

Records, scale flows, and Gaussian limits from lattice Poisson fields

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Version 2.1

This research received substantial assistance from AI systems; see the declaration at the end of the article.

Abstract

How does a rare-event field determine the history of its extremes? Starting from the complete signed run-field comparison in *Long runs and rare patterns*, we develop records, censoring, occupation and Gaussian limits for the resulting lattice Poisson target. The arithmetic input retains every site, full exact length and actual sign, at growing intensity under a one-factor information–intensity budget.

Strict integer records, their signed imbalance and logarithmic leadership time have a joint phase-uniform Brownian limit with explicit covariance, independent of the terminal critical field and its extremal paths. Right censoring gives scalar M_1 convergence for the visible maximum and J_1 convergence for the first-holder sign. Physical occupation instead has a nondegenerate phase-indexed limit. Canonical interpolation connects threshold thinning and a stationary scale flow without identifying their record rules or their Gaussian constants.

The signed spatial theory includes weighted exact levels, fixed-window Ornstein–Uhlenbeck paths, two correlated threshold-compression noises, and rectangle fields with explicit grid centering. Direct partition and quadratic-reconstruction proofs provide alternative routes. For each specified observation, the target calculations identify its limit and the dependencies within the joint constructions.

Keywords. Records; Poisson approximation; random multiplicative functions; occupation times; extremal processes; lattice effects.

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Contents

1	Introduction	2
2	Arithmetic input and observation maps	7
3	Integer records and logarithmic leadership	11
4	Maxima visible in a finite word	16
5	A Brownian history independent of terminal extremes	20
6	Two observation clocks, two occupation laws	22
7	One Poisson geometry, two process clocks	25
8	A geometric target theorem with finite colours	36
9	Observation scope and further questions	38
	Availability and artificial-intelligence assistance	40

1 Introduction

A long constant run can be viewed in three ways. It is a rare point, carrying a location and a length. It can establish a record among earlier runs. And, as the observation window grows, it can hold that record for a substantial time. A Poisson approximation for the first description does not automatically settle the other two: records use the order of arrivals, equal integer lengths create ties, and a finite observer does not know a run's final length when it starts.

This paper develops a common transfer for these descriptions. Its arithmetic application is a sequence with strong long-range dependence. Choose independent fair signs at the primes and define

$$f(1) = 1, \quad f(n) = \prod_p f(p)^{v_p(n)}. \quad (1.1)$$

In particular $f(n^2) = 1$ and $f(2n) = f(2)f(n)$. Nevertheless, the rare long runs admit a complete signed Poisson comparison on their exact grid [14, Theorem 7.7]. This arithmetic theorem is our input. The question here is what becomes of the field when one follows its history: which readout represents an intrinsic record, which topology absorbs censoring, and which dependence survives a Gaussian normalization?

1.1 The three distinctions that organize the paper

First, an integer record is not a continuously ranked record. For example, successive maximal runs with lengths and signs

$$4^+, \quad 2^-, \quad 7^+, \quad 7^-, \quad 11^+$$

produce strict records at lengths 4, 7, 11. The second run of length 7 does not replace the first holder. Adding independent fractional parts to the lengths can create additional records within a tied integer level. That interpolation is useful for a continuous flow, but changes the record-time observable.

Second, a record run is one episode, not a collection of unit improvements. The run of length 11 above produces one new episode, although its visible length raises the preceding maximum from 7 to 11 in four steps. At its start, an observer of the finite word knows neither that it will be a record nor its final length. The record is recognized at its eighth symbol.

Third, logarithmic and physical observation time weight the history differently. A logarithmic interval of length T contains T doublings. Its last few doublings contribute a negligible proportion of that interval, but a positive proportion of the physical prefixes. Consequently a law of large numbers for logarithmic leadership need not imply concentration of physical-time occupation.

The rules used throughout are strict improvement and retention of the earliest holder at a tie. Changing either rule defines a different process.

1.2 Main conclusions

Here is an overview; all arithmetic parameters and conditioning hypotheses are fixed in Section 2. We observe the finite prefixes from $m_0 + 1$ to $n + 1$, where

$$n = M - L, \quad T = \log_2(n/m_0) \longrightarrow \infty.$$

Although this interval moves towards the terminal scale, every record is compared with *all* previous runs, not merely with those in the interval. Let a_i, q_i, σ_i be the start, full length, and sign of the i th maximal run, and put $H_{i-1} = \max_{j < i} q_j$, with $H_0 = 0$. A strict record has $q_i > H_{i-1}$ and is recognized at prefix $c_i = a_i + H_{i-1}$.

Define the global recognition counts and the visible maximum by

$$E^\sigma(m) = \sum_{\substack{i: q_i > H_{i-1} \\ \sigma_i = \sigma}} \mathbf{1}_{\{c_i \leq m\}}, \quad R^{\text{obs}}(m) = \max_i (\min(q_i, m - a_i + 1))_+.$$

Let $Z^{\text{obs}}(m)$ be the sign of the earliest run attaining this visible maximum. For $m(v) = \lfloor m_0 2^v \rfloor + 1$, set

$$\begin{aligned} \Delta E_M^\sigma(v) &= E^\sigma(m(v)) - E^\sigma(m_0 + 1), \\ N_M^{\text{obs}}(v) &= \Delta E_M^+(v) + \Delta E_M^-(v), \quad D_M^{\text{obs}}(v) = \Delta E_M^+(v) - \Delta E_M^-(v), \\ A_M^{\text{obs}}(v) &= \int_0^v Z^{\text{obs}}(m(s)) ds. \end{aligned}$$

Thus the counters start at zero, but the historical maximum and its holder are not reset. A run begun before the initial prefix and recognized after it is counted in the window. These are observables of the finite word: an episode is recognized precisely when its visible length first exceeds the preceding maximum. The normalized history is

$$\mathcal{G}_M(t) = \frac{1}{\sqrt{T}} \left(N_M^{\text{obs}}(Tt) - \frac{Tt}{2}, D_M^{\text{obs}}(Tt), A_M^{\text{obs}}(Tt) \right), \quad 0 \leq t \leq 1. \quad (1.2)$$

One arithmetic input, before any limit. Let $\Lambda = (M - L)2^{-L} \rightarrow \infty$, and let the admissible small-prime conditioning event have information cost I_M . Under

$$I_M + \log \Lambda \leq V_M - c\nu_M,$$

Input 2.1 compares the full array of sites, exact lengths and signs with its independent Poisson target, with error $O(e^{-c'\nu_M} + M^{-1/3+\varepsilon})$ for every $0 < c' < c$ and $0 < \varepsilon < 1/3$. The target retains the exact spatial grid. Corollary 2.2 gives the common-observation rule used below.

A phase-free Brownian record history. Under the one-factor shell conditions of Theorem 2.3, with an initially populated retained field,

$$\mathcal{G}_M \implies \mathbb{B}_\Sigma \quad \text{in } D([0, 1], \mathbb{R}^3), J_1, \quad (1.3)$$

where

$$\Sigma = \begin{pmatrix} 1/4 & 0 & 0 \\ 0 & 1/2 & 1 \\ 0 & 1 & \kappa \end{pmatrix}, \quad \kappa = \frac{\pi^2}{6(\log 2)^2}. \quad (1.4)$$

Here $\mathbb{E}[\mathbb{B}_\Sigma(t)\mathbb{B}_\Sigma(u)^\top] = \min(t, u)\Sigma$. No lattice-phase subsequence is required. In particular the positive and negative *global* record counts, centered by $Tt/4$ and divided by $\sqrt{T}$, have limiting covariance

$$\begin{pmatrix} 3/16 & -1/16 \\ -1/16 & 3/16 \end{pmatrix}. \quad (1.5)$$

The negative off-diagonal entry expresses competition for a common record, not independence of two colour-specific record problems.

An independent terminal extreme field. Write $b = \lfloor \log_2 n \rfloor$ and $\theta_M = \log_2 n - b$. The terminal signed field converges, along $\theta_M \rightarrow \theta \in [0, 1]$, to

$$\Pi_\theta \sim \text{PPP} \left(du \sum_{r \in \mathbb{Z}} \sum_{\sigma \in \{-1, +1\}} 2^{\theta-r-2} \delta_{(r, \sigma)} \right) \quad \text{on } [0, 1] \times \mathbb{Z} \times \{-1, +1\}. \quad (1.6)$$

It has finitely many points above each integer level, though infinitely many towards $r = -\infty$. Let $Q_\theta(u)$ be its maximum mark before $u > 0$, and $Z_\theta(s)$ the sign of its earliest maximizer before 2^s .

For a fixed $S > 0$, let $Y_M(s)$ be the longest run actually visible at prefix $\lfloor n2^s \rfloor + 1$, minus b , and let $Z_M(s)$ be its holder's sign. The joint conclusion is

$$(\mathcal{G}_M, \Xi_M, Y_M, Z_M) \implies (\mathbb{B}_\Sigma, \Pi_\theta, Q_\theta(2^\cdot), Z_\theta). \quad (1.7)$$

The Brownian motion is independent of the entire terminal block. The last three objects are *not* independent: both paths are functions of the same field. Their topologies are, respectively, vague convergence, scalar M_1 , and scalar J_1 . Right censoring makes the visible maximum fail to be J_1 -tight; it is not an optional weakening of the conclusion. Theorems 4.5 and 5.2 give the precise statements and proofs.

A physical occupation fraction that does not concentrate. If O_M^+ is the fraction of the first n queried physical prefixes at which the visible holder is positive, then its limit is $(1 + V_\theta)/2$, where

$$V_\theta = \int_0^1 Z_\theta(\log_2 u) du.$$

The variable V_θ has a bounded, symmetric, nondegenerate law. Its variance $K(\theta)$ has the explicit series in Theorem 6.1, and

$$\int_0^1 K(\theta) d\theta = 2 - \frac{1}{\log 2}. \quad (1.8)$$

The variance is of order one; only its nonzero modulation with phase is small. By contrast, logarithmic positive leadership occupies an asymptotic fraction $1/2$. The phase integral in (1.8) is an identity about the family of limits, not an assumption that the arithmetic phase has been randomized.

A signed spatial hierarchy at the same cost. There is also a Gaussian question that requires spatial resolution but not the order of individual arrivals. At intensity $\Lambda = (M - L)2^{-L} \rightarrow \infty$, center each exact-length, sign-resolved measure by its deterministic Poisson target mean and divide by $\sqrt{\Lambda}$. Under the same one-factor information budget, these measures converge jointly to independent spatial white noises in every fixed weighted Hilbert space with finite Gaussian trace. Their threshold sums give a spatial Ornstein–Uhlenbeck process on a fixed time interval. Compressing the entire threshold range produces two correlated white noises, for integer and continuous sampling. The low-level fields, the compression pair, and the critical Poisson field are three independent limiting blocks.

These assertions require different target estimates: a norm tail for the exact spectrum, a maximal time estimate for paths, and an integrated estimate for compression. They are stated separately in Section 7.6; the rectangle limit retains its own centering and depth conditions. This three-block statement concerns spatial fluctuations, not the Brownian record history in (1.7).

1.3 One input, two probabilistic tasks

The principal route is

arithmetic run array $\xrightarrow[\text{Input 2.1}]{\text{total variation}}$ same-grid Poisson array $\xrightarrow{\text{common readout}}$ observed target.

The last object still depends on M . For ordered observables, the work is to retain strict integer ties, the initial holder and recognition delays while passing from a grid to continuous arrival times. For Gaussian fields, the work is to establish the correct centering, the tails in the declared state space and the joint separation from critical points.

The direct partition route remains available without microscopic coupling. For R cells of masses w_r , its ordered-edge inequality

$$\sum_{x \sim y} \frac{1}{\sqrt{w(x)w(y)}} \leq \sum_x \frac{\deg(x)}{w(x)} \leq nR\Delta$$

separates the number of cells from their smallest mass. A marked quadratic estimate then reconstructs weak spatial laws from fixed-cell comparisons. These arguments are collected as alternatives in the companion, rather than imposed on the main route. In particular, no mesh penalty remains for the count comparison itself under Input 2.1.

Section 9 compares these interfaces through a parity example, an exact disintegration identity and the arithmetic cumulant example of [14, technical companion, Theorem G.9].

1.4 Relation to earlier work

Rare-run probabilities, geometric overshoots and Poisson approximation have classical antecedents. Gordon–Schilling–Waterman [7] treat long independent head runs, including lattice effects. Our arithmetic input is the complete signed run-field comparison proved in *Long runs and rare patterns* [14, Theorem 7.7 and technical companion, Appendix F]. The present paper studies the observations and limiting processes obtained from that field. The direct partition and reconstruction routes retained here avoid that microscopic theorem but share arithmetic ingredients; they are not independent validations of all the arithmetic.

The theory of discrete records, including functional limits, predates this work: see Vervaat [17], Prodinger [15] and Bai–Hwang–Liang [1]. Gouet–López–Sanz [8, Example 1] explicitly recover the geometric constants $1/2$ and $1/4$. Our contribution is the particular phase-uniform joint vector of strict global record counts, signed imbalance and first-holder leadership time, with its covariance and arithmetic transfer, not a new functional central limit theorem for geometric counts alone. Gnedin’s continuously ranked Poisson layers [5, Proposition 2] concern a different record rule.

Goldie–Grübel [6] prove a joint functional limit theorem for record counts and time spent inside record exponential interarrival intervals. Our occupation instead integrates the sign of the earliest current maximizer until its next strict replacement, including time between later nonrecord arrivals; its normalization and limiting covariance differ. Grübel–Reimers [9] study discrete-uniform record sums with a renewal-time interpretation. The physical-time law and its nonconstant periodic variance concern this leadership observable. Mellin analysis is classical [4]; the contribution is the resulting occupation formula.

Holland–Todd [10] use point-process convergence to obtain extremal and record processes in non-uniformly hyperbolic dynamical systems, including limits for record times and values. Here integer ties, earliest-holder signs and finite-word recognition require the joint censoring theorem in scalar M_1 and scalar J_1 , coupled to history and terminal extremes. The general tools are classical: M_1 methods [18,12], mixing invariance [3], Poisson mapping and marking

[13], and martingale limits [19]. Immigration–death, OU processes and the stationary OU sheet [11] are established limits; the companion gives the cosine-space estimates for the white-noise theory [16] used here.

Gaussian–Poisson mixed limits and asymptotic independence have a general theory on Poisson space [2]; a finite-level product is already in [14, Theorem 5.12]. The additional result here is the signed functional hierarchy in Table 1: a fixed weighted spectrum, paths, and the joint discrete/continuous compression through the non-Gaussian critical threshold. Its three-block product is distinct from the record-history product, whose proof removes a recent spatial strip. The marked quadratic reconstruction in the companion gives an alternative route from fixed partitions, uniformly in the declared types and auxiliary marks.

Section 8 extends the marked record calculation within a finite-colour Poisson target. An arithmetic application beyond the binary model would additionally require macroscopic, tail and ordering bounds; no such extension is asserted here.

1.5 Conventions and proof inputs

The following conventions keep the different resolutions compatible.

Object	Convention used here
Base-contained starts	$2 \leq x \leq M - L + 1$, $j = x - 1$, $n = M - L$; position j/n and exact intensity $\Lambda = n2^{-L}$.
Terminal centering	$b = \lfloor \log_2 n \rfloor$ is constant along a terminal path. A phase limit 1 is the level-shifted phase-0 law.
Integer records	A strict global improvement; ties retain the first holder. Classifying global records by sign differs from taking records separately within each sign.
Visible episodes	One episode is counted at first recognition, even when several later unit increments raise the visible maximum. Counters start at the observation window; the historical maximum is not reset.
Clocks	Physical prefix length m ; logarithmic scale time $\log_2(j/m_0)$; and excess threshold t . Fractional lengths interpolate the last clock and are auxiliary random variables.
Gaussian centering	The deterministic target measure is subtracted, not an unproved formula for the actual conditional expectation.

Records and visible paths use the count consequence of Input 2.1 together with the shell reconstruction estimates; the direct proof of Theorem 2.3 is optional in this regime. Appendix F develops the signed target limits. The alternative quadratic route is Appendix G. Both routes keep the same target topologies and joint dependence.

1.6 Reading guide

Section 2 states the arithmetic input and the common-readout principle before introducing the geometric shells used for records. Readers interested in ordered observables can then proceed through Sections 3–6, using the companion’s Appendix B for reconstruction and censoring. Section 7 places threshold and scale processes in one Poisson geometry. Table 1 separates the spaces, horizons and limiting dependencies of the signed theory.

In the companion, Appendices C, D and F form the functional target route. Appendices A, E and G are the direct alternatives. The arithmetic pivot proof and its Palm/cumulant consequences are in *Long runs and rare patterns* [14, technical companion, Appendices F–G]. Section 9 discusses the comparison of proof interfaces and further process questions.

2 Arithmetic input and observation maps

Throughout $\beta = \log 2$; unlabelled logarithms are natural. We use $d_{TV}(\mu, \nu) = \sup_B |\mu(B) - \nu(B)|$. An exact run mark includes its full rightward length, even when that length is not yet visible in a finite prefix.

2.1 Runs, exact marks, and the terminal convention

For $x \geq 2$ and $L \geq 1$ put

$$J_{x,L} = \mathbf{1}_{\{f(x-1)=-f(x), f(x)=\dots=f(x+L-1)\}}.$$

On a start let ℓ_x be its full length. The source bounds imply that all these lengths are almost surely finite. Set

$$L = \lfloor \log_2 M \rfloor - a_M, \quad n = M - L, \quad p = 2^{-L}, \quad \Lambda = np, \quad a_M \in \mathbb{N}_0, \quad a_M = o(\log M). \quad (2.1)$$

All statements concern sufficiently large M , so $L \geq 1$. At $j = x - 1 \in \{1, \dots, n\}$ define

$$I_{j,e,\sigma} = \mathbf{1}_{\{j+1 \text{ is a run start, } \ell_{j+1}=L+e, f(j+1)=\sigma\}}, \quad \gamma_{e,\sigma} = 2^{-e-2}. \quad (2.2)$$

The comparison mean is $p\gamma_{e,\sigma}$. These are actual signs of runs, not labels added to an unsigned field.

The source field retains $2 \leq x \leq M - L + 1$: the base window of length L is contained in $[1, M]$, but the full run can continue beyond M . Our last queried physical prefix is *exactly* $n + 1 = M - L + 1$. This choice makes every queried start available in one terminal field. It does not change full marks into censored lengths; that change is proved in Section 4.

2.2 Information and the hard cutoff

We recall the exact deterministic scales from [14, Proposition 4.2]. Put $H = \log M$. For large ν , let $u(\nu) > 1$ solve $e^u/u = \nu$, and write

$$D(\nu) = e^{u(\nu)} - \text{Ei}(u(\nu)).$$

The hard scales are

$$V_M = D(H/V_M), \quad \nu_M = H/V_M, \quad Y_M^\# = \lfloor e^{V_M} \rfloor. \quad (2.3)$$

The relevant large root is uniquely specified for sufficiently large M . In particular

$$V_M \sim \sqrt{\frac{1}{2}H \log H}, \quad V_M = o(H), \quad \log H = o(\nu_M).$$

Let $\mathcal{A}_M$ have positive probability and be measurable with respect to $\mathcal{F}_{Y_M^\#} = \sigma(f(p) : p \leq Y_M^\#)$. Its information cost is $I_M = -\log \mathbb{P}(\mathcal{A}_M)$. The principal condition is

$$\Lambda \geq 1, \quad I_M + \log \Lambda \leq V_M - c\nu_M, \quad c > 0 \text{ fixed}. \quad (2.4)$$

All arithmetic probability limits below are under $\mathbb{P}(\cdot \mid \mathcal{A}_M)$; choosing $\mathcal{A}_M$ to be the whole space gives unconditional statements. Properties of a Poisson target are stated under its own law.

2.3 A complete arithmetic input

The following result is Theorem 7.7 of the published Version 3 of *Long runs and rare patterns* [14], restated in the terminal convention used here. Its selected-prime proof is in the technical companion of that source, Appendix F. The assumptions below are part of that theorem; in particular the intensity must diverge.

Write $\mathbf{I}_M = (I_{j,e,\sigma})_{j,e,\sigma}$ and let $\mathbf{P}_M = (P_{j,e,\sigma})_{j,e,\sigma}$ have independent coordinates $P_{j,e,\sigma} \sim \text{Pois}(p\gamma_{e,\sigma})$. Both fields take values in the countable space of arrays on $\{1, \dots, n\} \times \mathbb{N}_0 \times \{-1, +1\}$ with finite total mass. The target has total mean Λ ; the arithmetic field has at most one atom at each site.

Input 2.1 (Full same-grid signed comparison). Assume (2.1) and (2.4), with $\Lambda \rightarrow \infty$. For every $0 < c' < c$ and $0 < \varepsilon < 1/3$,

$$\Delta_M^{\text{mic}} := d_{\text{TV}}(\mathcal{L}(\mathbf{I}_M \mid \mathcal{A}_M), \mathcal{L}(\mathbf{P}_M)) \ll_{c,c',\varepsilon} e^{-c'\nu_M} + M^{-1/3+\varepsilon}. \quad (2.5)$$

All spatial labels, full exact lengths, and actual signs are retained simultaneously. No partition, mesh, or additional observation-window assumption is required for this comparison.

The word “Input” denotes the external theorem just cited. Its one-window, two-window, cutoff and scalar-tail estimates are proved in *Long runs and rare patterns*, independently of the process limits studied here.

Corollary 2.2 (Every measurable readout of the retained field). *Under Input 2.1, every measurable map Φ_M into a standard Borel space satisfies*

$$d_{\text{TV}}(\mathcal{L}(\Phi_M(\mathbf{I}_M) \mid \mathcal{A}_M), \mathcal{L}(\Phi_M(\mathbf{P}_M))) \leq \Delta_M^{\text{mic}}. \quad (2.6)$$

The same conclusion holds for any common Markov kernel, including auxiliary marking followed by a measurable readout. The map or kernel may depend on M . Equivalently,

$$\sup_{F: 0 \leq F \leq 1} |\mathbb{E}[F(\mathbf{I}_M) \mid \mathcal{A}_M] - \mathbb{E}F(\mathbf{P}_M)| = \Delta_M^{\text{mic}}.$$

Proof. Pull an output event back to the input space. For a kernel K_M , apply the variational formula to $K_M(\cdot, B) \in [0, 1]$. The variational formula follows from the layer-cake representation of a bounded nonnegative function; indicator tests give equality. $\square$

Scope of the common readout. There is no continuity, dimension, number-of-queries, or horizon restriction on *this comparison with the moving same-grid target*, provided the readout uses only the retained field and a common random seed. In particular every deterministic partition, even the singleton partition, inherits (2.5) without a cell-count or smallest-cell penalty. Adaptive queries determined by that same data are also one measurable readout. Multiple atoms in a target coordinate receive separate auxiliary marks; any rule requiring a simple configuration is extended measurably to other configurations.

This corollary does not create starts beyond $n + 1$, the missing run from 1, or information about the whole infinite multiplicative sequence. Full lengths already recorded may be inspected beyond M , but the rest of that future is not supplied. The finite-word record theorems retain their boundary, initial-contender, and censoring arguments precisely because their observables initially involve more than the retained interior field.

Equation (2.6) compares laws on a common prelimit space. The target limit, its joint dependence, and admissible horizons or weights are established for each observation below; the comparison alone does not control unbounded moments. The arithmetic source remains the binary model under (2.4), with $\Lambda \rightarrow \infty$.

For the binary application, Input 2.1 supplies the complete-field hypothesis of Theorem 7.5. Appendix F proves the signed target refinements, including their joint dependence, before applying this common readout. The alternative fixed-partition arguments are kept separately in Appendices A, E and G.

Grid and centering dictionary. The input in [14, Theorem 7.7] uses the same index $j = x - 1$ and $n = M - L$. Relabeling x as j is a bijection of array coordinates, not a total-variation approximation between point measures on two different grids. Here the position is j/n , not x/M . If $b_M = \lfloor \log_2 M \rfloor$ and $b = \lfloor \log_2 n \rfloor$, then its critical level is

$$L + e - b = e - a_M + (b_M - b).$$

In particular $J_\Lambda = \lfloor \log_2 \Lambda \rfloor = b - L$ and $e - J_\Lambda = L + e - b$. The possible one-level shift and the endpoint phase are retained throughout.

2.4 Shells for ordering; a direct comparison as an alternative

Choose integers $1 \leq m_0 < n$ and $J \geq 1$. Put

$$T = \log_2(n/m_0), \quad h = T/J \leq 1, \quad m_r = \lfloor m_0 2^{rh} \rfloor \quad (r < J), \quad m_J = n.$$

The cells are $A_0 = \{1, \dots, m_0\}$ and $A_r = \{m_{r-1} + 1, \dots, m_r\}$ for $1 \leq r \leq J$. The first cell summarizes the entire past of the observation interval. Write $R = J + 1$, $w_r = |A_r|/n$, $w_* = \min_r w_r$. The geometric conditions retained for the direct route and for the subsequent shell reconstruction are

$$T = O(V_M), \quad \log(2 + T + h^{-1}) = o(\nu_M), \quad m_0 h \rightarrow \infty. \quad (2.7)$$

They imply

$$w_* \gg h 2^{-T}, \quad \eta_M := \max_{r \geq 1} \log_2(m_r/m_{r-1}) \leq h + C/m_0. \quad (2.8)$$

There is no exceptionally short final shell: the mesh is chosen through J , rather than by appending a remainder interval.

For growing intensity, the count comparison for these shells follows immediately from Corollary 2.2, with $\delta_M \leq \Delta_M^{\text{mic}}$, without (2.7). Those conditions are used below to control within-shell order and passage to a diffuse target. The next theorem also retains an independent direct proof; unlike the microscopic input, its statement allows $\Lambda \geq 1$ without requiring divergence.

Theorem 2.3 (The complete signed shell comparison). *Under (2.1), (2.4), and (2.7), let $C_{r,e,\sigma} = \sum_{j \in A_r} I_{j,e,\sigma}$. For every $0 < c' < c$ and $0 < \varepsilon < 1/3$,*

$$\delta_M := d_{\text{TV}} \left(\mathcal{L}((C_{r,e,\sigma}) \mid \mathcal{A}_M), \bigotimes_{r,e,\sigma} \text{Pois}(|A_r| 2^{-L-e-2}) \right) \ll e^{-c' \nu_M} + M^{-1/3+\varepsilon}. \quad (2.9)$$

Every nonnegative exact excess and both signs are retained simultaneously.

Proof. When $\Lambda \rightarrow \infty$, this is the deterministic aggregation map of Corollary 2.2: sums over disjoint cells of independent Poisson coordinates have the displayed exact means. For the full stated range $\Lambda \geq 1$, Appendix A supplies a separate finite partition ledger and its logarithmic-shell specialization. That proof uses the signed two-window profile and scalar tail from *Long runs and rare patterns*, without using Input 2.1. $\square$

For the tail estimates and the alternative reconstructions, fix the internal cutoff

$$E_M^* = \left\lceil \frac{I_M + V_M + \log(2 + M2^{-L})}{\beta} \right\rceil. \quad (2.10)$$

The shifted scalar tail from [14, Theorem 7.4] gives the independent truncation bound used by the direct route. Under the microscopic input the same required law-level tail bound can instead be read directly from the Poisson target.

Comparison domains and conditioning fields. The graph-based comparisons in [14, Section 6 and technical companion, Appendix C] allow individual sites under a hard-cutoff budget $I + \log \Lambda + \log(1 + \Lambda) < V$, or an enlarged-cutoff budget $I + \log \Lambda < \hat{V}/2$, with $\hat{V} = 2D(H/\hat{V}) \sim \sqrt{2}V$. Each has its specified prime field and fixed negative margin. The direct shell proof reaches $I + \log \Lambda < V$. These inequalities describe different sufficient interfaces, not the best range available for every observable.

Input 2.1 reaches the same hard one-factor budget while retaining every site. At diverging intensity it therefore gives the same comparison range for individual sites and shell counts. The direct finite-partition proof also covers merely $\Lambda \geq 1$; the subsequent shell reconstruction controls order and observation times. None of these ranges is asserted optimal for the model.

For positive information cost, [14, Proposition 6.2] optimizes the graph-based labelled route with a saddle $2D(H/w_M(I)) = w_M(I) + I$ and allowance $\log \Lambda < (w_M(I) - I)/2$, together with a fixed margin and measurability in $\mathcal{F}_{[e^{w_M(I)}]}$. This remains a contextual comparison with a different admissible information field. The microscopic input uses only the hard field $\mathcal{F}_{Y_M^\#}$; it is not silently transplanted to an enlarged cutoff or the scalar soft-exception regime.

2.5 From a shell field to a history

For growing record histories we add

$$T \rightarrow \infty, \quad \Lambda 2^{-T} \rightarrow \infty, \quad (2.11)$$

$$Th \rightarrow 0, \quad \text{together with (2.7).} \quad (2.12)$$

A convenient choice is $\tilde{h} = \nu_M^{-1/4}/(1+T)^2$, $J = \lceil T/\tilde{h} \rceil$, $h = T/J$. These assumptions are nonvacuous. Unconditionally, take $a_M = \lfloor V_M/(2\beta) \rfloor$ and $m_0 = \lfloor ne^{-V_M/4} \rfloor$. Then $\log \Lambda = V_M/2 + O(1)$, $T \sim V_M/(4\beta)$, and $\Lambda 2^{-T} = e^{V_M/4 + O(1)}$. The window contains a growing number of doublings but begins at $M^{1-o(1)}$, not at the origin.

Theorem 2.4 (Observable signed record history). *Under Theorem 2.3 and (2.11)–(2.12), the finite-word history (1.2), with recognition counters measured from $m_0 + 1$ as in Section 1.2, satisfies (1.3). On a common coupling the announced signed record counters of the retained interior field agree with their Poisson counterparts at every shell endpoint, with at most one discrepancy per sign within a shell, except on an event of probability at most*

$$\delta_M + e^{-\Lambda 2^{-T}} + CTh. \quad (2.13)$$

The same coupling retains the terminal signed field used in (1.7).

Proof. Section 3 proves the phase-uniform target limit. Since (2.11) implies $\Lambda \rightarrow \infty$, the main route obtains the common count coupling from Input 2.1 and Corollary 2.2; no direct arithmetic partition proof is needed. Appendix B then uses these coupled counts and bounds order ambiguity, the possible initial-holder tie, and the delays from announcement to recognition. The coupling preserves actual signs and integer lengths. These discrepancies vanish after division by $\sqrt{T}$; no convergence of unbounded arithmetic moments is inferred from total variation. $\square$

Two elementary consequences will be used repeatedly. Outside an event of conditional probability $o(1)$, every full run starting by $n + 1$ has length at most

$$B_M = L + E_M^* = O(\log M). \quad (2.14)$$

The initial cell contains a retained contender with failure probability at most $\delta_M + e^{-\Lambda 2^{-T}}$. The run beginning at 1 is shorter than L , except on an event of probability at most

$$e^{I_M} 2^{-\pi(L)} = o(1). \quad (2.15)$$

The boundary probability is exact before conditioning. Its prime-rank cost is much larger than V_M in the present range. Thus neither endpoint convention is hidden in the process transfer.

For later reference, write $b = \lfloor \log_2 n \rfloor$ and $\theta_M = \log_2 n - b$, and define

$$\Xi_M = \sum_{j=1}^n \sum_{e \geq 0} \sum_{\sigma \in \{-1, +1\}} I_{j,e,\sigma} \delta_{(j/n, L+e-b, \sigma)}. \quad (2.16)$$

Using the visible maximum and earliest-holder sign defined in Section 1.2, set

$$Y_M(s) = R^{\text{obs}}(\lfloor n2^s \rfloor + 1) - b, \quad Z_M(s) = Z^{\text{obs}}(\lfloor n2^s \rfloor + 1), \quad -S \leq s \leq 0. \quad (2.17)$$

We always use the constant integer centre b . The possible subsequential endpoint $\theta = 1$ is retained: its law is the level shift of the law at $\theta = 0$, not the same law under this centering.

3 Integer records and logarithmic leadership

For $\theta \in [0, 1]$ let $\mathcal{P}_\theta$ be a Poisson random measure on $(0, \infty) \times \mathbb{Z} \times \{-1, +1\}$ with intensity

$$du \sum_{q \in \mathbb{Z}} \sum_{\sigma \in \{-1, +1\}} 2^{\theta-q-2} \delta_{(q, \sigma)}. \quad (3.1)$$

There are infinitely many low-level points, but only finitely many above each fixed integer level on bounded spatial intervals. Let $Q_\theta(u)$ be the largest integer mark observed by $u > 0$, and let $Z_\theta(s)$ be the sign of the earliest point with mark $Q_\theta(2^s)$. Let $N_\theta(s)$ count strict records on $(1, 2^s]$, let $D_\theta(s)$ sum their signs, and put $A_\theta(s) = \int_0^s Z_\theta(v) dv$. Records before time 1 determine the initial holder but are not counted.

The projection forgetting signs has intensity $du \sum_q 2^{\theta-q-1} \delta_q$. Conditional on this entire unsigned configuration, the signs of distinct atoms are independent and fair. This independent marking describes the *target*; actual signs in the multiplicative model are supplied by the signed arithmetic theorem, not created by such a kernel.

Proposition 3.1 (Maximum and record race). *For $m \in \mathbb{Z}$ and $u > 0$,*

$$\mathbb{P}(Q_\theta(u) \leq m) = \exp(-u 2^{\theta-m-1}). \quad (3.2)$$

Given a current maximum q , the next strict record has ordinary waiting time $2^{q+1-\theta} E$, with $E \sim \text{Exp}(1)$, jump height G with $\mathbb{P}(G = j) = 2^{-j}$ for $j \geq 1$, and a fair sign. These three variables are independent. Successive normalized races use independent triples.

Proof. The total intensity above m is $2^{\theta-m-1}$. Its void probability gives (3.2). Above a current record q , the total arrival rate is $2^{\theta-q-1}$, and the proportion at mark $q+j$ and sign σ is 2^{-j-1} . The exponential-race and independent-increment properties prove the claims. Equal-level arrivals are excluded from this race, as required by the tie convention. $\square$

The ordinary-time pair $(Q, \text{holder sign})$ is Markov, with generator

$$\mathcal{G}f(q, \sigma) = \sum_{j \geq 1} \sum_{\tau \in \{-1, +1\}} 2^{\theta - q - j - 2} \{f(q + j, \tau) - f(q, \sigma)\}.$$

The holder sign alone is not assigned a two-state Markov generator here.

3.1 Categorical record levels

Let $Q_0 = Q_\theta(1)$ and $H_\theta(s) = Q_\theta(2^s) - Q_0$. Mark every positive level relative to Q_0 by $+1$ or -1 if it is visited by a record of that sign, and by 0 if skipped.

Lemma 3.2 (Categorical representation). *These level labels $(C_j)_{j \geq 1}$ are iid, with*

$$\mathbb{P}(C_j = 0) = \frac{1}{2}, \quad \mathbb{P}(C_j = +1) = \mathbb{P}(C_j = -1) = \frac{1}{4}.$$

For every $s \geq 0$,

$$N_\theta^\sigma(s) = \sum_{j=1}^{H_\theta(s)} \mathbf{1}_{\{C_j = \sigma\}}. \quad (3.3)$$

The iid sequence is not asserted to be independent of H_θ .

Proof. By Proposition 3.1, the successive visited heights have iid geometric gaps of parameter $1/2$. The indicators of visited levels are therefore iid Bernoulli($1/2$): the probability of a prescribed binary string of length m is the product of the complete gap probabilities and the final gap tail, which equals 2^{-m} . Conditional on those visited levels, their record signs are independent fair marks by (3.1). This proves the iid categorical law and the counting identity. $\square$

Lemma 3.3 (Uniform height envelope). *Put $B_\theta(s) = Q_\theta(2^s) - \theta - s$. For $T \geq 1$ and $z \geq 1$,*

$$\sup_\theta \mathbb{P} \left(\sup_{0 \leq s \leq T} |B_\theta(s)| > z + 1 \right) \leq C(1 + T) \{2^{-z} + e^{-2^{z-1}}\}. \quad (3.4)$$

Consequently $\sup_\theta \mathbb{E} \sup_{s \leq T} |B_\theta(s)|^r \ll_r \log^r(2 + T)$ for every fixed $r \geq 1$.

Proof. Equation (3.2) gives the fixed-time upper bound 2^{-z} and lower bound $e^{-2^{z-1}}$. For $j \leq s \leq j + 1$, monotonicity gives $B_\theta(j) - 1 \leq B_\theta(s) \leq B_\theta(j + 1) + 1$. Apply the union bound to the integer times and integrate the resulting tail, using the bound one for small z . $\square$

Let $S_m^\sigma = \sum_{j \leq m} \mathbf{1}_{\{C_j = \sigma\}}$. Since all summands lie in $[0, 1]$,

$$\sup_{s \leq T} |N_\theta^\sigma(s) - S_{[s]}^\sigma| \leq 2 \sup_{s \leq T} |B_\theta(s)| + 1. \quad (3.5)$$

In particular, the signed-count pair has a Brownian functional limit with covariance (1.5), uniformly over deterministic phase sequences. For the unsigned target,

$$\mathbb{E} N_\theta(T) = T/2 + O(1), \quad \text{Var } N_\theta(T) = T/4 + O(\sqrt{T} \log(2 + T) + \log^2(2 + T)). \quad (3.6)$$

The variance estimate follows from the L^2 form of (3.5). For the bounded error in the mean, the logarithmic jump intensity is

$$\rho(\vartheta) = \frac{\beta}{2} \sum_{m \in \mathbb{Z}} 2^{\vartheta - m} e^{-2^{\vartheta - m}}, \quad \int_0^1 \rho(\vartheta) d\vartheta = \frac{1}{2}.$$

The intensity is periodic, so its integral over any interval of length T differs from $T/2$ by a uniformly bounded quantity. Neither (3.6) nor any later target moment formula is automatically a moment-convergence assertion for the arithmetic model.

3.2 The persistence of the first holder

The signed count does not describe how long its sign remains in charge. The next formula resolves that distinction while keeping all integer ties.

Proposition 3.4 (Persistence kernel). *The probability of no strict record on $(1, 2^h]$, for $h \geq 0$, is*

$$p_\theta(h) = \sum_{q \in \mathbb{Z}} \left\{ e^{-2^h 2^{\theta-q-1}} - e^{-(2^h+1)2^{\theta-q-1}} \right\}. \quad (3.7)$$

The series is absolutely convergent, $p_\theta(0) = 1$, and

$$0 \leq p_\theta(h) \leq C 2^{-h} \quad \text{uniformly in } \theta, h \geq 0. \quad (3.8)$$

Moreover

$$\mathbb{E} Z_\theta(s) = 0, \quad \mathbb{E}[Z_\theta(s) Z_\theta(s+h)] = p_{\{\theta+s\}}(h). \quad (3.9)$$

Its exact phase-period integral is

$$\int_0^1 p_\theta(h) d\theta = \frac{\log(1 + 2^{-h})}{\beta}. \quad (3.10)$$

Proof. Set $a_q = 2^{\theta-q-1}$. The mass of $Q_\theta(1) = q$ is $e^{-a_q} - e^{-2a_q}$. Given this event, absence of an arrival above q in the next strip has probability $e^{-(2^h-1)a_q}$. Multiplying and summing gives (3.7).

With $z_q = 2^h a_q$, the summand is $e^{-z_q}(1 - e^{-2^{-h}z_q}) \leq 2^{-h} z_q e^{-z_q}$. The dyadic sum $\sum_q z_q e^{-z_q}$ is bounded uniformly in its phase: sum the terms with $z_q \leq 1$ geometrically and the terms with $z_q > 1$ by exponential decay. This proves convergence and (3.8); at $h = 0$ the series telescopes to one.

Conditional on the unsigned cloud, equal holders give sign product one. If a strict update occurred, the final holder is a distinct atom, whose independent fair sign gives conditional mean product zero. This proves (3.9).

Finally unfold the period sum and substitute $a = 2^x$:

$$\int_0^1 p_\theta(h) d\theta = \frac{1}{\beta} \int_0^\infty \frac{e^{-2^h a} - e^{-(2^h+1)a}}{a} da = \frac{1}{\beta} \log \frac{2^h + 1}{2^h}.$$

The last identity follows by integrating $\int_{2^h}^{2^h+1} e^{-ta} dt$ first in a . All integrands are nonnegative. $\square$

The phase integral is a calculation, not an extra randomization of the arithmetic model. A long interval of logarithmic time traverses full phase periods; this is why the occupation variance below has no phase parameter.

Proposition 3.5 (Occupation variance and count–occupation cross term). *Uniformly in θ ,*

$$\frac{1}{T} \text{Var } A_\theta(T) \longrightarrow \kappa = \frac{\pi^2}{6\beta^2}, \quad (3.11)$$

$$\text{Cov}(D_\theta(T), A_\theta(T)) = T - \int_0^T p_\theta(s) ds = T + O(1). \quad (3.12)$$

Also $\text{Cov}(N_\theta(T), A_\theta(T)) = \text{Cov}(N_\theta(T), D_\theta(T)) = 0$ and $\text{Var } D_\theta(T) = \mathbb{E} N_\theta(T)$.

Proof. By (3.9),

$$\text{Var } A_\theta(T) = 2 \int_0^T \int_0^{T-s} p_{\{\theta+s\}}(h) dh ds.$$

For every fixed h , the mean over $s \in [0, T-h]$ converges to the period integral in (3.10), uniformly in the starting phase. The exponential bound (3.8) permits dominated convergence. Consequently the limit is

$$2 \int_0^\infty \frac{\log(1 + e^{-\beta h})}{\beta} dh = \frac{2}{\beta^2} \sum_{j \geq 1} \frac{(-1)^{j+1}}{j^2} = \frac{\pi^2}{6\beta^2}.$$

The integrable power-series expansion justifies the middle equality.

Let ξ_0 be the initial holder sign and $\xi_1, \dots, \xi_{N(T)}$ the subsequent record signs. Conditional on the unsigned configuration they are independent fair signs. The imbalance is $\sum_{i=1}^{N(T)} \xi_i$, while the occupation integral weights each holder sign by its duration in $[0, T]$. Its conditional covariance with the imbalance is therefore exactly the total time held by post-initial record holders. The expected excluded initial duration is $\int_0^T p_\theta(s) ds$, proving (3.12). The remaining identities follow by the same conditional sign symmetry. $\square$

3.3 The joint functional limit

A scalar central limit theorem for N does not by itself give a joint theorem with A . We verify a functional-limit interface for the joint block sequence.

Lemma 3.6 (Exponential forgetting and block moments). *For each fixed phase, the two-sided sequence*

$$W_k = \left(N([k, k+1]) - \frac{1}{2}, D([k, k+1]), \int_k^{k+1} Z(s) ds \right), \quad k \in \mathbb{Z},$$

is centered, strictly stationary, has an exponential moment in a neighbourhood of zero for its norm, and has strong-mixing coefficients bounded by $C2^{-j}$ at lag j . The constants can be chosen uniformly in phase.

Proof. The intensity (3.1) is invariant under $(u, q, \sigma) \mapsto (2u, q+1, \sigma)$. This proves stationarity under integer time shifts. The mean of its first coordinate is zero because the mean record intensity integrates to $1/2$ over one full period. The other means vanish by sign symmetry.

Each strict update raises the integer maximum by at least one, so the number in a unit block is bounded by $Q_\theta(2^{k+1}) - Q_\theta(2^k)$. The fixed-time bounds in Lemma 3.3 imply an exponentially decaying upper tail for this difference, uniformly in phase. The imbalance is bounded by the number of records and the absolute occupation by one. Thus the asserted moment bound holds.

For the dependence estimate, replace the whole Poisson past before physical time 1 by an independent copy, keeping all future strips identical. Once a future point strictly exceeds both initial maxima, both versions have the same integer maximum and the same first holder, and all their future record increments and holder paths coincide. The probability that either copy has not refreshed by 2^j is at most $2 \sup_\theta p_\theta(j) \leq C2^{-j}$. The modified future is independent of the original past. This coupling bounds the strong-mixing coefficient; shifting the splitting point by one unit covers the block endpoints. No independence between adjacent blocks is claimed. $\square$

Theorem 3.7 (Joint signed record and occupation invariance principle). *For every deterministic phase sequence $\theta_T \in [0, 1)$, as $T \rightarrow \infty$,*

$$\frac{1}{\sqrt{T}} (N_{\theta_T}(Tt) - Tt/2, D_{\theta_T}(Tt), A_{\theta_T}(Tt))_{0 \leq t \leq 1} \Longrightarrow \mathbb{B}_\Sigma$$

in $D([0, 1], \mathbb{R}^3)$ with the J_1 topology and covariance (1.4).

Proof. First fix phase zero. We use Theorem 1 and its multivariate remark in Doukhan–Massart–Rio [3, pp. 66–67]. In their notation, a sufficient condition is

$$\int_0^1 \alpha^{-1}(u) Q_{|W_0|}(u)^2 du < \infty.$$

By Lemma 3.6, $\alpha^{-1}(u) \ll 1 + |\log u|$ and $Q_{|W_0|}(u) \ll 1 + |\log u|$. The integral is finite. Thus the block partial sums satisfy a multivariate invariance principle, with covariance equal to the limiting covariance of their sums divided by the number of blocks. Equations (3.6) and (3.11)–(3.12) identify it as (1.4).

There are at most $O(T)$ unit blocks up to time T . The uniform exponential moment gives a maximal block count $O_{\mathbb{P}}(\log(2 + T))$ by a union bound. The difference between the block partial-sum process and the full continuous-time additive vector, divided by $\sqrt{T}$, therefore vanishes uniformly. This also handles noninteger T .

Finally couple all phases through one cloud $\mathcal{P}_0$. Multiplication of all arrival rates by 2^θ is exactly the ordinary-time change $u \mapsto 2^\theta u$. Thus the phase- θ additive vector over $[0, s]$ has the law of the phase-zero vector over $[\theta, \theta + s]$. A shift by at most one changes either count coordinate by at most a constant times the largest count in a unit block, and changes the occupation integral by at most two. These errors are $O_{\mathbb{P}}(\log(2 + T))$, uniformly for all $\theta \in [0, 1]$. This proves the theorem for arbitrary deterministic θ_T without a triangular-array mixing theorem or a phase-convergent subsequence. $\square$

An explicit realization is useful. With independent standard Brownian motions B_0, B_1, B_2 , the limit in coordinates (N, D, A) is

$$\left(\frac{1}{2} B_0, \frac{1}{\sqrt{2}} B_1, \sqrt{2} B_1 + \sqrt{\kappa - 2} B_2 \right). \quad (3.13)$$

Here $\kappa \simeq 3.4237147425 > 2$. This displays both the independence of the total count and the correlation of imbalance with leadership time.

Remark 3.8 (Two distinct signed record questions). The counts $N^\pm$ classify *global* strict records by sign. They are not the records within the positive and negative subsequences separately. In the target, the latter two processes depend on independent half-intensity clouds. Each has centering $T/2$ and variance rate $1/4$, and their limits are independent. In contrast, the global-record counts have centering $T/4$ each and covariance (1.5). This distinction should remain explicit in statements and simulations.

3.4 Why the strict count is not Poisson

The mean and variance in (3.6) already rule out a Poisson record process. A distributional distinction is stronger. The categorical coupling and the envelope give, uniformly in phase,

$$\sup_z \left| \mathbb{P} \left(\frac{N_\theta(T) - T/2}{\sqrt{T}/2} \leq z \right) - \Phi(z) \right| \leq C \frac{\log(2 + T)}{\sqrt{T}}. \quad (3.14)$$

Indeed choose the envelope cutoff $C_1 \log(2 + T)$ so that its exceptional probability is $O(T^{-2})$, compare with a binomial sum of $\lfloor T \rfloor$ fair Bernoulli variables, and use Berry–Esseen and the bounded normal density. Thus, for any deterministic phases θ_T ,

$$\liminf_{T \rightarrow \infty} d_{\text{TV}}(\mathcal{L}(N_{\theta_T}(T)), \text{Pois}(T/2)) \geq 2\{\Phi(1) - \Phi(1/\sqrt{2})\} > 0.$$

Use the event $|N - T/2| \leq \sqrt{T}/2$ and the two normal limits. These constants agree with classical geometric-record theory [8, Example 1]; the present coupling is used for the particular joint signed-and-occupation path, not as a claim that functional limits for discrete records are new.

For the target count N , independent fair record marks give

$$\text{Var}(N_+) = \frac{\mathbb{E}N + \text{Var} N}{4}, \quad \text{Cov}(N_+, N_-) = \frac{\text{Var} N - \mathbb{E}N}{4}.$$

Thus the two-count matrix (1.5) alone is an elementary marking consequence of the classical geometric rates [17,15,1,8]. The occupation coordinate and its cross term, retained jointly in Theorem 3.7, require the persistence calculation above.

4 Maxima visible in a finite word

Lemma 4.1 (Weak field transfer while retaining the common coupling). *Under (2.4)–(2.12), the signed field (2.16) is asymptotically equivalent, for bounded Lipschitz tests of the vague topology, to the moving target (1.6) with phase θ_M . If $\theta_M \rightarrow \theta \in [0, 1]$, it converges weakly to Π_θ .*

Proof. Couple the complete shell vectors using Theorem 2.3. For every atom give both copies a common synthetic position uniform in its exact continuous shell $(m_{r-1}/n, m_r/n]$, retaining the actual integer mark and sign. Poisson marking gives exactly

$$du \sum_{q \geq L, \sigma} n 2^{-q-2} \delta_{(q-b, \sigma)}.$$

Extend this target independently to levels below $L - b$. In active shells the displacement in normalized position is at most $Ch + C/n$; in the initial shell it is at most 2^{-T} . For any fixed finite mark set, the total number of target atoms is tight uniformly in phase. Therefore the vague difference between the true and synthetic fields tends to zero on the agreement event. The cutoff $L - b \rightarrow -\infty$ follows from (2.11). On finite mark sets the intensity in (1.6) varies continuously in θ , proving the final assertion. No total-variation statement to diffuse positions is made. $\square$

For $u > 0$ the target maximum and its finite-dimensional law are

$$\mathbb{P}(Q_\theta(u) \leq r) = e^{-u 2^{\theta-r-1}}, \quad (4.1)$$

$$\mathbb{P}(Q_\theta(u_i) \leq r_i, 1 \leq i \leq d) = \exp \left\{ - \sum_{i=1}^d (u_i - u_{i-1}) 2^{\theta - \min_{j \geq i} r_j - 1} \right\}, \quad (4.2)$$

where $0 = u_0 < u_1 < \dots < u_d$. This is the void probability of the union of the forbidden rectangles. Given $Q_\theta(u) = q$, the next strict jump has waiting time $2^{q+1-\theta}E$, jump size G with $\mathbb{P}(G = k) = 2^{-k}$, and a fair new sign, independently.

Let $P(j)$ be the maximum of the full lengths of runs starting at or before physical index $j + 1$, including the first run; let $Z^{\text{ann}}(j)$ be its earliest-holder sign. Put

$$Y_M^{\text{ann}}(s) = P(\lfloor n 2^s \rfloor) - b, \quad Z_M^{\text{ann}}(s) = Z^{\text{ann}}(\lfloor n 2^s \rfloor).$$

Proposition 4.2 (Announced maximum and first-holder continuity). *For each fixed $S > 0$, along $\theta_M \rightarrow \theta$,*

$$(\Xi_M, Y_M^{\text{ann}}, Z_M^{\text{ann}}) \Longrightarrow (\Pi_\theta, Q_\theta(2^\cdot), Z_\theta)$$

with a single vector J_1 topology for the two announced paths, jointly with the vague field.

Proof. First retain marks $-K \leq r \leq H$. In this finite target configuration, distinct positions are almost surely distinct and none is an endpoint 2^{-S} or 1. If a contender is present by 2^{-S} , vague convergence matches all atoms, with identical integer marks and signs and convergent positions. Their ordering is eventually unchanged. Hence the strict-record identities and the earliest choice among integer ties are preserved. An increasing piecewise-linear time change aligns all record times, proving vector J_1 convergence. Integer ties in heights do not cause a discontinuity because arrival positions are distinct.

The probability of an atom above H is at most $2^{\theta-H-1}$. The probability that the initial maximum is below $-K$ is $e^{-2^{\theta+K-S}}$. The same upper bounds, up to δ_M , hold under the common arithmetic shell coupling. Let $H, K \rightarrow \infty$ after the finite-configuration argument. The border contribution is negligible by (2.15). This proves the joint statement and explains why vague convergence alone, without the two tail controls, would not suffice for an extremal functional. $\square$

4.1 Right censoring and completed graphs

The standard scalar M_1 metric is defined by continuous nondecreasing parametrizations of the *completed graph*: the vertical segment between the left and right values is included at every jump. See [18] for the general theory. We prove the elementary version needed here.

Lemma 4.3 (Delayed integer staircases). *Let f, g be finite, nondecreasing, integer-valued, right-continuous step functions on $[a, b]$. Suppose they have the same initial and terminal values. For every integer level k with $f(a) < k \leq f(b)$, let*

$$t_f(k) = \inf\{t \in [a, b] : f(t) \geq k\}, \quad t_g(k) = \inf\{t \in [a, b] : g(t) \geq k\}.$$

If $0 \leq t_g(k) - t_f(k) \leq \delta$ for every such level, then

$$d_{M_1}(f, g) \leq \delta. \tag{4.3}$$

A sufficient condition is $f(t - \delta) \leq g(t) \leq f(t)$, with f extended constantly to the left, together with endpoint equality.

Proof. Traverse the completed graph one integer level at a time. The vertical segment from $k - 1$ to k occurs at $t_f(k)$ for f and at $t_g(k)$ for g , even when several such segments belong to one larger jump. Match their vertical coordinates exactly. At the plateau at integer height k , match the horizontal intervals from $t_f(k)$ to $t_f(k + 1)$ and from $t_g(k)$ to $t_g(k + 1)$ linearly. A zero-length interval is traversed by pausing while the other graph traverses its plateau. Initial and terminal horizontal pieces are treated in the same way. The resulting continuous graph parametrizations have identical value coordinates and time coordinates at distance at most δ . This proves (4.3). The sandwich condition implies the asserted inequalities for all level-crossing times. $\square$

Lemma 4.4 (A deterministic bound for visible maxima). *Suppose every full run beginning by physical prefix $n + 1$ has length at most an integer B . For $j \geq B$,*

$$P(j - B) \leq R^{\text{obs}}(j + 1) \leq P(j). \tag{4.4}$$

Moreover if τ_k^{ann} is the start of the first full run of length at least k , then its visible threshold is attained at exactly

$$\tau_k^{\text{obs}} = \tau_k^{\text{ann}} + k - 1. \tag{4.5}$$

Proof. Every run starting by $j - B + 1$ ends by j , so it is completely observed by $j + 1$. This gives the left inequality. Every visible constant segment belongs to a run already begun and cannot exceed that run's full length, giving the right inequality. For (4.5), runs shorter than k never hit the level, and among qualifying runs the first start is also the first time at which its k -th symbol is observed. $\square$

In logarithmic coordinates $s \in [-S, 0]$, take, for large M ,

$$\delta_{M,S} = -\log_2 \left(1 - \frac{B_M + 2}{n2^{-S}} \right) = O_S(B_M/n) = o(1).$$

The floors in (2.17) and (4.4) give

$$Y_M^{\text{ann}}(s - \delta_{M,S}) \leq Y_M(s) \leq Y_M^{\text{ann}}(s). \quad (4.6)$$

Here the announced path is taken on the slightly larger interval to the left. On the event that it has no strict jump in the two collars $[-S - \delta_{M,S}, -S]$ and $[-\delta_{M,S}, 0]$, the endpoints agree. Lemma 4.3 then applies. Proposition 4.2 on an enlarged fixed interval and the absence of fixed-time target jumps imply that the probability of either collar event tends to zero. Thus

$$d_{M_1, [-S, 0]}(Y_M, Y_M^{\text{ann}}) \longrightarrow 0 \quad \text{in probability.} \quad (4.7)$$

The events (2.14)–(2.15) and initial non-vacancy justify the deterministic assumptions used above.

For the holder sign, a record run from a_i replaces the preceding holder at the recognition time $a_i + H_{i-1}$, with $H_{i-1} < q_i \leq B_M$. Record recognition occurs before the next run begins. Thus the order and signs of all record episodes are preserved, and their logarithmic times move by at most $O_S(B_M/n)$. The same collars remove endpoint problems. Coupling with the finitely many limiting record times yields

$$d_{J_1, [-S, 0]}(Z_M, Z_M^{\text{ann}}) \longrightarrow 0 \quad \text{in probability.} \quad (4.8)$$

Equal signs at consecutive record updates simply remove the corresponding sign jump in both copies.

Theorem 4.5 (Visible extremal process and the necessary topology). *Under (2.4)–(2.12), for fixed $S > 0$ and $\theta_M \rightarrow \theta \in [0, 1]$,*

$$(\Xi_M, Y_M, Z_M) \Longrightarrow (\Pi_\theta, Q_\theta(2), Z_\theta) \quad (4.9)$$

in the product of vague topology, scalar M_1 , and scalar J_1 . In this normalization the sequence of visible maximum paths is not J_1 -tight.

Proof. Combine Proposition 4.2, (4.7) and (4.8). For the final assertion, every jump of Y_M has size at most one: appending a single symbol can increase the visible maximum by at most one. Every possible J_1 limit has this same jump-size bound, because J_1 convergence matches jumps. But the target has positive probability of a strict jump of size at least two in $(-S, 0)$: its next jump height has tail $\mathbb{P}(G \geq 2) = 1/2$ and its waiting time has a positive continuous density. Any J_1 -tight subsequence would have the same finite-dimensional laws as the already identified target, a contradiction. The target has no fixed discontinuities, so its dense-time laws determine its path law. $\square$

Remark 4.6 (A concrete obstruction and a product-space caution). Let $f_\epsilon(t) = \mathbf{1}_{\{t \geq 1/2\}} + \mathbf{1}_{\{t \geq 1/2 + \epsilon\}}$ and $f(t) = 2\mathbf{1}_{\{t \geq 1/2\}}$. Lemma 4.3 gives $d_{M_1}(f_\epsilon, f) \leq \epsilon$. Under any J_1 time change,

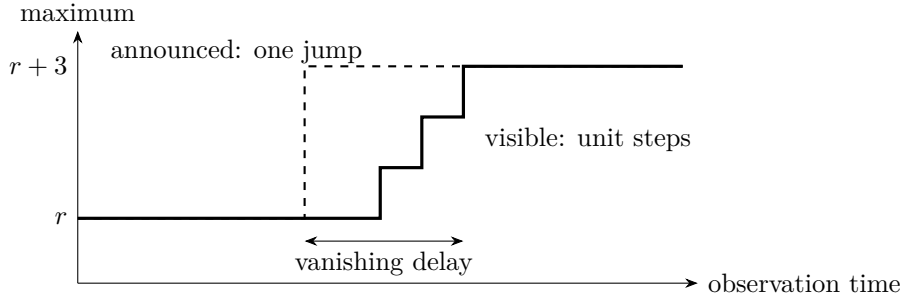


Figure 1: Schematic of right censoring. The unit staircases approach a single completed vertical segment in scalar M_1 . A J_1 time change cannot remove their intermediate values. The sign of the holder changes at the first recognized improvement, which explains the product, rather than strong vector M_1 , topology.

f_ϵ still takes the value 1 on a nonempty interval, while f only takes 0 and 2; their uniform value discrepancy is at least one. This is exactly the visible-record obstruction.

The product topology in (4.9) matters. During a staircase the holder may change at its first unit improvement, so the vector (Y_M, Z_M) can traverse an L-shaped path rather than the straight segment completing one vector jump. We do not assert strong vector M_1 convergence, nor vector J_1 convergence, for that pair.

4.2 A moving-target approximation on the whole growing window

The local weak limit is not the strongest censoring conclusion. The deterministic lemma also gives a coupling on the entire admissible logarithmic interval, whose length grows with M .

Theorem 4.7 (Whole-window visible maximum approximation). *Under (2.4)–(2.12), put $b_0 = \lfloor \log_2 m_0 \rfloor$, $\alpha_M = \{\log_2 m_0\}$, and*

$$\hat{Y}_M(v) = R^{\text{obs}}(\lfloor m_0 2^v \rfloor + 1) - b_0, \quad 0 \leq v \leq T_M.$$

There is a coupling with the integer Poisson maximum Q_{α_M} for which

$$d_{M_1, [0, T_M]}(\hat{Y}_M, Q_{\alpha_M}(2)) \longrightarrow 0 \quad \text{in probability.} \quad (4.10)$$

This is a moving-target approximation on a changing interval, not weak convergence to one fixed phase-free law. It does not assert a whole-window J_1 coupling of the holder sign at the initial time.

Proof. On the common shell-count coupling, if an initial retained contender exists and no shell contains two strictly higher candidates, the true and synthetic announced maxima have the same initial height and the same ordered record heights. Their record times lie in the same active shells, so they can be aligned with error at most $\eta_M \leq h_M + C/m_0$. There is no initial-height tie problem, though an initial *sign* tie is possible. The error probability so far is at most (2.13).

Restrict to the event of (2.14) and (2.15). The delayed-maximum sandwich holds throughout the interval, with time error $O(B_M/m_0)$. It remains to ensure endpoint equality. Exclude retained starts in physical index bands of width $B_M + 2$ immediately before $m_0 + 1$ and $n + 1$. Both bands are contained in the macroscopic range $[M^{1/2}, M)$ for large M . The masked first-moment bound in [14, Corollary 2.5] gives conditional probability at most

$$Ce^{I_M} 2^{-L_M} (B_M + M^{1/2+o(1)}) = o(1). \quad (4.11)$$

It also excludes an announced jump exactly at the terminal endpoint. On this event the initial maximum has already been revealed, and every terminal record was revealed before the endpoint. Lemma 4.3 gives completed-graph distance $O(B_M/m_0)$ from the observed to the announced maximum. Compose with the announced time alignment. Since $h_M + B_M/m_0 \rightarrow 0$, this proves (4.10). The omitted high-mark and border probabilities are $o(1)$ by Section 2. No union bound over all intermediate prefixes is used. $\square$

5 A Brownian history independent of terminal extremes

We first work entirely in the target. Let $\mathcal{P}_\alpha$ be (1.6) on all ordinary times $u > 0$, where $\alpha \in [0, 1]$ is the phase at the initial scale 1. Let $\mathcal{G}_T$ be its normalized additive history as in (1.2). Put

$$b_T = \lfloor \alpha + T \rfloor, \quad \theta_T = \alpha + T - b_T.$$

The terminal critical field formed by $(u, q, \sigma) \mapsto (u2^{-T}, q - b_T, \sigma)$ for $u \leq 2^T$ has law Π_{θ_T} . Its terminal paths on $[-S, 0]$ have the corresponding law in (4.9). Write $\mathcal{E}_{T,S,K}$ for this pair of paths together with the terminal field restricted to levels $r \geq -K$. This upper half-line has finite total mass.

Proposition 5.1 (A quantitative target separation bound). *Let $K \geq 0$, $S > 0$, and $S < g < T$. Let F be bounded by one and Lipschitz with constant one for a bounded J_1 metric of the history space dominated by uniform distance. Let G be any measurable function of $\mathcal{E}_{T,S,K}$ bounded by one. Uniformly in α ,*

$$|\mathbb{E}[F(\mathcal{G}_T)G] - \mathbb{E}F(\mathcal{G}_T)\mathbb{E}G| \leq C \frac{g+1}{\sqrt{T}} + C2^{S-g} + C2^{K-g}. \quad (5.1)$$

For fixed S, K , a choice $g = 2\log_2(T+2)$, eventually smaller than T , gives an $O_{S,K}(\log(2+T)/\sqrt{T})$ bound.

Proof. Stop the *centered* additive vector at logarithmic time $T - g$ and keep it constant thereafter. Call the resulting normalized path $\mathcal{G}_T^{\text{old}}$. Since $|D|$ -increments are bounded by record-count increments and $|A|$ -increments by elapsed time,

$$\mathbb{E}\|\mathcal{G}_T - \mathcal{G}_T^{\text{old}}\|_\infty \leq C(g+1)/\sqrt{T}.$$

The expected number of records in an interval of length g is $g/2 + O(1)$ uniformly in phase, by (3.6). The stopped path depends only on the Poisson past $u \leq 2^{T-g}$.

Construct $\mathcal{E}_{T,S,K}^{\text{new}}$ using only the independent strip $2^{T-g} < u \leq 2^T$. Its critical field differs from the original restricted field only if an old point has terminal mark at least $-K$. The expected number of such points is $2^{\theta_T+K-g} \leq 2^{K+1-g}$. Its maximum and holder paths agree on the whole terminal window once the new strip has produced a strict record above the old maximum before 2^{T-S} . The failure probability is at most $C2^{S-g}$ by (3.7). This also handles integer ties: the refreshing record must be strictly higher.

Thus $\mathcal{E}^{\text{new}}$ and $\mathcal{E}$ agree with probability at least $1 - C2^{S-g} - C2^{K-g}$. The former is independent of $\mathcal{G}_T^{\text{old}}$. Replacing first the history and then the endpoint object in both terms of the covariance proves (5.1). All comparisons in this proof are within the Poisson target. The displayed bound is not a normal-approximation rate for the arithmetic history. $\square$

The proposition also permits a terminal window $S = S_T = o(\sqrt{T})$ by taking, for example, $g = 2S_T + 2\log_2(T+2)$. This is a target decoupling assertion with a growing endpoint state

space. Our weak-limit theorem below keeps S fixed. Nor does (5.1) assert total-variation deletion of the whole infinite negative-mark field: the restriction $r \geq -K$ is essential before K is sent to infinity.

Theorem 5.2 (Joint Brownian history and terminal Poisson field). *Under (2.4)–(2.12), (1.7) holds along every sequence for which $\theta_M \rightarrow \theta \in [0, 1]$. The Brownian motion has covariance (1.4) and is independent of the entire terminal block.*

Without selecting the phase, the equivalent statement is approximation in bounded-Lipschitz distance by the moving product law

$$\mathcal{L}(\mathbb{B}_\Sigma) \otimes \mathcal{L}(\Pi_{\theta_M}, Q_{\theta_M}(2), Z_{\theta_M}), \quad (5.2)$$

with the product topologies specified after (1.7).

Proof. In the target, the phase-uniform additive FCLT and the exact terminal Poisson law give tight marginals. Proposition 5.1 gives independence in every subsequential limit for the terminal upper half-line $r \geq -K$ and the two terminal paths. Cylinder functions on finite mark sets determine the vague point-process law. Letting $K \rightarrow \infty$ therefore gives independence of the full locally finite field as well. The terminal paths are extracted by Proposition 4.2; its two tail controls justify their joint inclusion. Continuity of the target intensity in θ identifies the limit along a convergent phase subsequence.

For the arithmetic transfer use *one* common signed shell coupling. The additive-history discrepancy, including observable recognition, vanishes after normalization by Theorem 2.4 and Appendix B. Lemma 4.1 couples the critical field weakly on that same construction. Proposition 4.2 identifies its announced terminal paths, and the deterministic completion and recognition lemmas replace them by the observed paths. All failure probabilities tend to zero. Joint convergence, not a product of unrelated couplings, follows by Slutsky's theorem on the product space.

Finally, if (5.2) failed along some sequence, compactness of $[0, 1]$ would give a phase-convergent subsequence contradicting the first part. The target path laws are continuous in phase with the constant integer centering used here; $\theta = 1$ is included as specified in Section 2. $\square$

Corollary 5.3 (Conditioning on a terminal extreme). *Fix an integer r and a sign σ . Along a convergent phase sequence, the normalized history still converges to $\mathbb{B}_\Sigma$ after conditioning, in addition to $\mathcal{A}_M$, on*

$$R^{\text{obs}}(n+1) - b = r, \quad Z^{\text{obs}}(n+1) = \sigma.$$

The limiting probability of this event under $\mathcal{A}_M$ is

$$\frac{1}{2} \left(e^{-2^{\theta-r-1}} - e^{-2^{\theta-r}} \right) > 0. \quad (5.3)$$

Proof. The target holder sign is fair independently of its unsigned field at a fixed time. Equation (4.1) gives (5.3). Terminal evaluations are continuous at the target paths, which have no fixed-time jumps. Since the variables are integer and sign valued, small intervals around r and σ give continuity sets. Apply Theorem 5.2 and divide by the positive limiting probability. This is post-selection after a proved joint comparison, not a claim that the record event belongs to the small-prime field. Vanishing-probability post-selection would require a separate relative-error estimate. $\square$

The independence is asymptotic only. For example the target has the exact identity

$$\text{Cov}(D_\alpha(T), Z_\alpha(T)) = 1 - p_\alpha(T). \quad (5.4)$$

Indeed, conditional on the unsigned cloud, the terminal holder's sign is one of the signs summed in $D_\alpha(T)$ exactly when a strict update has occurred after the initial time. Independent fair marking then gives (5.4). Division by $\sqrt{T}$ removes this bounded correlation; independence is not an exact finite-window statement.

6 Two observation clocks, two occupation laws

The terminal field also gives an observable whose fluctuations do not shrink: the fraction of physical prefixes at which a given sign holds the record. Define

$$V_M^{\text{phys}} = \frac{1}{n} \sum_{j=1}^n Z^{\text{obs}}(j+1), \quad O_M^+ = \frac{1 + V_M^{\text{phys}}}{2}. \quad (6.1)$$

In the target let

$$V_\theta = \int_0^1 Z_\theta(\log_2 u) du, \quad O_\theta^+ = (1 + V_\theta)/2. \quad (6.2)$$

The bounded integrand makes the integral well-defined despite infinitely many low-level records approaching ordinary time zero.

Theorem 6.1 (Physical occupation limit). *Along $\theta_M \rightarrow \theta$, jointly with Theorem 5.2,*

$$V_M^{\text{phys}} \implies V_\theta,$$

and the entire endpoint block, including V_θ , is independent of $\mathbb{B}_\Sigma$. The law of V_θ is symmetric, supported on $[-1, 1]$, and nondegenerate. Its mean is zero and

$$\text{Var } V_\theta = K(\theta) := \sum_{q \in \mathbb{Z}} \frac{1 - e^{-2a_q} - 2a_q e^{-a_q}}{a_q^2}, \quad a_q = 2^{\theta-q-1}. \quad (6.3)$$

Consequently conditional means and variances of the bounded variables in (6.1) also converge along this sequence:

$$\mathbb{E}(O_M^+ | \mathcal{A}_M) \rightarrow \tfrac{1}{2}, \quad \text{Var}(O_M^+ | \mathcal{A}_M) \rightarrow K(\theta)/4 > 0.$$

Proof. First restrict physical time to $u \in [2^{-S}, 1]$. Substitution $u = 2^s$ turns the integral into $\int_{-S}^0 \beta 2^s Z_\theta(s) ds$. Integration against this continuous weight is continuous at every bounded J_1 path with finitely many jumps. The corresponding finite-word Riemann sum differs from its step-path integral by at most $2/n$: the sum in (6.1) and the integral of $Z^{\text{obs}}(\lfloor nu \rfloor + 1)$ telescope at their end indices. Apply Theorem 5.2 on this fixed window. The omitted initial physical interval has length 2^{-S} and absolute contribution at most that length for both models. Send $S \rightarrow \infty$. This proves joint convergence, including independence from the Brownian block.

Conditional on the unsigned cloud all holder signs are independent fair labels of distinct atoms. For $0 < u \leq v \leq 1$, the sign covariance is the probability of the same first holder, namely

$$\mathbb{E}[Z_\theta(\log_2 u) Z_\theta(\log_2 v)] = \sum_q (e^{-va_q} - e^{-(u+v)a_q}). \quad (6.4)$$

Each summand is nonnegative. Fubini–Tonelli on the triangle yields

$$\begin{aligned} \text{Var } V_\theta &= 2 \sum_q \int_0^1 \int_0^v (e^{-va_q} - e^{-(u+v)a_q}) du dv \\ &= \sum_q \frac{1 - e^{-2a_q} - 2a_q e^{-a_q}}{a_q^2}. \end{aligned}$$

Write $f(a) = (1 - e^{-2a} - 2ae^{-a})/a^2$. Its integral representation above is positive. It has $f(a) = a/3 + O(a^2)$ near zero and $f(a) = a^{-2} + O(e^{-a}/a)$ at infinity. The dyadic sum is therefore uniformly convergent in phase and positive, with a positive minimum on the compact phase period. Sign symmetry gives zero mean. Bounded support with positive variance rules out a Gaussian limit.

All variables V_M^{phys} are bounded by one. Uniform integrability is therefore automatic for every fixed moment, which justifies the displayed moment conclusions. This argument does not upgrade the unbounded record-count moments elsewhere. $\square$

The variance is of order one; only its variation with the lattice phase is small. The next calculation separates this nonconcentration from the much finer periodic modulation. Its Mellin unfolding belongs to the classical theory of harmonic sums [4]; the formula below is for the signed first-holder occupation, not the record-duration sums of [9,6].

Proposition 6.2 (The exact phase average and a genuine phase effect). *The function K is continuous and one-periodic. Its average is*

$$\int_0^1 K(\theta) d\theta = 2 - \frac{1}{\beta} \simeq 0.5573049591110366. \quad (6.5)$$

It is not constant. With Fourier convention $\widehat{K}(k) = \int_0^1 K(\theta) e^{-2\pi i k \theta} d\theta$ and $z_k = -2\pi i k / \beta$, for $k \neq 0$,

$$\widehat{K}(k) = \frac{2\Gamma(z_k + 1)}{\beta(z_k - 1)(2 - z_k)} \neq 0. \quad (6.6)$$

Proof. Average (6.4) over phase. For $u \leq v$ the result is $\beta^{-1} \log(1 + u/v)$ by the same period unfolding as in (3.7). Thus

$$\int_0^1 K(\theta) d\theta = \frac{2}{\beta} \int_0^1 \int_0^v \log(1 + u/v) du dv = \frac{1}{\beta} \int_0^1 \log(1 + r) dr = 2 - \frac{1}{\beta}.$$

For nonconstancy, unfold the dyadic sum: $\widehat{K}(k) = \beta^{-1} \int_0^\infty a^{z_k-1} f(a) da$. The Mellin integral is analytic on $-1 < \Re z < 2$. Initially for $1 < \Re z < 2$, integration by parts gives

$$\int_0^\infty a^{z-1} f(a) da = \frac{2\Gamma(z-1)}{2-z} \{2^{1-z} + z - 2\}.$$

Both sides continue to the stated strip after removable singularities are filled. At $z = z_k$, $k \neq 0$, $2^{1-z_k} = 2$, and the gamma recurrence gives (6.6). The gamma function has no zeros. This proves a nonzero phase dependence, rather than assuming the small oscillation vanishes. The phase integral is not an assertion that the arithmetic phase is random or averaged. $\square$

For orientation, high-precision evaluation gives

$$K(0) \simeq 0.5573050812702, \quad K(1/4) \simeq 0.5573046451398, \quad |\widehat{K}(1)| \simeq 1.6844945581 \cdot 10^{-7}.$$

These numerical values are diagnostics of the exact series, not proof inputs. The fluctuation scale itself is order one; only its phase modulation is small.

Logarithmic occupation behaves differently: Theorem 2.4 gives $T^{-1} A_M^{\text{obs}}(T) \rightarrow 0$ in probability. Therefore the positive holder occupies asymptotically half of the *logarithmic* observation time, but its fraction of *physical prefixes* does not concentrate at one half. The distinction comes from the weights: the final bounded logarithmic window carries a positive fraction of the physical interval, but a vanishing fraction of a long logarithmic interval.

6.1 First passage of critical integer thresholds

The completion argument has a direct threshold formulation, without applying an inverse map at a problematic integer plateau. Let $H_M(k)$ be the first physical prefix at which a run of length $b + k$ is visible, with infinity if it has not appeared by $n + 1$. Under the convention of this paragraph we only query times $t < 1$; at $t = 1$ the same formula follows by a terminal collar argument.

Corollary 6.3 (Joint critical thresholds). *For fixed integers $k_1, \dots, k_d$ and $0 = t_0 < t_1 < \dots < t_d \leq 1$, along $\theta_M \rightarrow \theta$,*

$$\mathbb{P}(H_M(k_i)/n > t_i \ (1 \leq i \leq d) \mid \mathcal{A}_M) \longrightarrow \exp \left\{ - \sum_{i=1}^d (t_i - t_{i-1}) 2^{\theta - \min_{j \geq i} k_j} \right\}. \quad (6.7)$$

These hitting-time coordinates are jointly asymptotically independent of $\mathcal{G}_M$ in the sense of Theorem 5.2. For one threshold the survival function is $e^{-t2^{\theta-k}}$ over the displayed range.

Proof. The first announced contender of length at least $b + k$ is a first point of (2.16) at mark at least k . Its recognition is delayed by exactly $b + k - 1$ symbols by (4.5), negligible after division by n . Threshold levels below the retained base are not encountered for fixed k and large M . The target event excludes the union of rectangles $[0, t_i] \times \{r \geq k_i\} \times \{-1, +1\}$, whose intensity gives (6.7). Diffuse arrival times remove boundary ambiguities. Apply the common field–history convergence. No assertion about an unobserved future beyond the terminal comparison range is needed. $\square$

6.2 Fixed smooth weights in logarithmic time

A smooth weight on the whole logarithmic history interpolates between different historical averages without concentrating on the last few doublings. For fixed $\gamma \geq 0$, define

$$w_\gamma(u) = \begin{cases} \frac{\gamma e^{-\gamma(1-u)}}{1 - e^{-\gamma}}, & \gamma > 0, \\ 1, & \gamma = 0, \end{cases} \quad 0 \leq u \leq 1.$$

Its integral is one. For the target set $V_{T,\gamma} = T^{-1} \int_0^T w_\gamma(s/T) Z_{\theta_T}(s) ds$. For the observed word use instead

$$V_{M,\gamma}^{\text{obs}} = \frac{1}{T} \int_0^T w_\gamma(s/T) Z^{\text{obs}}(\lfloor m_0 2^s \rfloor + 1) ds.$$

This is a weighted difference of leadership times; the positive fraction is $(1 + V_{M,\gamma}^{\text{obs}})/2$.

Corollary 6.4 (Fixed-weight occupation fluctuations). *For every fixed $\gamma \geq 0$ and every deterministic phase sequence in the target,*

$$\sqrt{T} V_{T,\gamma} \Longrightarrow \mathcal{N}(0, \kappa F(\gamma)), \quad F(\gamma) = \begin{cases} \frac{\gamma}{2} \coth(\gamma/2), & \gamma > 0, \\ 1, & \gamma = 0. \end{cases} \quad (6.8)$$

Under the hypotheses of Theorem 2.4, the same limit holds for $\sqrt{T} V_{M,\gamma}^{\text{obs}}$ under $\mathbb{P}(\cdot \mid \mathcal{A}_M)$. Under the joint endpoint hypotheses of Theorem 5.2, this limit is independent of the terminal block. Each assertion also holds jointly for any fixed finite collection of parameters γ .

Proof. The occupation coordinate of the target or observed history is $B_T(u) = A(Tu)/\sqrt{T} \Rightarrow \sqrt{\kappa} B(u)$, by Theorems 3.7 and 2.4. Integration by parts gives

$$\sqrt{T} V_{T,\gamma} = w_\gamma(1) B_T(1) - w_\gamma(0) B_T(0) - \int_0^1 w'_\gamma(u) B_T(u) du.$$

The same identity holds for the observed process. This functional is continuous at every continuous limiting path for the J_1 topology. The limit is the deterministic-weight Brownian integral $\sqrt{\kappa} \int_0^1 w_\gamma(u) dB(u)$, of variance

$$\kappa \int_0^1 w_\gamma(u)^2 du = \kappa \frac{\gamma}{2} \frac{1 + e^{-\gamma}}{1 - e^{-\gamma}} = \kappa F(\gamma) \quad (\gamma > 0).$$

At $\gamma = 0$ the integral equals κ , agreeing with $F(\gamma) \rightarrow 1$. Applying the same continuous map to the joint history–endpoint limit proves the independence assertion. A finite vector of these functionals proves the last claim; its covariances are $\kappa \int_0^1 w_\gamma w_{\gamma'}$. No unbounded arithmetic moment is inferred from this weak convergence. $\square$

The same integration-by-parts argument works for every deterministic weight $w \in C^1([0, 1])$ fixed before the limit: under the same target or observed-history hypotheses, $T^{-1/2} \int_0^T w(s/T) Z(s) ds$ converges to $\mathcal{N}(0, \kappa \int_0^1 w(u)^2 du)$, with the corresponding leadership-sign process Z . This consequence uses no uniformity for a varying weight w_T .

The parameter is fixed before $T \rightarrow \infty$. A weight representing physical time on this logarithmic window has $\gamma = (\log 2)T$, outside the corollary. The physical occupation theorem is therefore not obtained by sending γ to infinity in (6.8); a moving-parameter transition would require uniform estimates on the shrinking effective window.

7 One Poisson geometry, two process clocks

The preceding observables use integer lengths only. We now introduce an auxiliary fractional mark to interpolate the threshold process. This construction is useful precisely because its integer skeleton is unchanged; its continuously ranked records, however, are different records.

7.1 The canonical interpolation

Let E have probabilities 2^{-e-1} , $e \geq 0$, and independently let U have density

$$g(u) = 2\beta 2^{-u} \mathbf{1}_{(0,1)}(u). \quad (7.1)$$

Then $E + U$ has exponential density $\beta 2^{-t}$ on $(0, \infty)$, and $\{E \geq k\} = \{E + U > k\}$ for every integer $k \geq 0$ simultaneously. A uniform fractional mark would not give this exponential law.

Proposition 7.1 (Exact planarization). *Attach independent copies of U to every atom of the independent exact-length field with means 2^{-q-1} at $j \in \{1, \dots, n\}$, $q \geq L$, using separate marks for multiple atoms. Under*

$$(j, q, U) \mapsto (u, v) = (j/n, n2^{-(q+U)}), \quad (7.2)$$

the image is exactly

$$\text{PPP} \left(\frac{1}{n} \sum_{j=1}^n \delta_{j/n}(du) \otimes \mathbf{1}_{(0,\Lambda)}(v) dv \right).$$

For the signed field there is an additional fair sign mark. Applying the same kernel to two input laws cannot increase their total-variation distance.

Proof. At one site and $t = q + U$, the marked intensity is $2^{-q-1} g(U) dU = \beta 2^{-t} dt$. Since $v = n2^{-t}$, its image is dv/n . Sum the disjoint cells $q \geq L$. The Poisson mapping and marking theorems [13, Chapter 5] identify the image; kernel contraction gives the last statement. $\square$

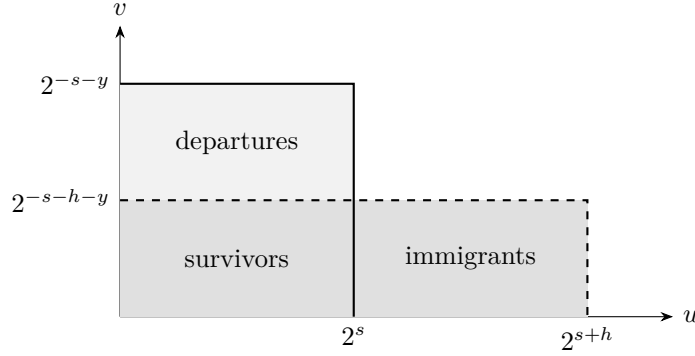


Figure 2: The critical scale moves both edges, keeping the area 2^{-y} fixed. Raising only the length threshold removes points; moving the spatial edge at the same time also brings new points. The drawing uses $2^h = 2$.

For a shell target, a uniform synthetic position in the exact continuous shell gives instead the homogeneous intensity $du dv$ on $(0, 1] \times (0, \Lambda)$. This assertion concerns the synthetic cloud. Its arithmetic counterpart remains coupled to its true positions, which are only known to lie in the same shell.

The density (7.1) is unique among *independent, phase-independent, identically applied within-cell densities* that turn the geometric cells into an exponential intensity: equating the density on each cell forces $g(u) = 2C2^{-u}$, and normalization gives $C = \beta$. This is not a uniqueness claim over all possible randomization kernels.

Let $\mathcal{P}$ be a homogeneous Poisson measure of unit intensity on $(0, \infty)^2$. Its restriction to $u \leq 1$ is the diffuse target for the terminal field. Extending it to $u > 1$ is useful for stationary target theory but does not assert an arithmetic comparison beyond the terminal prefix. Define

$$\mathcal{R}(a, b) = (0, 2^a] \times (0, 2^{-b}), \quad N(a, b) = \mathcal{P}(\mathcal{R}(a, b)). \quad (7.3)$$

The area gives the mean and the intersection gives the covariance:

$$\mathbb{E}N(a, b) = 2^{a-b}, \quad \text{Cov}(N(a, b), N(c, d)) = 2^{\min(a, c) - \max(b, d)}. \quad (7.4)$$

7.2 Stationary immigration–death dynamics

Set

$$X_s(y) = N(s, s + y), \quad c_y = 2^{-y}.$$

The map $(u, v) \mapsto (2^h u, 2^{-h} v)$ preserves area. It makes the scale flow stationary.

Theorem 7.2 (The scale semigroup). *For each fixed y , the process $X_\cdot(y)$ is stationary and reversible. It is an immigration–death chain of mean c_y . For $h \geq 0$,*

$$X_{s+h}(y) \mid X_s(y) = m \stackrel{d}{=} \text{Bin}(m, 2^{-h}) + \text{Pois}(c_y(1 - 2^{-h})), \quad (7.5)$$

with independent summands. Its generator is

$$\mathcal{L}_y f(m) = \beta c_y [f(m+1) - f(m)] + \beta m [f(m-1) - f(m)]. \quad (7.6)$$

For two thresholds, their joint covariance is

$$\text{Cov}(X_s(y), X_t(z)) = 2^{\min(s, t) - \max(s+y, t+z)}.$$

Proof. A point is active at threshold y on $[a, d_y)$, where $a = \log_2 u$ and $d_y = -\log_2 v - y$. The intensity in (a, d_y) is $\beta^2 2^{-y} 2^{a-d_y} da dd_y$. On $d_y > a$, this is a homogeneous birth process of rate βc_y with independent exponential lifetimes of rate β . This representation establishes the Markov property with respect to the count's past, not merely a two-time identity. Old particles survive with probability 2^{-h} ; independent new survivors have mean $c_y(1 - 2^{-h})$. The generator follows by differentiation. The Poisson stationary law satisfies detailed balance $\pi(m)\beta c_y = \pi(m+1)\beta(m+1)$. The covariance follows from (7.4). $\square$

The marked measure itself has a simple dynamics. At scale s , give a point the coordinates $w = 2^{-s}u$ and $z = -\log_2(2^s v)$. The resulting measure on $(0, 1] \times \mathbb{R}$ has stationary intensity $\beta 2^{-z} dw dz$. Particles drift with $\dot{w} = -\beta w$, $\dot{z} = -1$, while immigrants enter at $w = 1$ with intensity $\beta 2^{-z} dz ds$. These descriptions are valid on compactly supported cylinder tests, where all integrals are finite.

Quantitative path-metric convention. For a compact interval I , let $\mathcal{T}_I$ be the increasing continuous bijections of I onto itself. For vector-valued càdlàg paths we use

$$d_{J_1, I}^{(0)}(x, y) = \inf_{\lambda \in \mathcal{T}_I} \left(\|\lambda - \text{id}\|_\infty \vee \sup_{t \in I} \|x(t) - y(\lambda(t))\|_\infty \right).$$

Here the vector norm is the maximum norm. This metric induces the usual J_1 topology on each fixed interval. Quantitative estimates on changing intervals below refer to this uniform-displacement metric, not to a complete Skorokhod metric controlling logarithmic slope distortion.

Theorem 7.3 (Arithmetic scale paths). *Under Theorem 2.3 and (2.11)–(2.12), attach the canonical marks to the arithmetic runs and use their true positions in (7.2). For any fixed finite thresholds $y_1, \dots, y_d$, the count paths*

$$X_M(s, y) = \sum_{j, e, \sigma} I_{j, e, \sigma} \mathbf{1}_{\{j/n \leq 2^s, n 2^{-(L+e+U_{j, e, \sigma})} < 2^{-s-y}\}}, \quad -T \leq s \leq 0,$$

admit a coupling to $(X_s(y_i))_{i \leq d}$ for which their distance $d_{J_1, [-T_M, 0]}^{(0)}$ on the growing interval tends to zero in probability (with $T = T_M$). Keeping the signs gives independent half-intensity target flows.

Proof. Use the common shell coupling and the same vertical marks. Active birth times move by at most η_M ; death times do not move. The retained height cutoff contains all the queried rectangles eventually, because $\Lambda 2^{-T} \rightarrow \infty$. Appendix D controls both creation and loss of activity using a two-sided birth–death band in the full planar cloud. Grouping transitions of one particle, it bounds this event, collisions between distinct groups, and endpoint crossings by $C_y(1+T)\eta_M$. Outside that event one time change aligns the vector paths. This tends to zero under the mesh assumptions. Signs are retained by the arithmetic comparison; their independence is a property of the target. $\square$

This theorem concerns canonically interpolated counts. It does not give a finite- M Markov property to the original sequence.

7.3 The two Gaussian sections

A threshold increase at a fixed spatial edge gives pure death. If $P_\lambda(t) = \mathcal{P}((0, 1] \times (0, \lambda 2^{-t}))$, then $P_\lambda(t+h) \mid P_\lambda(t)$ is a binomial thinning. The standardized threshold process and scale

process have different limits:

$$\begin{aligned} \frac{P_\lambda(t) - \lambda 2^{-t}}{\sqrt{\lambda 2^{-t}}} &\Rightarrow H_t, & dH_t &= -\frac{\beta}{2} H_t dt + \sqrt{\beta} dB_t, \\ \frac{X_s(y) - c_y}{\sqrt{c_y}} &\Rightarrow G_s \quad (c_y \rightarrow \infty), & dG_s &= -\beta G_s ds + \sqrt{2\beta} dB_s. \end{aligned} \quad (7.7)$$

Both limits are stationary, with covariances $2^{-|t-u|/2}$ and $2^{-|s-r|}$, respectively, and convergence is in J_1 on fixed compact intervals. Their martingale brackets and tightness estimates are given in Appendix C.

The two limits are sections of a common Gaussian field. Define

$$H_J(a, b) = \frac{\mathcal{P}((0, 2^a] \times (0, 2^{J-b})) - 2^{J+a-b}}{2^{(J+a-b)/2}}.$$

Theorem 7.4 (The rectangle Gaussian field). *On each compact rectangle $K \subset \mathbb{R}^2$, as $J \rightarrow \infty$, H_J converges in finite-dimensional distributions and in $H^{-s_a, -s_b}(K)$, for $s_a, s_b > 1/2$, to a centered Gaussian field $\mathcal{H}$ with covariance*

$$\mathbb{E}[\mathcal{H}(a, b)\mathcal{H}(c, d)] = 2^{-(|a-c|+|b-d|)/2}. \quad (7.8)$$

The fixed- a section has the threshold OU law; the section $b = a + y$ has the scale OU law.

Proof. The covariance is exactly (7.8) by intersection of rectangles. A fixed test pairing is a compensated Poisson integral with integrand F_J satisfying $\|F_J\|_\infty = O_K(2^{-J/2})$ and bounded L^2 norm. Its absolute third cumulant is $O_K(2^{-J/2})$. Taylor expansion of the characteristic exponent gives the Gaussian limit. Uniform second moments of tensor Fourier coefficients and a compact negative-Sobolev embedding give tightness. Appendix D.3 supplies the details and a quantitative grid estimate for simultaneous arithmetic and high-intensity limits. The section *path* statements in (7.7) use their separate martingale proofs; they are not obtained by taking a diagonal trace in a negative Sobolev space. $\square$

The limit in Theorem 7.4 is the classical stationary *Ornstein–Uhlenbeck sheet*, obtained by exponential transformation of a Brownian sheet; see Khoshnevisan–Shi [11, Section 1]. More explicitly, if W is a standard Brownian sheet on $\mathbb{R}_+^2$, then

$$\mathcal{H}(a, b) \stackrel{\text{law}}{=} 2^{-(a-b)/2} W(2^a, 2^{-b}), \quad (a, b) \in \mathbb{R}^2.$$

The Brownian-sheet intersection covariance gives (7.8) directly, so equality of centered Gaussian laws identifies the limit. The theorem establishes this sheet as a limit of the Poisson rectangle field and specifies its arithmetic transfer; it does not introduce a new Gaussian process.

7.4 Spatial fields and the full threshold endpoint

The next results use the same geometry with two proof interfaces. We first state the abstract complete-field conclusions, then give the signed target refinements. Direct spatial reconstructions are collected separately in the companion. Neither result is needed for the record and occupation theorems. Let $n \rightarrow \infty$, let $\mu_n = n^{-1} \sum_{j=1}^n \delta_{j/n}$, and let the complete exact-level field take values almost surely in $\mathfrak{N}_f(\{j/n : 1 \leq j \leq n\} \times \mathbb{N}_0)$, the space of point configurations of *finite total mass* over all sites and levels, equipped with the sigma-field generated by coordinate counts. Suppose its law is at total-variation distance $\varepsilon_n \rightarrow 0$ from independent Poisson coordinates with intensity $\Lambda_n \mu_n(dx) 2^{-e-1}$, $e \geq 0$, where $\Lambda_n \rightarrow \infty$. Write $\Lambda = \Lambda_n$ in

the formulas below. The input law may be a specified conditional law; all convergence statements then refer to that law, with the same deterministic target centering. For the binary arithmetic field, Input 2.1 supplies this complete-field hypothesis under (2.4) and $\Lambda \rightarrow \infty$, including the actual signs. The labelled bounds of [14, Theorems 5.10, 7.6] are sufficient inputs with their own hypotheses. The shell theorem alone is not a complete-field theorem. Section 7.6 develops the signed and joint target laws; their calculation is not replaced by an assertion of independence from total variation.

Theorem 7.5 (Spatial and threshold Gaussian limits). *Under this complete-field hypothesis, the following conclusions hold.*

Exact levels and a fixed threshold interval. If $\mathcal{E}_{n,e}$ is the input measure at level e , define

$$\tilde{\mathcal{E}}_{n,e} = \frac{\mathcal{E}_{n,e} - \Lambda 2^{-e-1} \mu_n}{\sqrt{\Lambda}}. \quad (7.9)$$

The subtracted measure is the exact mean of the Poisson target, not an asserted conditional mean of the input field. The sequence converges to independent spatial white noises with variance weights 2^{-e-1} , in the space with squared norm $\sum_{e \geq 0} (1+e)^{2a} \|F_e\|_{-s}^2$, for every $a \geq 0$ and $s > 1/2$. After canonical interpolation, let $\mathcal{S}_n(t)$ be the input survivor measure and put

$$\mathbf{H}_\Lambda(t) = \frac{\mathcal{S}_n(t) - \Lambda 2^{-t} \mu_n}{\sqrt{\Lambda 2^{-t}}}. \quad (7.10)$$

For each fixed $T_0 < \infty$, this process converges in $D([0, T_0], H^{-s})$, with J_1 topology, to the stationary spatial OU process with drift $-\beta/2$ and covariance

$$2^{-|t-u|/2} \int_0^1 \varphi(x) \psi(x) dx.$$

Compression including the critical endpoint. Write $\Lambda = 2^{J+\theta}$, with integer $J \rightarrow \infty$, $\theta \in (0, 1)$. If $\mathbf{H}_\Lambda(t)$ denotes the standardized survivor measure, then

$$\begin{aligned} \frac{1}{\sqrt{J}} \sum_{j=0}^J \mathbf{H}_\Lambda(j) \otimes \delta_{j/J} &\Longrightarrow (1 + \sqrt{2}) \mathcal{W}, \\ \frac{1}{\sqrt{J}} \mathbf{H}_\Lambda(Ju) J du &\Longrightarrow \frac{2}{\sqrt{\beta}} \mathcal{W}. \end{aligned} \quad (7.11)$$

Here $\mathcal{W}$ is white noise on $[0, 1]^2$, with convergence in $H^{-s_x, -s_u}$ for $s_x, s_u > 1/2$. Spatial integration gives the scalar versions. Each convergence is joint, along a convergent phase, with the critical lattice field of levels $e - J$, and its Gaussian limit is independent of that field.

The white noises in the two lines are descriptions of their respective marginal limits; they are not asserted to be independent on a joint construction. Appendix C.5 proves the cubic remainder and tightness all the way to the critical endpoint. The endpoint has order-one Poisson counts, so a Gaussian approximation at every single threshold would be false. What makes (7.11) possible is averaging, not pointwise Gaussianity near $j = J$.

7.5 Spatial white noise at one-factor cost

Summing the exact types is a measurable readout of the complete field. For run starts, the Poisson target therefore gives the following spatial limit. A separate fixed-partition proof is retained in the companion. No growing observation window or auxiliary fractional marks are involved.

Theorem 7.6 (Spatial run-start white noise). *Let $L = \lfloor \log_2 M \rfloor - a_M$, with $a_M \in \mathbb{N}_0$ and $a_M = o(\log M)$, and put*

$$n = M - L, \quad p = 2^{-L}, \quad \Lambda = np \longrightarrow \infty, \quad \mu_n = \frac{1}{n} \sum_{j=1}^n \delta_{j/n}.$$

Let $\mathcal{A}_M \in \mathcal{F}_{Y_M^\#}$ have positive probability, set $I_M = -\log \mathbb{P}(\mathcal{A}_M)$, and suppose, for a fixed $c > 0$,

$$I_M + \log \Lambda \leq V_M - c\nu_M. \quad (7.12)$$

For the actual run-start measure $\mathcal{E}_M^{\text{start}} = \sum_{j=1}^n J_{j+1,L} \delta_{j/n}$, one has, under $\mathbb{P}(\cdot \mid \mathcal{A}_M)$,

$$\frac{\mathcal{E}_M^{\text{start}} - \Lambda \mu_n}{\sqrt{\Lambda}} \Longrightarrow \mathcal{W}_{\text{sp}} \quad \text{in } H^{-s}([0, 1]), \quad s > \frac{1}{2}. \quad (7.13)$$

Here H^{-s} has the cosine convention of (C.1), and $\mathcal{W}_{\text{sp}}$ is spatial white noise of covariance $\int_0^1 \varphi \psi$. The centering is the deterministic target measure; no formula for the original field's conditional expectation is assumed.

Proof. Sum all types in Input 2.1. The target is a Poisson measure of intensity $\Lambda \mu_n$. Its centered, $\sqrt{\Lambda}$ -normalized integrals have Gaussian limits with covariance $\int \varphi \psi$ by the compensated-Poisson characteristic exponent. The cosine coefficient variances are uniformly bounded. An intermediate exponent $1/2 < s_0 < s$ and the compact embedding $H^{-s_0} \hookrightarrow H^{-s}$ give tightness. Corollary 2.2 transfers the entire law with the same deterministic target centering. Appendix F.1 gives the signed version of this target argument. Appendix E retains an alternative proof from fixed partitions and a direct quadratic remainder, without using the microscopic input. $\square$

This weak result alone is not a microscopic total-variation comparison. Its base-support proof isolates a reconstruction mechanism that does not use Input 2.1. We next retain the signed types and supply the additional target estimates needed for the complete hierarchy; those conclusions do not follow from (7.13) alone.

7.6 The signed spatial hierarchy

The run-start theorem suppresses two kinds of information: the exact excess and the sign. Restoring them first identifies the independent Poisson type coordinates, and then raises three different approximation questions: the tail of all levels, the supremum over a fixed threshold interval, and the accumulation of fluctuations up to the critical endpoint. We treat these questions separately under the same budget (7.12).

Throughout this subsection, use the arithmetic parameters and conditioning of Theorem 7.6. Write $\mathbb{T} = \mathbb{N}_0 \times \{-1, +1\}$ and, for $\alpha = (e, \sigma)$, set

$$\gamma_\alpha = 2^{-e-2}, \quad \mathcal{E}_{M,\alpha} = \sum_{j=1}^n I_{j,e,\sigma} \delta_{j/n}, \quad Z_{M,\alpha} = \frac{\mathcal{E}_{M,\alpha} - \Lambda \gamma_\alpha \mu_n}{\sqrt{\Lambda}}. \quad (7.14)$$

The indicators are those in (2.2): they retain the full rightward length and the actual arithmetic sign. Every centering below is the deterministic target measure. Theorem 7.5 remains an abstract result for complete-field inputs, supplied by Input 2.1 in this model. The following results are proved by analyzing one complete signed Poisson target in Appendix F, then applying the common readout. Appendix G retains the independent direct reconstruction route.

Table 1: One-factor spatial limits. Write $H_x = H^{-s}([0, 1])$ and $H_{xu} = H^{-s_x, -s_u}([0, 1]^2)$; all displayed Sobolev exponents exceed $1/2$. The weighted spaces and compression parameters are defined below.

Result	Space or topology	Horizon or weights	Joint dependence
Finite types Theorem 7.7	$\prod_{\alpha \in F} H_x$	Nonempty finite F , fixed before the limit	Independent Gaussian type coordinates.
Full spectrum Theorem 7.8	$\mathcal{H}_{w,s}$	Fixed w ; $\Gamma(w) < \infty$	Independent types; joint with the threshold path below.
Integer thresholds Corollary 7.9	Polynomially weighted $\ell^2(H_x)$; standardized version in $\ell^2(v; H_x)$, both signs	All $j \geq 0$; fixed weights, with $\sum_j v_j < \infty$ for the standardized version	Correlated thresholds within each sign; independent sign blocks.
Threshold paths Theorem 7.10	$D([0, T_0], H_x^2), J_1$	Fixed $T_0 < \infty$	Independent signs; the paths and exact spectrum share the Gaussian measure in (7.19).
Compressions and critical field Theorem 7.11	H_{xu}^4 and vague topology for the point field	$J = \lfloor \log_2 \Lambda \rfloor \rightarrow \infty$; $\theta_M \rightarrow \theta$	Correlated compression pair per sign; independent signs and independent critical field.
Three blocks Corollary 7.12	Product of the spectrum, path, compression, and point-field spaces	Fixed w, T_0 ; the same compression and phase limits	The low-level block, compression block, and critical field are independent; internal dependencies remain.
Rectangles Theorem 7.13	$H^{-s_a, -s_b}(K)^2$ and finite-dimensional laws	Fixed rectangle K ; depth k_M satisfying (7.25)	Independent signs. No additional product with the preceding blocks is asserted.

Table 1 separates the spaces, horizons, and joint laws of the results below. All rows use the common one-factor arithmetic input. Their extra weights, horizons and depth conditions concern the target limit, not the resolution cost of the comparison; they remain part of the cited statements.

7.6.1 Finite types and the full weighted spectrum

Theorem 7.7 (Finite signed exact-level white noise). *For every deterministic nonempty finite $F \subset \mathbb{T}$, fixed before $M \rightarrow \infty$, and every $s > 1/2$,*

$$(Z_{M,\alpha})_{\alpha \in F} \implies (\sqrt{\gamma_\alpha} \mathcal{W}_\alpha)_{\alpha \in F} \quad \text{in} \quad \prod_{\alpha \in F} H^{-s}([0, 1]), \quad (7.15)$$

under $\mathbb{P}(\cdot \mid \mathcal{A}_M)$, where the $\mathcal{W}_\alpha$ are independent standard spatial white noises. No phase subsequence or auxiliary fractional mark is required.

Proof. For each type, the target has independent Poisson masses with intensity $\Lambda \gamma_\alpha \mu_n$. A joint compensated-Poisson expansion gives independent Gaussian finite projections; uniform cosine variances give H^{-s} tightness. Apply Corollary 2.2 to the finite family. Appendix F.1 gives the details. The categorical covariance calculation and spatial reconstruction in Appendix G.3 give an alternative proof without microscopic coupling. $\square$

More generally, fixed mark weights $h_1, \dots, h_d$ supported in F give joint Gaussian limits for $\sum_{\alpha \in F} h_i(\alpha) Z_{M,\alpha}$, with covariance

$$\left(\sum_{\alpha \in F} \gamma_\alpha h_i(\alpha) h_j(\alpha) \right) \int_0^1 \varphi \psi.$$

This is the continuous linear image of the finite-family limit.

For each fixed excess e , the total and imbalance

$$Z_{M,e,+} + Z_{M,e,-}, \quad Z_{M,e,+} - Z_{M,e,-}$$

therefore have independent spatial white-noise limits, both of variance weight 2^{-e-1} . This is a Gaussian independence of exact-level fluctuations, not a prelimit independence or a statement about competing global record counts.

For fixed weights $0 < w_\alpha < \infty$, define

$$\mathcal{H}_{w,s} = \left\{ (F_\alpha) : \sum_\alpha w_\alpha \|F_\alpha\|_{-s}^2 < \infty \right\}, \quad \Gamma(w) = \sum_\alpha w_\alpha \gamma_\alpha. \quad (7.16)$$

Theorem 7.8 (The complete signed spectrum). *If $s > 1/2$ and $\Gamma(w) < \infty$, then*

$$(Z_{M,\alpha})_{\alpha \in \mathbb{T}} \implies (\sqrt{\gamma_\alpha} W_\alpha)_{\alpha \in \mathbb{T}} \quad \text{in } \mathcal{H}_{w,s}. \quad (7.17)$$

The condition $\Gamma(w) < \infty$ is necessary and sufficient for the indicated Gaussian spectrum to be a random element of this fixed Hilbert space. It includes polynomial weights and

$$w_{e,\sigma} = 2^{\eta e} (1 + e)^{2a}, \quad 0 \leq \eta < 1, \quad a \geq 0.$$

This trace condition does not assert optimality of the arithmetic information–intensity budget.

Proof. The target obeys $\mathbb{E}_P \|Z_{M,\alpha}^P\|_{-s}^2 \leq \kappa_s \gamma_\alpha$. The trace condition makes its weighted level tail uniformly summable; finite-type convergence therefore completes the Hilbert space. Necessity follows from the independent constant spatial Gaussian modes. Appendix F.1 proves these claims. Transfer uses the measurable full spectrum and a probability bound, not a moment inequality for the original conditional field. The alternative modified-field estimate remains in Appendix G.4. $\square$

Corollary 7.9 (Integer threshold sequences). *For $j \geq 0$, put $G_{M,j,\sigma} = \sum_{e \geq j} Z_{M,e,\sigma}$. For every $a \geq 0$, these fields converge in the space of squared norm $\sum_{j,\sigma} (1+j)^{2a} \|F_{j,\sigma}\|_{-s}^2$ to a Gaussian sequence with covariance*

$$\text{Cov}(\langle G_{j,\sigma}, \varphi \rangle, \langle G_{k,\tau}, \psi \rangle) = \mathbf{1}_{\{\sigma=\tau\}} 2^{-\max(j,k)-1} \int_0^1 \varphi \psi.$$

The standardized sequence $2^{(j+1)/2} G_{M,j,\sigma}$ converges in $\ell^2(v; H^{-s})$ over both signs for every fixed strictly positive summable weight sequence v_j . Its covariance is $\mathbf{1}_{\{\sigma=\tau\}} 2^{-|j-k|/2} \int \varphi \psi$.

Proof. Tail summation is continuous from a positively exponentially weighted exact-level space into each polynomially weighted threshold space. For the standardized assertion, every target coordinate has uniformly bounded H^{-s} second moment; summability of v controls its tail. The complete-field readout transfers these laws. See Appendix F.2 for the target proof and Appendix G.4.1 for the direct alternative. $\square$

Summing both signs at $j = 0$ recovers Theorem 7.6. That deduction uses the full tail control, not an infinite sum of finite-dimensional convergence statements. The short base-support proof in Appendix E remains an independent route to this special case.

7.6.2 A fixed threshold interval

Attach the canonical independent mark (7.1) to each atom and define

$$\begin{aligned} S_M^\sigma(t) &= \sum_{j,e} I_{j,e,\sigma} \mathbf{1}_{\{e+U_{j,e,\sigma} > t\}} \delta_{j/n}, \\ H_M^\sigma(t) &= \frac{S_M^\sigma(t) - (\Lambda/2)2^{-t}\mu_n}{\sqrt{(\Lambda/2)2^{-t}}}. \end{aligned} \quad (7.18)$$

At integer t these are the intrinsic threshold fields. Noninteger times use only the declared interpolation.

Theorem 7.10 (Spatial threshold paths at one-factor cost). *For every fixed $T_0 < \infty$ and $s > 1/2$,*

$$(H_M^+, H_M^-) \implies (H^+, H^-) \quad \text{in } D([0, T_0], H^{-s} \times H^{-s}), \quad J_1,$$

where the limits are independent stationary spatial Ornstein–Uhlenbeck processes with drift $-\beta/2$ and covariance

$$\text{Cov}(\langle H_t^\sigma, \varphi \rangle, \langle H_u^\tau, \psi \rangle) = \mathbf{1}_{\{\sigma=\tau\}} 2^{-|t-u|/2} \int_0^1 \varphi \psi.$$

This convergence is joint with Theorem 7.8. Its joint limit is represented by independent Gaussian random measures $\mathcal{G}_\sigma$ on $[0, 1] \times (0, \infty)$ with control measure $(\beta/2)2^{-v} dx dv$:

$$Z_{e,\sigma} = \mathcal{G}_\sigma(\cdot \times [e, e+1)), \quad H_t^\sigma = \sqrt{2} 2^{t/2} \mathcal{G}_\sigma(\cdot \times (t, \infty)). \quad (7.19)$$

In particular the exact spectrum and the survivor path are not independent.

Proof. Under the target law each sign is a Poisson cloud with exponential lifetimes and spatial control measure μ_n . Appendix C.4 gives its Hilbert-valued martingale limit at intensity $\Lambda/2$. Joint finite-test Poisson exponents with the exact levels identify (7.19); the spectral tails and path tightness complete the joint space. Appendix F.3 records this signed joint argument. Canonical marking and the full readout then transfer the law. The empirical-bridge reconstruction in Appendix G.5 is an alternative proof, not a death martingale for the arithmetic field. $\square$

7.6.3 Two compressions and a critical field

Write $\Lambda = 2^{J+\theta_M}$, $J = \lfloor \log_2 \Lambda \rfloor$, and set

$$\mathcal{D}_M^\sigma = \frac{1}{\sqrt{J}} \sum_{j=0}^J H_M^\sigma(j) \otimes \delta_{j/J}, \quad \mathcal{C}_M^\sigma = \frac{1}{\sqrt{J}} \int_0^J H_M^\sigma(t) \otimes \delta_{t/J} dt. \quad (7.20)$$

The critical field is

$$\Xi_M = \sum_{j,e,\sigma} I_{j,e,\sigma} \delta_{(j/n, e-J, \sigma)}. \quad (7.21)$$

Since $J = \lfloor \log_2 n \rfloor - L$, this is the same integer centering as in the terminal record field, not a new choice of phase.

Theorem 7.11 (Joint compression and the critical field). *For $s_x, s_u > 1/2$, and along $\theta_M \rightarrow \theta \in [0, 1]$,*

$$(\mathcal{D}_M^+, \mathcal{C}_M^+, \mathcal{D}_M^-, \mathcal{C}_M^-, \Xi_M) \implies (\mathcal{D}^+, \mathcal{C}^+, \mathcal{D}^-, \mathcal{C}^-, \Pi_\theta)$$

in four copies of $H^{-s_x, -s_u}([0, 1]^2)$ and the vague topology for the point field, where

$$\Pi_\theta \sim \text{PPP} \left(dx \sum_{r \in \mathbb{Z}, \sigma} 2^{\theta-r-2} \delta_{(r, \sigma)} \right). \quad (7.22)$$

The critical field is independent of the whole Gaussian block, and the two sign blocks are independent. Within one sign, the joint covariance density of the discrete and continuous compressions is

$$\begin{pmatrix} c_d & c_c \\ c_c & c_c \end{pmatrix}, \quad c_d = 3 + 2\sqrt{2}, \quad c_c = 4/\beta. \quad (7.23)$$

Precisely, pairing the two-component field with tests f, g multiplies this matrix by $\int_{[0,1]^2} fg$.

Proof. For the single complete Poisson target, the absolute cubic remainders vanish jointly through the critical endpoint. The discrete/continuous cross covariance gives c_c , not zero. Deleting upper critical atoms together with their compensator proves independence from each upper restriction of the critical field; exhaustion gives the full vague field. Uniform tensor-Sobolev coefficient bounds complete the joint topology. These target calculations are printed in Appendix F.5. The common marked readout transfers the result. Appendix G.6 retains the direct spatial-remainder alternative. $\square$

The covariance has a useful interpretation. With independent tensor white noises $\mathcal{W}_c^\sigma, \mathcal{W}_\perp^\sigma$,

$$\mathcal{C}^\sigma = \sqrt{c_c} \mathcal{W}_c^\sigma, \quad \mathcal{D}^\sigma = \mathcal{C}^\sigma + \sqrt{c_d - c_c} \mathcal{W}_\perp^\sigma. \quad (7.24)$$

The difference has positive variance density $3 + 2\sqrt{2} - 4/\log 2 \simeq 0.05764696$ and is independent of the continuous compression. The two original compressions are neither independent nor proportional copies of the same noise. The unsigned marginal formulas (7.11) follow by summing signs and dividing by $\sqrt{2}$. The phase endpoint 1 in (7.22) retains its integer-level shift relative to phase 0.

Mixed Gaussian–Poisson approximation and asymptotic independence are classical in Poisson space; see [2]. The following conclusion specifies the joint functional blocks for the present marked threshold field, including the moving critical endpoint.

Corollary 7.12 (Three independent limiting blocks). *For every fixed admissible weight sequence and fixed T_0 , Theorems 7.8, 7.10, and 7.11 hold jointly. Their limits split into three mutually independent blocks:*

$$\underbrace{((Z_{e,\sigma})_{e,\sigma}, (H_t^\sigma)_{t \leq T_0, \sigma})}_{\text{exact levels and fixed thresholds}}, \quad \underbrace{(\mathcal{D}^\sigma, \mathcal{C}^\sigma)_{\sigma=\pm 1}}_{\text{compressed thresholds}}, \quad \Pi_\theta.$$

Dependence inside the first block is specified by (7.19), and inside the second by (7.23).

Proof. Use one complete marked target. Fixed exact levels and fixed threshold times have covariance $O(J^{-1/2})$ with either compression. Joint Gaussian identification makes the two Gaussian blocks independent. The compensated deletion of upper critical atoms also leaves the low block unchanged in probability. The norm tails and path tightness complete this target product; see Appendix F.6. One application of the full readout transfers the joint law. No independence of prelimit arithmetic objects is inferred. $\square$

This corollary does not add the Brownian *record history* to these three blocks. Its terminal product in Section 5 uses a different, recent-spatial-strip argument. A joint theorem combining the two products would require another common separation estimate.

7.6.4 The simultaneous rectangle limit

For a compact rectangle $K \subset (-\infty, 0] \times \mathbb{R}$ of positive side lengths, choose integers $k_M \rightarrow \infty$ satisfying

$$2^{k_M} \max_{(a,b) \in K} 2^{-b} < \Lambda \quad \text{eventually,} \quad 2^{k_M}/n^2 \rightarrow 0. \quad (7.25)$$

The second condition follows from the first in the arithmetic corridor, but is recorded to make the simultaneous grid error explicit. The symbol k_M denotes intensity depth, not the logarithmic shell width h_M . Put

$$\begin{aligned} N_M^\sigma(a, b) &= \sum_{j,e} I_{j,e,\sigma} \mathbf{1}_{\{j/n \leq 2^a, \Lambda 2^{-(e+U_{j,e,\sigma})} < 2^{k_M-b}\}}, \\ \mathcal{R}_M^\sigma(a, b) &= \frac{N_M^\sigma(a, b) - 2^{k_M+a-b}/2}{\sqrt{2^{k_M+a-b}/2}}. \end{aligned} \quad (7.26)$$

Theorem 7.13 (Rectangle fields at one-factor cost). *Under the common one-factor hypotheses and (7.25), the pair $(\mathcal{R}_M^+, \mathcal{R}_M^-)$ converges in finite-dimensional distributions and in $H^{-s_a, -s_b}(K)^2$, $s_a, s_b > 1/2$, to independent centered Gaussian fields of covariance*

$$\mathbb{E}[\mathcal{R}^\sigma(a, b) \mathcal{R}^\sigma(c, d)] = 2^{-(|a-c|+|b-d|)/2}.$$

The unsigned sum divided by $\sqrt{2}$ has the limit of Theorem 7.4.

Proof. For the marked target, keep the exact grid mean first. Intersection of the two rectangles gives the covariance, and compensated Poisson integrands have absolute cubic integral $O_K(2^{-k_M/2})$. Uniform coefficient variances give the tensor-Sobolev limit. Replacing the exact grid prefix mass by the continuous area costs squared bias $O_K(2^{k_M}/n^2)$. Appendix F.7 proves both the finite-point and distribution-space assertions, which transfer by the common readout. Appendix G.7 retains the direct spatial reconstruction. Neither argument uses a diagonal trace. $\square$

The resulting hierarchy concerns fixed weighted spaces, fixed threshold horizons, and the specified compressed or rectangular observables. Input 2.1 separately gives microscopic total variation to the moving grid target. Neither result gives an exact arithmetic Markov property, arbitrary growing-window Gaussian limits, or a path limit by tracing a negative-Sobolev field.

7.7 Sampling depends on direction

For the fixed-intensity stationary scale process let $Y_s = (X_s(y) - c_y)/\sqrt{c_y}$. Its long-time compressed fluctuations satisfy

$$\begin{aligned} \frac{1}{\sqrt{T}} \int_0^T Y_s \varphi(s/T) ds &\Rightarrow \mathcal{N}(0, (2/\beta) \|\varphi\|_2^2), \\ \frac{1}{\sqrt{J}} \sum_{j=0}^J Y_j \delta_{j/J} &\Rightarrow \sqrt{3} \mathcal{W} \quad \text{in } H^{-r}, \quad r > 1/2. \end{aligned} \quad (7.27)$$

In the first line the convergence holds jointly for finitely many tests; the corresponding random distributions converge to $\sqrt{2/\beta} \mathcal{W}$. The proof uses exponential particle lifetimes, not a high-intensity approximation.

For the arithmetic path of Theorem 7.3, which is available on $[-T_M, 0]$, the transferred observables are defined explicitly by

$$c_M^{\text{sc}}(\varphi) = \frac{1}{\sqrt{T_M}} \int_{-T_M}^0 \frac{X_M(s, y) - c_y}{\sqrt{c_y}} \varphi\left(\frac{s + T_M}{T_M}\right) ds$$

and, for $J_M^{\text{sc}} = \lfloor T_M \rfloor \geq 1$, by

$$\mathcal{D}_M^{\text{sc}} = \frac{1}{\sqrt{J_M^{\text{sc}}}} \sum_{j=0}^{J_M^{\text{sc}}} \frac{X_M(j - J_M^{\text{sc}}, y) - c_y}{\sqrt{c_y}} \delta_{j/J_M^{\text{sc}}}.$$

Both use only the available time interval. The superscript sc distinguishes the sampling length from the critical threshold index $J = \lfloor \log_2 \Lambda \rfloor$ in Section 7.6. For fixed y and $r > 1/2$, their respective limits in $H^{-r}([0, 1])$ are $\sqrt{2/\beta} \mathcal{W}$ and $\sqrt{3} \mathcal{W}$. These are marginal descriptions, not a claim that the two limiting noises are independent. Stationarity permits the displayed changes of origin on the target. On the common shell coupling, the continuous error is $O_{\mathbb{P}}(\sqrt{T_M} \eta_M)$ and the discrete discrepancy probability is $O(T_M \eta_M)$; both vanish under the mesh assumptions. Appendix D.4 proves these transfer estimates directly, without transferring unbounded moments.

For the Gaussian rectangle field, a line of nonzero direction (p, q) has OU rate $\alpha = \beta(|p| + |q|)/2$. Sampling at mesh h , with Riemann-sum normalization $h/\sqrt{T}$, has variance density

$$h \sum_{k \in \mathbb{Z}} e^{-\alpha h |k|} = h \coth \left(\frac{\beta(|p| + |q|)h}{4} \right), \quad (7.28)$$

whose continuous value is $4/[\beta(|p| + |q|)]$. Thus the threshold direction $(0, 1)$ gives $3 + 2\sqrt{2}$ at unit mesh and $4/\beta$ continuously; the scale direction $(1, 1)$ gives 3 and $2/\beta$. This single covariance calculation explains why the four constants differ.

7.8 What interpolation does to records

For the continuous cloud, let $V^{(k)}(u)$ be the k th smallest vertical coordinate before u . Then $uV^{(k)}(u)$ has the Gamma($k, 1$) law. The initial-rank- j record-time processes are independent Poisson processes of intensity du/u [5, Proposition 2]. In logarithmic time their rates are β , and the update count of the k th minimum has rate $k\beta$.

The shell coupling transfers this continuous hierarchy quantitatively. For fixed k , the distance of the first k layer counts on a window of length T from independent $\text{Pois}(\beta T)$ variables is at most

$$\delta_M + e^{-\Lambda 2^{-T}} \sum_{\ell=0}^{k-1} \frac{(\Lambda 2^{-T})^\ell}{\ell!} + Ck(k+1)Th. \quad (7.29)$$

Appendix D.2 proves the bound and the corresponding Brownian layer limits. These records depend on the auxiliary fractional lengths. Taking floors recovers integer order statistics exactly, but does not preserve their record-time counts. The contrast between a Poisson mean βT here and the intrinsic centering $T/2$ is therefore a distinction between observables, not a contradiction.

8 A geometric target theorem with finite colours

The arithmetic application fixes a binary tail and actual run signs. The target argument has a wider form, which makes clear where those choices enter. This section is a deduction within the Poisson model, with a freely specified geometric base and a finite colour law. It does not supply an arithmetic process comparison for another value group. Such a comparison would require its own boundary, macroscopic, tail and ordering estimates; the binary boundary estimate cannot simply be transferred by replacing the base.

Fix a geometric base $b_* > 1$, put $\beta_* = \log b_*$ and $p_* = 1 - b_*^{-1}$, and let π be a probability on a finite colour set $\mathcal{C}$. Consider

$$\mathcal{P}_{\theta,*} \sim \text{PPP} \left(du \sum_{q \in \mathbb{Z}} p_* b_*^{\theta-q} \delta_q \otimes \pi \right). \quad (8.1)$$

Let $\xi : \mathcal{C} \rightarrow \mathbb{R}^d$ have mean zero and covariance matrix $C = \mathbb{E}_\pi[\xi \xi^\top]$. Let $N_*(s)$ count strict records on $(1, b_*^s]$, let $D_*(s)$ sum their ξ -marks, and let $A_*(s)$ integrate the current first holder's ξ -mark over logarithmic base- b_* time. As before, ties retain the earlier holder.

Theorem 8.1 (Finite-colour record and occupation limit). *For any deterministic phase sequence $\theta_T \in [0, 1)$,*

$$\frac{1}{\sqrt{T}} (N_*(Tt) - p_* Tt, D_*(Tt), A_*(Tt))_{0 \leq t \leq 1} \Longrightarrow \mathbb{B}_{\Sigma_*}$$

in vector J_1 , where

$$\Sigma_* = \begin{pmatrix} p_*(1-p_*) & 0 & 0 \\ 0 & p_* C & C \\ 0 & C & \kappa_* C \end{pmatrix}, \quad \kappa_* = \frac{2}{\beta_*^2} \int_0^{b_*-1} \frac{\log(1+x)}{x} dx. \quad (8.2)$$

The persistence kernel has phase average

$$\int_0^1 p_{\theta,*}(h) d\theta = \frac{1}{\beta_*} \log(1 + (b_* - 1)b_*^{-h}). \quad (8.3)$$

For physical occupation $V_{\theta,*} = \int_0^1 \xi(\text{first holder at } u) du$, the phase-averaged covariance is

$$\int_0^1 \text{Cov}(V_{\theta,*}) d\theta = \left(\frac{b_*}{b_* - 1} - \frac{1}{\log b_*} \right) C. \quad (8.4)$$

Proof. The arrival rate strictly above q is $a_q = b_*^{\theta-q-1}$. The next height increment has probabilities $p_* b_*^{-(k-1)}$, $k \geq 1$, independently of the exponential holding variable and of its new colour. The indicators of visited relative heights are therefore independent Bernoulli variables of parameter p_* . Adding their independent colours gives categorical height labels. The maximum law is $\mathbb{P}(Q(u) \leq q) = e^{-ua_q}$, and the same integer-grid argument as Lemma 3.3 bounds the height clock error by $O_{\mathbb{P}}(\log(2+T))$, with moments of every fixed order. Consequently the count variance density is $p_*(1-p_*)$, while its mean is $p_* T + O(1)$. For the bounded mean error one may also integrate the periodic intensity obtained by subtracting and reindexing $\sum_q a_q (e^{-ua_q} - e^{-b_* u a_q})$.

Given the initial maximum q , absence of a strict improvement up to b_*^h has the usual exponential probability. Thus

$$p_{\theta,*}(h) = \sum_{q \in \mathbb{Z}} \{e^{-b_*^h a_q} - e^{-(b_*^h + b_* - 1)a_q}\}.$$

This is bounded by $C_{b_*} b_*^{-h}$, uniformly in phase. Unfolding the phase period and integrating in a proves (8.3). Conditional on the uncoloured cloud, distinct holder marks are independent with covariance C . Hence

$$\text{Var } D_*(T) = C \mathbb{E} N_*(T), \quad \text{Cov}(D_*(T), A_*(T)) = C \left(T - \int_0^T p_{\theta,*}(s) ds \right).$$

The count's covariance with either marked variable is zero. Integrating the holder covariance and using (8.3) gives

$$2 \int_0^\infty \frac{\log(1 + (b_* - 1)e^{-\beta_* h})}{\beta_*} dh = \kappa_*.$$

This identifies all entries of (8.2).

The functional assertion requires more than these covariances. Unit logarithmic blocks are stationary at fixed phase, have an exponential moment, and forget a replaced past at rate $O_{b_*}(b_*^{-j})$, exactly as in Lemma 3.6. The finite-colour marks are bounded. The strong-mixing criterion used in Theorem 3.7 therefore applies to the whole $(1 + 2d)$ -dimensional block vector. The maximal block remainder is $O_{\mathbb{P}}(\log(2 + T))$. A phase shift is a change of logarithmic origin by at most one and has the same negligible remainder. This proves the phase-uniform process limit, including when C is singular.

Finally, for $0 < u \leq v \leq 1$, the phase-averaged holder covariance is $C\beta_*^{-1} \log(1 + (b_* - 1)u/v)$. Twice its integral over the triangle is

$$\frac{C}{\beta_*} \int_0^1 \log(1 + (b_* - 1)r) dr = \left(\frac{b_*}{b_* - 1} - \frac{1}{\beta_*} \right) C,$$

which proves (8.4). Boundedness justifies every occupation integral. $\square$

For $b_* = 2$ and a fair sign mark, this recovers (1.4) and (1.8). The count constants alone are part of classical geometric-record theory [8]. The purpose of the formulation is to retain the joint marked occupation calculation and to expose the transferable hypotheses, not to reclassify those classical constants as new.

9 Observation scope and further questions

The original sequence supplies integer lengths, actual signs, and finite-word observations. A declared randomization supplies continuous lifetimes. The Poisson field on the same grid is a moving comparison target; diffuse Poisson and Gaussian objects are weak limits. Input 2.1 strengthens the first comparison without identifying these levels of description.

9.1 Moving-target universality is not universality of weak limits

Input 2.1 completes the same-grid comparison for the binary run field. Corollary 2.2 is uniform over measurable observations of that field. It does not replace any weak-limit topology by total variation to a diffuse or Gaussian object.

For example the fixed Borel event that all spatial atoms lie in $\mathbb{Q}$ has probability one for the arithmetic grid and probability $e^{-\Lambda}$ for a diffuse Poisson field of total intensity Λ . Their total-variation distance is therefore at least $1 - e^{-\Lambda}$. A standardized Poisson count has countable support and is at distance one in total variation from a nondegenerate Gaussian, despite its central limit theorem. On a fixed threshold interval $[0, T_0]$, the canonically marked target survivor path is continuous with probability $\exp\{-\Lambda(1 - 2^{-T_0})\}$, whereas its OU limit is continuous almost surely. These are measurable tests, not continuity sets for the corresponding weak convergences.

Even on the same grid, a readout need not have a limiting law: the constant map $\Phi_M = (-1)^M$ suffices. Nor does comparison of the laws of unbounded readouts compare their means. The laws δ_0 and $(1 - M^{-1})\delta_0 + M^{-1}\delta_{M^2}$ have distance M^{-1} but means zero and M . The coupling quantifiers are one coupling for each M , not a prescribed joint coupling of all terminal scales.

9.2 What fixed-cell and second-moment information does not determine

The following elementary construction isolates the missing information. It is not a counterexample in the multiplicative model.

Proposition 9.1 (A parity invisible to cell counts and second moments). *For even n , let $\mathbf{Q}$ be the full same-grid Poisson target with intensity Λ . There is a law $\mathbf{P}^{\text{par}}$ with exactly the same complete type counts on adjacent site pairs, and the same first and second factorial moment measures, such that*

$$d_{\text{TV}}(\mathbf{P}^{\text{par}}, \mathbf{Q}) = \frac{1}{2} \mathbb{P}\{\text{Pois}(\Lambda) \geq 3\} \longrightarrow \frac{1}{2}. \quad (9.1)$$

Independent auxiliary marks may be retained. A coupling changes at most one atom by one spatial grid step and preserves its type and marks.

Proof. Draw $N \sim \text{Pois}(\Lambda)$; for each labelled atom choose independently a uniform pair index $R_i \in \{1, \dots, n/2\}$, type α_i of law γ , and fair bit B_i . Its site is $2R_i - 1 + B_i$. For $N \leq 2$ do nothing. For $N \geq 3$, replace the bit vector by the uniform law of even total parity, leaving everything else unchanged. Any one or two bits are still independent and fair. This proves the first two factorial-moment assertions, while pair counts do not use the bits at all.

Flipping the last bit when the total parity is odd gives the required coupling, with disagreement probability $\frac{1}{2} \mathbb{P}(N \geq 3)$. The unlabelled event that $N \geq 3$ and the total multiplicity at even sites is odd has that probability under $\mathbf{Q}$ and zero under $\mathbf{P}^{\text{par}}$, proving equality in (9.1). $\square$

For a consecutive partition into R cells, the coupling changes cell counts with probability at most $(R-1)/n$. Thus fixed-cell comparisons and exact second-moment control can coexist with nonvanishing microscopic distance. When $\Lambda = n^{o(1)}$, excluding repeated sites costs $O(\Lambda^2/n) = o(1)$ and does not remove the obstruction. The proof of Input 2.1 avoids this inference by constructing the entire conditional law after forcing one word.

There is also an exact disintegration formulation. For laws $\mathbf{P}, \mathbf{Q}$ on the finite-configuration space, let π record cell counts and put $\delta = d_{\text{TV}}(\pi_* \mathbf{P}, \pi_* \mathbf{Q})$. Write $\mathbf{P}_c, \mathbf{Q}_c$ for the fibre laws and $\eta = \sum_c (\pi_* \mathbf{P})(c) d_{\text{TV}}(\mathbf{P}_c, \mathbf{Q}_c)$. For a Poisson target, every finite cell-count array has positive mass; its fibre law places atoms uniformly with replacement inside each cell and type.

Proposition 9.2 (Exact within-cell completion). *One has*

$$|d_{\text{TV}}(\mathbf{P}, \mathbf{Q}) - \eta| \leq \delta. \quad (9.2)$$

In particular, for the two laws in Input 2.1 and any deterministic partition, $\eta \leq 2\Delta_M^{\text{mic}} \rightarrow 0$.

Proof. Set $\tilde{\mathbf{Q}} = \sum_c (\pi_* \mathbf{P})(c) \mathbf{Q}_c$. Disjoint fibres give $d_{\text{TV}}(\mathbf{P}, \tilde{\mathbf{Q}}) = \eta$ and $d_{\text{TV}}(\mathbf{Q}, \tilde{\mathbf{Q}}) = \delta$. The triangle inequality proves (9.2); contraction gives $\delta \leq \Delta_M^{\text{mic}}$ in the last assertion. $\square$

9.3 Arithmetic complements and their logical status

The arithmetic example in [14, technical companion, Theorem G.9] has vanishing microscopic total variation and divergent global absolute cumulant activity. It shows that this absolute-smallness criterion is not necessary for comparison, without invalidating the finite cumulant inequality or supplying a process limit.

The signed Palm identities retain the conditioning environment, and the averaged one-sided relative-void deficit vanishes as a consequence of the microscopic theorem [14, technical companion, Propositions G.4 and G.6, and Corollary G.5]. This is neither uniform control

over prescribed configurations nor a vanishing bound on the stronger squared Walsh spectrum. These results are downstream of Input 2.1; no Palm deficit or all-order cancellation is assumed in the process proofs.

Independent prime signs are essential to the arithmetic input. Substitution of the deterministic Liouville sequence gives neither its random-origin Poisson law nor quartic cancellation or deterministic Palm avoidance. Likewise, the almost-sure empirical *count* law in [14, Corollary 7.8a] does not yield the present process laws after fixing the entire function.

9.4 Remaining process questions

The OU theorem treats fixed horizons, the spectral theorem fixed weights, and the rectangle theorem a distribution space without a diagonal trace. Moving horizons or weights require additional uniform estimates, even though the comparison with the moving Poisson target remains available. Corollary 6.4 also fixes its weight parameter: sending that parameter to infinity after the limit does not cover weights concentrating on a terminal window or give a physical-time theorem.

For other dictionaries, a one-length comparison supplies neither nested-word data nor geometric lifetimes. A finite-colour arithmetic process theorem requires its own macroscopic, tail and ordering bounds. Intrinsic higher-rank record *times* with integer ties need separate analysis, even though finite-rank integer order statistics are available from the lattice cloud. A direct description of the physical-leadership distribution would also complement its defining field and explicit moment formulas.

The independent-level target here does not cover fields whose points retain shared-germ dependence at distinct positions. A family of one-time compound-Poisson count laws alone would not specify its spatial increments or replace the joint target used in the proofs.

The limiting record history is measured on a late logarithmic window with $T = O(V_M)$, not on the full history from the first symbol. Its independence from the endpoint is asymptotic, and fixed-atom post-selection is justified by a positive limiting probability. Conditioning on a vanishing-probability extremal event would require a relative-error argument beyond the joint weak limit.

Availability and artificial-intelligence assistance

This article and its technical companion, both Version 2.1, form one proof package, dated 21 September 2026. The accompanying source archive contains the LaTeX files, figures, bibliographic data and build instructions. Input 2.1 and Appendix A identify the arithmetic statements used from the public Version 3 of [14], including its technical companion. No Lean formalization of the process results proved in the present article and companion is claimed. The separate formalization described in [14] concerns the arithmetic source.

Declaration on artificial-intelligence assistance. The author directed this research with substantial assistance from Claude Fable 5 and Claude Opus 4.8 (Anthropic), and GPT-5.6 Sol and GPT-6 Astra (OpenAI). Assistance covered candidate arguments, mathematical drafting and reorganization, reference checks, and executable diagnostics. GPT-6 Astra also assisted with the signed spatial hierarchy, the arithmetic-interface developments and their integration, and the editorial and bibliographic preparation. The author determines the claims and scope and accepts responsibility for the arguments, references, and remaining errors. AI-assisted checks are not independent human peer review. The claims rest on the printed proofs and stated inputs, not on model outputs or finite tests.

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Technical companion to Records, scale flows, and Gaussian limits from lattice Poisson fields

Brice Pouly

Version 2.1

This research received substantial assistance from AI systems; see the declaration in the article.

Abstract

This companion contains the probabilistic proofs and the alternative direct reconstructions used in the article. The arithmetic input is the full same-grid signed comparison in *Long runs and rare patterns*, Theorem 7.7, proved in its technical companion, Appendix F. Its Palm complements are consequences of that comparison.

The principal route consists of the record and censoring estimates in Appendix B, the functional Poisson theory in Appendix C, scale stability in Appendix D, and the signed target calculations and joint limits in Appendix F. The mixed compression covariance, the three-block separation, and the deterministic rectangle-centering error are proved here. Appendices A, E and G retain the direct partition and spatial-reconstruction routes without using the microscopic coupling. Their narrower interfaces remain useful, but are not additional hypotheses of the principal route.

This companion belongs to Version 2.1 of the article. Appendix letters refer to this document; numbered sections refer to the article. The article specifies the model and the exact scope of the transfer, and contains the declaration of artificial-intelligence assistance. No Lean formalization of the results proved here is claimed.

Contents

Proof routes and dependencies	2
A Direct partition comparison from arithmetic estimates	2
B A common coupling for records and observations	6
C Spatial Gaussian fields and the threshold endpoint	9
D Scale-path stability, record layers, and Gaussian geometry	14
E Direct spatial reconstruction from fixed partitions	18
F Signed Poisson targets and joint spatial limits	22
G Direct signed spatial reconstruction without microscopic coupling	26

Proof routes and dependencies

For the arithmetic application, read Input 2.1 and Corollary 2.2 in the article first. The input is a theorem of [5, Theorem 7.7], not an unproved hypothesis about the multiplicative model. It compares the complete retained array with its exact Poisson target, before any passage to a diffuse or Gaussian limit.

For records, use Appendix B and the target calculations in the article. The shell geometry in that appendix resolves the order of arrivals and the initial holder; under the complete-field input it adds no cost to the comparison of shell counts. Appendix D treats the distinct birth and death transitions of scale paths. For Gaussian fields use Appendix C, then Appendix F for the signed and joint refinements. Every moment estimate on this route is an estimate under the Poisson target law.

Appendix A proves the direct partition comparison, including its range with merely $\Lambda \geq 1$. Appendices E and G reconstruct spatial laws from fixed-cell comparisons and modified-field quadratic bounds. They are alternative sufficient proofs. None is needed to prove the microscopic input. The scalar tail used in the arithmetic work has its independent proof in [5, Theorem 7.4]; the Palm deficit is a consequence of its microscopic theorem, not an input to it or to the process limits here.

A Direct partition comparison from arithmetic estimates

This appendix proves an alternative signed partition comparison from the one-window and two-window estimates of *Long runs and rare patterns*. It does not use the microscopic input and is not needed for the principal route at diverging intensity. Its direct theorem also retains the range $\Lambda \geq 1$. The proof uses actual signed events, rather than an independent colouring of an unsigned comparison.

Let f have the completely multiplicative sign model of [5]. Set

$$b_M = \lfloor \log_2 M \rfloor, \quad L = L_M = b_M - a_M, \quad n = M - L, \quad p = 2^{-L}, \quad \Lambda = np, \quad \lambda = Mp.$$

Let $\mathcal{A}_M \in \mathcal{F}_{Y_M^\#}$ have positive probability, and put $I_M = -\log \mathbb{P}(\mathcal{A}_M)$. We import the exact hard cutoff scales from [5], Proposition 4.2: writing $H = \log M$, $u(\nu)$ for the larger solution of $e^u/u = \nu$ and $D(\nu) = e^{u(\nu)} - \text{Ei}(u(\nu))$, they satisfy

$$V_M = D(H/V_M), \quad \nu_M = H/V_M, \quad Y_M^\# = \lfloor e^{V_M} \rfloor.$$

In particular $V_M \asymp \sqrt{H \log H}$, $V_M = o(H)$, $\nu_M \rightarrow \infty$, and $\log H = o(\nu_M)$.

The base-contained starts are $x = j + 1$, $1 \leq j \leq n$. For $e \geq 0$ and $\sigma \in \{-1, +1\}$, let

$$I_{j,e,\sigma} = \mathbf{1}_{\{j+1 \text{ is a run start, } \ell_{j+1}=L+e, f(j+1)=\sigma\}}, \quad \gamma_{e,\sigma} = 2^{-e-2}.$$

The sign label is part of the arithmetic data. On a good maximal support its conditional marginal is $p\gamma_{e,\sigma}$.

Choose $1 \leq m_0 < n$, $J \geq 1$, and set

$$T = \log_2(n/m_0), \quad h = T/J \leq 1, \quad m_r = \lfloor m_0 2^{rh} \rfloor \quad (0 \leq r < J), \quad m_J = n.$$

The initial cell is $A_0 = \{1, \dots, m_0\}$, and $A_r = \{m_{r-1} + 1, \dots, m_r\}$ for $1 \leq r \leq J$. Let $R = J + 1$, $w_r = |A_r|/n$, and $w_* = \min_r w_r$. If $m_0 h \rightarrow \infty$, then

$$R = 1 + T/h, \quad w_* \gg h 2^{-T}, \quad \eta_M := \max_{r \geq 1} \log_2(m_r/m_{r-1}) \leq h + C/m_0. \quad (\text{A.1})$$

The initial cell has no within-window birth times to resolve.

A.1 The source interface, with its ranges

The following register identifies the arithmetic inputs in the article and technical companion of [5]. All row lengths below belong to a fixed positive logarithmic band. The retained starts satisfy $x \geq \lceil M^{1/2} \rceil$. A maximal support has $Q + 1$ vertices, from $x - 1$ to $x + Q - 1$.

Input used	Location in the arithmetic pair
Weighted one-window defects and masked first moment	Article Proposition 2.4 and Corollary 2.5
Full-value two-window profile for signed events	Article Proposition 3.24, from Theorem 3.1
Hard cutoff, bad supports and maximum degree	Article Proposition 4.2; companion B.1, (B.3)
Local signed pairs and maximal-support extension	Article Theorem 5.8; companion C.2
Directional Poisson second difference	Companion C.3, (C.3); Barbour, p. 179, Lemmas 2–3
Deep starts and shifted scalar tails	Article Proposition 7.3 and Theorem 7.4
Macroscopic containment and exact target mean	Article Theorem 7.6; companion C.6

An integer is Y -defective when all its prime valuations above Y are even. Let $\mathcal{D}_Y(Q)$ be the starts whose maximal support contains such an integer. On the complement, a prime above $Y > 2(Q + 1)$ supplies a private odd coordinate at each vertex of a local union. Conditional on $\mathcal{F}_Y$, exact prescribed signed words consequently have their fair-sign probabilities. Two maximal supports are joined in the deterministic dependency graph if they use a common prime above Y . Nonadjacent families depend on disjoint remaining prime signs, so this is an exact conditional graph.

The weighted one-window quantity $\mathcal{M}_{L+1}$ is the sum of the excess rank bounds $2^{m_{L+1}(x)} - 1$, where $m_{L+1}(x)$ counts defective vertices at the base defect threshold. Its bound is $M^{1/2+o(1)}$. The full-value relation sum is

$$R_{2,U}^{\text{val}}(M, Q + 1) = \sum_{x,y \in U, |x-y| > Q} (2^{\rho_{Q+1}^{\text{val}}(x,y)} - 1),$$

where ρ_{Q+1}^{val} is the nullity of the two stacked valuation-vector systems on their $Q + 1$ vertices, without the block-parity restrictions of an unsigned relative-sign system. The uniform envelope is (A.2); it is not replaced by $M^{5/3+o(1)}$ until the internal cutoff makes $Q = \log_2 M + O(V_M)$.

The macroscopic comparison in the source article, Theorem 7.6, has an unsigned field. The signed growing-shell extension proved here uses the signed finite comparison and the full-value profile, together with the same deep-start and tail arguments. The source theorem is not cited as already stating this extension. The complete microscopic comparison gives it in the growing-intensity range but is not used below. The proof below imports the directional factor, not the source's already aggregated conclusion: it establishes its own finite comparison with the full target means after deletion. The normalization by independent Poisson filling is also explicit in [5, technical companion, Theorem C.1]; below it is applied separately in each spatial category.

A.2 Finite comparison and the sign-specific inputs

Fix $E \geq 0$ and write $Q = L + E + 1$. Assume all lengths are in one fixed positive logarithmic band and $Y > 2(Q + 1)$. Let $C_{r,e,\sigma} = \sum_{j \in A_r} I_{j,e,\sigma}$, with target means $|A_r|p\gamma_{e,\sigma}$. Denote the

deleted bad maximal supports above $\lceil \sqrt{M} \rceil$ by $\mathcal{D}_Y(Q)$, their base weighted-defect mass by $\mathcal{M}_{L+1}$, and the large-prime graph's maximum degree by $\Delta_Y(Q)$.

The sign-specific arithmetic profile used here is

$$R_{2,U}^{\text{val}}(M, Q+1) \ll_{\varepsilon} M^{\varepsilon} \left(M^{5/3} + M^{2/3} 2^{Q+1} \right). \quad (\text{A.2})$$

It is Proposition 3.24 of [5], restricted to the retained macroscopic interval, not the unsigned profile applied without checking the extra value constraint. Its host term allows arbitrary nonempty square-product subsets of the full vertex occurrences, without either block-parity restriction.

Lemma A.1 (The directional factor in category coordinates). *For a finite product-Poisson target with total mean λ' and strictly positive proportions μ_i , the immigration–death Stein solution for an indicator test satisfies*

$$|\Delta_{ij}g(z)| \leq \min \left\{ 1, \frac{C(1 + \log^+(2\lambda'))}{\lambda' \sqrt{\mu_i \mu_j}} \right\}.$$

Proof. Use [5, technical companion, Section C.3, (C.3)], derived from Barbour's Lemmas 2–3 [2, p. 179] for the same immigration–death solution. To recall the normalization, set $\lambda_i = \lambda' \mu_i$, $H = (\Delta_{ij}g)$ and $D = \text{diag}(\sqrt{\lambda_i})$. The weighted quadratic bound gives $\|DHD\|_{2 \rightarrow 2} \leq \{1 + 2 \log^+(2\lambda')\}/2$; bounding an entry gives the second estimate. The bound by one is entrywise, not an unweighted operator-norm bound. Taking the minimum proves the assertion. For a single category, split the mean into two positive means and compose the test with their sum. The signed applications below already have at least two categories. The discussion in [6] concerns the logarithmic factor. $\square$

Proposition A.2 (Signed finite partition comparison). *For any deterministic partition into R nonempty cells, with $w_r = |A_r|/n$ and $w_* = \min_r w_r$, the mean conditional total-variation distance of the finite signed partition vector to its target is bounded, up to fixed-band constants, by*

$$p\mathcal{M}_{L+1} + p|\mathcal{D}_Y(Q)| + \Lambda M^{-1/2} + e^{-c_0 \log M / \log \log M} + \frac{\ell(\Lambda)p^2}{\Lambda} \left\{ \frac{nQ}{w_*} + nR\Delta_Y(Q) + \frac{R_{2,U}^{\text{val}}(M, Q+1)}{w_*} \right\}, \quad \ell(\Lambda) = 1 + \log^+(2\Lambda), \quad (\text{A.3})$$

provided $\Lambda \geq 1$.

Proof. The target's total truncated intensity is ΛS_E , with $S_E = \sum_{e \leq E, \sigma} \gamma_{e, \sigma} = 1 - 2^{-E-1} \in [1/2, 1)$. Its category proportions are $\mu_{r, e, \sigma} = w_r \gamma_{e, \sigma} / S_E$. Lemma A.1, equivalently (C.3) of [5, technical companion], therefore gives

$$\|\Delta_{(r, e, \sigma), (t, f, \tau)} g\|_{\infty} \ll \frac{\ell(\Lambda)}{\Lambda \sqrt{w_r w_t \gamma_{e, \sigma} \gamma_{f, \tau}}}. \quad (\text{A.4})$$

The factors S_E cancel in the denominator. The square-root sum is

$$\sum_{e \geq 0, \sigma} \sqrt{\gamma_{e, \sigma}} = \sqrt{2}(1 + \sqrt{2}).$$

Delete actual bad-site marks and fill every deleted target index with an independent Poisson variable of its full target mean, then aggregate. The deleted actual marks, summed over lengths and signs, are bounded by the base event $J_{x, L}$. Their cost is $O(p\mathcal{M}_{L+1} + p|\mathcal{D}_Y(Q)|)$;

deep sites cost the two remaining terms on the first line, by Proposition 7.3 of [5]. The independent filling has no larger total target cost. In the multivariate Stein identity its contribution cancels exactly by Poisson integration by parts. Hence the full target proportions in (A.4) remain legitimate even when most of a small category was deleted. More explicitly, let Z_i be the independent filling with mean ζ_i , and write the category total as $W_i = Y_i + Z_i$. Conditional on the good indicators,

$$\mathbb{E}[Z_i \Delta_i g(W - e_i)] = \zeta_i \mathbb{E}[\Delta_i g(W)].$$

In the Stein generator $\sum_i \{\lambda_i \Delta_i g(W) - W_i \Delta_i g(W - e_i)\}$, these terms cancel the ζ_i part of λ_i . Only the good indicators remain to be telescoped, while the Stein solution still has the full deterministic target parameters.

At a good local pair put $a = L + e$ and $d = y - x > 0$. The signed joint mass divided by $p^2 \gamma_{e,\sigma} \gamma_{f,\tau}$ is respectively

$$0 \ (d < a), \quad 4\mathbf{1}_{\{\tau=-\sigma\}} \ (d = a), \quad 2\mathbf{1}_{\{\tau=\sigma\}} \ (d = a + 1), \quad 1 \ (a + 1 < d \leq Q).$$

These are the raw-value union ranks in companion C.2. In particular the local constant must be four, not the unsigned constant two. Same-site and local terms number $O(nQ)$ and cost at most their unweighted mass times w_*^{-1} .

For separated starts, the averaged signed joint probability is at most $p^2 \gamma_{e,\sigma} \gamma_{f,\tau} 2^{\rho_{Q+1}^{\text{val}}(x,y)}$ by the zero extension in C.2. Average over the prime environment before using this bound. Split its relation factor into one plus its excess. The excess costs $w_*^{-1} R_{2,U}^{\text{val}}(M, Q + 1)$ after the type sum. The graph part uses the deterministic inequality

$$\sum_{x \sim y} \frac{1}{\sqrt{w(x)w(y)}} \leq \sum_x \frac{\deg(x)}{w(x)} \leq nR\Delta_Y(Q),$$

where edges are ordered and the graph is undirected. Marginal products obey the same bound. Insert (A.4), sum the finite square-root type weights, and include the deletion costs. This proves (A.3). $\square$

A.3 Specialization to logarithmic shells

We now prove Theorem 2.3. Its complete assumptions are those of Section 2. In the notation above, $R = 1 + T/h$, $w_*^{-1} \ll 2^T/h$ and the exact target intensity is $\Lambda = (M - L)2^{-L}$.

Proof. Use the internal cutoff

$$E_M^* = \left\lceil \frac{I_M + V_M + \log(2 + \lambda_M)}{\beta} \right\rceil. \quad (\text{A.5})$$

The budget implies $I_M, \log \Lambda_M, a_M, E_M^* = O(V_M)$, hence $Q = \log_2 M + O(V_M)$. Only now may (A.2) be specialized to $M^{5/3+o(1)}$. At $Y = Y_M^\sharp$ the companion gives

$$\mathcal{M}_{L+1} \leq M^{1/2+o(1)}, \quad |\mathcal{D}_Y(Q)|/M \leq e^{-V_M+o(\nu_M)}, \quad \Delta_Y(Q)/M \leq e^{-V_M+o(\nu_M)} + M^{-1+o(1)}.$$

For the last inequality a maximal support uses $O(Q \log M/V_M)$ large primes, each meeting $O(Q(M/Y + 1))$ other supports, as in companion (B.3).

Conditioning the averaged finite ledger on $\mathcal{A}_M$ costs e^{I_M} , by [5, Lemma 6.1]. Its degree term is

$$O\left(e^{I_M} \ell(\Lambda_M) \Lambda_M R [e^{-V_M+o(\nu_M)} + M^{-1+o(1)}]\right).$$

The logarithms of R and $\ell(\Lambda_M)$ are $o(\nu_M)$, so the sharp part is $O(e^{-c'\nu_M})$. The bad-site deletion has the same margin. The relation and local terms are at most

$$e^{I_M} \ell(\Lambda_M) \Lambda_M \frac{2^{T_M}}{h_M} M^{-1/3+o(1)} = M^{-1/3+o(1)}.$$

Here every extra logarithm is $O(V_M) + o(\nu_M) = o(\log M)$. Start with a smaller exponent in (A.2) to obtain the stated ε . The deep-start term times e^{I_M} is smaller than every fixed $e^{-c'\nu_M}$, since $\log M / \log \log M \gg V_M$.

An exact mark beyond E_M^* is equivalent to an unsigned threshold hit at $L' = L + E_M^* + 1$. The global scalar bound, Theorem 7.4 of [5], gives the unconditional union bound $O(\lambda' + e^{-c_0 \log M / \log \log M})$, with $\lambda' = M2^{-L'}$. The cutoff makes $e^{I_M} \lambda' \leq Ce^{-V_M}$. The sign sum changes neither tail event nor target tail mass. Couple both tails to zero and combine with the finite result. Finally $\lambda_M / \Lambda_M = M / (M - L_M) = 1 + o(1)$, which is absorbed by the strict budget margin. $\square$

B A common coupling for records and observations

The principal arithmetic route obtains the signed shell-count coupling from Input 2.1 by the aggregation map in Corollary 2.2. Thus its error δ_M can be bounded by Δ_M^{mic} independently of the mesh. The construction below then deals only with order, the missing initial-holder information in the aggregated field and recognition delays. The direct proof of Theorem 2.3 in Appendix A is an alternative source of the same coupling.

The initial shell contains all past record contenders, but does not retain their order. This is harmless for record counts and requires a separate argument for the initial holder's sign.

For $s \in [0, T_M]$ put $j(s) = \lfloor m_0 2^s \rfloor$, with $j(T_M) = n$. Among all interior run starts $2 \leq x \leq j(s)+1$, let the earliest run of largest *full rightward integer length* hold the record. Let $K_M^\pm(s)$ count new strict records of each sign with $m_0 < x-1 \leq j(s)$, put $D_M = K_M^+ - K_M^-$, and let $A_M^{\text{lead}}(s)$ integrate the corresponding holder sign over logarithmic time. To avoid confusion, the conditioning event is always denoted $\mathcal{A}_M$, not A_M^{lead} .

Under a non-vacancy event, lengths below L_M are irrelevant. Its failure probability under the coupled target is

$$\exp(-\Lambda_M m_0 / n) = \exp(-\Lambda_M 2^{-T_M}).$$

Given the signed shell counts, give every atom a synthetic position uniform in its exact continuous shell $(m_{r-1}/n, m_r/n]$, keeping both its integer length and sign. For the independent Poisson target the resulting cloud is exactly

$$du \sum_{q \geq L_M} \sum_{\sigma} n 2^{-q-2} \delta_{(q, \sigma)}. \quad (\text{B.1})$$

The kernel preserves total variation. A disintegration of the coupling keeps each arithmetic atom at its true site as well. In an active shell its true and synthetic logarithmic birth times differ by at most η_M in (A.1). No artificial sign or fractional length is introduced.

Lemma B.1 (Signed shell-order ambiguity). *In the complete lattice target, the probability that some active shell contains two or more arrivals strictly above its initial integer maximum is $O(T_M h_M)$. Outside this event, every internal ordering of each shell produces the same strict-record heights and signs. The signed counters agree at all shell boundaries and differ in each coordinate by at most one at every intermediate time.*

Proof. At an edge u with maximum $Q_\theta(u)$, set $Z = u 2^{\theta - Q_\theta(u) - 1}$. A planar minimum representation gives $Z \leq uV(u)$ with $uV(u) \sim \text{Exp}(1)$, so $\mathbb{E}Z^2 \leq 2$. The count above the old

maximum in the next shell $(u, v]$ is conditionally Poisson of mean $(v/u - 1)Z$. Its probability of being at least two is at most $(v/u - 1)^2$. Equation (A.1) and $m_0 h_M \rightarrow \infty$ make the sum $O(Jh_M^2) = O(T_M h_M)$. With at most one strictly higher candidate, equal or lower arrivals cannot change the first holder and cannot affect that candidate's record sign. Induct over shells. $\square$

Lemma B.2 (Initial ties and occupation error). *On the successful signed-count coupling, the true and synthetic initial holders can have different signs if their common maximum length has a tie. Let τ_1 be the target waiting time to its first strict update after the baseline. Then, outside the ambiguity event, the occupation paths can be coupled with*

$$\sup_{s \leq T_M} |A_M^{\text{lead}}(s) - A_{\vartheta_M}(s)| \leq 2(\tau_1 \wedge T_M) + 2\eta_M \{N_{\vartheta_M}(T_M) + 1\}, \quad (\text{B.2})$$

where $\vartheta_M = \{\log_2 m_0\}$. Uniformly in phase, $\mathbb{P}(\tau_1 > h) \leq C2^{-h}$.

Proof. The common shell vector determines the same initial maximum, but not its earliest sign. We do not pretend to reconstruct that sign. Until the first strict update the occupation difference is at most twice the elapsed time. After that update, Lemma B.1 identifies every new record sign. Paired change times are in the same shell and differ by at most η_M . The set of disagreement after synchronization has total length at most η_M times the number of subsequent transitions, with one additional interval for the first synchronization. This gives (B.2). The persistence bound is Proposition 3.4. $\square$

Theorem B.3 (Arithmetic signed records and leadership times). *Assume Theorem 2.3 and*

$$T_M \rightarrow \infty, \quad \Lambda_M 2^{-T_M} \rightarrow \infty, \quad T_M h_M \rightarrow 0. \quad (\text{B.3})$$

Under the conditional law given $\mathcal{A}_M$,

$$\left(\frac{K_M^+(T_M t) + K_M^-(T_M t) - T_M t/2}{\sqrt{T_M}}, \quad \frac{D_M(T_M t)}{\sqrt{T_M}}, \quad \frac{A_M^{\text{lead}}(T_M t)}{\sqrt{T_M}} \right)_{0 \leq t \leq 1} \implies \mathbb{B}_\Sigma. \quad (\text{B.4})$$

No phase subsequence is required. The signed counter coupling fails with probability at most

$$\delta_M + e^{-\Lambda_M 2^{-T_M}} + C T_M h_M. \quad (\text{B.5})$$

Proof. Couple the complete signed shell counts with error δ_M , then apply the common synthetic-position kernel. If there is a retained initial atom, deleting all lower marks changes no subsequent record. After scaling positions by n/m_0 and subtracting an integer from the length coordinate, the synthetic target has phase ϑ_M in (3.1).

Lemma B.1 gives equality of terminal signed counts and uniform counter discrepancy at most one per sign, outside (2.13). For occupation use Lemma B.2. Its first term is $O_{\mathbb{P}}(1)$ uniformly in phase, and $N_{\vartheta_M}(T_M) = O_{\mathbb{P}}(T_M)$. Thus the occupation discrepancy divided by $\sqrt{T_M}$ tends to zero: $\sqrt{T_M} \eta_M \rightarrow 0$ follows from $T_M h_M \rightarrow 0$, $m_0 h_M \rightarrow \infty$. Theorem 3.7 and Slutsky's theorem complete the proof. $\square$

A convenient mesh is obtained by setting $\tilde{h}_M = \nu_M^{-1/4}/(1 + T_M)^2$ and $J_M = \lceil T_M / \tilde{h}_M \rceil$, then taking the effective $h_M = T_M / J_M$. The logarithmic horizon may satisfy

$$\log \Lambda_M = \beta T_M + r_M + O(1), \quad r_M \rightarrow \infty, \quad I_M + \beta T_M + r_M \leq V_M - c\nu_M.$$

The theorem proves a conditional limit law with deterministic target normalizations. It does not assert convergence of actual conditional means and variances without uniform integrability.

B.1 Observable episodes and leadership

For any deterministic one-sided word with finite maximal run lengths q_i , starts a_i and signs σ_i , put $H_{i-1} = \max_{j < i} q_j$, with $H_0 = 0$. A strict record run $q_i > H_{i-1}$ starts at a_i and is first recognized from the observed word at

$$c_i = a_i + H_{i-1} \leq a_i + q_i - 1.$$

It is counted once at this recognition time. A later equal-length run does not replace the holder.

Lemma B.4 (Signed pending-episode and occupation bounds). *Let $B_\sigma(m)$ count record runs of sign σ which have started by m , and let $E^\sigma(m)$ be the global recognition count defined in the article. Then*

$$0 \leq B_\sigma(m) - E^\sigma(m) \leq 1, \quad \sum_{\sigma \in \{-1, +1\}} (B_\sigma(m) - E^\sigma(m)) \leq 1.$$

Consequently each signed increment over any observation interval changes by at most two when starts are replaced by recognition times.

Suppose every run relevant to the current record or a subsequent record has length at most Q_* . On prefixes $m(s) = \lfloor m_0 2^s \rfloor + 1$, the integral difference between the announced-holder sign and the observable-holder sign obeys

$$\sup_{u \leq T} \left| \int_0^u (Z^{\text{ann}}(s) - Z^{\text{obs}}(s)) ds \right| \leq C \frac{Q_*}{m_0} \{K(T) + 1\}, \quad (\text{B.6})$$

where $K(T)$ is the number of record episodes started in the window. The extra one covers a record pending at the initial prefix.

Proof. Every announced record is recognized before the next run starts. Thus at most the currently active run is pending, and it has one sign. This proves the count bounds. The two holder signs can disagree only between the announcement and recognition of a record. Each such physical interval has length at most Q_* , and its logarithmic duration inside the window is at most CQ_*/m_0 by monotonicity of $\log$. The integrand has magnitude at most two. Sum these disjoint or overcounted disagreement intervals to obtain (B.6). $\square$

Corollary B.5 (Joint law for observables of the finite word). *Under Theorem B.3, replace each signed counter by the increment $\Delta E_M^\sigma(s) = E^\sigma(m(s)) - E^\sigma(m_0 + 1)$ of recognized global record episodes and replace the holder in the occupation integral by the earliest holder of the longest run actually visible in each finite prefix. Then (B.4) remains valid, with terminal physical prefix $n_M + 1 = M - L_M + 1 \sim M$.*

Proof. The special run from 1 has length at least L_M with exact probability $2^{-\pi(L_M)}$ by [5, (7.1)]. Its conditional probability is at most $e^{I_M} 2^{-\pi(L_M)} = o(1)$, since $L_M \sim \log_2 M$, $\pi(L_M) \asymp \log M / \log \log M$ and $I_M = O(V_M)$. On its complement, initial non-vacancy at length L_M makes the border irrelevant to subsequent announced records.

The tail argument using (A.5) also shows that, outside an event of probability $o(1)$, every retained full run length is at most $Q_* = L_M + E_M^* = O(\log M)$. Every run relevant to the interior baseline or to a later record is retained. The border is shorter than L_M on the preceding good event. The signed episode discrepancies in Lemma B.4 vanish after division by $\sqrt{T_M}$. Its occupation bound does also, because $K(T_M) = O_{\mathbb{P}}(T_M)$ and

$$\frac{Q_* \sqrt{T_M}}{m_0} \rightarrow 0, \quad m_0 = n_M 2^{-T_M} = M^{1-o(1)}.$$

Therefore all three observable coordinates have the same limit. The statement concerns record episodes, not the individual unit increases of the visible maximum. No J_1 convergence of the censored maximum itself is used. $\square$

C Spatial Gaussian fields and the threshold endpoint

This appendix retains the functional threshold results in a form that can be used with either a dyadic or a macroscopic lattice comparison. Its input is a complete exact-level field, including positions when spatial conclusions are wanted. The conclusions in this appendix use that complete-field input. Appendix E treats the narrower run-start spatial limit by fixed partitions and a separate quadratic estimate.

C.1 The field input and the state spaces

Let $\mu_n = n^{-1} \sum_{j=1}^n \delta_{j/n}$, let $\gamma_e = 2^{-e-1}$, and let $\Pi_{n,e}$, $e \geq 0$, be independent Poisson measures on $[0, 1]$ with intensities $\Lambda \gamma_e \mu_n$. Suppose $\mathcal{E}_n = (\mathcal{E}_{n,e})_{e \geq 0}$ is a point configuration on the same grid and levels with

$$\sum_{e \geq 0} \mathcal{E}_{n,e}([0, 1]) < \infty \quad \text{almost surely,}$$

and its law is within total variation $\varepsilon_n \rightarrow 0$ of this target. The finite-configuration space carries the sigma-field generated by its coordinate counts, as in Section 2.3. Finiteness at each level separately would not suffice. This is a domain assumption on the input, not a conclusion from a small total-variation error on a larger space. For a spatial mask, replace μ_n by its specified deterministic measure and assume the required weak convergence; a general mask has no universal Riemann-sum rate. We state the results for the uniform grid.

For the present binary model, Input 2.1 supplies the complete signed field under $I_M + \log \Lambda \leq V_M - c\nu_M$ and $\Lambda \rightarrow \infty$. The graph-based labelled estimates [5, Theorems 5.10, 7.6 and Corollary 5.11] remain sufficient alternatives, with their own cutoffs and budgets. Under an aggregate-only input in another application, the endpoint product is only with aggregate critical counts; a spatial endpoint field still requires a common spatial input. Every centering below is the deterministic *target* measure, not an unproved formula for the actual conditional expectation.

For signed input the exact-level target means are $\Lambda \gamma_{e,\sigma} \mu_n$, with $\gamma_{e,\sigma} = 2^{-e-2}$. The base normalization remains $\sqrt{\Lambda}$. The target survivor mean for one sign is $(\Lambda/2)2^{-t} \mu_n$, and its own standardization uses $\sqrt{(\Lambda/2)2^{-t}}$. These are target parameters under the signed complete-field hypothesis; they are not conditional expectations inferred for the original sequence.

Put $q_0(x) = 1$ and $q_k(x) = \sqrt{2} \cos(\pi kx)$ for $k \geq 1$. For a distribution F define the spectral negative-Sobolev norm

$$\|F\|_{-s}^2 = \sum_{k \geq 0} (1 + k^2)^{-s} |\langle F, q_k \rangle|^2. \quad (\text{C.1})$$

This fixes the boundary convention of $H^{-s}([0, 1])$ used here. Tensor norms use the product of the two weights. For $s > 1/2$, the norms of all δ_x are bounded by a finite constant $\kappa_s^{1/2}$. A spatial white noise is the Gaussian series $\sum_k g_k q_k$, with independent standard normal g_k , converging in every such H^{-s} . Tensor white noise is defined similarly. The maps between these separable distribution spaces are measurable. For broader white-noise regularity results see [7]; the cosine-space estimates used here are proved directly below.

The compensated-Poisson identities below use the standard Poisson exponential formula and isometry; see [4]. We repeatedly use two elementary criteria. If $\tilde{\Pi}_n$ is compensated

Poisson measure and

$$\int F_n^2 d\lambda_n \rightarrow v, \quad \int |F_n|^3 d\lambda_n \rightarrow 0, \quad (\text{C.2})$$

then $\int F_n d\tilde{\Pi}_n \Rightarrow \mathcal{N}(0, v)$. Indeed its characteristic exponent is $\int (e^{itF_n} - 1 - itF_n) d\lambda_n = -t^2 \int F_n^2/2 + O_t(\int |F_n|^3)$. Cramér–Wold gives the finite-dimensional version. Second, uniformly bounded second moments of all cosine coefficients give tightness in H^{-s} , $s > 1/2$: first sum the variances with an exponent $s_0 \in (1/2, s)$, then use the compact embedding $H^{-s_0} \hookrightarrow H^{-s}$. The same argument applies to tensor spaces, with an intermediate exponent in each coordinate. We use finite-test convergence to identify the unique limit. These criteria give proofs for the spaces used here, without an unspecified distribution-valued central limit theorem.

Input 2.1 supplies the complete-field hypothesis at one-factor cost. Appendix F develops the signed joint target refinements. Appendix G retains the fixed-cell reconstruction alternative. Every route keeps the stated target spaces and parameter conditions.

C.2 Exact levels and the discrete threshold chain

Let

$$W_{n,e} = \frac{\Pi_{n,e} - \Lambda \gamma_e \mu_n}{\sqrt{\Lambda}}.$$

Proposition C.1 (Exact-level white noise). *As $n, \Lambda \rightarrow \infty$, $(W_{n,e})_{e \geq 0}$ converges to $(\sqrt{\gamma_e} \mathcal{W}_e)_{e \geq 0}$, with independent spatial white noises, in the space with squared norm*

$$\sum_{e \geq 0} (1 + e)^{2a} \|W_e\|_{-s}^2, \quad a \geq 0, \quad s > 1/2.$$

The same limit holds for $\mathcal{E}_n$ centered by the target means.

Proof. For finitely many levels and cosine coefficients, (C.2) gives independent Gaussian limits, since the third integrals are $O(\Lambda^{-1/2})$ and $\mu_n \Rightarrow dx$. For $a_0 > a$ and $1/2 < s_0 < s$,

$$\mathbb{E} \sum_e (1 + e)^{2a_0} \|W_{n,e}\|_{-s_0}^2 \leq C_{s_0} \sum_e (1 + e)^{2a_0} \gamma_e < \infty.$$

The stronger weighted space embeds compactly into the asserted space: truncate the level and Fourier indices, then use the strict inequalities on both weights. This proves tightness. A maximal coupling identifies the entire input fields with probability at least $1 - \varepsilon_n$, so it transfers the law, not these target moment estimates, to the arithmetic field. $\square$

Spatial integration and tail summation give the centered integer threshold sequence

$$G(j) = \sum_{e \geq j} \sqrt{\gamma_e} g_e, \quad \text{Cov}(G(j), G(k)) = 2^{-\max(j,k)}.$$

It belongs almost surely to every polynomially weighted ℓ^2 space. The standardized sequence $H(j) = 2^{j/2} G(j)$ has covariance $2^{-|j-k|/2}$. Its Gaussian Markov representation is $H(j+1) = 2^{-1/2} H(j) + 2^{-1/2} \zeta_{j+1}$, with independent standard normal innovations. This is a limiting representation, not an exact autoregression for a finite-intensity Poisson staircase.

C.3 Canonical marking, the critical field, and the whole path

Independently give every target atom a fractional mark with density (7.1). Its lifetime $\tau = e + U$ is exponential of rate β . The survivor measure

$$\Xi_\Lambda(t) = \sum_{\tau_i > t} \delta_{x_i}, \quad t \geq 0,$$

has deterministic mean $m(t)\mu_n$, where $m(t) = \Lambda 2^{-t}$. By the total-mass condition in Appendix C.1, it is a finite-jump path for every admissible input configuration, locally in time, and the evaluation maps at rational times show that the map from the marked configuration into the local Skorokhod space is measurable. Marking and this path map are Markov kernels. Therefore a *single* complete-field error ε_n bounds the total-variation distance of the entire survivor paths. There is no union bound over thresholds or time points.

The scalar semigroup is binomial thinning:

$$\Xi_\Lambda(t+h)([0,1]) \mid \Xi_\Lambda(t)([0,1]) = k \stackrel{d}{=} \text{Bin}(k, 2^{-h}).$$

Its generator is $\beta k[f(k-1) - f(k)]$. The spatial process independently kills every existing particle at rate β . At integer times its counts coincide exactly with the original threshold counts, since $e \geq j$ if and only if $e + U > j$.

For the critical lattice cloud of marks $r \in \mathbb{Z}$ and phase θ , the transformation $y = r - \theta + U$ gives intensity $dx \beta 2^{-y} dy$, independently of θ . Its first few upper order statistics are well-defined despite the infinite low-level mass. Under $v = 2^{-y}$ this is homogeneous intensity $dx dv$. The ordered vertical coordinates are $\Gamma_k = E_1 + \dots + E_k$ for independent $E_i \sim \text{Exp}(1)$. Hence the maximum has distribution

$$\mathbb{P}(Y_{[1]} \leq y) = e^{-2^{-y}}, \quad Y_{[k]} = -\log_2 \Gamma_k.$$

For each finite k , beta–gamma change of variables gives independent ratios $\Gamma_j/\Gamma_{j+1} \sim \text{Beta}(j, 1)$, $1 \leq j < k$, independently of Γ_k . Thus the adjacent extreme gaps are independent exponential variables of rates $j\beta$. This is a statement about continuously marked *heights*, not about intrinsic integer record times. Integer order statistics are restored exactly by flooring the uncentered fractional lengths. Grid-to-diffuse convergence and the two extremal tail bounds in Proposition 4.2 give the full-sequence phase-free critical limit after this declared interpolation.

C.4 Spatial Ornstein–Uhlenbeck paths

Define

$$H_\Lambda(t) = \frac{\Xi_\Lambda(t) - m(t)\mu_n}{\sqrt{m(t)}}.$$

We prove the fixed-window spatial claim in Theorem 7.5. For a finite test family, the death martingale gives

$$H_\Lambda(t) = H_\Lambda(0) - \frac{\beta}{2} \int_0^t H_\Lambda(s) ds + M_\Lambda(t), \quad (\text{C.3})$$

with

$$\langle M_\Lambda(\varphi), M_\Lambda(\psi) \rangle_t = \beta \int_0^t \frac{\Xi_\Lambda(s)(\varphi\psi)}{m(s)} ds.$$

The normalized particle measure $\Xi_\Lambda(s)/m(s)$ is a martingale on each bounded test. Its variance at T_0 is $\mu_n(\varphi^2)/m(T_0)$, so Doob's inequality gives uniform convergence of these brackets to $\beta t \int \varphi\psi$ as $\Lambda, n \rightarrow \infty$. All normalized jumps tend to zero.

Here are the Hilbert-space tightness details. Fix $1/2 < s_0 < s$. For $R_\Lambda(t) = \Xi_\Lambda(t)([0, 1])/m(t)$,

$$\mathbb{E} R_\Lambda(T_0)^2 = 1 + 1/m(T_0), \quad \mathbb{E} \sup_{t \leq T_0} R_\Lambda(t)^2 \leq 4(1 + 1/m(T_0)).$$

The H^{-s_0} martingale bracket has trace at most $\beta \kappa_{s_0} \int_0^t R_\Lambda(s) ds$. The initial second moment is bounded by κ_{s_0} . Doob's inequality and the linear equation (C.3) give

$$\sup_{\Lambda \geq 1} \mathbb{E} \sup_{t \leq T_0} \|H_\Lambda(t)\|_{-s_0}^2 < \infty, \quad (\text{C.4})$$

with a constant depending on T_0, s_0 . The balls in H^{-s_0} are relatively compact in H^{-s} , so this is compact containment. For a bounded stopping time τ and $0 \leq \delta \leq 1$, the mean square of the martingale increment in H^{-s} is at most $C\delta \mathbb{E} \sup_{t \leq T_0+1} R_\Lambda(t)$; the drift increment has mean square $O(\delta^2)$ by (C.4). This is the stopping-time tightness criterion [1]. The maximum jump is at most $\kappa_s^{1/2}/\sqrt{m(T_0)}$, and therefore the family is C -tight.

Identification can be done without assuming independence of the prelimit initial state and the death martingale. For any fixed times and finite test family, their joint vector is a compensated Poisson integral on the initial position–lifetime cloud. Its third integral is $O_{T_0}(\Lambda^{-1/2})$ and its covariance tends to

$$2^{-|t-u|/2} \int \varphi \psi.$$

Criterion (C.2) identifies the unique Gaussian finite-dimensional limit. It is the stationary H^{-s} -valued OU solution

$$dH_t = -\frac{\beta}{2} H_t dt + \sqrt{\beta} dB_t,$$

where the initial spatial white noise is independent of the cylindrical L^2 Brownian noise. The inclusion $L^2 \hookrightarrow H^{-s}$ is Hilbert–Schmidt for $s > 1/2$, so this realization is a genuine H^{-s} -valued process. Pairings are first defined on the positive-Sobolev dual; the L^2 covariance formulation is its isonormal extension, not a pathwise pairing with every L^2 function. Kernel contraction transfers the result to the original field. Finite-test martingale central limit theorems provide an alternative identification [8].

For the scalar scale process of mean c , write $dX_s = \beta(c - X_s) ds + dM_s$ and $d\langle M \rangle_s = \beta(c + X_s) ds$. After division by $\sqrt{c}$, stationarity, Doob's inequality and the linear equation give a bounded second moment of the supremum on each fixed interval. Thus $\sup |X_s/c - 1| \rightarrow 0$ in probability. The bracket tends uniformly to $2\beta s$, jumps have size $c^{-1/2}$, and the same tightness and finite-test identification give the second OU limit in (7.7). No spatial trace is used.

C.5 Compression over the full threshold interval

Write $\Lambda = 2^{J+\theta}$, $J \in \mathbb{N}$, $\theta \in [0, 1)$, and let $J \rightarrow \infty$. For bounded spatial and temporal tests φ, ψ , the discrete pairing is

$$\frac{1}{\sqrt{J}} \sum_{j=0}^J \langle H_\Lambda(j), \varphi \rangle \psi(j/J).$$

An atom at exact excess e contributes the compensated integrand

$$F_{J,e}(x) = \frac{\varphi(x)}{\sqrt{J\Lambda}} \sum_{j=0}^{\min(e,J)} 2^{j/2} \psi(j/J). \quad (\text{C.5})$$

The third integral, including all $e > J$, satisfies

$$\begin{aligned} \sum_{e \geq 0} \Lambda 2^{-e-1} \int |F_{J,e}|^3 d\mu_n &\leq \frac{C \|\varphi\|_\infty^3 \|\psi\|_\infty^3}{J^{3/2} \sqrt{\Lambda}} \left(\sum_{e=0}^J 2^{e/2} + 2^{3J/2} \sum_{e>J} 2^{-e} \right) \\ &\leq C \|\varphi\|_\infty^3 \|\psi\|_\infty^3 J^{-3/2}. \end{aligned} \quad (\text{C.6})$$

This is the smallness needed for a Gaussian pairing even though the terminal count is not large.

The covariance of two discrete pairings is

$$\mu_n(\varphi \tilde{\varphi}) \frac{1}{J} \sum_{j,k=0}^J 2^{-|j-k|/2} \psi(j/J) \tilde{\psi}(k/J).$$

For continuous tests, first truncate $|j - k|$ and take the Riemann limit, then use summability of the geometric tail. The result is

$$(3 + 2\sqrt{2}) \int_0^1 \varphi \tilde{\varphi} dx \int_0^1 \psi \tilde{\psi} du.$$

Equations (C.2) and (C.6) give Gaussian finite-test convergence. Every tensor cosine coefficient has bounded variance, uniformly in J, n, θ , since the covariance kernel is summable and the basis functions are bounded. The Fourier compactness criterion proves tightness in every stated tensor negative-Sobolev space. This proves the first line of (7.11).

For continuous compression, the lifetime τ is exponential of rate β and the integrand is

$$F_{J,\tau}(x) = \frac{\varphi(x)}{\sqrt{J\Lambda}} \int_0^{\min(\tau,J)} e^{\beta t/2} \psi(t/J) dt. \quad (\text{C.7})$$

Its absolute third integral against $\Lambda \mu_n(dx) \beta e^{-\beta \tau} d\tau$ is again $O(J^{-3/2})$: integrate separately $\tau \leq J$ and $\tau > J$, using $|F_{J,\tau}| \leq C(J\Lambda)^{-1/2} e^{\beta \min(\tau,J)/2}$ times the two test suprema. The covariance is

$$\frac{\mu_n(\varphi \tilde{\varphi})}{J} \int_0^J \int_0^J e^{-\beta |t-u|/2} \psi(t/J) \tilde{\psi}(u/J) dt du,$$

which tends to $(4/\beta) \int \varphi \tilde{\varphi} \int \psi \tilde{\psi}$. Uniform cosine variances and compactness give the second line of (7.11). The proof treats both endpoints of the interval; it does not discard the critical layer before proving tightness.

For completeness, a terminal strip of $K_J + 1 = o(J)$ discrete sample indices has expected squared $H^{-s_x, -s_u}$ norm $O((K_J + 1)/J)$, by the same summable covariance bound. A continuous strip of length $K_J = o(J)$ has expected squared norm $O(K_J/J)$. Thus such strips vanish as random distributions, even though a fixed number of their individual counts remains non-Gaussian.

C.6 Independence of the critical Poisson layer

For fixed $R > 0$, remove from a compressed Gaussian pairing all target atoms at levels $e - J \geq -R$, *together with their compensator*. The remaining field is independent of that upper critical field. The number of removed atoms is tight, with mean at most 2^{R+1} , uniformly in phase. By (C.5), the norm of one removed atom's contribution to the compressed distribution is at most $C_R J^{-1/2}$; the geometric sum gives the same bound in every tensor space with exponents greater than $1/2$. Its compensator has the same order. Formula (C.7) gives the continuous counterpart. Therefore the deletion changes the Gaussian random distribution by $o_{\mathbb{P}}(1)$.

The critical field, on each fixed mark set, converges to the phase-indexed Poisson measure with means $2^{\theta-r-1} dx$. Signs can be retained by using the corresponding split means. The deletion proves independence of the Gaussian limit from every upper restriction $r \geq -R$. These restrictions determine the vague law on $[0, 1] \times \mathbb{Z}$, and tightness of both marginals gives the joint product theorem by exhaustion. After canonical marking the critical limit is phase-free, but its independence was already established before that extra kernel. The argument does not claim total-variation deletion of the entire infinite negative-level field.

This completes the proof of Theorem 7.5. Its product theorem concerns threshold-compressed Gaussian fluctuations. The Brownian record-history product in Theorem 5.2 instead removes a *recent spatial strip*. The two independence arguments use different decompositions of the same underlying Poisson cloud.

D Scale-path stability, record layers, and Gaussian geometry

The shell comparison does not reconstruct sites. Its process consequences use stability under a small perturbation of birth times. This appendix makes that step explicit, including simultaneous transitions and endpoint events.

D.1 Grouped transitions for a finite threshold family

Fix distinct thresholds $y_1, \dots, y_d$; repeated thresholds may first be identified. For a point (u, v) let $a = \log_2 u$ and $d_i = -\log_2 v - y_i$. It contributes to coordinate i on $[a, d_i)$ when this interval is nonempty. In the true and synthetic configurations the d_i are identical, while active-shell birth times differ by at most η_M . Initial-shell points have already arrived in both configurations, so only their heights are needed at the left endpoint.

Lemma D.1 (Stability with event groups). *On a logarithmic interval $I = [s_0, s_0 + T]$, two such configurations with paired active-shell birth coordinates at distance at most η (including atoms inactive at the queried thresholds) have distance $d_{J_1, I}^{(0)}$ at most $C\eta$, with the uniform-displacement convention of Theorem 7.3, outside an event of target probability at most $C_{\mathbf{y}}(1+T)\eta$, provided their initial vertical configurations agree and η is sufficiently small. The exceptional event includes crossings of either endpoint. Births of the same particle in several coordinates are one event group, not a collision.*

Proof. Write $I = [s_0, s_0 + T]$ and let $I^{3\eta}$ denote its enlargement by 3η at both ends. We first work in the *full planar cloud*, not just its active immigration–death population. In coordinates (a, d_i) its intensity is

$$du dv = \beta^2 2^{-y_i} 2^{a-d_i} da dd_i.$$

If moving a to a' with $|a' - a| \leq \eta$ creates or removes a nonempty interval $[a, d_i)$, then $|a - d_i| \leq \eta$. If the change meets I , its birth is in $I^{3\eta}$. Exclude all atoms in the larger two-sided band

$$\mathcal{B}_i = \{a \in I^{3\eta}, |a - d_i| \leq 3\eta\}.$$

Its expected count is exactly

$$\mathbb{EP}(\mathcal{B}_i) = \beta 2^{-y_i} (T + 6\eta) (2^{3\eta} - 2^{-3\eta}) \leq C_{y_i} (T + 1) \eta \quad (0 < \eta \leq 1).$$

The union over the fixed threshold family has the same order. This covers both target-active intervals that disappear and target-inactive atoms that become active; the latter need not belong to the target's active population. It also excludes a birth and death of the same particle within 3η .

Off these bands, activity at every queried threshold is preserved. The remaining relevant particles are in the stationary immigration–death population at $y_{\min}$. Their births have finite constant rate and their initial count has finite mean. Each particle has one birth group and at most d death groups. Distinct deaths of the same particle are separated by a fixed nonzero $|y_i - y_j|$, hence by more than 3η for sufficiently small η .

For different particles, the Poisson second factorial-moment formula bounds the expected number of pairs of event groups within 3η in $I^{3\eta}$ by $C_{\mathbf{y}}(T+1)\eta$: the marginal birth and death intensities are bounded, and the two-time integration is over a strip of width 6η . Events within 3η of an endpoint have expected number $O_{\mathbf{y}}(\eta)$. Include these events in the exceptional set. No conditioning on band avoidance is used in these expectation bounds; a union bound combines the exclusions.

Outside the resulting exceptional event, the active coordinates in each birth group and the order of all distinct groups are unchanged. A single increasing piecewise-linear time change aligns paired birth groups, fixes the death groups and both endpoints, and has displacement at most $C\eta$. The vector values then agree. Simultaneous births of one particle in several coordinates are aligned as one group, not excluded as collisions. This proves the assertion. $\square$

For the arithmetic application, the initial state agrees on the shell coupling. The continuous target is initially truncated at $v < \Lambda$. All rectangles on $[-T, 0]$ and at the fixed thresholds lie below $C_{\mathbf{y}}2^T$, which is eventually less than Λ under (2.11). Thus the vertical extension introduces no relevant particle. The mesh assumptions give $T\eta_M \rightarrow 0$, and the shell failure probability is $\delta_M \rightarrow 0$. This proves Theorem 7.3. It is an approximation on a changing interval to a stationary target, not weak convergence on one fixed space containing all intervals.

D.2 Continuously ranked record layers

Let $V^{(k)}(u)$ be the k th smallest vertical coordinate in the full planar cloud before u . Poisson counting gives

$$uV^{(k)}(u) \sim \text{Gamma}(k, 1).$$

The stationary centered height $R_s^{(k)} = -\log_2(2^s V^{(k)}(2^s))$ has distribution

$$\mathbb{P}(R_s^{(k)} \leq r) = e^{-2^{-r}} \sum_{\ell=0}^{k-1} \frac{2^{-\ell r}}{\ell!}.$$

The record-layer theorem [3, Proposition 2] identifies the initial-rank- j time projections as independent Poisson processes of intensity du/u . One must not infer independence between the time and value projections of the entire record configuration; the theorem used here concerns independence *between rank layers* in one projection.

For completeness, the scalar k th-minimum dynamics has holding time E/x at current ordinary minimum x , and the new value is Bx with independent $B \sim \text{Beta}(k, 1)$. Its Markov property is with respect to the history of that order-statistic process. It follows from the conditional uniform-order-statistic property in [3, Lemma 4]; it is not a statement that the lower $k-1$ values remain unobserved after conditioning on the *whole* point configuration. In logarithmic centered coordinates the generator is

$$\mathcal{G}_k f(r) = -f'(r) + \beta 2^{-r} \int_0^\infty [f(r+z) - f(r)] k\beta e^{-k\beta z} dz.$$

At a fixed time the location of the first minimum is uniform on the available horizontal interval and independent of its value, by the product intensity. Its normalized location decays at speed $\dot{w} = -\beta w$ between records and resets to 1 at an update.

We now prove (7.29). At a shell edge u , the number of arrivals in $(u, v]$ below the current k th minimum is conditionally Poisson of mean $(v/u - 1)uV^{(k)}(u)$. Hence

$$\mathbb{P}(\text{at least two candidates in the shell}) \leq \frac{1}{2}(v/u - 1)^2 k(k+1).$$

Summing over the rounded shells gives $Ck(k+1)Th$. Outside this ambiguity event, every within-shell ordering produces the same first- k updates and rank labels: if there is only one candidate, every other arrival remains outside the top k after that update. The initial cloud has at least k retained atoms except with probability

$$p_k(\Lambda 2^{-T}) = e^{-\Lambda 2^{-T}} \sum_{\ell=0}^{k-1} \frac{(\Lambda 2^{-T})^\ell}{\ell!}.$$

On that event the omitted higher vertical part cannot affect any subsequent top- k record. Couple the shell vectors and their common fractional heights. The full target layer counts have independent means βT , so the stated total-variation bound follows.

Here the arithmetic counts are defined from the retained interior field with its canonical marks. Including the run beginning at 1 in a full-word version costs in addition the boundary error in (2.15). Neither interpretation is an intrinsic count of integer records. Paired logarithmic record times lie in the same shells; after time rescaling to $[0, 1]$, the discrepancies vanish. The functional central limit theorem for the independent Poisson layers therefore gives independent standard Brownian limits after centering each count by $\beta T t$ and scaling by $\sqrt{\beta T}$. The update count of the k th minimum has centering $k\beta T t$ and variance scale $k\beta T$.

D.3 The tensor field and a simultaneous grid estimate

We fill in the functional and arithmetic parts of Theorem 7.4. For a smooth test φ supported on a compact rectangle K ,

$$\langle H_J, \varphi \rangle = \int F_{J,\varphi}(u, v)(\mathcal{P} - du dv),$$

where

$$F_{J,\varphi}(u, v) = \int_K \varphi(a, b) 2^{-(J+a-b)/2} \mathbf{1}_{\{u \leq 2^a, v < 2^{J-b}\}} da db.$$

The sup norm is at most $C_K 2^{-J/2} \|\varphi\|_1$. The $L^2(du dv)$ norm squared is the double integral of the exact covariance kernel against $\varphi \otimes \varphi$, hence is bounded. The third integral is at most the sup norm times that squared L^2 norm, and tends to zero. Tensor cosine coefficients have uniformly bounded variances; use intermediate exponents in $(1/2, s_a)$ and $(1/2, s_b)$ for compactness. This proves the distribution-space convergence. The same Poisson criterion at finitely many rectangles proves the separately stated finite-dimensional convergence.

For an arithmetic application let $K \subset (-\infty, 0] \times \mathbb{R}$ be compact. Suppose the complete labelled comparison has error tending to zero, and choose $J = J_M \rightarrow \infty$ so that

$$2^J \max_{(a,b) \in K} 2^{-b} < \Lambda \quad \text{eventually,} \quad 2^J/n^2 \rightarrow 0. \quad (\text{D.1})$$

Couple the diffuse cloud to the gridded target by $u \mapsto \lceil nu \rceil/n$, keeping v . At a rectangle with right edge 2^a , the difference is exactly the count in a strip of width less than $1/n$ and height 2^{J-b} . Its mean d is at most $C_K 2^J/n$. Both standardized fields are centred by the same continuous rectangle mean, so

$$\mathbb{E}|H_J^{\text{grid}}(a, b) - H_J(a, b)|^2 \leq C_K \left(\frac{1}{n} + \frac{2^J}{n^2} \right). \quad (\text{D.2})$$

The second term is the squared deterministic mean discrepancy; it must not be discarded. Integrating over K gives the same order for the squared $L^2(K)$ distance, and therefore for every weaker negative-Sobolev norm. Equations (D.1)–(D.2) justify the *simultaneous* high-intensity and grid limit. In the arithmetic corridor $J = O(V_M) = o(\log M)$ and $n \sim M$, the second condition is automatic; a depth margin ensures the first. The complete-field coupling then transfers the target limit to the arithmetic planarized field.

This quantitative step is needed: an unqualified “first pass from the grid to the continuum, then let intensity grow” would be an iterated limit, not the asserted triangular-array limit. Nor does convergence in this Sobolev space by itself establish path convergence along a diagonal. Theorem 7.13 and Appendix G.7 give the separate one-factor signed route by fixed-cell reconstruction; they keep the squared grid bias and do not infer any diagonal trace.

D.4 Scale-time compression at fixed intensity

Fix $c = 2^{-y} > 0$. The immigration–death process has the stationary Poisson-interval representation

$$X_s(y) = \int \mathbf{1}_{\{a \leq s < a+\ell\}} \Pi(da, d\ell), \quad \mathbb{E}\Pi(da, d\ell) = c\beta da \beta e^{-\beta\ell} d\ell.$$

Its standardized covariance is $e^{-\beta|s-t|}$. For continuous compression against a bounded test φ , each interval contributes

$$\frac{1}{\sqrt{cT}} \int_0^T \mathbf{1}_{\{a \leq s < a+\ell\}} \varphi(s/T) ds.$$

Writing $L(a, \ell)$ for the intersection length with $[0, T]$, we have $\int_{\mathbb{R}} L(a, \ell)^3 da \leq \ell^2 \int L(a, \ell) da = T\ell^3$. Consequently its absolute third Poisson integral is $O_{c,\varphi}(T^{-1/2})$. The covariance tends to

$$\frac{2}{\beta} \int_0^1 \varphi(u)\psi(u) du.$$

The cosine variances are uniformly bounded, so the Poisson criterion and Fourier compactness give $\sqrt{2/\beta} \mathcal{W}$ in H^{-r} , $r > 1/2$.

For integer sampling, replace the intersection length by the number of sample times hit. This is bounded by $\ell + 1$, and integrating it over the birth coordinate counts at most $J + 1$ intervals of length ℓ . Its third Poisson integral is $O_c(J^{-1/2})$ because exponential lifetimes have a finite third moment. The Toeplitz covariance has sum $\sum_{k \in \mathbb{Z}} 2^{-|k|} = 3$. The same tightness argument proves the discrete limit in (7.27).

For the arithmetic observables $\mathcal{C}_M^{\text{sc}}$ and $\mathcal{D}_M^{\text{sc}}$ defined in Section 7.7, apply the stationary target argument on $[-T_M, 0]$ and on the integer grid $j - J_M^{\text{sc}}$, respectively. Translation gives exactly the target laws above, with continuous coordinate $(s+T_M)/T_M$ and discrete coordinate j/J_M^{sc} . No arithmetic path outside $[-T_M, 0]$ is used.

To transfer the continuous random distribution, first exclude the two-sided activation event in Lemma D.1; its probability is $O_{\mathbf{y}}((T+1)\eta_M) = o(1)$. The relevant active particles are then the same in both configurations. Pair their true and synthetic births. Each has displacement at most η_M , so the L^1 difference of the count paths is at most $C\eta_M$ times the number of relevant births, which is $O_{\mathbb{P}}(T)$ under the coupled target. For $r > 1/2$ the norm of any signed measure is bounded by a constant times its total variation. After normalization the error is $O_{\mathbb{P}}(\sqrt{T}\eta_M)$. For integer samples, a discrepancy requires a target birth within η_M of a sampling time; the expected number is $O(T\eta_M)$. The mesh in Section 2 makes both errors vanish. This proves the arithmetic versions without transferring any unbounded moments from total variation alone.

E Direct spatial reconstruction from fixed partitions

This is an alternative proof of Theorem 7.6 without a site-labelled total-variation input. The cell limits come from Proposition A.2; the missing spatial control comes from a direct second-moment calculation. All expectations and probabilities carrying a subscript $\mathcal{A}$ below are under $\mathbb{P}(\cdot | \mathcal{A}_M)$, enlarged by independent Poisson variables when required.

Throughout, use the hypotheses of Theorem 7.6. Put $H_M = \lceil \sqrt{M} \rceil$ and

$$U_M = \{x \in \mathbb{Z} : H_M \leq x \leq M - L + 1\}, \quad Y = Y_M^\#.$$

For the quadratic calculation we use *only the start event*: set $Q = L$, so its support is $\{x - 1, \dots, x + L - 1\}$, with $Q + 1$ vertices. Define the deterministic good indices

$$G_M = \{j : 1 \leq j \leq n, j + 1 \in U_M, \{j, \dots, j + L\} \text{ contains no } Y\text{-defective integer}\}.$$

Here defective means that every prime valuation above Y is even, as in Appendix A. The bad retained starts form $\mathcal{D}_Y(L)$. Let $\mathcal{M}_{L+1}$ denote the base weighted-defect sum on U_M and let $\Delta_Y(L)$ be the maximum degree of the large-prime graph on these base supports.

The one-factor budget implies $I_M, \log \Lambda, a_M = O(V_M)$ and hence $L = \log_2 M + O(V_M)$. Thus $n \sim M$, $Y > 2(L + 1)$ eventually, and all the source bounds used here hold in one fixed logarithmic band. The signs at primes above Y remain independent after the small-prime environment is fixed.

E.1 A modification with exact conditional means

On an extension of the probability space, take mutually independent $P_j \sim \text{Pois}(p)$, independent of the whole multiplicative function, and set

$$\hat{X}_j = \begin{cases} J_{j+1,L}, & j \in G_M, \\ P_j, & j \notin G_M. \end{cases} \quad (\text{E.1})$$

The filling is a proof device; it does not redefine the arithmetic observable in the theorem.

Lemma E.1 (Deletion, filling, and conditioning). *There is a coupling, given by (E.1), for which*

$$\begin{aligned} \chi_M &:= \mathbb{P}_{\mathcal{A}}(\exists j : \hat{X}_j \neq J_{j+1,L}) \\ &\ll e^{I_M} \left[p\mathcal{M}_{L+1} + p|\mathcal{D}_Y(L)| + \Lambda M^{-1/2} + \exp\left(-c_0 \frac{\log M}{\log \log M}\right) \right] = o(1). \end{aligned} \quad (\text{E.2})$$

Moreover, $\mathbb{E}_{\mathcal{A}} \hat{X}_j = p$ for every j . Distinct filling coordinates, and a filling coordinate and a good coordinate, have zero covariance. Two good coordinates not joined by the large-prime graph also have zero covariance under $\mathbb{P}_{\mathcal{A}}$.

Proof. A discrepancy at a deleted index requires either an actual start or a nonzero filling. The filling probabilities sum to at most $p|G_M^c|$. The actual mass on retained bad supports is at most $p(|\mathcal{D}_Y(L)| + \mathcal{M}_{L+1})$, by [5, Corollary 2.5]. The mass of starts below H_M is $O(MpM^{-1/2} + e^{-c_0 \log M / \log \log M})$ by [5, Proposition 7.3]. Their filling costs $O(pH_M)$. Condition these nonnegative actual bounds at cost e^{I_M} , use $Mp \asymp \Lambda$, and include the filling in the same upper bound. This proves the inequality.

The input bounds are $\mathcal{M}_{L+1} \leq M^{1/2+o(1)}$ and $|\mathcal{D}_Y(L)|/M \leq e^{-V_M+o(\nu_M)}$. Consequently the sharp term is bounded by $e^{-c\nu_M+o(\nu_M)}$, the polynomial terms by $M^{-1/2+o(1)}$, and the last term tends to zero since $\log M / \log \log M \gg V_M$. This proves $\chi_M \rightarrow 0$.

At a good index private prime pivots give $\mathbb{E}(J_{j+1,L} \mid \mathcal{F}_Y) = p$ on every environment. Since $\mathcal{A}_M \in \mathcal{F}_Y$, the same mean holds under $\mathbb{P}_{\mathcal{A}}$. The filling also has that mean and remains independent after conditioning. For two nonadjacent good indices, conditional independence gives

$$\mathbb{E}(J_{j+1,L} J_{k+1,L} \mid \mathcal{F}_Y) = p^2.$$

Averaging over $\mathcal{A}_M$ preserves this identity. The constancy of both marginal means is essential: independence on each environment alone would not imply zero covariance after mixing environments. $\square$

E.2 A direct quadratic estimate

Write $R_{2,U_M}^{\text{val}}(M, L+1)$ for the full-value relation sum over ordered separated pairs in U_M , as defined in Appendix A. Define

$$r_M^{\text{cov}} = \frac{e^{I_M} \Lambda}{n^2} \left[nL + n\Delta_Y(L) + R_{2,U_M}^{\text{val}}(M, L+1) \right]. \quad (\text{E.3})$$

This symbol is distinct from the shell-rounding error η_M . In particular, L here is the support diameter of a start event, not the larger support used below to truncate exact marks.

Lemma E.2 (Bounded-coefficient quadratic control). *For arbitrary deterministic real coefficients $c_1, \dots, c_n$,*

$$\mathbb{E}_{\mathcal{A}} \left| \frac{1}{\sqrt{\Lambda}} \sum_{j=1}^n c_j (\hat{X}_j - p) \right|^2 \leq (1 + C r_M^{\text{cov}}) \|c\|_{\infty}^2. \quad (\text{E.4})$$

Here C depends only on the fixed logarithmic band. For every fixed $0 < c' < c$ and $0 < \varepsilon < 1/3$,

$$r_M^{\text{cov}} \ll_{c', \varepsilon} e^{-c' \nu_M} + M^{-1/3+\varepsilon} = o(1). \quad (\text{E.5})$$

Proof. The diagonal contribution is at most $\Lambda^{-1} \sum_j c_j^2 p \leq \|c\|_{\infty}^2$: a good coordinate has variance $p(1-p)$ and a filling has variance p . No information factor is charged to this calculation.

All remaining covariances except good–good graph pairs vanish by Lemma E.1. Local pairs at distance at most L number $O(nL)$. For two good starts at distance $d < L$ the events are incompatible; at distance $d = L$ the good union has conditional joint mass p^2 . In particular their unconditional joint mass is at most p^2 . For separated good starts, the affine identity and the inclusion of the relative-sign relation kernel into the full-value kernel give

$$\mathbb{P}(J_{x,L} J_{y,L} = 1) \leq p^2 2^{\rho_{L+1}^{\text{val}}(x,y)}.$$

This is an unconditional estimate, not a claim uniform on each small-prime environment. On any good pair,

$$|\text{Cov}_{\mathcal{A}}(J_{x,L}, J_{y,L})| \leq \mathbb{E}_{\mathcal{A}}(J_{x,L} J_{y,L}) + p^2 \leq e^{I_M} \{\mathbb{P}(J_{x,L} J_{y,L} = 1) + p^2\}.$$

The first inequality uses the exact conditional marginal p . The second bounds a nonnegative quantity before conditioning. There are at most $n\Delta_Y(L)$ ordered graph pairs. Splitting the relation weight into $1 + (2^{\rho^{\text{val}}} - 1)$, and bounding the excess by the full nonnegative sum over separated pairs in U_M , bounds the sum of absolute off-diagonal covariances by

$$C e^{I_M} p^2 \{nL + n\Delta_Y(L) + R_{2,U_M}^{\text{val}}(M, L+1)\}.$$

Since $p^2/\Lambda = \Lambda/n^2$, this proves (E.4). Negative overlap covariances have been bounded in absolute value, so the argument also covers coefficients of both signs.

By [5, Proposition 4.2] and [5, technical companion, (B.3)], $\Delta_Y(L)/M \leq e^{-V_M+o(\nu_M)} + M^{-1+o(1)}$. Thus the degree term is at most $e^{-c\nu_M+o(\nu_M)} + M^{-1+o(1)}$. The local term is $M^{-1+o(1)}$. Proposition 3.24 of [5] gives, for any fixed $\varepsilon_0 > 0$,

$$R_{2,U_M}^{\text{val}}(M, L+1) \ll_{\varepsilon_0} M^{\varepsilon_0} (M^{5/3} + M^{2/3} 2^{L+1}).$$

Here $2^{L+1} = 2n/\Lambda \leq 2n$ eventually. Multiplication by $e^{I_M \Lambda/n^2} = M^{-2+o(1)}$ leaves $M^{-1/3+\varepsilon_0+o(1)}$. Choose $\varepsilon_0 < \varepsilon$ and absorb the subpolynomial factors. This proves (E.5). $\square$

E.3 Fixed spatial partitions

For an integer $R \geq 1$, partition $[0, 1]$ into $B_1 = [0, 1/R]$ and $B_r = ((r-1)/R, r/R]$ for $2 \leq r \leq R$. Let $A_{n,r} = \{j : j/n \in B_r\}$ and $w_{n,r} = |A_{n,r}|/n$. They are nonempty for $n \geq R$, and $w_{n,r} \rightarrow 1/R$.

Lemma E.3 (Independent Gaussian masses on fixed cells). *For each fixed R , under $\mathbb{P}_A$,*

$$\left(\frac{\sum_{j \in A_{n,r}} J_{j+1,L} - \Lambda w_{n,r}}{\sqrt{\Lambda}} \right)_{r=1}^R \implies (R^{-1/2} g_r)_{r=1}^R, \quad (\text{E.6})$$

where the g_r are independent standard normals. Replacing $J_{j+1,L}$ by $\hat{X}_j$ gives the same limit. No uniformity for growing R is asserted here.

Proof. Apply Proposition A.2 to these cells. This proof uses an internal exact-excess cutoff

$$E_M^* = \left\lceil \frac{I_M + V_M + \log(2 + Mp)}{\beta} \right\rceil, \quad Q_* = L + E_M^* + 1 = \log_2 M + O(V_M).$$

The support $Q_* + 1$ in this finite-mark comparison is larger than the base support $L + 1$ used in Lemma E.2; the two arguments need not use the same deleted set. At fixed R , $w_*^{-1} = O(R)$ and the conditional graph term in (A.3) is bounded by $C_R e^{I_M \Lambda \ell(\Lambda)} e^{-V_M+o(\nu_M)}$ plus a polynomial remainder. Since $\log \ell(\Lambda) = o(\nu_M)$, the budget absorbs it. The local and relation terms are $M^{-1/3+o(1)}$, after starting with an arbitrarily small fixed exponent in the source profile. Deletion and deep starts are controlled as in Appendix A.3.

A mark beyond E_M^* is a hit at threshold $L + E_M^* + 1$. The global scalar tail bound gives actual conditional cost

$$O \left(e^{I_M} \left[M 2^{-L-E_M^*-1} + e^{-c_0 \log M / \log \log M} \right] \right) = o(1).$$

The target tail is at most $\Lambda 2^{-E_M^*-1} = o(1)$. Coupling each tail to zero and summing all exact marks and signs therefore shows that the original cell counts are at distance $o(1)$ from independent Poisson variables of means $\Lambda w_{n,r}$. These are the full start means, not the truncated means $\Lambda w_{n,r}(1 - 2^{-E_M^*-1})$.

For fixed real $t_1, \dots, t_R$, the logarithm of the characteristic function of the centered target is

$$\sum_{r=1}^R \Lambda w_{n,r} \left(e^{it_r/\sqrt{\Lambda}} - 1 - \frac{it_r}{\sqrt{\Lambda}} \right) = -\frac{1}{2} \sum_{r=1}^R w_{n,r} t_r^2 + O_{\mathbf{t}}(\Lambda^{-1/2}).$$

It converges to $-(2R)^{-1} \sum_r t_r^2$. Total variation transfers this convergence of bounded tests to the original cell vector. Lemma E.1 then transfers its law to the modified vector. No moment convergence is deduced from either coupling. $\square$

E.4 The spatial remainder and completion of the proof

Set

$$\widehat{Z}_M = \frac{1}{\sqrt{\Lambda}} \sum_{j=1}^n (\widehat{X}_j - p) \delta_{j/n}.$$

Let $\pi_R(x) = r/R$ on B_r and let $(\pi_R)_* \widehat{Z}_M$ denote the pushforward of this finite signed measure. In particular it pushes the deterministic compensator forward as well. We do not apply this discontinuous map to an arbitrary distribution in H^{-s} .

For the basis in (C.1), the constant coefficient is unchanged and

$$\|q_k - q_k \circ \pi_R\|_\infty \leq C \min\{1, k/R\}, \quad k \geq 1. \quad (\text{E.7})$$

This follows from the bounded cosine amplitude and the derivative bound $\|q'_k\|_\infty \leq \sqrt{2}\pi k$. Apply Lemma E.2 to these coefficient differences at j/n , and sum the nonnegative terms. For $s > 1/2$,

$$\mathbb{E}_{\mathcal{A}} \|\widehat{Z}_M - (\pi_R)_* \widehat{Z}_M\|_{-s}^2 \leq C(1 + r_M^{\text{cov}}) S_s(R), \quad S_s(R) = \sum_{k \geq 1} (1 + k^2)^{-s} \min\{1, k/R\}^2. \quad (\text{E.8})$$

The series is dominated by the summable sequence $(1 + k^2)^{-s}$ and tends to zero as $R \rightarrow \infty$. More explicitly, splitting at $k = R$ gives

$$S_s(R) \ll_s \begin{cases} R^{1-2s}, & 1/2 < s < 3/2, \\ R^{-2} \log(2 + R), & s = 3/2, \\ R^{-2}, & s > 3/2. \end{cases}$$

The compensator in the projected vector is exactly $\sqrt{\Lambda} \sum_r w_{n,r} \delta_{r/R}$; replacing it by a continuum mean before this estimate would introduce a separate deterministic error.

Construct a spatial white noise $\mathcal{W}_{\text{sp}}$ as the isonormal Gaussian process on $L^2([0, 1])$, realized in H^{-s} through its cosine coefficients, and define

$$\mathcal{W}_R = \sum_{r=1}^R \mathcal{W}_{\text{sp}}(\mathbf{1}_{B_r}) \delta_{r/R}.$$

The notation on cell indicators uses the L^2 isonormal extension, not pathwise evaluation of an arbitrary distribution at a discontinuous test. Its coefficients are independent normals of variance $1/R$. The isometry and (E.7) give

$$\mathbb{E} \|\mathcal{W}_{\text{sp}} - \mathcal{W}_R\|_{-s}^2 \leq C S_s(R) \rightarrow 0. \quad (\text{E.9})$$

For each fixed R , Lemma E.3 proves $(\pi_R)_* \widehat{Z}_M \Rightarrow \mathcal{W}_R$ in the finite-dimensional span of the measures $\delta_{r/R}$, hence in H^{-s} .

Here is the two-step convergence argument explicitly. Let F have absolute value at most one and Lipschitz constant at most one on H^{-s} , and write $Z_M = (\mathcal{E}_M^{\text{start}} - \Lambda \mu_n) / \sqrt{\Lambda}$. The coupling, Cauchy–Schwarz, and the last two displays give

$$\begin{aligned} |\mathbb{E}_{\mathcal{A}} F(Z_M) - \mathbb{E} F(\mathcal{W}_{\text{sp}})| &\leq 2\chi_M + [C(1 + r_M^{\text{cov}}) S_s(R)]^{1/2} \\ &\quad + |\mathbb{E}_{\mathcal{A}} F((\pi_R)_* \widehat{Z}_M) - \mathbb{E} F(\mathcal{W}_R)| + [C S_s(R)]^{1/2}. \end{aligned}$$

At fixed R , the middle difference tends to zero, also in the bounded-Lipschitz metric, because its finite-dimensional laws converge. First take $M \rightarrow \infty$, then $R \rightarrow \infty$. The right side tends to zero. Bounded-Lipschitz convergence on this separable Hilbert space proves (7.13).

The only second moments used were calculated directly for the filled field and for the Gaussian target. The final coupling transfers laws to the original arithmetic field, not its unbounded conditional moments. The proof supplies spatial compactness for the start measure at one threshold; it supplies no temporal tightness, exact-mark tail norm, or common coupling with a critical Poisson field.

Appendix G supplies the additional reconstruction estimates on a common signed modification. Appendix F instead proves the signed Poisson-target limits used by the principal complete-field route; it constructs no arithmetic modification. The extended conclusions require these separate arguments, not this single-threshold proof alone.

F Signed Poisson targets and joint spatial limits

This appendix is the principal proof of the signed spatial results. All expectations denoted $\mathbb{E}_P$ are under the complete independent Poisson target. The only arithmetic step is to apply Corollary 2.2 once to the entire collection of derived observables. There is no modified arithmetic field, dependency graph or Palm condition on this route. Appendix G retains the alternative reconstruction from fixed-cell comparisons.

Write $\mu_n = n^{-1} \sum_{j=1}^n \delta_{j/n}$ and $\gamma_{e,\sigma} = 2^{-e-2}$. Let $\mathcal{E}_{M,\alpha}^P$ have independent Poisson masses of means $p\gamma_\alpha$ at these sites, and set

$$Z_{M,\alpha}^P = \frac{\mathcal{E}_{M,\alpha}^P - \Lambda \gamma_\alpha \mu_n}{\sqrt{\Lambda}}.$$

Attaching the canonical independent fractional mark makes the sign- σ target a Poisson measure $\mathcal{Q}_M^\sigma$ on $[0, 1] \times (0, \infty)$ of intensity

$$\frac{\Lambda}{2} \mu_n(dx) \beta 2^{-v} dv. \quad (\text{F.1})$$

The two sign measures are independent. Exact excess e corresponds to $v \in [e, e+1)$, and a survivor at threshold t has $v > t$. This standard Poisson marking construction [4] represents all the target observables on one probability space.

Use the cosine norm (C.1) and put $\kappa_s = 1 + 2 \sum_{k \geq 1} (1+k^2)^{-s} < \infty$ for $s > 1/2$. Then $\sup_x \|\delta_x\|_{-s}^2 \leq \kappa_s$. Covariance formulas are first interpreted on the positive-Sobolev dual and then, when stated on L^2 , by their isonormal extensions. They do not define pathwise pairings of an arbitrary distribution with every square-integrable function.

F.1 Finite types and the complete weighted spectrum

For any fixed finite set of types and bounded continuous tests φ_α , the joint characteristic exponent is

$$\sum_{j,\alpha} p\gamma_\alpha \left(\exp \left\{ \frac{i\varphi_\alpha(j/n)}{\sqrt{\Lambda}} \right\} - 1 - \frac{i\varphi_\alpha(j/n)}{\sqrt{\Lambda}} \right) = -\frac{1}{2} \sum_{\alpha} \gamma_\alpha \mu_n(\varphi_\alpha^2) + O(\Lambda^{-1/2}). \quad (\text{F.2})$$

The implicit constant depends on the fixed tests and types. Riemann sums identify independent spatial Gaussian limits of variance weights γ_α . Poisson isometry gives, exactly,

$$\text{Cov}_P(\langle Z_{M,\alpha}^P, \varphi \rangle, \langle Z_{M,\alpha'}^P, \psi \rangle) = \mathbf{1}_{\{\alpha=\alpha'\}} \gamma_\alpha \mu_n(\varphi\psi), \quad \mathbb{E}_P \|Z_{M,\alpha}^P\|_{-s}^2 \leq \kappa_s \gamma_\alpha.$$

At finitely many types, take $1/2 < s_0 < s$ and use the compact embedding $H^{-s_0} \hookrightarrow H^{-s}$ for tightness. This proves the target assertion of Theorem 7.7, including the sum/imbalance consequence.

For the fixed weights of (7.16), the target has the uniform tail bound

$$\mathbb{E}_P \sum_{e>K,\sigma} w_{e,\sigma} \|Z_{M,e,\sigma}^P\|_{-s}^2 \leq \kappa_s \sum_{e>K,\sigma} w_{e,\sigma} \gamma_{e,\sigma}. \quad (\text{F.3})$$

Finite-type tightness and this vanishing tail give convergence in $\mathcal{H}_{w,s}$, by approximation with finitely many coordinates. The Gaussian limit has the same tail bound. The original and target centered spectra are measurable random elements of this space: their atom configurations are finite, and the compensator belongs to it because $\sum_{\alpha} w_{\alpha} \gamma_{\alpha}^2 \leq \Gamma(w)/4$. Consequently the arithmetic tail is controlled at the level of probabilities by

$$\mathbb{P} \left(\sum_{e>K,\sigma} w_{e,\sigma} \|Z_{M,e,\sigma}\|_{-s}^2 > \epsilon \middle| \mathcal{A}_M \right) \leq \Delta_M^{\text{mic}} + \frac{\kappa_s}{\epsilon} \sum_{e>K,\sigma} w_{e,\sigma} \gamma_{e,\sigma}. \quad (\text{F.4})$$

This is not a transfer of an unbounded moment.

For necessity of the trace condition, restrict to the constant spatial mode. With independent standard normal variables g_{α} , its squared norm is $\sum_{\alpha} w_{\alpha} \gamma_{\alpha} g_{\alpha}^2$. If $\Gamma(w) = \infty$, then for every $t > 0$,

$$\mathbb{E} \exp \left(-t \sum_{\alpha} w_{\alpha} \gamma_{\alpha} g_{\alpha}^2 \right) = \prod_{\alpha} (1 + 2t w_{\alpha} \gamma_{\alpha})^{-1/2} = 0.$$

Indeed the sum of the logarithms diverges, whether infinitely many terms exceed a fixed positive number or all terms eventually tend to zero. The squared norm is infinite almost surely. This proves the target trace assertion and, with the common-readout transfer, Theorem 7.8.

F.2 Integer threshold sequences

For $\eta_0 > 0$, the inequality

$$\left\| \sum_{e \geq j} F_e \right\|_{-s}^2 \leq C_{\eta_0} 2^{-\eta_0 j} \sum_e 2^{\eta_0 e} \|F_e\|_{-s}^2$$

makes tail summation continuous from a positively exponentially weighted exact-level space into every polynomially weighted threshold space. Taking $0 < \eta_0 < 1$ keeps the trace finite. Poisson isometry gives the target threshold covariance

$$\mathbf{1}_{\{\sigma=\tau\}} 2^{-\max(j,k)-1} \mu_n(\varphi\psi).$$

For the standardized sequence, each sign-specific coordinate $2^{(j+1)/2} \sum_{e \geq j} Z_{M,e,\sigma}^P$ has expected squared H^{-s} norm at most κ_s , uniformly in j and M . A fixed summable v_j therefore controls the tail in $\ell^2(v; H^{-s})$. Fixed-coordinate convergence and that bound prove the target version of Corollary 7.9; the full readout transfers the law. No Gaussian approximation of a single moving critical threshold is asserted.

F.3 Fixed threshold paths and their joint low-level law

For the target measure (F.1), the sign-specific survivor path has exponential lifetimes and initial mean $\Lambda/2$. Apply the martingale proof of Appendix C.4 with that intensity. It yields tightness and convergence in $D([0, T_0], H^{-s})$ for each fixed T_0 ; the same Hilbert-valued proof applies to the direct sum of the two independent sign spaces. The limiting drift is $-\beta/2$ and the covariance is $2^{-|t-u|/2} \int \varphi\psi$ within a sign.

At fixed exact levels and fixed times, every joint test is one compensated integral against (F.1). Its integrand is $O_{T_0}(\Lambda^{-1/2})$ and has bounded squared integral, so its cubic remainder vanishes. This identifies the joint limit by independent Gaussian measures $\mathcal{G}_\sigma$ with control measure $(\beta/2)2^{-v} dx dv$:

$$Z_{e,\sigma} = \mathcal{G}_\sigma(\cdot \times [e, e+1)), \quad H_t^\sigma = \sqrt{2} 2^{t/2} \mathcal{G}_\sigma(\cdot \times (t, \infty)).$$

The target spectral tails from (F.3), together with the path tightness just proved, complete the joint space and establish Theorem 7.10 after common marking and transfer. The paths and spectrum share this Gaussian measure and are not independent. No martingale is assigned to the arithmetic field.

F.4 Compression coefficients through the critical endpoint

Write $\Lambda = 2^{J+\theta_M}$ with $J = \lfloor \log_2 \Lambda \rfloor \rightarrow \infty$ and $\theta_M \in [0, 1)$. For a bounded temporal test ψ , put

$$h_{J,e}^d(\psi) = \frac{1}{\sqrt{J}} \sum_{j=0}^{\min(e,J)} 2^{j/2} \psi(j/J), \quad (F.5)$$

$$h_{J,e}^c(u; \psi) = \frac{1}{\sqrt{J}} \int_0^{\min(e+u,J)} 2^{t/2} \psi(t/J) dt. \quad (F.6)$$

A sign-specific target pairing is a compensated Poisson integral, divided by $\sqrt{\Lambda}$, with coefficients $\sqrt{2} \varphi(j/n) h_{J,e}^\bullet$ on that sign and zero on the other. For either choice,

$$|h_{J,e}^\bullet| \leq \frac{C}{\sqrt{J}} 2^{\min(e,J)/2} \|\psi\|_\infty. \quad (F.7)$$

For the continuous choice, replacing e by $e+1$ costs only a fixed factor. With $\gamma_e = 2^{-e-1}$, geometric summation gives

$$\sum_e \gamma_e \mathbb{E} |h_{J,e}^\bullet|^2 \leq C \|\psi\|_\infty^2, \quad (F.8)$$

$$\sum_e \gamma_e \mathbb{E} |h_{J,e}^\bullet| \leq \frac{C}{\sqrt{J}} \|\psi\|_\infty, \quad (F.9)$$

$$\Lambda^{-1/2} \sum_e \gamma_e \mathbb{E} |h_{J,e}^\bullet|^3 \leq C J^{-3/2} \|\psi\|_\infty^3. \quad (F.10)$$

The last bound includes every $e > J$; use $2^{J/2}/\sqrt{\Lambda} \leq 1$. It is the absolute cubic Poisson integral. These estimates involve only the Poisson target and its canonical marks. They include the order-one critical layer, rather than approximating every threshold individually by a Gaussian variable.

F.5 The joint compression law and critical-field independence

Use (7.20) for the target. In the coefficient formulas above, the expectation over U is under the canonical mark law. A finite linear combination of discrete and continuous compression tests has an absolute cubic Poisson integral tending to zero by (F.10); the bound includes all levels above J . The characteristic-exponent criterion (C.2) gives joint Gaussian finite-test limits, not only two marginal Gaussian laws.

Put $a = \beta/2$. The target survivor covariance is exactly $e^{-a|t-u|} \mu_n(\varphi \tilde{\varphi})$ within one sign. Appendix C.5 gives the discrete/discrete and continuous/continuous temporal constants

$$c_d = 3 + 2\sqrt{2}, \quad c_c = 4/\beta, \quad c_d - c_c > 0. \quad (F.11)$$

The last inequality follows from $\coth x > 1/x$ at $x = \beta/4$. The mixed covariance is determined by

$$\frac{1}{J} \sum_{j=0}^J \int_0^J e^{-a|j-t|} \psi(j/J) \tilde{\psi}(t/J) dt \longrightarrow \frac{2}{a} \int_0^1 \psi(u) \tilde{\psi}(u) du. \quad (\text{F.12})$$

For continuous bounded tests, first restrict to $|j - t| \leq A$, pass to Riemann sums in j/J , and then let $A \rightarrow \infty$. The tails are uniformly exponentially small, and the endpoint strips contribute $o(1)$. This identifies the limiting covariance on smooth tests. Its continuous bilinear form has the stated isonormal L^2 extension; no evaluation of arbitrary L^2 representatives at grid points is required. Thus the cross constant is $2/a = c_c$, and the matrix is precisely (7.23).

The coefficient bound (F.8), bounded cosine functions and Poisson isometry give uniform variances for all tensor coefficients. Choose intermediate exponents $1/2 < s'_x < s_x$ and $1/2 < s'_u < s_u$. The expected squared $H^{-s'_x, -s'_u}$ norm is bounded, so compact embedding gives tightness in the stated tensor spaces. The spatial control μ_n converges to Lebesgue measure on continuous tests. These facts identify the full joint compression law for both independent signs.

For independence from the critical field, fix K and delete all target atoms with $e \geq J - K$ from each compression, together with their compensator. Their total mean is at most 2^{K+1} uniformly in phase. One deleted atom contributes at most $C_{s_x, s_u}/\sqrt{J}$ in the compression norm: use (F.7), divide by $\sqrt{\Lambda}$, and bound the spatial and temporal Dirac norms. The compensator has the same expected bound. The change is therefore $o_P(1)$. The remaining compression pair is independent of the entire deleted Poisson block.

At each fixed set of critical levels $r = e - J$, the grid target converges vaguely, along $\theta_M \rightarrow \theta$, to the intensity in (7.22). This follows directly from its Poisson Laplace functional and $\mu_n \Rightarrow dx$. The preceding deletion proves independence from every upper restriction $r \geq -K$. Exhaustion by K identifies independence from the entire vague field; it does not delete the infinite negative-level field in total variation. Joint tightness and finite restrictions now prove the target version of Theorem 7.11. One common marked readout transfers the complete joint law.

F.6 The common three-block limit

Use the same target measure for the exact spectrum, the fixed-threshold paths, both compressions and the critical field. For every fixed exact level e , its covariance with a compression test is $O_e(J^{-1/2})$. For a fixed threshold time $t \leq T_0$, the covariance is $O_{T_0}(J^{-1/2})$, since both $\sum_{j \geq 0} e^{-a|t-j|}$ and $\int_0^\infty e^{-a|t-u|} du$ are bounded on that interval. A joint linear combination still has vanishing cubic remainder, by (F.10) and the finite-test bounds in Appendix F.3. Its limiting Gaussian covariance is therefore block diagonal. This proves independence between the low-level Gaussian block and the compression block at finite projections.

Delete the upper critical atoms as in Appendix F.5. Fixed exact levels are eventually untouched. On $[0, T_0]$, the upper block and its compensator change the standardized survivor path by $O_P(\Lambda^{-1/2})$ at each fixed K , uniformly in time, because its count is tight and the denominator is bounded below by a constant times $\sqrt{\Lambda}$. Thus this deletion leaves the entire low block unchanged in probability while making the retained Gaussian approximants independent of the upper critical block.

Complete the exact levels using (F.3), the paths by their J_1 tightness, and the compressions by the tensor tightness above. Joint tightness, finite projections and exhaustion of critical levels prove the three-block target law in Corollary 7.12. The common-readout inequality transfers that joint law, without transferring unbounded moments. This argument does not add the record-history Brownian motion to the three blocks; that remains a different joint

theorem with its own separation argument.

F.7 Rectangle targets with exact grid centering

For a fixed compact rectangle $K \subset (-\infty, 0] \times \mathbb{R}$, assume the depth conditions (7.25). Under $v = \Lambda 2^{-\tau}$ the marked sign target becomes a Poisson measure of intensity $\frac{1}{2}\mu_n(dx) dv$ on $0 < v < \Lambda$. Every queried rectangle lies below that cutoff. With $k = k_M$, the exact mean of its count is

$$m_M^P(a, b) = \frac{1}{2} 2^{k-b} \mu_n([0, 2^a]).$$

First subtract this mean but keep the denominator $\sqrt{2^{k+a-b}/2}$ from the article. Intersecting rectangles gives the exact covariance of the centered target:

$$2^{-(a+c)/2} \mu_n([0, 2^{\min(a,c)}]) 2^{-|b-d|/2}. \quad (\text{F.13})$$

Uniformly on $K \times K$, it tends to $2^{-(|a-c|+|b-d|)/2}$, since the grid prefix error is at most $1/n$.

After the vertical rescaling $v = 2^k z$, every fixed finite-point or integrated test is a compensated Poisson integral with integrand of sup norm $O_K(2^{-k/2})$ and bounded squared integral. Its absolute cubic integral is $O_K(2^{-k/2})$. This proves the joint Gaussian finite-test limit. Uniform pointwise variances also bound every tensor cosine coefficient variance on K . Intermediate negative-Sobolev exponents give tightness, exactly as above. The signs are independent under the target law. These arguments prove the finite-dimensional and distribution-space limits separately; no diagonal restriction of a negative-Sobolev distribution is used.

Finally the difference between the exact grid mean and the continuous area mean, after normalization, obeys

$$\sup_{(a,b) \in K} \frac{|m_M^P(a, b) - 2^{k+a-b}/2|}{\sqrt{2^{k+a-b}/2}} \leq C_K \frac{2^{k/2}}{n}. \quad (\text{F.14})$$

Its squared pointwise and integrated errors are $O_K(2^k/n^2)$ and vanish by the depth condition. Restoring the stated centering and applying the common readout proves Theorem 7.13. The same calculation with a fixed projected spatial measure $\mu_{n,R}$ is available to the alternative direct reconstruction in Appendix G.7.

G Direct signed spatial reconstruction without microscopic coupling

We prove Theorems 7.7–7.13 under the hypotheses of Theorem 7.6 and with the conditional-law convention of Appendix E. Types, exact means, and centered fields are those of (7.14); independent filling and auxiliary marks are introduced where required.

This alternative proof does not use Input 2.1; it retains direct moment and spatial-remainder information that a law-level comparison would not itself provide. A common growing-support modification preserves the categorical covariance of all retained types and fills the remaining coordinates with their exact Poisson means. Its marked quadratic estimate supports three distinct completions: a weighted level tail, a maximum over fixed time intervals, and an integrated compression estimate. The rectangle argument retains its separate grid-centering calculation.

We use the cosine norm (C.1) and write $\kappa_s = 1 + 2 \sum_{k \geq 1} (1 + k^2)^{-s}$, so $\sup_x \|\delta_x\|_{-s}^2 \leq \kappa_s$ for $s > 1/2$. The covariance statements on square-integrable tests are isonormal extensions of the pairings first defined on the positive-Sobolev dual. They are not pathwise pairings of an arbitrary distribution with every L^2 function. The arithmetic estimates are those of the

source identified in Appendix A. The common target calculations are collected in Appendix F; the estimates here show how to reach them without a microscopic input.

G.1 A full modification and complete fixed-cell comparisons

Let $\lambda = Mp$, and take the internal cutoff (A.5) already used in Appendix A.3:

$$E_M^* = \left\lceil \frac{I_M + V_M + \log(2 + \lambda)}{\beta} \right\rceil, \quad Q_M = L + E_M^* + 1. \quad (\text{G.1})$$

The budget gives $I_M, \log \Lambda, E_M^*, a_M = O(V_M)$, hence $Q_M = \log_2 M + O(V_M)$. All maximal supports therefore stay in a fixed positive logarithmic band. In particular $Y_M^\# > 2(Q_M + 1)$ eventually.

Write $Y = Y_M^\#$, $H_M = \lceil \sqrt{M} \rceil$, $U_M = \{x : H_M \leq x \leq n + 1\}$, and

$$G_M = \{j : 1 \leq j \leq n, j + 1 \in U_M, \{j, \dots, j + Q_M\} \text{ has no } Y\text{-defective integer}\}.$$

Defectiveness and the deterministic large-prime graph are exactly those of Appendix A. The rightmost vertex checks the end of an exact run. Every retained type uses this same good set.

Take mutually independent $P_{j,\alpha} \sim \text{Pois}(p\gamma_\alpha)$, independent of the multiplicative function, and define the *complete* modification

$$\hat{X}_{j,e,\sigma} = \begin{cases} I_{j,e,\sigma}, & j \in G_M, \quad e \leq E_M^*, \\ P_{j,e,\sigma}, & \text{otherwise.} \end{cases} \quad (\text{G.2})$$

The expected total filling mass is finite for every M . In particular, for each M , every original or modified prelimit atom configuration is finite almost surely. Filling also above the mark cutoff is intentional: it preserves exact target means in *every* coordinate, rather than leaving a deterministic tail compensation problem.

Lemma G.1 (Full coupling and uniform arithmetic remainder). *The modification has $\mathbb{E}_{\mathcal{A}} \hat{X}_{j,\alpha} = p\gamma_\alpha$ for every j, α . Put*

$$\chi_M = \mathbb{P}_{\mathcal{A}}(\exists j, \alpha : \hat{X}_{j,\alpha} \neq I_{j,\alpha}), \quad (\text{G.3})$$

$$r_M = \frac{e^{I_M} \Lambda}{n^2} \left[nQ_M + n\Delta_Y(Q_M) + \mathcal{R}_{2,U_M}^{\text{val}}(M, Q_M + 1) \right]. \quad (\text{G.4})$$

For every $0 < c' < c$ and $0 < \varepsilon < 1/3$,

$$\chi_M + r_M \ll_{c',\varepsilon} e^{-c'\nu_M} + M^{-1/3+\varepsilon} = o(1). \quad (\text{G.5})$$

The constants do not depend on the number of retained types. Here r_M is a covariance remainder; it is unrelated to the shell displacement η_M .

Proof. The private-prime pivots of Appendix A give $\mathbb{E}(I_{j,\alpha} \mid \mathcal{F}_Y) = p\gamma_\alpha$ at a good site for $e \leq E_M^*$. Filling supplies the same mean elsewhere. These identities survive conditioning on $\mathcal{A}_M$.

On deleted sites the sum of actual marks, over all signs and lengths, is at most the base start indicator. The masked first-moment and deep-start bounds used in Appendices A and E, together with the shifted scalar tail in Appendix A.3, give

$$\chi_M \ll e^{I_M} [p\mathcal{M}_{L+1} + p|D_Y(Q_M)| + \Lambda M^{-1/2} + \frac{\epsilon_M}{\log M} + \lambda 2^{-E_M^*-1}], \quad \epsilon_M = \exp\left(-c_0 \frac{\log M}{\log \log M}\right). \quad (\text{G.6})$$

The nonzero filling probabilities on deleted indices are bounded by their total target mass. Above the cutoff that mass is $\Lambda 2^{-E_M^*-1}$; it is included in (G.6), since $\Lambda \leq \lambda$ and $e^{I_M} \geq 1$. The event of any actual mark above the cutoff is controlled by the scalar shifted threshold bound. This is a probability-of-disagreement argument, not a weighted moment estimate for discarded arithmetic marks.

The inputs printed in Appendix A are

$$\mathcal{M}_{L+1} \leq M^{1/2+o(1)}, \quad |D_Y(Q_M)|/M \leq e^{-V_M+o(\nu_M)}, \quad (\text{G.7})$$

$$\Delta_Y(Q_M)/M \leq e^{-V_M+o(\nu_M)} + M^{-1+o(1)}, \quad (\text{G.8})$$

$$\mathcal{R}_{2,U_M}^{\text{val}}(M, Q_M + 1) \ll_{\varepsilon_0} M^{\varepsilon_0} \left(M^{5/3} + M^{2/3} 2^{Q_M+1} \right). \quad (\text{G.9})$$

The cutoff makes $e^{I_M} \Lambda 2^{-E_M^*-1} \leq C e^{-V_M}$. The sharp deletion and degree terms are at most $\exp(I_M + \log \Lambda - V_M + o(\nu_M))$. All other factors $e^{I_M}, \Lambda, 2^{Q_M+1}/M$ are $M^{o(1)}$. Thus the relation contribution is $M^{-1/3+\varepsilon_0+o(1)}$. Choose $\varepsilon_0 < \varepsilon$; use (7.12) for the sharp terms. Finally $\log M / \log \log M \gg V_M$ handles $e^{I_M} \epsilon_M$. This proves (G.5) uniformly at the growing cutoff. $\square$

Fix an integer $R \geq 1$. Partition $[0, 1]$ into $B_1 = [0, 1/R]$, $B_r = ((r-1)/R, r/R]$, and put

$$A_{n,r} = \{j : j/n \in B_r\}, \quad w_{n,r} = |A_{n,r}|/n, \quad \pi_R(x) = r/R \ (x \in B_r), \quad \mu_{n,R} = (\pi_R)_* \mu_n.$$

For large n , the cells are nonempty and $w_{n,r} \rightarrow 1/R$.

Lemma G.2 (All levels on a fixed spatial partition). *For every fixed R ,*

$$d_{\text{TV}} \left(\mathcal{L}_{\mathcal{A}} \left(\sum_{j \in A_{n,r}} I_{j,\alpha} \right)_{r,\alpha}, \bigotimes_{r,\alpha} \text{Pois}(\Lambda w_{n,r} \gamma_\alpha) \right) \rightarrow 0. \quad (\text{G.10})$$

The same holds for $\widehat{X}$. It also holds after attaching any fixed independent mark kernel to every atom; therefore one coupling of the projected complete field controls all its measurable derived observables jointly.

Proof. Apply Proposition A.2 with $E = E_M^*$, $Q = Q_M$. At fixed R , $w_*^{-1} = O(R)$. Apart from deletion and the scalar tail already included in (G.6), its conditioned ledger is bounded by $CR\ell(\Lambda)r_M$, where $\ell(\Lambda) = 1 + \log^+(2\Lambda)$. Because $\log \ell(\Lambda) = o(\nu_M)$, the strict budget absorbs this factor. The actual tail above E_M^* and the target tail have vanishing probabilities of any atom. Couple each tail to zero. The resulting comparison has the *full* target means, not truncated means. Lemma G.1 gives the assertion for the modified field. Independent marking and measurable mapping contract total variation. No mesh $R = R_M$ is needed. $\square$

G.2 A quadratic inequality uniform in levels and marks

Let η be a probability law for an auxiliary mark U , independent for every atom. The canonical interpolation later uses

$$\eta(du) = 2\beta 2^{-u} \mathbf{1}_{(0,1)}(u) du. \quad (\text{G.11})$$

For deterministic real functions $h_{j,\alpha}(u)$, write $\bar{h}_{j,\alpha} = \int h_{j,\alpha} d\eta$, and define

$$T_M(h) = \frac{1}{\sqrt{\Lambda}} \left[\sum_{j,\alpha} \sum_{\ell=1}^{\widehat{X}_{j,\alpha}} h_{j,\alpha}(U_{j,\alpha,\ell}) - p \sum_{j,\alpha} \gamma_\alpha \bar{h}_{j,\alpha} \right], \quad (\text{G.12})$$

$$D_M(h) = \frac{1}{n} \sum_{j,\alpha} \gamma_\alpha \int |h_{j,\alpha}|^2 d\eta, \quad B_M(h) = \sum_{\alpha: e \leq E_M^*} \gamma_\alpha \sup_j |\bar{h}_{j,\alpha}|. \quad (\text{G.13})$$

Assume $\sum_{\alpha} \gamma_{\alpha} \sup_j \int |h_{j,\alpha}|^2 d\eta < \infty$. This implies the first-moment summability needed in (G.12).

Proposition G.3 (Marked quadratic bound). *Under (7.12),*

$$\mathbb{E}_{\mathcal{A}} |T_M(h)|^2 \leq D_M(h) + Cr_M B_M(h)^2. \quad (\text{G.14})$$

The constant depends only on the fixed logarithmic band, not on the type support or on the size of the coefficients. One may enlarge B_M by summing over all α . For unmarked coefficients $c_{j,\alpha}$, this gives

$$\mathbb{E}_{\mathcal{A}} \left| \frac{1}{\sqrt{\Lambda}} \sum_{j,\alpha} c_{j,\alpha} (\hat{X}_{j,\alpha} - p\gamma_{\alpha}) \right|^2 \leq \frac{1}{n} \sum_{j,\alpha} \gamma_{\alpha} c_{j,\alpha}^2 + Cr_M \left(\sum_{\alpha} \gamma_{\alpha} \|c_{\cdot,\alpha}\|_{\infty} \right)^2. \quad (\text{G.15})$$

For unmarked arrays c, d , put $b(c) = \sum_{\alpha} \gamma_{\alpha} \|c_{\cdot,\alpha}\|_{\infty}$. If $b(c), b(d) < \infty$ and the displayed diagonal sums are finite, then

$$\left| \mathbb{E}_{\mathcal{A}} [T_M(c)T_M(d)] - \frac{1}{n} \sum_{j,\alpha} \gamma_{\alpha} c_{j,\alpha} d_{j,\alpha} \right| \leq (p + Cr_M) b(c) b(d). \quad (\text{G.16})$$

Proof. At one good site the retained low types are mutually exclusive. With independent auxiliary marks their variance is exactly

$$p \sum_{e \in E_M^*, \sigma} \gamma_{e,\sigma} \int h_{j,e,\sigma}^2 d\eta - p^2 \left(\sum_{e \in E_M^*, \sigma} \gamma_{e,\sigma} \bar{h}_{j,e,\sigma} \right)^2.$$

The filling has independent compound-Poisson variances equal to its corresponding diagonal terms. Thus the total same-site contribution is at most the first term of (G.14). The negative quantity is a *whole square*; discarding individual cross covariances would not be justified for coefficients of arbitrary signs. No information cost is charged to the diagonal calculation.

For two unmarked arrays, the difference from the Poisson diagonal is exactly

$$-\frac{p^2}{\Lambda} \sum_{j \in G_M} \left(\sum_{e \in E_M^*, \sigma} \gamma_{e,\sigma} c_{j,e,\sigma} \right) \left(\sum_{e \in E_M^*, \sigma} \gamma_{e,\sigma} d_{j,e,\sigma} \right).$$

Its absolute value is at most $p b(c) b(d)$. Thus the same-site correlations $-p^2 \gamma_{\alpha} \gamma_{\alpha'}$ do not vanish identically; their normalized sum vanishes because $p \rightarrow 0$.

For two good nonadjacent sites, independence conditional on $\mathcal{F}_Y$, together with constant marginal means $p\gamma_{\alpha}$, gives zero covariance after conditioning on $\mathcal{A}_M$. The constancy is essential: conditional independence alone would not justify this after mixing environments. Independent filling has no off-site covariance.

For a local pair the signed joint mass is at most $4p^2 \gamma_{\alpha} \gamma_{\alpha'}$, including all overlap cases. For separated sites, Appendix A gives the *unconditional* bound

$$\mathbb{P}(I_{j,\alpha} I_{k,\alpha'} = 1) \leq p^2 \gamma_{\alpha} \gamma_{\alpha'} 2^{\rho_{Q_M+1}^{\text{val}}(j+1, k+1)}.$$

This is not asserted uniformly on the small-prime environment. Conditioning nonnegative joint masses costs at most e^{I_M} . Subtracting the exact marginal product and using absolute values, the sum of off-site covariance contributions is bounded by

$$C e^{I_M} p^2 B_M(h)^2 \left[n Q_M + n \Delta_Y(Q_M) + \mathcal{R}_{2,U_M}^{\text{val}}(M, Q_M + 1) \right].$$

The auxiliary marks at different atoms replace each coefficient by its mean $\bar{h}$ in this calculation. Divide by Λ and use $p^2/\Lambda = \Lambda/n^2$. The same off-site calculation with two arrays is bounded by $Cr_M b(c)b(d)$; combining it with the exact diagonal correction proves (G.16). Infinite type sums follow by truncation in L^2 ; the high types are independent Poisson coordinates, and the assumed summability controls the tails. $\square$

Why one factor survives. The sharp term in r_M is

$$e^{I_M} \Lambda \frac{\Delta_Y(Q_M)}{n} \lesssim e^{I_M + \log \Lambda - V_M + o(\nu_M)} + M^{-1+o(1)}.$$

The growing support contributes only the factors already present in the fixed-band arithmetic inputs. There is no raw factor equal to the number of types, and no second factor of Λ . Later coefficients may grow with e or M ; the quantities to control are D_M and B_M , not an unweighted maximum over types.

For future spatial approximation define

$$S_s(R) = \sum_{k \geq 1} (1+k^2)^{-s} \min(1, k/R)^2 \longrightarrow 0 \quad (s > 1/2). \quad (\text{G.17})$$

The finite-family spatial estimate also has the quantitative form

$$S_s(R) \ll_s \begin{cases} R^{1-2s}, & 1/2 < s < 3/2, \\ R^{-2} \log(2+R), & s = 3/2, \\ R^{-2}, & s > 3/2. \end{cases} \quad (\text{G.18})$$

Indeed split the sum at $k = R$; the first part is bounded by $R^{-2} \sum_{k \leq R} k^{2-2s}$ and the second by $\sum_{k > R} k^{-2s}$. This rate concerns the spatial reconstruction term, not a uniform rate for the full two-step arithmetic limit.

The cosine estimate in Appendix E is $\|q_k - q_k \circ \pi_R\|_\infty \leq C \min(1, k/R)$. Whenever a finite signed measure is projected, its compensator is projected as well. The map $(\pi_R)_*$ is never applied without definition to an arbitrary negative-Sobolev distribution.

G.3 Finite signed levels

Proof of Theorem 7.7. Fix a finite set F of types. A proof restricted to F could use the fixed diameter $L + 1 + \max_{\alpha \in F} e$. We use the larger common support Q_M instead, so that all the limits below are derived on one construction. Every fixed type belongs to the retained range eventually.

At fixed R , Lemma G.2 compares the complete cell field to independent Poisson masses; restriction to F contracts total variation. Their means are exactly $\Lambda w_{n,r} \gamma_\alpha$. For real $(t_{r,\alpha})$, the characteristic exponent of the centered, $\sqrt{\Lambda}$ -normalized target is

$$\sum_{r,\alpha \in F} \Lambda w_{n,r} \gamma_\alpha \left(e^{it_{r,\alpha}/\sqrt{\Lambda}} - 1 - \frac{it_{r,\alpha}}{\sqrt{\Lambda}} \right) = -\frac{1}{2} \sum_{r,\alpha \in F} w_{n,r} \gamma_\alpha t_{r,\alpha}^2 + O_{F,R,t}(\Lambda^{-1/2}).$$

Since $w_{n,r} \rightarrow 1/R$, this identifies independent Gaussian cell coordinates. No rounding error in a mean is multiplied by $\sqrt{\Lambda}$; the exact grid mean was subtracted before the limit.

Set $\hat{Z}_{M,\alpha} = (\sum_j \hat{X}_{j,\alpha} \delta_{j/n} - \Lambda \gamma_\alpha \mu_n) / \sqrt{\Lambda}$. Apply (G.15) to $(q_k(j/n) - q_k(\pi_R(j/n))) \mathbf{1}_{\{\alpha=\alpha_0\}}$. Summing the cosine weights and the finitely many types gives

$$\mathbb{E}_A \sum_{\alpha \in F} \|\hat{Z}_{M,\alpha} - (\pi_R)_* \hat{Z}_{M,\alpha}\|_{-s}^2 \leq C S_s(R) \sum_{\alpha \in F} (\gamma_\alpha + Cr_M \gamma_\alpha^2).$$

The projection moves both the atom measure and its compensator. The constant spatial mode is unchanged. The right-hand side tends to zero as $R \rightarrow \infty$, uniformly in the preceding M -limit.

The limiting cell noises can be constructed from independent isonormal white noises by $W_{\alpha,R} = \sum_r W_\alpha(\mathbf{1}_{B_r})\delta_{r/R}$. Gaussian isometry gives the same $C\gamma_\alpha S_s(R)$ bound for $\sqrt{\gamma_\alpha}(W_\alpha - W_{\alpha,R})$; this defines the cell approximation without asserting a pushforward of a general distribution. Apply the triangle inequality to bounded Lipschitz tests of the Hilbert product, first at fixed R and then let $R \rightarrow \infty$. This proves weak convergence of the modified vector. Equivalently, tightness follows by using an exponent $1/2 < s_0 < s$ and the compact embedding $H^{-s_0} \hookrightarrow H^{-s}$ on its finitely many components. The complete agreement event in Lemma G.1 transfers the laws to the original vector; it does not transfer its unbounded moments.

For a fixed ϵ , the two sign limits have equal variance weight 2^{-e-2} and are independent. Their sum and difference are jointly Gaussian, uncorrelated, and each has variance weight 2^{-e-1} . This proves the stated total/imbalance consequence, jointly over every finite family of excesses. $\square$

G.4 The complete weighted signed spectrum

Use the fixed positive finite weights, space $\mathcal{H}_{w,s}$ and trace $\Gamma(w)$ of (7.16), and assume $\Gamma(w) < \infty$.

For each M , both configurations have only finitely many atoms almost surely. The deterministic compensator belongs to $\mathcal{H}_{w,s}$ because $\sum_\alpha w_\alpha \gamma_\alpha^2 \leq \Gamma(w)/4$. Thus the original and modified centered fields are measurable random elements of this space before any limiting assertion is made.

Proof of Theorem 7.8. Let $\hat{\mathcal{E}}_{M,\alpha} = \sum_j \hat{X}_{j,\alpha} \delta_{j/n}$ and $\hat{Z}_{M,\alpha} = (\hat{\mathcal{E}}_{M,\alpha} - \Lambda \gamma_\alpha \mu_n)/\sqrt{\Lambda}$. Apply (G.15) to one type and one cosine coefficient, then sum the nonnegative terms. It gives, for every α ,

$$\mathbb{E}_{\mathcal{A}} \left\| \hat{Z}_{M,\alpha} \right\|_{-s}^2 \leq \kappa_s(\gamma_\alpha + Cr_M \gamma_\alpha^2). \quad (\text{G.19})$$

Consequently

$$\mathbb{E}_{\mathcal{A}} \sum_{e>K,\sigma} w_{e,\sigma} \left\| \hat{Z}_{M,e,\sigma} \right\|_{-s}^2 \leq \kappa_s(1 + Cr_M) \sum_{e>K,\sigma} w_{e,\sigma} \gamma_{e,\sigma}. \quad (\text{G.20})$$

In particular, for every $\epsilon > 0$, the original field satisfies the explicit law-level estimate

$$\mathbb{P}_{\mathcal{A}} \left(\sum_{e>K,\sigma} w_{e,\sigma} \|Z_{M,e,\sigma}\|_{-s}^2 > \epsilon \right) \leq \chi_M + \frac{\kappa_s(1 + Cr_M)}{\epsilon} \sum_{e>K,\sigma} w_{e,\sigma} \gamma_{e,\sigma}. \quad (\text{G.21})$$

This is the required uniform tail control. It is not obtained by transferring an unbounded target moment.

For any finite collection of types, Lemma G.2 gives independent Poisson cell masses and hence independent Gaussian cell limits. The spatial remainder follows from (G.15) and (G.17), as in the finite-family proof in Appendix G.3: for one type it is bounded by $CS_s(R)(\gamma_\alpha + r_M \gamma_\alpha^2)$. At fixed R , pass to the Gaussian cell vector; then send $R \rightarrow \infty$. This proves the finite-type convergence in products of H^{-s} .

Finite-type tightness together with (G.20) proves tightness in $\mathcal{H}_{w,s}$, by approximation with the finitely many coordinates $e \leq K$. The Gaussian limit has the same vanishing tails, since its expected squared norm is at most $\kappa_s \Gamma(w)$. The finite projections identify the unique limit. Lemma G.1 transfers the complete modified spectrum to the original one, proving (7.17).

For necessity, look only at the constant spatial mode. Its squared weighted Gaussian norm is $\sum_{\alpha} w_{\alpha} \gamma_{\alpha} g_{\alpha}^2$ with independent standard normals. If $\sum_{\alpha} w_{\alpha} \gamma_{\alpha} = \infty$, then for every $t > 0$

$$\mathbb{E} \exp \left(-t \sum_{\alpha} w_{\alpha} \gamma_{\alpha} g_{\alpha}^2 \right) = \prod_{\alpha} (1 + 2t w_{\alpha} \gamma_{\alpha})^{-1/2} = 0.$$

The series is therefore infinite almost surely. No tight limit in this weighted Hilbert space can have the asserted finite-coordinate laws. $\square$

G.4.1 Integer thresholds

Proof of Corollary 7.9. Tail summation is continuous from a positively exponentially weighted exact-level space into every polynomially weighted threshold space: for $\eta_0 > 0$,

$$\left\| \sum_{e \geq j} F_e \right\|_{-s}^2 \leq C_{\eta_0} 2^{-\eta_0 j} \sum_e 2^{\eta_0 e} \|F_e\|_{-s}^2.$$

Apply Theorem 7.8. For the standardized assertion, use (G.15) with the coefficients $2^{(j+1)/2} \mathbf{1}_{\{e \geq j, \text{sign}=\sigma\}} q_k(x)$. The diagonal type sum equals one, and the squared first-weight sum is 2^{-j-1} . Thus the modified threshold coordinates have uniformly bounded H^{-s} second moments. Summability of v , the finite-threshold limits, and the full coupling prove the claim. This does not assert Gaussianity of individual moving thresholds near $j = \log_2 \Lambda$. $\square$

G.5 Fixed-window spatial Ornstein–Uhlenbeck paths

Attach canonical independent marks (G.11) to each actual atom, and write $\tau = e + U$. For one sign let

$$S_M^{\sigma}(t) = \sum_{j,e} I_{j,e,\sigma} \mathbf{1}_{\{e+U_{j,e,\sigma} > t\}} \delta_{j/n}, \quad H_M^{\sigma}(t) = \frac{S_M^{\sigma}(t) - (\Lambda/2)2^{-t}\mu_n}{\sqrt{(\Lambda/2)2^{-t}}}. \quad (\text{G.22})$$

The marks for every occurrence of a filling atom are independent as well. Sharing marks at equal count coordinates preserves the full coupling on its success event. At integer times these survivors equal the intrinsic integer threshold fields; noninteger times use the declared auxiliary interpolation.

Lemma G.4 (A Hilbert-valued empirical-bridge bound). *Let $a_1, \dots, a_N$ be fixed vectors in a real Hilbert space, and let V_i be independent uniforms on $(0, 1)$. Then*

$$\mathbb{E} \sup_{0 \leq v \leq 1} \left\| \sum_{i=1}^N a_i (\mathbf{1}_{\{V_i \leq v\}} - v) \right\|^2 \leq 16 \sum_{i=1}^N \|a_i\|^2. \quad (\text{G.23})$$

The assertion holds conditionally when the finite vector list is random but independent of the uniforms.

Proof. Introduce independent ghost uniforms V'_i . Jensen's inequality replaces the centered sum by the difference of the two empirical sums. Independently symmetrize each difference with a Rademacher sign. The squared triangle inequality and exchangeability of the two copies bound the result by $4\mathbb{E} \sup_v \left\| \sum_i \epsilon_i a_i \mathbf{1}_{\{V_i \leq v\}} \right\|^2$. Conditional on the uniforms, order their distinct values. The partial sums of the correspondingly ordered $\epsilon_i a_i$ form a Hilbert martingale. Doob's L^2 inequality bounds its squared maximum by $4 \sum_i \|a_i\|^2$. This proves (G.23). $\square$

Proposition G.5 (Uniform-in-time spatial reconstruction). *For each fixed $T < \infty$, $s > 1/2$, and sign σ , the modified survivor process obeys*

$$\mathbb{E}_{\mathcal{A}} \sup_{0 \leq t \leq T} \left\| \hat{H}_M^\sigma(t) - (\pi_R)_* \hat{H}_M^\sigma(t) \right\|_{-s}^2 \leq C_{T,s}(1 + r_M) S_s(R). \quad (\text{G.24})$$

Proof. Let $q_e(t) = \mathbb{P}(e + U > t)$, and put

$$B_{M,e,\sigma}(t) = \frac{1}{\sqrt{\Lambda}} \sum_{\text{modified atoms of type } (e,\sigma)} (\mathbf{1}_{\{e+U>t\}} - q_e(t)) \delta_x.$$

Since $\sum_e \gamma_{e,\sigma} q_e(t) = 2^{-t}/2$, there is the exact decomposition

$$\hat{H}_M^\sigma(t) = \sqrt{2} 2^{t/2} \sum_{e \geq 0} \left[q_e(t) \hat{Z}_{M,e,\sigma} + B_{M,e,\sigma}(t) \right]. \quad (\text{G.25})$$

The sum of the L^2 norms of its exact-level terms is finite by $\sum_e \sqrt{\gamma_{e,\sigma}} < \infty$. For the projected exact-level difference, (G.15) gives

$$\mathbb{E}_{\mathcal{A}} \left\| \hat{Z}_{M,e,\sigma} - (\pi_R)_* \hat{Z}_{M,e,\sigma} \right\|_{-s}^2 \leq C(\gamma_{e,\sigma} + r_M \gamma_{e,\sigma}^2) S_s(R).$$

Conditional on the entire modified unmarked field, apply Lemma G.4 after transforming U by its continuous distribution function, with vectors $(\delta_x - \delta_{\pi_R x})/\sqrt{\Lambda}$. Their squared norms are at most $CS_s(R)/\Lambda$. The exact modified expected number of type- (e, σ) atoms is $\Lambda \gamma_{e,\sigma}$. Hence

$$\mathbb{E}_{\mathcal{A}} \sup_t \|B_{M,e,\sigma}(t) - (\pi_R)_* B_{M,e,\sigma}(t)\|_{-s}^2 \leq C \gamma_{e,\sigma} S_s(R).$$

Minkowski's inequality in L^2 of the time supremum, $0 \leq q_e(t) \leq 1$, and $2^{t/2} \leq 2^{T/2}$ now prove (G.24). This uses independence of the auxiliary marks only. It gives no death martingale for the arithmetic field itself. $\square$

Proof of Theorem 7.10. At fixed R , Lemma G.2 and the canonical mark kernel give a total-variation comparison of the *entire projected survivor path* to a Poisson death field at the points r/R , with mean weights $w_{n,r}/2$ for each sign. The finite-dimensional martingale proof in Appendix C.4 gives its OU limit, with spatial masses of variance $1/R$. The argument uses the standard martingale functional CLT for this *target*, not for the arithmetic process; see also [8, Theorem 2.1(ii)].

Let $H_R^\sigma(t) = \sum_r H^\sigma(t; \mathbf{1}_{B_r}) \delta_{r/R}$, using the isonormal Gaussian realization on cell indicators. To compare it to H^σ , the spatial kernel

$$K_R f = \int_0^1 f(x) (\delta_x - \delta_{\pi_R x}) dx$$

has squared Hilbert–Schmidt norm at most $CS_s(R)$. Its application to the OU initial noise and Brownian driver, followed by Doob's inequality, gives $\mathbb{E} \sup_{t \leq T} \|H^\sigma(t) - H_R^\sigma(t)\|_{-s}^2 \leq C_{T,s} S_s(R)$. Together with Proposition G.5, this proves convergence by taking $M \rightarrow \infty$ at fixed R , then $R \rightarrow \infty$. The J_1 metric is bounded by uniform distance when the identity time change is used. The full coupling returns to the original field.

On a fixed partition, finitely many exact levels and times are compensated Poisson integrals on the common position–lifetime cloud. Their cubic integrals tend to zero, and their joint covariances are precisely those in (7.19). The weighted level tails and the path remainder already proved give joint tightness and completion. This proves the last assertion without inventing independent couplings for the two blocks. $\square$

G.6 Compression through the critical endpoint

Write $\Lambda = 2^{J+\theta_M}$, $J = \lfloor \log_2 \Lambda \rfloor \rightarrow \infty$, and $\theta_M \in [0, 1)$. With the sign-specific normalization (G.22), define

$$\mathcal{D}_M^\sigma = \frac{1}{\sqrt{J}} \sum_{j=0}^J H_M^\sigma(j) \otimes \delta_{j/J}, \quad \mathcal{C}_M^\sigma = \frac{1}{\sqrt{J}} \int_0^J H_M^\sigma(t) \otimes \delta_{t/J} dt. \quad (\text{G.26})$$

The second expression is equivalently $\sqrt{J} H_M^\sigma(Ju) du$. Let

$$\Xi_M = \sum_{j,e,\sigma} I_{j,e,\sigma} \delta_{(j/n, e-J, \sigma)}. \quad (\text{G.27})$$

Since $J = \lfloor \log_2 n \rfloor - L$, this is the critical centering of the article. Along $\theta_M \rightarrow \theta \in [0, 1]$, the target is

$$\Pi_\theta \sim \text{PPP} \left(dx \sum_{r \in \mathbb{Z}, \sigma} 2^{\theta-r-2} \delta_{(r, \sigma)} \right). \quad (\text{G.28})$$

The endpoint $\theta = 1$ has its usual integer level shift; it is not identified with phase zero without that shift.

G.6.1 Coefficient estimates and the spatial remainder

The coefficient kernels and their complete geometric bounds are (F.5)–(F.10) in Appendix F.4. They are target calculations, and so belong to the principal route as well. Applied here to the modified-field inequality (G.14), their first-weight bound charges $O(r_M/J)$ to the off-site covariance error, not $O(\Lambda r_M)$.

Proposition G.6 (Spatial reconstruction for compression). *For $s_x, s_u > 1/2$, either modified compression satisfies*

$$\mathbb{E}_{\mathcal{A}} \left\| \widehat{\mathcal{D}}_M^\sigma - (\pi_R)_* \widehat{\mathcal{D}}_M^\sigma \right\|_{-s_x, -s_u}^2 \leq C_{s_x, s_u} (1 + r_M/J) S_{s_x}(R), \quad (\text{G.29})$$

and the same bound holds with $\mathcal{C}$ in place of $\mathcal{D}$. Projection acts on the spatial coordinate and its compensator only.

Proof. Test with $(q_k - q_k \circ \pi_R)(x) q_\ell(u)$. Apply Proposition G.3, (F.8), and (F.9). The resulting second moment is at most $C \min(1, k/R)^2 (1 + r_M/J)$, uniformly in ℓ ; the constant spatial mode is unchanged. Sum with the tensor weights. The temporal series is summable and the spatial series is $S_{s_x}(R)$. $\square$

G.6.2 Direct completion of the joint compression law

Alternative proof of Theorem 7.11. At fixed R , Lemma G.2 and the common canonical mark kernel compare the projected derived variables jointly. The Poisson target has spatial control measure $\mu_{n,R}$, tending to $\mu_R = R^{-1} \sum_r \delta_{r/R}$. The target calculation in Appendix F.5, including (F.12) and the compensated deletion of critical atoms, applies with this spatial measure. It gives the mixed compression law and its independence from the critical layer on every fixed partition. The covariance kernels are the corresponding spatial projections of the limits there.

Spatial completion for the joint law. Proposition G.6 controls the two Gaussian remainders uniformly in J . The limiting tensor white noises have the corresponding spatial projection error $CS_{s_x}(R)$ by Gaussian isometry. For the critical field, on a fixed compact mark set

$F \subset \mathbb{Z} \times \{-1, +1\}$, a spatial Lipschitz test changes by at most $R^{-1}\Xi_M([0, 1] \times F)$. These counts are tight by Lemma G.2 with $R = 1$, not by a transferred moment estimate. Hence the critical spatial remainder vanishes in probability as $R \rightarrow \infty$, uniformly in the preceding M -limit. An exhaustion by finite mark sets gives the vague metric assertion. Now take $M \rightarrow \infty$ at fixed R , and then $R \rightarrow \infty$, for bounded Lipschitz tests of the *joint* product space. Lemma G.1 returns to the original marked arithmetic field. This proves Theorem 7.11 and its independence statements. $\square$

G.6.3 The common three-block limit

Proof of Corollary 7.12. At fixed R , use the same marked cell coupling. For any fixed exact level, its target covariance with either compressed pairing is $O_e(J^{-1/2})$. For a fixed threshold time $t \leq T$, the covariance is also $O_T(J^{-1/2})$, since $\sum_{j \geq 0} e^{-a|t-j|}$ and $\int_0^\infty e^{-a|t-u|} du$ are bounded on fixed t -intervals. Joint compensated-Poisson characteristic exponents have vanishing cubic remainders; hence these zero limiting cross covariances identify independent Gaussian blocks.

For the critical product, delete the upper atoms as in Theorem 7.11. Fixed exact levels are unaffected eventually. On $[0, T]$, the upper atoms and their compensator change the standardized survivor path by $O_{\mathbb{P}}(\Lambda^{-1/2})$ for fixed K . Thus they do not affect the low Gaussian block. Complete the exact-level coordinates using (G.20), the path using (G.24), and the compressed fields using (G.29). Joint tightness and identification of finite projections prove the result. This argument does not claim independence of prelimit objects. $\square$

G.7 The simultaneous rectangle transfer

This section gives an alternative one-factor proof of the arithmetic rectangle limit without using Input 2.1. It complements the complete-field route in Appendix D.3; it does not improve that route's current information budget. Neither argument uses a trace map in a negative-Sobolev space. Let $K \subset (-\infty, 0] \times \mathbb{R}$ be a compact rectangle of positive side lengths, and choose integers $k_M \rightarrow \infty$ such that

$$2^{k_M} \max_{(a,b) \in K} 2^{-b} < \Lambda \quad \text{eventually,} \quad 2^{k_M}/n^2 \rightarrow 0. \quad (\text{G.30})$$

The second condition is automatic in the present arithmetic range but is displayed to preserve the simultaneous-grid bookkeeping. Put $d_M = \log_2 \Lambda - k_M$, and define

$$\begin{aligned} N_M^\sigma(a, b) &= \sum_{j,e} I_{j,e,\sigma} \mathbf{1}_{\{j/n \leq 2^a, \Lambda 2^{-(e+U_{j,e,\sigma})} < 2^{k_M-b}\}}, \\ \mathcal{R}_M^\sigma(a, b) &= \frac{N_M^\sigma(a, b) - \frac{1}{2} 2^{k_M+a-b}}{\sqrt{\frac{1}{2} 2^{k_M+a-b}}}. \end{aligned} \quad (\text{G.31})$$

Proof of Theorem 7.13. First keep the exact grid compensator. Replace the continuous mean in (G.31) temporarily by $\frac{1}{2} 2^{k_M-b} \mu_n([0, 2^a])$, keeping its denominator unchanged. For the modified field the resulting centered variable is $T_M(h)$ with, on sign σ ,

$$h_{j,e,\sigma}(u) = \sqrt{2} 2^{(d_M+b-a)/2} \mathbf{1}_{\{j/n \leq 2^a\}} \mathbf{1}_{\{e+u > d_M+b\}}.$$

The condition (G.30) makes $t = d_M + b > 0$. Since $\sum_e \gamma_{e,\sigma} \mathbb{P}(e + U > t) = 2^{-t}/2$, its diagonal bound is $2^{-a} \mu_n([0, 2^a]) \leq C_K$, and its squared first-weight bound is $2^{-d_M-b-a-1} \leq C_K$. The latter is uniformly bounded because $d_M + b > 0$.

Project the centered field, not the raw count alone. After moving j/n to $\pi_R(j/n)$, the difference of the two prefix indicators is supported in one cell, containing at most $n/R + 1$

sites. Use the projected target prefix mass for the projected compensator. Proposition G.3 gives, uniformly on K ,

$$\mathbb{E}_{\mathcal{A}} |\widehat{\mathcal{R}}_M^{\text{grid},\sigma}(a,b) - \widehat{\mathcal{R}}_{M,R}^{\text{grid},\sigma}(a,b)|^2 \leq C_K \left(R^{-1} + n^{-1} + r_M 2^{-d_M} \right) \leq C_K \left(R^{-1} + n^{-1} + r_M \right). \quad (\text{G.32})$$

The first-weight error need not tend to zero with R ; it is multiplied by $r_M \rightarrow 0$. Integrating (G.32) over K gives an $L^2(K)$ remainder, and hence a remainder in every stated weaker tensor Sobolev norm. It also gives the required finite-point remainder directly.

Identify the projected target. At fixed R , Lemma G.2 supplies the common marked Poisson target with spatial control $\mu_{n,R}$. The target calculation of Appendix F.7 applies with this measure: the intensity is $\frac{1}{2}\mu_{n,R}(dx)dv$ for $v < \Lambda$, and all queried rectangles lie below the cutoff. The same $O_K(2^{-k_M/2})$ cubic remainder and uniform coefficient variances give joint Gaussian finite-test limits and tensor-Sobolev tightness.

At fixed R , the limiting covariance is

$$2^{-(a+c)/2} \mu_R([0, 2^{\min(a,c)}]) 2^{-|b-d|/2}, \quad \mu_R = \frac{1}{R} \sum_{r=1}^R \delta_{r/R}.$$

It tends to the displayed covariance. More strongly, realize these Gaussian targets from a common white noise on (x, z) by replacing x with $\pi_R x$ in the rectangle indicator. Gaussian isometry bounds their squared pointwise and integrated discrepancies by C_K/R . This proves their finite-dimensional and distribution-space convergence as $R \rightarrow \infty$.

Restore the stated continuous mean. The difference between the exact grid mean and $\frac{1}{2}2^{k_M+a-b}$, after normalization, has magnitude at most $C_K 2^{k_M/2}/n$. Its squared pointwise and integrated errors vanish by (G.30). This is the deterministic squared-bias term that must not be dropped. Combining this estimate, (G.32), the fixed-cell coupling, and Lemma G.1, first at fixed R and then as $R \rightarrow \infty$, proves the theorem. $\square$

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