

Three-Element Lucky Sets of Stirling Permutations

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Abstract

We characterize the three-element sets of lucky car indices that occur when a Stirling permutation of order n is used as a parking function. A set $\{1, x, y\}$, with $1 < x < y < 2n$, is admissible precisely when x is odd, or x is even and y is odd with $x < n$, or both are even with $x + y \leq 2n$. Explicit Stirling words realize every permitted case. The number of admissible sets is $(n-1)(n-2) + 2nh - 3h(h+1)/2$, where $h = \lfloor (n-1)/2 \rfloor$. This answers the three-element characterization and enumeration question in Problem 46 of Colmenarejo et al. The complete classification, parking constructions, and enumeration have been checked in Lean.

1 Statement and scope

A *Stirling permutation of order n* is a word of length $2n$ containing two copies of each label in $[n] = \{1, \dots, n\}$, such that all letters strictly between the two copies of a label a are greater than a . Denote this set of words by Q_n .

Interpret a word $w = w_1 \cdots w_{2n}$ as preferences of cars arriving in the order $1, \dots, 2n$. Car j parks at the first unoccupied position at least w_j . All cars park in $[2n]$: the nondecreasing rearrangement of the preferences is $1, 1, 2, 2, \dots, n, n$, whose j th entry $\lceil j/2 \rceil$ is at most j , the usual parking-function criterion. Car j is *lucky* if it parks at w_j itself. A set S is *n -admissible* if $S = \text{Lucky}(w)$ for some $w \in Q_n$.

Colmenarejo et al. [1] studied these sets, including selected three-element families. Their Problem 46 first asks for a complete characterization and enumeration of the three-element admissible sets, and then asks the more general question for arbitrary cardinalities. The following theorem answers the first question.

Theorem 1.1. *Let $n \geq 1$ and let S have three elements. Then S is n -admissible if and only if $S = \{1, x, y\}$ with $1 < x < y < 2n$ and one of the following disjoint conditions holds:*

- (A) x is odd;
- (B) x is even, y is odd, and $x < n$;
- (C) x and y are even, and $x + y \leq 2n$.

Corollary 1.2. *Let $N(n)$ be the number of three-element n -admissible sets and put $h = \lfloor (n-1)/2 \rfloor$. For $n \geq 1$,*

$$N(n) = (n-1)(n-2) + 2nh - \frac{3h(h+1)}{2}. \quad (1)$$

Equivalently,

$$N(2r) = \frac{13r^2 - 17r + 4}{2} \quad (r \geq 1), \quad N(2r + 1) = \frac{13r^2 - 3r}{2} \quad (r \geq 0). \quad (2)$$

The first nine values, for $n = 1, \dots, 9$, are 0, 0, 5, 11, 23, 35, 54, 72, 98. We count distinct lucky sets, not the number of words realizing each set. No characterization of sets with four or more elements is asserted.

2 Necessary conditions

Lemma 2.1. *In any successful parking word with two copies of each label in $[n]$, if car j is lucky with preference b , then $j + b \leq 2n$.*

Proof. Immediately before car j , position b is still free. Each earlier car with preference at most b must have parked below b : at its arrival b was available, but it did not occupy b . Thus at most $b - 1$ of the $2b$ occurrences of labels in $[b]$ appeared before position j . At least $b + 1$ remain in a suffix of length $2n - j + 1$. This gives $b + 1 \leq 2n - j + 1$. $\square$

Lemma 2.2. *Let x be the second lucky index and put $a = w_1$. After the first j cars, for $1 \leq j < x$, the occupied positions are exactly $[a, a + j - 1]$. Every preference in that prefix is at least a .*

Proof. Car 1 occupies a . Each subsequent car before x is unlucky, so its preferred position is in the current occupied interval. It parks at the first position just to the right of that interval. Induction gives both assertions. $\square$

Lemma 2.3. *A new strict record minimum in a Stirling word occurs at an odd position. In particular, the first occurrence of 1 is odd-indexed and lucky.*

Proof. If w_j is smaller than every earlier letter, every label seen before j must have both copies before j . Otherwise its pair would enclose the smaller label w_j , violating the Stirling condition. The prefix has even length, so j is odd. No preference greater than 1 can occupy position 1, proving the last assertion. $\square$

Suppose now that $\text{Lucky}(w) = \{1, x, y\}$ with $1 < x < y$. Car 1 is always lucky, and Lemma 2.1 gives $y < 2n$. If x is odd, condition (A) holds. Assume x is even, set $a = w_1$, and put $c = w_x$. By Lemma 2.2, a lucky preference c must satisfy either $c < a$ or $c \geq a + x - 1$. The former would be a new strict record minimum, contrary to Lemma 2.3. Therefore $a + x - 1 \leq c \leq n$, and $x \leq n$. Equality $x = n$ forces $a = 1$ and $c = n$. Immediately after car x , every preferred position in $[n]$ would then be occupied, leaving no possible third lucky car. Hence

$$x < n \quad \text{when the second lucky index is even.} \quad (3)$$

If y is odd this proves (B). If y is even, the only odd lucky index is 1. Lemma 2.3 forces $w_1 = 1$. Positions $1, \dots, x - 1$ are occupied before car x and remain occupied, so the lucky preference $b = w_y$ satisfies $b \geq x$. Lemma 2.1 now gives $x + y \leq b + y \leq 2n$, proving (C).

3 Words realizing every permitted triple

For integers $a \leq b$, let

$$D(a, b) = a a (a + 1) (a + 1) \cdots b b,$$

and let $D(a, b)$ be empty for $a > b$. If U is an increasing list of integers, $D(U)$ doubles each entry in its list order. Juxtaposition means word concatenation; an arithmetic expression such as $(2r)$ between blocks denotes one letter of that value.

We use two elementary parking observations. A consecutive doubled run whose first preference is free has exactly one lucky car, its first. After the second copy of a label a has parked, position $a + 1$ is occupied, so the first copy of the next label is unlucky. This remains true if some larger positions were occupied before the run. Second, if the occupied positions form $[L, H]$, an increasing doubled run starting at an occupied preference and with consecutive label gaps at most 2 is entirely unlucky. Each pair extends the right endpoint by 2, so the next preference remains within the occupied interval. Both observations follow directly by induction from the first-free rule.

3.1 Odd x , odd y

Write $x = 2r + 1$ and $y = 2r + 2t + 1$, where $r, t \geq 1$, and let $u = n - r - t \geq 1$. Use

$$D(u + t + 1, n) D(u + 1, u + t) D(1, u). \quad (4)$$

Each of the three runs starts with a preference below all earlier preferences, hence at a free position. By the first observation, only the first car of each run is lucky. Their indices are $1, 2r + 1, 2r + 2t + 1$.

3.2 Odd x , even y

Write $x = 2r + 1$ and $y = 2r + 2t + 2$, where $r \geq 1, t \geq 0$, and $r + t + 2 \leq n$. Use

$$D(3, r + 2) 1 D(r + 3, r + t + 2) 2 2 1 D(r + t + 3, n). \quad (5)$$

The initial run has only its first car lucky and occupies $[3, 2r + 2]$. The first 1, at position x , is lucky. The following t doubled labels extend the upper interval to $[3, y]$, since their first preference $r + 3$ is at most $2r + 2$. The first 2, at position y , is lucky. The second 2 and the closing 1 extend the occupied interval to $[1, y + 2]$. The final doubled tail is unlucky: its first preference $r + t + 3$ is at most $y + 2$, and its labels are consecutive.

3.3 Even x , even y

Write $x = 2r, y = 2s$, where $1 \leq r < s$ and $r + s \leq n$. Use

$$\begin{aligned} D(1, r - 1) r D(2r + 1, r + s) (2r) (2r) \\ D(r + 1, 2r - 1) r D(r + s + 1, n). \end{aligned} \quad (6)$$

Before position x , the occupied interval is $[1, x - 1]$, and only car 1 has been lucky. For $r = 1$, the displayed initial letter r is itself car 1. The nonempty upper doubled block starts at preference $x + 1$ and occupies $[x + 1, y]$; only its first car, at position x , is lucky. The next letter $2r = x$ is at position y and fills the gap at x , giving the third lucky car. All subsequent preferences in the middle block and the closing r are occupied. At that point the occupied interval is $[1, 2(r + s)]$. The final tail is also unlucky, since its initial preference is $r + s + 1$.

3.4 Even x , and $y = x + 1$

Write $x = 2r$; the condition $x < n$ gives $x + 1 \leq n$. Use

$$\begin{aligned} D(1, r-1) \ r \ x \ (x+1) \ (x+1) \ x \\ D(r+1, x-1) \ r \ D(x+2, n). \end{aligned} \tag{7}$$

The first $x-1$ cars occupy $[1, x-1]$ with only car 1 lucky. The next two preferences, x and $x+1$, are free and occur at positions x and y . The next two cars extend occupancy to $[1, x+3]$. The middle doubled run and closing r extend it to $[1, 2x+2]$, and the final tail stays behind that frontier. Thus there are exactly three lucky cars. When $r = 1$ and $n = 3$, this construction is the word 123321.

3.5 Even x , odd $y \geq x + 3$

Write $x = 2r$, where $r \geq 1$ and $n \geq x + 1$. Let U be the increasing list $[r+2, n]$ with $x+1$ omitted, and put $t = (y - x - 3)/2$. Then $0 \leq t \leq n - r - 2 = |U|$. Split U into its first t entries U_0 and the remaining entries U_1 . Use

$$D(2, r) \ (r+1) \ (x+1) \ (x+1) \ (r+1) \ D(U_0) \ 1 \ 1 \ D(U_1). \tag{8}$$

After the first $x-1$ cars, the occupied interval is $[2, x]$, with only car 1 lucky. The first $x+1$ occurs at position x and is lucky; its second copy and the closing $r+1$ extend occupancy to $[2, x+3]$. The least element of U , if present, is at most $x+3$, and successive elements differ by at most 2. The frontier observation shows that $D(U_0)$ extends occupancy to $[2, y]$ and has no lucky car. The first 1 occurs at position y and is lucky. Its second copy leaves $[1, y+1]$ occupied. The remaining run $D(U_1)$ is unlucky by the same frontier bound, including when U_0 is empty.

3.6 Checking the Stirling condition

Every construction uses each label in $[n]$ exactly twice: the displayed blocks partition the unused labels, and the separately displayed letters supply the omitted pairs. In (4) all pairs are adjacent. In (5), only the pair of 1's can enclose other letters, all greater than 1. In (6), the pair of r 's encloses only labels greater than r . In (7), the pair of x 's encloses $x+1$, and the pair of r 's encloses only larger letters. In (8), the pair of $r+1$'s encloses only $x+1 > r+1$. Thus all five words belong to Q_n . The preceding parking analyses give precisely $\{1, x, y\}$ in each case. Together with the necessary conditions, they prove Theorem 1.1.

4 Enumeration

The cases of Theorem 1.1 are disjoint. In case (A), put $x = 2a + 1$ for $1 \leq a \leq n - 2$. There are $2n - 2a - 2$ choices of y , giving $(n-1)(n-2)$ triples when $n \geq 2$; the count is also zero at $n = 1$. In case (B), put $x = 2a$ and $y = 2b + 1$. Then $1 \leq a \leq h$ and $a \leq b \leq n - 1$, so the count is

$$\sum_{a=1}^h (n-a) = nh - \frac{h(h+1)}{2}.$$

In case (C), put $x = 2a$ and $y = 2b$. The constraints are $1 \leq a \leq h$ and $a < b \leq n - a$, giving

$$\sum_{a=1}^h (n-2a) = nh - h(h+1).$$

Their sum proves (1). Substituting $h = r - 1$ for $n = 2r$ and $h = r$ for $n = 2r + 1$ gives (2). In particular, $n = 1, 2$ have no admissible triples, as required.

5 Formal verification

The project `stirling-lucky-triples` defines Stirling words, the actual least-free parking process, and lucky-index equality. It proves the necessity lemmas and all five constructions, including their nesting, multiplicities, parking success, and absence of further lucky cars. The declarations `StirlingLucky.classification`, `StirlingLucky.enumeration`, `StirlingLucky.exact_count_even`, and `StirlingLucky.exact_count_odd` are the unconditional classification and counting endpoints.

The 25-module build uses Lean 4.27.0 and mathlib revision `a3a10db0e9d66acbebf76c5e6a135066525ac900`. With the dependencies pinned by `lake-manifest.json` provisioned, run `python3 scripts/verify.py`. The successful build and transitive axiom-audit logs are recorded in `verification/result.json`; the definition correspondence is documented in `STATEMENT_MATCH_AUDIT.txt` and `SEMANTIC_STATEMENT_AUDIT.txt`. Only `propext`, `Classical.choice`, and `Quot.sound` occur in the final transitive axiom lists. The proof uses no placeholders, additional mathematical axioms, or native computation. The pinned compiler and cached dependency proof objects are trusted. Formal checking verifies the declarations rather than this manuscript’s prose or its bibliographic status.

References

- [1] L. Colmenarejo, A. Dawkins, J. Elder, P. E. Harris, K. J. Harry, S. Kara, D. Smith, and B. E. Tenner, *On the lucky and displacement statistics of Stirling permutations*, J. Integer Sequences **27** (2024), Article 24.6.7. <https://cs.uwaterloo.ca/journals/JIS/VOL27/Tenner/tenner12.pdf>.