

A Nonunimodal Acyclic Polynomial with All Roots in the Left Half-Plane

Yuran Xiu
IIIS, Tsinghua University
xiuyr23@mails.tsinghua.edu.cn

14 October 2026

Abstract

Let $A(G, x)$ count vertex subsets of a graph G that induce forests, by their cardinalities. Barton, Brown, and Pike asked whether failure of coefficient unimodality forces $A(G, x)$ to have a root with positive real part. We give a negative answer. The join of K_{46} with three disjoint copies of P_3 has a nonunimodal acyclic polynomial of degree nine, yet all its complex roots have strictly negative real part. We derive the polynomial by classifying induced forests and give its exact positive Hurwitz determinants. A Lean formalization verifies the graph-to-polynomial identity and the full counterexample, using an exact rational Lyapunov certificate for strict stability.

1 Introduction

For a finite simple graph G , define its *acyclic polynomial* by

$$A(G, x) = \sum_{\substack{S \subseteq V(G) \\ G[S] \text{ is a forest}}} x^{|S|} = \sum_{j=0}^{|V(G)|} a_j(G) x^j. \quad (1)$$

The induced subgraph on the empty set is included. A finite sequence $(a_0, \dots, a_N)$ is *unimodal* if there is an index m such that it is weakly increasing up to m and weakly decreasing thereafter. A real polynomial is *strictly Hurwitz stable* if each of its complex roots has strictly negative real part.

Barton, Brown, and Pike [1] study the coefficients and root locations of (1). In Question 3.4 of the published version, they ask whether a nonunimodal acyclic polynomial must have a root with positive real part. The following 55-vertex graph disproves this implication, even when the conclusion is weakened to a root with nonnegative real part.

Theorem 1.1. *Let H be the disjoint union of three copies of the three-vertex path $P_3 = K_{1,2}$, and let $G = K_{46} \vee H$ be the join. Then*

$$\begin{aligned} A(G, x) = & 1 + 55x + 1485x^2 + 1464x^3 + 2196x^4 \\ & + 1506x^5 + 498x^6 + 82x^7 + 9x^8 + x^9. \end{aligned} \quad (2)$$

Its coefficient sequence is not unimodal, but every root z satisfies $\operatorname{Re} z < 0$.

The construction uses a clique and is not bipartite. The theorem addresses Question 3.4 alone; it does not establish a classification of nonunimodal acyclic polynomials or a minimum order for counterexamples.

2 Counting the induced forests

For a graph H , let $I(H, x) = \sum_{S \subseteq V(H), S \text{ independent}} x^{|S|}$ be its independence polynomial. The join $K_c \vee H$ adds all edges between the clique and H .

Lemma 2.1. *For every finite simple graph H and integer $c \geq 0$,*

$$A(K_c \vee H, x) = A(H, x) + cxI(H, x) + \binom{c}{2}x^2. \quad (3)$$

Proof. Partition a selected vertex set by the number of vertices it contains from the clique. If that number is zero, its induced subgraph is a forest precisely when the selected subgraph of H is a forest. This contributes $A(H, x)$.

If exactly one clique vertex is selected, any selected edge of H would form a triangle with it. Thus the selected vertices of H must be independent. Conversely, an independent selection together with one clique vertex induces a star, so the contribution is $cxI(H, x)$.

If exactly two clique vertices are selected, any selected vertex of H forms a triangle with them. Hence H must contribute no vertex, giving $\binom{c}{2}x^2$. Three or more clique vertices already contain a triangle. These cases are disjoint and exhaust all possibilities. $\square$

For a forest on nine vertices every induced subgraph is a forest, so $A(H, x) = (1 + x)^9$ for the H in Theorem 1.1. The independent sets of P_3 give $I(P_3, x) = 1 + 3x + x^2$. Independence polynomials multiply over disjoint unions, hence

$$A(G, x) = (1 + x)^9 + 46x(1 + 3x + x^2)^3 + 1035x^2. \quad (4)$$

Expansion gives (2). In particular,

$$a_2 = 1485 > 1464 = a_3 < 2196 = a_4.$$

A descent at indices 2, 3 forces any unimodal mode to be at most 2, whereas an ascent at indices 3, 4 forces it to be at least 4. This contradiction proves nonunimodality. Including the zero coefficients above degree nine does not alter this conclusion.

3 Exact strict stability

We recall the Hurwitz criterion [2]. For a real polynomial

$$p(x) = b_0x^d + b_1x^{d-1} + \cdots + b_d, \quad b_0 > 0,$$

extend the coefficient list by $b_j = 0$ for $j < 0$ or $j > d$. Its Hurwitz matrix has entries $H_{ij} = b_{2j-i}$ for $1 \leq i, j \leq d$. All roots of p have negative real part if and only if all leading principal determinants $\Delta_r = \det H[1:r, 1:r]$, $1 \leq r \leq d$, are positive.

For the polynomial in (2), the descending coefficient list is

$$(b_0, \dots, b_9) = (1, 9, 82, 498, 1506, 2196, 1464, 1485, 55, 1).$$

In particular, the matrix to be checked is explicitly

$$H = \begin{pmatrix} 9 & 498 & 2196 & 1485 & 1 & 0 & 0 & 0 & 0 \\ 1 & 82 & 1506 & 1464 & 55 & 0 & 0 & 0 & 0 \\ 0 & 9 & 498 & 2196 & 1485 & 1 & 0 & 0 & 0 \\ 0 & 1 & 82 & 1506 & 1464 & 55 & 0 & 0 & 0 \\ 0 & 0 & 9 & 498 & 2196 & 1485 & 1 & 0 & 0 \\ 0 & 0 & 1 & 82 & 1506 & 1464 & 55 & 0 & 0 \\ 0 & 0 & 0 & 9 & 498 & 2196 & 1485 & 1 & 0 \\ 0 & 0 & 0 & 1 & 82 & 1506 & 1464 & 55 & 0 \\ 0 & 0 & 0 & 0 & 9 & 498 & 2196 & 1485 & 1 \end{pmatrix}.$$

Exact determinant evaluation gives the following certificate.

r	Δ_r
1	9
2	240
3	17298
4	10581516
5	17654863239
6	3903814488441
7	4104634643529867
8	221412068364709627
9	221412068364709627

Every entry is positive, so the criterion proves strict stability. Together with Section 2, this completes the proof of Theorem 1.1. These are integer determinant identities, not approximations to roots near the imaginary axis.

4 The formal root certificate

The Lean development uses a rational Lyapunov certificate rather than assuming the Hurwitz criterion as a formal premise. We explain the argument to specify exactly what is checked.

Let c_j be the ascending coefficients of p in (2), so $c_9 = 1$. Define $C \in \mathbb{R}^{9 \times 9}$ by

$$C_{0j} = -c_{8-j} \quad (0 \leq j \leq 8), \quad C_{i,i-1} = 1 \quad (1 \leq i \leq 8),$$

with other entries zero. At a root z of p the vector $v = (z^8, z^7, \dots, z, 1)^\top$ satisfies $Cv = zv$. The formal certificate supplies a rational symmetric matrix P , a rational unit lower triangular matrix L , and a positive diagonal matrix D such that

$$P = LDL^\top, \quad C^\top P + PC = -\delta e_8 e_8^\top, \quad \delta = 442824136729419254. \quad (5)$$

The matrices and exact entrywise identities are in `Acyclic/Certificate.lean`; their rational data are also recorded in `documentation/direct-certificate.json`.

For completeness, the diagonal entries used there are

$$\begin{array}{ccc} 4104634643529867, & 3903814488441, & \\ \hline 434333309604785204292411317, & 780958447998089584484844 & \\ \hline 456070515947763, & 1301271496147 & \\ \hline 425553995396971903094, & 4428241367294192540 & \\ \hline 1961651471, & 881793 & \\ \hline 221412068364709627, & 221412068364709627, & 221412068364709627 \\ \hline 1922, & 240 & 9 \end{array}.$$

They are strictly positive, and the factorization in (5) implies $v^*Pv \geq 0$. Multiplying the second identity by v^* and v , and using $Cv = zv$ and $v_8 = 1$, yields

$$2\operatorname{Re}(z)v^*Pv = -\delta < 0.$$

If $\operatorname{Re} z \geq 0$, the left side is nonnegative, a contradiction. Thus the certificate proves $\operatorname{Re} z < 0$ for every root. All certificate identities and positivity statements are checked in Lean over exact rationals and complex numbers.

5 Formal scope and reproducibility

The project `acyclic-hurwitz-counterexample` defines the actual 55-vertex graph, vertex-induced acyclicity, and the coefficient counts in (1). Its graph classification proves the cases of Lemma 2.1 for the witness, and `AcyclicCounterexample.candidateGraph_polynomial` proves (2) with the actual graph semantics. The endpoints `AcyclicCounterexample.question34_counterexample` and `AcyclicCounterexample.question34_false` combine nonunimodality with the strict root bound. They do not assume a graph-counting bridge or use numerical root finding.

The 15-module verification uses Lean 4.27.0 and mathlib revision `a3a10db0e9d66acbebf76c5e6a135066525ac900`; all package revisions are pinned in `lake-manifest.json`. With the dependencies provisioned, run `python3 scripts/verify.py -packages /path/to/packages`. Successful logs, source identities, and transitive axiom audits are recorded in `verification/result.json`. Only `propext`, `Classical.choice`, and `Quot.sound` occur in the audited final dependencies. There are no proof placeholders, additional mathematical axioms, or native-computation certificates. The pinned compiler and cached library proofs remain trusted. This formal result does not assert a minimum counterexample order or bibliographic priority.

References

- [1] C. Barton, J. I. Brown, and D. A. Pike, *Acyclic polynomials of graphs*, Australas. J. Combin. **82**(2) (2022), 146–181. https://ajc.maths.uq.edu.au/pdf/82/ajc_v82_p146.pdf.
- [2] A. Hurwitz, *Ueber die Bedingungen, unter welchen eine Gleichung nur Wurzeln mit negativen reellen Theilen besitzt*, Math. Ann. **46** (1895), 273–284. <https://doi.org/10.1007/BF01446812>.