

A construction of the sphere-packing magic functions in dimensions 8 and 24

Abstract

We develop an alternative route to the solutions of the sphere-packing problem in dimensions 8 and 24, through the auxiliary functions in the Cohn–Elkies Fourier bound. In each dimension we construct a radial Schwartz function with the prescribed lattice zeros by solving a countable system of linear equations in absolutely summable coefficients. We prove existence and uniqueness after normalization, without using the known magic functions or Fourier interpolation. Through a Cauchy boundary problem, solutions of the homogeneous equations are identified with global sections of a rank-six holomorphic vector bundle over the compactified modular curve of the full modular group, with its orbifold structure retained. Riemann–Roch shows that this bundle has Euler characteristic one, forcing a nonzero global section and hence a nontrivial solution of the equations. A separate argument proves uniqueness after normalization. An equivalent positive operator equation has a bounded inverse. The construction is motivated by the coefficient method for planar circle packing. A subsequent comparison identifies our solutions with the functions of Viazovska and of Cohn, Kumar, Miller, Radchenko, and Viazovska. The route to sphere packing is not yet independent in its entirety: we do not prove the required global sign inequalities from the coefficient equations. Those inequalities are inherited from the known functions after identification.

1 Introduction

The Fourier bound of Cohn and Elkies [2] converts sphere packing into a problem about a function and its Fourier transform. A suitable auxiliary function gives an upper bound on the density of every packing. In dimensions 8 and 24, specially chosen functions make this bound equal to the densities of the E_8 and Leech lattices, respectively, and hence prove their optimality. Such sharp auxiliary functions are often called *magic functions*. This paper gives a construction of those functions from countably many linear equations.

There are two distinct tasks in a sharp Fourier bound: constructing the function with the required zeros and normalization, and proving its signs between those zeros. We give existence and uniqueness proofs for the coefficient systems in both dimensions. We then identify the resulting functions with the known magic functions. We do not give independent proofs of the global sign inequalities. Thus the construction part is independent of the earlier magic functions, but a complete sphere-packing proof still uses their established signs. Section 12 describes this remaining task and the methods used to address it in the planar case.

1.1 The functions to be constructed

Fix $d \in \{8, 24\}$. Let $\mathbf{y}, \boldsymbol{\xi} \in \mathbb{R}^d$ denote a spatial vector and a frequency vector, respectively. For a Schwartz function $f : \mathbb{R}^d \rightarrow \mathbb{C}$, define

$$\hat{f}(\boldsymbol{\xi}) = \int_{\mathbb{R}^d} f(\mathbf{y}) e^{-2\pi i \mathbf{y} \cdot \boldsymbol{\xi}} d\mathbf{y}. \quad (1.1)$$

Here $\mathbf{y} \cdot \boldsymbol{\xi}$ is the Euclidean inner product. A Schwartz function is smooth and, together with each derivative, decays faster than every inverse power of $|\mathbf{y}|$. We consider radial functions: $f(\mathbf{y})$ depends only on $|\mathbf{y}|$. Their Fourier transforms are radial as well.

For a radial f , define $\Phi, G : [0, \infty) \rightarrow \mathbb{C}$ by

$$f(\mathbf{y}) = \Phi(|\mathbf{y}|^2), \quad \hat{f}(\boldsymbol{\xi}) = G(|\boldsymbol{\xi}|^2). \quad (1.2)$$

The letters u, v denote independent real arguments in $[0, \infty)$ of Φ, G , respectively. In the displayed identities we evaluate these functions at $|\mathbf{y}|^2, |\boldsymbol{\xi}|^2$. Primes denote ordinary derivatives of the one-variable functions; in particular $\frac{d}{dr}\Phi(r^2) = 2r\Phi'(r^2)$.

Put $n_8 = 1$ and $n_{24} = 2$. In the normalization where each lattice has covolume one, its first nonzero squared length is $2n_d$. The Cohn–Elkies bound asks for a real radial Schwartz function satisfying

$$\Phi(0) = G(0) = 1, \quad \Phi(u) \leq 0 \quad (u \geq 2n_d), \quad G(v) \geq 0 \quad (v \geq 0). \quad (1.3)$$

It then bounds the packing density by the volume of a ball of radius $\sqrt{2n_d}/2$, which is precisely the density of the corresponding lattice packing.

The lattices are even, so their nonzero squared lengths belong to $2\mathbb{Z}$, starting at $2n_d$; see [16, Section 1] and [4, Section 1]. Equality in the Fourier bound forces both profiles to vanish at the lattice lengths. We impose these conditions at every node $2n$, $n \geq n_d$. The sign inequalities also force their derivatives to vanish, except possibly at the first zero of Φ , the endpoint of its required sign interval. The equalities to be imposed are therefore

$$\begin{aligned} \Phi(2n) &= 0 & (n \geq n_d), & & \Phi'(2n) &= 0 & (n > n_d), \\ G(2n) &= 0 & (n \geq n_d), & & G'(2n) &= 0 & (n \geq n_d). \end{aligned} \quad (1.4)$$

Our coefficient theorems construct normalized functions satisfying these equalities. From this point on, our proofs concern their construction and uniqueness; we do not return to an independent proof of the more general inequalities (1.3). The later comparisons with the known magic functions transfer the established signs to our solutions.

There are well-documented methods for proving the required inequalities. Viazovska’s original dimension-eight proof combines interval arithmetic with asymptotic expansions and rigorous remainder bounds near the two ends of the integral parameter range [16, Section 5]. In dimension 24, Cohn, Kumar, Miller, Radchenko, and Viazovska bound the errors in truncated Fourier series of modular forms and use Sturm’s theorem for real polynomials, with exact rational arithmetic, to certify the remaining signs [4, Appendix A]. Subsequent proofs replace these numerical certifications by structural arguments: Dan Romik proves the dimension-eight modular-form inequalities directly using modular identities, coefficient positivity, and special values [11]; Seewoo Lee gives algebraic proofs in both dimensions using differential identities, positivity under Serre differentiation, and monotonicity of ratios of modular forms [8, Sections 3, 5, and 6]. The recent planar construction instead uses certified polynomial approximations in the Bernstein basis, together with coefficient error bounds and estimates for the unbounded region [10, Section 4 and Appendices A–B]. Section 12 describes that Bernstein approach and what would be needed to apply it to the present coefficient construction.

1.2 The planar construction as the starting point

The guiding method comes from the planar circle-packing preprint of OpenAI [10, Sections 2, 3, and 5]. That paper starts with a trigonometric polynomial having double zeros at a periodic set

of nodes. Dividing by linear and quadratic factors permits changes to a value or first derivative at a chosen node. Gaussian damping turns these expressions into radial Schwartz functions. Their coefficients are determined by imposing Fourier conditions. Estimates on the entire coefficient space control the infinite system, and Poisson summation determines the normalization.

We adopt this sequence of ideas: choose a family with the required zeros, express the Fourier conditions as linear equations, control the infinite coefficient space, and recover normalization by Poisson summation. In both dimensions studied here, the evenly spaced nodes allow the common factor

$$P(u) = \sin^2(\pi u/2).$$

We combine it with integrals of Gaussians over $t \geq 1$ and $0 < t \leq 1$, where t is the Gaussian parameter. Additional functions permit the values and derivatives that must remain unconstrained at the first few nodes. There are five such correction functions in dimension 24 and three in dimension 8. Sections 2 and 11 specify the families, including the parity restrictions on their coefficients. These choices are part of the construction; they are not asserted to follow from the zero set alone.

Our method for solving the resulting system differs from that of the planar paper. Fourier samples become moments of two measures on a fixed contour. Their Cauchy integrals give a boundary problem, from which we derive translation and inversion laws. These laws lead to an index calculation and a vanishing argument. The planar paper thus supplies the starting construction strategy; the boundary and index analysis proves existence and uniqueness for the systems studied here.

1.3 Earlier constructions and the scope of the result

Viazovska constructed the eight-dimensional magic function from modular and quasimodular forms [16]. Cohn, Kumar, Miller, Radchenko, and Viazovska adapted this method to dimension 24 [4]. Earlier computations of Cohn and Miller [6] had revealed striking special values and suggested convergence of finite approximations.

Two subsequent constructions are especially close to the present work. Feigenbaum, Grabner, and Hardin [7] start with a holomorphic Laplace input satisfying specified endpoint conditions. Their Proposition 2.4 characterizes Fourier eigenfunctions by functional equations for that input. Proposition 3.1 and Section 4 identify the resulting quasimodular and level-two structures; Sections 3.1 and 4.1 use dimension counts, and Section 5 gives recurrences. Thus deriving modular structure from Fourier conditions is already part of that work. Rolin and Wagner [12, Theorems 1.1–1.2], with Theorem 1.2 read in its corrected form [13], construct families of radial Schwartz functions from modular and quasimodular forms, control their prescribed zeros, and recover the known functions in dimensions 8 and 24.

Our starting data are different: square-summable coefficients in an explicit elementary family, with discrete Fourier values and derivatives required to vanish. We do not assume a holomorphic Laplace input on the whole upper half-plane or an eigenfunction identity. The contribution proved here is the equivalence of this entire coefficient kernel with a space of global sections, including endpoint continuation and geometric coefficient decay. Once that equivalence is established, the familiar quasimodular and level-two structures reappear in its two blocks. The rank-six bundle and its index organize the resulting finite-dimensional problem; the novelty claimed is not a new source of modular structure or a replacement for the earlier modular classifications. The compact regularizer adds an explicitly specified bounded operation on arbitrary coefficient vectors, rather than merely an abstract projection onto the finite-dimensional solution space.

The subsequent Fourier interpolation theorem [5] determines a radial Schwartz function from its values and radial derivatives, and those of its Fourier transform, at the relevant lattice radii. Its proof develops the modular integral machinery of the sphere-packing constructions. The interpolation

paper first appeared in 2019 and was published in 2022; the sphere-packing constructions first appeared in 2016.

Our existence and uniqueness proofs in the chosen coefficient spaces use neither the known magic functions nor the interpolation theorem. They do use modular geometry: the boundary equations produce transformation laws, Riemann–Roch gives existence, and low-weight consequences of the valence formula give uniqueness. The known functions enter only in the later comparisons in Sections 10 and 11.6. Those comparisons transfer their signs to our functions. Interpolation also gives uniqueness in the larger class of all radial Schwartz functions with the prescribed data, whereas our independent uniqueness statements concern the specified coefficient families.

1.4 Strategy of the proof

The problem is to solve a countable system of linear equations: choose coefficients so that the Fourier values and derivatives in (1.4) vanish, and impose one scalar normalization. The zeros of Φ are already built into the chosen functions. Solving successively larger finite systems does not by itself produce a solution of all the equations, nor prove uniqueness. We control the entire coefficient family by interpreting the equations as an analytic continuation problem.

1. Build in the zeros of the function. Section 2 gives the explicit functions for dimension 24 and assigns a coefficient to each. The trigonometric factor supplies the zeros, and five correction functions permit the unconstrained values and derivatives. We seek absolutely summable coefficients but initially work in the larger Hilbert space of square-summable families. Section 3 proves that each family defines a radial Schwartz function. Poisson summation relates its normalization to the derivative at the first lattice radius. Dimension 8 uses the same steps with three correction functions.

2. Replace Fourier samples by moments. Lemma 4.1 expresses $G(2n)$ and $G'(2n)$ as integrals of w^{n-1} against two complex measures depending linearly on the coefficients. Their supports are fixed. In dimension 24, where $n \geq 2$, the equations prescribe the moments of positive degree. The constant moments remain unrestricted: setting them to zero would add equations at $v = 2$. In dimension 8, the samples start at $n = 1$, so the constant moments are prescribed as well. This distinction will change the allowed poles in the global problem.

3. Turn vanishing moments into a boundary problem. A Cauchy integral expanded at infinity reads off the moments as Laurent coefficients. Subtracting the contribution of any unprescribed constant moment gives two functions that vanish outside the contour. Inside it they are holomorphic except along a real segment. Their boundary values and their jumps across that segment are explicit expressions in four generating series of coefficients. We prove both implications: coefficient solutions give this boundary problem, and sufficiently regular solutions of the boundary problem give the original equations.

4. Recover the generating series and continue them. Four boundary values determine the four generating series through an invertible matrix. Away from two endpoints this gives holomorphic continuation immediately. Section 5 removes the endpoint singularities using area estimates and square summability. Continuation beyond the unit circle gives geometric decay of the coefficients. Thus every solution in the Hilbert space belongs to the required absolutely summable space.

5. Count solutions of the global continuation problem. Section 6 joins the continued functions over a fundamental domain. The four generating series and two Cauchy integrals give six holomorphic functions with translation and inversion laws. We prove equivalence with this global problem; the laws are consequences of the boundary equations, not extra constraints. In each dimension they define a rank-six holomorphic vector bundle over the compactified modular curve of the full modular group, retaining its orbifold structure. In particular, the stabilizers at the elliptic points enter the index calculation. The cusp conditions specify how the bundle extends across the added cusp. Solutions of the homogeneous coefficient equations correspond exactly to global holomorphic sections of this bundle. If h^0 is the dimension of its space of global sections and h^1 is the dimension of its first sheaf cohomology group, Riemann–Roch gives

$$h^0 - h^1 = 1.$$

Since $h^1 \geq 0$, a nonzero section exists, and the correspondence gives a nontrivial coefficient solution. A separate vanishing argument shows that a solution with zero normalization is zero. The solution space is therefore one-dimensional and has a unique normalized element. Both dimensions have Euler characteristic one, although their scalar weights and cusp conditions differ. The index is that of the bundle’s cohomology, not an index of the compact moment operator.

6. Obtain a stable equation and identify the function. The moment operator is compact, so a small output alone cannot control the coefficient norm. The continuation estimates supply a second compact operator that fixes every exact solution. Section 9 combines the operators and the normalization into a positive equation with a bounded inverse. Its exact Galerkin approximations converge; a certified numerical implementation would require additional quantitative estimates. Only after these proofs do we compare the coefficients with the known magic functions and inherit their sign inequalities.

We give the detailed argument in dimension 24 in Sections 2–9. Section 11 then proves the dimension-eight theorem: it specifies the changed coefficient system, checks the boundary and global equivalences, recomputes the index, and supplies its vanishing proof. Section 12 returns to the missing independent sign estimates in both dimensions.

The variables used along the way belong to different spaces. The real arguments u, v describe Φ, G . A complex parameter z describes Gaussians. A complex variable q is the argument of a generating series, while w is a coordinate in the contour plane. Maps between these planes have separate names, and each integral specifies which parameters are held fixed. Later $\tau \in \mathbb{H}$ is the upper-half-plane coordinate and $\mathfrak{q} = e^{2\pi i\tau}$ is the cusp coordinate. For reference, the coordinate maps are

u, v	independent real arguments of the two radial profiles
z	complex Gaussian parameter
$q = Q(z) = e^{-\pi(z-1)}$	argument of the coefficient series
$w = \omega(z) = e^{-2\pi/z}$	coordinate in the contour plane
$\tau = iz, \quad \mathfrak{q} = e^{2\pi i\tau}$	upper-half-plane and modular cusp coordinates.

In particular, $q = e^{\pi}\mathfrak{q}^{1/2}$ under these substitutions. It is not an identification of the two independent series variables.

Bold lowercase letters denote vectors; sans serif uppercase letters denote linear operators and finite matrices. Calligraphic letters denote the principal indexing sets and geometric objects. Scalar functions use ordinary or Greek letters, retaining the standard notation for Eisenstein series and the elementary families P, U, V . A superscript T denotes transpose, $*$ denotes an adjoint, and

a subscript $*$ denotes a normalized solution. Hats denote Fourier transforms and primes denote derivatives. Tildes mark candidate functions; the superscript $\sharp$, used only in Appendix B, marks a holomorphic average. Auxiliary matrices have descriptive subscripts: pair for paired traces, frm for the six-component frame, cell for recovery of four components, and rec, match for recovery and matching of six components. A parenthesized dimension distinguishes the latter matrices when their powers depend on d . The permanent gluing matrices are $J_{\text{tr}}, J_{\text{inv}}$.

2 The coefficient system in dimension 24

From here through Section 9, the dimension is fixed at 24. Section 11 gives the dimension-eight version.

2.1 Elementary functions and the zeros they enforce

We now choose the functions whose coefficients will be the unknowns. Throughout, $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{N}_0 = \{0, 1, 2, \dots\}$. For $u \geq 0$, $m \in \mathbb{N}$, and $k = 0, 1, 2$, define

$$U_{m,k}(u) = e^{\pi m} P(u) \int_1^\infty t^k e^{-\pi(m+u)t} dt, \quad (2.1)$$

$$V_m(u) = e^{\pi m} P(u) \int_0^1 t^{10} e^{-\pi(m/t+ut)} dt. \quad (2.2)$$

Here $P(u) = \sin^2(\pi u/2)$ as above. The index m labels the exponential factor in the integral; it does not label a point where a zero is prescribed. We reserve n for the sample points $2n$. The factors $e^{\pi m}$ rewrite the coefficient damping as $e^{-\pi m(t-1)}$ in the first integral and $e^{-\pi m(1/t-1)}$ in the second. This will allow us to collect coefficients into power series on a unit disk.

There is no convergence exception at $u = 0$ in these definitions: $m \geq 1$, so $\int_1^\infty t^k e^{-\pi m t} dt$ is finite. The reciprocal integral also converges there, since $e^{-\pi m/t}$ controls its integrand as $t \downarrow 0$. Both integrals are analytic in a neighborhood of each $u = 2n$, including $n = 0$. Multiplication by P therefore gives a zero of order at least two at every nonnegative even integer. The zero index $n = 0$ does not introduce a mode $m = 0$. We use the following combinations:

$$U_{m,0} - 2V_m \quad (m \text{ odd}), \quad U_{m,0}, U_{m,1}, U_{m,2} + V_m \quad (m \text{ even}).$$

These are the functions to which we assign independent coefficients. In particular, one coefficient multiplies the whole expression $U_{m,0} - 2V_m$ when m is odd, and one multiplies $U_{m,2} + V_m$ when m is even. Appendix A explains the choice: Gaussian inversion suggests the two integration ranges, and cancellation of the unbounded contour integrals at the Fourier sample points gives exactly these parity restrictions and combinations. Section 10 subsequently compares the family with the known integral formula.

The common factor P imposes more zeros than we want: $\Phi(0)$ must be nonzero, neither $\Phi(2)$ nor $\Phi'(2)$ is prescribed, and $\Phi'(4)$ need not vanish. To permit precisely these changes, together with an unrestricted derivative at 0, define five correction functions:

$$\begin{aligned} R_{\ell,0}(u) &= \frac{P(u)e^{-\pi(u-\ell)}}{\pi(u-\ell)}, & \ell = 0, 2, 4, \\ R_{\ell,1}(u) &= P(u)e^{-\pi(u-\ell)} \left(\frac{1}{\pi(u-\ell)} + \frac{1}{\pi^2(u-\ell)^2} \right), & \ell = 0, 2. \end{aligned} \quad (2.3)$$

These are definitions by explicit formulas and removable continuation, not by possibly divergent integrals. For $u > \ell$ they agree with the expressions obtained formally from (2.1) at $m = -\ell$, after the corresponding exponential rescaling. At $u = \ell$ their singularities are removable, since P has a double zero. In particular,

$$\lim_{u \rightarrow 0} \frac{R_{0,0}(u)}{u} = \frac{\pi}{4}, \quad R_{0,1}(0) = \frac{1}{4}.$$

At the respective removable points one also has

$$R_{\ell,0}(\ell) = 0, \quad R'_{\ell,0}(\ell) = \frac{\pi}{4}, \quad R_{\ell,1}(\ell) = \frac{1}{4}, \quad R'_{\ell,1}(\ell) = 0.$$

Thus the correction functions deliberately change the zero order; they are not included in the double-zero assertion for $m \geq 1$. A simple denominator permits a nonzero first derivative; a double denominator permits a nonzero value. These five functions therefore allow a value and a derivative at 0, a value and a derivative at 2, and a derivative at 4. All other zeros required of Φ in (1.4) remain built into the family. The point 2 is allowed because the Leech lattice has no vectors of squared length 2.

2.2 Unknown coefficients and their norms

The coefficients are indexed by pairs (j, m) . The first subscript specifies a generating series and the second its exponent. Define the common set of positive-mode indices by

$$\mathcal{I}_+ = (\{0, 1, 2\} \times 2\mathbb{N}) \cup (\{3\} \times (2\mathbb{N} - 1)),$$

and the dimension-24 indexing set by

$$\mathcal{I}_{24} = \{(0, -4), (0, -2), (0, 0), (1, -2), (1, 0)\} \cup \mathcal{I}_+.$$

A coefficient vector is a family $\mathbf{c} = (c_{j,m})_{(j,m) \in \mathcal{I}_{24}}$ of complex numbers. Its five nonpositive modes multiply the correction functions. Its positive modes give three sequences for even m and one for odd m . This indexing will also be used in the generating series of Section 4.

For $p = 1, 2$, define the norm by

$$\begin{aligned} \|\mathbf{c}\|_p^p &= |c_{0,-4}|^p + |c_{0,-2}|^p + |c_{0,0}|^p + |c_{1,-2}|^p + |c_{1,0}|^p \\ &\quad + \sum_{m \in 2\mathbb{N}} \sum_{k=0}^2 |c_{k,m}|^p + \sum_{m \in 2\mathbb{N}-1} |c_{3,m}|^p. \end{aligned} \tag{2.4}$$

For $p = 1, 2$, write $\ell^p(\mathcal{I}_{24})$ for the space of these coefficient families with finite norm $\|\mathbf{c}\|_p$. Thus

$$\begin{aligned} \ell^1(\mathcal{I}_{24}) &= \mathbb{C}^5 \oplus \ell^1(2\mathbb{N})^3 \oplus \ell^1(2\mathbb{N} - 1), \\ \ell^2(\mathcal{I}_{24}) &= \mathbb{C}^5 \oplus \ell^2(2\mathbb{N})^3 \oplus \ell^2(2\mathbb{N} - 1). \end{aligned}$$

We equip $\ell^2(\mathcal{I}_{24})$ with its standard coefficient inner product associated with the sum in (2.4). Every sequence space below displays its indexing set. Lebesgue spaces of functions, used later for boundary values, are denoted by L^p . The construction seeks a vector in $\ell^1(\mathcal{I}_{24})$. Working temporarily in the larger Hilbert space $\ell^2(\mathcal{I}_{24})$ permits the use of adjoints and compact operators. We will prove that every exact solution in $\ell^2(\mathcal{I}_{24})$ already belongs to $\ell^1(\mathcal{I}_{24})$.

For $\mathbf{c} \in \ell^2(\mathcal{I}_{24})$, define a function $\Phi_{\mathbf{c}} : [0, \infty) \rightarrow \mathbb{C}$ by the following series, with independent argument $u \geq 0$:

$$\begin{aligned}\Phi_{\mathbf{c}}(u) &= c_{1,0}R_{0,1}(u) + c_{0,0}R_{0,0}(u) \\ &\quad + e^{-2\pi}c_{1,-2}R_{2,1}(u) + e^{-2\pi}c_{0,-2}R_{2,0}(u) + e^{-4\pi}c_{0,-4}R_{4,0}(u) \\ &\quad + \sum_{m \in 2\mathbb{N}-1} c_{3,m}(U_{m,0}(u) - 2V_m(u)) \\ &\quad + \sum_{m \in 2\mathbb{N}} (c_{0,m}U_{m,0}(u) + c_{1,m}U_{m,1}(u) + c_{2,m}(U_{m,2}(u) + V_m(u))).\end{aligned}\tag{2.5}$$

Lemma 3.1 proves that this series defines a radial Schwartz function

$$f_{\mathbf{c}}(\mathbf{y}) = \Phi_{\mathbf{c}}(|\mathbf{y}|^2), \quad \mathbf{y} \in \mathbb{R}^{24}.$$

Using (1.1), let $G_{\mathbf{c}}$ be the unique function on $[0, \infty)$ satisfying

$$\widehat{f}_{\mathbf{c}}(\boldsymbol{\xi}) = G_{\mathbf{c}}(|\boldsymbol{\xi}|^2), \quad \boldsymbol{\xi} \in \mathbb{R}^{24}.$$

Thus $\Phi_{\mathbf{c}}$ is specified by the coefficients, whereas $G_{\mathbf{c}}$ is obtained from it by Fourier transformation. Both depend linearly on $\mathbf{c}$.

Only $R_{0,1}$ in (2.5) is nonzero at $u = 0$, and $R_{0,1}(0) = 1/4$. It is therefore convenient to name the normalizing linear functional

$$\lambda(\mathbf{c}) = c_{1,0}/4 = \Phi_{\mathbf{c}}(0).\tag{2.6}$$

2.3 The equations and the main theorem

The required zeros of $\Phi_{\mathbf{c}}$ are automatic from the choice of functions. The unknown coefficients must make the Fourier profile satisfy

$$G_{\mathbf{c}}(2n) = G'_{\mathbf{c}}(2n) = 0 \quad (n \geq 2), \quad \lambda(\mathbf{c}) = 1.\tag{2.7}$$

Here is the linear system explicitly. In this paragraph only, enumerate the entries of $\mathbf{c}$ by $j \in \mathbb{N}$, write c_j for the corresponding scalar, and let $\mathbf{e}_j$ be the coefficient family with that entry equal to one and all other entries zero. Put $g_j = G_{\mathbf{e}_j}$; these are fixed functions, obtained by Fourier transforming the individual terms of (2.5). The equations are

$$\begin{aligned}\sum_{j \geq 1} g_j(2n)c_j &= 0, \quad \sum_{j \geq 1} g'_j(2n)c_j = 0 \quad (n \geq 2), \\ \sum_{j \geq 1} \lambda(\mathbf{e}_j)c_j &= 1.\end{aligned}\tag{2.8}$$

Thus $g_j(2n)$ and $g'_j(2n)$ are matrix entries, not unknowns. Only the c_j are unknown. The last row has a single nonzero entry, $1/4$, in the position of $c_{1,0}$. Lemma 3.1 justifies the series and evaluation term by term. Lemma 4.1 will express the two types of Fourier rows as contour integrals.

Theorem 2.1 (Coefficient construction). *The system (2.7) has exactly one solution $\mathbf{c}_*$ in $\ell^2(\mathcal{I}_{24})$. Its entries are real, and it belongs to $\ell^1(\mathcal{I}_{24})$. More precisely, for every $1 < r < e^\pi$,*

$$\sum_{m \in 2\mathbb{N}} (|c_{0,m}| + |c_{1,m}| + |c_{2,m}|)r^m + \sum_{m \in 2\mathbb{N}-1} |c_{3,m}|r^m < \infty.$$

The associated profiles satisfy

$$\begin{aligned}\Phi_{\mathbf{c}_*}(0) &= G_{\mathbf{c}_*}(0) = 1, & \Phi_{\mathbf{c}_*}(2n) &= 0 & (n \geq 2), \\ \Phi'_{\mathbf{c}_*}(4) &= -\frac{1}{65520}, & \Phi'_{\mathbf{c}_*}(2n) &= 0 & (n \geq 3).\end{aligned}$$

Before imposing $\lambda(\mathbf{c}) = 1$, the Fourier equations have a one-dimensional solution space in $\ell^2(\mathcal{I}_{24})$.

Poisson summation will force $c_{0,-4} = -e^{4\pi}/(16380\pi)$ for the normalized solution. Its correction part can therefore be written

$$4R_{0,1} - \frac{R_{4,0}}{16380\pi} + c_{0,0}R_{0,0} + e^{-2\pi}c_{1,-2}R_{2,1} + e^{-2\pi}c_{0,-2}R_{2,0}.$$

Only the three coefficients $c_{0,0}, c_{1,-2}, c_{0,-2}$ and the four positive-mode sequences remain unknown.

Section 3 proves the regularity of $f_{\mathbf{c}}$ and the Poisson identities. Sections 4–8 prove the coefficient theorem. Section 9 gives an equivalent equation with a bounded inverse. The comparison with the known magic function follows in Section 10; Section 11 then proves the dimension-eight theorem.

3 The coefficient map and Poisson normalization

Before manipulating Fourier samples, we justify the series defining $f_{\mathbf{c}}$. We also show that distinct coefficient vectors give distinct functions. The second part of this section explains why the two normalizations at the origin and the first derivative at $u = 4$ are tied together.

Lemma 3.1. *The map $\mathbf{c} \mapsto f_{\mathbf{c}}$ defined by (2.5) is a continuous injective linear map from $\ell^2(\mathcal{I}_{24})$ to the radial Schwartz space on $\mathbb{R}^{24}$. For every $\mathbf{c} \in \ell^2(\mathcal{I}_{24})$,*

$$\begin{aligned}\Phi_{\mathbf{c}}(0) &= \lambda(\mathbf{c}), & \Phi_{\mathbf{c}}(2n) &= 0 & (n \geq 2), \\ \Phi'_{\mathbf{c}}(4) &= \frac{\pi}{4}e^{-4\pi}c_{0,-4}, & \Phi'_{\mathbf{c}}(2n) &= 0 & (n \geq 3).\end{aligned}\tag{3.1}$$

Proof. For $k = 0, 1, 2$, integration gives

$$U_{m,k}(u) = P(u)e^{-\pi u} \sum_{j=0}^k \frac{k!}{(k-j)!} \frac{1}{[\pi(m+u)]^{j+1}}.$$

For $u \geq 0$, each derivative of the factor after P is bounded by a constant times $e^{-\pi u}/m$. Cauchy–Schwarz and $\sum_{m \geq 1} m^{-2} < \infty$ control all corresponding series.

Write $d_m = c_{2,m}$ for even m and $d_m = -2c_{3,m}$ for odd m . The remaining sum is

$$P(u) \int_0^1 t^{10} e^{-\pi ut} \sum_{m \geq 1} d_m e^{-\pi m(1/t-1)} dt.$$

The inner sum is bounded by $C\|\mathbf{c}\|_2 e^{-\pi/t}$ for $0 < t \leq 1/2$, and by $C\|\mathbf{c}\|_2(1-t)^{-1/2}$ for $1/2 < t < 1$. Differentiation in u adds only powers of t and bounded derivatives of P . The inequality $t^{-1} + ut \geq 2\sqrt{u}$ then bounds each derivative, for $u \geq 1$, by

$$C_j\|\mathbf{c}\|_2(e^{-2\pi\sqrt{u}} + e^{-\pi u/2}).$$

The five functions $R_{\ell,k}$ in the finite sum have removable singularities and Gaussian decay. Each spatial derivative of $\Phi_{\mathbf{c}}(|\mathbf{y}|^2)$ is a finite sum of a polynomial in $\mathbf{y}$ times a derivative of $\Phi_{\mathbf{c}}$. The

displayed decay absorbs all such polynomials and all polynomial weights in the Schwartz seminorms. These estimates therefore prove continuity into Schwartz space. The double zeros of P , together with $R_{0,1}(0) = 1/4$ and $R'_{4,0}(4) = \pi/4$, prove (3.1).

For injectivity, continue the profile meromorphically in u . For each compact set $K \subset \mathbb{C}$, the inequality $1/t - 1 \geq 1 - t$, $0 < t \leq 1$, gives

$$\sup_{u \in K} |V_m(u)| \leq C_K \int_0^1 e^{-\pi m(1-t)} dt \leq C_K/(\pi m).$$

Thus the V_m series converges normally by Cauchy–Schwarz and is entire. The rational formula for $U_{m,k}$ gives the same C_K/m bound on compact sets avoiding its poles, with finitely many small m treated separately. Its series therefore converges normally away from the negative integers. If the profile vanishes, its residues at $u = -m$, for odd and even positive m , respectively, give

$$\frac{e^{\pi m}}{\pi} c_{3,m} = 0, \quad \frac{e^{\pi m}}{2\pi} c_{2,m} = 0.$$

No V_m terms remain. Divide by $P(u)e^{-\pi u}$. The double and simple principal parts at each negative even integer give $c_{1,m} = c_{0,m} = 0$; the principal parts at $0, 2, 4$ give the five coefficients of the correction functions. Thus $\mathbf{c} = 0$. $\square$

Let $\Lambda \subset \mathbb{R}^{24}$ be the Leech lattice, scaled so that a fundamental domain has volume one. For $n \in \mathbb{N}$, let $N_n = \#\{\mathbf{y} \in \Lambda : |\mathbf{y}|^2 = 2n\}$. Its dual lattice is $\Lambda^* = \{\boldsymbol{\xi} \in \mathbb{R}^{24} : \boldsymbol{\xi} \cdot \mathbf{y} \in \mathbb{Z} \text{ for all } \mathbf{y} \in \Lambda\}$. We use $\Lambda^* = \Lambda$, the fact that all squared norms are even integers, and $N_1 = 0$, $N_2 = 196560$; see [3, Appendix B.1].

Lemma 3.2 (Poisson normalization). *For every $\mathbf{c} \in \ell^2(\mathcal{I}_{24})$,*

$$G_{\mathbf{c}}(0) + \sum_{n \geq 2} N_n G_{\mathbf{c}}(2n) = \lambda(\mathbf{c}), \quad (3.2)$$

$$\pi N_2 e^{-4\pi} c_{0,-4} + 12\lambda(\mathbf{c}) = - \sum_{n \geq 2} 2n N_n G'_{\mathbf{c}}(2n). \quad (3.3)$$

Consequently every solution of the homogeneous sample equations satisfies

$$G_{\mathbf{c}}(0) = \lambda(\mathbf{c}), \quad c_{0,-4} = - \frac{e^{4\pi}}{16380\pi} \lambda(\mathbf{c}). \quad (3.4)$$

Proof. For a real dilation parameter $t > 0$, apply Poisson summation to $\mathbf{y} \mapsto f_{\mathbf{c}}(\sqrt{t}\mathbf{y})$:

$$\Phi_{\mathbf{c}}(0) + \sum_{n \geq 2} N_n \Phi_{\mathbf{c}}(2nt) = t^{-12} \left(G_{\mathbf{c}}(0) + \sum_{n \geq 2} N_n G_{\mathbf{c}}(2n/t) \right).$$

Schwartz regularity justifies differentiation locally uniformly in $t > 0$. Evaluate at $t = 1$, and then differentiate and evaluate there, using (3.1). The identities follow; $N_2/12 = 16380$ gives (3.4). $\square$

4 From Fourier samples to a boundary problem

This section replaces the Fourier equations by a problem in one complex variable. Its first objective is the following identity. The measures in the statement are defined explicitly in (4.7); their supports are independent of the coefficient vector. Their dependence on that vector is linear.

Lemma 4.1 (Fourier samples as contour integrals). *For each $\mathbf{c} \in \ell^2(\mathcal{I}_{24})$, let μ_0, μ_1 be the finite complex measures constructed in (4.7). For every integer $n \geq 0$,*

$$G_{\mathbf{c}}(2n) = \frac{1}{8\pi} \int_{\mathbb{C}} w^{n-1} d\mu_0(w), \quad G'_{\mathbf{c}}(2n) = -\frac{1}{8} \int_{\mathbb{C}} w^{n-1} d\mu_1(w). \quad (4.1)$$

The integrals converge absolutely, including when $n = 0$.

Thus the equations for $n \geq 2$ are exactly the vanishing of the moments of positive degree of these measures. Once the lemma has been proved, we package the moments as an operator and use Cauchy integrals to express their vanishing as boundary conditions for holomorphic functions.

4.1 Gaussian parameters and generating series

For each fixed $z \in \mathbb{C}$ with $\Re z > 0$, define $g_z(\mathbf{y}) = e^{-\pi z |\mathbf{y}|^2}$, $\mathbf{y} \in \mathbb{R}^{24}$. Gaussian Fourier inversion gives, for every fixed $\boldsymbol{\xi} \in \mathbb{R}^{24}$,

$$\widehat{g}_z(\boldsymbol{\xi}) = \int_{\mathbb{R}^{24}} e^{-\pi z |\mathbf{y}|^2} e^{-2\pi i \mathbf{y} \cdot \boldsymbol{\xi}} d\mathbf{y} = z^{-12} e^{-\pi |\boldsymbol{\xi}|^2 / z}. \quad (4.2)$$

Only $\mathbf{y}$ is integrated in this formula. Equivalently, the one-variable function giving its radial Fourier values is $v \mapsto z^{-12} e^{-\pi v / z}$, for an independent real argument $v \geq 0$. In the contour calculations below we hold v fixed and integrate in z . We then evaluate at $v = 2n$, or differentiate with respect to v and evaluate there. No frequency vector $\boldsymbol{\xi}$ occurs in those calculations.

Define two maps, with different roles:

$$Q(z) = e^{-\pi(z-1)} \quad (z \in \mathbb{C}), \quad \omega(z) = e^{-2\pi/z} \quad (z \in \mathbb{C} \setminus \{0\}). \quad (4.3)$$

The map Q supplies arguments for generating series; the map ω supplies points of the contour plane. In particular $e^{-\pi(2n)/z} = \omega(z)^n$, which is the source of the powers in (4.1). We reserve q for an independent complex argument of a generating series and w for an independent complex coordinate in the contour plane. Neither letter denotes a map.

Fix $\mathbf{c} \in \ell^2(\mathcal{I}_{24})$ for the definitions that follow. All generating series, densities, and measures depend on this fixed vector; we omit that dependence from their subscripts. For $0 < |q| < 1$, define

$$\begin{aligned} A_0(q) &= c_{0,-4} q^{-4} + c_{0,-2} q^{-2} + c_{0,0} + \sum_{m \in 2\mathbb{N}} c_{0,m} q^m, \\ A_1(q) &= c_{1,-2} q^{-2} + c_{1,0} + \sum_{m \in 2\mathbb{N}} c_{1,m} q^m, \\ A_2(q) &= \sum_{m \in 2\mathbb{N}} c_{2,m} q^m, \\ A_3(q) &= \sum_{m \in 2\mathbb{N}-1} c_{3,m} q^m. \end{aligned} \quad (4.4)$$

These series already contain four of the six functions that will represent a global section of the bundle. More precisely, for a complex variable τ with $\Im \tau > 1$, substitute $q = Q(-i\tau) = e^\pi e^{\pi i \tau}$ and form

$$(A_0(q), -iA_1(q), -A_2(q), A_3(q)).$$

When the Fourier equations are satisfied, these four functions continue holomorphically to the upper half-plane and become the first four components of a section of the rank-six bundle over the

compactified modular curve, with the orbifold and cusp conditions described in Section 6. The other two components come from the Cauchy integrals. This assertion is proved in Proposition 6.4; it is not an additional assumption on the series. The components are coordinate functions of a bundle section, rather than four separately invariant functions on the modular curve.

Set $A_-(q) = A_2(q) - 2A_3(q)$ and $A_+(q) = A_2(q) + 2A_3(q)$. The first three series are even functions of q ; the fourth is odd. Define, for $\Re z > 1$,

$$\begin{aligned} b_0(z) &= A_0(Q(z)) + zA_1(Q(z)) + z^2A_2(Q(z)) + A_3(Q(z)), \\ b_1(z) &= A_1(Q(z)) + 2zA_2(Q(z)). \end{aligned} \tag{4.5}$$

These evaluations are within the convergence disk because $|Q(z)| < 1$ precisely when $\Re z > 1$.

We also need values on the line $\Re z = 1$. They are initially boundary limits, not values of a continuation already known to exist. Indeed, $Q(\sigma + it) = e^{-\pi(\sigma-1)}e^{-\pi it}$. As $\sigma \downarrow 1$, square summability and Parseval's identity give convergence in L^2 on bounded t -intervals of each series and of b_0, b_1 . These L^2 limits are called *boundary traces*. Evaluations of these functions on $\Re z = 1$ below mean these traces, defined almost everywhere.

4.2 The parametrized contours and the measures

Define paths in the Gaussian-parameter plane by

$$z_+(t) = 1 + it, \quad z_-(t) = 1 - i + it, \quad 0 \leq t \leq 1,$$

and paths in the contour plane by

$$\gamma_+(t) = \omega(z_+(t)), \quad \gamma_-(t) = \omega(z_-(t)), \quad 0 \leq t \leq 1.$$

Put $w_0 = e^{-2\pi}$ and $w_c = -e^{-\pi}$. As t increases, γ_+ runs from w_0 to w_c in the upper half-plane, and γ_- runs from w_c to w_0 in the lower half-plane. Explicitly, $\gamma_-(t) = \overline{\gamma_+(1-t)}$; this formula specifies the reversal of direction in the conjugate path. Let $\Gamma_\pm = \gamma_\pm([0, 1])$ be their images, and let $\Gamma = \Gamma_+ \cup \Gamma_-$ be the resulting set.

The set Γ is a Jordan curve. For example, along γ_+ the argument $2\pi t/(1+t^2)$ increases from 0 to π , which proves simplicity on that half. Let Ω be the bounded component of $\mathbb{C} \setminus \Gamma$. It contains 0. Traversing γ_+ followed by γ_- gives the positive orientation of its boundary. Whenever we integrate over Γ , we mean the sum of the integrals along these two parametrized paths.

The two analytic arcs meet tangentially at w_c , with the direction of traversal reversing along their common tangent. This is the nonsmooth point of the Jordan curve, called its *contour cusp* below. The curve is smooth at w_0 , but an additional segment will meet it there. Set $I = (0, w_0) \subset \mathbb{R}$. Integrals along this segment use the path $s \mapsto s$, $0 < s < w_0$, directed from 0 towards w_0 . Thus ds in those integrals is ordinary real integration with the displayed limits.

Figure 1 shows the two arcs and the segment, with arrows indicating these directions of integration.

Define the boundary densities $\rho_{0,\Gamma}, \rho_{1,\Gamma}$ almost everywhere on Γ by the following four formulas. Every occurrence of $b_0, b_1, A_j, A_\pm$ has its argument displayed:

$$\begin{aligned} \rho_{0,\Gamma}(\gamma_+(t)) &= z_+(t)^{-10} [b_0(z_+(t)) - ib_1(z_+(t))], \\ \rho_{1,\Gamma}(\gamma_+(t)) &= z_+(t)^{-11} [b_0(z_+(t)) - ib_1(z_+(t)) + iz_+(t)A_+(Q(z_+(t)))], \\ \rho_{0,\Gamma}(\gamma_-(t)) &= -z_-(t)^{-10} [b_0(z_-(t)) + ib_1(z_-(t))], \\ \rho_{1,\Gamma}(\gamma_-(t)) &= -z_-(t)^{-11} [b_0(z_-(t)) + ib_1(z_-(t)) - iz_-(t)A_+(Q(z_-(t)))]. \end{aligned} \tag{4.6}$$

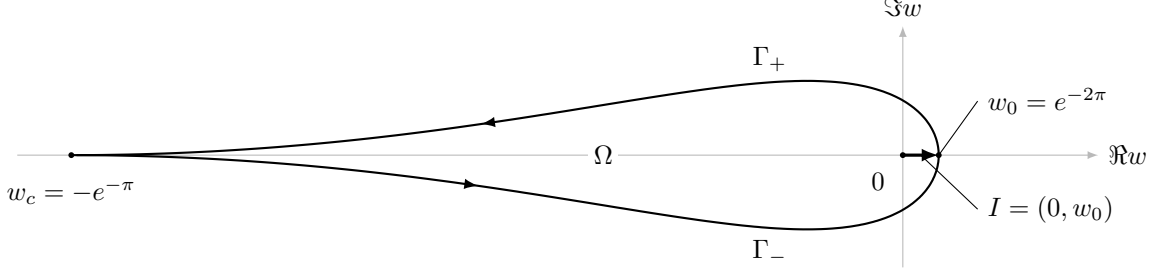


Figure 1: The contour $\Gamma = \Gamma_+ \cup \Gamma_-$ and the segment I , drawn with a uniform enlargement. The upper arc runs from w_0 to w_c , the lower arc returns to w_0 , and I is directed from 0 towards w_0 . The arcs meet tangentially at the cusp w_c .

Here $0 < t < 1$, and all boundary evaluations have the trace meaning specified above. Define densities on the real segment by

$$\rho_{0,I}(s) = 2A_-(e^\pi \sqrt{s}), \quad \rho_{1,I}(s) = -\pi^{-1}(\log s) A_-(e^\pi \sqrt{s}), \quad 0 < s < w_0,$$

using the positive square root and the real logarithm.

We define the complex Borel measures μ_ν , $\nu = 0, 1$, by specifying their integrals against bounded Borel functions $\varphi : \mathbb{C} \rightarrow \mathbb{C}$:

$$\begin{aligned} \int_{\mathbb{C}} \varphi(w) d\mu_\nu(w) &= \sum_{\sigma \in \{+, -\}} \int_0^1 \varphi(\gamma_\sigma(t)) \rho_{\nu, \Gamma}(\gamma_\sigma(t)) \gamma'_\sigma(t) dt \\ &\quad + \int_0^{w_0} \varphi(s) \rho_{\nu, I}(s) ds. \end{aligned} \quad (4.7)$$

In the left integral w is a dummy complex variable; on the right t, s are real integration variables. This definition includes all orientation signs and makes the meaning of the contour integrals precise. The measures have no atoms and are supported in the compact set $\Gamma \cup [0, w_0]$.

The traces on the arcs are square integrable. On the segment, Cauchy–Schwarz gives

$$|A_-(x)| \leq 2\|\mathbf{c}\|_2 \frac{x}{\sqrt{1-x^2}} \quad (0 < x < 1).$$

These estimates prove that the measures are finite and, for a constant C_μ independent of $\mathbf{c}$,

$$\|\mu_0\|_{\text{TV}} + \|\mu_1\|_{\text{TV}} \leq C_\mu \|\mathbf{c}\|_2. \quad (4.8)$$

The symbol $\|\mu\|_{\text{TV}}$ denotes the total variation norm of a complex measure. Near $s = 0$, the two segment densities are bounded by a constant times $\sqrt{s}$ and $\sqrt{s} |\log s|$, respectively. Their products with s^{-1} are integrable, which explains the convergence at $n = 0$ in Lemma 4.1.

4.3 Proof of the sample identity

Proof of Lemma 4.1. Use the identity $P(u) = (2 - e^{\pi i u} - e^{-\pi i u})/4$ in the defining integrals of the elementary functions. Apply (4.2) to the three Gaussian terms in each elementary function and shift their integration paths by i and $-i$. The argument $v \geq 0$ remains fixed during these operations. To record the result, define $J_{m,k}(v)$ for integers $m, k = 0, 1, 2$, and $W_m(v)$ for positive integers m , by

$$J_{m,k}(v) = \int_1^{1+i} (z-i)^k z^{-12} e^{-\pi m(z-1) - \pi v/z} dz, \quad W_m(v) = \frac{e^{-\pi v}}{2\pi(m+v)}.$$

The first integral follows the straight path z_+ defined above; z is its integration variable. For an elementary function $h(u)$ in the left column below, Fourier transform the radial function $\mathbf{y} \mapsto h(|\mathbf{y}|^2)$. Its value at any frequency vector of squared length $2n$ is the right column:

$h(u)$	Fourier value at $v = 2n$
$U_{m,0} - 2V_m \quad (m \text{ odd})$	$\frac{1}{2}\Re J_{m,0}(2n) - 2W_m(2n)$
$U_{m,0} \quad (m \text{ even})$	$\frac{1}{2}\Re J_{m,0}(2n)$
$U_{m,1} \quad (m \text{ even})$	$\frac{1}{2}\Re J_{m,1}(2n)$
$U_{m,2} + V_m \quad (m \text{ even})$	$\frac{1}{2}\Re(J_{m,2}(2n) + J_{m,0}(2n)) + W_m(2n)$

For the Fourier derivative, first replace each $2n$ in the right column by the independent variable v , differentiate that expression, and then set $v = 2n$. Add $\pi\Im J_{m,0}(2n)$ in the first row and $\frac{\pi}{2}\Im J_{m,0}(2n)$ in the last row. These extra terms come from differentiating the phase factors before evaluating at an even integer; the other rows need no correction. They account for A_+ in (4.6). The identities before specialization to $v = 2n$, including the phases that produce these corrections, are (A.4)–(A.5) in Appendix A. Real and imaginary parts are taken for the individual real elementary functions before multiplying by complex coefficients.

For the five correction functions, direct integration gives

$$R_{\ell,k}(u) = \frac{1}{2}\Re \int_1^{1+i} (z-i)^k e^{-\pi(u-\ell)z} dz.$$

Here u is fixed and z is again integrated along z_+ . Their Fourier values are $e^{\pi\ell}\Re J_{-\ell,k}(2n)/2$; differentiating the same Fourier integral gives their Fourier derivatives.

Now substitute $w = \omega(z)$ on each path. The differential identity $\omega'(z) = 2\pi\omega(z)z^{-2}$ turns these path integrals into the arc terms of (4.7). For the remaining terms set $d_m = c_{2,m}$ when m is even and $d_m = -2c_{3,m}$ when m is odd. Then

$$\sum_{m \geq 1} d_m W_m(2n) = \frac{1}{4\pi} \int_0^{w_0} s^{n-1} A_-(e^\pi \sqrt{s}) ds.$$

Before evaluation at $v = 2n$, the power is $s^{v/2-1}$. Differentiation with respect to v contributes $(\log s)/2$, giving the second segment density. This proves the lemma for coefficient families with finite support. Continuity into Schwartz space, (4.8), and the integrable bounds near $s = 0$ extend it to every $\mathbf{c} \in \ell^2(\mathcal{I}_{24})$. In particular, for $n = 0$ the segment contribution is bounded by $C\|\mathbf{c}\|_2 \int_0^{w_0/2} s^{-1/2}(1+|\log s|) ds$ near zero, with a similar uniform bound on the remaining segment. Thus this limiting step is uniform in the coefficient norm even for the negative power w^{-1} in that sample. $\square$

4.4 The moment operator and the unprescribed moments

Allow $\mathbf{c}$ to vary in the preceding construction. Set $\mathcal{J}_{24} = \{0, 1\} \times \mathbb{N}$. The first index records which measure is used and the second its moment degree. Thus $k \in \mathbb{N}$ means exactly $k \geq 1$; the two moments of degree zero are omitted. Define the moment operator $\mathbf{M} : \ell^2(\mathcal{I}_{24}) \rightarrow \ell^2(\mathcal{J}_{24})$ by

$$(\mathbf{M}\mathbf{c})_{\nu,k} = \int_{\mathbb{C}} w^k d\mu_\nu(w), \quad \nu = 0, 1, \quad k \geq 1. \quad (4.9)$$

Lemma 4.1 shows that $\mathbf{M}\mathbf{c} = 0$ is equivalent to the Fourier equations in (2.7), without normalization: $k = n - 1$. The zeroth moments

$$m_{\nu,0} = \int_{\mathbb{C}} d\mu_\nu(w), \quad \nu = 0, 1,$$

are not among the equations. More specifically, $m_{0,0} = 8\pi G_{\mathbf{c}}(2)$ and $m_{1,0} = -8G'_{\mathbf{c}}(2)$. They are determined by $\mathbf{c}$, but the system does not prescribe them as additional data. This is the meaning of leaving them unrestricted.

The claimed target space follows from the total variation bound. Both measures are supported in $\{|w| \leq \rho\}$, where $\rho = e^{-\pi} < 1$, and hence

$$|(\mathbf{M}\mathbf{c})_{\nu,k}| \leq \rho^k \|\mu_\nu\|_{\text{TV}}.$$

Consequently

$$\|\mathbf{M}\mathbf{c}\|_{\ell^2(\mathcal{J}_{24})}^2 \leq \frac{\rho^2}{1-\rho^2} (\|\mu_0\|_{\text{TV}}^2 + \|\mu_1\|_{\text{TV}}^2) \leq \frac{C_\mu^2 \rho^2}{1-\rho^2} \|\mathbf{c}\|_{\ell^2(\mathcal{I}_{24})}^2.$$

This proves both square summability of the moments and boundedness of the linear operator.

Write $\text{Id}_{\mathcal{J}_{24}}$ for the identity on $\ell^2(\mathcal{J}_{24})$. For an integer $J \geq 1$, define the orthogonal projection $\mathbf{Q}_J$ on the moment sequence space by

$$(\mathbf{Q}_J \mathbf{b})_{\nu,k} = \begin{cases} b_{\nu,k}, & 1 \leq k \leq J, \\ 0, & k > J. \end{cases}$$

Here $\mathbf{b} = (b_{\nu,k}) \in \ell^2(\mathcal{J}_{24})$. This projection has rank $2J$. The same geometric estimate gives

$$\|(\text{Id}_{\mathcal{J}_{24}} - \mathbf{Q}_J)\mathbf{M}\| \leq C_\mu \frac{\rho^{J+1}}{\sqrt{1-\rho^2}}. \quad (4.10)$$

Thus $\mathbf{M}$ is a compact linear operator: the finite-rank operators $\mathbf{Q}_J \mathbf{M}$ converge to it in operator norm. This compactness is useful, but is not an existence theorem for the normalized equations. The continuation and index arguments below supply existence.

4.5 Cauchy functions, traces, and jumps

Fix $\mathbf{c} \in \ell^2(\mathcal{I}_{24})$, and form its measures and their zeroth moments as above. For an evaluation point $\zeta \in \mathbb{C} \setminus (\Gamma \cup [0, w_0])$, define

$$h_\nu(\zeta) = \frac{1}{2\pi i} \left(\int_{\mathbb{C}} \frac{d\mu_\nu(w)}{w - \zeta} + \frac{m_{\nu,0}}{\zeta} \right), \quad \nu = 0, 1. \quad (4.11)$$

We call these the *corrected Cauchy transforms*, or *Cauchy functions*, of the coefficient vector. In this integral ζ is fixed and w is integrated. The correction term removes the zeroth moment from the expansion at infinity. Write $\mathbf{h} = (h_0, h_1)^T$.

If $\mathbf{c} \in \ker \mathbf{M}$, then for $|\zeta| > \rho$ the convergent expansion of $(w - \zeta)^{-1}$ gives

$$\int_{\mathbb{C}} \frac{d\mu_\nu(w)}{w - \zeta} = - \sum_{k \geq 0} \zeta^{-k-1} \int_{\mathbb{C}} w^k d\mu_\nu(w) = -m_{\nu,0}/\zeta.$$

Hence $h_\nu = 0$ on the unbounded component of $\mathbb{C} \setminus \Gamma$, by analytic continuation. Inside Ω , it is holomorphic away from the closed segment $[0, w_0]$. Removing that segment gives a *slit domain*. At a point $s \in I$ there are two boundary approaches within this domain: from above and from below the real axis. We denote their traces by $h_{\nu,+}(s)$ and $h_{\nu,-}(s)$. On Γ , the interior trace is the limit approached from Ω ; the exterior trace is approached from the unbounded component. All assertions about these limits are local on open arcs away from $0, w_0, w_c$. More precisely, after straightening an arc by a holomorphic coordinate, the restrictions to parallel lines converge in L^2 on every smaller

closed interval. We use this meaning of local L^2 trace convergence throughout. The limits also exist almost everywhere in the nontangential sense. A nontangential approach stays inside a cone that makes a positive angle with the boundary tangent.

The Cauchy jump formula then gives

$$\begin{aligned} h_{\nu, \text{int}}(w) &= \rho_{\nu, \Gamma}(w) & (w \in \Gamma \setminus \{w_0, w_c\}), \\ h_{0,+}(s) - h_{0,-}(s) &= 2A_-(e^\pi \sqrt{s}) & (s \in I), \\ h_{1,+}(s) - h_{1,-}(s) &= -\pi^{-1}(\log s) A_-(e^\pi \sqrt{s}) & (s \in I). \end{aligned} \quad (4.12)$$

The first line uses the zero exterior trace. The next two give the jump, meaning the upper trace minus the lower trace, across the segment oriented towards w_0 .

For clarity, we recall the analytic facts used here. On the real line directed from left to right, the Cauchy integral of $g \in L^2(\mathbb{R})$ is

$$(\mathbb{C}g)(\zeta) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{g(x)}{x - \zeta} dx, \quad \Im \zeta \neq 0.$$

Its upper and lower traces are $\mathbb{C}_+g = \mathbb{P}_+g$ and $\mathbb{C}_-g = -\mathbb{P}_-g$, where $\mathbb{P}_\pm$ are the orthogonal projections onto the subspaces with Fourier transform supported in the positive and negative half-lines. They are called *Hardy projections*. Their ranges are the boundary values of the Hardy spaces: holomorphic functions on the respective half-planes whose squared integrals along horizontal lines are uniformly bounded. Thus $\mathbb{C}_+ - \mathbb{C}_- = \mathbb{P}_+ + \mathbb{P}_- = \text{Id}_{L^2(\mathbb{R})}$. See [14, Sections 8.2–8.3 and 8.7] for the boundary-value description and Cauchy representation.

For an open analytic arc, choose a smaller subarc, cut off the density outside a slightly larger subarc, and straighten the arc by a holomorphic coordinate. The transformed Cauchy kernel differs from the real-line kernel by a bounded smooth kernel; the omitted part of the contour contributes a holomorphic function near the smaller subarc. The real-line result therefore proves the local L^2 trace and jump statements used in (4.12). This is the local form of the Plemelj formula; see [15, Section 3.5, especially (3.5.3)–(3.5.5)]. No trace theorem at the nonsmooth point w_c is being invoked.

Lemma 4.2 (Gluing across an open arc). *Let an analytic arc divide a neighborhood into two sides, and let h_+, h_- be holomorphic on those sides. Suppose their traces converge locally in L^2 , in the sense just specified, to the same function. Then they are restrictions of one holomorphic function on a neighborhood of the open arc.*

Proof. Straighten the arc and restrict to a smaller rectangle. Trace convergence bounds the L^2 norms on parallel intervals uniformly near the real axis. Cauchy–Schwarz followed by integration in the perpendicular coordinate gives local area integrability of the function h_{glue} defined by the two pieces. For a smooth compactly supported test function, integrate by parts above the line of height r and below the line of height $-r$. Holomorphy leaves only the two boundary terms. Their difference tends to zero by L^2 trace convergence, so $\bar{\partial} h_{\text{glue}} = 0$ in distributions. Local convolution with a smooth approximate identity gives holomorphic functions converging to h_{glue} in local L^1 . The area mean inequality makes this convergence uniform on smaller disks. Its limit is the required extension. $\square$

For continuous matching values this is also the consequence of Morera’s theorem obtained by splitting triangles along the arc; see [17, Section 1]. In our applications the original coefficient series have the asserted trace convergence by the Hardy-space theorem, and the Cauchy functions have it by the localized Plemelj formula above. We require no trace theorem at a contour endpoint.

4.6 Reconstructing the equations from boundary data

We have proved that a coefficient solution gives holomorphic Cauchy functions with the boundary conditions (4.12). The reverse implication, needed for the existence proof, is the following statement. Its endpoint assumptions prevent additional singular terms that the jumps on open arcs cannot detect. For $\zeta_0 \in \mathbb{C}$ and $r > 0$, write $D(\zeta_0, r) = \{w : |w - \zeta_0| < r\}$.

Lemma 4.3 (Reconstruction from boundary values). *Fix $\mathbf{c} \in \ell^2(\mathcal{I}_{24})$, with densities and measures as in (4.6)–(4.7). Suppose $\tilde{h}_0, \tilde{h}_1$ are holomorphic on $\Omega \setminus [0, w_0]$, have boundary limits converging in L^2 on every compact subarc away from the three endpoints, and satisfy the outer trace and segment jump conditions (4.12), with $\tilde{h}_\nu$ in place of h_ν . Suppose there are constants κ_0, κ_1 such that, for $\nu = 0, 1$,*

$$\begin{aligned} \lim_{r \downarrow 0} \int_{\partial D(\zeta_0, r) \cap (\Omega \setminus [0, w_0])} |\tilde{h}_\nu(w)| |dw| &= 0 & (\zeta_0 = w_0, w_c), \\ \lim_{r \downarrow 0} \int_{\partial D(0, r) \setminus [0, w_0]} \left| \tilde{h}_\nu(w) - \frac{\kappa_\nu}{2\pi i w} \right| |dw| &= 0. \end{aligned}$$

Then $\kappa_\nu = \int d\mu_\nu$, $\mathbf{c} \in \ker \mathbf{M}$, and $\tilde{h}_\nu$ equals the Cauchy function (4.11) on the slit domain.

Proof. Fix an evaluation point ζ away from the contour and the slit. First fix small disjoint disks about $0, w_0, w_c$ that avoid ζ , choosing their boundaries transverse to the incident arcs. Exhaust the remaining slit domain from inside by curves parallel to its smooth boundary pieces. For fixed disk radii, Cauchy's formula passes to the boundary by local L^2 , hence local L^1 , trace convergence. The two banks of the slit are combined at this stage: their difference is $\rho_{\nu, I}$. There is no need to integrate either bank separately all the way to an endpoint.

Now let the disk radii decrease to zero. Finite total variation controls the omitted pieces of the arc and slit integrals. The small-circle integrals at w_0, w_c tend to zero by hypothesis, since $(w - \zeta)^{-1}$ is bounded there. At zero, subtract the indicated pole before taking the limit. The remainder has a vanishing circle integral, and the pole contributes $\kappa_\nu / (2\pi i \zeta)$. Cauchy's formula has value zero at an exterior evaluation point and value $\tilde{h}_\nu(\zeta)$ at an interior one. It therefore gives

$$\frac{1}{2\pi i} \left(\int_{\mathbb{C}} \frac{d\mu_\nu(w)}{w - \zeta} + \frac{\kappa_\nu}{\zeta} \right) = \begin{cases} \tilde{h}_\nu(\zeta), & \zeta \in \Omega \setminus [0, w_0], \\ 0, & \zeta \in \mathbb{C} \setminus \bar{\Omega}. \end{cases}$$

Expansion of the exterior expression at infinity first gives $\kappa_\nu = \int d\mu_\nu$ and then gives every positive moment equal to zero. The interior expression is therefore (4.11). The argument uses Cauchy's formula with boundary limits, justified by the same local L^2 approximation used for the jump formula. $\square$

We have reached the promised boundary formulation. To use it for existence, we still need endpoint regularity, a global continuation problem, and a proof that this global problem has a nonzero solution. The next three sections address those tasks.

5 Regularity at the boundary endpoints

We will prove that a solution in $\ell^2(\mathcal{I}_{24})$ has generating functions that extend beyond the unit circle. Cauchy's coefficient estimates will then give absolute summability. The open arcs permit continuation by an invertible matrix. What remains is to justify passage through the contour

endpoints w_0, w_c . Most of this section concerns those two points. The estimates will also supply the compact regularizing operator used in Section 9.

Fix $\mathbf{c} \in \ker \mathbf{M}$, and use its Cauchy functions $\mathbf{h}$ from (4.11). We now work on open sets of the Gaussian-parameter plane, not just along the paths z_+, z_- . The letter z continues to denote an independent complex variable; write $\sigma = \Re z$, $t = \Im z$. Define the open set

$$\mathcal{D} = \{\sigma + it : 0 < \sigma < 1, |t| < 1 - \sqrt{1 - \sigma^2}\}.$$

The composite Cauchy functions will be needed on a larger set than $\mathcal{D}$. Define

$$\mathcal{W} = \{z \in \mathbb{C} \setminus \{0\} : \omega(z) \in \Omega \setminus [0, w_0]\}.$$

On this entire open set define

$$\mathbf{h}_{\text{par}}(z) = \begin{pmatrix} h_{0,\text{par}}(z) \\ h_{1,\text{par}}(z) \end{pmatrix} = z^{10} \mathbf{h}(\omega(z)).$$

The upper and lower halves of $\mathcal{D}$ lie in $\mathcal{W}$ and map to the upper and lower halves of Ω under ω . We initially use these portions of the domain; later occurrences of $\mathbf{h}_{\text{par}}(1/z)$ use the same composite on $\mathcal{W}$, with a separate domain check. Thus the arguments of the Cauchy functions lie in their holomorphic domains. Boundary values of $\mathbf{h}_{\text{par}}$ mean traces from these halves. For each fixed $t \in (0, 1)$, evaluate at $z = z_+(t)$ and $z_1 = z_-(t) = z - i$. Then $Q(z_1) = -Q(z)$. At this point it is convenient to collect the four series into the column vector

$$\mathbf{a}_{\text{ser}}(q) = (A_0(q), A_1(q), A_2(q), A_3(q))^T.$$

Using their parity to pair the outer traces at z and z_1 gives

$$\begin{pmatrix} h_{0,\text{par}}(z) \\ h_{1,\text{par}}(z) \\ h_{0,\text{par}}(z_1) \\ h_{1,\text{par}}(z_1) \end{pmatrix} = \mathbf{S}_{\text{pair}}(z) \mathbf{a}_{\text{ser}}(Q(z)), \quad (5.1)$$

$$\mathbf{S}_{\text{pair}}(z) = \begin{pmatrix} 1 & z_1 & z^2 - 2iz & 1 \\ z^{-1} & z_1/z & z_1 & z^{-1} + 2i \\ -1 & -z & -z^2 - 1 & 1 \\ -z_1^{-1} & -z/z_1 & -z & z_1^{-1} - 2i \end{pmatrix}, \quad \det \mathbf{S}_{\text{pair}} = 4i.$$

In this matrix z_1 always means $z - i$, including when z varies off the boundary. The determinant is nonzero, so its inverse recovers all four generating functions from the Cauchy values. In a neighborhood to the left of each open boundary point, those values are holomorphic. They have the same traces as the original series on the right. The gluing lemma 4.2 therefore gives continuation across $z_+((0, 1))$. Parity gives the corresponding continuation across its translates by integer multiples of i . The points at the ends of these segments require separate work.

We also need two matching identities for $\mathbf{h}_{\text{par}}$. In the first line below, z ranges over the open upper curved boundary of $\mathcal{D}$; the point $z/(1 + iz)$ is on the lower curved boundary and has the same image in the w -plane. In the second line, $+$ and $-$ denote limits from above and below the real interval:

$$\mathbf{h}_{\text{par}}(z) = (1 + iz)^{10} \mathbf{h}_{\text{par}}\left(\frac{z}{1 + iz}\right) \quad (|z - i| = 1), \quad (5.2)$$

$$\mathbf{h}_{\text{par},+}(z) - \mathbf{h}_{\text{par},-}(z) = 2z^{10} A_-(Q(1/z)) \begin{pmatrix} 1 \\ z^{-1} \end{pmatrix} \quad (0 < z < 1). \quad (5.3)$$

The first identity follows because $\mathbf{h}$ is holomorphic across the negative real axis in Ω ; it also identifies neighborhoods of the two open circle arcs. The second is the segment jump (4.12). Its argument $1/z > 1$ lies in the half-plane where the coefficient series already converge.

5.1 Area bounds and removal of a puncture

We record the estimates before constructing any continuation. They apply to arbitrary coefficient vectors; matching traces will be used only when constructing a solution's holomorphic extension. Throughout this section dA denotes ordinary planar area.

Lemma 5.1 (Local area estimates). *For a bounded measurable set E whose closure avoids zero,*

$$\int_E |h_\nu(\zeta)| dA(\zeta) \leq C_E \|\mathbf{c}\|_2, \quad \nu = 0, 1.$$

The same assertion holds for the compositions used below on fixed coordinate neighborhoods, provided the coordinate maps are one-to-one with nonzero derivatives and have images avoiding zero. For each fixed $0 < r < 1/2$, the original coefficient functions also satisfy

$$\int_{\substack{|z-1| < r \\ \Re z > 1}} \sum_{j=0}^3 |A_j(Q(z))| dA(z) \leq C_r \|\mathbf{c}\|_2.$$

Proof. For a finite measure μ , Tonelli's theorem and the layer-cake formula for the radially decreasing kernel give

$$\int_E \left| \int \frac{d\mu(w)}{w - \zeta} \right| dA(\zeta) \leq \|\mu\|_{\text{TV}} \sup_w \int_E |w - \zeta|^{-1} dA(\zeta) \leq 2\sqrt{\pi \text{area}(E)} \|\mu\|_{\text{TV}}.$$

The correction term $m_{\nu,0}/\zeta$ has the required bound because E avoids zero. Use (4.8) to obtain the first assertion. Changes of variables with bounded Jacobian and bounded multipliers give the assertion about compositions. One may split a fixed neighborhood into finitely many coordinate patches when necessary.

Writing $x = \Re z - 1 > 0$, Cauchy-Schwarz gives

$$\left| \sum_{m \geq 1} \beta_m e^{-\pi m(z-1)} \right| \leq \frac{\|\beta\|_2}{\sqrt{e^{2\pi x} - 1}} \leq \frac{\|\beta\|_2}{\sqrt{2\pi x}}. \quad (5.4)$$

The finitely many nonpositive modes are uniformly bounded by $C_r \|\mathbf{c}\|_2$ on this disk. Since $x^{-1/2}$ is area-integrable on its right half, the last assertion follows. $\square$

Lemma 5.2 (Removal using one radius). *Let h be holomorphic and area-integrable on $0 < |z - z_0| < r_0$. Then h has at most a simple pole at z_0 . If $t h(z_0 + te^{i\theta_0}) \rightarrow 0$ on one radius as $t \downarrow 0$, the singularity is removable. After removal, an area-norm bound on the disk gives a uniform bound on every fixed smaller disk.*

Proof. Translate z_0 to zero and let $m(r) = (2\pi)^{-1} \int_0^{2\pi} |h(re^{i\theta})| d\theta$. Area integrability implies that some $r_j \downarrow 0$ has $r_j^2 m(r_j) \rightarrow 0$; otherwise $\int_0^{r_0} r m(r) dr$ would diverge. The Laurent coefficient of z^{-k} has absolute value at most $r_j^k m(r_j)$. All coefficients with $k \geq 2$ therefore vanish. The condition on one radius excludes the remaining coefficient of z^{-1} . The final claim is the area mean inequality applied to the resulting holomorphic function on disks of a fixed smaller radius. $\square$

5.2 Continuation through the two endpoints

Proposition 5.3 (Uniform continuation). *There are constants $R > 1$ and $C < \infty$, independent of $\mathbf{c}$, such that every $\mathbf{c} \in \ker_{\ell^2(\mathcal{I}_{24})} \mathbf{M}$ has $q^4 \mathbf{a}_{\text{ser}}(q)$ holomorphic on a neighborhood of $|q| \leq R$, and*

$$\sup_{|q|=R} \|q^4 \mathbf{a}_{\text{ser}}(q)\|_1 \leq C \|\mathbf{c}\|_2. \quad (5.5)$$

In particular $\ker_{\ell^2(\mathcal{I}_{24})} \mathbf{M} \subset \ell^1(\mathcal{I}_{24})$.

Proof. The contour cusp. Use a local parameter z near 1; the arguments $z + i, z - i$ approach the two ends of the contour at $w_c = -e^{-\pi}$. Write

$$D_r = D(1, r), \quad D_r^{\text{out}} = D_r \cap \{|z| > 1\}, \quad D_r^{\text{in}} = D_r \cap \{|z| < 1\}.$$

Initially, on $\Re z > 1$, put

$$\begin{aligned} b_{\text{end}}(z) &= A_0(Q(z)) + zA_1(Q(z)) + (z^2 + 1)A_2(Q(z)) - A_3(Q(z)), \\ b_-(z) &= A_-(Q(z)). \end{aligned} \quad (5.6)$$

On the two portions of $D_{1/4}^{\text{out}}$ with $\Re z < 1$, define these two functions by solving

$$\begin{aligned} \mathbf{h}_{\text{par}}(z + i) &= \begin{pmatrix} b_{\text{end}}(z) \\ b_{\text{end}}(z)/(z + i) + ib_-(z) \end{pmatrix} & (\Im z < 0), \\ \mathbf{h}_{\text{par}}(z - i) &= \begin{pmatrix} -b_{\text{end}}(z) \\ -b_{\text{end}}(z)/(z - i) + ib_-(z) \end{pmatrix} & (\Im z > 0). \end{aligned} \quad (5.7)$$

These are legitimate Cauchy evaluations. In the lower portion, $|z| > 1$ implies $\Im z < -\sqrt{1 - (\Re z)^2}$, so $z + i$ lies in the upper half of $\mathcal{D}$; the upper portion is analogous. The two equations are triangular and uniquely determine the two functions. On each open segment $\Re z = 1$, substitution of the outer densities, using the parity of A_j , gives exactly the traces of (5.6). Lemma 4.2 therefore joins the three definitions into holomorphic functions on $D_{1/4}^{\text{out}}$. Lemma 5.1 bounds their area norms there by $C\|\mathbf{c}\|_2$: it covers both the original-series portion and the two Cauchy portions. All denominators $z \pm i$ stay away from zero.

On the two open arcs $|z| = 1, z \neq 1$, the circle matching law (5.2) yields

$$b_{\text{end}}(z) = z^{10} b_{\text{end}}(1/z), \quad b_-(z) + z^{10} b_-(1/z) = b_{\text{end}}(z). \quad (5.8)$$

Here is the algebra at the lower arc. The two arguments in (5.7) are $z + i$ and $1/z - i$. Their circle-transfer multiplier is $(iz)^{10} = -z^{10}$. The first component gives the first identity, and the second gives the second, using $(1/z - i)^{-1} - (z + i)^{-1} = i$. The upper arc gives the inverse identities. These identities hold for the analytic germs at the open arcs, because their Cauchy arguments lie on the interior negative real interval of Ω , across which $\mathbf{h}$ is holomorphic.

For $z \in D_{1/8}^{\text{in}}$ define

$$b_{\text{end}}(z) = z^{10} b_{\text{end}}(1/z), \quad b_-(z) = b_{\text{end}}(z) - z^{10} b_-(1/z).$$

The right sides have already been defined: $|1/z| > 1$ and

$$|1/z - 1| \leq \frac{|z - 1|}{1 - |z - 1|} < \frac{1}{7} < \frac{1}{4}.$$

The transformation on the ordered pair is z^{10} times $\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$; applying it at z and then $1/z$ is the identity. The arc identities just proved match these inner pieces to the outer pieces. We thus have a single holomorphic pair on the whole punctured disk $D_{1/8} \setminus \{1\}$. This specifies both the cover and all its overlaps.

Inversion has bounded Jacobian on these disks, and z^{10} is bounded there. Lemma 5.1 consequently bounds the area norm of the whole pair, including the inner pieces. On the radius $z = 1 + t$, (5.4) bounds each component by $C\|\mathbf{c}\|_2(1 + t^{-1/2})$. Lemma 5.2 removes both punctures and gives uniform bounds on $D_{1/16}$. The involution identities continue wherever both sides are defined. In particular, b_- continues $A_-(Q(z))$ through $z = 1$.

The real endpoint and the remaining components. At the other contour endpoint $w_0 = e^{-2\pi}$, which corresponds to $z = 1$, define on the left of $z = 1$

$$b_+(z) = -i(h_{1,\text{par}}(z) - h_{0,\text{par}}(z)/z). \quad (5.9)$$

Use this formula separately on the upper and lower left half-disks of $D_{1/8}$, both contained in the corresponding halves of $\mathcal{D}$. Its real-interval jump is zero: the jump vector in (5.3) has second component equal to its first divided by z . On the right half-disk use $b_+(z) = A_+(Q(z))$. The outer trace formulas give this same value on the two open vertical segments. Applying Lemma 4.2 to the real interval and the vertical segments yields a holomorphic function on $D_{1/8} \setminus \{1\}$. Its left pieces have the Cauchy area bound and its right piece has the series area bound of Lemma 5.1. On $z = 1 + t$ it satisfies (5.4). Lemma 5.2 removes the puncture and bounds the extension on $D_{1/16}$. Consequently

$$A_2(Q(z)) = (b_+(z) + b_-(z))/2, \quad A_3(Q(z)) = (b_+(z) - b_-(z))/4$$

are holomorphic near that point.

It remains to recover A_1, A_0 . Work on $D_{1/32}$, where all three functions just constructed are bounded by $C\|\mathbf{c}\|_2$. In the upper and lower left half-disks, the subscripts $+$ and $-$ on $h_{0,\text{par}}$ designate its holomorphic restrictions to those half-disks; on the intervening real interval they designate the corresponding traces. Use, respectively,

$$\begin{aligned} A_1(Q(z)) &= i[h_{0,\text{par},+}(z) - b_{\text{end}}(z) + (1 + 2iz)A_2(Q(z)) - 2A_3(Q(z))], \\ A_1(Q(z)) &= i[h_{0,\text{par},-}(z) + b_{\text{end}}(z) - (1 - 2iz)A_2(Q(z)) + 2A_3(Q(z))]. \end{aligned} \quad (5.10)$$

Their difference is $i[h_{0,\text{par},+}(z) - h_{0,\text{par},-}(z) - 2b_{\text{end}}(z) + 2b_-(z)]$. On the real interval, (5.3) identifies the jump of $h_{0,\text{par}}$ with $2z^{10}b_-(1/z)$. The second identity of (5.8) identifies this with $2b_{\text{end}}(z) - 2b_-(z)$. The difference is therefore zero. On $\Re z = 1$, direct substitution of the outer traces gives $A_1(Q(z))$ from either formula. Glue the two left pieces to that original right-hand piece by Lemma 4.2. Their area norms are bounded: use Lemma 5.1 for the Cauchy terms and the original series, and the bounds on $D_{1/32}$ for the other terms. The right-hand radius again satisfies (5.4). Lemma 5.2 now removes the puncture and bounds $A_1(Q(z))$ on $D_{1/64}$. Finally,

$$A_0(Q(z)) = b_{\text{end}}(z) - zA_1(Q(z)) - (z^2 + 1)A_2(Q(z)) + A_3(Q(z)). \quad (5.11)$$

This formula agrees with the original $A_0(Q(z))$ on the right half-disk by its definition in (5.6); it is holomorphic throughout $D_{1/32}$ and bounded on $D_{1/64}$. All four functions now extend across $q = 1$, and parity gives the same conclusion at $q = -1$.

A common annulus and its norm bound. For an explicit common neighborhood, put $\delta_{\text{strip}} = 2^{-18}$ and $\delta_{\text{end}} = 2^{-7}$. If $1 - \delta_{\text{strip}} < \Re z < 1$ and $\delta_{\text{end}} \leq \Im z \leq 1 - \delta_{\text{end}}$, both $z, z - i$ lie in the required

halves of $\mathcal{D}$, since $(1 - \delta_{\text{strip}})^2 > 1 - \delta_{\text{end}}^2$. Use $S_{\text{pair}}(z)^{-1}$ there and the endpoint disks elsewhere. Parity covers a full period. Hence $q^4 \mathbf{a}_{\text{ser}}$ is holomorphic for $|q| < e^{\pi \delta_{\text{strip}}}$. On the circle

$$R = e^{\pi \delta_{\text{strip}}/2},$$

the middle-arc Cauchy evaluations range over fixed compact sets away from the contours, and the endpoint evaluations have the area-mean bounds already proved. This proves (5.5). Cauchy's coefficient estimate gives geometric coefficient tails, and hence absolute summability. $\square$

6 The global holomorphic problem

The preceding section gives continuation a little beyond the unit circle. We now describe a global continuation problem. It involves six scalar holomorphic functions: four encode the coefficients, and two encode the Cauchy integrals after their explicit jumps have been subtracted. Proposition 6.4 will prove equivalence in both directions. In particular, none of the global continuation conditions is being assumed for a coefficient solution.

6.1 Six functions and their gluing matrices

Let $\mathbb{H} = \{\tau \in \mathbb{C} : \Im \tau > 0\}$. The linear change of coordinate $z \mapsto \tau = iz$ identifies the right half-plane in the z -plane with $\mathbb{H}$. In this section τ is an independent variable in $\mathbb{H}$, and the corresponding Gaussian parameter is $-i\tau$. The map Q satisfies

$$Q(-i\tau) = e^\pi e^{\pi i\tau}. \quad (6.1)$$

For $\Im \tau > 1$, define

$$\begin{aligned} a_0(\tau) &= A_0(Q(-i\tau)), & a_1(\tau) &= -iA_1(Q(-i\tau)), \\ a_2(\tau) &= -A_2(Q(-i\tau)), & a_3(\tau) &= A_3(Q(-i\tau)). \end{aligned}$$

The first three are periodic under $\tau \mapsto \tau + 1$; the fourth changes sign. The global problem seeks continuations of these four functions to all of $\mathbb{H}$, together with two more holomorphic functions a_4, a_5 that are periodic.

To explain the additional functions, first suppose such global functions are given. Define the branch

$$\log_{\text{cut}} w = \log |w| + i \arg w, \quad 0 < \arg w < 2\pi,$$

on $\mathbb{C} \setminus [0, \infty)$. Its restriction to $\Omega \setminus [0, w_0]$ is used throughout this construction. For w in that slit domain, put $\tau_w = \log_{\text{cut}} w / (2\pi i)$. The upper and lower traces on the positive real segment use arguments 0 and 2π , respectively. Define candidate Cauchy functions by

$$\tilde{h}_0(w) = 2\tau_w a_2(\tau_w) - 2a_3(\tau_w) + a_4(\tau_w), \quad (6.2)$$

$$\tilde{h}_1(w) = i[(-\tau_w^2 + \tau_w)a_2(\tau_w) + (2\tau_w - 1)a_3(\tau_w) + a_5(\tau_w)]. \quad (6.3)$$

Indeed, the lower bank uses $\tau_w + 1$; subtracting its value from the upper bank gives exactly (4.12). Conversely, subtracting these explicit jumping terms from Cauchy functions with (4.12) leaves periodic remainders a_4, a_5 . Their definition for an actual coefficient solution is carried out on a fundamental domain in Lemma 6.3.

Let I_r denote the r -by- r identity matrix. Collect the six functions as $\mathbf{a} = (a_0, a_1, a_2, a_3, a_4, a_5)^T$. The translation matrix is $\mathbf{J}_{\text{tr}}$. To specify the inversion matrix, define the frame matrix $\mathbf{S}_{\text{frm}}(\tau)$ and the matrix $\mathbf{S}_{\text{swap}}$ that interchanges two triples:

$$\mathbf{J}_{\text{tr}} = \text{diag}(1, 1, 1, -1, 1, 1), \quad \mathbf{S}_{\text{swap}} = \begin{pmatrix} 0 & I_3 \\ I_3 & 0 \end{pmatrix},$$

$$\mathbf{S}_{\text{frm}}(\tau) = \begin{pmatrix} 1 & \tau & \tau^2 & 1 & 0 & 0 \\ 0 & 1 & 2\tau & 0 & 0 & 0 \\ 0 & \tau^{-1} & 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & -1 - 2\tau & 0 & -1 & 0 \\ 0 & 0 & \tau^2 & 1 & 0 & -1 \end{pmatrix}, \quad \mathbf{J}_{\text{inv}}(\tau) = \mathbf{S}_{\text{frm}}(-1/\tau)^{-1} \mathbf{S}_{\text{swap}} \mathbf{S}_{\text{frm}}(\tau). \quad (6.4)$$

The gluing laws are

$$\mathbf{a}(\tau + 1) = \mathbf{J}_{\text{tr}} \mathbf{a}(\tau), \quad \mathbf{a}(-1/\tau) = \tau^{-10} \mathbf{J}_{\text{inv}}(\tau) \mathbf{a}(\tau). \quad (6.5)$$

All matrices are holomorphic and invertible in the upper half-plane: $\det \mathbf{S}_{\text{frm}} = -4$ and $\det \mathbf{J}_{\text{inv}} = -1$. Writing $S\tau = -1/\tau$, $T\tau = \tau + 1$, $R_3 = ST$, and $\mathbf{J}_{ST}(\tau) = \mathbf{J}_{\text{inv}}(\tau + 1) \mathbf{J}_{\text{tr}}$, direct multiplication gives

$$\mathbf{J}_{\text{inv}}(S\tau) \mathbf{J}_{\text{inv}}(\tau) = I_6, \quad \mathbf{J}_{ST}(R_3^2\tau) \mathbf{J}_{ST}(R_3\tau) \mathbf{J}_{ST}(\tau) = I_6. \quad (6.6)$$

The scalar weight factors also have product 1. The transformations S, T generate the modular group $\text{PSL}_2(\mathbb{Z}) = \text{SL}_2(\mathbb{Z})/\{\pm I_2\}$, acting on $\mathbb{H}$ by fractional linear transformations. The group relations say that applying (6.5) along two words representing the same transformation gives the same result. This consistency is what is meant here by a holomorphic automorphy factor.

The inversion matrix $\mathbf{J}_{\text{inv}}(\tau)$ in (6.4) can be recovered directly from the boundary data. The first three entries of $\mathbf{S}_{\text{frm}}(\tau) \mathbf{a}(\tau)$ are the following expressions, in which all six scalar functions are evaluated at τ :

$$a_0 + \tau a_1 + \tau^2 a_2 + a_3, \quad a_1 + 2\tau a_2, \quad a_1/\tau + a_2 + 2a_3.$$

The last three are

$$a_2 + 2a_3, \quad -(1 + 2\tau)a_2 - a_4, \quad \tau^2 a_2 + a_3 - a_5.$$

Substituting (6.2)–(6.3) in both outer traces pairs the two triples by inversion. More explicitly, write $\tilde{\mathbf{h}} = (\tilde{h}_0, \tilde{h}_1)^T$. For this check, the four expressions $A_j(Q(z))$ are obtained from $a_j(iz)$ by reversing their definitions, and $b_0(z), b_1(z)$ are formed using (4.5). For $z \in z_+((0, 1))$ or $z \in z_-((0, 1))$, evaluate the candidate Cauchy functions at $\omega(z)$. Equation (6.5) gives the first identity below on the first path and the second identity on the second path:

$$z^{10} \tilde{\mathbf{h}}(\omega(z)) = \begin{pmatrix} b_0(z) - ib_1(z) \\ z^{-1}(b_0(z) - ib_1(z)) + iA_+(Q(z)) \end{pmatrix},$$

$$z^{10} \tilde{\mathbf{h}}(\omega(z)) = \begin{pmatrix} -b_0(z) - ib_1(z) \\ -z^{-1}(b_0(z) + ib_1(z)) + iA_+(Q(z)) \end{pmatrix}. \quad (6.7)$$

For these evaluations, put $\tau = iz$. The logarithmic parameter of $\omega(z)$ is $-1/\tau$ on the first arc and $-1/\tau + 1$ on the second. These are algebraic row identities in $\mathbf{J}_{\text{inv}}$ and $\mathbf{J}_{\text{tr}}$. To show that every coefficient solution produces these transformation laws, we must extend its functions over a full fundamental domain. Lemma 6.3 establishes this direction.

6.2 Cusp conditions and equivalence

The end $\Im\tau \rightarrow \infty$ is called the modular cusp. Let $\mathfrak{q}$ be a local complex coordinate at this cusp: substitution of $\mathfrak{q} = e^{2\pi i\tau}$ converts Laurent series in $\mathfrak{q}$ into periodic functions of τ . The half-power $\mathfrak{q}^{1/2}$ means $e^{\pi i\tau}$ when pulled back to the upper half-plane. Thus the cusp corresponds to $\mathfrak{q} = 0$. It is distinct from the contour cusp w_c treated in Section 5. At this modular cusp, prescribe the following bounds:

function	a_0	a_1	a_2	a_3	a_4	a_5
lowest allowed exponent of $\mathfrak{q}$	-2	-1	1	$\frac{1}{2}$	-1	-1.

(6.8)

Each function is $\mathfrak{q}$ to the displayed exponent times a convergent power series in $\mathfrak{q}$. In particular, a_4, a_5 may each have a simple pole. Write their principal parts as $\kappa_0 \mathfrak{q}^{-1}/(2\pi i)$ and $-\kappa_1 \mathfrak{q}^{-1}/(2\pi)$, respectively. The constants κ_0, κ_1 are not prescribed in this global problem. Lemma 4.3 will identify them with the zeroth moments $m_{0,0}, m_{1,0}$ of the recovered coefficient vector. The constant terms of a_4, a_5 are also unrestricted. Neither $\lambda = 1$ nor $G_c(0) = 1$ has been imposed here.

Here is the geometric meaning of this problem. Form the quotient of $\mathbb{H}$ by $\mathrm{SL}_2(\mathbb{Z})$, retaining the stabilizers, and add the cusp. In this quotient the central element $-I_2$ is retained as a stabilizer even though it acts trivially on $\mathbb{H}$; the stabilizers are larger at the two elliptic points. This is the compactified modular orbifold $\overline{\mathcal{X}}$, identified with the weighted projective line $\mathbb{P}(4, 6)$ in [1]. The six-dimensional vectors are identified over equivalent points by (6.5). At the cusp, divide each component by its displayed power in (6.8); these quotients are the coordinates required to be holomorphic there. Explicitly, write $\kappa_{\mathrm{cusp},j}$, $0 \leq j \leq 5$, for the six exponents in (6.8), and put

$$\mathbf{S}_{\mathrm{cusp}}(\tau) = \mathrm{diag}(e^{2\pi i \kappa_{\mathrm{cusp},j} \tau})_{j=0}^5.$$

Then $\mathbf{S}_{\mathrm{cusp}}(\tau + 1) = \mathbf{J}_{\mathrm{tr}} \mathbf{S}_{\mathrm{cusp}}(\tau)$. Thus $\mathbf{S}_{\mathrm{cusp}}^{-1} \mathbf{a}$ is a single-valued vector function of $\mathfrak{q}$, and requiring it to be holomorphic at zero defines the cusp trivialization. In particular the half-integer exponent cancels the fourth component's antiperiodicity. These rules define a rank-six holomorphic vector bundle $\mathcal{E}$.

For the arguments below, a global section of $\mathcal{E}$ means exactly a vector of six holomorphic functions on $\mathbb{H}$ satisfying (6.5) and (6.8). We denote the vector space of these sections by $H^0(\overline{\mathcal{X}}, \mathcal{E})$, or simply $H^0(\mathcal{E})$. The geometric description allows us to apply Riemann–Roch in Section 7.

6.3 From a section to the coefficient equations

Lemma 6.1. *Taking the Laurent coefficients of the first four components defines an injective linear map*

$$H^0(\overline{\mathcal{X}}, \mathcal{E}) \longrightarrow \ker_{\ell^2(\mathcal{I}_{24})} \mathbf{M}.$$

Every coefficient vector in its image is absolutely summable with weight r^m for every $1 < r < e^\pi$. Its Cauchy functions are (6.2)–(6.3).

Proof. Let $\mathbf{a}$ be a section. Recover A_0, A_1, A_2, A_3 by reversing the four definitions above: for $q \neq 0$, $|q| < e^\pi$, choose $\tau \in \mathbb{H}$ with $Q(-i\tau) = q$, and set

$$A_0(q) = a_0(\tau), \quad A_1(q) = ia_1(\tau), \quad A_2(q) = -a_2(\tau), \quad A_3(q) = a_3(\tau).$$

Any two choices of τ differ by an even integer, so periodicity makes these definitions independent of the choice. The translation law and (6.8) give exactly the allowed exponents in (4.4). These four functions are holomorphic for $|q| < e^\pi$, apart from the specified poles at 0. For a given $r < e^\pi$,

choose r_1 with $r < r_1 < e^\pi$. Cauchy's coefficient estimates on $|q| = r_1$ bound each positive coefficient by a constant times r_1^{-m} ; summing $(r/r_1)^m$ gives the stated weighted absolute summability and, in particular, membership in ℓ^2 .

Define the candidate functions $\tilde{h}_\nu$ by (6.2)–(6.3). Their cut jumps and outer traces are (4.12) and (6.7). At w_0, w_c , their arguments have finite limits in the upper half-plane, so the functions are bounded on each incident side. At 0, the terms $a_2(\tau_w)$ and $a_3(\tau_w)$ are bounded by constant multiples of $|w|$ and $|w|^{1/2}$, respectively. Even after multiplication by the displayed logarithmic factors, they tend to zero. Only the specified simple poles remain. The endpoint hypotheses of Lemma 4.3 therefore hold. That lemma gives all the moment equations. It also identifies these candidate functions with the actual Cauchy integrals (4.11). This map is injective: if the four a_j vanish, the first triple in the six expressions following (6.6) vanishes; inversion forces the last triple to vanish, and hence $a_4 = a_5 = 0$. □

6.4 From the coefficient equations to a section

We will need one elementary endpoint estimate for Cauchy integrals. It is separate from the endpoint continuation theorem: that theorem will make the densities analytic, and the following lemma will then control the two remaining components at the elliptic vertices.

Lemma 6.2 (Analytic densities at an endpoint). *Suppose finitely many finite analytic arcs end at w_* , with integrable densities. Suppose the density with respect to the directed differential dw on each arc extends holomorphically to a neighborhood of w_* . The sum of their Cauchy integrals is bounded, on each incident sector near w_* , by*

$$C(1 + |\log |w - w_*||).$$

Adding a function holomorphic near w_* preserves this bound.

Proof. Restrict an arc to a small neighborhood where its density g is holomorphic. For an evaluation point ζ there, write

$$\frac{g(w)}{w - \zeta} = \frac{g(\zeta)}{w - \zeta} + \frac{g(w) - g(\zeta)}{w - \zeta}.$$

The second quotient is uniformly bounded, including its removable diagonal value. The integral of the first term is $g(\zeta)$ times the difference of logarithms at the two arc endpoints. On each incident sector their arguments can be chosen in a fixed bounded interval. The endpoint different from w_* stays away from ζ , and the endpoint w_* gives the asserted logarithm. The portions of all arcs outside the neighborhood give holomorphic bounded terms. Sum over the finitely many arcs. □

Lemma 6.3. *Every $\mathbf{c} \in \ker_{\ell^2(\mathcal{I}_{24})} \mathbf{M}$ determines a holomorphic section of $\mathcal{E}$ whose first four components have the prescribed coefficient series.*

Proof. Fix $\mathbf{c} \in \ker \mathbf{M}$. Proposition 5.3 gives its local holomorphic continuation. We construct the section on a fundamental cell, check every side identification, remove the vertices in local orbifold charts, and then verify the cusp conditions.

The cell and its analytic domains.

Use the Gaussian-parameter cell

$$\mathcal{R} = \{z \in \mathbb{C} : 0 < \Im z < 1, \Re z > 0, |z| > 1, |z - i| > 1\}.$$

Its image under $\tau = iz$ is the fundamental domain

$$-1 < \Re \tau < 0, \quad |\tau| > 1, \quad |\tau + 1| > 1,$$

with its two halves related to the usual fundamental domain by a translation. For the following domain checks write $\sigma = \Re z$ and $t = \Im z$. On the part $\sigma > 1$ the four coefficient functions already exist. On the cap $\sigma < 1$, both z and $z - i$ lie in the two respective halves of $\mathcal{D}$. Indeed,

$$\sqrt{1 - \sigma^2} < t < 1 - \sqrt{1 - \sigma^2}.$$

Consequently the inverse of the matrix in (5.1) defines all four coefficient functions on the cap. Its determinant is $4i$. The original boundary identities glue this definition to the original functions across $\sigma = 1$. Thus all four a_j are holomorphic on this entire cell.

The two other functions can also be defined there. For each $z \in \mathcal{R}$, evaluate the Cauchy functions at $e^{-2\pi z} = \omega(1/z)$. This point lies in $\Omega \setminus [0, w_0]$. To check the only nontrivial case, put $r = \min(t, 1 - t) \leq 1/2$. The upper boundary height of the logarithmic image of Ω is

$$h_\Omega(r) = \frac{1 + \sqrt{1 - 4r^2}}{2},$$

whereas the cell has $\sigma > \sqrt{1 - r^2} \geq h_\Omega(r)$. For the last inequality put $x = \sqrt{1 - r^2}$. Then $2x - 1 \geq 0$, and $(2x - 1)^2 - (1 - 4r^2) = 4(1 - x) \geq 0$. This also proves $1/z \in \mathcal{W}$, the domain on which $\mathbf{h}_{\text{par}}(1/z)$ was defined. Since $\tau = iz$ has $-1 < \Re \tau < 0$, the chosen logarithm of $e^{-2\pi z}$ corresponds to $\tau + 1$. Therefore set

$$a_4(\tau) = h_0(e^{2\pi i \tau}) - 2(\tau + 1)a_2(\tau) - 2a_3(\tau), \quad (6.9)$$

$$a_5(\tau) = h_1(e^{2\pi i \tau})/i + (\tau^2 + \tau)a_2(\tau) + (2\tau + 1)a_3(\tau). \quad (6.10)$$

The translation sides. The sides $t = 0, 1$ of $\mathcal{R}$ have $\sigma > 1$. There the original series converge normally. Their parity gives the translation law for the first four components. The two traces of $h_\nu(e^{2\pi i \tau})$ differ by (4.12); subtracting the polynomial terms displayed in (6.10) cancels that difference. For example, across the positive slit the logarithmic parameter changes from τ to $\tau + 1$; $a_2(\tau + 1) = a_2(\tau)$ and $a_3(\tau + 1) = -a_3(\tau)$. The difference of the terms $2\tau a_2 - 2a_3$ is $-2a_2 - 4a_3 = 2A_-$, exactly the first jump. Subtracting the terms in (6.3) similarly gives $2i\tau(a_2 + 2a_3) = -\pi^{-1}(\log s)A_-$. The remaining components a_4, a_5 consequently match periodically. Lemma 4.2 glues the functions across these open sides, giving the full translation law.

The curved sides. We specify the ordered Cauchy inputs and the recovery matrices to check these identifications without assuming any continuation law. Write

$$\mathbf{h}_{\text{cell}}(z) = \begin{pmatrix} h_{0,\text{par}}(z) \\ h_{1,\text{par}}(z)/i \end{pmatrix}.$$

On the cap the four coefficient functions are recovered from

$$\begin{pmatrix} \mathbf{h}_{\text{cell}}(z) \\ \mathbf{h}_{\text{cell}}(z - i) \end{pmatrix} = \mathbf{S}_{\text{cell}}(\tau) \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix}, \quad \tau = iz, \quad (6.11)$$

where

$$\mathbf{S}_{\text{cell}}(\tau) = \begin{pmatrix} 1 & \tau + 1 & \tau^2 + 2\tau & 1 \\ \tau^{-1} & 1 + \tau^{-1} & \tau + 1 & 2 + \tau^{-1} \\ -1 & -\tau & 1 - \tau^2 & 1 \\ -(\tau + 1)^{-1} & -\tau/(\tau + 1) & -\tau & -(2\tau + 1)/(\tau + 1) \end{pmatrix}, \quad \det \mathbf{S}_{\text{cell}} = -4.$$

The relation to the earlier paired-trace matrix is

$$\mathbf{S}_{\text{cell}}(\tau) = \text{diag}(1, -i, 1, -i) \mathbf{S}_{\text{pair}}(-i\tau) \text{diag}(1, i, -1, 1).$$

The two diagonal factors simply implement the changes from $\mathbf{h}_{\text{par}}$ to $\mathbf{h}_{\text{cell}}$ and from $\mathbf{a}_{\text{ser}}$ to $(a_0, a_1, a_2, a_3)^T$. Define a six-by-six rational recovery matrix $\mathbf{S}_{\text{rec}}^{(24)}(\tau)$ by the following recipe on three two-component inputs $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$, where $\mathbf{b}_j = (b_{j,0}, b_{j,1})^T$:

$$\begin{aligned} (a_0, a_1, a_2, a_3)^T &= \mathbf{S}_{\text{cell}}(\tau)^{-1} (\mathbf{b}_1^T, \mathbf{b}_2^T)^T, \\ a_4 &= -\tau^{10} b_{3,0} - 2(\tau + 1)a_2 - 2a_3, \\ a_5 &= -\tau^{10} b_{3,1} + (\tau^2 + \tau)a_2 + (2\tau + 1)a_3. \end{aligned}$$

The actual six functions on the cap are thus

$$\mathbf{a}(\tau) = \mathbf{S}_{\text{rec}}^{(24)}(\tau) \begin{pmatrix} \mathbf{h}_{\text{cell}}(z) \\ \mathbf{h}_{\text{cell}}(z - i) \\ \mathbf{h}_{\text{cell}}(1/z) \end{pmatrix}.$$

For the third input, $\mathbf{h}_{\text{par}}(1/z) = z^{-10} \mathbf{h}(e^{-2\pi z})$, so this expression agrees with (6.10).

Consider the curved side $|z| = 1$, $0 < \Im z < 1/2$, and put $z_1 = z - i$, $z_2 = 1/z + i$. Its partner is z_2 on the other curved side. The elementary identities

$$\frac{1}{z_2} = \frac{1}{z_1} - i, \quad e^{-2\pi/(1/z_2)} = e^{-2\pi/z}$$

and the single-valuedness of $\mathbf{h}$ across the negative real axis give

$$\mathbf{h}_{\text{cell}}(z_2) = \tau^{-10} \mathbf{h}_{\text{cell}}(z_1), \quad \mathbf{h}_{\text{cell}}(z_2 - i) = \mathbf{h}_{\text{cell}}(1/z), \quad \mathbf{h}_{\text{cell}}(1/z_2) = (1 + \tau)^{-10} \mathbf{h}_{\text{cell}}(z). \quad (6.12)$$

These identities hold in a neighborhood of each open side point. For example, on that side $e^{-2\pi/z_1}$ and $e^{-2\pi/z_2}$ are the same negative real point strictly inside Ω , so this matching does not cross the cut or appeal to an unknown continuation.

Let $\mathbf{S}_{\text{match}}^{(24)}(\tau)$ denote the block matrix

$$\begin{pmatrix} 0 & \tau^{-10} I_2 & 0 \\ 0 & 0 & I_2 \\ (1 + \tau)^{-10} I_2 & 0 & 0 \end{pmatrix}.$$

Substituting the displayed recovery recipe gives the rational identity

$$\mathbf{J}_{\text{tr}} \mathbf{S}_{\text{rec}}^{(24)}(-1/\tau - 1) \mathbf{S}_{\text{match}}^{(24)}(\tau) = \tau^{-10} \mathbf{J}_{\text{inv}}(\tau) \mathbf{S}_{\text{rec}}^{(24)}(\tau). \quad (6.13)$$

All entries are specified above, so this can be checked by direct multiplication; no identities of special modular forms enter this multiplication. Together with (6.12), it proves precisely the inversion law (6.5) across the paired curved sides. This verifies the gluing for the actual solution, rather than imposing that law on it.

The elliptic vertices and extension to $\mathbb{H}$. At $z = e^{i\pi/6}$, all three Cauchy arguments in the recovery formula equal $-e^{-\pi\sqrt{3}}$, an interior point of the negative real interval in Ω . The matrices and coordinate maps are nonsingular there. The six recovered functions therefore extend holomorphically to a full neighborhood of this vertex in the parameter plane.

At $z = 1$ and $1+i$, Proposition 5.3 makes the four coefficient functions holomorphic on neighborhoods of the vertices. The outer densities near w_0 are consequently holomorphic functions on each incident arc: the map ω has nonzero derivative at $z = 1$, and all the factors in (4.6) are holomorphic there. The segment density is also holomorphic near $w_0 > 0$, using the local square root and real logarithm in (4.6). Lemma 6.2, with the correction $1/w$ holomorphic near w_0 , gives

$$|h_\nu(w)| \leq C(1 + |\log |w - w_0||)$$

in each incident sector. Since $e^{-2\pi z} - w_0$ has a simple zero at $z = 1$, the definitions (6.10) give the same logarithmic bound for a_4, a_5 at the corresponding vertex.

For completeness, the gluing at a vertex takes place on the upper-half-plane cover, not merely on the quotient. Take a sufficiently small disk about the elliptic fixed point. Finitely many translates of the incident cell sectors fill the punctured disk. At the point of order two, translation first joins the sectors represented by $z = 1$ and $1 + i$; inversion supplies the remaining sectors. At the point of order three use the three sectors related by the order-three transformation. Assign functions on successive sectors by the stated side laws. The open-side identities already proved glue adjacent assignments holomorphically. A complete turn composes either the order-two relation or the order-three relation in (6.6), including the scalar factors; the product is the identity. Thus the assignment is single-valued on this punctured disk.

All transition matrices and their inverses are holomorphic and bounded on a smaller disk. The sector bounds therefore give a common bound $C(1 + |\log |\tau - \tau_*||)$ in all directions. For every $k \geq 1$, the coefficient of $(\tau - \tau_*)^{-k}$ has absolute value at most $Cr^k(1 + |\log r|)$, which tends to zero. Every negative Laurent coefficient vanishes. The six functions extend holomorphically across the fixed point, and equivariance there follows by continuity. This is precisely holomorphy in the local orbifold chart.

Now extend over all translates of the cell. Across an open side the assignments agree by the identities already checked; at vertices they agree by the local argument above. The relations of $\mathrm{PSL}_2(\mathbb{Z})$ ensure independence of the word used to reach a translated cell. We obtain a holomorphic vector on $\mathbb{H}$ satisfying (6.5). The retained central element $-I_2$ acts trivially, so this vector defines a section on the modular orbifold away from the cusp.

The modular cusp. The first four components still have their original series for $\Im \tau > 1$, and hence the first four exponents of (6.8). The periodic components a_4, a_5 are single-valued holomorphic functions of $\mathfrak{q}$ on a punctured disk. On a slit version of that disk their expressions are the Cauchy functions minus the explicit jumping terms. The Cauchy kernel area estimate in Lemma 5.1 holds also on a disk containing zero; the separate correction $1/w$ is itself area-integrable there. The subtracted terms are bounded by constants times $|w| |\log w|^2 + |w|^{1/2}(1 + |\log w|)$. Thus a_4, a_5 , as functions of $\mathfrak{q}$, are area-integrable. Lemma 5.2 leaves at most a simple pole in each.

We compute those poles without any unproved boundary estimate at zero. Split the measure in (4.11) into its arc and segment parts, and take $w = -r < 0$, $r \downarrow 0$. The arc stays a positive distance from zero, so multiplying its Cauchy integral by w makes it tend to zero. On the segment, with variable $s > 0$,

$$\left| \frac{w}{s - w} \right| = \frac{r}{s + r} \leq 1, \quad \frac{w}{s - w} \longrightarrow 0.$$

Finite total variation and the absence of an atom at zero give the same conclusion by dominated convergence. The explicit correction contributes exactly $m_{\nu,0}/(2\pi i)$ to $\lim_{w \rightarrow 0} w h_\nu(w)$. The jumping terms contribute zero to that limit. Hence the principal parts of a_4, a_5 are, respectively,

$$\frac{m_{0,0}}{2\pi i} \mathfrak{q}^{-1}, \quad -\frac{m_{1,0}}{2\pi} \mathfrak{q}^{-1}.$$

No restriction is imposed on their constant terms. These are exactly the last two conditions in (6.8), so the section extends across the cusp in the prescribed frame and recovers the original coefficients. $\square$

Proposition 6.4 (Boundary–bundle correspondence). *The coefficient map is an isomorphism*

$$H^0(\overline{\mathcal{X}}, \mathcal{E}) \simeq \ker_{\ell^2(\mathcal{I}_{24})} \mathbf{M}.$$

For every vector in this kernel, the infinite coefficient sequences are absolutely summable with weight r^m , for each $1 < r < e^\pi$.

Proof. Lemma 6.1 gives an injective map to the coefficient kernel and its weighted summability. Lemma 6.3 gives a section over every kernel vector, so that map is surjective. Because the first map is injective, the recovered section is unique and the two constructions are inverse. No known magic function or Fourier interpolation theorem enters either construction. $\square$

7 The index and existence

The boundary problem has now become a problem about holomorphic sections of $\mathcal{E}$. Write $h^j(\mathcal{E})$ for the dimension of its sheaf cohomology group $H^j(\mathcal{E})$, for $j = 0, 1$. In particular, h^0 counts the independent holomorphic sections. Riemann–Roch computes their difference, the Euler characteristic $\chi(\mathcal{E}) = h^0(\mathcal{E}) - h^1(\mathcal{E})$, from the transition laws. A positive value will force a nonzero section and hence a nonzero coefficient solution. This is the index we compute; it is not an index of the compact moment map $\mathbf{M}$.

Proposition 7.1. *The bundle $\mathcal{E}$ has Euler characteristic 1. In particular, $\ker_{\ell^2(\mathcal{I}_{24})} \mathbf{M} \neq \{0\}$.*

Proof. The determinant automorphy factor has scalar weight -60 and character

$$\varepsilon_{\det}(S) = \varepsilon_{\det}(T) = -1, \quad \varepsilon_{\det}(ST) = 1.$$

The sum of the cusp exponents in (6.8) is $-7/2$. Let $\mathcal{L}_k$ denote the line bundle whose sections are scalar modular forms of weight k , holomorphic at the cusp. The determinant bundle $\det \mathcal{E}$ is a line bundle: its transition functions are the determinants of those of $\mathcal{E}$. The line-bundle extension formula [1, Theorem 3.4] gives

$$\det \mathcal{E} \simeq \mathcal{L}_{-60-12(-7/2)} = \mathcal{L}_{-18}. \quad (7.1)$$

For the character convention, Dedekind’s eta function is $\eta(\tau) = \mathfrak{q}^{1/24} \prod_{n \geq 1} (1 - \mathfrak{q}^n)$, where $\mathfrak{q}^{1/24} = e^{2\pi i \tau / 24}$. The character $\varepsilon_{\det}$ is the sixth power of the character of η^2 . Its allowed exponent is $1/2 - 4$. Thus (7.1) includes both the character and the integer shift of the cusp frame. One can see the isomorphism directly: divide a determinant section by η^{12} , of weight 6, character $\varepsilon_{\det}$, and leading exponent $1/2$. The result has trivial character, weight -66 , and lowest allowed exponent -4 . Set $\Delta_{\text{mod}} = \eta^{24}$, the modular discriminant. Multiplication by Δ_{mod}^4 gives weight -18 and a holomorphic cusp frame. These multipliers have no zeros in $\mathbb{H}$, and their leading cusp factors are nonzero after the specified powers have been removed; hence this is an isomorphism of the extended line bundles.

It remains to account for the elliptic fibers. Put $\rho_3 = e^{2\pi i / 3}$. The elliptic points are fixed by S and ST , respectively. The actions of these stabilizers on the fibers, including the scalar weight factor, are the following two matrices, denoted by

$$\mathbf{S}_{\text{ell},2}^{(24)} = -\mathbf{J}_{\text{inv}}(i), \quad \mathbf{S}_{\text{ell},3}^{(24)} = \rho_3 \mathbf{J}_{\text{inv}}(\rho_3 + 1) \mathbf{J}_{\text{tr}}.$$

Using (6.4), with an independent polynomial variable x , gives

$$\det(xI_6 - \mathbf{S}_{\text{ell},2}^{(24)}) = (x-1)^3(x+1)^3, \quad \det(xI_6 - \mathbf{S}_{\text{ell},3}^{(24)}) = (x-1)^2(x-\rho_3)^2(x-\rho_3^2)^2.$$

These identities can be verified by reducing the rational matrices modulo $\tau^2 + 1$ and $\tau^2 + \tau + 1$, respectively, and using (6.6). All nontrivial elliptic traces are zero, and the central element $-I_2$ acts trivially.

The Riemann–Roch formula for a general holomorphic bundle on $\mathbb{P}(4, 6)$, in this case, is

$$\chi(\mathcal{E}) = \frac{5 \operatorname{rank} \mathcal{E} + \deg_{\mathcal{L}_1}(\det \mathcal{E})}{12}.$$

Here the degree is normalized by $\deg_{\mathcal{L}_1}(\mathcal{L}_k) = k$. The elliptic correction terms vanish because of the traces just computed. The identity and the central stabilizer contribute equal terms, since the latter acts trivially; together they give the denominator 12 displayed here. This is the general bundle formula in [1, Theorem 6.1]; no constant matrix representation is required. It gives $\chi(\mathcal{E}) = (30 - 18)/12 = 1$. Since $h^0 - h^1 = 1$, a nonzero holomorphic section exists. Proposition 6.4 sends it to a nonzero coefficient solution. $\square$

The cusp count has a useful independent check. With the baseline exponents $(0, 0, 0, 1/2, 0, 0)$, the determinant is $\mathcal{L}_{-66}$ and the index is -3 . Allowing the poles in a_0, a_1, a_4, a_5 adds $2+1+1+1 = 5$; removing the constant term of a_2 subtracts 1. The resulting index is again $-3 + 5 - 1 = 1$.

8 Vanishing and uniqueness

The index proves existence but does not by itself determine the dimension. To prove uniqueness we show that normalization is injective on the kernel. The following constant change of variables separates the system of six functions into two blocks of three. We analyze the blocks separately.

$$\begin{aligned} a_0^+ &= (a_0 + a_5)/2, & a_1^+ &= (a_1 + a_2 + a_4)/2, & a_2^+ &= a_2, \\ a_0^- &= (a_0 - a_5)/2, & a_1^- &= (a_1 - a_2 - a_4)/2, & a_2^- &= a_3. \end{aligned}$$

Here all functions on the right are evaluated at the same argument τ as the function on the left. The superscripts $+$ and $-$ distinguish the two invariant blocks of the gluing laws. They do not assert that these scalar functions are Fourier eigenfunctions. This is an invertible constant change of variables. Direct substitution in $\mathbf{J}_{\text{inv}}$ gives two separate systems:

$$a_0^+(-1/\tau) = \tau^{-12} a_0^+(\tau), \tag{8.1}$$

$$a_1^+(-1/\tau) = \tau^{-10} a_1^+(\tau) + 2\tau^{-11} a_0^+(\tau), \tag{8.2}$$

$$a_2^+(-1/\tau) = \tau^{-8} a_2^+(\tau) + \tau^{-9} a_1^+(\tau) + \tau^{-10} a_0^+(\tau), \tag{8.3}$$

$$\begin{pmatrix} a_0^- \\ a_1^- \\ a_2^- \end{pmatrix}(-1/\tau) = \tau^{-10} \begin{pmatrix} -1/2 & \tau^{-1} - \tau/2 & 3/2 \\ 0 & 1 & 0 \\ 1/2 & \tau/2 & 1/2 \end{pmatrix} \begin{pmatrix} a_0^- \\ a_1^- \\ a_2^- \end{pmatrix}(\tau). \tag{8.4}$$

All these functions except a_2^- are periodic; a_2^- is antiperiodic. These identities follow from the derived matrix; none of the functions is selected from a known magic-function formula.

Proposition 8.1 (Vanishing at zero normalization). *If $\mathbf{c} \in \ker \mathbf{M}$ and $\lambda(\mathbf{c}) = 0$, then $\mathbf{c} = 0$.*

Proof. By Proposition 6.4, $\mathbf{c}$ gives global holomorphic functions with the transformation laws above. We use $h \ll g$, with $g \geq 0$, to mean $|h| \leq Cg$ near the indicated limit for a fixed constant C . The Poisson identity (3.4) gives $c_{0,-4} = 0$. Consequently the largest pole in a_0 drops from order two to order one in $\mathfrak{q}$. The cusp bounds are now, as $\mathfrak{q} \rightarrow 0$,

$$\begin{aligned} |a_0^+| + |a_1^+| + |a_0^-| + |a_1^-| &\ll |\mathfrak{q}|^{-1}, \\ |a_2^+| &\ll |\mathfrak{q}|, \quad |a_2^-| \ll |\mathfrak{q}|^{1/2}. \end{aligned} \tag{8.5}$$

Recall the product expansion of the modular discriminant:

$$\Delta_{\text{mod}}(\tau) = \mathfrak{q} \prod_{n \geq 1} (1 - \mathfrak{q}^n)^{24} = \mathfrak{q} - 24\mathfrak{q}^2 + \sum_{n \geq 3} \delta_n \mathfrak{q}^n.$$

The coefficients δ_n are defined by the product; the displayed series converges for $|\mathfrak{q}| < 1$. This is a weight-12 cusp form with no zeros in the upper half-plane. Let Mod_k be the space of scalar modular forms of weight k for $\text{SL}_2(\mathbb{Z})$, holomorphic on the upper half-plane and at the cusp. Let $E_4 = 1 + 240 \sum_{n \geq 1} (\sum_{\ell|n} \ell^3) \mathfrak{q}^n$ be the normalized Eisenstein series of weight 4. We use only $\text{Mod}_0 = \mathbb{C}$, $\text{Mod}_2 = \{0\}$, $\text{Mod}_4 = \mathbb{C}E_4$, and the weight-12 bound $\text{ord}_\infty f \leq 1$ for a nonzero holomorphic form. These are immediate consequences of the usual valence formula. See [9, Examples 4.13–4.15 and Proposition 4.16]; his weight indexing is half the weight used here.

Since a_0^+ has weight -12 and at most a simple cusp pole, $\Delta_{\text{mod}} a_0^+$ is a holomorphic weight-zero form. Thus

$$a_0^+ = \frac{c}{\Delta_{\text{mod}}} \quad \text{for a constant } c.$$

Write $\partial_\tau = d/d\tau$. Differentiating its weight- -12 law gives

$$\begin{aligned} (\partial_\tau a_0^+)(-1/\tau) &= \tau^{-10} \partial_\tau a_0^+ - 12\tau^{-11} a_0^+, \\ (\partial_\tau^2 a_0^+)(-1/\tau) &= \tau^{-8} \partial_\tau^2 a_0^+ - 22\tau^{-9} \partial_\tau a_0^+ + 132\tau^{-10} a_0^+. \end{aligned}$$

It follows that $a_1^+ + \frac{1}{6} \partial_\tau a_0^+$ is a weight- -10 modular form with at most a simple cusp pole. Multiplication by Δ_{mod} puts it in $\text{Mod}_2 = \{0\}$, so

$$a_1^+ = -\frac{1}{6} \partial_\tau a_0^+.$$

Using this in the third law in (8.3) shows that $a_2^+ - \frac{1}{132} \partial_\tau^2 a_0^+$ is a weight- -8 modular form with at most a simple cusp pole. Multiplying by Δ_{mod} now puts it in Mod_4 :

$$a_2^+ = \frac{c}{132} \partial_\tau^2 \left(\frac{1}{\Delta_{\text{mod}}} \right) + b \frac{E_4}{\Delta_{\text{mod}}} \tag{8.6}$$

for a constant b .

Only two elementary coefficients are needed to finish this block. The coefficient of $\mathfrak{q}$ in E_4 is 240, and its constant coefficient is 1. Using the product for Δ_{mod} , we obtain, as $\mathfrak{q} \rightarrow 0$,

$$\partial_\tau^2 (1/\Delta_{\text{mod}}) - (2\pi i)^2 \mathfrak{q}^{-1} \ll |\mathfrak{q}|, \quad \frac{E_4}{\Delta_{\text{mod}}} - \mathfrak{q}^{-1} - 264 \ll |\mathfrak{q}|.$$

The absence of both a pole and a constant term in a_2^+ , by (8.5), gives

$$\frac{(2\pi i)^2}{132} c + b = 0, \quad 264b = 0.$$

Therefore $b = c = 0$ and all three a_j^+ vanish.

For the other block, a_1^- is a weight-10 modular form with at most a simple cusp pole. Again $\Delta_{\text{mod}} a_1^- \in \text{Mod}_2 = \{0\}$, so $a_1^- = 0$. For this part of the proof, define the two-by-two matrices $S_{\text{lev},S}, S_{\text{lev},T}$ in the following display. The remaining two functions obey

$$\begin{pmatrix} a_0^- \\ a_2^- \end{pmatrix}(-1/\tau) = \tau^{-10} S_{\text{lev},S} \begin{pmatrix} a_0^- \\ a_2^- \end{pmatrix}(\tau), \quad S_{\text{lev},S} = \frac{1}{2} \begin{pmatrix} -1 & 3 \\ 1 & 1 \end{pmatrix}, \quad S_{\text{lev},T} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The matrices satisfy $S_{\text{lev},S}^2 = S_{\text{lev},T}^2 = (S_{\text{lev},S} S_{\text{lev},T})^3 = I_2$; their representation factors through $\text{PSL}_2(\mathbb{F}_2)$. Thus

$$h(\tau) = \Delta_{\text{mod}}(\tau) a_2^-(\tau)$$

is a weight-2 scalar form for the principal congruence subgroup $\Gamma(2) = \{\gamma \in \text{SL}_2(\mathbb{Z}) : \gamma \equiv I_2 \pmod{2}\}$. Write γ_{jk} for the entries of $\gamma \in \text{SL}_2(\mathbb{Z})$. The weight- k slash operator is

$$(h|_k \gamma)(\tau) = (\gamma_{21}\tau + \gamma_{22})^{-k} h\left(\frac{\gamma_{11}\tau + \gamma_{12}}{\gamma_{21}\tau + \gamma_{22}}\right).$$

All its cusp transforms are bounded: every weight-2 slash transform of h is a constant linear combination of $\Delta_{\text{mod}} a_0^-$, which is bounded, and $\Delta_{\text{mod}} a_2^-$, whose absolute value is at most a constant times $|\mathbf{q}|^{3/2}$.

Here is a reduction to the level-one valence bound, avoiding any dimension formula for level two. If $h \neq 0$, form the product

$$p_h = \prod_{\gamma \in \Gamma(2) \backslash \text{SL}_2(\mathbb{Z})} (h|_2 \gamma).$$

It is a nonzero holomorphic weight-12 form for the full modular group. Among the six cosets choose I_2 and T ; their factors are h and $-h$. Since $|h| \ll |\mathbf{q}|^{3/2}$ and the other factors are bounded, $|p_h| \ll |\mathbf{q}|^3$. Thus its order of vanishing at the cusp is at least three. This contradicts the weight-12 valence bound. Therefore $a_2^- = 0$. The inversion formula then gives $a_0^- = 0$ also.

All six functions vanish. Inverting their constant change of variables gives $a_0 = a_1 = a_2 = a_3 = 0$, hence all original coefficients vanish. $\square$

Proof of Theorem 2.1. Proposition 7.1 gives a nonzero homogeneous solution, and Proposition 8.1 makes λ injective on the homogeneous kernel. The kernel is therefore one-dimensional, and its unique vector with $\lambda = 1$ is $\mathbf{c}_*$. Complex conjugation preserves the real coefficient equations and the normalization, so $\mathbf{c}_*$ is real. Proposition 6.4 proves the weighted absolute summability. Lemmas 3.1 and 3.2 give the claimed values and derivatives. $\square$

The proof also shows $h^0(\mathcal{E}) = 1$ and $h^1(\mathcal{E}) = 0$. No explicit coefficient of the known magic function has entered the existence or uniqueness argument.

9 A boundedly invertible coefficient equation

The sample equations already specify the solution uniquely. For approximation it is useful to replace them by an equivalent equation whose inverse is bounded on the full Hilbert space. Write $\text{Id}_{\mathcal{I}_{24}}$ for the identity on $\ell^2(\mathcal{I}_{24})$. The extra ingredient is a compact linear operator that fixes every exact solution. Appendix B defines such an operator $\mathbf{K} : \ell^2(\mathcal{I}_{24}) \rightarrow \ell^2(\mathcal{I}_{24})$ by fixed Cauchy integrals, local disk averages, and coefficient extraction. It uses no unknown kernel vector and satisfies

$$\mathbf{K}\mathbf{c} = \mathbf{c} \quad (\mathbf{c} \in \ker \mathbf{M}), \quad \|(\text{Id}_{\mathcal{I}_{24}} - \mathbf{P}_J)\mathbf{K}\| \leq C_K R^{-J}, \quad (9.1)$$

where $R > 1$ is fixed and $J \in \mathbb{N}$. The orthogonal projection $P_J : \ell^2(\mathcal{I}_{24}) \rightarrow \ell^2(\mathcal{I}_{24})$ is defined to be the identity on the five entries with nonpositive exponent and to act on each infinite sequence by setting its entries with $m > J$ to zero.

Use the usual coefficient inner products on $\ell^2(\mathcal{I}_{24})$, on the moment space $\ell^2(\mathcal{J}_{24})$, and on $\mathbb{C}$. A superscript $*$ denotes the adjoint with respect to these inner products. Define

$$\begin{aligned} \mathbf{L} &= (\text{Id}_{\mathcal{I}_{24}} - \mathbf{K})^*(\text{Id}_{\mathcal{I}_{24}} - \mathbf{K}) + \mathbf{M}^*\mathbf{M}, \\ \mathbf{N} &= \mathbf{L} + \lambda^*\lambda. \end{aligned} \tag{9.2}$$

Both are bounded positive self-adjoint operators. They are the identity plus compact operators, and

$$\ker \mathbf{L} = \ker \mathbf{M},$$

because $\langle \mathbf{L}\mathbf{c}, \mathbf{c} \rangle = \|(\text{Id}_{\mathcal{I}_{24}} - \mathbf{K})\mathbf{c}\|_2^2 + \|\mathbf{M}\mathbf{c}\|_2^2$.

Theorem 9.1 (Bounded inverse). *There is a constant $c_{\text{coer},24} > 0$ such that, for every $\mathbf{c} \in \ell^2(\mathcal{I}_{24})$,*

$$\begin{aligned} \langle \mathbf{N}\mathbf{c}, \mathbf{c} \rangle &= \|(\text{Id}_{\mathcal{I}_{24}} - \mathbf{K})\mathbf{c}\|_2^2 + \|\mathbf{M}\mathbf{c}\|_2^2 + |\lambda(\mathbf{c})|^2 \\ &\geq c_{\text{coer},24} \|\mathbf{c}\|_2^2. \end{aligned} \tag{9.3}$$

Consequently $\mathbf{N}$ is boundedly invertible and

$$\mathbf{c}_* = \mathbf{N}^{-1}\lambda^*(1). \tag{9.4}$$

Proof. If the inequality failed, there would be unit vectors $\mathbf{c}_j$ with

$$(\text{Id}_{\mathcal{I}_{24}} - \mathbf{K})\mathbf{c}_j \rightarrow 0, \quad \mathbf{M}\mathbf{c}_j \rightarrow 0, \quad \lambda(\mathbf{c}_j) \rightarrow 0.$$

Compactness of $\mathbf{K}$ gives a convergent subsequence of $\mathbf{K}\mathbf{c}_j$; the same subsequence of $\mathbf{c}_j$ converges to a unit vector $\mathbf{c}$. Then $\mathbf{M}\mathbf{c} = 0$ and $\lambda(\mathbf{c}) = 0$, contradicting Proposition 8.1. Coercivity gives closed range and zero kernel. Self-adjointness then gives dense range, hence surjectivity.

Let $\mathbf{b} \neq 0$ be a vector in $\ker \mathbf{M}$. For $\mathbf{c} = \mathbf{N}^{-1}\lambda^*(1)$, pair the equation with $\mathbf{b}$. Since $\lambda(\mathbf{b}) \neq 0$ and $\mathbf{L}\mathbf{b} = 0$, this gives $\lambda(\mathbf{c}) = 1$. The equation becomes $\mathbf{L}\mathbf{c} = 0$. Thus $\mathbf{c}$ is the normalized coefficient solution. $\square$

This also gives convergence of the exact Galerkin approximations. On $P_J\ell^2(\mathcal{I}_{24})$ solve

$$P_J\mathbf{N}P_J\mathbf{c}_J = P_J\lambda^*(1).$$

Every compression is coercive with the same constant $c_{\text{coer},24}$. Galerkin orthogonality and (9.3) give

$$\|\mathbf{c}_J - \mathbf{c}_*\|_2 \leq \frac{\|\mathbf{N}\|}{c_{\text{coer},24}} \|(\text{Id}_{\mathcal{I}_{24}} - P_J)\mathbf{c}_*\|_2 \rightarrow 0.$$

The right side has a geometric tail by Theorem 2.1. This is a convergence statement for the positive equation (9.2); it does not assert stability of arbitrary finite truncations of the original compact moment map $\mathbf{M}$. The proof establishes $c_{\text{coer},24} > 0$ without supplying a numerical value. The compressed matrix still contains integrals and infinite inner products, through $\mathbf{K}^*\mathbf{K}$ and $\mathbf{M}^*\mathbf{M}$. A certified numerical implementation would additionally require error bounds for their evaluation, a computable coercivity bound, and control of the resulting perturbation of the finite equations. No such implementation or numerical efficiency claim is needed for the existence theorem.

10 Comparison with the known function in dimension 24

Existence and uniqueness in the coefficient space have now been proved. We next identify the answer. This comparison is not used in either dimension's existence or uniqueness proof.

10.1 Coefficient identification

Use $\mathfrak{q} = e^{2\pi i\tau}$, and normalize

$$E_2 = 1 - 24 \sum_{n \geq 1} \sigma_1(n) \mathfrak{q}^n, \quad E_4 = 1 + 240 \sum_{n \geq 1} \sigma_3(n) \mathfrak{q}^n, \quad E_6 = 1 - 504 \sum_{n \geq 1} \sigma_5(n) \mathfrak{q}^n,$$

where $\sigma_j(n) = \sum_{\ell|n} \ell^j$. Here $\Delta_{\text{mod}} = (E_4^3 - E_6^2)/1728$. For the comparison only, set

$$\begin{aligned} r_0 &= -49E_4^3 + 25E_6^2, & r_1 &= 48E_6E_4^2, & r_2 &= 25E_4^4 - 49E_4E_6^2, \\ \varphi_{24,2} &= \frac{r_2 + r_1E_2 + r_0E_2^2}{\Delta_{\text{mod}}^2}, & \varphi_{24,1} &= \frac{-6r_1 - 12E_2r_0}{\Delta_{\text{mod}}^2}, & \varphi_{24,0} &= \frac{-36r_0}{\Delta_{\text{mod}}^2}. \end{aligned}$$

Let

$$\theta_2(\tau) = \sum_{n \in \mathbb{Z}} e^{\pi i(n+1/2)^2\tau}, \quad \theta_4(\tau) = \sum_{n \in \mathbb{Z}} (-1)^n e^{\pi i n^2\tau}.$$

Writing $x = \theta_4^4$, $y = \theta_2^4$, define

$$\psi_{24,I} = \frac{7x^5y^2 + 7x^6y + 2x^7}{\Delta_{\text{mod}}^2}, \quad \psi_{24,S} = -\frac{7y^5x^2 + 7y^6x + 2y^7}{\Delta_{\text{mod}}^2}.$$

These are the ingredients of [4, Sections 2–4]: $\varphi_{24,2}$ is a weakly holomorphic quasimodular form of weight -8 and depth 2, and $\psi_{24,I}$ has weight -10 for $\Gamma(2)$. Let f_{known} be the normalized magic function constructed in [4], and define $\Phi_{\text{known}} : [0, \infty) \rightarrow \mathbb{R}$ by $f_{\text{known}}(\mathbf{y}) = \Phi_{\text{known}}(|\mathbf{y}|^2)$. For $u > 4$, it is given by

$$\Phi_{\text{known}}(u) = P(u) \int_0^\infty \left(\frac{\pi}{K_{24}^\varphi} t^{10} \varphi_{24,2}(i/t) - \frac{1}{K_{24}^\psi \pi} \psi_{24,I}(it) \right) e^{-\pi u t} dt, \quad (10.1)$$

Here $K_{24}^\varphi = 28304640$ and $K_{24}^\psi = 65520$. The profile extends to all $u \geq 0$. The required transformation identities are

$$\begin{aligned} t^{10} \varphi_{24,2}(i/t) &= t^2 \varphi_{24,2}(it) + \frac{t}{\pi} \varphi_{24,1}(it) - \frac{1}{\pi^2} \varphi_{24,0}(it), \\ \psi_{24,I}(it) &= -t^{10} \psi_{24,S}(i/t), & \psi_{24,S}(\tau) &= \psi_{24,I}(\tau) - \psi_{24,I}(\tau + 1). \end{aligned} \quad (10.2)$$

Here is an explicit translation into our coefficients. Use the cusp coordinate $\mathfrak{q} = e^{2\pi i\tau}$, with $\mathfrak{q}^{1/2} = e^{\pi i\tau}$. Define the scalar coefficients by

$$\begin{aligned} \varphi_{24,j}(\tau) &= \sum_m \varphi_{24,j,m} \mathfrak{q}^{m/2} & (j = 0, 1, 2), \\ \psi_{24,I}(\tau) &= \sum_m \psi_{24,I,m} \mathfrak{q}^{m/2}, & \psi_{24,S}(\tau) &= \sum_m \psi_{24,S,m} \mathfrak{q}^{m/2}. \end{aligned}$$

The indices m are integers and only finitely many negative powers occur. For example, $\varphi_{24,2,m} = [\mathfrak{q}^{m/2}] \varphi_{24,2}$ denotes the scalar coefficient of $\mathfrak{q}^{m/2}$ in this expansion. The brackets denote coefficient

extraction. The relation to our generating-series variable is $q = e^\pi \mathbf{q}^{1/2}$. The second subscript j in $\varphi_{24,j}$ is chosen so that its weight is $2j - 12$, matching the three polynomial degrees in the large-parameter integrals.

Split (10.1) at $t = 1$, and use (10.2) on the appropriate halves. The large-parameter coefficients before damping are

$$\begin{aligned}\gamma_{24,0,m} &= -\frac{\varphi_{24,0,m} + 432\psi_{24,I,m}}{K_{24}^\varphi \pi}, \\ \gamma_{24,1,m} &= \frac{\varphi_{24,1,m}}{K_{24}^\varphi}, \\ \gamma_{24,2,m} &= \frac{\pi \varphi_{24,2,m}}{K_{24}^\varphi},\end{aligned}\tag{10.3}$$

and the small-parameter coefficient is

$$\beta_{24,m} = \frac{\pi \varphi_{24,2,m}}{K_{24}^\varphi} + \frac{\psi_{24,S,m}}{K_{24}^\psi \pi}.$$

Each of $\varphi_{24,0}, \varphi_{24,1}, \varphi_{24,2}$ has only even powers of $\mathbf{q}^{1/2}$, and $\psi_{24,S,m} = (1 - (-1)^m)\psi_{24,I,m}$. Thus $\beta_{24,m} = \gamma_{24,2,m}$ for even m , and $\beta_{24,m} = -2\gamma_{24,0,m}$ for odd m . These are exactly the parity relations in (2.5). Because the elementary profiles $U_{m,k}$ and V_m include the factor $e^{\pi m}$, the corresponding sequence coefficients in (2.5) are

$$c_{j,m} = e^{-\pi m} \gamma_{24,j,m} \quad (m \text{ even}), \quad c_{3,m} = e^{-\pi m} \gamma_{24,0,m} \quad (m \text{ odd}).$$

In particular the two functions from the splitting in Section 8 satisfy

$$a_2^+ = -\frac{\pi}{K_{24}^\varphi} \varphi_{24,2}, \quad a_2^- = -\frac{1}{2K_{24}^\psi \pi} \psi_{24,S}.$$

The quasimodular and level-two structures thus agree with the structures derived from the boundary equations. The five coefficients of the correction functions are $c_{j,m} = e^{-\pi m} \gamma_{24,j,m}$ at $(j,m) = (0,-4), (0,-2), (0,0), (1,-2), (1,0)$. There are no other nonpositive modes. In particular the resulting finite part is

$$4R_{0,1} - \frac{R_{4,0}}{16380\pi} - \frac{14347}{1365\pi} R_{0,0} + \frac{1}{39} R_{2,1} - \frac{292}{4095\pi} R_{2,0}.\tag{10.4}$$

These identities follow by multiplication of the displayed series; they also check the normalization of the coefficient coordinates. Convergence of these expansions for $|\mathbf{q}^{1/2}| < 1$, with only the stated poles at 0, gives weighted absolute summability after the damping factor.

The split integral first identifies the two profiles for $u > 4$. Here is the analytic domain needed to extend this identity. Subtract the finitely many nonpositive exponential modes from the large- t integrand and integrate them explicitly, giving the entire correction functions in (10.4). The remaining large- t integrand decays at least as a polynomial times $e^{-\pi t}$, and the small- t integrand decays exponentially in $1/t$. The resulting integrals converge normally for complex u on compact subsets of $\Re u > -1$. They give the analytic continuation used in [4, Sections 2–4]. Our coefficient profile is holomorphic on this same half-plane by the normal-convergence estimates in Lemma 3.1; its possible negative integer poles lie outside it. The identity theorem gives equality throughout $\Re u > -1$, and in particular at all nonnegative real u . The known Fourier zeros and normalization now show that the coefficient vector solves (2.7). Theorem 2.1 identifies it with $\mathbf{c}_*$. Consequently the sign inequalities proved in [4] hold for $f_{\mathbf{c}_*}$. In this sense the coefficient construction recovers the Leech lattice magic function.

10.2 The alternative uniqueness argument from interpolation

The interpolation theorem of [5, Theorems 1.7 and 1.9] states that a radial Schwartz function on $\mathbb{R}^{24}$ is determined by the values and radial derivatives of itself and its Fourier transform at $\sqrt{2n}$, $n \geq 2$. For our normalized function, all these data vanish except the radial derivative

$$\left. \frac{d}{dr} \Phi_{\mathbf{c}_*}(r^2) \right|_{r=2} = 4\Phi'_{\mathbf{c}_*}(4) = -\frac{1}{16380}.$$

In the interpolation basis of that paper this says

$$f_{\mathbf{c}_*} = -\frac{1}{16380}b_2,$$

where b_2 has radial derivative 1 at 2 and all other interpolation data zero. It follows at once that the function is unique in the full radial Schwartz class with these data. Injectivity in Lemma 3.1 then gives coefficient uniqueness as well. This is an alternative argument, not a premise of our proof. Our independent uniqueness statement is within the coefficient class (2.5); the assertion about all radial Schwartz functions uses the interpolation theorem.

11 The construction in dimension 8

We now prove the analogous theorem for E_8 . The first nonzero squared length is 2, so the Fourier equations start one node earlier. This removes two correction functions and forces the constant moments to vanish. Gaussian inversion changes the scalar power from 10 to 2. We give the resulting system explicitly, check the continuation problem and its index, and prove uniqueness. None of these steps uses the comparison with the known function in Section 11.6.

11.1 The coefficient system and its theorem

Keep $P, U_{m,k}, R_{\ell,k}$ as defined in (2.1) and (2.3), and define

$$V_m^{(8)}(u) = e^{\pi m} P(u) \int_0^1 t^2 e^{-\pi(m/t+ut)} dt, \quad u \geq 0, \quad m \in \mathbb{N}. \quad (11.1)$$

The power is $d/2 - 2 = 2$: Gaussian inversion contributes $t^{-d/2}$, and the substitution $t \mapsto 1/t$ contributes its Jacobian. The large-parameter integrals $U_{m,k}$ do not change.

Define

$$\mathcal{I}_8 = \{(0, -2), (0, 0), (1, 0)\} \cup \mathcal{I}_+.$$

An eight-dimensional coefficient vector $\mathbf{c} = (c_{j,m})_{(j,m) \in \mathcal{I}_8}$ consists of the three numbers $c_{0,-2}, c_{0,0}, c_{1,0}$, three sequences $(c_{k,m})_{m \in 2\mathbb{N}}$, $k = 0, 1, 2$, and one sequence $(c_{3,m})_{m \in 2\mathbb{N}-1}$. For $p = 1, 2$, define

$$\|\mathbf{c}\|_{p,8}^p = |c_{0,-2}|^p + |c_{0,0}|^p + |c_{1,0}|^p + \sum_{m \in 2\mathbb{N}} \sum_{k=0}^2 |c_{k,m}|^p + \sum_{m \in 2\mathbb{N}-1} |c_{3,m}|^p. \quad (11.2)$$

Write $\ell^p(\mathcal{I}_8)$ for the space of these families with finite norm. Thus $\ell^p(\mathcal{I}_8) = \mathbb{C}^3 \oplus \ell^p(2\mathbb{N})^3 \oplus \ell^p(2\mathbb{N}-1)$. These are new coefficient vectors; they are not the dimension-24 solutions with two entries removed.

For $\mathbf{c} \in \ell^2(\mathcal{I}_8)$, set

$$\begin{aligned}\Phi_{\mathbf{c}}^{(8)}(u) &= c_{1,0}R_{0,1}(u) + c_{0,0}R_{0,0}(u) + e^{-2\pi}c_{0,-2}R_{2,0}(u) \\ &\quad + \sum_{m \in 2\mathbb{N}-1} c_{3,m}(U_{m,0}(u) - 2V_m^{(8)}(u)) \\ &\quad + \sum_{m \in 2\mathbb{N}} (c_{0,m}U_{m,0}(u) + c_{1,m}U_{m,1}(u) + c_{2,m}(U_{m,2}(u) + V_m^{(8)}(u))).\end{aligned}\quad (11.3)$$

Define $f_{\mathbf{c}}^{(8)}(\mathbf{y}) = \Phi_{\mathbf{c}}^{(8)}(|\mathbf{y}|^2)$ for $\mathbf{y} \in \mathbb{R}^8$, and define $G_{\mathbf{c}}^{(8)}$ on $[0, \infty)$ by $\hat{f}_{\mathbf{c}}^{(8)}(\boldsymbol{\xi}) = G_{\mathbf{c}}^{(8)}(|\boldsymbol{\xi}|^2)$. The Fourier transform here is taken in $\mathbb{R}^8$. Finally put $\lambda_8(\mathbf{c}) = c_{1,0}/4$.

Theorem 11.1 (Eight-dimensional coefficient construction). *The countable system of linear equations*

$$G_{\mathbf{c}}^{(8)}(2n) = (G_{\mathbf{c}}^{(8)})'(2n) = 0 \quad (n \geq 1), \quad \lambda_8(\mathbf{c}) = 1 \quad (11.4)$$

has exactly one solution $\mathbf{c}_{*,8} \in \ell^2(\mathcal{I}_8)$. Its entries are real, and for every $1 < r < e^\pi$,

$$\sum_{m \in 2\mathbb{N}} \sum_{k=0}^2 |c_{k,m}| r^m + \sum_{m \in 2\mathbb{N}-1} |c_{3,m}| r^m < \infty.$$

In particular it belongs to $\ell^1(\mathcal{I}_8)$. Its radial Schwartz function satisfies

$$\begin{aligned}\Phi_{\mathbf{c}_{*,8}}^{(8)}(0) &= G_{\mathbf{c}_{*,8}}^{(8)}(0) = 1, & \Phi_{\mathbf{c}_{*,8}}^{(8)}(2n) &= 0 & (n \geq 1), \\ (\Phi_{\mathbf{c}_{*,8}}^{(8)})'(2) &= -\frac{1}{120}, & (\Phi_{\mathbf{c}_{*,8}}^{(8)})'(2n) &= 0 & (n \geq 2).\end{aligned}$$

The homogeneous Fourier equations have a one-dimensional solution space in $\ell^2(\mathcal{I}_8)$.

As in (2.8), the matrix entries are the Fourier values and derivatives of the explicitly specified individual functions in (11.3); only their coefficients are unknown. The following subsections prove the theorem.

11.2 Regularity and Poisson normalization

The proof of Lemma 3.1 applies to (11.3). To check the change at small t , the relevant estimate is now

$$t^2 \left| \sum_{m \geq 1} d_m e^{-\pi m(1/t-1)} \right| \leq C \|\mathbf{c}\|_{2,8} \begin{cases} t^2 e^{-\pi/t}, & 0 < t \leq 1/2, \\ (1-t)^{-1/2}, & 1/2 < t < 1, \end{cases}$$

where $d_m = c_{2,m}$ for even m and $d_m = -2c_{3,m}$ for odd m . The exponential at zero still controls every power introduced by differentiation. The same estimate $t^{-1} + ut \geq 2\sqrt{u}$ proves continuity into the radial Schwartz space. The meromorphic argument in Lemma 3.1 proves injectivity: the small-parameter part is entire, and the large-parameter rational functions are unchanged. The three correction functions are linearly independent by their values and derivatives at 0, 2.

The built-in values are

$$\Phi_{\mathbf{c}}^{(8)}(0) = \lambda_8(\mathbf{c}), \quad (\Phi_{\mathbf{c}}^{(8)})'(2) = \frac{\pi}{4} e^{-2\pi} c_{0,-2}. \quad (11.5)$$

All values at $2n$, $n \geq 1$, and derivatives there for $n \geq 2$, vanish.

For Poisson summation use the even self-dual lattice

$$E_8 = \{\mathbf{y} \in \mathbb{Z}^8 : \sum_j y_j \in 2\mathbb{Z}\} \cup \{\mathbf{y} \in (\mathbb{Z} + 1/2)^8 : \sum_j y_j \in 2\mathbb{Z}\}.$$

It has covolume one and 240 vectors of squared length 2. Indeed, $112 = 4\binom{8}{2}$ have two integer coordinates equal to ± 1 , and 128 have all coordinates equal to $\pm 1/2$ with an even number of minus signs. See also [16, Section 1] for these lattice conventions.

If $\mathbf{c}$ solves the homogeneous Fourier equations, Poisson summation and its differentiated dilation identity, as in Lemma 3.2, give

$$G_{\mathbf{c}}^{(8)}(0) = \lambda_8(\mathbf{c}), \quad 2 \cdot 240 (\Phi_{\mathbf{c}}^{(8)})'(2) = -4\lambda_8(\mathbf{c}).$$

The factor 4 is $d/2$. Thus

$$(\Phi_{\mathbf{c}}^{(8)})'(2) = -\frac{\lambda_8(\mathbf{c})}{120}, \quad c_{0,-2} = -\frac{e^{2\pi}}{30\pi} \lambda_8(\mathbf{c}). \quad (11.6)$$

After normalization, the correction terms are consequently

$$4R_{0,1} - \frac{R_{2,0}}{30\pi} + c_{0,0}R_{0,0}.$$

Only $c_{0,0}$ and the four positive-mode sequences remain unknown.

11.3 The boundary problem and continuation

Fix $\mathbf{c} \in \ell^2(\mathcal{I}_8)$. In this section the symbols $A_j, A_{\pm}, b_0, b_1$ refer to the series and functions formed from this eight-dimensional vector. Specifically, replace (4.4) by

$$\begin{aligned} A_0(q) &= c_{0,-2}q^{-2} + c_{0,0} + \sum_{m \in 2\mathbb{N}} c_{0,m}q^m, \\ A_1(q) &= c_{1,0} + \sum_{m \in 2\mathbb{N}} c_{1,m}q^m, \\ A_2(q) &= \sum_{m \in 2\mathbb{N}} c_{2,m}q^m, \\ A_3(q) &= \sum_{m \in 2\mathbb{N}-1} c_{3,m}q^m, \quad 0 < |q| < 1. \end{aligned} \quad (11.7)$$

The definitions of $A_{\pm}, b_0, b_1$ in (4.5), the maps Q, ω , and all the parametrized contours of Section 4 remain the same.

For each fixed z with $\Re z > 0$, Gaussian inversion in $\mathbb{R}^8$ is

$$(\mathbf{y} \mapsto \widehat{e^{-\pi z |\mathbf{y}|^2}})(\boldsymbol{\xi}) = z^{-4} e^{-\pi |\boldsymbol{\xi}|^2 / z}.$$

Consequently define $\rho_{0,\Gamma}^{(8)}, \rho_{1,\Gamma}^{(8)}$ by the four formulas in (4.6), replacing the powers $z_{\pm}(t)^{-10}, z_{\pm}(t)^{-11}$ by $z_{\pm}(t)^{-2}, z_{\pm}(t)^{-3}$, respectively. The segment densities are unchanged:

$$\rho_{0,I}^{(8)}(s) = 2A_{-}(e^{\pi}\sqrt{s}), \quad \rho_{1,I}^{(8)}(s) = -\pi^{-1}(\log s) A_{-}(e^{\pi}\sqrt{s}) \quad (0 < s < w_0).$$

Equation (4.7), with these densities, defines $\mu_0^{(8)}, \mu_1^{(8)}$. All arguments and orientations in that definition are retained.

The calculation proving Lemma 4.1 now gives

$$G_{\mathbf{c}}^{(8)}(2n) = \frac{1}{8\pi} \int_{\mathbb{C}} w^{n-1} d\mu_0^{(8)}(w), \quad (G_{\mathbf{c}}^{(8)})'(2n) = -\frac{1}{8} \int_{\mathbb{C}} w^{n-1} d\mu_1^{(8)}(w). \quad (11.8)$$

This holds for every integer $n \geq 0$. In that calculation the path integral $J_{m,k}$ uses z^{-4} in place of z^{-12} . The rational term W_m , the phase-derivative corrections, and the substitution $\omega'(z) = 2\pi\omega(z)z^{-2}$ do not change. They give exactly the densities just specified. The same total variation bound and integrability at $s = 0$ extend the calculation from finite coefficient families to all of $\ell^2(\mathcal{I}_8)$.

Set $\mathcal{J}_8 = \{0, 1\} \times \mathbb{N}_0$, so that both moments of degree zero are now included. Define the moment operator by

$$(\mathbf{M}_8 \mathbf{c})_{\nu,k} = \int_{\mathbb{C}} w^k d\mu_{\nu}^{(8)}(w), \quad \nu = 0, 1, \quad k = 0, 1, 2, \dots \quad (11.9)$$

Its domain is $\ell^2(\mathcal{I}_8)$ and its target is $\ell^2(\mathcal{J}_8)$. The total variation estimate gives $\|\mathbf{M}_8 \mathbf{c}\|_{\ell^2(\mathcal{J}_8)} \leq C_{\mu,8}(1-\rho^2)^{-1/2}\|\mathbf{c}\|_{\ell^2(\mathcal{I}_8)}$, where $C_{\mu,8}$ is the dimension-eight total variation constant. It is a compact linear operator by the geometric moment estimate (4.10); the two additional constant-moment rows have finite rank. By (11.8), its kernel is exactly the homogeneous Fourier system in (11.4).

For $\mathbf{c} \in \ker \mathbf{M}_8$, both zeroth moments are zero. The Cauchy functions therefore have no explicit pole correction:

$$h_{\nu}^{(8)}(\zeta) = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{d\mu_{\nu}^{(8)}(w)}{w - \zeta}, \quad \mathbf{h}_{\text{par}}^{(8)}(z) = z^2 \mathbf{h}^{(8)}(\omega(z)). \quad (11.10)$$

Here $\mathbf{h}^{(8)} = (h_0^{(8)}, h_1^{(8)})^T$. The first formula is defined off $\Gamma \cup [0, w_0]$; the second is used wherever its argument lies in that set's complement. The exterior Cauchy functions vanish, and their interior traces and segment jumps obey the same formulas as before with the new densities. The reconstruction statement of Lemma 4.3 applies with both pole constants equal to zero. It then gives the vanishing of *all* nonnegative moments, including the two new rows.

The local continuation proof also applies. Here are the changes needed to check this assertion. The paired matrix $\mathbf{S}_{\text{pair}}(z)$ in (5.1) is unchanged, because rescaling by z^2 cancels the new density power. Its determinant is still $4i$. The circle transfer factor is $(1 + iz)^2$, and the real-interval jump is

$$2z^2 A_{-}(Q(1/z)) \begin{pmatrix} 1 \\ z^{-1} \end{pmatrix}.$$

For the functions b_{end}, b_{-} defined by (5.6), with the present coefficient series, the endpoint involutions are

$$b_{\text{end}}(z) = z^2 b_{\text{end}}(1/z), \quad b_{-}(z) + z^2 b_{-}(1/z) = b_{\text{end}}(z).$$

Every other recovery identity in the proof of Proposition 5.3 is unchanged. In particular, the punctured-disk cover, the bound $|1/z - 1| < 1/7$, and every trace-overlap check remain valid. The lower circle multiplier is now $(iz)^2 = -z^2$, so the same two-by-two involution still squares to the identity. Lemmas 5.1 and 5.2 apply with the dimension-eight total variation bound. Powers of z are bounded, with bounded inverses, on the fixed endpoint neighborhoods used there. The Cauchy area estimates and the coefficient bound by a constant times the inverse square root of radial distance therefore give the same removable-singularity argument. In particular there are constants $R > 1$ and $C_8 < \infty$, independent of the solution, such that the four functions continue to $0 < |q| < R$ and

$$\sup_{|q|=R} \sum_{j=0}^3 |A_j(q)| \leq C_8 \|\mathbf{c}\|_{2,8}. \quad (11.11)$$

At $q = 0$, multiplying by q^2 removes the sole allowed pole. This proves absolute summability already at the local stage.

11.4 The global problem and its index

Define a_0, a_1, a_2, a_3 from (11.7) exactly as in Section 6, using $Q(-i\tau) = e^\pi e^{\pi i\tau}$. Define a_4, a_5 on the fundamental cell by (6.10), with $h_\nu^{(8)}$ in place of h_ν . For $\mathbf{a} = (a_0, a_1, a_2, a_3, a_4, a_5)^T$, the global laws are

$$\mathbf{a}(\tau + 1) = \mathbf{J}_{\text{tr}} \mathbf{a}(\tau), \quad \mathbf{a}(-1/\tau) = \tau^{-2} \mathbf{J}_{\text{inv}}(\tau) \mathbf{a}(\tau). \quad (11.12)$$

Here $\mathbf{J}_{\text{tr}}, \mathbf{J}_{\text{inv}}$ are the same permanent gluing matrices as in (6.4); they have no dimension parameter. The allowed cusp exponents are now

function	a_0	a_1	a_2	a_3	a_4	a_5
lowest allowed exponent of $\mathbf{q}$	-1	0	1	$\frac{1}{2}$	0	0.

(11.13)

As before, each component divided by its displayed power is holomorphic at $\mathbf{q} = 0$. Denote this holomorphic bundle on $\overline{\mathcal{X}}$ by $\mathcal{E}_8$.

Lemma 11.2. *The coefficient map gives an isomorphism*

$$H^0(\overline{\mathcal{X}}, \mathcal{E}_8) \simeq \ker_{\ell^2(\mathcal{I}_8)} \mathbf{M}_8.$$

Every vector in this kernel has the weighted absolute summability asserted in Theorem 11.1.

Proof. We apply the two directions proved in Lemmas 6.1 and 6.3, checking each change with the dimension. The open cell and its Cauchy evaluation points are unchanged. The four coefficient functions are recovered by the four-by-four matrix $\mathbf{S}_{\text{cell}}$ in (6.11), whose determinant is -4 . Write $\mathbf{S}_{\text{rec}}^{(8)}, \mathbf{S}_{\text{match}}^{(8)}$ for the dimension-eight recovery and matching matrices. In the six-by-six recovery recipe, replace the two factors $-\tau^{10}$ by $-\tau^2$; call the resulting matrix $\mathbf{S}_{\text{rec}}^{(8)}(\tau)$. Its last two output coordinates are explicitly

$$a_4 = -\tau^2 b_{3,0} - 2(\tau + 1)a_2 - 2a_3, \quad a_5 = -\tau^2 b_{3,1} + (\tau^2 + \tau)a_2 + (2\tau + 1)a_3.$$

The inputs $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ are the three two-component Cauchy values used in (6.11), now formed from $\mathbf{h}_{\text{par}}^{(8)}$. Curved-side matching is expressed by

$$\mathbf{S}_{\text{match}}^{(8)}(\tau) = \begin{pmatrix} 0 & \tau^{-2} I_2 & 0 \\ 0 & 0 & I_2 \\ (1 + \tau)^{-2} I_2 & 0 & 0 \end{pmatrix}.$$

The same direct multiplication, with the new powers, gives

$$\mathbf{J}_{\text{tr}} \mathbf{S}_{\text{rec}}^{(8)}(-1/\tau - 1) \mathbf{S}_{\text{match}}^{(8)}(\tau) = \tau^{-2} \mathbf{J}_{\text{inv}}(\tau) \mathbf{S}_{\text{rec}}^{(8)}(\tau). \quad (11.14)$$

The boundary trace identities in (6.7) use z^2 on the left. Their row identities in $\mathbf{J}_{\text{inv}}, \mathbf{J}_{\text{tr}}$ are unchanged, since $i^2 = i^{10} = -1$. Equations (11.14) and (6.6) prove the side identifications (11.12). For the open sides the domain checks and the trace-convergence hypotheses are identical to those in Lemma 6.3. At the order-three vertex the three Cauchy arguments are still $-e^{-\pi\sqrt{3}}$, and the matrices are regular there. At the order-two vertex the endpoint continuation makes the incident densities analytic, so Lemma 6.2 gives the logarithmic bound. The finite-sector construction on

the upper-half-plane cover applies because the scalar weight -2 , like weight -10 , has product one in both group relations. Its Laurent coefficient estimate removes the vertices in all sectors. These arguments establish holomorphy on $\mathbb{H}$, with the required side laws.

At the cusp, the first four exponents follow directly from (11.7). To check the last two, the subtracted cut-jump terms are bounded by constants times $|w| |\log w|^2$ and $|w|^{1/2} |\log w|$. After these terms have been removed, a_4, a_5 are single-valued and area-integrable near zero. They can therefore have at most simple poles. The residue calculation in Lemma 6.3 identifies these residues with the two zeroth moments, which now vanish. Thus the two singularities are removable, giving the last two exponents in (11.13).

Conversely, a section gives four generating functions on $0 < |q| < e^\pi$ with exactly the powers in (11.7). Define candidate Cauchy functions by (6.2)–(6.3). They have the required jumps and outer traces. The cusp conditions give no pole at zero, and their endpoint values are bounded on each incident side. Lemma 4.3 with zero pole constants proves all nonnegative moment equations, including the two zeroth moments. It also identifies the candidates with (11.10). Injectivity follows by the same triple-pairing argument in the six expressions following (6.6). Finally, holomorphy of the recovered generating functions up to $|q| < e^\pi$, apart from their specified pole at zero, gives weighted absolute summability for every $r < e^\pi$ by Cauchy's estimates. $\square$

Lemma 11.3. *The Euler characteristic of $\mathcal{E}_8$ is 1. In particular $\ker M_8 \neq \{0\}$.*

Proof. The determinant factor in (11.12) has scalar weight -12 and the same character $\varepsilon_{\det}(S) = \varepsilon_{\det}(T) = -1$ as before. The sum of the exponents in (11.13) is $1/2$. Thus the line-bundle extension formula used in (7.1) gives

$$\det \mathcal{E}_8 \simeq \mathcal{L}_{-12-12(1/2)} = \mathcal{L}_{-18}.$$

The character's exponent here is $1/2$, without the integer shift -4 required in dimension 24. Equivalently, division of a determinant section by η^{12} already gives weight -18 , trivial character, and a holomorphic cusp frame, so no power of Δ_{mod} is needed.

At the elliptic fixed points the fiber actions are

$$-J_{\text{inv}}(i), \quad \rho_3^2 J_{\text{inv}}(\rho_3 + 1) J_{\text{tr}}, \quad \rho_3 = e^{2\pi i/3}.$$

The first is unchanged. The second is ρ_3 times the dimension-24 action, so its three eigenvalues are cyclically permuted; each still has multiplicity two. The eigenvalue multiplicities are therefore $(3, 3)$ and $(2, 2, 2)$. All nontrivial elliptic traces vanish, and the central element acts trivially. The Riemann–Roch formula used in Proposition 7.1 now gives

$$\chi(\mathcal{E}_8) = \frac{5 \cdot 6 - 18}{12} = 1.$$

Hence a nonzero section exists, and Lemma 11.2 gives a nonzero coefficient solution. $\square$

The equality of the two indices has a simple accounting explanation. Changing the scalar factor from weight -10 to weight -2 raises the determinant weight by 48. Removing the two coefficient poles and the two Cauchy poles raises the sum of cusp exponents by 4, subtracting exactly 48 from the extended determinant weight. Both determinant bundles are therefore $\mathcal{L}_{-18}$.

11.5 Vanishing, uniqueness, and the bounded inverse

Lemma 11.4. *If $\mathbf{c} \in \ker M_8$ and $\lambda_8(\mathbf{c}) = 0$, then $\mathbf{c} = 0$.*

Proof. Poisson summation in (11.6) gives $c_{0,-2} = 0$. Consequently every component in (11.13) is bounded at the cusp; moreover a_2 vanishes there and a_3 is bounded by a constant times $|\mathbf{q}|^{1/2}$.

Apply the same constant splitting as in Section 8:

$$\begin{aligned} a_0^+ &= (a_0 + a_5)/2, & a_1^+ &= (a_1 + a_2 + a_4)/2, & a_2^+ &= a_2, \\ a_0^- &= (a_0 - a_5)/2, & a_1^- &= (a_1 - a_2 - a_4)/2, & a_2^- &= a_3. \end{aligned}$$

All functions are holomorphic on $\mathbb{H}$. All except a_2^- are periodic under $\tau \mapsto \tau + 1$; a_2^- is antiperiodic. The first block obeys

$$\begin{aligned} a_0^+(-1/\tau) &= \tau^{-4} a_0^+(\tau), \\ a_1^+(-1/\tau) &= \tau^{-2} a_1^+(\tau) + 2\tau^{-3} a_0^+(\tau), \\ a_2^+(-1/\tau) &= a_2^+(\tau) + \tau^{-1} a_1^+(\tau) + \tau^{-2} a_0^+(\tau). \end{aligned}$$

These identities follow by multiplying the splitting matrix against J_{inv} , as in (8.3). The valence formula admits no nonzero holomorphic full-group modular form of negative weight. It first gives $a_0^+ = 0$, at weight -4 , and then $a_1^+ = 0$, at weight -2 . Now a_2^+ is a holomorphic form of weight zero, hence constant. Its vanishing at the cusp makes it zero.

For the second block, a_1^- is a holomorphic full-group form of weight -2 , so it too is zero. The remaining vector satisfies the following laws, using the two-by-two matrices $S_{\text{lev},S}, S_{\text{lev},T}$ already used in Section 8:

$$\begin{pmatrix} a_0^- \\ a_2^- \end{pmatrix}(-1/\tau) = \tau^{-2} S_{\text{lev},S} \begin{pmatrix} a_0^- \\ a_2^- \end{pmatrix}(\tau), \quad \begin{pmatrix} a_0^- \\ a_2^- \end{pmatrix}(\tau + 1) = S_{\text{lev},T} \begin{pmatrix} a_0^- \\ a_2^- \end{pmatrix}(\tau),$$

where

$$S_{\text{lev},S} = \frac{1}{2} \begin{pmatrix} -1 & 3 \\ 1 & 1 \end{pmatrix}, \quad S_{\text{lev},T} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Their representation factors through the group modulo 2. Thus $h = a_2^-$ is a weight -2 form for $\Gamma(2)$, holomorphic at every cusp: each slash transform is a constant linear combination of the bounded functions a_0^-, a_2^- . If $h \neq 0$, the product over the six cosets,

$$\prod_{\gamma \in \Gamma(2) \backslash \text{SL}_2(\mathbb{Z})} (h|_{-2}\gamma),$$

would be a nonzero holomorphic full-group form of weight -12 . The valence formula forbids it. Hence $a_2^- = 0$, and its inversion law gives $a_0^- = 0$. All six split components vanish. Inverting the splitting and recovering the coefficients gives $\mathbf{c} = 0$. $\square$

Proof of Theorem 11.1. Lemma 11.3 gives a nonzero element of $\ker M_8$, and Lemma 11.4 makes λ_8 injective on that kernel. The kernel is therefore one-dimensional, with a unique vector of normalization one. Complex conjugation of the real coefficient equations shows that its entries are real. Lemma 11.2 gives the weighted absolute summability, and (11.5)–(11.6) give all the asserted values and derivatives. $\square$

There is also a boundedly invertible equation, with the same purpose as in Section 9. Define K_8 by the recipe in Appendix B, using the eight-dimensional measures, replacing the powers 10 by 2, and extracting only the powers allowed in (11.7). For arbitrary coefficient vectors use the corrected

Cauchy formula (4.11) with their actual zeroth moments; on $\ker \mathbf{M}_8$ those moments are zero. The disk averaging and coefficient extraction then give a bounded compact operator satisfying

$$\mathbf{K}_8 \mathbf{c} = \mathbf{c} \quad (\mathbf{c} \in \ker \mathbf{M}_8).$$

Indeed the estimates for arbitrary raw assignments in the appendix use only the total variation and area bounds, which remain valid, and reproduction on solutions follows from (11.11).

With adjoints in the coefficient and moment Hilbert spaces, put

$$\mathbf{N}_8 = (\text{Id}_{\mathcal{I}_8} - \mathbf{K}_8)^*(\text{Id}_{\mathcal{I}_8} - \mathbf{K}_8) + \mathbf{M}_8^* \mathbf{M}_8 + \lambda_8^* \lambda_8. \quad (11.15)$$

Here $\text{Id}_{\mathcal{I}_8}$ is the identity on $\ell^2(\mathcal{I}_8)$. The compactness argument in Theorem 9.1, using Lemma 11.4, proves

$$\langle \mathbf{N}_8 \mathbf{c}, \mathbf{c} \rangle \geq c_{\text{coer},8} \|\mathbf{c}\|_{2,8}^2 \quad \text{for some } c_{\text{coer},8} > 0.$$

Thus $\mathbf{N}_8$ has a bounded inverse and $\mathbf{c}_{*,8} = \mathbf{N}_8^{-1} \lambda_8^*(1)$. The exact finite-dimensional positive compressions converge by the same Galerkin argument, subject to the distinction between exact entries and certified numerical evaluation made in Section 9. This completes the coefficient construction before any comparison with Viazovska's function.

11.6 Identification with Viazovska's function

We now identify the answer, using [16, Sections 4–5]. This comparison is not part of the preceding existence or uniqueness proof. Use the Eisenstein series, discriminant, and theta functions from Section 10, and also define $\theta_3(\tau) = \sum_{n \in \mathbb{Z}} e^{\pi i n^2 \tau}$. Set

$$\varphi_{8,0} = \frac{E_4^2}{\Delta_{\text{mod}}}, \quad \varphi_{8,1} = \frac{E_4^2 E_2 - E_4 E_6}{\Delta_{\text{mod}}}, \quad \varphi_{8,2} = \frac{(E_4 E_2 - E_6)^2}{\Delta_{\text{mod}}}.$$

$$\begin{aligned} \psi_{8,I} &= 128 \left(\frac{\theta_3^4 + \theta_4^4}{\theta_2^8} + \frac{\theta_4^4 - \theta_2^4}{\theta_3^8} \right), \\ \psi_{8,S} &= -128 \left(\frac{\theta_3^4 + \theta_2^4}{\theta_4^8} + \frac{\theta_2^4 - \theta_4^4}{\theta_3^8} \right). \end{aligned}$$

All functions on the right are evaluated at $\tau \in \mathbb{H}$. The second subscript $j = 0, 1, 2$ in $\varphi_{8,j}$ has the same role as in dimension 24: the weight is $2j - d/2$, now $-4, -2, 0$. The first subscript specifies the dimension.

Let $\Phi_{\text{known}}^{(8)}$ be the profile of Viazovska's normalized function. For $u > 2$, its integral formula is

$$\Phi_{\text{known}}^{(8)}(u) = P(u) \int_0^\infty \left(-\frac{\pi}{2160} t^2 \varphi_{8,2}(i/t) - \frac{1}{60\pi} \psi_{8,I}(it) \right) e^{-\pi u t} dt. \quad (11.16)$$

The transformation identities needed to split this integral are

$$\begin{aligned} t^2 \varphi_{8,2}(i/t) &= t^2 \varphi_{8,2}(it) - \frac{12t}{\pi} \varphi_{8,1}(it) + \frac{36}{\pi^2} \varphi_{8,0}(it), \\ \psi_{8,I}(it) &= -t^2 \psi_{8,S}(i/t), \\ \psi_{8,S}(\tau) &= \psi_{8,I}(\tau) - \psi_{8,I}(\tau + 1). \end{aligned} \quad (11.17)$$

Define the scalar coefficients of the cusp expansions by

$$\begin{aligned}\varphi_{8,j}(\tau) &= \sum_m \varphi_{8,j,m} \mathfrak{q}^{m/2} & (j = 0, 1, 2), \\ \psi_{8,I}(\tau) &= \sum_m \psi_{8,I,m} \mathfrak{q}^{m/2}, & \psi_{8,S}(\tau) = \sum_m \psi_{8,S,m} \mathfrak{q}^{m/2}.\end{aligned}$$

Here again $\mathfrak{q}^{1/2} = e^{\pi i \tau}$, and the coefficients are numbers, defined by coefficient extraction. Splitting (11.16) at $t = 1$ and applying (11.17) gives the undamped large-parameter coefficients

$$\begin{aligned}\gamma_{8,0,m} &= -\frac{\varphi_{8,0,m} + \psi_{8,I,m}}{60\pi}, \\ \gamma_{8,1,m} &= \frac{\varphi_{8,1,m}}{180}, \\ \gamma_{8,2,m} &= -\frac{\pi \varphi_{8,2,m}}{2160},\end{aligned}\tag{11.18}$$

and the small-parameter coefficients

$$\beta_{8,m} = -\frac{\pi \varphi_{8,2,m}}{2160} + \frac{\psi_{8,S,m}}{60\pi}.$$

All three $\varphi_{8,j}$ have only even powers, and $\psi_{8,S,m} = (1 - (-1)^m) \psi_{8,I,m}$. Hence $\beta_{8,m} = \gamma_{8,2,m}$ for even m , and $\beta_{8,m} = -2\gamma_{8,0,m}$ for odd m . The coefficients with positive exponent m in (11.3) are

$$c_{k,m} = e^{-\pi m} \gamma_{8,k,m} \quad (m \in 2\mathbb{N}), \quad c_{3,m} = e^{-\pi m} \gamma_{8,0,m} \quad (m \in 2\mathbb{N} - 1).$$

The permitted coefficients with exponent $m \leq 0$ are obtained by the same rule $c_{k,m} = e^{-\pi m} \gamma_{8,k,m}$.

For clarity, the coefficients at powers $\mathfrak{q}^{-1}, \mathfrak{q}^0$ of $(\varphi_{8,2}, \varphi_{8,1}, \varphi_{8,0}, \psi_{8,I})$ are respectively

$$(0, 0, 1, 1), \quad (0, 720, 504, 144).$$

There are no other nonpositive terms. Thus the correction part is exactly

$$4R_{0,1} - \frac{R_{2,0}}{30\pi} - \frac{54}{5\pi} R_{0,0}.\tag{11.19}$$

In particular its normalization is one. These Laurent series, viewed in the variable $\mathfrak{q}^{1/2}$, converge for $|\mathfrak{q}^{1/2}| < 1$, apart from the stated poles at zero. Cauchy's estimates give weighted absolute summability after the factor $e^{-\pi m}$. The split integral gives equality of the profiles for $u > 2$. Subtracting its nonpositive large- t modes gives the entire correction functions in (11.19). The remaining large- t integrand decays at least as a polynomial times $e^{-\pi t}$, and the small- t part decays exponentially in $1/t$. As in the preceding comparison, these expressions define holomorphic profiles on the common connected domain $\Re u > -1$; this is the continuation of the original contour formula in [16, Sections 4–5]. The identity theorem therefore gives equality for every nonnegative real u . The known Fourier zeros and normalization make the coefficient vector a solution of (11.4), and Theorem 11.1 identifies it with $\mathbf{c}_{*,8}$. The signs proved in [16, Theorem 4] therefore hold for our function.

Fourier interpolation gives the parallel uniqueness statement in the full radial Schwartz class. By [5, Theorems 1.7 and 1.9], the data at $\sqrt{2n}$, $n \geq 1$, determine such a function in $\mathbb{R}^8$. All our interpolation data are zero except

$$\left. \frac{d}{dr} \Phi_{\mathbf{c}_{*,8}}^{(8)}(r^2) \right|_{r=\sqrt{2}} = -\frac{1}{30\sqrt{2}}.$$

If b_1^{int} denotes the interpolation basis function whose radial derivative at $\sqrt{2}$ is one and whose other data vanish, then $f_{\mathbf{c}_{*,8}}^{(8)} = -(30\sqrt{2})^{-1} b_1^{\text{int}}$. The superscript marks an interpolation basis function. This identification is an additional consequence, not an input to the construction.

12 The remaining sign problem

Theorems 2.1 and 11.1 solve the coefficient systems: they produce real radial Schwartz functions with the required zeros, normalization, and first derivative. To complete an independent Cohn–Elkies proof, one must still establish

$$\Phi_{\mathbf{c}_*,8}^{(8)}(u) \leq 0 \ (u \geq 2), \quad G_{\mathbf{c}_*,8}^{(8)}(v) \geq 0 \ (v \geq 0),$$

and

$$\Phi_{\mathbf{c}_*}(u) \leq 0 \ (u \geq 4), \quad G_{\mathbf{c}_*}(v) \geq 0 \ (v \geq 0).$$

Prescribing values and first derivatives at a discrete set does not establish these inequalities on the intervals between nodes. Nor does an index or a uniqueness argument establish a sign. Our comparisons prove that the inequalities hold by transferring the earlier sign theorems. Without that comparison, the global sign proof in both dimensions remains unfinished. Thus we have an independent construction of the functions, not yet a complete independent proof of the sphere-packing bounds.

It is useful to distinguish this situation from the planar paper, which proves its signs within the coefficient construction [10, Section 4 and Appendices A–B]. Its finite coefficient approximations come with rigorous bounds for the error in each ℓ^1 coefficient list. On bounded intervals, the exact prescribed zeros are factored out. Near a zero, the lower Taylor terms of the approximation error are subtracted before division by the vanishing factor, so the error bound does not become singular. Polynomial approximations in the Bernstein basis then certify signs on entire intervals through positive Bernstein coefficients, with rigorous interval enclosures and quadrature remainder bounds.

The unbounded region is controlled separately by an explicit barrier built from the trigonometric product, together with second-derivative and tail estimates. These analytic estimates are what let finitely many certified computations prove signs for the infinite coefficient series on the entire required half-lines. They are not a numerical sampling argument. Adapting that program here would require effective coefficient error bounds and sign margins suited to the dimension-eight and dimension-24 families. Our bounded-inverse theorems provide convergence, but do not supply those quantitative certificates.

A Why these elementary functions?

The factor P builds in the desired zeros of the physical profile. It does not determine the remaining factors in $U_{m,k}$ and V_m , nor does it explain their parity restrictions. This appendix motivates those choices by Gaussian inversion and an explicit cancellation in the Fourier samples. The calculation applies to both $d = 8$ and $d = 24$. It uses no modular form and does not assume that a nonzero coefficient solution exists.

A.1 Gaussian inversion and the two integration ranges

For a radial profile h whose associated function is Schwartz, write $\mathcal{F}_d h$ for its radial Fourier profile:

$$(\mathbf{y} \mapsto \widehat{h(|\mathbf{y}|^2)})(\boldsymbol{\xi}) = (\mathcal{F}_d h)(|\boldsymbol{\xi}|^2).$$

This is the Fourier transform in $\mathbb{R}^d$, expressed in one-variable profiles; it is not the one-dimensional Fourier transform of h . For $m \geq 1$, start with

$$g_m(u) = e^{\pi m} \int_1^\infty e^{-\pi m t} e^{-\pi t u} dt.$$

Gaussian inversion and the substitution $t = 1/s$ give

$$\begin{aligned} (\mathcal{F}_d g_m)(v) &= e^{\pi m} \int_1^\infty t^{-d/2} e^{-\pi(mt+v/t)} dt \\ &= e^{\pi m} \int_0^1 s^{d/2-2} e^{-\pi(m/s+vs)} ds. \end{aligned} \quad (\text{A.1})$$

Consequently

$$U_{m,0} = P g_m, \quad V_m^{(d)} = P \mathcal{F}_d g_m,$$

where $V_m^{(24)} = V_m$ and $V_m^{(8)}$ is (11.1). The power $d/2 - 2$ is forced by Gaussian inversion and the Jacobian. The split at 1 is natural because 1 is fixed by inversion of the Gaussian parameter. Multiplication by P does not commute with $\mathcal{F}_d$, so these identities alone do not determine the Fourier samples of $U_{m,0}$ or $V_m^{(d)}$. That is the purpose of the next calculation.

The factor $e^{\pi m}$ normalizes the exponential mode at the splitting point:

$$e^{\pi m} e^{-\pi m t} = e^{-\pi m(t-1)}.$$

It therefore makes the coefficient series converge in the variable $q = Q(t) = e^{-\pi(t-1)}$ on the unit disk. For each fixed m , it is a scalar normalization of the profile; the coefficient norm is then prescribed in these coordinates. Finally, polynomial weights in t give elementary rational factors:

$$U_{m,k}(u) = P(u) e^{-\pi u} \sum_{j=0}^k \frac{k!}{(k-j)! [\pi(m+u)]^{j+1}}. \quad (\text{A.2})$$

Thus $k = 0, 1, 2$ permits successively higher principal parts at the negative modes while preserving all the physical zeros.

A.2 The parity restrictions and the coupled terms

Let p be a polynomial with real coefficients and define

$$U_m[p](u) = e^{\pi m} P(u) \int_1^\infty p(t) e^{-\pi(m+u)t} dt.$$

For $p(t) = t^k$ this is $U_{m,k}$. Put $\sigma_m = (-1)^m$ and define the polynomial

$$(\delta_m p)(t) = 2p(t) - \sigma_m(p(t+i) + p(t-i)). \quad (\text{A.3})$$

The sign σ_m comes from shifting the exponential mode by i or $-i$. To see this precisely, use

$$P(u) e^{-\pi t u} = \frac{1}{4} (2e^{-\pi t u} - e^{-\pi(t-i)u} - e^{-\pi(t+i)u}).$$

Hold $v \geq 0$ fixed, apply Gaussian inversion, and shift the two displaced horizontal paths back to the positive real axis. For $m \geq 1$, the integrands decay exponentially at the right ends, so the connecting integrals there tend to zero. Define the remaining integral along the straight path from 1 to $1+i$ by

$$J_m[p](v) = e^{\pi m} \int_1^{1+i} p(z-i) z^{-d/2} e^{-\pi(mz+v/z)} dz.$$

The lower path contributes its complex conjugate. The result is

$$(\mathcal{F}_d U_m[p])(v) = \frac{\sigma_m}{2} \Re J_m[p](v) + \frac{e^{\pi m}}{4} \int_1^\infty (\delta_m p)(t) t^{-d/2} e^{-\pi(mt+v/t)} dt. \quad (\text{A.4})$$

The notation $J_m[1]$ means that the polynomial is the constant 1, not the monomial of degree one. For $p(t) = t^k$, this path integral is $J_{m,k}$ from the sample calculation, with the exponent appropriate to d .

There is a companion identity for the reciprocal family. Let $W_m(v) = e^{-\pi v}/(2\pi(m+v))$, as in the sample calculation. Then

$$(\mathcal{F}_d V_m^{(d)})(v) = W_m(v) + \frac{\sigma_m}{2} \Re(e^{-\pi i v} J_m[1](v)) - \frac{\sigma_m}{2} \cos(\pi v) (\mathcal{F}_d g_m)(v). \quad (\text{A.5})$$

For completeness, the unshifted Gaussian term in this calculation is

$$\frac{e^{\pi m}}{2} \int_0^1 t^{-2} e^{-\pi(m+v)/t} dt = W_m(v).$$

In the other two terms, first substitute $s = 1/t$. The transformed Gaussian parameters involve $s+i$ and $s-i$; the identity $s/(1+is) = -i + 1/(s-i)$ and its conjugate separate the factors $e^{\pi i v}$ and $e^{-\pi i v}$. Shifting the resulting paths back to the real axis gives the last two terms of (A.5). The constant phase $i^{-d/2}$ equals 1 in both dimensions under consideration. All these identities can be differentiated in v on compact real intervals by dominated convergence.

Suppose now that $\delta_m p$ is a constant b . For the combination

$$U_m[p] + \frac{b}{2\sigma_m} V_m^{(d)},$$

the sum of the unbounded real-path terms in (A.4)–(A.5) is

$$\frac{b}{4} (1 - \cos(\pi v)) (\mathcal{F}_d g_m)(v) = \frac{b}{2} P(v) (\mathcal{F}_d g_m)(v).$$

This has a double zero at every $v = 2n$, including $n = 0$. Its contribution to both Fourier sample equations is therefore zero. Only the fixed finite path integrals and W_m remain. The latter also becomes a bounded-interval moment:

$$W_m(2n) = \frac{1}{4\pi} \int_0^{w_0} s^{n-1} e^{\pi m} s^{m/2} ds, \quad w_0 = e^{-2\pi}.$$

The elementary polynomial differences are

parity of m	$p(t)$	$\delta_m p$	$U_m[p] + \frac{\delta_m p}{2\sigma_m} V_m^{(d)}$
even	1	0	$U_{m,0}$
even	t	0	$U_{m,1}$
even	t^2	2	$U_{m,2} + V_m^{(d)}$
odd	1	4	$U_{m,0} - 2V_m^{(d)}$

These are precisely the four coefficient families used in the paper. Among polynomial weights, the requirement that $\delta_m p$ be constant permits degree at most two for even m , and degree zero for odd m . For even m , the difference kills constants and linear polynomials, and sends degree $k \geq 2$ to degree $k-2$; for odd m , it preserves the leading degree. Higher-degree weights could be considered, but their real-path terms would require additional reciprocal families for this cancellation.

A.3 What the calculation explains

The construction now has a sequence of reasons: the lattice nodes suggest P ; inversion of Gaussians pairs the two integration ranges; exponential modes organize the unknowns into series; and the polynomial difference calculation leaves three even series and one odd series. The resulting Fourier values and derivatives become two moment families on fixed supports. Their two Cauchy transforms, together with the four generating series, account for the six functions in the boundary and bundle problems.

These observations motivate a tractable coefficient space. They neither characterize every family with the prescribed zeros nor prove that the chosen system has a nonzero solution. Existence and uniqueness require the continuation, index, and vanishing arguments in the body of the paper. The later comparison with the known magic functions confirms which function this construction produces; it is not used to justify the choices in this appendix.

B A compact regularizing operator

For completeness we specify the operator used in Section 9. The construction turns the local continuation proof into a bounded linear operation on arbitrary coefficient vectors, including vectors that do not solve the equations.

For arbitrary $\mathbf{c} \in \ell^2(\mathcal{I}_{24})$, use (4.7) and (4.11) to define $\mathbf{h}$ and $\mathbf{h}_{\text{par}}$ on the same interior domains. No vanishing of the exterior Cauchy function is needed for these definitions. For a center $\zeta_0 \in \mathbb{C}$, a radius $r > 0$, and an area-integrable function h on the disk $D(\zeta_0, r)$, define its holomorphic average by

$$(\mathbf{B}_{\zeta_0, r} h)(z) = \frac{1}{\pi r^2} \int_{D(\zeta_0, r)} \frac{h(\zeta)}{(1 - (z - \zeta_0)(\bar{\zeta} - \bar{\zeta}_0)/r^2)^2} dA(\zeta). \quad (\text{B.1})$$

For $0 < \sigma < 1$,

$$\sup_{|z - \zeta_0| \leq \sigma r} |\mathbf{B}_{\zeta_0, r} h(z)| \leq \frac{\|h\|_{L^1(D(\zeta_0, r))}}{\pi r^2 (1 - \sigma)^2}.$$

Expanding the kernel and integrating angularly proves that $\mathbf{B}_{\zeta_0, r}$ reproduces every holomorphic area-integrable function on the disk.

The continuation formulas will first assign functions separately on open pieces. For arbitrary coefficient vectors those assignments need not agree across the separating curves. We denote them by the subscript raw, and use a superscript $\sharp$ for the holomorphic functions obtained by applying B. Values on the separating curves do not affect the area integrals.

1. On $\{|z - 1| < 1/4, |z| > 1\}$, assign functions $b_{\text{end, raw}}, b_{-, \text{raw}}$ from (5.6) where $\Re z > 1$, from the first identity of (5.7) in the lower left lens, and from the second in the upper left lens. Inside $|z| < 1, |z - 1| < 1/8$, assign

$$b_{\text{end, raw}}(z) = z^{10} b_{\text{end, raw}}(1/z), \quad b_{-, \text{raw}}(z) = b_{\text{end, raw}}(z) - z^{10} b_{-, \text{raw}}(1/z).$$

Apply $\mathbf{B}_{1, 1/8}$ to get $b_{\text{end}}^\sharp, b_-^\sharp$.

2. For $z \in D(1, 1/8)$ with $\Re z > 1$, set $b_{+, \text{raw}}(z) = A_+(Q(z))$. For $\Re z < 1$, use the expression in (5.9). Apply $\mathbf{B}_{1, 1/8}$ to obtain $b_+^\sharp$, and define

$$a_{2, \text{par}}^\sharp(z) = (b_+^\sharp(z) + b_-^\sharp(z))/2, \quad a_{3, \text{par}}^\sharp(z) = (b_+^\sharp(z) - b_-^\sharp(z))/4.$$

These averaged functions have argument z , whereas the original A_j have argument q . For an exact coefficient solution their relation is $a_{j, \text{par}}^\sharp(z) = A_j(Q(z))$.

3. For $z \in D(1, 1/32)$ with $\Re z > 1$, set $a_{1,\text{par,raw}}(z) = A_1(Q(z))$. On the two left half-disks use (5.10), with the sharp functions in place of b_{end}, A_2, A_3 . Apply $B_{1,1/32}$ to get $a_{1,\text{par}}^\sharp$, and use (5.11) to define $a_{0,\text{par}}^\sharp$. Write $\mathbf{a}_{\text{par}}^\sharp = (a_{0,\text{par}}^\sharp, a_{1,\text{par}}^\sharp, a_{2,\text{par}}^\sharp, a_{3,\text{par}}^\sharp)^T$.

Bounds for arbitrary coefficients. These estimates use no matching identity. In the first two stages, Lemma 5.1 bounds the Cauchy pieces, the original series pieces, and their pullbacks by inversion. In particular

$$\|b_{\text{end,raw}}\|_{L^1(D(1,1/8))} + \|b_{-, \text{raw}}\|_{L^1(D(1,1/8))} + \|b_{+, \text{raw}}\|_{L^1(D(1,1/8))} \leq C\|\mathbf{c}\|_2.$$

The averaging estimate bounds the three sharp functions, and hence $a_{2,\text{par}}^\sharp, a_{3,\text{par}}^\sharp$, on $D(1, 1/16)$ by $C\|\mathbf{c}\|_2$. In the third stage these functions are consequently bounded on all of $D(1, 1/32)$. The remaining terms in (5.10) are Cauchy pieces on the left and the original series on the right. Lemma 5.1 therefore gives

$$\|a_{1,\text{par,raw}}\|_{L^1(D(1,1/32))} \leq C\|\mathbf{c}\|_2.$$

A second application of the averaging estimate, followed by (5.11), yields

$$\sup_{|z-1| \leq 1/64} \|\mathbf{a}_{\text{par}}^\sharp(z)\|_1 \leq C_0\|\mathbf{c}\|_2.$$

These are bounds for the piecewise raw assignments of an arbitrary vector, not conclusions about its analytic continuation.

Reproduction for exact solutions. If $\mathbf{c} \in \ker \mathbf{M}$, the overlap checks and removals in Proposition 5.3 show that each raw assignment is the restriction of the actual holomorphic continuation. Disk averaging reproduces it. The first two stages therefore recover $b_{\text{end}}, b_\pm$ and $A_2(Q(z)), A_3(Q(z))$; the third recovers $A_1(Q(z))$ and then $A_0(Q(z))$. Thus $\mathbf{a}_{\text{par}}^\sharp(z) = \mathbf{a}_{\text{ser}}(Q(z))$ for an exact solution. This reproduction statement is separate from the arbitrary-input bounds above.

Coefficient extraction. Put $R = e^{\pi 2^{-19}}$, $\sigma_0 = 1 - 2^{-19}$, $\delta_{\text{end}} = 2^{-7}$, and $\mathbf{S}_{\text{par}} = \text{diag}(1, 1, 1, -1)$. Here $\mathbf{S}_{\text{pair}}$ is the paired-trace matrix of (5.1), and $\mathbf{S}_{\text{par}}$ records the parity of the four series. Define paths $z_R(t) = \sigma_0 + it$ and $\gamma_R(t) = Re^{-\pi it}$, $0 \leq t \leq 1$. Then $Q(z_R(t)) = \gamma_R(t)$, and γ_R parametrizes the lower semicircle in the q -plane from R to $-R$. Define a vector-valued function on this semicircle by

$$\tilde{\mathbf{a}}_{\text{ser}, \mathbf{c}}(\gamma_R(t)) = \begin{cases} \mathbf{a}_{\text{par}}^\sharp(z_R(t)), & 0 \leq t < \delta_{\text{end}}, \\ \mathbf{S}_{\text{pair}}(z_R(t))^{-1} \begin{pmatrix} \mathbf{h}_{\text{par}}(z_R(t)) \\ \mathbf{h}_{\text{par}}(z_R(t) - i) \end{pmatrix}, & \delta_{\text{end}} \leq t \leq 1 - \delta_{\text{end}}, \\ \mathbf{S}_{\text{par}} \mathbf{a}_{\text{par}}^\sharp(z_R(t) - i), & 1 - \delta_{\text{end}} < t \leq 1. \end{cases}$$

Define its values on the upper semicircle by $\tilde{\mathbf{a}}_{\text{ser}, \mathbf{c}}(-q) = \mathbf{S}_{\text{par}} \tilde{\mathbf{a}}_{\text{ser}, \mathbf{c}}(q)$ for q on the lower semicircle. The endpoint definitions agree because $\mathbf{S}_{\text{par}}^2 = I_4$. The middle Cauchy arguments lie in fixed compact sets off the contours. It follows that

$$\sup_{|q|=R} \|\tilde{\mathbf{a}}_{\text{ser}, \mathbf{c}}(q)\|_1 \leq C_{\text{patch}}\|\mathbf{c}\|_2. \quad (\text{B.2})$$

Write $\tilde{\mathbf{a}}_{\text{ser}, \mathbf{c}} = (\tilde{A}_{\mathbf{c},0}, \tilde{A}_{\mathbf{c},1}, \tilde{A}_{\mathbf{c},2}, \tilde{A}_{\mathbf{c},3})^T$. The four components are ordered as A_0, A_1, A_2, A_3 and are numbered $j = 0, 1, 2, 3$. The allowed powers are precisely those in (4.4): the five specified nonpositive

powers in the first two components, positive even powers in the first three, and positive odd powers in the fourth. Extract their coefficients by

$$(\mathbf{K}\mathbf{c})_{j,m} = \frac{R^{-m}}{2\pi} \int_0^{2\pi} \tilde{A}_{\mathbf{c},j}(Re^{i\theta})e^{-im\theta} d\theta. \quad (\text{B.3})$$

The output index (j, m) ranges over $\mathcal{I}_{24}$. Thus (B.3) produces the coefficient vector in the same coordinates as $\mathbf{c}$; coefficients at all other powers are discarded. This defines a linear map from $\ell^2(\mathcal{I}_{24})$ to coefficient families on $\mathcal{I}_{24}$. For $m \geq 1$, let $(\mathbf{K}\mathbf{c})_m$ be the tuple of entries at exponent m : three entries if m is even, and one if m is odd. By (B.2),

$$\|(\mathbf{K}\mathbf{c})_m\|_1 \leq C_{\text{patch}} R^{-m} \|\mathbf{c}\|_2 \quad (m \geq 1),$$

so

$$\|(\text{Id}_{\mathcal{I}_{24}} - \mathbf{P}_J)\mathbf{K}\| \leq \frac{C_{\text{patch}} R^{-J-1}}{\sqrt{1 - R^{-2}}}.$$

The five remaining coefficients are bounded linear functionals by the same integral estimate. Hence $\mathbf{K} : \ell^2(\mathcal{I}_{24}) \rightarrow \ell^2(\mathcal{I}_{24})$ is bounded, and the tail estimate proves compactness and (9.1). For $\mathbf{c} \in \ker \mathbf{M}$, the boundary assignments coincide with the actual continuation, and Cauchy's coefficient formula gives $\mathbf{K}\mathbf{c} = \mathbf{c}$. Every operation defining $\mathbf{K}$ has been specified before any coefficient of $\mathbf{c}_*$ is known.

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