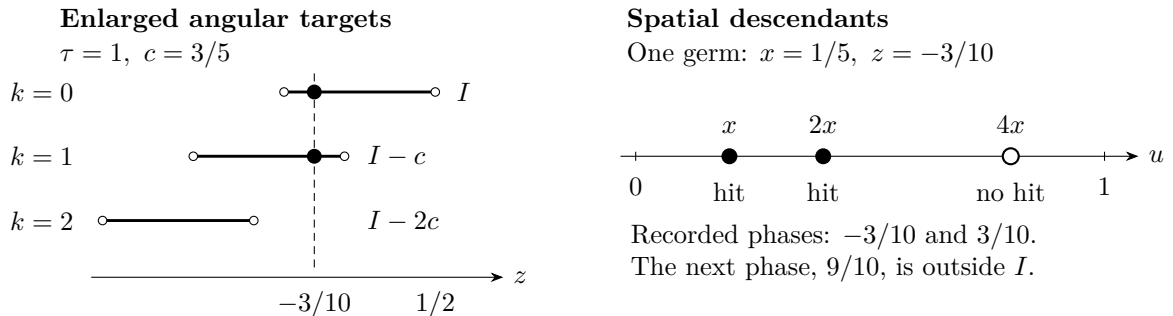


Figure 1: A target-level activation diagram, not a simulation. With the displayed germ, exactly the first two of the eligible multipliers 1, 2, 4 hit the target. Angular overlap does not make their normalized positions coincide. Open endpoints have zero germ mass.

Three questions organize the results. When do the corrected observations become Poisson without assuming a phase limit? What are the possible cluster fields when overlaps persist? Which parts of those fields are visible through a spatial window or through low-order statistics?

1.2 Pair overlaps, the Poisson criterion, and actual variances

For a nonzero integer vector $v \in \mathbb{Z}^{\mathcal{E}}$, let

$$A(v) = \prod_{p \in \mathcal{E}} p^{(v_p)_+}, \quad B(v) = \prod_{p \in \mathcal{E}} p^{(-v_p)_+}, \quad H(v) = \max(A(v), B(v)).$$

The coprime ratio $A(v)/B(v)$ is its primitive multiplicative channel. Define

$$\beta_N(v) = (\tau - N \|\xi_N^v\|_{\mathbb{T}})_+, \quad \mathcal{R}_N = \sum_{v \neq 0} \frac{\beta_N(v)}{H(v)},$$

where circular distance is measured in turns. Both orientations are included. The height weights are summable, so channels whose heights grow with N are controlled rather than discarded.

Theorem 3.1 gives a necessary and sufficient criterion for every deterministic phase sequence:

$$W_N \xrightarrow{d_{\text{TV}}} \text{Pois}(\tau) \iff \mathcal{R}_N \rightarrow 0 \iff \liminf_N N \|\xi_N^v\|_{\mathbb{T}} \geq \tau \quad (v \neq 0 \text{ fixed}).$$

Here W_N is the corrected count. The same criterion gives a total-variation comparison of the complete marked field with a Poisson field on its *same spatial grid*.

A separate arithmetic argument proves

$$\text{Var}(W_N) - \mathbb{E}W_N = \mathcal{R}_N + O_{\mathcal{E}, \tau}(N^{-1/2}(1 + \log N)^{2|\mathcal{E}|+2}).$$

It counts distinct regular cores whose valuation vectors are proportional; it does not infer second moments from total variation. Thus vanishing variance excess characterizes Poisson convergence in this particular model, not for arbitrary random variables.

The arithmetic comparison is proved directly in Theorem 2.2. Corollary 2.3 gives the uniform error

$$\varepsilon_H(N) \leq \exp\left[-\left(1/\sqrt{2} - o(1)\right) \sqrt{\log N \log \log N}\right]$$

against the moving germ kernel. Persistent pair overlaps are retained in that kernel, not absorbed into this error. For quantitative Poisson control, the point-minus-germ deficit D_N counts descendants beyond the first per active germ. Theorem 3.3 bounds the Poisson distance above and below in terms of D_N ; Section 4 evaluates it by finite overlap graphs with a multiplier-tail certificate.

1.3 Effective profiles and a complete spatial invariant

For a centered argument, put $r_N(v) = N \text{Arg}_0(\xi_N^v)/(2\pi)$. Every sequence has a subsequence described by an integer subgroup L and an additive map $h : L \rightarrow \mathbb{R}$:

$$r_N(v) \rightarrow h(v) \quad (v \in L), \quad |r_N(v)| \rightarrow \infty \quad (v \notin L).$$

Proposition 5.2 identifies the realizable subgroups: the torsion part of $\mathbb{Z}^{\mathcal{E}}/L$ must be cyclic, and this condition is sufficient. Nonsaturated subgroups are retained, not replaced by rational spans. Theorem 5.4 constructs every corresponding spatial and angularly marked cluster limit by moving angular charts, core counting, and a summable configuration tail.

The effective subgroup need not be the relation lattice of a coarse limiting phase vector. Intermediate phase scales can remove directions before scale $1/N$ is reached. On a full prefix,

effective rank at least two forces unbounded cluster-size support. At rank one, explicit second differences give the jump intensities; at fixed target width τ and displacement $d = |h(v_0)|$, the count law depends on the larger channel integer A but not on the smaller B , which remains visible in space.

For a deterministic profile with an interval target, the complete spatial covariance measure determines every limiting coefficient

$$\beta(v) = \begin{cases} (\tau - |h(v)|)_+, & v \in L \setminus \{0\}, \\ 0, & v \notin L. \end{cases}$$

Conversely these coefficients determine the full unmarked spatial law (Theorem 8.4). The positive forest reconstruction avoids an infinite cumulant expansion, and Appendix A gives a total-variation modulus with quadratic finite-depth cost. The covariance is an entire measure containing all primitive rational rays, not a scalar variance or a finite covariance matrix. Spatial convergence is equivalent to convergence of each $\beta_N(v)$ (Theorem 8.6); count convergence alone is weaker.

With the angular marks retained, a visible channel also reveals its signed displacement. Theorem 12.2 shows that the complete covariance on positions and marks determines the fully marked law for any fixed known bounded Borel target. In the interval case the full-sequence criterion becomes convergence of every pair $(\beta_N(v), r_N(v)\beta_N(v))$. These marked and unmarked inverse statements use different data.

1.4 Observation windows and environmental mixing

On an interior window $U = (a, b)$, $0 < a < b \leq 1$, a relation must also fit spatially. Define

$$\ell(v) = \log \frac{A(v)}{B(v)}.$$

The restricted profile field is Poisson exactly when no nonzero $v \in L$ satisfies both

$$|\ell(v)| < \log(b/a), \quad |h(v)| < \tau.$$

The maximal possible number of descendants of one germ is a rectangle-occupancy number for $\Gamma_{L,h} = \{(\ell(v), h(v)) : v \in L\} \subset \mathbb{R}^2$. It is bounded exactly when this subgroup is discrete (Theorems 10.2–10.3). This distinguishes a rank-two profile with an interior Poisson window from the unavoidable unbounded local cluster support at rank at least three.

The Poisson criterion survives arbitrary random exceptional phases independent of the regular source. It even gives asymptotic independence from the retained exceptional parameter in the Poisson regime. The identification theorem does not survive arbitrary mixing: Section 11 gives two spatial laws with identical factorial moment measures through order three but different void probabilities. A negative eighth factorial cumulant rules out a positive-integer compound-Poisson representation for the mixed count, although each conditional count has one.

1.5 Relation to earlier work

Kernels and comparison. The finite-environment germ kernel, its exact environment-only correction and its fixed-phase count classification are established in *Arithmetic clusters* [25, Theorems 4.1 and 5.1]. Its Section 6.4 and Theorem 6.1 give the point-minus-germ deficit, first-active-site anchoring and void lower bound; Section 5.8 integrates fixed-environment limits. Its Section 6.5 also gives the scale $\exp[-c\sqrt{\log N \log \log N}]$, $c < 1/\sqrt{2}$, for rational observations. The direct Haar comparison proved here obtains that scale for the moving exceptional-environment kernel through a weighted core budget. It retains all descendants of each germ,

using the stronger Haar hypothesis in place of the density-smoothing step of the general L^2 comparison.

The comparison tools are classical: process Chen–Stein approximation [2,3], Haar characters as in Najnudel [21, arXiv:1702.01470v2, Lemma 2.1], and Rankin’s smooth-number bound [11]. Poisson mapping, marking and Laplace functionals are treated in [17,16]; finite-cluster and void formulas appear in [17, author manuscript, 21 August 2017, Exercises 5.6–5.7]. Compound-Poisson rare-return laws and geometric clusters have antecedents in Haydn–Vaienti [10], Kifer–Rapaport [15] and Rapaport [27]. Spatio-temporal observations [23], multiple-word clumps [28] and point-process metrics [29] provide related perspectives on what an occurrence records. For random dynamical systems, compound-Poisson limits arise through spectral perturbation [4] and probabilistic block approximation [1]. Here target-scale integer profiles describe every subsequential marked limit, while the pair-overlap array decides Poisson convergence without assuming a phase limit.

Interval geometry. Hunter–Worsley [12,30] give optimized tree bounds for unions. The exact tree sieve of Naiman–Wynn [19], in the form reproduced with proof by Dohmen [7, Corollary 2, p. 64], is directly applicable; the abstract-tube framework is developed in [20]. Lemma 4.1 specializes this identity to intervals and gives a short proof of its maximum-weight form. Grouping the arithmetic descendants and controlling the multiplier tail then produce finite deficit certificates, a positive Laplace reconstruction and the quadratic finite-depth stability bound of Appendix A.

Identification. First- and second-order statistics identify laws in several structured families, including those of Newman [22] and Gauss–Poisson processes [14, Section III]. Bacry–Muzy [5, arXiv:1401.0903v2, Theorem 1 and Corollary 1] use causal Wiener–Hopf inversion for stable multivariate linear Hawkes processes; the Hawkes–Oakes representation [9] allows clusters larger than two. Log Gaussian Cox processes [18] are another family determined by intensity and pair correlation. Model-set correlations [6] and homometry [8] concern related inverse questions with different observed data.

The inversion in this paper separates primitive multiplicative rays, recovers their overlap coefficients and reconstructs the spatial law by finite interval geometry, without an infinite cumulant expansion. With a fixed known bounded Borel target, the complete covariance on positions and angular marks also recovers signed channel shifts and independent overlap components. These are statements about deterministic profiles and complete covariance measures. Together with realizability of the target-scale integer profiles and exact subgroup-rectangle occupancy in interior windows, they identify the arithmetic structures that remain visible in the limit.

1.6 Reading guide

After the model and direct comparison in Section 2, the main route is $3 \rightarrow 5 \rightarrow 8 \rightarrow 10$: the Poisson criterion, all effective profiles, spatial identification, and interior-window geometry. Section 4 provides the forest tool used twice along this route. Sections 6–7 and 9 give explicit models and prefix observables; Sections 11–12 examine mixing, Borel targets and fully marked identification. Appendices A and B give quantitative stability and proof inputs.

Scope at a glance. The following conventions apply throughout; local statements specify any additional hypotheses.

Object	Scope retained in this article
Source and environment	A fixed finite exceptional set; Haar at the other primes. Any random exceptional parameter is independent of that regular source.
Arithmetic comparison	Exact removal of environment-only points; total variation on one prelimit grid, followed separately by weak diffuse limits.
Moments	Arithmetic covariances have a direct proof. Higher cumulants of a target do not provide higher arithmetic moments.
Identification	Deterministic profiles and complete covariance measures: spatial for intervals, fully marked for a known bounded Borel target. No finite-data inversion or arbitrary-mixture identification is asserted.
Paths and cluster size	Prefix marginals do not imply independent increments. Unbounded size support is per germ, not an infinite realization.
Rates and parameters	The stretched-exponential rate is to the moving kernel. Profile convergence keeps finite depth fixed first and uses a target-dependent translation modulus; the exceptional set does not grow.

2 Model, notation, and the arithmetic comparison

Fix a finite set of primes $\mathcal{E}$, possibly empty. Throughout it is fixed before N . Write

$$b_0 = |\mathcal{E}|, \quad q = \prod_{p \in \mathcal{E}} p, \quad \delta = \prod_{p \in \mathcal{E}} (1 - p^{-1}), \quad \mathcal{D} = \{s \geq 1 : p \mid s \Rightarrow p \in \mathcal{E}\}.$$

Let $B_{\mathcal{E}}(X) = |\mathcal{D} \cap [1, X]|$, with $B_{\mathcal{E}}(X) = 0$ for $X < 1$, and set

$$M(X) = \lfloor X \rfloor - B_{\mathcal{E}}(X) \quad (X \geq 0).$$

Thus $M(X)$ counts the positive integers at most X having a prime factor outside $\mathcal{E}$. For $s \in \mathcal{D}$, $\mathbf{v}(s) = (v_p(s))_{p \in \mathcal{E}}$ is its valuation vector.

For each N choose an *arbitrary deterministic* vector $\xi_N \in \mathbb{T}^{\mathcal{E}}$. No convergence, base phase, differentiability, or rate of phase motion is assumed. Fix these prime values, take mutually independent Haar values at the other primes, and extend to a completely multiplicative function F_N . A coupling between different N is neither required nor asserted. Put

$$\chi_N(s) = \prod_{p \in \mathcal{E}} \xi_{N,p}^{v_p(s)}.$$

For a fixed $\tau > 0$ let $I = (-\tau/2, \tau/2)$, $J_N = \exp(2\pi i I/N)$, and $K = [0, 1] \times \bar{I}$. For all sufficiently large N , define

$$\Xi_N = \sum_{\substack{2 \leq n \leq N \\ n \notin \mathcal{D}}} \mathbf{1}_{\{F_N(n) \in J_N\}} \delta_{(n/N, N \operatorname{Arg}(F_N(n))/(2\pi))}, \quad W_N = \Xi_N(K). \quad (2.1)$$

The principal argument is in $(-\pi, \pi]$. The target interval is open, while the ambient rectangle K is closed. The exclusion $n \notin \mathcal{D}$ removes the *exact environment-only configuration*. It is not subtraction of an expectation. The uncorrected field need not have a finite limit.

Every retained site contains a regular prime. A nonzero power of a Haar variable is Haar, so the exact mean measure and its total mass are

$$\mu_N = \frac{1}{N} \sum_{\substack{2 \leq n \leq N \\ n \notin \mathcal{D}}} \delta_{n/N} \otimes \mathbf{1}_I(w) dw, \quad m_N = \mu_N(K) = \mathbb{E} W_N = \tau \left(1 - \frac{B_{\mathcal{E}}(N)}{N}\right). \quad (2.2)$$

Write X_N for the spatial projection and let $P_N = \text{PPP}(\mu_N)$. This is a Poisson comparison on the *same spatial grid*, not a diffuse process substituted in total variation.

2.1 The finite-environment germ kernel

Write $n = sd$ uniquely, with $s \in \mathcal{D}$ and $(d, q) = 1$. The core d is not a squarefree kernel. We retain the kernel of *Arithmetic clusters* [25, Theorem 4.1], specialized to Haar regular prime values and with its exact $d = 1$ part removed. An angular germ at a regular core produces all of its activated descendants, with their phases. The direct argument below proves

$$d_{\text{TV}}(\mathcal{L}(\Xi_N), \mathcal{L}(\tilde{\Xi}_N)) \leq \varepsilon_H(N), \quad \varepsilon_H(N) \longrightarrow 0, \quad (2.3)$$

uniformly in ξ_N , with the finite bound of Theorem 2.2 and the choice in Corollary 2.3. Every core reference law is Haar, including prime cores.

The general L^2 comparison in [25, Theorem 4.1] gives an alternative bound with $C = 1$, mass ceiling τ and $K_2 = 1$; its admissible choice $\theta = 1/2$, $\kappa = 1/32$ yields $O_{\mathcal{E}, \tau}((\log N)^{-1/32})$. The proof given here exploits the stronger Haar hypothesis instead of using that theorem as its arithmetic input. It neither changes the cluster target nor replaces the general source-density result.

For precision, at a core $d \geq 2$ with $(d, q) = 1$, put

$$S_{N,d} = \{s \in \mathcal{D} : sd \leq N\}, \quad K_{N,d} = \bigcup_{s \in S_{N,d}} \chi_N(s)^{-1} J_N.$$

An independent Poisson measure of intensity $\mathbf{1}_{K_{N,d}} m$ on the circle, with m normalized Haar measure, is mapped by

$$u \longmapsto \sum_{s \in S_{N,d}} \mathbf{1}_{\{\chi_N(s)u \in J_N\}} \delta_{(sd/N, N \operatorname{Arg}(\chi_N(s)u)/(2\pi))}.$$

Summing these configurations defines $\tilde{\Xi}_N$. Its exact mean is μ_N . Write $\tilde{X}_N$ and $\tilde{W}_N$ for its spatial projection and total count. Its positive-integer jump intensities are $\lambda_{N,j}$; define

$$\Lambda_N = \sum_{j \geq 1} \lambda_{N,j}, \quad D_N = m_N - \Lambda_N = \sum_{j \geq 1} (j-1) \lambda_{N,j}, \quad K_{2,N} = \sum_{j \geq 1} j(j-1) \lambda_{N,j}. \quad (2.4)$$

Here Λ_N is the mean number of active germs, D_N is the point-minus-germ deficit, and $K_{2,N}$ is the target's variance excess. Every sum is finite at fixed N .

We use $d_{\text{TV}}(P, Q) = \sup_B |P(B) - Q(B)|$. Total variation is used on a common prelimit space; a diffuse spatial limit is a subsequent weak limit. Applying the same measurable restriction or projection to a field and its target contracts the comparison error. No relation between fields at different N is imposed.

2.2 Primitive channels and scaled overlaps

For $v \in \mathbb{Z}^{\mathcal{E}}$ define

$$A(v) = \prod_p p^{(v_p)_+}, \quad B(v) = \prod_p p^{(-v_p)_+}, \\ H(v) = \max(A(v), B(v)), \quad H_-(v) = \min(A(v), B(v)).$$

The integers $A(v), B(v)$ are coprime. Let $\|z\|_{\mathbb{T}}$ be the circular distance in turns from $z \in \mathbb{T}$ to 1. Put

$$r_N(v) = \frac{N}{2\pi} \operatorname{Arg}_0(\xi_N^v), \quad \beta_N(v) = (\tau - |r_N(v)|)_+ = (\tau - N \|\xi_N^v\|_{\mathbb{T}})_+, \quad \mathcal{R}_N = \sum_{v \neq 0} \frac{\beta_N(v)}{H(v)}. \quad (2.5)$$

Here $\operatorname{Arg}_0 \in [-\pi, \pi)$; its branch convention has no effect on β_N . Both orientations v and $-v$ occur in the sum. These are ordinary integer relations, not vectors modulo saturation.

The same symbol v_p denotes a coordinate of an integer vector here, while $v_p(n)$ is a prime valuation. Define also the additive logarithmic ratio

$$\ell(v) = \log \frac{A(v)}{B(v)} = \sum_{p \in \mathcal{E}} v_p \log p. \quad (2.6)$$

Unique factorization implies $\ell(v) = 0$ only for $v = 0$. All logarithms without a specified base are natural.

2.3 Summability and finite-height control

For $\alpha > 0$,

$$Z(\alpha) = \sum_{s \in \mathcal{D}} s^{-\alpha} = \prod_{p \in \mathcal{E}} (1 - p^{-\alpha})^{-1} < \infty, \quad \delta Z(1) = 1. \quad (2.7)$$

We have $B_{\mathcal{E}}(X) \ll_{\mathcal{E}} (1 + \log X)^{b_0}$ for $X \geq 1$, since every exponent is bounded by $\log X / \log p$. Similarly,

$$\#\{v : H(v) \leq X\} \ll_{\mathcal{E}} (1 + \log X)^{b_0}, \quad \sum_{H(v) > Q} \frac{1}{H(v)} \ll_{\mathcal{E}} \frac{(1 + \log Q)^{b_0}}{Q} \quad (Q \geq 2). \quad (2.8)$$

The second estimate follows by dyadic summation of the first. In particular $\mathcal{R}_N$ is absolutely convergent and bounded uniformly in N and the phases.

An explicit, less sharp alternative is useful for finite certificates. Since $H(v) \geq \prod_p p^{|v_p|/2}$,

$$\sum_{H(v) > Q} \frac{1}{H(v)} \leq Q^{-1/2} \left\{ \prod_{p \in \mathcal{E}} \frac{1 + p^{-1/4}}{1 - p^{-1/4}} - 1 \right\}. \quad (2.9)$$

Thus one can bound the full energy using finitely many precisely specified characters and a deterministic tail. A finite search alone does not recognize exact relations of arbitrary real phases; this bound is a tail certificate once the finite overlap values have themselves been supplied reliably.

Throughout, $S_H = \mathcal{D} \cap [1, H]$, $k_H = |S_H|$, and

$$T_H = \sum_{s > H, s \in \mathcal{D}} \frac{1}{s} = \delta^{-1} - \sum_{s \leq H, s \in \mathcal{D}} \frac{1}{s} \leq H^{-1/2} Z(1/2). \quad (2.10)$$

These quantities always use a fixed exceptional prime set. No estimate uniform in a growing set is inferred.

2.4 Direct comparison for Haar regular primes

We use the exact-marginal form of the classical process Chen–Stein bound [3, Theorem 2 and Section 3.1], whose theorems are proved in [2]. The conditional marked formulation and its normalization are also stated in *Arithmetic clusters* [25, Theorem 2.2]. We recall the reduction to finite marks before making the arithmetic estimates.

Lemma 2.1 (Conditional marked comparison with exact marginals). *On a finite index set, let an event at index i carry a mark in a standard Borel space, and let its reference subprobability be ν_i , of mass a_i . On a deterministic subset G , suppose that the complete marked variables have an exact dependency graph conditionally on $\mathcal{F}$, and that their conditional marked subprobabilities*

equal ν_i . For each $i \in G$, let $B_i \subseteq G$ contain the closed neighbourhood of i in that graph, including i itself. Then the full marked field η satisfies

$$\begin{aligned} \mathbb{E} d_{\text{TV}} \left(\mathcal{L}(\eta \mid \mathcal{F}), \text{PPP} \left(\sum_i \delta_i \otimes \nu_i \right) \right) &\leq \sum_{i \notin G} (\mathbb{P}(A_i) + a_i) + 2 \sum_{i \in G} \sum_{j \in B_i} a_i a_j \\ &\quad + 2 \sum_{i \in G} \sum_{j \in B_i \setminus \{i\}} \mathbb{P}(A_i \cap A_j). \end{aligned} \quad (2.11)$$

The pair probabilities on the last line are unconditional. The constants do not depend on the number or resolution of the marks.

Proof. Delete real and target points outside G , at the first displayed cost after averaging. On each conditional fibre partition the mark space into finitely many cells and use the indicators of activation in each cell. Different cells at one index are mutually exclusive. The product sum over cells is exactly the middle sum in (2.11), and the distinct-index joint sum is its last sum before averaging. The conditional dependency graph for complete marks makes the third Chen–Stein term zero. Theorem 2 of [3] gives $2(b_1 + b_2)$ in our convention $d_{\text{TV}} = \sup_A |P(A) - Q(A)|$, half the variation norm used there. Take increasing finite partitions generating the Borel sigma-field. Their configuration counts generate the sigma-field of finite marked measures. Total variation restricted to these increasing sigma-fields increases to full total variation, as follows from martingale convergence of the two Radon–Nikodym densities relative to the sum of the laws. Monotone convergence permits averaging the conditional distances. The tower property then gives the unconditional pair probabilities in the last sum. $\square$

Let $\Psi(N, Y)$ count the positive integers at most N all of whose prime factors are at most Y , including 1. Put $L_N = \lfloor \log_2 N \rfloor$. For $\max(\mathcal{E} \cup \{2\}) \leq Y \leq N$, expose only

$$\mathcal{F}_Y^{\text{reg}} = \sigma(F_N(p) : p \leq Y, p \notin \mathcal{E}).$$

For this comparison the exceptional phases are fixed, not exposed at random.

Characters of arbitrary regular cores. For any finite family of integers $d_i \geq 1$ coprime to q and $k_i \in \mathbb{Z}$, independence and Haar orthogonality give

$$\mathbb{E}_{\text{reg}} \prod_i F_N(d_i)^{k_i} = \prod_{p \notin \mathcal{E}} \int_{\mathbb{T}} z^{\sum_i k_i v_p(d_i)} dm(z) = \mathbf{1}_{\{\sum_i k_i v_p(d_i) = 0 \text{ for every } p \notin \mathcal{E}\}}.$$

Only finitely many factors are nontrivial. This is the character proof of [21, arXiv:1702.01470v2, Lemma 2.1], written here for arbitrary regular cores rather than consecutive arguments. Full rational rank of the valuation vectors therefore gives product Haar measure. In particular, two nonproportional nonzero vectors give independent phases *after averaging all regular prime values*, including the exposed small ones. Conditional independence of whole marked collections instead comes from disjoint unexposed prime supports, as required in Lemma 2.1.

Theorem 2.2 (Uniform finite Haar germ comparison). *For integers $N > \tau$ and the preceding range of Y , define*

$$\begin{aligned} \mathcal{E}_{\text{H}}(N, Y) &= \frac{2\tau\Psi(N, Y)}{N} + \frac{2\tau^2 Z(1/2)^2}{N} + 4\tau^2 Z(1)^2 \sum_{Y < p \leq N} \frac{1}{p^2} \\ &\quad + \frac{2\tau B_{\mathcal{E}}(N) \lfloor \sqrt{N} \rfloor L_N^2}{N}. \end{aligned} \quad (2.12)$$

The corrected arithmetic field and the cluster target of Section 2.1 obey

$$\sup_{\xi \in \mathbb{T}^\mathcal{E}} \mathbb{E}_{\text{reg}} d_{\text{TV}} \left(\mathcal{L}(\Xi_N^\xi \mid \mathcal{F}_Y^{\text{reg}}), \mathcal{L}(\Xi_N^\xi) \right) \leq \min\{1, \mathcal{E}_H(N, Y)\}. \quad (2.13)$$

The same estimate holds for deterministic site masks and deterministic site-dependent circle targets of Haar mass at most τ/N , with their corresponding germ–descendant map. In particular it applies to the bounded Borel targets of Section 12 once their scaled circle images are injective.

Proof. Write $b_d = m(K_{N,d})$. The union bound gives $b_d \leq (\tau/N)B_\mathcal{E}(N/d)$. Before the descendant map the real process is

$$\eta_N = \sum_{\substack{2 \leq d \leq N \\ (d,q)=1}} \mathbf{1}_{\{F_N(d) \in K_{N,d}\}} \delta_{(d, F_N(d))},$$

with independent Poisson measures of intensity $\mathbf{1}_{K_{N,d}}m$ as reference.

Deletion. Delete the cores with no prime divisor above Y . Each $F_N(d)$, $d \geq 2$, is Haar before conditioning, so the two deletions cost at most $2 \sum_d \text{deleted } b_d$. Unique decomposition $n = sd$ and the condition $\mathcal{E} \subseteq [2, Y]$ give the exact identity

$$\sum_{\substack{d \geq 2, (d,q)=1 \\ P^+(d) \leq Y}} B_\mathcal{E}(N/d) = \Psi(N, Y) - B_\mathcal{E}(N). \quad (2.14)$$

Thus the first term of (2.12) pays for deletion. There is no extra factor $B_\mathcal{E}(N)$ in this term.

Conditional marginals and graph. Every retained core contains a regular prime $p > Y$. A nonzero integer power of its Haar value is Haar, independently of the other prime values; hence the complete conditional marked subprobability is exactly $\mathbf{1}_{K_{N,d}}m$. Join two retained cores if they share a prime above Y . Nonadjacent collections use disjoint sets of unexposed coordinates, so this is an exact conditional graph for the complete phases, not merely a pairwise-independence assertion after averaging.

Weighted diagonal and product sums. Dropping coprimality restrictions and extending nonnegative sums,

$$\sum_d b_d^2 \leq \frac{\tau^2}{N^2} \sum_{s,t \in \mathcal{D}} \left\lfloor \frac{N}{\max(s,t)} \right\rfloor \leq \frac{\tau^2}{N} Z(1/2)^2, \quad (2.15)$$

$$\sum_{p|d} b_d \leq \frac{\tau}{N} \sum_{s \in \mathcal{D}} \left\lfloor \frac{N}{ps} \right\rfloor \leq \frac{\tau Z(1)}{p} \quad (p > Y). \quad (2.16)$$

The last line bounds the sum over regular cores divisible by p . Therefore, with distinct ordered edges and permissible overcounting,

$$\sum_{d \sim e} b_d b_e \leq \tau^2 Z(1)^2 \sum_{Y < p \leq N} p^{-2}. \quad (2.17)$$

Joint probabilities after averaging. If the complete regular valuation vectors of d, e are not proportional, the character criterion [21, arXiv:1702.01470v2, Lemma 2.1] gives independent Haar phases, so $\mathbb{P}(F_N(d) \in K_{N,d}, F_N(e) \in K_{N,e}) = b_d b_e$. This identity is used only after averaging the small regular coordinates. For distinct proportional vectors, divide their common rational direction by the gcd of its integer coordinates. Unique factorization gives $d = c^a$, $e = c^{a'}$ with integers $c \geq 2$, $a \neq a' \geq 1$. At least one exponent is at least two, so $c \leq \sqrt{N}$, and both exponents are at most L_N . The number of ordered such pairs is at most $\lfloor \sqrt{N} \rfloor L_N^2$, even allowing repeated representations. Each joint probability is at most $\min(b_d, b_e) \leq \tau B_\mathcal{E}(N)/N$.

Apply Lemma 2.1. Its first Stein sum contains the diagonal and the products on ordered edges. Its second sum contributes another copy of the products outside the proportional pairs, plus the last bound on proportional pairs. Thus the costs are deletion, twice the diagonal, four times (2.17), and twice the proportional-pair bound. This is (2.12). The deterministic map from a germ to all its descendants contracts total variation, without a cluster-size factor. The estimates are uniform in ξ . Masks and site targets only change the sets $K_{N,d}$ and preserve the displayed upper bounds, which proves the last assertion. $\square$

2.5 Choice of cutoff and comparison rate

Corollary 2.3 (A stretched-exponential comparison to the moving kernel). *For fixed $\mathcal{E}$ and τ , choose, for sufficiently large N ,*

$$H_N = \log N, \quad w_N = \sqrt{H_N \log H_N / 2}, \quad Y_N = e^{w_N}, \quad \varepsilon_H(N) = \min\{1, \mathcal{E}_H(N, Y_N)\}.$$

For the finitely many smaller N one may set $\varepsilon_H(N) = 1$. Then

$$\varepsilon_H(N) \leq \exp\left[-\left(\frac{1}{\sqrt{2}} - o(1)\right) \sqrt{\log N \log \log N}\right]. \quad (2.18)$$

In particular (2.3) holds with this error, uniformly in the exceptional phases. The mean conditional comparison (2.13) has the same rate at the chosen cutoff.

Proof. Set $H = \log N$, $w = \sqrt{H \log H / 2}$, and $\alpha = \log H / (2w)$, so $\alpha H = w$ and $\alpha w = (\log H) / 2$. For large N , $0 < \alpha < 1/4$. Rankin's inequality [11, Section 1, (1.3), p. 414] gives

$$\frac{\Psi(N, e^w)}{N} \leq N^{-\alpha} \prod_{p \leq e^w} (1 - p^{-1+\alpha})^{-1}. \quad (2.19)$$

The logarithm of the product is $\sum_{p \leq e^w} p^{-1+\alpha} + O(1)$, because all higher prime-power terms are bounded uniformly for $\alpha < 1/4$. Chebyshev's bound and partial summation give

$$\sum_{p \leq e^w} p^{-1+\alpha} \ll 1 + \frac{e^{\alpha w}}{w} + \int_{\log 2}^w \frac{e^{\alpha t}}{t} dt \ll \log H + \frac{\sqrt{H}}{\log H} = o(w).$$

For the last estimate substitute $u = \alpha t$, integrate near zero using $\int du/u$, and use $\int_1^U e^u/u du \ll e^U/U$ for $U \geq 2$. Since $\alpha H = w$, (2.19) is $e^{-w+o(w)}$. Even the integer tail $\sum_{p > Y} p^{-2} \leq \sum_{n > Y} n^{-2} \leq 1/\lfloor Y \rfloor$ gives the same leading exponent for the graph term. The remaining terms in (2.12) are $O_{\mathcal{E}, \tau}(N^{-1/2}(1 + \log N)^{b_0+2})$, which is smaller. This proves (2.18). $\square$

The smooth-number inequality and this optimization use classical estimates; the gain is the weighted direct comparison in the Haar submodel. The error controls the arithmetic field against its *moving cluster kernel*. It is not a uniform rate to a fixed profile law, and a simple Poisson approximation still has to pay for the overlaps in Section 3. The finite budget has explicit constants and may be evaluated with rigorous bounds on its sums and products; the asymptotic $o(1)$ alone is not a finite-size numerical error bar. The exact Haar marginals and independent nonproportional pairs are essential; no general L^2 extension is inferred.

3 A uniform Poisson criterion and direct second moments

Theorem 3.1 (Uniform overlap criterion). *Under the model of Section 2, for all sufficiently large N ,*

$$d_{\text{TV}}(\mathcal{L}(\Xi_N), \mathcal{P}_N) \leq \varepsilon_H(N) + \mathcal{R}_N, \quad (3.1)$$

$$d_{\text{TV}}(\mathcal{L}(W_N), \text{Pois}(\tau)) \leq \varepsilon_H(N) + \mathcal{R}_N + \tau B_{\mathcal{E}}(N)/N. \quad (3.2)$$

For every $v \neq 0$, without any restriction on its height,

$$d_{\text{TV}}(\mathcal{L}(W_N), \text{Pois}(\tau)) \geq e^{-\tau} \left[\exp \left\{ \frac{M(N/H(v))}{N} \beta_N(v) \right\} - 1 \right] - \varepsilon_H(N). \quad (3.3)$$

There is also a direct, uniform arithmetic estimate

$$\text{Var}(W_N) = m_N + \mathcal{R}_N + O_{\varepsilon, \tau} \left(N^{-1/2} (1 + \log N)^{2b_0+2} \right). \quad (3.4)$$

Consequently the following are equivalent:

$$\begin{aligned} d_{\text{TV}}(\mathcal{L}(\Xi_N), \mathbf{P}_N) &\rightarrow 0; & d_{\text{TV}}(\mathcal{L}(W_N), \text{Pois}(\tau)) &\rightarrow 0; \\ \mathcal{R}_N &\rightarrow 0; & \liminf_{N \rightarrow \infty} N \|\xi_N^v\|_{\mathbb{T}} &\geq \tau \quad \text{for every fixed } v \neq 0; \\ \text{Var}(W_N) - \mathbb{E}W_N &\rightarrow 0. \end{aligned}$$

The last equivalence is particular to this arithmetic model and its proved error bounds. It is not a general two-moment criterion for arbitrary random variables.

3.1 An exact primitive-channel identity

Let $\widetilde{W}_N = \widetilde{\Xi}_N(K)$. A finite Poisson cluster sum with germ sizes j and intensities $\lambda_{N,j}$ satisfies

$$\mathbb{E}\widetilde{W}_N = \sum_j j \lambda_{N,j}, \quad \text{Var}(\widetilde{W}_N) - \mathbb{E}\widetilde{W}_N = \sum_j j(j-1) \lambda_{N,j} =: K_{2,N}.$$

These identities follow by independence of the Poisson numbers of germs of each size. They concern the target at this point.

Lemma 3.2 (Exact pair energy). *For $N > 2\tau$,*

$$K_{2,N} = \sum_{v \neq 0} \frac{M(N/H(v))}{N} \beta_N(v). \quad (3.5)$$

In particular $0 \leq K_{2,N} \leq \mathcal{R}_N$ and

$$0 \leq \mathcal{R}_N - K_{2,N} \ll_{\varepsilon, \tau} \frac{(1 + \log N)^{2b_0}}{N}. \quad (3.6)$$

Proof. For two translates of an arc of mass $\tau/N < 1/2$, the overlap is exactly

$$m(\chi_N(s)^{-1} J_N \cap \chi_N(t)^{-1} J_N) = \frac{1}{N} (\tau - N \|\chi_N(s/t)\|_{\mathbb{T}})_+.$$

Every ordered distinct pair of multipliers has a unique form $(s, t) = (gA(v), gB(v))$, where $g \in \mathcal{D}$ and $v \neq 0$. The phase difference does not depend on g . For fixed v , the allowed pairs (g, d) satisfy $g \in \mathcal{D}$, $d \geq 2$, $(d, q) = 1$, and $gd \leq N/H(v)$. Unique decomposition of the integer gd counts them *exactly* by $M(N/H(v))$. Summing the ordered offspring pairs gives (3.5).

For $H(v) \leq N$,

$$0 \leq \frac{1}{H(v)} - \frac{M(N/H(v))}{N} \leq \frac{1 + B_{\mathcal{E}}(N/H(v))}{N}.$$

There are $O_{\mathcal{E}}((1 + \log N)^{b_0})$ such vectors and $B_{\mathcal{E}}(N/H(v)) \ll_{\mathcal{E}} (1 + \log N)^{b_0}$. The omitted height tail is bounded by (2.8). Multiply by $\beta_N(v) \leq \tau$ to prove (3.6). $\square$

The unique decomposition eliminates both the regular-core density and the common smooth multiplier. It will also underlie the limiting primitive-tuple formulas in Section 8.

3.2 The deficit and an anchor coupling

The point-minus-germ deficit and the first-active-site coupling already appear in *Arithmetic clusters* [25, Section 6.4, Theorem 6.1, equations (6.8)–(6.9)], together with the void lower bound. We state the finite-cluster mechanism in a general form and retain the exponential upper bound instead of its linear relaxation $2D$. The following proof makes this inherited argument available for every moving-environment kernel and every deterministic restriction.

Theorem 3.3 (Two-sided Poisson control by the deficit). *Consider a Poisson random measure on the space of nonempty finite point configurations over a standard Borel observation space, and superpose its configurations to form the cluster field. Let λ_j be the intensity of configurations with exactly $j \geq 1$ descendants. Assume the total point mean $m = \mu(\text{observation space}) = \sum_{j \geq 1} j\lambda_j$ is finite, where μ is the point mean measure. Set $\Lambda = \sum_{j \geq 1} \lambda_j$, $D = m - \Lambda$, and $\lambda_{\geq 2} = \sum_{j \geq 2} \lambda_j$. Then*

$$d_{\text{TV}}(\mathcal{L}(\text{field}), \text{PPP}(\mu)) \leq 1 - \exp[-D - \lambda_{\geq 2}] \leq 1 - e^{-2D}, \quad (3.7)$$

$$d_{\text{TV}}(\mathcal{L}(\text{count}), \text{Pois}(m)) \geq e^{-m}(e^D - 1). \quad (3.8)$$

In particular, for the actual arithmetic field,

$$d_{\text{TV}}(\mathcal{L}(\Xi_N), \text{PPP}(\mu_N)) \leq \varepsilon_{\text{H}}(N) + 1 - e^{-2D_N}, \quad (3.9)$$

$$d_{\text{TV}}(\mathcal{L}(W_N), \text{Pois}(m_N)) \geq e^{-m_N}(e^{D_N} - 1) - \varepsilon_{\text{H}}(N). \quad (3.10)$$

These bounds apply to any deterministic restriction, with its own exact mean and deficit. They concern Poisson comparison on the common prelimit space, not total variation to diffuse positions.

Proof. Choose a Borel injection of the observation space into $[0, 1]$ and retain one atom with least image in each nonempty finite offspring configuration. This is a measurable selection; multiplicities at the selected location do not affect the chosen point. Finite m gives $\Lambda \leq m < \infty$ and finitely many active germs almost surely. These anchors form a Poisson field with intensity $\mu_0 \leq \mu$. Add an independent Poisson field of intensity $\mu - \mu_0$, whose total mass is D . The completed anchor field has law $\text{PPP}(\mu)$. If no multi-descendant germ occurs and no point was added, both configurations agree. These two absence events are independent and have probability $e^{-\lambda_{\geq 2}}e^{-D}$. Since $\lambda_{\geq 2} \leq D$, this proves (3.7). The original field is empty exactly when there is no active germ, of probability $e^{-\Lambda} = e^{-m+D}$. Comparing to e^{-m} proves (3.8). Finally use (2.3) and contraction under restrictions. $\square$

In particular the deficit bound implies the ordered-sibling-pair bound used in the uniform criterion, since

$$2D_N \leq \sum_j j(j-1)\lambda_{N,j} = K_{2,N} \leq \mathcal{R}_N.$$

A large cluster contributes linearly to D_N and quadratically to $K_{2,N}$. These bounds give comparability to D_N , up to constants depending on bounded m_N , without comparing unbounded moments.

As $1 - e^{-2D_N} \leq 2D_N \leq K_{2,N} \leq \mathcal{R}_N$, Theorem 3.3 proves (3.1). Projection to counts and a coupling of Poisson variables of means m_N and τ prove (3.2).

3.3 A lower bound from one primitive channel

Lemma 3.4 (A single-channel deficit). *For every $v \neq 0$,*

$$D_N \geq \frac{M(N/H(v))}{N} \beta_N(v).$$

Proof. Replace v by $-v$ if necessary so that $A = A(v) > B = B(v)$, hence $H(v) = A$. At a fixed regular core, partition the eligible smooth multipliers into chains with ratios powers of A/B , ordered increasingly. A multiplier s has its immediately preceding chain member sB/A if and only if $A \mid s$. That predecessor is smaller and thus is eligible whenever s is. This uses $(A, B) = 1$, not saturation of a relation lattice.

For any finite ordered family of sets (T_i) ,

$$\sum_i m(T_i) - m\left(\bigcup_i T_i\right) \geq \sum_{i \geq 2} m(T_i \cap T_{i-1}).$$

Add the sets successively and observe that the previous union contains T_{i-1} . Apply this within each chain to $T_s = \chi_N(s)^{-1} J_N$; merging different chains can only increase the total deficit. Every consecutive overlap has mass $\beta_N(v)/N$, even if a long chain winds around the circle. Finally the pairs (s, d) with $A \mid s$, $sd \leq N$, and regular core $d \geq 2$ are counted by $M(N/A)$. Summing proves the claim. $\square$

Since $\mathbb{P}(\widetilde{W}_N = 0) = e^{-\Lambda_N} = e^{-m_N + D_N}$ and $m_N \leq \tau$, the void event and (2.3) prove (3.3). The bound counts all common-multiplier chains; it is not restricted to one chosen pair.

3.4 Direct second moments of the original arithmetic field

The following argument is independent of total-variation moment transfer. It uses only Haar characters and a count of powers.

Proposition 3.5 (Uniform covariance comparison). *For arbitrary bounded real Borel tests f, g on K with sup norms at most one,*

$$\left| \text{Cov}(\Xi_N(f), \Xi_N(g)) - \text{Cov}(\widetilde{\Xi}_N(f), \widetilde{\Xi}_N(g)) \right| \ll_{\mathcal{E}, \tau} N^{-1/2} (1 + \log N)^{2b_0+2}. \quad (3.11)$$

The estimate is uniform in ξ_N and in the tests, which may depend on N .

Proof. Extend the tests by zero outside K . Put $p_N = \tau/N$ and

$$X_n(f) = \mathbf{1}_{\{F_N(n) \in J_N\}} f(n/N, N \text{Arg } F_N(n)/(2\pi)).$$

The pairs of regular cores have three types.

Two linearly independent regular valuation vectors. The two regular phases are independent Haar. Indeed, every nontrivial integer character of the pair has expectation zero: its exponent at some regular prime is nonzero. This is the standard Haar character criterion, as in [21, arXiv:1702.01470v2, Lemma 2.1], applied to the regular coordinates. Deterministic exceptional rotations preserve independence. These pairs have zero covariance, just as distinct cores do in the target.

One and the same core. For $n = sd$, $m = td$, the true uncentred mixed moment of the two site contributions is exactly the mixed offspring contribution of one target germ. This includes $n = m$. The target covariance is that uncentred contribution, whereas the actual covariance also subtracts the product of the two means. The total absolute discrepancy of this type is at most

$$p_N^2 \sum_{\substack{d \geq 2 \\ (d, q)=1}} |\{s \in \mathcal{D} : sd \leq N\}|^2 \leq p_N^2 N \sum_{s, t \in \mathcal{D}} \frac{1}{\max(s, t)} \leq \frac{\tau^2}{N} Z(1/2)^2.$$

No count of individual sites is replaced by an independence assertion here.

Distinct but proportional regular valuation vectors. If the nonzero integer vectors of d and e are proportional, there are a common integer $c \geq 2$ and distinct positive integers a, a' such that $d = c^a$, $e = c^{a'}$. To see this, divide one vector by the gcd of its entries; an integer vector

proportional to this primitive vector is an integer multiple of it. Since $a \neq a'$, their maximum is at least two, so $c \leq \sqrt{N}$. Each exponent is at most $\log_2 N$. For each core, at most $B_{\mathcal{E}}(N)$ exceptional multipliers can occur. Hence the number of ordered site pairs of this type is at most

$$C_{\mathcal{E}} \sqrt{N} (1 + \log N)^{2b_0+2}.$$

Each mixed absolute moment is at most p_N , and each product of means at most p_N^2 . Their total contribution is therefore $O_{\mathcal{E},\tau}(N^{-1/2}(1 + \log N)^{2b_0+2})$. These types exhaust all pairs. Summing proves (3.11). $\square$

Taking $f = g = 1$, and then using (3.6), proves (3.4). In particular the actual variances are bounded uniformly in the phases. Higher arithmetic moments are not covered by this pair argument.

3.5 Completion of the criterion

If $\beta_N(v) \rightarrow 0$ for every fixed nonzero v , the summable majorant $\tau/H(v)$ gives $\mathcal{R}_N \rightarrow 0$, hence the field and count Poisson conclusions. Conversely, if the count converges in total variation to $\text{Pois}(\tau)$, (3.3) forces $\beta_N(v) \rightarrow 0$ for each fixed v , because $M(N/H(v))/N \rightarrow 1/H(v)$. This last assertion follows from $B_{\mathcal{E}}(X) = O_{\mathcal{E}}((1 + \log X)^{b_0})$ with $H(v)$ fixed. The nonnegative part in (2.5) tends to zero exactly when the displayed liminf separation condition holds. Finally (3.4) equates vanishing variance excess with $\mathcal{R}_N \rightarrow 0$. This proves Theorem 3.1 in full.

Remark 3.6 (What “every fixed relation” means). There is no unproved interchange with all integer vectors of size growing with N . Those vectors are covered by the summable height tail. Large-order exact roots of unity can therefore be invisible: their new exact relations have large height. Conversely, even one fixed channel with a persistent positive target-scale overlap forces a positive void-probability discrepancy.

4 Spanning forests and finite certificates

For a finite vertex set S and nonnegative symmetric weights b_{ij} on its unordered pairs, let

$$\mathsf{T}(S; b) = \max_{F \text{ a forest on } S} \sum_{\{i,j\} \in F} b_{ij}. \quad (4.1)$$

It is zero when $|S| \leq 1$. Since weights are nonnegative, one may equivalently take a spanning tree of the complete graph, using zero-weight edges between components. No random graph is being added to the multiplicative model.

The exact tree sieve of Naiman–Wynn [19], in the form proved by Dohmen [7, Corollary 2], applies when every endpoint intersection along a tree path is contained in each intermediate set. Translates of a common interval, ordered by their left endpoints, satisfy this condition on a path. For unequal interval lengths, the elimination below constructs a suitable forest; combined with the Hunter–Worsley bound [12,30], it yields the optimized form we need.

Lemma 4.1 (Tree-sieve specialization for intervals). *Let $T_1, \dots, T_k$ be bounded intervals, not necessarily of equal length. Set $b_{ij} = \text{Leb}(T_i \cap T_j)$. Then*

$$\text{Leb}\left(\bigcup_i T_i\right) = \sum_i \text{Leb}(T_i) - \mathsf{T}(\{1, \dots, k\}; b). \quad (4.2)$$

For arbitrary measurable sets the right side is only an upper bound in general. The equality also holds for finitely many circle arcs if their union has a gap, using arc length for every measure.

Proof. For any forest F and point z , the induced graph on the $r(z)$ sets containing z has at most $(r(z) - 1)_+$ edges. Integrating gives

$$\sum_{\{i,j\} \in F} \text{Leb}(T_i \cap T_j) \leq \sum_i \text{Leb}(T_i) - \text{Leb}\left(\bigcup_i T_i\right).$$

This is the usual forest form of the second-order union bound.

Discard empty intervals. For the remaining intervals, remove an interval $T_i = [l_i, r_i]$ with smallest right endpoint. Its intersections with all remaining intervals have the same right endpoint r_i when nonempty, and are nested. Hence

$$\text{Leb}\left(T_i \cap \bigcup_{j \neq i} T_j\right) = \max_{j \neq i} \text{Leb}(T_i \cap T_j).$$

Choose an edge to a maximizing remaining interval if this maximum is positive, and continue by induction. Each edge joins a removed vertex to a later vertex, so the chosen edges form a forest. Their sum equals the total overlap deficit. The preceding upper bound on every forest proves equality. Endpoint conventions do not change any length. Cutting circle arcs at a gap reduces them to intervals without changing lengths or intersections. $\square$

The chosen forest also exhibits the tree-sieve condition: for each point, the intervals containing it form a connected subgraph whenever nonempty. The proof is included to expose the interval and uncovered-arc hypotheses, not to claim a new general inclusion–exclusion identity.

Lemma 4.2 (Positive thinning formula and stability). *Independently retain vertex i with probability $p_i \in [0, 1]$, and denote the random set by S_p . For the intervals of Lemma 4.1,*

$$\int \left[1 - \prod_i (1 - p_i \mathbf{1}_{T_i}(z)) \right] dz = \sum_i p_i \text{Leb}(T_i) - \mathbb{E}_p \mathsf{T}(S_p; b). \quad (4.3)$$

For two weight arrays on the same vertices,

$$|\mathsf{T}(S; b) - \mathsf{T}(S; b')| \leq \sum_{\{i,j\} \subset S} |b_{ij} - b'_{ij}|. \quad (4.4)$$

The last inequality also holds after averaging over any common independent thinning.

Proof. The integrand in (4.3) is the probability that at least one retained interval contains z . Fubini and (4.2) give the formula. For (4.4), evaluate the difference on a maximizing forest for one array, bound the sum of changed edge weights, and exchange the arrays. $\square$

The Bernoulli variables in (4.3) are auxiliaries for evaluating a functional. They are not extra marks in the observed arithmetic field. Neither this expectation nor a general Laplace functional is claimed to have a polynomial-time exact evaluation; the unthinned forest maximum does.

4.1 The exact finite deficit

Theorem 4.3 (Exact finite forest expression). *For $d \geq 2$, $(d, q) = 1$, let $S_{N,d} = \{s \in \mathcal{D} : sd \leq N\}$. For all sufficiently large N , uniformly in the phases,*

$$D_N = \frac{1}{N} \sum_{\substack{2 \leq d \leq N \\ (d,q)=1}} \mathsf{T}(S_{N,d}; \beta_N(\mathbf{v}(s) - \mathbf{v}(t))). \quad (4.5)$$

The same formula with $S_{N,d}$ restricted to $sd/N \in U$ gives the deficit for a deterministic spatial mask $U \subset [0, 1]$.

Proof. At a fixed core, every angular target has mass τ/N . The sum of their masses is at most $\tau B_{\mathcal{E}}(N)/N < 1$ for all large N , since $B_{\mathcal{E}}(N) = O_{\mathcal{E}}((1 + \log N)^{b_0})$. Thus their union has a gap, regardless of phase motion or wrapping around the circle. Their pair intersection masses are exactly $\beta_N(\mathbf{v}(s) - \mathbf{v}(t))/N$ for $N > 2\tau$. The point-minus-germ mass at that core is the sum of lengths minus the length of the union. Lemma 4.1, homogeneity of the maximum, and summation over cores give (4.5). Removing vertices proves the mask assertion. $\square$

The condition on the total arc mass cannot be dropped from this argument for arbitrary circular families: a fully covered circle has no cut reducing the family to intervals. Here it is automatic and uniform because the exceptional set is fixed.

Proposition 4.4 (Finite-height certificate and compressed core sum). *Delete all multipliers $s > H$ from the target and let $D_N^{(H)}$ be the resulting deficit. Then*

$$0 \leq D_N - D_N^{(H)} \leq \tau T_H, \quad T_H = \delta^{-1} - \sum_{s \leq H, s \in \mathcal{D}} \frac{1}{s}. \quad (4.6)$$

For the full prefix, list $\mathcal{D} \cap [1, \min(H, N)]$ as $s_1 < \dots < s_k$, set $s_{k+1} = \infty$, and put

$$c_j(N) = \#\{d \geq 2 : (d, q) = 1, N/s_{j+1} < d \leq N/s_j\}.$$

Then the exact finite expression is

$$D_N^{(H)} = \frac{1}{N} \sum_{j=1}^k c_j(N) \mathbb{T}(\{s_1, \dots, s_j\}; \beta_N(\mathbf{v}(s) - \mathbf{v}(t))). \quad (4.7)$$

The $c_j(N)$ are evaluated by finite inclusion–exclusion over the divisors of q . Thus the certificate uses at most k finite forest problems, rather than a sum over N cores or over all multiplier subsets.

Proof. Couple full and truncated targets using the same germs. If the number of active descendants of a germ changes from r to $r' \leq r$, its deficit changes from $(r-1)_+$ to $(r'-1)_+$ by a number in $[0, r-r']$. Integration gives the first inequality and bounds the change by the removed first mass. That mass is at most

$$\frac{\tau}{N} \sum_{s > H} \#\{d \geq 2 : (d, q) = 1, sd \leq N\} \leq \tau \sum_{s > H} \frac{1}{s}.$$

The Euler product in (2.10) gives the exact value of this tail. For (4.7), the eligible multiplier set is precisely the first j elements when $N/s_{j+1} < d \leq N/s_j$. Group identical graphs in (4.5). $\square$

For rationally supplied finite overlap coefficients, the geometric certificate can be evaluated in rational arithmetic. A floating-point phase list does not certify exact phase relations. A numerical error bar for the original arithmetic field must additionally evaluate or rigorously bound the finite budget (2.12); the leading asymptotic exponent alone does not provide that error bar.

For example, (4.6) and the target void identity give

$$e^{-m_N + D_N^{(H)}} - \varepsilon_H(N) \leq \mathbb{P}(W_N = 0) \leq e^{-m_N + \min(m_N, D_N^{(H)}) + \tau T_H} + \varepsilon_H(N). \quad (4.8)$$

Both probabilities can be clipped to $[0, 1]$.

5 All subsequential marked limits

The uniform criterion does not require a limit law to be chosen first. When clusters persist, the following profile identifies the possible limits. Diagonal compactness and the structure of finitely generated abelian groups are classical; their role here is to classify precisely the character displacements realizable at scale $1/N$, including the integer torsion.

Definition 5.1 (Effective microscopic profile). A sequence has profile (L, h) if $L \leq \mathbb{Z}^\mathcal{E}$ is a subgroup, $h : L \rightarrow \mathbb{R}$ is additive, and

$$r_N(v) \rightarrow h(v) \quad (v \in L), \quad |r_N(v)| \rightarrow \infty \quad (v \notin L). \quad (5.1)$$

The subgroup records characters remaining within bounded distance after magnification by N . It is not the exact relation lattice at finite N , or necessarily the relation lattice of a coarse limiting phase vector.

Proposition 5.2 (Compactness and realizability). *Every deterministic phase sequence has a subsequence satisfying (5.1). The possible subgroups L are precisely those for which $\text{Tor}(\mathbb{Z}^\mathcal{E}/L)$ is cyclic, including the trivial group. For every such L and every additive $h : L \rightarrow \mathbb{R}$, the profile is realized by a fixed-base first-order family.*

Proof. Compactify $\mathbb{R}$ by one point ∞ , where convergence to that point means absolute value tending to infinity. This is a compact metrizable space. A diagonal subsequence makes $r_N(v)$ converge in this compactification for every member of the countable group $\mathbb{Z}^\mathcal{E}$. Define L to be the vectors with finite limits. If v, w have finite limits, then both corresponding angular turns are $O(N^{-1})$. For all large N no branch crossing occurs in their sum, so

$$r_N(v + w) = r_N(v) + r_N(w), \quad r_N(-v) = -r_N(v).$$

It follows that L is a subgroup and its finite limit h is additive. All other magnitudes tend to infinity, by the definition of the compactification.

For necessity of the torsion condition let

$$S = \{u \in \mathbb{Z}^\mathcal{E} : ku \in L \text{ for some } k \geq 1\}$$

be the saturation of L . Pass, if needed, to a further subsequence with $\xi_N \rightarrow \xi_*$. For $u \in S$, choose k with $ku \in L$. Then $(\xi_*^u)^k = 1$. The character $u \mapsto \xi_*^u$ has kernel exactly L on S . Inclusion of L is immediate. For the converse, if $\xi_*^u = 1$, then $\xi_N^u \rightarrow 1$ and, for all large N , $r_N(ku) = kr_N(u)$. Thus $r_N(u) \rightarrow h(ku)/k$, so $u \in L$. The finite group $S/L = \text{Tor}(\mathbb{Z}^\mathcal{E}/L)$ therefore embeds into the circle and must be cyclic.

For sufficiency, the structure theorem for finitely generated abelian groups writes $\mathbb{Z}^\mathcal{E}/L \cong \mathbb{Z}^d \oplus C_m$ with C_m cyclic. Choose real numbers $\alpha_1, \dots, \alpha_d$ such that $1, \alpha_1, \dots, \alpha_d$ are linearly independent over $\mathbb{Q}$. The map

$$(n_1, \dots, n_d, j) \mapsto \exp \left(2\pi i \left[\sum_i n_i \alpha_i + j/m \right] \right)$$

is injective (omit the last term for trivial torsion). Composing with the quotient map gives a character ξ_0 whose kernel is exactly L . Extend h to a real linear form $\eta \cdot v$ on $\mathbb{R}^\mathcal{E}$: an integer basis of L is linearly independent over $\mathbb{R}$, so this is a finite linear system of full row rank. Now put $\xi_{N,p} = \xi_{0,p} e^{2\pi i \eta_p / N}$. Inside L the scaled turns tend to h ; outside it the base character stays away from 1. This realizes the required profile. $\square$

Remark 5.3 (A constraint that mere nonsaturation does not express). The subgroup $2\mathbb{Z} \oplus 2\mathbb{Z} \leq \mathbb{Z}^2$ cannot occur as an effective profile: its torsion quotient is $C_2 \oplus C_2$, not cyclic. In contrast $\mathbb{Z}(2, -2)$ is allowed, since its torsion quotient has a single C_2 factor. We do not replace an allowable subgroup by its saturation; the proposition instead characterizes which unsaturated subgroups are realizable.

5.1 The cluster kernel attached to a profile

Partition $\mathcal{D}$ by

$$s \sim_L t \iff \mathbf{v}(s) - \mathbf{v}(t) \in L.$$

For each class C choose its smallest member s_C and set

$$a_s = h(\mathbf{v}(s) - \mathbf{v}(s_C)) \quad (s \in C).$$

For $x \in (0, 1]$, $z \in \mathbb{R}$, define

$$G_C(x, z) = \sum_{\substack{s \in C \\ sx \leq 1}} \mathbf{1}_I(z + a_s) \delta_{(sx, z + a_s)}, \quad r_C = G_C(K). \quad (5.2)$$

Take independent Poisson processes of intensity $\delta dx dz$ on the domains $\{r_C > 0\}$, and sum their offspring configurations. Denote the resulting marked field by $\Xi_{L,h}$ and its spatial projection by $X_{L,h}$.

Changing the anchor in a class translates the common z coordinate and preserves its intensity. The construction therefore depends only on (L, h) and the fixed prime set. Its marked Laplace functional is

$$\mathbb{E} e^{-\langle f, \Xi_{L,h} \rangle} = \exp \left\{ -\delta \sum_C \int_0^1 \int_{\mathbb{R}} (1 - e^{-\langle f, G_C(x,z) \rangle}) dz dx \right\}, \quad f \in C(K), \quad f \geq 0. \quad (5.3)$$

Finiteness and the mean identity are proved below.

Theorem 5.4 (Arbitrary-array marked limit). *Under (5.1) and the input (2.3),*

$$\Xi_N \implies \Xi_{L,h}$$

weakly as finite point measures on K . The limit has mean $du dw$ on $(0, 1) \times I$, is spatially simple, and has no point on any fixed spatial boundary. Its total count converges in total variation to

$$W_{L,h} = \sum_{j \geq 1} j P_j, \quad P_j \sim \text{Pois}(\lambda_j) \text{ independently}, \quad \lambda_j = \delta \sum_C \int_0^1 \int_{\mathbb{R}} \mathbf{1}_{\{r_C(x,z)=j\}} dz dx. \quad (5.4)$$

Here $\sum_j j \lambda_j = \tau$. Write $\Lambda_g = \sum_j \lambda_j$ for the active-germ intensity of a profile; then $\delta \tau \leq \Lambda_g \leq \tau$. For the same terminal- N model and target, the prefix-count paths converge in J_1 to $t \mapsto X_{L,h}([0, t])$. Counts on any fixed finite family of intervals converge jointly in total variation. Every subsequential weak limit of the marked fields is represented by an admissible profile from Proposition 5.2, and every such profile occurs. Distinct profiles need not give distinct observable laws at one fixed target width.

Proof. Put $S_H = \mathcal{D} \cap [1, H]$, $k_H = |S_H|$, and $T_H = \sum_{s > H, s \in \mathcal{D}} s^{-1}$. Deleting multipliers $s > H$ from the target changes it with probability at most

$$\frac{\tau}{N} \sum_{s > H} \#\{d \geq 2 : (d, q) = 1, sd \leq N\} \leq \tau T_H. \quad (5.5)$$

The proposed limit has total expected mass

$$\delta \sum_C \int r_C dx dz = \delta \tau \sum_{s \in \mathcal{D}} \frac{1}{s} = \tau.$$

Thus its active-germ intensity is finite, it has finitely many points almost surely, and its multiplier tail costs at most $\delta \tau T_H$. The class containing 1 gives active-germ mass at least $\delta \tau$.

Fix H . If a class is represented in S_H , its chosen smallest anchor also lies in S_H . At each core use the *moving* chart

$$u = \chi_N(s_C)^{-1} \exp(2\pi iz/N).$$

For $s \in C \cap S_H$, the relative scaled shift is $a_{N,s} = r_N(\mathbf{v}(s) - \mathbf{v}(s_C)) \rightarrow a_s$. All relevant z therefore lie in a fixed bounded set. Distinct represented classes have anchor separation tending to infinity after scaling by N , so these charts are disjoint for all large N . This remains true when their unscaled phases approach the same point. Haar measure is exactly dz/N in each chart, and the recorded phase is exactly $z + a_{N,s}$ on an active target. Only the group-relative shifts, not absolute convergence of the anchors, are used.

Elementary inclusion–exclusion over the primes of q gives

$$\#\{2 \leq d \leq Y : (d, q) = 1\} = \delta Y + O_{\mathcal{E}}(1), \quad Y \geq 0.$$

Consequently the core measures $N^{-1} \sum_{d \geq 2, (d, q) = 1} \delta_{d/N}$ on $[0, 1]$ converge weakly to δdx . At fixed H , integrate the Poisson Laplace exponent of (5.2). Membership changes caused by $a_{N,s} - a_s$ have angular length at most $2|a_{N,s} - a_s|$. Outside them, continuous test functions change uniformly by a quantity tending to zero. The remaining core integrand is piecewise continuous, with finitely many spatial breaks $x = 1/s$. The core estimate proves the finite- H marked limit.

For clarity, count total variation has a separate finite bound. Write $\omega_N(H) = \max_{s \in S_H} |a_{N,s} - a_s|$. Once the charts are disjoint, translation coupling costs at most $2Z(1)\omega_N(H)$. Partition x at the k_H reciprocal multipliers; the total angular length of the active charts on each piece is at most τk_H . The core-counting errors in the jump intensities sum to at most $C_{\mathcal{E}}\tau k_H^2/N$. Coupling their independent Poisson variables and then restoring both tails gives

$$d_{\text{TV}}(\mathcal{L}(W_N), \mathcal{L}(W_{L,h})) \leq \varepsilon_H(N) + 2\tau T_H + 2Z(1)\omega_N(H) + C_{\mathcal{E}}\tau k_H^2/N. \quad (5.6)$$

Its threshold depends on H and the particular profile sequence. Send $N \rightarrow \infty$ first, then $H \rightarrow \infty$. The same tails complete marked weak convergence. The Poisson exponential identity on the active germ space gives (5.4).

The mean measure follows by $u = sx$, $w = z + a_s$ and $\delta \sum_s 1/s = 1$. Within a germ distinct multipliers give distinct positive positions. Between different germs, coincidences $sx = ty$ lie in null sets for the diffuse core coordinates. Finiteness gives spatial simplicity. Mean absolute continuity excludes fixed boundaries and angular endpoints.

A finite simple configuration without points at $0, 1$ is a continuity point of the counting-path map into J_1 . Under a convergent deterministic sequence of finite point measures on the compact space, the integer total mass is eventually equal. Match the finitely many ordered spatial points with the convergent points; an increasing piecewise-linear homeomorphism fixing 0 and 1 aligns them, with displacement tending to zero. Fixed interval boundaries have zero limiting mass, giving convergence of the associated count vector. On a countable discrete space, weak convergence is total variation: retain a finite set of limiting probability at least $1 - \epsilon$ and sum its atom-probability convergence.

The exact mean bound $\mathbb{E}W_N \leq \tau$ gives tightness of the fields on the compact space K . Proposition 5.2 gives a profile refinement of every subsequence. Applying the proved limit on that refinement represents every weak limit. Its realization assertion proves the converse. $\square$

6 Rank, multiple scales, and growing displacements

For a profile write

$$\beta(v) = \begin{cases} (\tau - |h(v)|)_+, & v \in L \setminus \{0\}, \\ 0, & v \notin L. \end{cases} \quad (6.1)$$

Then $\beta_N(v) \rightarrow \beta(v)$ for every $v \neq 0$, and $\mathcal{R}_N \rightarrow \sum_{v \neq 0} \beta(v)/H(v)$ by summability.

Corollary 6.1 (The effective rank, not a coarse rank). *The profile limit is Poisson if and only if $|h(v)| \geq \tau$ for all $0 \neq v \in L$. If $L = \mathbb{Z}v_0$ has rank one, this reduces to $|h(v_0)| \geq \tau$ for an actual integer generator. If $\text{rank } L \geq 2$, the limit is non-Poisson and its jump-size support is unbounded. These statements concern the effective subgroup L in (5.1).*

Proof. Apply Theorem 3.1, or observe that two descendants of a germ overlap on a positive-measure angular set exactly when their displacement difference has magnitude less than τ . For $\text{rank } L \geq 2$, two independent vectors have a nonzero integer combination u with $|h(u)|$ arbitrarily small: either a kernel vector exists, or pigeonhole the fractional parts of the ratio of their two real images. Fix $m \geq 2$ and choose $(m-1)|h(u)| < \tau$. The m distinct integers

$$s_k = A(u)^k B(u)^{m-1-k}, \quad 0 \leq k < m,$$

lie in one class. Their shifts are consecutive with step $h(u)$, and $s_k \leq H(u)^{m-1}$. On a core interval of length $H(u)^{-(m-1)}$, their common angular intersection has length $\tau - (m-1)|h(u)|$. Consequently

$$\sum_{j \geq m} \lambda_j \geq \delta H(u)^{-(m-1)} (\tau - (m-1)|h(u)|) > 0. \quad (6.2)$$

There is no lower bound uniform in the displacement form, since $H(u)$ may be very large. $\square$

A coarse limiting phase vector can have many relations that escape before the target scale is reached. The rank in this corollary is the effective rank at the target scale.

6.1 First-order families and stability

For a fixed base environment ξ_0 and $\eta \in \mathbb{R}^{\mathcal{E}}$, suppose

$$\xi_{N,p} = \xi_{0,p} \exp\left(\frac{2\pi i}{N}(\eta_p + e_{N,p})\right), \quad \max_{p \in \mathcal{E}} |e_{N,p}| \rightarrow 0. \quad (6.3)$$

Then $L = L_0 = \{v : \xi_0^v = 1\}$ and $h(v) = \eta \cdot v$ on L_0 . For each fixed vector in L_0 , the centered scaled turn eventually has no winding and converges to that value; outside L_0 the base character is separated from 1. This is a direct instance of Theorem 5.4.

The class shifts may equivalently be chosen as $a_s = \eta \cdot \mathbf{v}(s)$ rather than relative to the smallest member. Their difference is constant within a class and is removed by translating the germ coordinate. Two tangent vectors with the same restriction to L_0 therefore give the same full marked law.

Proposition 6.2 (Stability of counts and unmarked spatial laws). *For fixed base environment and two tangent vectors η, η' , let $X_\eta = X_{L_0, \eta}$ be the spatial field constructed from (5.2). Then*

$$d_{\text{TV}}(\mathcal{L}(X_\eta), \mathcal{L}(X_{\eta'})) \leq \min \left\{ 1, 2 \sum_{p \in \mathcal{E}} \frac{|\eta_p - \eta'_p|}{p-1} \right\}.$$

The same bound holds for total counts. It is not asserted for the full angularly marked law.

Proof. Use the same Poisson germs on the union of both active domains, a finite-intensity space. The spatial configurations agree unless an interval membership differs. This probability is at most

$$2\delta \sum_{s \in \mathcal{D}} \frac{|(\eta - \eta') \cdot \mathbf{v}(s)|}{s} \leq 2\delta \sum_p |\eta_p - \eta'_p| \sum_{s \in \mathcal{D}} \frac{v_p(s)}{s}.$$

The geometric-series identity $\delta \sum_s v_p(s)/s = (p-1)^{-1}$ proves the result. Angular marks need not agree under this coupling, even when all activations do. $\square$

6.2 Intermediate scales and growing displacements

Proposition 6.3 (A finite hierarchy of phase scales). *Suppose in angular turns*

$$\xi_{N,p} = \xi_{0,p} \exp \left(2\pi i \left[\sum_{j=1}^k N^{-\alpha_j} \eta_{j,p} + N^{-1} \eta_{*,p} + o(N^{-1}) \right] \right), \quad 0 < \alpha_1 < \dots < \alpha_k < 1,$$

with fixed vectors. Let $L_0 = \{v : \xi_0^v = 1\}$. The effective profile is

$$L = L_0 \cap \bigcap_{j=1}^k \ker(\eta_j \cdot), \quad h(v) = \eta_* \cdot v \quad (v \in L). \quad (6.4)$$

Proof. Outside L_0 the unscaled character stays away from 1. Inside L_0 , all unscaled perturbations tend to zero, so there is eventually no branch crossing for a fixed vector. If j is the first index with $\eta_j \cdot v \neq 0$, the scaled turn is asymptotic to $N^{1-\alpha_j} \eta_j \cdot v$, whose absolute value diverges. If all those forms vanish, the finite limit is $\eta_* \cdot v$. $\square$

A common intermediate drift. Take $\mathcal{E} = \{2, 3\}$ and

$$F_N(2) = \exp(2\pi i N^{-1/2}), \quad F_N(3) = \exp(2\pi i [N^{-1/2} + c/N]). \quad (6.5)$$

The coarse limit is $(1, 1)$ with relation lattice $\mathbb{Z}^2$. Nevertheless $L = \mathbb{Z}(-1, 1)$ and $h(-1, 1) = c$. Its rank-one channel has $(A, B) = (3, 2)$, and the limit is Poisson precisely when $|c| \geq \tau$. The common drift is not $O(N^{-1})$ and does not satisfy (6.3). The example demonstrates why the rank must be measured at the observation scale.

Growing displacements. More generally let $R_N \rightarrow \infty$, $R_N = o(N)$, and $\xi_{N,p} = \xi_{0,p} \exp(2\pi i R_N \eta_p / N)$. For every fixed character the profile is

$$L = L_0 \cap \ker(\eta \cdot), \quad h = 0 \text{ on } L.$$

Thus, whenever $\eta \cdot$ is injective on L_0 , the arithmetic field itself has the Poisson comparison in Theorem 3.1, at every such rate R_N . No uniform extension of the first-order error in (6.3) is being assumed. If a nonzero kernel remains, its exact effective relations instead produce the corresponding zero-displacement cluster law.

Nonsaturated example. At one exceptional prime with base phase -1 , the first-order subgroup is $2\mathbb{Z}$ and the displacement of its generator is 2η . Its threshold remains $2|\eta| \geq \tau$, not $|\eta| \geq \tau$. At the exceptional prime 2, taking $\eta_2 = 3\tau/5$ gives a Poisson limit under the correct lattice $2\mathbb{Z}$, whereas saturation would falsely predict clustering. Intermediate-scale reduction and torsion are different operations.

6.3 Two directions with no finite-scale exact relation

Take

$$\mathcal{E} = \{2, 3\}, \quad \xi_0 = (1, 1), \quad \tau = 1, \quad \eta = (\sqrt{2}, \sqrt{3}), \quad e_N = 0. \quad (6.6)$$

At each N , the exceptional phase vector has *no nonzero exact relation*. Indeed

$$m\sqrt{2} + n\sqrt{3} = kN, \quad m, n, k \in \mathbb{Z}$$

forces $mn = 0$ upon squaring, since otherwise $\sqrt{6}$ would be rational. The remaining irrational square root then forces $m = n = k = 0$.

The two one-prime models in isolation have displacement greater than the target width and therefore Poisson limits. Together, however, the multipliers 2 and 3 have shift difference $\sqrt{3} - \sqrt{2} < 1$. A common core can activate both. The effective lattice is $\mathbb{Z}^2$, so Corollary 6.1 gives a non-Poisson limit and unbounded jump support. The single-channel bound of Lemma 3.4, with $v = (-1, 1)$ and $H(v) = 3$, gives

$$1 - \Lambda_g \geq \frac{1 - \sqrt{3} + \sqrt{2}}{3} > 0. \quad (6.7)$$

Indeed, $D_N \geq M(N/3)\beta_N(-1, 1)/N$, where $M(N/3)/N \rightarrow 1/3$ and $\beta_N(-1, 1) \rightarrow 1 - \sqrt{3} + \sqrt{2}$. The target void probabilities converge to $e^{-\Lambda_g}$ and $m_N \rightarrow 1$, so $D_N \rightarrow 1 - \Lambda_g$. This uses all common-multiplier chains and requires no transfer of unbounded moments.

This example is not a superposition of independent one-prime pilot laws: the same regular core supplies both multiplicative channels. The irrationality statements above are algebraic arguments, not floating-point relation tests.

Remark 6.4 (A qualitative obstruction, not a uniform numerical gap). The rank obstruction does not assert that clustering stays bounded away from zero as a tangent vector varies. For example, if $h(v) \neq 0$ for every nonzero $v \in L_0$, dilating η by $R \rightarrow \infty$ separates every finite collection of distinct same-class shifts. At fixed multiplier cutoff the target then becomes an ordinary Poisson field. The multiplier tail bounds show that the full limiting marked laws approach the Poisson law in total variation as $R \rightarrow \infty$: the truncated single-offspring field has intensity $\delta \sum_{s \leq H} s^{-1} du dw$, and the two missing-tail costs vanish with H . This statement concerns the family of already constructed limits. The preceding growing-displacement calculation separately gives an arithmetic conclusion when $R_N \rightarrow \infty$ and $R_N = o(N)$.

7 Rank-one formulas and the one-prime pilot

Assume an effective profile with $L = \mathbb{Z}v_0 \neq 0$, and abbreviate $W = W_{L,h}$. Orient the generator so that

$$A = \prod_p p^{(v_0, p)_+} > B = \prod_p p^{(-v_0, p)_+}, \quad (A, B) = 1, \quad d = |h(v_0)|.$$

The capital integer A is the larger channel multiplier. Put

$$U_j = A^{-(j-1)}(\tau - (j-1)d)_+, \quad j \geq 1. \quad (7.1)$$

Theorem 7.1 (The rank-one count law). *The jump intensities in Theorem 5.4 are*

$$\lambda_j = U_j - 2U_{j+1} + U_{j+2}, \quad \Lambda_g = \tau - \frac{(\tau - d)_+}{A}. \quad (7.2)$$

For $d > 0$ the jump support is contained in $\{1, \dots, \lceil \tau/d \rceil\}$. For $d = 0$,

$$\lambda_j = \tau(1 - A^{-1})^2 A^{-(j-1)},$$

recovering Arithmetic clusters, Corollary 5.3 [25]. The smaller multiplier B does not enter the count law, but remains in spatial cluster ratios $(A/B)^k$.

Proof. In each profile class the multipliers form consecutive powers of the ratio A/B , over an integer interval, possibly one-sided infinite. Integrality of an intermediate multiplier follows prime by prime from coprimality of A, B . In the increasing ordering, a multiplier s has exactly $\nu_A(s) = \max\{k : A^k \mid s\}$ predecessors. This is the divisibility argument already used in *Arithmetic clusters*, Corollary 5.3 [25]; it does not require a primitive ambient-lattice generator.

For fixed x , eligibility $sx \leq 1$ keeps an initial segment of a class. Its shifts form an arithmetic progression of step $h(v_0)$. At any angular coordinate z , the active indices in that segment are

consecutive. Hence $(r_C(x, z) - j + 1)_+$ counts the consecutive blocks of j active multipliers. Such a block ending at s exists exactly when $\nu_A(s) \geq j - 1$ and $sx \leq 1$. Its angular intersection length is $(\tau - (j - 1)d)_+$. Integrating and summing all classes gives

$$\delta \sum_C \int (r_C - j + 1)_+ dx dz = (\tau - (j - 1)d)_+ \delta \sum_{s \in A^{j-1}\mathcal{D}} s^{-1} = U_j.$$

For every nonnegative integer r ,

$$\mathbf{1}_{\{r=j\}} = (r - j + 1)_+ - 2(r - j)_+ + (r - j - 1)_+.$$

This proves the intensity formula. Telescoping gives $\sum_j \lambda_j = U_1 - U_2$ and $\sum_j j \lambda_j = U_1 = \tau$. The remaining assertions follow directly. This argument also proves nonnegativity of the second differences. $\square$

The same profile has variance excess

$$\text{Var}(W_{L,h}) - \tau = 2 \sum_{k \geq 1} A^{-k} (\tau - kd)_+. \quad (7.3)$$

This follows from the primitive pair formula of Theorem 8.1, or from the limiting array supported on $\mathbb{Z}v_0$.

7.1 The exact one-prime model

Take $\mathcal{E} = \{2\}$, $\xi_0 = 1$, $\eta_2 = c$, and $e_N = 0$. Then $A = 2$, $B = 1$, $d = |c|$, and

$$W_N = \sum_{n=2}^N \mathbf{1}_{\{F_N(n) \in J_N\}} - \sum_{\substack{k \geq 1 \\ 2^k \leq N}} \mathbf{1}_{\{F_N(2^k) \in J_N\}}.$$

At a regular core there are exactly l eligible powers when

$$N/2^l < d_{\text{core}} \leq N/2^{l-1}, \quad d_{\text{core}} \geq 3 \text{ odd}.$$

Their number is $N2^{-l-1} + O(1)$. With

$$r_l(z) = \sum_{k=0}^{l-1} \mathbf{1}_{I-kc}(z),$$

the general formula becomes the orbit formula

$$\lambda_j(c) = \sum_{l \geq 1} 2^{-l-1} \text{Leb}\{z : r_l(z) = j\}. \quad (7.4)$$

Discarding limiting orbits with $l > L$ costs at most

$$\tau \sum_{l > L} l 2^{-l-1} = \tau \frac{L+2}{2^{L+1}},$$

uniformly in c . Fixed-depth core counting followed by this tail proves the same count limit as Theorem 5.4; no interchange of an uncontrolled infinite sum is needed.

An independent derivation of the coefficients uses

$$\int_{\mathbb{R}} (r_l(z) - j + 1)_+ dz = (l - j + 1)_+ (\tau - (j - 1)d)_+,$$

because a string of j consecutive intervals has that common-intersection length. Summing its factor $l - j + 1$ gives $\sum_{l \geq j} 2^{-l-1}(l - j + 1) = 2^{1-j}$, precisely (7.1) at $A = 2$.

For $d = 0$, $\lambda_j = \tau 2^{-j-1}$. For $d \geq \tau$, $\lambda_1 = \tau$ and all higher rates vanish. When $0 < d < \tau$, write $q = \lfloor \tau/d \rfloor$ and $r = \tau - qd$. Then

$$\begin{aligned} \lambda_j &= 2^{-j-1}[\tau + (3-j)d], & 1 \leq j < q, \\ \lambda_q &= 2^{1-q}d, & \lambda_{q+1} = 2^{-q}r, \end{aligned} \tag{7.5}$$

with all remaining rates zero. A zero last rate is omitted. The preceding proof derives these coefficients from the germ kernel. At $d = \tau/k$, the formulas from both adjacent branches agree, since (7.1) is continuous in d . In particular,

$$\tau/2 \leq d \leq \tau \implies W \stackrel{d}{=} P_1 + 2P_2, \quad P_1 \sim \text{Pois}(d), \quad P_2 \sim \text{Pois}((\tau - d)/2)$$

with independence. This is the Hermite law [13], including its Poisson boundary case. The new content of this formula is the arithmetic identification of its parameters. Also,

$$\sum_j j \lambda_j = \tau, \quad \Lambda_g = \frac{\tau + \min(d, \tau)}{2}.$$

For $c \notin \mathbb{Q}$ the exact relation lattice of ξ_N is zero at every finite N , but the triangular limit is non-Poisson when $0 < |c| < \tau$. The base lattice, together with the displacement, is the relevant data.

7.2 The environment-only correction is indispensable

For fixed c , eventually $|c| \lfloor \log_2 N \rfloor / N < 1/2$, so there is no winding at any power $2^k \leq N$. The removed count is then exactly

$$B_N(c) = \#\{1 \leq k \leq \lfloor \log_2 N \rfloor : |kc| < \tau/2\}.$$

For $c = 0$ it equals $\lfloor \log_2 N \rfloor$ and diverges. For $d > 0$ it eventually equals

$$\max\{0, \lceil \tau/(2d) \rceil - 1\}.$$

The strict inequality matters at an angular endpoint. For general perturbations with nonzero e_N , an endpoint can be approached from either side; no convergence of the exact correction is assumed. Theorems concern its exact removal.

8 Primitive tuples and a complete spatial invariant

The primitive pair cancellation extends to every fixed factorial order. Its second-order case will then do more than compute a covariance: for interval targets it identifies the whole unmarked spatial law.

Three intervals first. The geometric reason is already visible for three translates of the target interval. In this illustration let $T_i = I - b_i$, where I has length τ and $b_1 \leq b_2 \leq b_3$. The two extreme shifts determine the common intersection:

$$\text{Leb}(T_1 \cap T_2 \cap T_3) = (\tau - b_3 + b_1)_+ = \min_{i < j} \text{Leb}(T_i \cap T_j).$$

Indeed, the middle translate cannot shorten the intersection of the two extremes; if the extremes are disjoint, both sides vanish. Thus pair-overlap masses determine the triple activation mass, including the case of zero overlap.

The reconstruction below uses this geometry in two steps. First the complete spatial covariance separates the primitive multiplicative rays and recovers each pair coefficient in (8.7). At fixed multiplier depth, interval intersections then determine all finite activation masses, as made explicit in Appendix A. The Poisson germ construction and the summable multiplier tail recover the full law. The positive forest formula of Theorem 8.2 performs this reconstruction without an infinite moment expansion.

This illustration concerns interval translates and unsigned overlap masses. For a general known Borel target the fully marked reconstruction in Section 12 instead uses signed displacement channels; the minimum-of-pairs identity is not assumed there.

8.1 The hierarchy of fixed-order formulas

Let $\mathfrak{P}_k$ be the ordered k -tuples of pairwise distinct members of $\mathcal{D}$ with

$$\gcd(a_1, \dots, a_k) = 1.$$

This is a *collective gcd*, not pairwise coprimality. For $k = 1$, set $\mathfrak{P}_1 = \{(1)\}$. Write $H(\mathbf{a}) = \max_i a_i$. If all differences $\mathbf{v}(a_i) - \mathbf{v}(a_1)$ belong to L , put

$$b_i = h(\mathbf{v}(a_i) - \mathbf{v}(a_1)), \quad b_1 = 0, \quad \vartheta(\mathbf{a}) = \left(\tau - (\max_i b_i - \min_i b_i) \right)_+.$$

Otherwise set $\vartheta(\mathbf{a}) = 0$ and ignore the associated summand.

Theorem 8.1 (Primitive factorial cumulants). *All fixed-order moments of the limit count are finite. Its factorial cumulants are*

$$\kappa_k^{\text{F}}(W_{L,h}) = \sum_{\mathbf{a} \in \mathfrak{P}_k} \frac{\vartheta(\mathbf{a})}{H(\mathbf{a})}, \quad k \geq 1. \quad (8.1)$$

The k th spatial factorial cumulant measure is

$$\mathsf{K}_k^{\text{sp}} = \sum_{\mathbf{a} \in \mathfrak{P}_k} \vartheta(\mathbf{a}) \int_0^{1/H(\mathbf{a})} \delta_{(a_1 x, \dots, a_k x)} dx. \quad (8.2)$$

With angular marks retained, replace each summand by

$$\int_0^{1/H(\mathbf{a})} \int_{\cap_i (I - b_i)} \delta_{((a_1 x, z + b_1), \dots, (a_k x, z + b_k))} dz dx. \quad (8.3)$$

The formulas are absolutely convergent for each fixed k ; they are not an assertion of an infinite cumulant expansion at the origin.

Proof. For a finite Poisson cluster process, differentiating its generating functional to a fixed order, or first doing so for bounded germ sizes, shows that its k th factorial cumulant measure is the integral of the ordered distinct offspring k -tuples of one germ. This is the finite-order Poisson identity being used.

In the class construction such a tuple of multipliers has a unique form $s_i = ga_i$ with collective $\gcd g \in \mathcal{D}$ and $\mathbf{a} \in \mathfrak{P}_k$. The same-class condition depends only on the differences of the primitive valuation vectors. Subtract the first shift from the common angular variable. Then change x to gx , which contributes a factor $1/g$. The spatial restriction becomes $x \leq 1/H(\mathbf{a})$, with no additional restriction because $g \geq 1$. Summing $\delta \sum_{g \in \mathcal{D}} 1/g = 1$ proves (8.3) and its projections.

To justify all interchanges, the number of primitive tuples of height at most X is at most $B_{\mathcal{E}}(X)^k \ll_{\mathcal{E},k} (1 + \log X)^{b_0 k}$. Dyadic summation makes $\sum_{\mathbf{a}} H(\mathbf{a})^{-1}$ finite. Every angular

intersection is bounded by τ . Thus each fixed factorial cumulant integral is finite. Alternatively the germ power integrals are bounded using $r^k \leq \sum_{j=1}^k c_{k,j}(r)_j$ for fixed nonnegative constants. A Poisson number of germs with these finite mark moments has finite count moments. This also validates differentiation at each finite order by truncation. $\square$

There is a useful exact prelimit version. Define

$$\vartheta_N(\mathbf{a}) = N m \left(\bigcap_i \chi_N(a_i)^{-1} J_N \right).$$

At every finite N the k th factorial cumulant of the *target* equals

$$K_{k,N} = \sum_{\mathbf{a} \in \mathfrak{P}_k} \frac{M(N/H(\mathbf{a}))}{N} \vartheta_N(\mathbf{a}). \quad (8.4)$$

The proof is exactly the unique decomposition *gd* used in Lemma 3.2. Under a profile, a fixed tuple has $\vartheta_N(\mathbf{a}) \rightarrow \vartheta(\mathbf{a})$: different classes are eventually separated at target scale, and same-class shifts converge. Since $\vartheta_N \leq \tau$ and $M(N/H)/N \leq 1/H$, dominated convergence proves convergence of all target factorial cumulants to (8.1). It does *not* transfer all actual arithmetic moments. Proposition 3.5 supplies that missing argument at order two only.

8.2 Recovering every pair coefficient from the covariance measure

The $k = 2$ specialization can be written

$$\mathsf{K}_2^{\text{sp}} = \sum_{v \neq 0} \beta(v) \int_0^{1/H(v)} \delta_{(B(v)x, A(v)x)} dx. \quad (8.5)$$

The covariance of two spatial counts is

$$\text{Cov}(X_{L,h}(U), X_{L,h}(V)) = \tau \text{Leb}(U \cap V) + \mathsf{K}_2^{\text{sp}}(U \times V). \quad (8.6)$$

For $v \neq 0$ let

$$R_v = \{(B(v)x, A(v)x) : 0 < x \leq 1/H(v)\}.$$

Different v give disjoint Borel rays away from $(0,0)$, because their ratios are distinct reduced positive rationals. Therefore

$$\beta(v) = H(v) \mathsf{K}_2^{\text{sp}}(R_v). \quad (8.7)$$

The rays can be dense in slope; their exact Borel restrictions remain disjoint. This is an identifiability statement from an ideal probability law, not a numerically stable estimator based on finite data.

8.3 Positive forest reconstruction

Let $X = X_{L,h}$ be a profile spatial field and β its coefficient array. For a nonnegative bounded Borel test f on $[0,1]$, put $p_f(u) = 1 - e^{-f(u)}$. At core position $x > 0$, independently retain each $s \in \mathcal{D}$, $sx \leq 1$, with probability $p_f(sx)$; denote the resulting finite random vertex set by $S_{x,f}$. Only this finite set is thinned at each x .

Theorem 8.2 (Positive forest Laplace formula). *The spatial Laplace functional has the exact representation*

$$\log \mathbb{E} e^{-X(f)} = -\tau \int_0^1 p_f(u) du + \delta \int_0^1 \mathbb{E}_f \mathsf{T}(S_{x,f}; \beta(\mathbf{v}(s) - \mathbf{v}(t))) dx. \quad (8.8)$$

The correction is nonnegative and integrable. The identical formula with the empirical regular-core sum and the exact grid mean holds for $\tilde{X}_N$, using β_N , once $\tau B_{\mathcal{E}}(N) < N$.

Consequently the spatial law is reconstructed from the pair coefficients without an infinite alternating expansion, a moment-determinacy assumption, or an exponential-moment hypothesis.

Proof. The Poisson germ exponential formula gives, at core position x and class C , a Laplace cost

$$\int_{\mathbb{R}} \left[1 - \prod_{\substack{s \in C \\ sx \leq 1}} (1 - p_f(sx) \mathbf{1}_{I-a_s}(z)) \right] dz.$$

There are finitely many eligible multipliers at each $x > 0$. Apply Lemma 4.2 and sum over represented classes. Between classes the weights are zero, so the maximum forest over their union is the sum of their separate maximum weights. The first term integrates to

$$\delta\tau \sum_s \int_0^{1/s} p_f(sx) dx = \delta\tau \sum_s \frac{1}{s} \int_0^1 p_f(u) du = \tau \int_0^1 p_f(u) du.$$

The forest weight is bounded by τ times the number of eligible vertices. Its integral is at most τ by the same calculation. This validates all nonnegative integrals. Applying the Poisson exponential identity with its minus sign proves (8.8). Alternatively one can truncate multipliers first and use the first-mass tail; no infinite inclusion–exclusion series is used. The circle-gap argument of Theorem 4.3 proves the prelimit formula. $\square$

The general Poisson exponential and marking identities used here are classical; see [17, author manuscript, 21 August 2017, Theorems 3.9 and 5.6]. The arithmetic contribution is the particular overlap array, its finite core grouping, and its summable multiplier structure.

Theorem 8.3 (A weighted, non-exponential continuity bound). *For two admissible profile laws with the same $\mathcal{E}, \tau$ and pair arrays β, β' , define*

$$d_{\text{pair}}(\beta, \beta') = \sum_{v \neq 0} \frac{|\beta(v) - \beta'(v)|}{H(v)}.$$

Then for every nonnegative bounded Borel f ,

$$\left| \log \mathbb{E} e^{-X(f)} - \log \mathbb{E} e^{-X'(f)} \right| \leq \frac{1}{2} d_{\text{pair}}(\beta, \beta'). \quad (8.9)$$

The same bound holds for the difference of the Laplace functionals themselves. If f is supported on $U = (a, b) \subset (0, 1]$, replace $H(v)^{-1}$ by

$$w_U(v) = \left(\frac{b}{H(v)} - \frac{a}{H_-(v)} \right)_+. \quad (8.10)$$

In particular these coefficients also control the log-void probabilities on U .

Proof. The first terms of (8.8) agree. Use the same auxiliary thinning and (4.4). Dropping thinning probabilities bounds the remaining difference by the integral of all changed unordered pair weights. Every ordered pair of distinct multipliers has the unique form $(s, t) = (gB(v), gA(v))$, with $g \in \mathcal{D}$. Its common core interval has length $1/(gH(v))$, or $w_U(v)/g$ after masking. The cancellation $\delta \sum_g 1/g = 1$ proves the ordered-pair sum; dividing by two accounts for unordered edges. Absolute convergence follows from the height sum in (2.8). Both log-Laplace values are nonpositive, and the exponential function is 1-Lipschitz there. Finally let f increase to $+\infty$ on U to obtain the void assertion. $\square$

Two different metrics. This linear bound controls log-Laplace functionals. Appendix A proves a separate total-variation modulus with quadratic finite-depth cost and a global square-root bound with logarithmic factors. Neither bound is a linear marked total-variation estimate, or a stable procedure for recovering infinitely many rational rays from finite data.

For a window U , the limiting deficit is obtained by taking all selected probabilities equal to one there:

$$D_U = \delta \int_0^1 \mathbb{T}(\{s \in \mathcal{D} : sx \in U\}; \beta(\mathbf{v}(s) - \mathbf{v}(t))) dx. \quad (8.11)$$

It satisfies $0 \leq D_U \leq \tau|U|$ and $\mathbb{P}(X(U) = 0) = \exp[-\tau|U| + D_U]$. Thus the same pair array yields a first-order Poisson correction, exact avoidance probabilities, and the full Laplace functional.

8.4 Identification from the complete spatial covariance

The classical precedents include Newman [22], Gauss–Poisson processes [14, Section III], stable multivariate linear Hawkes processes [5, arXiv:1401.0903v2, Theorem 1 and Corollary 1], and log Gaussian Cox processes [18]. For Hawkes, the first two orders determine the causal excitation kernel through Wiener–Hopf equations; its cluster representation is not confined to pairs [9]. For a log Gaussian Cox process, pair correlation is $g(u, v) = \exp(C(u, v))$, where C is the latent Gaussian covariance, and intensity recovers the latent mean.

The arithmetic inversion below reads a singular covariance measure on primitive multiplicative rays, recovers a countable overlap array, and reconstructs the law by the positive finite-geometry formula. It does not require that clusters have at most two points or that higher factorial cumulants vanish. The target is known and the profile deterministic; these restrictions are part of the assertion.

Theorem 8.4 (Second order determines the spatial law). *Fix $\mathcal{E}$ and τ . For two admissible deterministic profiles, the following are equivalent:*

$$\mathcal{L}(X_{L,h}) = \mathcal{L}(X_{L',h'}), \quad \mathbf{K}_2^{\text{sp}} = \mathbf{K}_2'^{\text{sp}}, \quad \beta(v) = \beta'(v) \quad \text{for every } v \neq 0.$$

This concerns the complete unmarked spatial field, not merely its total count. These unsigned data do not determine angularly marked laws or arbitrary mixtures; the fully marked counterpart, with richer data, is Theorem 12.2.

Proof. Equality of laws implies equality of their covariance measures by the finite second moments in Theorem 8.1. Subtracting the known diagonal mean term gives the second factorial cumulant measure. Equation (8.7) then recovers every pair coefficient. Conversely, equal pair coefficients give equal spatial Laplace functionals by Theorem 8.2, hence equal laws. Appendix A supplies a finite exact-subset reconstruction and its separate total-variation bound. $\square$

Remark 8.5 (Why infinite cumulants would be an unsafe shortcut). The limit can have all moments but no positive exponential moment. For example take $\mathcal{E} = \{2, 3\}$, $L = \mathbb{Z}^2$, and $h = 0$. For every integer $m \geq 1$, the m^2 distinct integers $2^i 3^j$, $0 \leq i, j < m$, are at most 6^{m-1} . Their distinctness follows from unique factorization, and the upper bound is attained at $i = j = m - 1$. A germ with $x \leq 6^{-(m-1)}$ and $z \in I$ generates at least m^2 offspring. Thus

$$\mathbb{P}(W_{L,0} \geq m^2) \geq 1 - \exp\{-\delta\tau 6^{-(m-1)}\}.$$

For every $t > 0$, e^{tm^2} times this lower bound tends to infinity, proving $\mathbb{E}e^{tW_{L,0}} = \infty$. We therefore do not deduce Theorem 8.4 from equality of all cumulants or claim an analytic generating function around 1. The positive forest formula gives one replacement; finite inclusion–exclusion followed by a summable configuration tail gives another.

8.5 A full-sequence spatial convergence criterion

Theorem 8.6 (Pair profiles classify spatial convergence). *For an arbitrary deterministic phase sequence in Section 2, the unmarked spatial fields Ξ_N^{sp} converge weakly if and only if $\beta_N(v)$ converges for every fixed nonzero $v \in \mathbb{Z}^E$. In that case the limits $\beta(v)$ determine the spatial law uniquely by Theorem 8.4. Prefix-count paths converge in J_1 , and fixed finite interval-count vectors converge in total variation to those of that law.*

The total count alone may converge even when this condition fails. Nor is convergence of the unsigned β_N claimed sufficient for convergence of the angularly marked field.

Proof. Suppose every coefficient converges. The spatial fields are tight by the exact bounded first moment on a compact space. Every subsequence admits a signed profile refinement by Proposition 5.2. Theorem 5.4 identifies its spatial limit, and that profile has precisely the given pair coefficients. Theorem 8.4 makes all these limits equal. Thus the full sequence converges. The same subsequence argument, using the J_1 and discrete-vector conclusions of Theorem 5.4, proves the other convergences.

Conversely suppose the spatial fields converge but a coefficient $\beta_N(v)$ does not. It is bounded in $[0, \tau]$, so two subsequences give different limits for that coefficient. Refine each to a full signed profile. Their spatial laws must both be the assumed full-sequence limit, but (8.7) would give different values of the chosen coefficient. This contradiction proves necessity. No unbounded moment is transferred from weak convergence in this argument. $\square$

Corollary 8.7 (Actual covariance convergence). *Under the hypotheses of Theorem 8.6, actual arithmetic covariances on any fixed pair of spatial intervals converge to (8.6). Under a signed profile, the corresponding assertion also holds for bounded continuous tests on K .*

Proof. For a fixed primitive pair the prelimit same-germ covariance, including its angular integral, converges to the corresponding term of (8.3); the core-product integer gd supplies an empirical measure with the $\mathcal{D}$ -only integers deleted, and this converges to Lebesgue measure. Angular shifts converge in a fixed chart; fixed spatial interval boundaries have zero ray measure. The absolute contribution of each primitive pair is at most $\tau/H(v)$, times the test sup norms. Dominated convergence gives convergence of target second cumulants; the diagonal mean terms converge separately. Proposition 3.5 transfers these second moments to the arithmetic field. When only unsigned coefficients converge, apply this argument on every signed-profile refinement and use the unique covariance measure (8.5). $\square$

The additional direct covariance argument is what permits arithmetic moment conclusions in the examples below. Neither weak convergence nor the total-variation comparison alone supplies them.

8.6 What spatial pairs forget and angular pairs retain

At a fixed width τ , all channels with $|h(v)| \geq \tau$ have coefficient zero. Their exact displacements, and even their membership in L , need not be identifiable from the unmarked field at that width. Moreover h and $-h$ give the same spatial law, by reflecting the common angular coordinate. In contrast, for a visible channel the angularly marked second cumulant lies on the deterministic phase-difference relation $w_2 - w_1 = h(v)$, so angular marks can retain the sign.

The complete covariance contains infinitely many structured coefficients. The rational ray slopes can be dense: Borel separation identifies their ideal measures, but does not make estimation from a finite sample stable. Neither a scalar variance nor finitely many measured covariances suffice for the asserted reconstruction.

Knowing the spatial laws for *all* widths $\tau > 0$ identifies L and $|h(v)|$ for each member: a nonzero coefficient first appears at $\tau > |h(v)|$. This is a theorem about a family of ideal

probability laws, not a finite-resolution phase-recognition procedure. Arbitrary random mixtures of environments are outside the second-order completeness assertion; their averaged pair coefficients need not preserve the nonlinear finite intersection reconstruction. The signed channel information is made precise in Section 12.2: connected components of positive angular overlap reconstruct the shifts up to harmless common translations. This yields a fully marked identification theorem even for a known disconnected Borel target.

9 Prefix counts, first occurrences, and spatial dependence

In the decorated limit of Pène–Saussol [23, arXiv:1803.06865, Theorem 5.2], one germ produces points $(t, D(T^{-kp})_{x_0}y)$ with the same limiting time t and transformed internal coordinates (the derivative is taken at the periodic point). Here one germ produces distinct macroscopic positions sx . Keeping this difference is essential: a compound-Poisson marginal does not give independent spatial increments or simultaneous jumps in the prefix path.

A different arithmetic example is the signed binary run field of *Long runs and rare patterns* [24, Theorem 7.7, Corollary 7.8 and technical companion, Appendix F], whose same-grid comparison has an independent Poisson target. *Records, scale flows, and Gaussian limits* [26, Input 2.1, Corollary 2.2 and Section 7.2] studies observations of that field. A common observation map preserves the target’s dependencies: in the present model those include siblings at distinct positions. The joint prefix law below makes that dependence explicit.

Let $X = X_{L,h}$ be a profile spatial field. Set $C_N(t) = X_N([0, t])$ and $C(t) = X([0, t])$ for $0 \leq t \leq 1$. These counts use the same terminal- N model and the same target J_N throughout the path: neither is resampled as t varies. This differs from the immigration–death scale flow in [26, Section 7.2]: no coupling across different terminal values of N is imposed. By Theorem 5.4, $C_N \Rightarrow C$ in J_1 , with finitely many unit jumps in the limit. Reading a prefix path is a common measurable observation of the arithmetic field and its germ target, but the limiting joint law still retains every sibling produced by a common germ. The joint exponent below, rather than a product of one-time marginals, records that dependence.

Proposition 9.1 (Joint interval laws and one-time prefix marginals). *For fixed intervals $B_1, \dots, B_m \subseteq [0, 1]$ and $u_i \geq 0$,*

$$\mathbb{E} \exp \left(- \sum_i u_i X(B_i) \right) = \exp \left\{ -\delta \sum_C \int_0^1 \int_{\mathbb{R}} \left(1 - e^{-\sum_i u_i n_{C,i}(x,z)} \right) dz dx \right\}, \quad (9.1)$$

where $n_{C,i}(x, z) = \sum_{s \in C, sx \in B_i} \mathbf{1}_I(z + a_s)$. The arithmetic interval-count vectors converge to this vector in total variation. For each fixed t ,

$$C(t) \stackrel{d}{=} \sum_{j \geq 1} j P_{t,j}, \quad P_{t,j} \sim \text{Pois}(t\lambda_j) \quad \text{independently in } j. \quad (9.2)$$

This specifies one-time marginals, not independent increments in t .

Proof. Equation (9.1) is the Poisson exponential formula for the common germ configuration. It is not a product of marginal void probabilities. The total-variation convergence on the countable discrete vector space follows from Theorem 5.4. For a prefix $[0, t]$, substituting $x = ty$ in the germ exponent changes $sx \leq t$ to $sy \leq 1$ and dx to $t dy$. Every jump intensity is multiplied by t , proving (9.2). $\square$

If any pair overlap survives, choose small disjoint neighborhoods of two descendant positions produced by that channel. Their covariance is positive by (8.6). Thus the path does not have independent increments. It is not the compound-Poisson Lévy process sharing its one-time marginals: the present path has unit spatial jumps, and the siblings arrive at distinct positions.

9.1 Identical prefix laws, different spatial laws

Consider two environments with $\mathcal{E} = \{2, 3\}$, $\tau = 1$, $d = 3/4$, and a fixed phase ζ of infinite order:

	$F_N(2)$	$F_N(3)$
I	ζ	$e^{2\pi id/N}$
II	ζ	$\zeta e^{2\pi id/N}$

Their profiles have respectively $L = \mathbb{Z}(0, 1)$ and $L = \mathbb{Z}(-1, 1)$, and the same $A = 3, d = 3/4$ but $B = 1$ and $B = 2$. For a fixed prefix t , the change of core variable $x = ty$ multiplies every jump intensity by t while leaving the eligibility condition $sy \leq 1$ unchanged. Thus both limiting counts have law

$$\text{Pois}(5t/6) + 2\text{Pois}(t/12).$$

Set $V_1 = (2/5, 1/2)$ and $V_2 = (3/5, 3/4)$. A ratio from V_1 to V_2 lies in $(6/5, 15/8)$. In model I no allowed ratio (a positive power of 3) lies there. In model II the only active ratio in that range is $3/2$, and

$$(1 - d) \text{Leb}(V_1/2 \cap V_2/3) = \frac{1}{4} \cdot \frac{1}{20} = \frac{1}{80}.$$

Corollary 8.7 proves for the *original corrected arithmetic fields*

$$\text{Cov}(X_{N,\text{I}}(V_1), X_{N,\text{I}}(V_2)) \rightarrow 0, \quad \text{Cov}(X_{N,\text{II}}(V_1), X_{N,\text{II}}(V_2)) \rightarrow \frac{1}{80}. \quad (9.3)$$

The zero covariance in model I comes from disjoint contributing germ domains: no germ can visit both intervals, so the two limiting counts are independent. The arithmetic covariance conclusions in (9.3) use Corollary 8.7.

Alternating between these two environments on even and odd N gives a count converging in total variation to the same compound-Poisson law, while the spatial fields do not converge. Indeed the coefficient of channel $3/1$ is positive on one subsequence and zero on the other. This is a concrete boundary between the count law and the full spatial criterion of Theorem 8.6.

For the two-direction profile in (6.6), the same primitive channel gives

$$\text{Cov}(X_{L,h}(V_1), X_{L,h}(V_2)) \geq \frac{1 - \sqrt{3} + \sqrt{2}}{20} > 0.$$

Other channels may also contribute; this is a lower bound, not an equality for that profile.

9.2 First occurrence, including no occurrence

Let $T = \inf\{t : C(t) > 0\}$ with $T = \infty$ when the corrected field is empty. The same definition gives T_N . As a map into $[0, 1] \sqcup \{\infty\}$, with ∞ isolated, first occurrence is continuous at the almost surely finite simple limit configurations, including the empty one. Therefore $T_N \Rightarrow T$, and

$$\mathbb{P}(T > t) = e^{-t\Lambda_g} \quad (0 \leq t \leq 1), \quad \mathbb{P}(T = \infty) = e^{-\Lambda_g}.$$

On $(0, 1)$ the density is $\Lambda_g e^{-t\Lambda_g}$. This exponential first-arrival law does not imply independent spatial increments. The two models in Section 9 also have the same first-arrival law, despite their distinct two-interval laws.

10 Observation windows and local cluster sizes

A fixed interior window tests angular closeness and spatial ratios simultaneously. The geometric facts about discrete subgroups of $\mathbb{R}^2$ used below are classical; the result is their exact identification with the support of arithmetic cluster sizes.

Fix $U = (a, b)$ with $0 < a < b \leq 1$, and let

$$w = \log(b/a) > 0, \quad m_U = \tau(b - a).$$

The restriction of the marked profile field has the same germs as before but keeps only descendants $sx \in U$. Let $\lambda_{j,U}$ be the resulting positive cluster-size intensities. The count $X(U)$ is a compound-Poisson variable of mean m_U ; all its fixed moments are finite by domination by the full count.

Theorem 10.1 (Two simultaneous visibility constraints). *The following are equivalent:*

1. *The restricted marked field is $\text{PPP}(\mathbf{1}_U(u) du \otimes \mathbf{1}_I(z) dz)$.*
2. *$X(U) \sim \text{Pois}(m_U)$.*
3. *There is no $v \in L \setminus \{0\}$ such that*

$$|\ell(v)| < w \quad \text{and} \quad |h(v)| < \tau. \quad (10.1)$$

Equivalently, every $\beta(v)w_U(v)$ vanishes. Its nonnegative second factorial cumulant is

$$\text{Var}(X(U)) - m_U = \sum_{v \neq 0} \beta(v)w_U(v). \quad (10.2)$$

Under an arithmetic profile sequence these are genuine arithmetic distributional limits on the mask. The actual variances also converge to (10.2), by Corollary 8.7.

Proof. Restrict (8.5) to U^2 . Both descendants lie in U precisely on the core interval of length $w_U(v)$ in (8.10), proving (10.2). This length is positive precisely when $|\ell(v)| < \log(b/a)$. The angular overlap is positive precisely when $v \in L$ and $|h(v)| < \tau$.

If no such pair exists, the integrated number of sibling pairs is zero. Almost every active germ therefore has exactly one retained descendant, so the mapping theorem gives a Poisson marked field with its known mean. Conversely a positive pair contribution means a positive intensity of multi-descendant germs. Its deficit D_U is positive, and the void identity after (8.11) differs from e^{-m_U} ; the count is not Poisson. The implication from a Poisson marked field to a Poisson count is immediate. Distributional transfer follows from the profile theorem and the zero boundary mass of U . The final moment assertion uses Proposition 3.5, not weak convergence alone. $\square$

The extra condition $|\ell(v)| < w$ is absent from a full prefix: there the core can be sufficiently close to zero to fit any fixed multiplicative ratio. This is why the global rank-two obstruction does not automatically apply in a bulk interval.

10.1 The exact maximum cluster size

For a real-valued function g on a nonempty finite set V , write $\text{span}_V g = \max_V g - \min_V g$. Define

$$\mathfrak{M}_U(L, h) = \sup \{ |V| : V \subset L \text{ finite nonempty, } \text{span}_V \ell < w, \text{span}_V h < \tau \} \in \mathbb{N} \cup \{\infty\}. \quad (10.3)$$

This is a rectangle-occupancy number for the subgroup

$$\Gamma_{L,h} = \{(\ell(v), h(v)) : v \in L\} \subset \mathbb{R}^2. \quad (10.4)$$

The map to $\Gamma_{L,h}$ is injective because ℓ is injective on integer vectors.

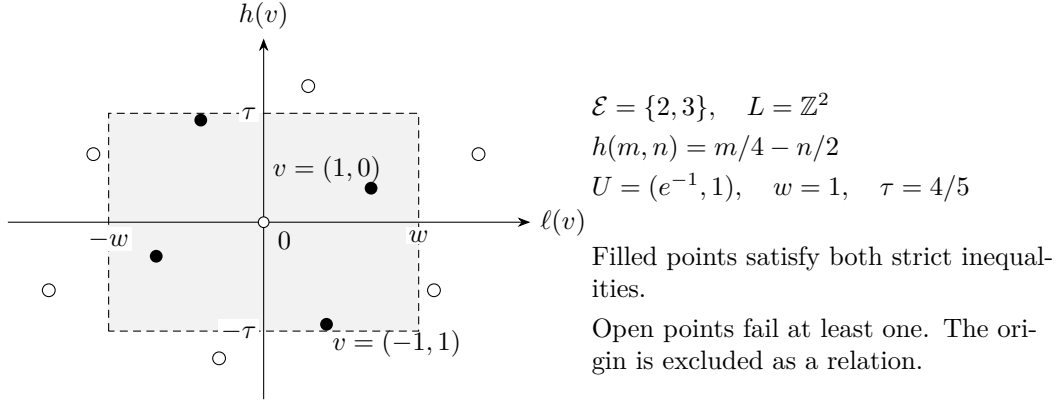


Figure 2: Pair visibility in the subgroup $\Gamma_{L,h}$. The shaded rectangle is $|\ell| < w$, $|h| < \tau$, where $w = \log(b/a)$. The drawing shows only a finite part of the specified group; it is not a numerical certificate of exact relations. For *whole-cluster* occupancy in Theorem 10.2, the translated rectangles instead have side lengths w and τ .

Theorem 10.2 (Exact local cluster support). *The essential maximum number of descendants of one active germ in U equals $\mathfrak{M}_U(L, h)$. More precisely, for every integer $j \geq 1$,*

$$\sum_{k \geq j} \lambda_{k,U} > 0 \iff \text{there is a set } V \text{ of size } j \text{ satisfying (10.3)}. \quad (10.5)$$

If $\mathfrak{M}_U$ is finite, $\lambda_{\mathfrak{M}_U, U} > 0$ and all larger jump intensities vanish. The total count is still unbounded, because the number of germs is Poisson; only the number of descendants per germ is bounded.

Proof. Suppose j distinct descendants $s_1x, \dots, s_jx$ of one germ lie in U , with all $z + a_{s_i} \in I$. Set $v_i = \mathbf{v}(s_i) - \mathbf{v}(s_1) \in L$. Then

$$\text{span}_i \ell(v_i) = \log \frac{\max_i s_i}{\min_i s_i} < \log(b/a), \quad \text{span}_i h(v_i) < \tau.$$

Thus every realized j -tuple yields the required V , including on all positive-measure activation regions.

Conversely, take such a finite V . Choose an integer exponent vector c with all $c + v$, $v \in V$, nonnegative coordinatewise, and define $s_v = \prod_p p^{c_p + v_p}$. These are distinct exceptional multipliers in one class. Their common spatial core interval has length

$$\left(\frac{b}{\max_{v \in V} s_v} - \frac{a}{\min_{v \in V} s_v} \right) > 0,$$

and their common angular activation interval has length $\tau - \text{span}_V h > 0$. This core interval lies in $(0, 1]$ since $b \leq 1$ and $s_v \geq 1$. Hence a positive-measure set of germs has at least j descendants. In particular,

$$\sum_{k \geq j} \lambda_{k,U} \geq \delta \left(\frac{b}{\max s_v} - \frac{a}{\min s_v} \right) (\tau - \text{span}_V h) > 0. \quad (10.6)$$

Additional descendants can only help this lower bound. If the supremum in (10.3) is finite, it is an attained integer. Applying both directions at that integer and one above proves the final claims. $\square$

All inequalities in the rectangle definition are strict. At equality, a common spatial or angular interval can collapse to a null set and contributes no cluster intensity.

10.2 A discreteness dichotomy, rather than rank alone

Theorem 10.3 (Local boundedness and the rank transition). *For every fixed nonempty bulk interval U , the cluster sizes are bounded if and only if $\Gamma_{L,h}$ is discrete in $\mathbb{R}^2$. Consequently:*

1. *For rank $L \leq 1$, bulk cluster sizes are bounded.*
2. *For rank $L = 2$, bulk cluster sizes are bounded exactly when the two real forms ℓ and h are linearly independent on $\text{span}_{\mathbb{R}} L$. If h is proportional to ℓ there, every bulk interval has unbounded cluster sizes.*
3. *For rank $L \geq 3$, every nonempty bulk interval has unbounded cluster sizes, hence a non-Poisson count.*

The existence of a Poisson bulk interval at rank two is determined by the empty-rectangle criterion in Theorem 10.1, not by rank alone.

Proof. If $\Gamma_{L,h}$ is discrete, some neighborhood of zero contains no other subgroup element. Hence distinct elements have a positive uniform separation. A rectangle of the fixed side lengths w, τ , at any translate, contains only a bounded number of such separated points. This bounds (10.3).

If it is not discrete, it contains nonzero vectors arbitrarily close to zero. For any $j \geq 2$, choose $u \in L \setminus \{0\}$ such that

$$(j-1)|\ell(u)| < w, \quad (j-1)|h(u)| < \tau.$$

The set $\{0, u, \dots, (j-1)u\}$ satisfies (10.3). Theorem 10.2 gives unbounded cluster support.

To verify the rank classification directly, an injective image of $\mathbb{Z}$ in $\mathbb{R}^2$ generated by one nonzero vector is discrete. At rank two, independent image vectors form a lattice; dependent image vectors lie on a line and remain integer-independent by injectivity. Among $(Q+1)^2$ image points with two integer coordinates in $[0, Q]$, contained in a line segment of length $O(Q)$, two fall into the same interval of any prescribed small length when Q is large. Their nonzero difference tends to zero. For rank $r \geq 3$, the $(Q+1)^r$ images of an integer basis box lie in a planar box of side $O(Q)$. Partition it into squares of any prescribed positive side length. There are only $O(Q^2)$ squares. Two distinct images share a square for large Q , again giving a nonzero arbitrarily small difference. This proves nondiscreteness without a Diophantine approximation rate. $\square$

The threshold three has a precise origin: a bulk observation imposes two smallness constraints, angular and logarithmic-spatial. It is not a claim that rank three is the smallest rank ever producing a bulk cluster; ranks one and two can certainly do so.

Corollary 10.4 (Rank-one maximum on a bulk interval). *If $L = \mathbb{Z}v_0 \neq 0$, set $l = |\ell(v_0)| > 0$ and $d = |h(v_0)|$. Then*

$$\mathfrak{M}_U(L, h) = \left\lceil \min \left(\frac{w}{l}, \frac{\tau}{d} \right) \right\rceil, \quad (10.7)$$

where $\tau/d = \infty$ for $d = 0$. In particular an exact global relation ($d = 0$) can have unbounded clusters in the full prefix but only bounded clusters in any fixed bulk interval.

Proof. A set of j distinct integer multiples of v_0 has index span at least $j-1$, attained by consecutive multiples. It fits if and only if $(j-1)l < w$ and $(j-1)d < \tau$. The largest such integer is (10.7), including integral threshold ratios. $\square$

The fact that a mask can remove repeated phases was already explicitly noted in *Arithmetic clusters*, Section 5.8 [25]. The new statement here is the simultaneous rectangle criterion, its exact cluster-support characterization, and the resulting local discreteness/rank dichotomy.

10.3 A certified two-prime example: global clusters, Poisson bulk

Take $\mathcal{E} = \{2, 3\}$, $\tau = 1$, Haar at all other primes, and

$$F_N(2) = e^{2\pi i 100/N}, \quad F_N(3) = e^{2\pi i 100\sqrt{2}/N}. \quad (10.8)$$

The effective profile is $L = \mathbb{Z}^2$, $h(m, n) = 100(m + \sqrt{2}n)$. The map h is injective on $\mathbb{Z}^2$. By Corollary 6.1, the corrected full-prefix limit is non-Poisson and has unbounded cluster sizes. That does not give a positive lower bound uniform in the size of the displacement vector; the non-Poisson effect in this example may be small.

Now observe only

$$U = (e^{-1/10}, 1). \quad (10.9)$$

Proposition 10.5 (A certified empty observation rectangle). *For (10.8)–(10.9), the restricted marked field converges to the ordinary Poisson field of intensity $\mathbf{1}_U(u) du \otimes \mathbf{1}_{(-1/2, 1/2)}(z) dz$. In particular,*

$$X_N(U) \xrightarrow{d_{\text{TV}}} \text{Pois}(1 - e^{-1/10}). \quad (10.10)$$

Proof. Suppose an integer pair (m, n) satisfies the two strict visibility conditions. Put

$$L_* = m \log 2 + n \log 3, \quad H_* = m + \sqrt{2}n.$$

Then $|L_*| < 1/10$ and $|H_*| < 1/100$. Eliminating m gives

$$n = \frac{L_* - H_* \log 2}{\log 3 - \sqrt{2} \log 2}.$$

The elementary bounds

$$\log 2 < 0.694, \quad \log 3 > 1.098, \quad \sqrt{2} < 1.415$$

imply $\log 3 - \sqrt{2} \log 2 > 0.11$. Therefore

$$|n| < \frac{0.1 + 0.694/100}{0.11} < 1,$$

so $n = 0$. Then $|m| < 1/100$ gives $m = 0$. The nonzero observation rectangle is empty, and Theorem 10.1 applies.

For an entirely rational check of the logarithmic inequalities, use $z = (x - 1)/(x + 1)$ and

$$2 \sum_{k=0}^{M-1} \frac{z^{2k+1}}{2k+1} < \log x < 2 \sum_{k=0}^{M-1} \frac{z^{2k+1}}{2k+1} + \frac{2z^{2M+1}}{(2M+1)(1-z^2)}, \quad x > 1.$$

At $x = 2, 3$ and $M = 24$ these rational enclosures certify the stated strict margins. Rational bounds on $\sqrt{2}$ are checked by squaring. The accompanying script performs these checks without floating-point phase recognition. $\square$

This example separates the global and local statements under one and the same arithmetic family and one and the same shrinking target. It does not re-randomize the function at different spatial positions. The independent-increment bulk limit follows from the marked field restriction, not only from its marginal count.

11 Random exceptional environments

For an exceptional law fixed before the limit, *Arithmetic clusters* [25, Section 5.8] already integrates its conditional cluster laws and uses excess void probability to exclude a Poisson mixture of genuinely clustered fibers. For random dynamical systems, Atnip and coauthors [4] use spectral perturbation, while Amorim–Haydn–Vaienti [1] obtain quenched compound-Poisson hitting limits by probabilistic approximation under correlation-decay hypotheses. The noise independence in the latter result is distinct from the comparison considered here: a uniform arithmetic criterion for *moving* exceptional laws, with no convergence of those laws assumed, and a joint total-variation comparison retaining the environment. The actual variance criterion uses the separate calculation of Proposition 3.5.

Let Z_N be a random exceptional parameter in a standard Borel space, independent of all regular Haar prime coordinates. Let $\xi_N = \xi_N(Z_N)$ be measurable, with arbitrary joint law across the exceptional primes and arbitrary dependence across N . The comparison target is the conditional kernel obtained by retaining Z_N ; it is not the kernel evaluated at averaged phases. The corrected mean m_N in (2.2) does not depend on Z_N . Define $D_N(Z_N)$ and $\mathcal{R}_N(Z_N)$ on each exceptional fibre.

Theorem 11.1 (Averaged and environment-retaining Poisson criterion). *Under this model, the following are equivalent:*

$$\begin{aligned} d_{\text{TV}}(\mathcal{L}(Z_N, \Xi_N), \mathcal{L}(Z_N) \otimes \text{PPP}(\mu_N)) &\rightarrow 0; \\ d_{\text{TV}}(\mathcal{L}(W_N), \text{Pois}(\tau)) &\rightarrow 0; \\ \mathbb{E}D_N \rightarrow 0; \quad \mathbb{E}\mathcal{R}_N &\rightarrow 0; \\ \beta_N(v) &\rightarrow 0 \text{ in probability for every fixed } v \neq 0; \\ \text{Var}(W_N) - \mathbb{E}W_N &\rightarrow 0. \end{aligned}$$

In particular the annealed count cannot become Poisson by cancellation between non-Poisson fibres of nonvanishing overlap mass. Quantitatively,

$$d_{\text{TV}}(\mathcal{L}(Z_N, \Xi_N), \mathcal{L}(Z_N) \otimes \text{PPP}(\mu_N)) \leq \varepsilon_H(N) + \mathbb{E}(1 - e^{-2D_N}), \quad (11.1)$$

$$d_{\text{TV}}(\mathcal{L}(W_N), \text{Pois}(m_N)) \geq e^{-m_N}(\mathbb{E}e^{D_N} - 1) - \varepsilon_H(N). \quad (11.2)$$

Proof. On each fibre, (2.3) and Theorem 3.3 have the same uniform error. Integration of the conditional total-variation bounds proves (11.1). The target conditional void probability is $e^{-m_N + D_N}$. Integrating it and using the actual-to-target comparison gives (11.2). Since $0 \leq D_N \leq \tau$, vanishing $\mathbb{E}D_N$ is equivalent to the two displayed Poisson conclusions; replace m_N by τ at cost $\tau B_{\mathcal{E}}(N)/N$.

The elementary bound $2D_N \leq K_{2,N} \leq \mathcal{R}_N$ gives one implication from the energy. For the converse, Lemma 3.4 gives for each fixed nonzero v

$$D_N \geq \frac{M(N/H(v))}{N} \beta_N(v), \quad M(t) = \lfloor t \rfloor - B_{\mathcal{E}}(t).$$

It counts every adjacent overlap in the multiplicative chains of that channel, not merely one pair. The prefactor tends to $1/H(v)$. Thus $\mathbb{E}D_N \rightarrow 0$ forces $\mathbb{E}\beta_N(v) \rightarrow 0$ for every fixed v . Boundedness of $\beta_N(v)$ equates this to convergence in probability. The summable majorant $\tau/H(v)$ then gives $\mathbb{E}\mathcal{R}_N \rightarrow 0$.

Finally condition the direct estimate (3.4) on Z_N . The conditional mean is the deterministic m_N , so the conditional-mean variance term is exactly zero. Therefore

$$\text{Var}(W_N) - m_N = \mathbb{E}\mathcal{R}_N + O_{\mathcal{E},\tau}(N^{-1/2}(1 + \log N)^{2b_0+2}), \quad (11.3)$$

which proves the last equivalence. This uses the additional direct arithmetic moment proof, not a conclusion from the integrated total-variation bound. $\square$

The first line is a comparison with a moving product law, since Z_N itself need not converge. It states asymptotic independence from the retained exceptional parameter in the Poisson regime. Arbitrary dependence between that parameter and the regular prime source is not covered.

Proposition 11.2 (Mixing a pointwise profile limit). *Suppose instead a parameter Z in a standard Borel space is sampled once, independently of the regular source, and for almost every $Z = z$ the phase sequence has a measurable profile $(L(z), h_z)$. Then the conditional profile laws of Theorem 5.4 can be integrated. The corrected counts satisfy*

$$d_{\text{TV}}(\mathcal{L}(Z, W_N), \mathcal{L}(Z, W_{L(Z), h_Z})) \rightarrow 0,$$

where the second member denotes the actual profile kernel, not an independently averaged compound-Poisson law. For marked or spatial fields, bounded continuous tests converge jointly with every bounded measurable function of Z to the corresponding conditional profile kernel.

Proof. The conditional count distances tend to zero for almost every parameter and are bounded by one. Dominated convergence, followed by integration of kernels, proves the joint total-variation assertion. The field version follows in the same way for conditional bounded continuous tests, multiplied by any bounded measurable parameter test. Kernel measurability can be read either from the countable profile coordinates and finite truncations, or as the pointwise weak limit of the measurable prelimit kernels. No common rate in z , and no cross- N independent coupling, is needed. $\square$

This is the formulation proved here. A general convergence-in-distribution theorem for laws of random profiles without a pointwise realization is not needed for the examples and is not silently substituted for it.

11.1 Three matching moment orders do not identify a mixed law

Theorem 8.4 is for deterministic profile laws. Low-order correlations need not identify a law in general: Deng–Moody [6, arXiv:0901.4381v3, Section 7] give distinct model sets with identical two- and three-point correlations. The example below concerns a different mechanism, a mixture of arithmetic profile laws, and shows why our deterministic identification excludes arbitrary mixtures. Take $\mathcal{E} = \{2\}$ and $\tau = 1$, with $F_N(2) = \exp(2\pi i d/N)$ and Haar at odd primes. For a fixed $d \in [1/2, 1]$, Theorem 7.1 gives the limiting count

$$W_d \stackrel{d}{=} P_1 + 2P_2, \quad P_1 \sim \text{Pois}(d), \quad P_2 \sim \text{Pois}((1-d)/2), \quad (11.4)$$

with independent summands. These are Hermite laws [13], with a Poisson law at $d = 1$. Each conditional mean is exactly one in the limit. Consider two experiments:

$$\text{fixed: } d = 3/4; \quad \text{mixed: } d = 1/2 \text{ or } 1 \text{ with probability } 1/2 \text{ each.}$$

The value of d in the second experiment is chosen once for the entire field, not independently for each germ.

Theorem 11.3 (Three matching moment orders, different laws). *The two limiting spatial fields have identical factorial moment measures through order three, including identical complete spatial covariance measures. Their laws are different. If W_{fix} and W_{mix} denote their total counts, then*

$$\mathbb{P}(W_{\text{mix}} = 0) - \mathbb{P}(W_{\text{fix}} = 0) = e^{-7/8}(\cosh(1/8) - 1) > 0, \quad (11.5)$$

$$\kappa_4^F(W_{\text{mix}}) - \kappa_4^F(W_{\text{fix}}) = 3/16, \quad (11.6)$$

$$\kappa_8^F(W_{\text{mix}}) = -105/128. \quad (11.7)$$

In particular the mixed count is not a compound-Poisson law with positive-integer jumps, although every conditional count in (11.4) is such a law. Both arithmetic experiments converge to the indicated limits. Higher actual arithmetic moments, beyond the separately established second order, are not claimed to converge.

Proof. For $d \in [1/2, 1]$, the pair coefficients are

$$\beta_d(1) = \beta_d(-1) = 1 - d, \quad \beta_d(k) = 0 \quad (|k| \geq 2).$$

The averaged array at the two endpoints is exactly the array at $d = 3/4$. Each conditional field has the same deterministic first intensity. Its only nonzero factorial cumulant measures have orders one and two, because every germ has at most two descendants. The second is linear in $1 - d$. The moment-cumulant identities through order three use this coefficient only linearly: the third factorial moment measure is the cube of the first measure plus the three products of the first and second cumulant measures. Averaging therefore gives identical factorial moment measures through order three. Equivalently this follows by finite-order expansion of the bounded-cluster generating functional against arbitrary finite bounded test families.

For the count, put $t = z - 1$. The generating functions from (11.4) are

$$\begin{aligned} G_{\text{fix}}(z) &= \exp(t + t^2/8), \\ G_{\text{mix}}(z) &= \frac{1}{2} \exp(t) + \frac{1}{2} \exp(t + t^2/4) = \exp(t + t^2/8) \cosh(t^2/8). \end{aligned}$$

Setting $z = 0$ proves (11.5). In this specific bounded-cluster mixture all exponential moments exist, so expansion near $t = 0$ is justified. Since

$$\log \cosh(t^2/8) = \frac{t^4}{128} - \frac{t^8}{49152} + O(t^{12}),$$

the fourth difference is $4!/128 = 3/16$, and the eighth cumulant is $-8!/49152 = -105/128$. A positive-integer compound-Poisson law with finite eighth moment has

$$\kappa_8^F = \sum_{j \geq 1} \lambda_j(j)_8 \geq 0,$$

so the negative value rules it out. The passage from the two conditional arithmetic limits to their mixture is Proposition 11.2; (3.4) also transfers the common limiting variance $5/4$. $\square$

The example does not contradict Theorem 11.1: both laws are non-Poisson. It shows that averaged pair data can still decide the *vanishing-overlap Poisson question*, but cannot reconstruct a non-Poisson mixed law. One must average the conditional Laplace functionals, not exponentiate an averaged logarithm. This is global environmental dependence despite constant conditional means, not merely a random Poisson intensity.

12 Known Borel targets and fully marked identification

Our reconstruction from unsigned spatial pair masses uses the interval assumption. It is unnecessary for profile limits, the Poisson criterion, or reconstruction from the richer fully marked covariance when the Borel target is known.

12.1 The covariogram and profile limits

Let $I \subset \mathbb{R}$ be a bounded Borel set of finite positive measure τ . Define $J_N = \exp(2\pi i I/N)$ for large N and retain the same normalised coordinates in a compact rectangle containing $[0, 1] \times I$. Put

$$g_I(t) = \text{Leb}(I \cap (I - t)), \quad \beta_N^I(v) = g_I(r_N(v)), \quad \mathcal{R}_N^I = \sum_{v \neq 0} \frac{\beta_N^I(v)}{H(v)}.$$

Covariograms and correlation-based inverse problems also occur in homometry and model sets [8,6]. Here I is fixed and known: we use its covariogram to specify a probabilistic limit, not to reconstruct an unknown window from autocorrelation data. The covariogram g_I is even, bounded by τ , continuous, and supported in a bounded interval. Continuity follows from translation continuity in L^1 ; the latter can be proved by approximating $\mathbf{1}_I$ in L^1 by continuous compactly supported functions.

For $t \geq 0$, write the translation modulus as

$$\omega_I(t) = \sup_{|u| \leq t} \|\mathbf{1}_{I-u} - \mathbf{1}_I\|_{L^1(\mathbb{R})}, \quad \omega_I(t) \longrightarrow 0 \quad (t \downarrow 0).$$

For the finite-depth charts in the proof of Theorem 5.4, put $\rho_N(H) = \max_{s \in S_H} |a_{N,s} - a_s|$. This is the quantity denoted $\omega_N(H)$ in that proof, not a modulus of the target. Let $W_{L,h}^I$ be the count of the profile field with target I .

Theorem 12.1 (Borel profiles and the covariogram criterion). *For this target the marked profile limit of Theorem 5.4 holds with I in the offspring rule. No boundary-null assumption on I is required. Theorem 3.1 also holds with $\beta_N, \mathcal{R}_N$ replaced by $\beta_N^I, \mathcal{R}_N^I$, except that its circular-distance threshold is replaced by*

$$g_I(r_N(v)) \longrightarrow 0 \quad \text{for every fixed } v \neq 0.$$

For a profile, the Poisson criterion is $g_I(h(v)) = 0$ for every nonzero $v \in L$. If $\text{rank } L \geq 2$, clustering and unbounded jump support still follow. The unmarked second-order theorem is not asserted for general Borel targets; Theorem 12.2 gives a fully marked version with richer observations.

More precisely, along a fixed profile sequence, for each fixed H and all sufficiently large N so that its finite-depth charts are separated,

$$d_{\text{TV}}(\mathcal{L}(W_N), \mathcal{L}(W_{L,h}^I)) \leq \varepsilon_H(N) + 2\tau T_H + Z(1)\omega_I(\rho_N(H)) + C_\varepsilon \tau k_H^2/N.$$

The threshold may depend on I , H and the profile sequence. This is a count bound, not a marked-field total-variation estimate. For intervals, $\omega_I(t) \leq 2t$ recovers (5.6); a general Borel target has no universal linear translation bound.

Proof. The direct germ comparison of Theorem 2.2 applies with the same mass ceiling τ/N and the rate of Corollary 2.3. Choose R with $I \subset [-R, R]$ and take $N > 4R$. On overlap, the relative centred scaled turn must lie in $[-2R, 2R]$; the circle has no other contributing lift. Consequently the scaled two-target intersection is exactly $g_I(r_N(v))$. The exact primitive pair identity, anchor coupling, and chain-deficit proof use only pair intersection mass and therefore apply unchanged. Every retained site's mass is still τ/N , so the direct covariance proof and its error are unchanged. The summable majorant is again $\tau/H(v)$.

In the finite-chart proof of the profile limit, replace the interval estimate $2|a_{N,s} - a_s|$ by $\|\mathbf{1}_{I-a_{N,s}} - \mathbf{1}_{I-a_s}\|_1 \rightarrow 0$. There are finitely many shifts at fixed depth. On the complement of their membership discrepancies, the continuous recorded-mark tests converge uniformly. Thus the marked Laplace and count-intensity proofs need no topological boundary regularity of I . First-mass tails are still τT_H . For the displayed count bound, the total intensity of membership discrepancies at depth H is at most

$$\begin{aligned} & \frac{1}{N} \sum_{s \in S_H} \#\{d \geq 2 : (d, q) = 1, sd \leq N\} \|\mathbf{1}_{I-a_{N,s}} - \mathbf{1}_{I-a_s}\|_1 \\ & \leq \sum_{s \in S_H} \frac{\omega_I(\rho_N(H))}{s} \leq Z(1)\omega_I(\rho_N(H)). \end{aligned}$$

After these shifts are coupled, partitioning the core coordinate at $1/s$ gives exactly the same $C_\varepsilon \tau k_H^2/N$ counting error as before: the total angular measure on each piece is at most τk_H , regardless of target boundary geometry. The two restored tails cost at most $2\tau T_H$. Add the arithmetic comparison $\varepsilon_H(N)$, then let N tend to infinity at fixed H before sending H to infinity.

Finally, for any fixed m , translation continuity gives

$$\text{Leb} \left(\bigcap_{k=0}^{m-1} (I - kt) \right) \longrightarrow \tau \quad \text{as } t \rightarrow 0,$$

by bounding the missing part of I by the sum of the $m - 1$ symmetric differences. In rank at least two choose a nonzero u with $h(u)$ sufficiently small, and use the same positive-exponent multiplier string as in (6.2). It has a common activation set of positive measure and a positive core interval. This proves unbounded jump support. $\square$

For a rank-one subgroup one must now check *all* multiples $kh(v_0)$ through $g_I(kh(v_0))$, not only the first multiple against the measure τ . Target measure and target diameter are different.

The simple finite intersection rule (A.1) genuinely uses intervals. For instance, let

$$I = \bigcup_{a \in \{0,1,3\}} (a - \epsilon, a + \epsilon), \quad 0 < \epsilon < 1/4.$$

The three sets $I, I - 1, I - 2$ have pairwise intersection measures 2ϵ but empty triple intersection. Thus their common-intersection length is not the minimum of their pair overlaps. This example identifies the failure of the interval reconstruction argument; no stronger non-identifiability claim for all Borel families is needed here.

12.2 Signed channels and the fully marked covariance

For this subsection I is fixed and known. Write $\Xi_{L,h}^I$ for the profile field with both normalized positions and enlarged angular marks. Its mean is $\mu^I(du, dw) = \mathbf{1}_{[0,1]}(u) du \mathbf{1}_I(w) dw$, independent of the profile. For $v \neq 0$ define the finite measure on the angular coordinate pair by

$$\mathcal{Q}_v^{L,h} = \begin{cases} \int_{\mathbb{R}} \mathbf{1}_I(w) \mathbf{1}_I(w + h(v)) \delta_{(w, w+h(v))} dw, & v \in L, \\ 0, & v \notin L. \end{cases} \quad (12.1)$$

Its mass is $g_I(h(v))$ when $v \in L$. A nonzero measure is supported on $w_2 - w_1 = h(v)$, and therefore specifies the signed displacement. A zero measure says that this pair can never be observed simultaneously, up to a null set of germs.

Let $\mathbf{C}_2^{\text{mk}}$ be the full covariance measure on pairs of (position, mark) coordinates. Subtract the known same-point diagonal to obtain $\mathbf{K}_2^{\text{mk}} = \mathbf{C}_2^{\text{mk}} - \Delta_* \mu^I$, where $\Delta(y) = (y, y)$. The fixed-order argument in Theorem 8.1 applies with the actual intersection sets of I ; in particular all second moments remain finite. Its primitive pair formula is

$$\mathbf{K}_2^{\text{mk}} = \sum_{v \neq 0} \int_0^{1/H(v)} \int \delta_{((B(v)x, w_1), (A(v)x, w_2))} \mathcal{Q}_v^{L,h}(dw_1, dw_2) dx. \quad (12.2)$$

The sum is absolutely convergent, since each angular mass is at most τ and $\sum_{v \neq 0} H(v)^{-1} < \infty$. Let ρ project such a coordinate pair onto its two positions and let π_w project onto its two marks. The oriented rays R_v of Section 8 are disjoint away from the origin, where the measure has no atom. Consequently

$$\mathcal{Q}_v^{L,h} = H(v) (\pi_w)_* \left(\mathbf{K}_2^{\text{mk}}|_{\rho^{-1}(R_v)} \right). \quad (12.3)$$

No factor δ is missing: the common-multiplier sum in (8.3) cancels it through $\delta \sum_{g \in \mathcal{D}} g^{-1} = 1$.

Theorem 12.2 (Fully marked identification for a known target). *Fix $\mathcal{E}$ and a known bounded Borel target I of positive measure. For two admissible deterministic profiles, the following are equivalent:*

$$\begin{aligned}\mathcal{L}(\Xi_{L,h}^I) &= \mathcal{L}(\Xi_{L',h'}^I), \\ \mathcal{C}_2^{\text{mk}} &= \mathcal{C}_2'^{\text{mk}}, \\ \mathcal{Q}_v^{L,h} &= \mathcal{Q}_v^{L',h'} \quad \text{for every } v \neq 0.\end{aligned}$$

The covariance is the complete measure on positions and angular marks. The assertion does not require an interval or a boundary-null condition on I , but it does require deterministic profiles and knowledge of I itself.

Proof. Equal laws have equal covariances because their second moments are finite. Formula (12.3) recovers every $\mathcal{Q}_v$ from that covariance. We prove the converse at finite multiplier depth and then remove the tail.

Fix $H \geq 1$ and $x > 0$, and let $S(x) = \{s \in S_H : sx \leq 1\}$. In each profile class the activation sets are $I - a_s$. Form a graph on $S(x)$ in which s, t are adjacent precisely when their activation sets have an intersection of positive measure. Equivalently, $\mathcal{Q}_{\mathbf{v}(t) - \mathbf{v}(s)}$ has positive mass. Thus the graph is determined by the channel measures. On each oriented edge these measures also determine $a_t - a_s$. Along paths in a connected component, they recover every a_s relative to a chosen root. The sums are path-independent because each array comes from an admissible profile.

For two profiles with the same channel measures the connected components therefore coincide, and the shifts in each component differ by a common translation. The change of germ variable by the opposite translation leaves all recorded marks $z + a_s$ and positions sx unchanged. Different components inside one profile class have activation unions with intersection of measure zero: all pair intersections across them have measure zero and there are only finitely many pairs. Restrictions of the class Poisson measure to these unions are consequently independent, up to an immaterial null set. Components from different classes were already independent. The observed law is thus independent of the invisible distinction between these two reasons for independence.

We have reconstructed the intensity of every finite marked offspring configuration at this x , and hence its Poisson cluster kernel. The eligible set changes only at the finitely many values $1/s$, $s \in S_H$. The deterministic component decompositions and germ translations are therefore measurable in x . Integrating against δdx proves that the two full truncated marked fields have the same law. Restoring multipliers $s > H$ changes either field with probability at most $\delta\tau T_H$. Letting $H \rightarrow \infty$ proves equality of the untruncated laws. The argument uses no infinite moment inversion and no exponential moment. $\square$

The information in (12.1) is strictly richer than its mass. For a disconnected target it supplies the signed translations needed to reconstruct intersections using the known set I , instead of assuming that intersection measures are minima of pair masses. The theorem therefore does not extend unmarked covariance identification to arbitrary Borel targets. Nor does averaging the channel measures identify an arbitrary mixture of profile laws.

12.3 A full-sequence criterion with angular marks

For an arbitrary deterministic phase sequence put

$$\gamma_N^I(v) = r_N(v) g_I(r_N(v)), \quad \beta_N^I(v) = g_I(r_N(v)). \quad (12.4)$$

Both coordinates are uniformly bounded: $g_I(t)$ vanishes outside a fixed bounded interval of displacements. For the original interval target we write $\gamma_N(v) = r_N(v)\beta_N(v)$.

Corollary 12.3 (Signed pair coefficients classify marked convergence). *For the model with fixed known bounded Borel target I , the fully marked arithmetic fields Ξ_N converge weakly if and only if*

$$(\beta_N^I(v), \gamma_N^I(v)) \text{ converges for every fixed } v \neq 0. \quad (12.5)$$

In the interval model this is the convergence of $(\beta_N(v), r_N(v)\beta_N(v))$ in each channel.

Proof. For a real displacement t , denote the measure in (12.1) with shift t by $\mathcal{Q}(t)$. Its mass is $g_I(t)$, and

$$\int (w_2 - w_1) \mathcal{Q}(t)(dw_1, dw_2) = t g_I(t).$$

The map $t \mapsto \mathcal{Q}(t)$ is weakly continuous: for a continuous test on a fixed compact rectangle, translation continuity of $\mathbf{1}_I$ in L^1 controls the membership discrepancy, and uniform continuity of the test controls the change from $(w, w + t)$ to $(w, w + t')$. If the mass tends to zero the measure tends to zero. If its limiting mass is positive, the ratio of the second coordinate in (12.4) to the first recovers the limiting signed shift. Thus a limiting pair of coefficients uniquely specifies its channel measure.

The exact bounded first moment gives tightness of the fields on a fixed compact rectangle. Refine any subsequence to an effective profile by Proposition 5.2 and apply Theorem 12.1. Under (12.5), all such profiles have the same channel measures; Theorem 12.2 makes their marked laws equal. This proves sufficiency. For necessity, a nonconvergent bounded coefficient pair has two different subsequential limits. Refine both subsequences to profiles. Their channel measures differ, as is seen from their masses and signed-displacement integrals, so Theorem 12.2 makes their limiting laws different. This contradicts full-sequence convergence. Only the finite moments of the profile laws are used; no unbounded arithmetic moment is transferred from weak convergence. $\square$

For example, let $\mathcal{E} = \{2\}$ and alternate $F_N(2) = e^{2\pi ic/N}$ and $e^{-2\pi ic/N}$ with $0 < c < \tau$. Every unmarked coefficient $\beta_N(k)$ has the same limit on the two subsequences, but $\gamma_N(1)$ alternates between $c(\tau - c)$ and $-c(\tau - c)$. The spatial field converges and the angularly marked field does not. This uses the same observed angular coordinate, not a reflection applied after observation.

A small change of signed displacement moves a positive channel measure onto a different angular graph. Such measures can be mutually singular. The identification and weak-convergence conclusions above do not give a total-variation continuity modulus for marked laws; the modulus in Appendix A concerns the unmarked spatial laws only.

13 Scope and conclusion

The theory separates the accuracy of the arithmetic comparison, the information retained by an observation, and the geometry of a cluster. The direct Haar comparison controls the moving germ kernel uniformly. The pair-overlap array then decides whether the corrected count and marked field are Poisson; the point-minus-germ deficit gives a more economical quantitative parameter than the number of sibling pairs.

For deterministic interval profiles, the complete spatial covariance determines the unmarked law. Forest geometry yields both its positive Laplace formula and a quadratic finite-depth total-variation modulus. For a fixed known bounded Borel target, the complete covariance on positions and angular marks reconstructs the fully marked law through signed channels. These inverse statements use ideal measures on all rational rays, not finitely many estimated covariances. They do not recover invisible relations, and they do not identify arbitrary mixtures by averaging their covariances.

An interior window selects relations by both angular precision and logarithmic spatial ratio. Its per-germ cluster-size support is bounded exactly when the corresponding subgroup of $\mathbb{R}^2$ is

discrete. Unbounded support does not mean an infinite realization: the corrected field is finite almost surely. The terminal prefix-count path retains shared-germ dependence and must be distinguished from an independent-increment process or a scale flow.

The scope table in Section 1.6 and the input register in Appendix B delimit these conclusions. The broader source-density theorem in *Arithmetic clusters* [25, Theorem 4.1] retains prime-core intensities absent from the Haar model treated here. The present results keep the exceptional set fixed and finite and the regular source Haar. Growing exceptional sets, uniform high-intensity limits and ordered record or occupation laws for shared-germ clusters require their own hypotheses and estimates; they are outside this article. The direct arithmetic moment calculation stops at second order, and no Liouville cancellation is deduced from these random models.

Availability and formalization status

The source archive contains the complete LaTeX manuscript, its frozen bibliography, build instructions and finite diagnostic programs. The proofs of the results specific to this article are given here; Appendix B identifies external tools and antecedent constructions. The cited editions of the corpus are identified by their version-specific publication DOIs.

The results of this article are not claimed to be formalized in Lean. The formal development associated with *Long runs and rare patterns* has the scope stated in that work; it does not formalize the arguments of this article. Finite diagnostics detect algebraic and implementation errors but do not replace the asymptotic proofs.

Declaration on artificial-intelligence assistance

The research was organized and directed by the author with substantial assistance from GPT-6 Astra (OpenAI), the only language-model system used for this article. Assistance included exploring proof routes, drafting and reorganizing mathematical arguments, writing finite diagnostic programs, checking references, producing the manuscript and conducting adversarial reviews through separate instances. The author determines the claims and scope and accepts responsibility for the arguments, references and remaining errors. AI-assisted checks are not independent human peer review. The mathematical claims rest on the printed arguments and specified external results, not on model output or the success of finite tests.

A Finite reconstruction and total-variation stability

Pair overlaps determine the unmarked spatial law for interval targets. Here interval geometry also controls the dependence on the pair data in total variation. Both laws compared below are diffuse profile laws, not an arithmetic grid and a diffuse limit.

For a finite nonempty subset $J \subset \mathcal{D}$, define

$$\beta_J = \begin{cases} \tau, & |J| = 1, \\ \min_{s \neq t \in J} \beta(\mathbf{v}(s) - \mathbf{v}(t)), & |J| \geq 2. \end{cases} \quad (\text{A.1})$$

If J contains different classes, its common-intersection measure is zero. If it lies in one class, the intersection of the translated equal-length intervals has measure $(\tau - \max_{s,t \in J} |a_s - a_t|)_+$. In both cases its measure is exactly β_J .

Proposition A.1 (Quadratic finite-depth total-variation modulus). *Let the deterministic profiles and interval pair arrays be as in Theorem 8.4. With $S_H = \mathcal{D} \cap [1, H]$ and $k_H = |S_H|$,*

put

$$e_H = \max_{\substack{s, t \in S_H \\ s \neq t}} |\beta(\mathbf{v}(s) - \mathbf{v}(t)) - \beta'(\mathbf{v}(s) - \mathbf{v}(t))|,$$

with maximum zero when there is no pair. Then

$$d_{\text{TV}}(\mathcal{L}(X_{L,h}), \mathcal{L}(X_{L',h'})) \leq 2\delta\tau T_H + \delta \min\{3^{k_H}, 8k_H^2\} e_H. \quad (\text{A.2})$$

The right side may also be capped by 1. The quadratic coefficient does not assert optimality or a polynomial-time statistical reconstruction procedure.

Proof. Fix H and a core coordinate x . The eligible set is $S(x) = \{s \in S_H : sx \leq 1\}$. For nonempty $J \subseteq S(x)$, let $q_J(x)$ be the angular measure, summed over the represented class germ spaces, of producing exactly the descendant subset J . Finite inclusion–exclusion gives

$$q_J(x) = \sum_{J \subseteq U \subseteq S(x)} (-1)^{|U|-|J|} \beta_U. \quad (\text{A.3})$$

If J spans different classes every term is zero; otherwise this is the usual identity in its class. The truncated spatial field is a Poisson sum of configurations $\sum_{s \in J} \delta_{sx}$ with intensities $\delta q_J(x) dx$. Since finite minima are 1-Lipschitz in the sup norm, summing absolute differences in (A.3) bounds $\sum_{J \neq \emptyset} |q_J(x) - q'_J(x)|$ by $3^{k_H} e_H$.

A quadratic interval bound. Consider k labelled intervals T_i of common length τ , allowing separate copies of the line. For a finite family these copies may be represented by intervals far enough apart on a single line. Set $c_J = |\bigcap_{i \in J} T_i|$ and let q_J be its exact-subset measure. For a nonempty J , put $O = \{1, \dots, k\} \setminus J$. Apply Lemma 4.1 to the intervals $(\bigcap_{j \in J} T_j) \cap T_i$, $i \in O$. They may be empty and need not have equal lengths. The exact forest identity gives

$$q_J = c_J - \sum_{i \in O} c_{J \cup \{i\}} + \mathsf{T}\left(O; (c_{J \cup \{i,j\}})_{i \neq j \in O}\right). \quad (\text{A.4})$$

For another family write $e = \max_{i < j} ||T_i \cap T_j| - |T'_i \cap T'_j||$. Every c_J changes by at most e ; singleton measures agree. A forest on $|O|$ vertices has at most $(|O| - 1)_+$ edges, so

$$|q_J - q'_J| \leq \{1 + |O| + (|O| - 1)_+\} e \leq 2ke.$$

There are at most $2k$ nonempty exact subsets of positive measure in either family: its active index set is constant between consecutive endpoints, and there are at most $2k$ endpoints. Endpoint coincidences do not affect Lebesgue measure. The union of the two supports has cardinality at most $4k$. Hence

$$\sum_{J \neq \emptyset} |q_J - q'_J| \leq 8k^2 e. \quad (\text{A.5})$$

For $k = 1$ all differences are zero, and for $k = 0$ the assertion is empty. This proof also covers disjoint profile classes, which were represented on separate lines.

Apply (A.5) at each x to its eligible intervals. There are at most k_H of them and their pair discrepancy is at most e_H . Couple the two Poisson measures on (x, J) by their common intensity, then map a germ to its spatial configuration. The coupling cost is at most $\delta \int_0^1 \sum_J |q_J(x) - q'_J(x)| dx$. Taking the better of the two finite bounds and restoring the two omitted multiplier tails, each of mass at most $\delta\tau T_H$, proves (A.2). $\square$

Corollary A.2 (A global modulus in the weighted pair distance). *Let $\mathfrak{d} = d_{\text{pair}}(\beta, \beta')$ be the distance of Theorem 8.3. For $0 < \mathfrak{d} \leq 1/4$,*

$$d_{\text{TV}}(\mathcal{L}(X_{L,h}), \mathcal{L}(X_{L',h'})) \ll_{\varepsilon, \tau} \sqrt{\mathfrak{d}} (1 + \log(1/\mathfrak{d}))^{2b_0}. \quad (\text{A.6})$$

At $\mathfrak{d} = 0$ the laws are equal. This is a modulus for unmarked spatial profile laws; it is neither a grid-to-diffuse comparison nor a marked total-variation estimate.

Proof. If $s, t \leq H$ then $H(\mathbf{v}(s) - \mathbf{v}(t)) \leq H$, so $e_H \leq H\mathfrak{d}$. The elementary multiplier count and dyadic summation give

$$k_H \ll_{\varepsilon} (1 + \log H)^{b_0}, \quad T_H \ll_{\varepsilon} \frac{(1 + \log H)^{b_0}}{H}.$$

Use $H = \mathfrak{d}^{-1/2} \geq 2$ in the quadratic part of (A.2). The two terms are bounded by the right side of (A.6). At zero distance, equality follows either by sending $H \rightarrow \infty$ in that proposition or by Theorem 8.4. $\square$

For completeness, the finite inversion also gives the truncated Laplace exponent, for continuous nonnegative spatial f :

$$\log \mathbb{E} e^{-X^{(H)}(f)} = \delta \sum_{\emptyset \neq J \subseteq S_H} (-1)^{|J|} \beta_J \int_0^{1/\max J} \prod_{s \in J} (1 - e^{-f(sx)}) dx. \quad (\text{A.7})$$

Only a finite alternating expansion is used. The tail is removed at the level of configurations, without any moment-determinacy assumption. The linear Laplace bound of Theorem 8.3 and the total-variation bounds here retain their different metrics.

B Proof inputs and internal deductions

This register separates classical tools and antecedent constructions from the deductions proved here. The regular-prime source is Haar throughout its arithmetic comparison. The broader source-density theorem cited below is an alternative comparison, not an additional hypothesis of the direct Haar proof.

Input or tool	Form used	Internal deduction or role
Classical process Chen–Stein comparison [3, Theorem 2, Section 3.1]; conditional form [25, Theorem 2.2]	Exact graph for complete marks, $2(b_1 + b_2)$ in the supremum convention	Lemma 2.1 recalls mark refinement; Theorem 2.2 verifies all arithmetic costs.
Haar character criterion [21, arXiv:1702.01470v2, Lemma 2.1]	Character formula for arbitrary regular cores; full rank gives product Haar	Direct germ comparison and Proposition 3.5; proportional power pairs are counted explicitly.
Rankin bound and elementary prime estimates [11, Section 1]	Euler-product upper bound for $\Psi(N, Y)$ and Chebyshev partial summation	Corollary 2.3 derives the kernel budget; the same rate scale already occurs in [25, Section 6.5].
<i>Arithmetic clusters</i> [25]	Germ kernel, exact correction and deficit/anchor method; alternative comparison	Section 2.1 retains the kernel. Theorem 2.2 proves the Haar transfer directly; the broader Theorem 4.1 of [25] is not required.
Poisson mapping, marking and factorial identities [17, 16]	Finite-intensity germ measures and fixed-order moments	Profile laws, anchor coupling and covariance formulas.

Input or tool	Form used	Internal deduction or role
Tree sieve and forest bounds [19,7,12,30]	Exact interval union identity and finite forest weights	Deficit certificates, positive Laplace reconstruction and Appendix A's sparse-subset bound.
Finitely generated abelian groups	Structure of $\mathbb{Z}^{\mathcal{E}}/L$ and its torsion	Proposition 5.2 proves profile realizability; Section 10 identifies the observation-window geometry.

The Hawkes and log Gaussian Cox comparisons [5,9,18], and the name of the Hermite law [13], provide context, not new proof inputs. All limits keep their stated order of truncation, conditioning and averaging. Section 12.2 uses known-target overlap components; Appendix A uses interval geometry and summable tails. Finite diagnostics in the archive do not certify these asymptotic proofs.

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