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Values of the Huang–Ma invariant

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Abstract

We prove the binary-value conjecture of Huang and Ma for integer matrices admitting a fixed positive diagonal skew-symmetrizer. We determine the exact set of values in every size and construct matrices realizing each value.

1 Introduction

Skew-symmetrizable integer matrices are the exchange matrices of cluster algebra theory. Numerical invariants of matrix mutation provide obstructions to mutation equivalence. Casals introduced an invariant obtained by reducing a symmetrized determinant modulo four in the skew-symmetric case [2, Definition 1.3 and Theorem 1.6]. Seven and Ünal classified symmetric integer matrices with even diagonal modulo four under integral congruence [4, Theorems 1.1–1.2]. Huang and Ma extended Casals' construction to skew-symmetrizable matrices and conjectured that, for a fixed size and a fixed skew-symmetrizer, its image has at most two elements [3, Remark 2.7].

Fix an integer $n \geq 1$ and positive integers $d_1, \dots, d_n$. All matrix indices lie in $\{1, \dots, n\}$. Let $M_n(\mathbb{Z})$ denote the set of $n \times n$ integer matrices and $^{\mathsf{T}}$ denote transpose. Write

$$D = \operatorname{diag}(d_1, \dots, d_n), \qquad \mathcal{B}_D = \{B \in M_n(\mathbb{Z}) : DB = -B^{\mathsf{T}} D\},$$

Thus D is a skew-symmetrizer of each $B \in \mathcal{B}_D$. If $B = (b_{ij}) \in \mathcal{B}_D$, then $b_{ii} = 0$ and $d_i b_{ij} = -d_j b_{ji}$.

Let $\operatorname{sgn}(x)$ equal 1, 0, or -1 according as x is positive, zero, or negative. Define the real symmetric matrix $\mathfrak{S}(B) = (s_{ij})$ by

$$s_{ii} = 2, \qquad s_{ij} = s_{ji} = \operatorname{sgn}(b_{ij}) \sqrt{|b_{ij} b_{ji}|} \quad (i < j), \tag{1}$$

Its determinant is an integer by Lemma 2.1. Following [3, Definition 1.3], the Huang–Ma invariant is

$$\delta(B) = \det \mathfrak{S}(B) \pmod 4 \ \in \mathbb{Z}/4\mathbb{Z}.$$

We identify the elements of $\mathbb{Z}/4\mathbb{Z}$ with their representatives $0, 1, 2, 3$ when listing sets, and put

$$\operatorname{Im} \delta_D = \{\delta(B) : B \in \mathcal{B}_D\}.$$

For example, if $D = \operatorname{diag}(1, 2)$, then every member of $\mathcal{B}_D$ has the form

$$B = \begin{pmatrix} 0 & 2t \\ -t & 0 \end{pmatrix}, \qquad t \in \mathbb{Z},$$

and $\det \mathfrak{S}(B) = 4 - 2t^2$, so that $\operatorname{Im} \delta_D = \{0, 2\}$.

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Theorem 1.1. *Suppose $B \in \mathcal{B}_D$. If $n = 2m$ for a positive integer m, then*

$$\delta(B) \equiv (-1)^m \det B \pmod 4. \tag{2}$$

In this even-dimensional case, write $\prod_{i=1}^n d_i = sa^2$, where a is a positive integer and s is a positive squarefree integer. There is an integer z such that $\det B = sz^2$. Consequently,

$$\operatorname{Im} \delta_D \subseteq \{0, (-1)^m s \pmod 4\}.$$

If n is odd, then $\operatorname{Im} \delta_D \subseteq \{0, 2\}$. In particular, $|\operatorname{Im} \delta_D| \leq 2$ for every n and D.

For a prime ℓ and a nonzero integer x, let $v_\ell(x)$ be the largest nonnegative integer r such that ℓ^r divides x; set $v_\ell(0) = +\infty$. The exact image depends on the two-adic valuations of the diagonal entries. Write

$$d_i = 2^{\nu_i} u_i, \qquad \nu_i = v_2(d_i), \quad u_i \text{ odd and positive}, \qquad U = \prod_{i=1}^n u_i.$$

When $n = 2m$, list the valuations in nondecreasing order as $\nu_{(1)}, \ldots, \nu_{(2m)}$ and define

$$\mu(D) = \sum_{j=1}^{m} \bigl(\nu_{(2j)} - \nu_{(2j-1)}\bigr). \tag{3}$$

The valuation level $r \geq 0$ consists of the indices i with $\nu_i = r$. Writing $\#$ for cardinality, let

$$o(D) = \#\{r \in \mathbb{Z}_{\geq 0} : \#\{i : \nu_i = r\} \text{ is odd}\}.$$

Thus $o(D)$ counts the levels of odd cardinality.

Theorem 1.2. *For $n = 2m \geq 2$, the exact image is*

$$\operatorname{Im} \delta_D = \begin{cases} \{0, (-1)^m U \pmod 4\}, & \mu(D) = 0, \\ \{0, 2\}, & \mu(D) = 1, \\ \{0\}, & \mu(D) \geq 2. \end{cases}$$

For odd $n \geq 3$, the exact image is

$$\operatorname{Im} \delta_D = \begin{cases} \{0, 2\}, & o(D) = 1, \\ \{0\}, & o(D) \geq 3. \end{cases}$$

For $n = 1$, the image is $\{2\}$.

2 The determinant congruence

Lemma 2.1. *For $B \in \mathcal{B}_D$, define the integer matrix $C = (c_{ij})$ by*

$$c_{ij} = \begin{cases} 2, & i = j, \\ b_{ij}, & i < j, \\ -b_{ij}, & i > j. \end{cases}$$

Then C is similar over $\mathbb{R}$ to $\mathfrak{S}(B)$ and satisfies

$$C^{\mathsf{T}} = DCD^{-1}, \qquad C \equiv B \pmod 2. \tag{4}$$

In particular, $\det \mathfrak{S}(B) = \det C \in \mathbb{Z}$.

Proof. Let $H = \operatorname{diag}(\sqrt{d_1}, \ldots, \sqrt{d_n})$. For $i < j$, the relation $d_i b_{ij} = -d_j b_{ji}$ gives

$$s_{ij} = \sqrt{d_i/d_j}\, b_{ij}.$$

For $i < j$, the (i,j)-entry of $H^{-1}\mathfrak{S}(B)H$ is b_{ij}, while its (j,i)-entry is $(d_i/d_j)b_{ij} = -b_{ji}$. Its diagonal entries are 2, so it equals C. The relation $d_i c_{ij} = d_j c_{ji}$ gives the first identity in (4); the second follows by reducing the entries modulo two. □

For $X \in M_n(\mathbb{Z})$, its (i,j)-cofactor is $(-1)^{i+j}$ times the determinant obtained by deleting row i and column j, with the empty determinant equal to 1. Let $\operatorname{adj}(X)$ denote the transpose of the cofactor matrix, and let $\operatorname{tr}(X)$ denote the sum of the diagonal entries. Cofactor expansion gives

$$\operatorname{adj}(X)X = X \operatorname{adj}(X) = (\det X) I_n,$$

where I_n is the identity matrix. The next lemma is a weighted version of the adjugate–trace argument of Auel, Biesel, and Voight [1, Proposition 3.20 and Lemma 3.21].

Lemma 2.2. *Let $n = 2m$, and let $B, C \in M_n(\mathbb{Z})$ satisfy*

$$B^{\mathsf{T}} = -DBD^{-1}, \qquad C^{\mathsf{T}} = DCD^{-1}, \qquad C \equiv B \pmod 2.$$

Then $\det C \equiv (-1)^m \det B \pmod 4$.

Proof. Write $C = B + 2L$ with $L \in M_n(\mathbb{Z})$ and set $A = \operatorname{adj}(B)$. Cofactors give $\operatorname{adj}(B^{\mathsf{T}}) = A^{\mathsf{T}}$ and $\operatorname{adj}(-B) = (-1)^{n-1}A = -A$. For an invertible matrix, $\operatorname{adj}(B) = \det(B)B^{-1}$ gives

$$\operatorname{adj}(DBD^{-1}) = D \operatorname{adj}(B) D^{-1}.$$

For an indeterminate t, the matrix $B + tI_n$ is invertible over the rational-function field $\mathbb{Q}(t)$. Applying the identity there and setting $t = 0$ proves it for every B. The assumed relation for B^{T} therefore yields

$$A^{\mathsf{T}} = -DAD^{-1}.$$

Taking the trace and using its invariance under transpose and similarity gives

$$\operatorname{tr}(AC) = \operatorname{tr}(C^{\mathsf{T}}A^{\mathsf{T}}) = -\operatorname{tr}(DCAD^{-1}) = -\operatorname{tr}(CA) = -\operatorname{tr}(AC).$$

It follows over $\mathbb{Q}$ that $\operatorname{tr}(AC) = 0$. Substitution of $C = B + 2L$ yields the integer equality

$$0 = n \det B + 2 \operatorname{tr}(AL). \tag{5}$$

In the column expansion of $\det(B + 2L)$, the terms using exactly one column from $2L$ sum to

$$2 \sum_{i,j=1}^{n} A_{ji} L_{ij} = 2 \operatorname{tr}(AL),$$

since A_{ji} is the (i,j)-cofactor of B. Every term using at least two columns from $2L$ is divisible by four. Thus

$$\det(B + 2L) \equiv \det B + 2 \operatorname{tr}(\operatorname{adj}(B)L) \pmod 4,$$

as in [1, Proposition 3.10]. Combining this with (5), we obtain

$$\det C \equiv (1 - n) \det B \equiv (-1)^m \det B \pmod 4.$$ □

For an integer $m \geq 1$, a perfect matching M of $\{1, \ldots, 2m\}$ is a partition into unordered pairs. Write its pairs as $\{i_1, j_1\}, \ldots, \{i_m, j_m\}$ with $i_r < j_r$ and $i_1 < \cdots < i_m$, and let $\epsilon(M)$ be the sign of the permutation with one-line notation $(i_1, j_1, \ldots, i_m, j_m)$. For a skew-symmetric integer matrix $K = (k_{ij})$ of size $2m$, define its Pfaffian by summing over all perfect matchings:

$$\operatorname{Pf}(K) = \sum_{M} \epsilon(M) \prod_{\{i,j\} \in M,\ i<j} k_{ij}.$$

For size four this is $k_{12}k_{34} - k_{13}k_{24} + k_{14}k_{23}$. We use the Pfaffian identity $\det K = \mathrm{Pf}(K)^2$; see [1, p. 10].

Proof of Theorem 1.1. Let C be the integer matrix in Lemma 2.1. When $n = 2m$, Lemma 2.2 proves (2). Since DB is an integer skew-symmetric matrix, its Pfaffian is an integer and

$$\mathrm{Pf}(DB)^2 = \det(DB) = sa^2 \det B.$$

For every prime ℓ, integrality of $\det B$ implies

$$v_\ell(\mathrm{Pf}(DB)) \geq \left\lceil \frac{v_\ell(sa^2)}{2} \right\rceil = v_\ell(sa)$$

when the Pfaffian is nonzero. The last equality uses that s is squarefree. Thus sa divides $\mathrm{Pf}(DB)$, and $\mathrm{Pf}(DB) = saz$ for an integer z, including $z = 0$ when the Pfaffian vanishes. Substitution gives $\det B = sz^2$. Squares modulo four are 0 or 1, which proves the image bound.

When n is odd, skew-symmetry gives $\det(DB) = \det(-(DB)^{\mathsf{T}}) = -\det(DB)$, hence $\det B = 0$. Lemma 2.1 gives $\det C \equiv \det B \equiv 0 \pmod 2$, so $\delta(B)$ is 0 or 2. □

3 Values in even size

For $i < j$, the integers $d_i/\gcd(d_i, d_j)$ and $d_j/\gcd(d_i, d_j)$ are coprime. Hence $d_i b_{ij} = -d_j b_{ji}$ forces $d_j/\gcd(d_i, d_j)$ to divide b_{ij}. There is a unique integer t_{ij} such that

$$b_{ij} = \frac{d_j}{\gcd(d_i, d_j)} t_{ij}, \qquad b_{ji} = -\frac{d_i}{\gcd(d_i, d_j)} t_{ij}. \tag{6}$$

Conversely, arbitrary choices of the t_{ij} satisfy the skew-symmetrizer relation independently on each pair. In particular,

$$(DB)_{ij} = \mathrm{lcm}(d_i, d_j) t_{ij} \quad (i < j). \tag{7}$$

Lemma 3.1. *For any nondecreasing list of real numbers $x_1 \leq \cdots \leq x_{2m}$,*

$$\min_M \sum_{\{i,j\} \in M} |x_i - x_j| = \sum_{r=1}^{m} (x_{2r} - x_{2r-1}),$$

where M runs over the perfect matchings of $\{1, \ldots, 2m\}$.

Proof. For a matching M and $1 \leq k < 2m$, let N_k be the number of pairs with one index at most k and the other larger than k. Expanding each distance across the consecutive gaps gives

$$\sum_{\{i,j\} \in M} |x_i - x_j| = \sum_{k=1}^{2m-1} N_k (x_{k+1} - x_k).$$

The first k indices have N_k partners outside that set and all other indices paired inside, so $N_k \equiv k \pmod 2$. Thus $N_k \geq 1$ for odd k and $N_k \geq 0$ for even k. The adjacent matching $\{1, 2\}, \{3, 4\}, \ldots, \{2m-1, 2m\}$ attains all these bounds. □

Proposition 3.2. *If $n = 2m$, then $v_2(\det B) \geq \mu(D)$ for every $B \in \mathcal{B}_D$. There exists $B \in \mathcal{B}_D$ with equality.*

Proof. Using (7), expand $\mathrm{Pf}(DB)$ as an integer polynomial in the parameters t_{ij}. Distinct matchings give distinct monomials. For a perfect matching M, the coefficient of

$\prod_{\{i,j\}\in M, i<j} t_{ij}$ is $\epsilon(M)\prod_{\{i,j\}\in M}\operatorname{lcm}(d_i,d_j)$ and has valuation

$$\sum_{\{i,j\}\in M}\max(\nu_i,\nu_j)=\frac{1}{2}\left(\sum_i \nu_i+\sum_{\{i,j\}\in M}|\nu_i-\nu_j|\right).$$

The minimum of this valuation is $(\sum_i \nu_i+\mu(D))/2=\sum_{j=1}^{m}\nu_{(2j)}$ by Lemma 3.1. Thus every matching coefficient, and hence $\operatorname{Pf}(DB)$ after any integer choice of the parameters, is divisible by $2^{\sum_j \nu_{(2j)}}$. If the Pfaffian vanishes, then $\det B=0$. Otherwise, $\operatorname{Pf}(DB)^2=(\prod_i d_i)\det B$ gives

$$v_2(\det B)=2v_2(\operatorname{Pf}(DB))-\sum_i \nu_i\geq 2\sum_j \nu_{(2j)}-\sum_i \nu_i=\mu(D).$$

Choose a matching of the original indices by pairing adjacent entries in the sorted valuation list. Set $t_{ij}=1$ on its pairs, using $i<j$ in (6), and set every other parameter to zero. Only the chosen matching contributes to the Pfaffian, and its coefficient has the minimum valuation. Hence $v_2(\det B)=\mu(D)$. □

Proof of Theorem 1.2 in even size. If $\mu(D)\geq 2$, Proposition 3.2 and (2) give $\delta(B)=0$ for all B. If $\mu(D)=1$, the same results give $\delta(B)\in\{0,2\}$, and the matrix constructed in that proposition realizes 2.

Suppose $\mu(D)=0$. Adjacent valuations in the sorted list are then equal, so $\sum_i \nu_i$ is even. Write $\prod_i d_i=sa^2$ as in Theorem 1.1. Then s is odd and $a/2^{\frac{1}{2}\sum_i \nu_i}$ is an odd positive integer. Comparing odd parts gives

$$U=s\left(\frac{a}{2^{\frac{1}{2}\sum_i \nu_i}}\right)^2\equiv s \pmod 4.$$

Theorem 1.1 therefore bounds the image by $\{0,(-1)^m U\}$. The constructed matrix has odd determinant and realizes the nonzero residue. Finally, $B=0$ realizes 0 in every even size, since $\det\mathfrak{S}(0)=2^n$ is divisible by four. □

4 Values in odd size

We use the integer matrix C from Lemma 2.1. The parameterization (6) implies, for $i\neq j$,

$$v_2(c_{ij})=\max(\nu_j-\nu_i,0)+v_2(t_{\min(i,j),\max(i,j)}). \tag{8}$$

In particular, $c_{ij}c_{ji}$ can be odd only if $\nu_i=\nu_j$. For distinct indices $i_1,\dots,i_r$ with $r\geq 3$, the cyclic product

$$c_{i_1i_2}c_{i_2i_3}\cdots c_{i_ri_1}$$

can be odd only if $\nu_{i_1}=\cdots=\nu_{i_r}$: otherwise some step around the cycle strictly increases the valuation and contributes an even entry by (8).

Lemma 4.1. *If $n\geq 3$ is odd and $o(D)\geq 3$, then* $\det C\equiv 0 \pmod 4$.

Proof. Expand $\det C$ over permutations. For each permutation, let f be its number of fixed points and h its number of cycles of length at least three. Group permutations by independently reversing these h cycles. Each differs from its inverse, and their supports are disjoint, so every group has exactly 2^h elements. For a cycle $(i_1\cdots i_r)$, put $i_{r+1}=i_1$. Weighted symmetry gives

$$\prod_{a=1}^{r}c_{i_{a+1}i_a}=\left(\prod_{a=1}^{r}\frac{d_{i_a}}{d_{i_{a+1}}}\right)\prod_{a=1}^{r}c_{i_ai_{a+1}}=\prod_{a=1}^{r}c_{i_ai_{a+1}}.$$

Reversal preserves the permutation sign. Since each fixed point contributes 2, the group contributes 2^{f+h} times a signed integer product.

Only groups with $f + h \leq 1$ can survive modulo four. The case $f + h = 0$ consists entirely of transpositions and has even size. Hence the remaining types are $(f, h) = (1, 0)$ and $(f, h) = (0, 1)$. The first has one fixed point and all other cycles of length two. Its contribution is twice a signed product of the factors $c_{ij}c_{ji}$. For that product to be odd, each pair must lie in one valuation level. After removing the fixed point, every level would then have even cardinality, which forces $o(D) = 1$.

In the second type the unique long cycle has odd length. The contribution is twice its signed cyclic product times the products for the pairs. For this product to be odd, the long cycle must lie in one valuation level, as must each pair. The long cycle gives that level odd cardinality, while every other level has even cardinality. Thus both surviving types require $o(D) = 1$, so every group vanishes modulo four under the stated hypothesis. □

Proof of Theorem 1.2 in odd size. The sum of the cardinalities of the valuation levels is odd, so $o(D)$ is a positive odd integer. If $o(D) \geq 3$, Lemma 4.1 gives the claimed image. The zero matrix realizes 0 since $n \geq 3$.

Suppose $o(D) = 1$. Leave one index from the unique level of odd cardinality unpaired, and pair the remaining indices within their levels. Set the parameters of these pairs to one in (6) and every other parameter to zero. After simultaneously reordering rows and columns, C consists of one block (2) and blocks of the form

$$\begin{pmatrix} 2 & d_j/\gcd(d_i, d_j) \\ d_i/\gcd(d_i, d_j) & 2 \end{pmatrix}, \qquad \nu_i = \nu_j.$$

The ratios $d_i/\gcd(d_i, d_j)$ and $d_j/\gcd(d_i, d_j)$ are odd because $\nu_i = \nu_j$. Each block therefore has odd determinant $4 - d_i d_j/\gcd(d_i, d_j)^2$, so $\delta(B) = 2$. The zero matrix and Theorem 1.1 give the image $\{0, 2\}$. For $n = 1$, the only admissible matrix is (0) and its symmetrization is (2). □

5 Examples

The following even-size examples illustrate the three cases of Theorem 1.2. A tuple in the first column specifies the diagonal of D.

(d_1, d_2, d_3, d_4)	Sorted valuations	$\mu(D)$	$\operatorname{Im}\delta_D$
$(1, 1, 1, 1)$	$(0, 0, 0, 0)$	0	$\{0, 1\}$
$(1, 2, 3, 5)$	$(0, 0, 0, 1)$	1	$\{0, 2\}$
$(1, 3, 3, 3)$	$(0, 0, 0, 0)$	0	$\{0, 3\}$
$(1, 2, 4, 8)$	$(0, 1, 2, 3)$	2	$\{0\}$

For $D = \operatorname{diag}(1, 3, 3, 3)$, take $b_{12} = 3$, $b_{21} = -1$, $b_{34} = 1$, $b_{43} = -1$, and every other entry zero. Then $\det B = 3$, so $\delta(B) = 3$.

In odd size, $D = \mathrm{diag}(1, 2, 4)$ has three valuation levels of odd cardinality, and hence δ vanishes identically. For $D = \mathrm{diag}(1, 2, 3)$, the level cardinalities are two and one, so the image is $\{0, 2\}$. Taking $b_{13} = 3$, $b_{31} = -1$, and all other entries zero gives $\det \mathfrak{S}(B) = 2(4 - 3) = 2$.

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