

The Willow Transform and the Blind Zone Theorem: A Sharp Obstruction in the Turán Hierarchy for the Riemann ξ -Function

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Statement on the use of generative artificial intelligence. Generative AI assistance was used for literature retrieval, notation cross-checking, $\text{\LaTeX}$ typesetting, and numerical recomputation of the constants recorded in Section 9. The mathematical content—the reduction of the Toeplitz minors to Schur functions, the two-variable Chebyshev closed form, the index bound, the blind zone theorem, and the derivation of the threshold $r \leq 43$ —was produced by the author. Steps that are not proved are identified in Section 10.

Abstract

Let $\Xi(t) = \xi(\frac{1}{2} + it)$ denote the Riemann ξ -function in the critical-line parametrisation, let $\Xi(t) = \sum_{k \geq 0} (-1)^k c_k t^{2k}$ be its even Taylor expansion at the origin, and let $D(n, r) = \det[c_{n+i-j}]_{i,j=0}^{r-1}$ be the associated Toeplitz minors. By the Schoenberg–Edrei characterisation of Pólya frequency sequences, positivity of $D(n, r)$ for all $n \geq 0$ and all $r \geq 1$ is equivalent to the Riemann hypothesis (RH); for each fixed r the inequalities $D(n, r) \geq 0$ supply a necessary condition for RH, and for $r = 2$ they were proved unconditionally by Csordas, Norfolk and Varga [5], so that the second level of the resulting hierarchy is not discriminating. We show that this failure of discrimination persists far beyond level 2, and we determine the range over which it does so: every level $r \leq 43$ is non-discriminating.

Parametrising the zeros of the generating function $G(z) = \sum_{k \geq 0} c_k z^k$ by $\beta_j = -1/z_j$, the dual Jacobi–Trudi identity yields the exact identity $D(n, r) = c_0^r s_{(r^n)}(\beta)$, where $s_{(r^n)}$ denotes the Schur function of the rectangular partition (r^n) in the alphabet β . A zero of ζ off the critical line contributes to β a conjugate pair $\rho e^{\pm i\theta}$ whose argument θ coincides with the vertical angle of the corresponding violating pair, whereas zeros on the critical line contribute positive real variables. For an alphabet of cardinality two we prove the closed form

$$s_{(a,b)}(\rho e^{i\theta}, \rho e^{-i\theta}) = \rho^{a+b} \frac{\sin((a+1-b)\theta)}{\sin \theta} = \rho^{a+b} U_{a-b}(\cos \theta),$$

where U_k is the Chebyshev polynomial of the second kind; the sign is therefore governed by the criterion $U_k(\cos \theta) > 0$ if and only if $(k+1)\theta < \pi$.

Through the coproduct expansion $s_\lambda(X \sqcup Y) = \sum_\mu s_\mu(X) s_{\lambda/\mu}(Y)$, the Littlewood–Richardson expansion $s_{\lambda/\mu} = \sum_\nu c_{\mu\nu}^\lambda s_\nu$ of each skew factor, and the positivity of Schur functions on positive real alphabets, the sign analysis reduces to two-variable constituents. Since $c_{\mu\nu}^\lambda > 0$ forces $\nu \subseteq \lambda = (r^n)$, the index $k = \nu_1 - \nu_2$ of each constituent is bounded by the number of columns r and is independent of the shift n . All constituents then have the same sign, so no cancellation occurs. The same bound persists for an alphabet containing several conjugate pairs: iterating the coproduct over the

pairs and using the fact that the entry 1 occurs at most once in each column of a semistandard tableau bounds every constituent index by $\nu_1 \leq r$, independently of the number of pairs. This yields the following result.

Blind Zone Theorem. *Let the alphabet contain any number of violating pairs, each subtending a vertical angle at most $\theta_{\max}$, and suppose $\theta_{\max} < \pi/(r+1)$. Then every minor $D(n, r')$ with $r' \leq r$ is strictly positive, notwithstanding the presence of non-real zeros.*

The critical strip together with $\gamma_1 = 14.134725141 \dots$ forces $\theta \leq \theta_{\max} = 2 \arctan(1/(2\gamma_1)) = 0.07071826 \dots$ radians, whence $\pi/\theta_{\max} = 44.424 \dots$; consequently every order $r \leq 43$ is non-discriminating: for all $n \geq 0$ and all $r \leq 43$ the minor $D(n, r)$ is positive if RH fails and non-negative if RH holds, so no minor of order $r \leq 43$ can certify either the truth or the falsity of RH. The bound is on the order r and not on the shift n ; the corresponding n -based assertion is false, and the mechanism responsible is exhibited in Section 12. The threshold is established for an arbitrary number of violating pairs; the case of several pairs is settled in Section 5 and requires no separation condition on the moduli and no bound on the number of pairs. No claim of a proof of the Riemann hypothesis is made.

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1 Introduction

1.1 Background

Let $\zeta(s)$ denote the Riemann zeta function, initially defined for $\Re(s) > 1$ by the Dirichlet series $\sum_{n \geq 1} n^{-s}$ and extended to $\mathbb{C}$ by meromorphic continuation with a single simple pole at $s = 1$. Let

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) \quad (1)$$

be the completed zeta function. The factor $\frac{1}{2}s(s-1)$ removes the pole, and ξ is an entire function of order 1 and maximal type satisfying the functional equation

$$\xi(s) = \xi(1-s). \quad (2)$$

The Riemann hypothesis (RH) is the assertion that every non-trivial zero of ζ satisfies $\Re(s) = \frac{1}{2}$.

By (2), the transformation $s \mapsto \frac{1}{2} + it$ renders ξ a function of a real variable. Define

$$\Xi(t) = \xi\left(\frac{1}{2} + it\right), \quad t \in \mathbb{R}. \quad (3)$$

Then Ξ is an even, real-valued entire function of t , and RH is equivalent to the statement that all zeros of Ξ are real.

Because Ξ is even, its Taylor expansion at the origin has only even powers:

$$\Xi(t) = \sum_{k=0}^{\infty} (-1)^k c_k t^{2k}. \quad (4)$$

The coefficients admit the integral representation

$$c_k = \frac{1}{(2k)!} \int_0^{\infty} u^{2k} \Phi(u) du, \quad (5)$$

where Φ is the positive, rapidly decreasing kernel

$$\Phi(u) = 4\pi^2 \sum_{n \geq 1} n^2 \left(2\pi^2 n^2 e^{9u/2} - 3\pi n^2 e^{5u/2} \right) \exp(-\pi n^2 e^{2u}), \quad (6)$$

up to the normalisation of (1). Positivity $c_k > 0$ for all $k \geq 0$ follows from (5) together with $\Phi > 0$, and is classical. The sequence $(c_k)_{k \geq 0}$ is the object of study throughout this paper.

1.2 The Turán hierarchy

For integers $n \geq 0$ and $r \geq 1$ define

$$D(n, r) = \det [c_{n+i-j}]_{i,j=0}^{r-1}, \quad (7)$$

with the convention $c_k = 0$ for $k < 0$. Thus $D(n, 1) = c_n > 0$, and

$$D(n, 2) = c_n^2 - c_{n-1}c_{n+1}.$$

The inequalities $D(n, 2) \geq 0$ are the Turán inequalities, introduced in this context by Turán [3] and studied extensively by Csordas, Norfolk and Varga [5].

Theorem 1.1 (The hierarchy and RH). *The following assertions are equivalent.*

- (i) All zeros of Ξ are real, that is, RH holds.
- (ii) $D(n, r) \geq 0$ for all $n \geq 0$ and all $r \geq 1$.

The implication (i) $\Rightarrow$ (ii) follows from the representation (5) together with the Schoenberg–Edrei characterisation of Pólya frequency sequences; see Karlin [4]. The implication (ii) $\Rightarrow$ (i) is the criterion of Karlin [4, Thm. 5.3, p. 412], in the form employed by the determinantal approach to RH; see Nuttall [10]. We record the equivalence in the form stated because the filtration

$$r = 1, \quad r = 2, \quad r = 3, \quad \dots$$

constitutes the natural hierarchy along which discriminating power is conventionally expected to increase.

For a *fixed* order r , the system of inequalities $D(n, r) \geq 0$ for all $n \geq 0$ is a necessary condition for RH, by (i) $\Rightarrow$ (ii) above, but it is in general not sufficient. The case $r = 2$ is that of the Turán inequalities. Pólya [7] identified them as a necessary condition for RH and left open whether they hold unconditionally; Csordas, Norfolk and Varga [5] proved that they do hold for the ξ -function, thereby settling a question open since 1927, and their argument yields in particular $D(n, 2) > 0$ for all n ; see Nuttall [10], who records this consequence explicitly. Their theorem shows that the second level of the hierarchy carries no information about RH whatsoever: $D(n, 2) > 0$ is true independently of whether RH is true. No analogous unconditional result is known at any level $r \geq 3$; determining how far this phenomenon extends is the subject of the present paper.

Remark 1.2. *Theorem 1.1 converts RH into a countable family of determinantal inequalities. The minors $D(n, r)$ are the contiguous minors of the Toeplitz array $(c_{i-j})_{i,j \geq 0}$, with the convention $c_k = 0$ for $k < 0$: the entry in row i and column j of the submatrix defining $D(n, r)$ is c_{n+i-j} . This is the array whose total positivity expresses the property that (c_k) be a Pólya frequency sequence, and it is the convention used in the determinantal approach; cf. [10]. It should not be confused with the Hankel array $(c_{i+j})_{i,j \geq 0}$, whose minors constitute a different family; see Subsection 11.5. We use the Toeplitz form (7) throughout.*

1.3 Motivation and the question addressed

Theorem 1.1 invites two questions.

- (Q1) For which r is the r -th level of the hierarchy *discriminating*, in the sense that the failure of RH would force $D(n, r) < 0$ for some n ?
- (Q2) Does there exist a finite r_0 such that positivity of $D(n, r_0)$ for all $n \geq 0$ already implies RH?

Level 2 is known not to be discriminating: that is precisely the content of the theorem of Csordas, Norfolk and Varga [5] recalled above. Question (Q2) has a trivial affirmative answer if and only if RH is decidable at finite order, a statement for which no evidence is available. The purpose of this paper is to quantify the obstruction: we show that the discriminating power of level r is governed by an angle threshold $\pi/(r+1)$, and that the geometry of the critical strip places every hypothetically violating pair below this threshold for all $r \leq 43$. Thus the non-discrimination established for level 2 by [5] persists through level 43; the argument given here does not extend beyond that.

The mechanism is not analytic but combinatorial-algebraic: it is a consequence of the identification of $D(n, r)$ with a Schur function of rectangular shape, of the two-variable closed form for such Schur functions on conjugate pairs, and of the fact that the relevant index is bounded by the number of columns.

1.4 Overview of the method

The argument proceeds in five stages.

Stage 1: the Willow transform. The zeros z_j of the generating function $G(z) = \sum_{k \geq 0} c_k z^k$ are converted into an alphabet $\beta_j = -1/z_j$. Section 3 shows that the Hadamard factorisation of G identifies $e_k(\beta) = c_k/c_0$, after which the dual Jacobi–Trudi identity yields $D(n, r) = c_0^r s_{(r^n)}(\beta)$.

Stage 2: the two-variable closed form. Section 4 evaluates Schur functions of two-part partitions on a conjugate pair and obtains $s_{(a,b)} = \rho^{a+b} U_{a-b}(\cos \theta)$. The sign therefore depends on the angle and on the index $k = a - b$, and on nothing else.

Stage 3: coproduct reduction. Section 5 splits the alphabet into its real part X and its non-real part Y . Since $s_\mu(X) > 0$ for $X \subset \mathbb{R}_{>0}$, sign changes can originate only in the skew factors $s_{\lambda/\mu}(Y)$.

Stage 4: the index bound. The index $k = a - b$ arising in Stage 2 is bounded by r , the number of columns of (r^n) , and not by the shift n . This is Lemma 5.7 and is the structural core of the paper.

Stage 5: the sector bound and the threshold. Section 7 bounds the angle of a violating pair by $2 \arctan(1/2\gamma_1)$ using only the critical strip and the value of γ_1 . Section 8 combines this with the threshold $\pi/(r+1)$ and obtains $r \leq 43$.

1.5 Main results

- (1) **The Willow correspondence.** An exact identity between the determinantal minors $D(n, r)$ and the Schur functions $s_{(r^n)}$ evaluated on the alphabet of reciprocal zeros (Theorem 3.1).
- (2) **A two-variable closed form.** An explicit evaluation in terms of Chebyshev polynomials of the second kind (Corollary 4.2), together with the resulting sign criterion (Lemma 4.4).
- (3) **An index bound depending on r and not on n** (Lemma 5.7), which localises the discriminating power of the hierarchy in the column count.
- (4) **The blind zone theorem** (Theorem 6.1), the numerical threshold $r \leq 43$ (Theorem 8.1), and the unconditional consequence $D(n, r) > 0$ for all $n \geq 0$ and all $r \leq 43$ (Corollary 8.2), which extends the theorem of Csordas, Norfolk and Varga [5] from level 2 to level 43.
- (5) **A refutation** of an auxiliary covering conjecture (Appendix C) and an analysis of a floating-point artefact that arose during verification (Appendix D).

1.6 Organisation

Section 2 fixes notation and recalls the required background on symmetric functions and total positivity. Section 3 constructs the Willow transform. Section 4 establishes the Chebyshev closed form. Section 5 performs the coproduct reduction and proves the index bound. Section 6 states and proves the blind zone theorem. Section 7 derives the sector

bound. Section 8 obtains the threshold. Section 9 records numerical verification. Section 10 lists the status of each step. Section 12 contains remarks on scope. Appendices A–G contain supporting material.

2 Notation and preliminaries

2.1 Generating functions

Let

$$G(z) = \sum_{k=0}^{\infty} c_k z^k \quad (8)$$

denote the ordinary generating function of $(c_k)_{k \geq 0}$. Comparing (4) and (8),

$$\Xi(t) = G(-t^2). \quad (9)$$

Since Ξ is entire of order 1, the coefficients c_k decay faster than any geometric sequence and G is entire. Relation (9) gives a bijection between zeros: t is a zero of Ξ if and only if $z = -1/t^2$ is a zero of G .

Definition 2.1 (Willow alphabet). *Let $(z_j)_{j \geq 1}$ enumerate the zeros of G with multiplicity. The Willow alphabet is the multiset*

$$\beta = \{\beta_j\}_{j \geq 1}, \quad \beta_j = -\frac{1}{z_j}.$$

The assignment $z \mapsto \beta = -1/z$ is the Willow transform.

Remark 2.2. *The sign convention in Definition 2.1 is chosen so that $\beta_j = t^2$ whenever $z_j = -1/t^2$. Hence:*

- *if $t = \gamma \in \mathbb{R} \setminus \{0\}$, then $\beta = \gamma^2 \in \mathbb{R}_{>0}$;*
- *if $t = \gamma \pm i\delta$ with $\delta \neq 0$, then $\beta = (\gamma \pm i\delta)^2 \in \mathbb{C} \setminus \mathbb{R}_{\geq 0}$.*

Real zeros of Ξ therefore contribute positive real variables, and non-real zeros contribute non-real conjugate pairs.

2.2 Partitions and symmetric functions

We follow the conventions of Macdonald [1] and Sagan [2]. A *partition* $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_\ell > 0)$ is identified with its Young diagram. We write $|\lambda| = \sum_i \lambda_i$ for the size, $\ell(\lambda) = \ell$ for the length, and λ' for the conjugate partition. The rectangular partition with n rows and r columns is denoted (r^n) .

For an alphabet $x = (x_1, \dots, x_N)$ the Schur function is

$$s_\lambda(x) = \frac{\det[x_i^{\lambda_j + N - j}]_{i,j=1}^N}{\det[x_i^{N-j}]_{i,j=1}^N}. \quad (10)$$

Equivalently, $s_\lambda(x) = \sum_T x^{\text{wt}(T)}$, the sum over semistandard Young tableaux T of shape λ with entries in $\{1, \dots, N\}$.

Lemma 2.3 (Tableau expansion). *$s_\lambda(x)$ is a polynomial in $x_1, \dots, x_N$ with non-negative integer coefficients. In particular, if $x_i > 0$ for all i and $\ell(\lambda) \leq N$, then $s_\lambda(x) > 0$.*

Proof. Each tableau contributes a monomial with coefficient 1, and the sum is non-empty when $\ell(\lambda) \leq N$. Positivity follows. $\square$

The complete homogeneous and elementary symmetric functions are denoted h_k and e_k , with generating series

$$H(u) = \sum_{k \geq 0} h_k u^k = \prod_i (1 - x_i u)^{-1}, \quad E(u) = \sum_{k \geq 0} e_k u^k = \prod_i (1 + x_i u).$$

They satisfy $E(u)H(-u) = 1$.

Theorem 2.4 (Jacobi–Trudi identity). $s_\lambda = \det[h_{\lambda_i - i + j}]_{i,j=1}^{\ell(\lambda)}$.

Theorem 2.5 (Dual Jacobi–Trudi identity). $s_\lambda = \det[e_{\lambda'_i - i + j}]_{i,j=1}^{\lambda_1}$.

A derivation of Theorem 2.5 from Theorem 2.4 is recorded in Appendix A.

Definition 2.6 (Skew Schur function). For $\mu \subseteq \lambda$, the skew Schur function $s_{\lambda/\mu}$ is defined by $s_{\lambda/\mu} = \sum_\nu c_{\mu\nu}^\lambda s_\nu$, where $c_{\mu\nu}^\lambda$ are the Littlewood–Richardson coefficients.

Theorem 2.7 (Coproduct expansion). For alphabets X and Y ,

$$s_\lambda(X \sqcup Y) = \sum_{\mu \subseteq \lambda} s_\mu(X) s_{\lambda/\mu}(Y).$$

Proof. Expand $s_\lambda(X \sqcup Y)$ in the Schur basis and apply the Littlewood–Richardson rule to the resulting product; the standard branching rule for the symmetric group yields the stated sum. $\square$

2.3 Total positivity

Definition 2.8 (Pólya frequency sequence). A sequence $(c_k)_{k \geq 0}$ of non-negative reals is a Pólya frequency sequence if the infinite matrix $(c_{i-j})_{i,j \geq 0}$ is totally positive, i.e. all its finite minors are non-negative.

Theorem 2.9 (Schoenberg–Edrei). Let $c_0 > 0$ and let $\sum_{k \geq 0} c_k z^k$ have radius of convergence $\varrho > 0$. Then (c_k) is a Pólya frequency sequence if and only if

$$\sum_{k \geq 0} c_k z^k = c_0 e^{\gamma z} \frac{\prod_{j \geq 1} (1 + \alpha_j z)}{\prod_{j \geq 1} (1 - \beta_j z)},$$

with $\alpha_j, \beta_j, \gamma \geq 0$ and $\sum_j (\alpha_j + \beta_j) < \infty$.

Remark 2.10. Theorem 2.9 is the structural reason why the minors $D(n, r)$ detect non-real zeros: total positivity of the coefficient sequence is equivalent to reality of the zeros of Ξ , and each minor is one of the defining inequalities.

2.4 Chebyshev polynomials

Definition 2.11. The Chebyshev polynomials of the second kind are defined by

$$U_k(\cos \theta) = \frac{\sin((k+1)\theta)}{\sin \theta}, \quad k \geq 0.$$

They satisfy $U_0 = 1$, $U_1(x) = 2x$, and the recurrence $U_{k+1}(x) = 2x U_k(x) - U_{k-1}(x)$.

Lemma 2.12. *For $k \geq 0$:*

- (i) U_k has degree k and leading coefficient 2^k ;
- (ii) the zeros of U_k are $\cos(j\pi/(k+1))$, $j = 1, \dots, k$;
- (iii) U_k is positive on $(\cos(\pi/(k+1)), 1)$.

Proof. (i) follows from the recurrence. (ii): $U_k(\cos \theta) = 0$ if and only if $\sin((k+1)\theta) = 0$ and $\sin \theta \neq 0$, i.e. $\theta = j\pi/(k+1)$ with $j = 1, \dots, k$. (iii) follows from (ii) and $U_k(1) = k+1 > 0$. $\square$

3 The Willow transform

3.1 The correspondence

Theorem 3.1 (Willow correspondence). *Let G be as in (8) with $c_0 \neq 0$, and let β be the Willow alphabet of Definition 2.1. Then for all $n \geq 0$ and $r \geq 1$,*

$$D(n, r) = c_0^r s_{(r^n)}(\beta). \quad (11)$$

Proof. Since G is entire and $G(0) = c_0 \neq 0$, the Hadamard factorisation gives

$$G(z) = c_0 \prod_{j \geq 1} (1 - \beta_j z), \quad \beta_j = -\frac{1}{z_j}, \quad (12)$$

the product converging locally uniformly. Comparing (12) with $E(u) = \prod_j (1 + x_j u)$ at $x = \beta$ and $u = -z$ yields

$$e_k(\beta) = \frac{c_k}{c_0}, \quad k \geq 0. \quad (13)$$

Apply Theorem 2.5 to $\lambda = (r^n)$. The conjugate partition is $\lambda' = (n^r)$, i.e. n repeated r times, and $\lambda_1 = r$. Hence

$$s_{(r^n)}(\beta) = \det [e_{n-i+j}(\beta)]_{i,j=1}^r.$$

Substituting (13) and extracting c_0^{-1} from each of the r rows,

$$s_{(r^n)}(\beta) = c_0^{-r} \det [c_{n-i+j}]_{i,j=1}^r.$$

The matrix $(c_{n-i+j})_{i,j=1}^r$ is obtained from $(c_{n+i-j})_{i,j=0}^{r-1}$ by reversing the order of the rows and of the columns; each reversal contributes a factor -1 to the determinant, and the two factors cancel. Hence $\det [c_{n-i+j}]_{i,j=1}^r = D(n, r)$, and (11) follows. $\square$

Remark 3.2. *Since $c_0 > 0$, the factor c_0^r in (11) is positive and does not affect sign questions. In Sections 4 through 6 we suppress it and work with $s_{(r^n)}(\beta)$.*

Remark 3.3 (Geometric dictionary). *Theorem 3.1 establishes the following correspondence between the zeros of Ξ and the alphabet variables:*

zero of Ξ	$\beta = t^2$	type
$t = \gamma, \gamma \in \mathbb{R} \setminus \{0\}$	$\beta = \gamma^2$	positive real
$t = \gamma + i\delta, \delta \neq 0$	$\beta = (\gamma + i\delta)^2$	non-real, $\arg \beta = 2 \arctan(\delta/\gamma)$
$t = \gamma - i\delta, \delta \neq 0$	$\beta = (\gamma - i\delta)^2$	non-real, $\arg \beta = -2 \arctan(\delta/\gamma)$

Zeros on the critical line are therefore inert with respect to sign, while off-line zeros enter in conjugate pairs carrying an angle.

3.2 Algebraic consequences

Lemma 3.4. $s_{(r^n)}$ is homogeneous of degree rn . Consequently $D(n, r)$ is homogeneous of degree rn in the alphabet β and of degree r in the coefficients c_k .

Proof. Homogeneity of Schur functions is standard; the second assertion follows from (11) and $e_k(\beta) = c_k/c_0$. $\square$

Lemma 3.5 (Reality). $s_{(r^n)}(\beta) \in \mathbb{R}$ whenever β is closed under conjugation.

Proof. $s_{(r^n)}$ is a symmetric polynomial with rational coefficients; conjugation acts on β as a permutation, and symmetric polynomials are invariant under permutation of the variables. Hence the value is fixed by conjugation and is real. $\square$

Remark 3.6. Lemma 3.5 ensures that the sign of $D(n, r)$ is well defined. This is consistent with the fact that $D(n, r)$ is a determinant of real numbers.

3.3 The inertness of real zeros

Lemma 3.7. Let $\beta = X \sqcup Y$ with $X \subset \mathbb{R}_{>0}$ and Y closed under conjugation. Then

$$s_\lambda(\beta) = \sum_{\mu \subseteq \lambda} s_\mu(X) s_{\lambda/\mu}(Y),$$

and each factor $s_\mu(X)$ is non-negative, with strict positivity when $\ell(\mu) \leq |X|$.

Proof. The identity is Theorem 2.7. Non-negativity of $s_\mu(X)$ follows from Lemma 2.3. $\square$

Lemma 3.7 isolates the source of sign changes: only the skew factors $s_{\lambda/\mu}(Y)$ can be negative. Sections 4 and 5 analyse these factors.

4 The two-variable closed form

4.1 The closed form

Lemma 4.1. Let $a \geq b \geq 0$ be integers, $\rho > 0$, and $\theta \in (0, \pi)$. Put $x_1 = \rho e^{i\theta}$ and $x_2 = \rho e^{-i\theta}$. Then

$$s_{(a,b)}(x_1, x_2) = \rho^{a+b} \frac{\sin((a+1-b)\theta)}{\sin \theta}.$$

Proof. For an alphabet of cardinality 2, the Schur function of a partition with at most two parts is given by the alternant quotient

$$s_{(a,b)}(x_1, x_2) = \frac{x_1^{a+1}x_2^b - x_2^{a+1}x_1^b}{x_1 - x_2}.$$

Substituting the polar forms,

$$x_1^{a+1}x_2^b = \rho^{a+b+1}e^{i(a+1-b)\theta}, \quad x_2^{a+1}x_1^b = \rho^{a+b+1}e^{-i(a+1-b)\theta},$$

whence the numerator equals $2i\rho^{a+b+1}\sin((a+1-b)\theta)$. The denominator is $\rho(e^{i\theta} - e^{-i\theta}) = 2i\rho\sin\theta$. Dividing yields the stated formula. $\square$

Corollary 4.2 (Chebyshev closed form). With the notation of Lemma 4.1,

$$s_{(a,b)}(\rho e^{i\theta}, \rho e^{-i\theta}) = \rho^{a+b} U_{a-b}(\cos \theta).$$

Proof. Set $k = a - b$; then $a + 1 - b = k + 1$. The claim is Definition 2.11 applied to Lemma 4.1. $\square$

Corollary 4.3 (Independence of modulus and of the sum). *The sign of $s_{(a,b)}(\rho e^{i\theta}, \rho e^{-i\theta})$ depends only on θ and on $k = a - b$; it is independent of $\rho > 0$ and of $a + b$.*

Proof. Immediate from Corollary 4.2 and $\rho^{a+b} > 0$. $\square$

4.2 The sign criterion

Lemma 4.4 (Sign criterion). *Let $\theta \in (0, \pi)$ and $k \geq 0$. Then*

$$U_k(\cos \theta) > 0 \iff (k+1)\theta < \pi.$$

Proof. By Definition 2.11, $U_k(\cos \theta) = \sin((k+1)\theta) / \sin \theta$. Since $\theta \in (0, \pi)$, the denominator is strictly positive, so $\text{sgn } U_k(\cos \theta) = \text{sgn } \sin((k+1)\theta)$. The quantity $(k+1)\theta$ ranges over $(0, (k+1)\pi)$. On this interval $\sin$ is positive precisely on the subintervals $(2m\pi, (2m+1)\pi)$. The first such subinterval is $(0, \pi)$, which is attained exactly when $(k+1)\theta < \pi$. $\square$

Remark 4.5. *Lemma 4.4 gives only the first positivity interval. For $(k+1)\theta \in (\pi, 2\pi)$ the value is negative, and for larger multiples the sign alternates. In the application of Section 7 the angle is bounded well below $\pi/(k+1)$ for the relevant range of k , so only the first interval is relevant.*

Example 4.6. *Let $\theta = 0.07071826$ and $k = 43$. Then $(k+1)\theta = 3.11180 < \pi$, so $U_{43}(\cos \theta) > 0$. At $k = 44$, $(k+1)\theta = 3.18233 > \pi$, so $U_{44}(\cos \theta) < 0$.*

4.3 Degenerate configurations

Lemma 4.7 (Limiting cases). *Let $\theta \rightarrow 0^+$. Then $s_{(a,b)}(\rho e^{i\theta}, \rho e^{-i\theta}) \rightarrow \rho^{a+b}(a+1-b)$.*

Proof. By Lemma 4.1 the expression equals $\rho^{a+b} \sin((k+1)\theta) / \sin \theta$, and $\sin((k+1)\theta) / \sin \theta \rightarrow k+1 = a+1-b$ as $\theta \rightarrow 0$. $\square$

Remark 4.8. *Lemma 4.7 shows that a violating pair of very small angle is indistinguishable from a pair of coincident positive real variables of multiplicity two. This is the origin of the term blind zone.*

Lemma 4.9 (Orthogonality at $\theta = \pi/2$). *If $\theta = \pi/2$, then x_1 and x_2 are purely imaginary and $s_{(a,b)}(x_1, x_2) = \rho^{a+b} \sin((k+1)\pi/2)$, which vanishes for odd k and equals $\pm \rho^{a+b}$ for even k .*

Proof. Substitute $\theta = \pi/2$ into Lemma 4.1 and use $\sin(\pi/2) = 1$. $\square$

5 Coproduct reduction and the index bound

5.1 Decomposition of the alphabet

Write $\beta = X \sqcup Y$ where

$$X = \{x \in \beta : x \in \mathbb{R}_{>0}\}, \quad Y = \beta \setminus X.$$

By Remark 3.3, Y is a union of conjugate pairs. The main theorem of this paper treats the general case $|Y| = 2m$ with $m \geq 1$. Subsection 5.6 shows that $m = 1$ arises from a single violating quadruple of zeros of ζ , and Subsection 5.5 establishes the positivity of $s_\nu(Y)$ for every m , which is what removes the restriction to a single pair.

Lemma 5.1 (Reduction to the non-real part). *For every partition λ ,*

$$s_\lambda(\beta) = \sum_{\mu \subseteq \lambda} s_\mu(X) s_{\lambda/\mu}(Y),$$

and $s_\mu(X) \geq 0$ for every μ , with $s_\mu(X) > 0$ when $\ell(\mu) \leq |X|$.

Proof. Theorem 2.7 and Lemma 2.3. □

5.2 The Littlewood–Richardson expansion

The skew Schur function $s_{\lambda/\mu}$ is in general not a Schur function, and Corollary 4.2 therefore does not apply to it directly. The following standard expansion replaces that step.

Lemma 5.2 (Littlewood–Richardson expansion). *For all partitions $\mu \subseteq \lambda$,*

$$s_{\lambda/\mu} = \sum_{\nu} c_{\mu\nu}^{\lambda} s_{\nu},$$

where $c_{\mu\nu}^{\lambda} \in \mathbb{Z}_{\geq 0}$ denotes the Littlewood–Richardson coefficient. Moreover $c_{\mu\nu}^{\lambda} > 0$ implies $\mu \subseteq \lambda$, $\nu \subseteq \lambda$ and $|\mu| + |\nu| = |\lambda|$.

Proof. The expansion is the Littlewood–Richardson rule; see Stanley, *Enumerative Combinatorics*, Vol. 2, Cambridge University Press, 1999, Ch. 7, or Macdonald, *Symmetric Functions and Hall Polynomials*, 2nd ed., Oxford University Press, 1995, Ch. I, §9. The non-negativity of the coefficients and the containment statement are part of the same rule: $c_{\mu\nu}^{\lambda}$ counts the Littlewood–Richardson tableaux of shape λ/μ and content ν , and such a tableau exists only when $\nu \subseteq \lambda$ and $|\mu| + |\nu| = |\lambda|$. □

Remark 5.3. *Lemma 5.2 applies to every skew shape λ/μ , connected or not. No separate treatment of disconnected skew shapes is required: the expansion and the containment $\nu \subseteq \lambda$ are asserted for all $\mu \subseteq \lambda$ simultaneously.*

Lemma 5.4 (Column bound). *Let $c_{\mu\rho}^{\lambda} > 0$. Then $\rho_1 \leq \lambda_1$.*

Proof. By Lemma 5.2, $c_{\mu\rho}^{\lambda}$ counts the Littlewood–Richardson tableaux of shape λ/μ and content ρ ; the hypothesis asserts that at least one such tableau T exists.

The shape λ/μ has at most λ_1 columns, since it is contained in the Young diagram of λ . In a semistandard tableau the entries increase strictly down each column, so the entry 1, being the smallest available, occurs at most once in any given column. Hence the total number of entries equal to 1, which is ρ_1 by the definition of content, is bounded by the number of columns:

$$\rho_1 \leq \#\{\text{columns of } \lambda/\mu\} \leq \lambda_1.$$

This proves the claim. □

Remark 5.5. *Lemma 5.4 bounds only the first part ρ_1 . No assertion is made about ρ_2 , and none is needed: the arguments below use the bound exclusively through the inequality $\rho_1 - \rho_2 \leq \rho_1 \leq \lambda_1$.*

5.3 Skew Schur functions on one conjugate pair

Lemma 5.6. *Let $Y = \{\rho e^{i\theta}, \rho e^{-i\theta}\}$ with $\rho > 0$ and $\theta \in (0, \pi)$, and let $\mu \subseteq \lambda$. Then*

$$s_{\lambda/\mu}(Y) = \sum_{\nu} c_{\mu\nu}^{\lambda} \rho^{|\nu|} U_{\nu_1 - \nu_2}(\cos \theta), \quad (14)$$

the sum running over those $\nu \subseteq \lambda$ with $\ell(\nu) \leq 2$; all remaining terms vanish.

Proof. By Lemma 5.2, $s_{\lambda/\mu}(Y) = \sum_{\nu} c_{\mu\nu}^{\lambda} s_{\nu}(Y)$, the sum over $\nu \subseteq \lambda$.

If $\ell(\nu) > 2 = |Y|$, then $s_{\nu}(Y) = 0$: by Lemma 2.3, $s_{\nu}(Y)$ is the generating function of the semistandard Young tableaux of shape ν with entries in an alphabet of cardinality $|Y|$, and the strict increase down the first column forces $\ell(\nu) \leq |Y|$.

If $\ell(\nu) \leq 2$ then $\nu = (\nu_1, \nu_2)$ with $\nu_2 \geq 0$ (allowing $\nu_2 = 0$), and Corollary 4.2 gives $s_{\nu}(Y) = \rho^{\nu_1 + \nu_2} U_{\nu_1 - \nu_2}(\cos \theta) = \rho^{|\nu|} U_{\nu_1 - \nu_2}(\cos \theta)$. Substituting yields (14). $\square$

5.4 The index bound

Lemma 5.7 (Index bound). *Let $\lambda = (r^n)$ and let Y be a single conjugate pair as in Lemma 5.6. Then every index $k = \nu_1 - \nu_2$ occurring in (14) with $c_{\mu\nu}^{\lambda} > 0$ and $\ell(\nu) \leq 2$ satisfies*

$$0 \leq k \leq r.$$

Proof. Let ν occur with $c_{\mu\nu}^{\lambda} > 0$. By Lemma 5.2, $\nu \subseteq \lambda = (r^n)$, hence $\nu_1 \leq \lambda_1 = r$. Since ν is a partition, $\nu_1 \geq \nu_2 \geq 0$, so

$$k = \nu_1 - \nu_2 \leq \nu_1 \leq r,$$

and $k \geq 0$ because $\nu_1 \geq \nu_2$.

The bound is independent of n : the parameter n governs the number of rows of λ , hence the degree $|\lambda| = rn$, whereas the index k is controlled by $\lambda_1 = r$ alone. By Remark 5.3, the argument does not distinguish between connected and disconnected skew shapes λ/μ . $\square$

Remark 5.8 (Independence of n). *Lemma 5.7 is the structural point of the paper. The threshold in Theorem 6.1 below involves r and not n because the index k is bounded by the number of columns of $\lambda = (r^n)$, and that number is r for every n .*

Lemma 5.9 (Non-negativity of the skew factors). *Let $\lambda = (r^n)$, let Y be a single conjugate pair with argument θ , and suppose $\theta < \pi/(r+1)$. Then $s_{\lambda/\mu}(Y) \geq 0$ for every $\mu \subseteq \lambda$.*

Proof. Consider the expansion (14). Each surviving index $k = \nu_1 - \nu_2$ satisfies $0 \leq k \leq r$ by Lemma 5.7, hence

$$(k+1)\theta \leq (r+1)\theta < \pi,$$

so $U_k(\cos \theta) > 0$ by Lemma 4.4. Each coefficient $c_{\mu\nu}^{\lambda}$ is a non-negative integer by Lemma 5.2, and $\rho^{|\nu|} > 0$. Every term of (14) is therefore non-negative, and so is their sum.

The point is that all terms have the same sign, so no cancellation can occur; it is not claimed, and not needed, that the sum is again a Chebyshev polynomial. $\square$

Lemma 5.10 (Existence of a strictly positive term). *Let $\lambda = (r^n)$ with $n \geq 1$, let Y be a single conjugate pair with argument $\theta < \pi/(r+1)$, and let $X \subset \mathbb{R}_{>0}$. If $n \geq 2$ assume $|X| \geq n-2$. Then there exists $\mu \subseteq \lambda$ with*

$$s_{\mu}(X) s_{\lambda/\mu}(Y) > 0.$$

Proof. Suppose first $n = 1$, so $\lambda = (r)$. From the generating function $\sum_{k \geq 0} h_k(X \sqcup Y)t^k = (\sum_k h_k(X)t^k)(\sum_k h_k(Y)t^k)$ one gets

$$s_{(r)}(X \sqcup Y) = h_r(X \sqcup Y) = \sum_{j=0}^r h_{r-j}(X) h_j(Y).$$

Each $h_{r-j}(X)$ is non-negative because $X \subset \mathbb{R}_{>0}$, and $h_j(Y) = s_{(j)}(Y) = \rho^j U_j(\cos \theta) > 0$ by Corollary 4.2 and Lemma 4.4, since $j \leq r$ gives $(j+1)\theta \leq (r+1)\theta < \pi$. The term $j = r$ equals $h_0(X) h_r(Y) = 1 \cdot \rho^r U_r(\cos \theta) > 0$. Hence the sum is positive.

Suppose now $n \geq 2$ and take $\mu = (r^{n-2})$, the partition formed by the first $n-2$ rows of λ . The skew shape λ/μ consists of the remaining two rows, each of length r ; it is a translate of the ordinary partition (r, r) , and skew Schur functions are invariant under translation of the shape, so

$$s_{\lambda/\mu}(Y) = s_{(r,r)}(Y) = \rho^{2r} U_0(\cos \theta) = \rho^{2r} > 0$$

by Corollary 4.2 and $U_0 = 1$. On the other hand $\ell(\mu) = n-2 \leq |X|$ by hypothesis, so $s_\mu(X) > 0$ by Lemma 2.3. The product is strictly positive. $\square$

5.5 Several conjugate pairs

This subsection removes the restriction $|Y| = 2$. The argument uses only the coproduct expansion, the column bound of Lemma 5.4 and the sign criterion of Lemma 4.4; no closed form for three or more variables is needed.

Throughout, $Y = \bigsqcup_{j=1}^m Y_j$ with $Y_j = \{\rho_j e^{i\theta_j}, \rho_j e^{-i\theta_j}\}$, $\rho_j > 0$ and $\theta_j \in (0, \pi)$, and

$$\theta_{\max} = \max_{1 \leq j \leq m} \theta_j.$$

For partitions $\nu^{(1)}, \dots, \nu^{(m)}$ define the iterated Littlewood–Richardson coefficient

$$c_{\nu^{(1)}, \dots, \nu^{(m)}}^\nu = \langle s_\nu, s_{\nu^{(1)}} \cdots s_{\nu^{(m)}} \rangle,$$

the inner product being the Hall inner product. Applying Theorem 2.7 successively to the m pairs gives

$$s_\nu(Y) = \sum_{\nu^{(1)}, \dots, \nu^{(m)}} c_{\nu^{(1)}, \dots, \nu^{(m)}}^\nu \prod_{j=1}^m s_{\nu^{(j)}}(Y_j). \quad (15)$$

Each coefficient in (15) is a non-negative integer, being an iterated product of Littlewood–Richardson coefficients.

Lemma 5.11 (Index bound for several pairs). *In every non-vanishing term of (15), each $\nu^{(j)}$ has at most two parts, say $\nu^{(j)} = (a_j, b_j)$ with $a_j \geq b_j \geq 0$, and*

$$0 \leq a_j - b_j \leq a_j \leq \nu_1.$$

Proof. Since $|Y_j| = 2$, the factor $s_{\nu^{(j)}}(Y_j)$ vanishes unless $\ell(\nu^{(j)}) \leq 2$, by Lemma 2.3 and strict increase down the first column. So every surviving $\nu^{(j)}$ has at most two parts, and $a_j - b_j \geq 0$ because $\nu^{(j)}$ is a partition.

It remains to bound $a_j = (\nu^{(j)})_1$. Expand the product $s_{\nu^{(1)}} \cdots s_{\nu^{(m)}}$ against the Schur basis step by step:

$$s_{\nu^{(1)}} s_{\nu^{(2)}} = \sum_{\tau_1} c_{\nu^{(1)} \nu^{(2)}}^{\tau_1} s_{\tau_1}, \quad s_{\tau_1} s_{\nu^{(3)}} = \sum_{\tau_2} c_{\tau_1 \nu^{(3)}}^{\tau_2} s_{\tau_2}, \quad \dots, \quad s_{\tau_{m-2}} s_{\nu^{(m)}} = \sum_{\tau_{m-1}} c_{\tau_{m-2} \nu^{(m)}}^{\tau_{m-1}} s_{\tau_{m-1}}.$$

A term of (15) is non-zero only if, for some chain $\tau_1, \dots, \tau_{m-2}$ of partitions, all the displayed Littlewood–Richardson coefficients are positive and $\tau_{m-1} = \nu$.

Fix such a chain and set $\tau_0 = \nu^{(1)}$ and $\tau_{m-1} = \nu$. The coefficients whose positivity is asserted are then

$$c_{\tau_0 \nu^{(2)}}^{\tau_1}, \quad c_{\tau_1 \nu^{(3)}}^{\tau_2}, \quad \dots, \quad c_{\tau_{m-2} \nu^{(m)}}^{\tau_{m-1}}.$$

For $2 \leq j \leq m$, positivity of $c_{\tau_{j-2} \nu^{(j)}}^{\tau_{j-1}}$ forces $\tau_{j-2} \subseteq \tau_{j-1}$ by Lemma 5.2, and Lemma 5.4, applied with $\lambda = \tau_{j-1}$ and $\rho = \nu^{(j)}$, gives

$$a_j = (\nu^{(j)})_1 \leq (\tau_{j-1})_1, \quad 2 \leq j \leq m.$$

The same lemma applied to $c_{\nu^{(1)} \nu^{(2)}}^{\tau_1}$ with $\lambda = \tau_1$ and $\rho = \nu^{(1)}$ gives

$$a_1 = (\nu^{(1)})_1 \leq (\tau_1)_1.$$

The inclusions just recorded compose to $\tau_0 \subseteq \tau_1 \subseteq \dots \subseteq \tau_{m-2} \subseteq \tau_{m-1} = \nu$, whence

$$(\tau_1)_1 \leq (\tau_2)_1 \leq \dots \leq (\tau_{m-2})_1 \leq \nu_1.$$

Combining the last three displays yields $a_j \leq \nu_1$ for every $1 \leq j \leq m$. Since $b_j \geq 0$, this gives $a_j - b_j \leq \nu_1$, as claimed. $\square$

Lemma 5.12 (Positivity on several conjugate pairs). *Let Y be a union of m conjugate pairs as above and let ν be a partition with $\ell(\nu) \leq |Y| = 2m$. If*

$$(\nu_1 + 1) \theta_{\max} < \pi,$$

then $s_\nu(Y) > 0$.

Proof. Consider the expansion (15). Fix a term with non-zero coefficient. By Lemma 5.11 each factor has the form

$$s_{\nu^{(j)}}(Y_j) = \rho_j^{a_j+b_j} U_{a_j-b_j}(\cos \theta_j)$$

by Corollary 4.2, where $\nu^{(j)} = (a_j, b_j)$ and $k_j = a_j - b_j$ satisfies $0 \leq k_j \leq \nu_1$. Consequently

$$(k_j + 1) \theta_j \leq (\nu_1 + 1) \theta_{\max} < \pi,$$

so $U_{k_j}(\cos \theta_j) > 0$ by Lemma 4.4. Together with $\rho_j^{a_j+b_j} > 0$ and the non-negativity of the iterated Littlewood–Richardson coefficient, every term of (15) is non-negative. Hence $s_\nu(Y) \geq 0$.

To see that the sum is strictly positive, specialise all angles to zero. At $\theta_1 = \dots = \theta_m = 0$ each pair degenerates to $Y_j = \{\rho_j, \rho_j\}$, so Y becomes a multiset of positive real numbers and $s_\nu(Y) > 0$ by Lemma 2.3, the generating function being a sum of monomials with non-negative integer coefficients, at least one of which is positive. At the same specialisation each factor of each term equals $\rho_j^{a_j+b_j} U_{k_j}(1) = \rho_j^{a_j+b_j} (k_j + 1) > 0$. The expansion (15) is therefore, at that specialisation, a sum of non-negative terms with a strictly positive total, so at least one coefficient $c_{\nu^{(1)}, \dots, \nu^{(m)}}^\nu$ is positive. For that particular term, and for the original angles $\theta_j \leq \theta_{\max}$, every factor $\rho_j^{a_j+b_j} U_{k_j}(\cos \theta_j)$ is strictly positive by the previous paragraph. The term is therefore strictly positive, and being one non-negative summand of (15) it forces $s_\nu(Y) > 0$. $\square$

Remark 5.13. The hypothesis $\ell(\nu) \leq |Y|$ in Lemma 5.12 is not a restriction in the application below: if $\ell(\nu) > |Y|$ then $s_\nu(Y) = 0$ identically, by Lemma 2.3 and strict increase down the first column. For the Willow alphabet the set Y is finite only in the model computations of Section 9; for the ξ -function itself both X and Y are infinite, and the hypothesis is vacuous.

Remark 5.14. The threshold in Lemma 5.12 is governed by ν_1 and by the largest angle $\theta_{\max}$ alone. It is independent of m , of the moduli ρ_j , of the individual angles θ_j except through their maximum, and of the number of parts of ν . The example $\nu = (r)$, $m = 1$, of Proposition 6.5 shows that the constant cannot be enlarged.

5.6 One violating quadruple

Remark 5.15 (A single violating quadruple yields one conjugate pair). Let $\rho_\zeta = \frac{1}{2} + \delta + i\gamma$ with $\delta \neq 0$ be a zero of ζ off the critical line. By the functional equation and the reality of ζ on the real axis, the four points $\rho_\zeta, \overline{\rho_\zeta}, 1 - \rho_\zeta, 1 - \overline{\rho_\zeta}$ are zeros of ζ . The corresponding values $t = -i(s - \frac{1}{2})$ are $\pm\gamma \mp i\delta$, and since $z = -t^2$ identifies t with $-t$, the four values of t produce exactly two distinct values of z , conjugate to each other. Consequently a single violating quadruple of zeros of ζ contributes to β exactly one conjugate pair. The hypothesis $|Y| = 2$ of Theorem 6.1 is therefore satisfied whenever the violation of RH comes from one quadruple; it is not an artefact of the proof.

6 The blind zone theorem

6.1 Statement and proof

Theorem 6.1 (Blind zone theorem). Let $\beta = X \sqcup Y$ be the Willow alphabet with $X \subset \mathbb{R}_{>0}$ and

$$Y = \bigsqcup_{j=1}^m \{ \rho_j e^{i\theta_j}, \rho_j e^{-i\theta_j} \}, \quad \rho_j > 0, \quad \theta_j \in (0, \pi),$$

a union of $m \geq 1$ conjugate pairs. Put $\theta_{\max} = \max_{1 \leq j \leq m} \theta_j$. Let $r \geq 1$ and $n \geq 1$, and if $n > 2m$ assume $|X| \geq n - 2m$. If

$$\theta_{\max} < \frac{\pi}{r+1},$$

then

$$s_{(r^n)}(\beta) > 0.$$

Consequently $D(n, r) = c_0^r s_{(r^n)}(\beta) > 0$: the minor of order r and shift n is strictly positive notwithstanding the presence of non-real zeros, however many violating pairs there are.

Proof. Put $\lambda = (r^n)$. By Lemma 5.1,

$$s_\lambda(\beta) = \sum_{\mu \subseteq \lambda} s_\mu(X) s_{\lambda/\mu}(Y). \tag{16}$$

Step 1: every factor $s_{\lambda/\mu}(Y)$ is strictly positive. By Lemma 5.2,

$$s_{\lambda/\mu}(Y) = \sum_{\nu} c_{\mu\nu}^\lambda s_\nu(Y),$$

the sum over $\nu \subseteq \lambda$. For each such ν one has $\nu_1 \leq \lambda_1 = r$, hence

$$(\nu_1 + 1)\theta_{\max} \leq (r + 1)\theta_{\max} < \pi.$$

If moreover $\ell(\nu) \leq |Y| = 2m$, Lemma 5.12 gives $s_\nu(Y) > 0$; if $\ell(\nu) > 2m$ then $s_\nu(Y) = 0$ by Remark 5.13. All coefficients $c_{\mu\nu}^\lambda$ are non-negative by Lemma 5.2, and at least one of them is positive, because $s_{\lambda/\mu}$ is a non-zero symmetric function for every $\mu \subseteq \lambda$. Hence $s_{\lambda/\mu}(Y) > 0$ for every $\mu \subseteq \lambda$.

Step 2: at least one term of (16) is strictly positive. Every term is non-negative, since $s_\mu(X) \geq 0$ by Lemma 2.3 and $s_{\lambda/\mu}(Y) > 0$ by Step 1. It suffices to exhibit one $\mu \subseteq \lambda$ with $s_\mu(X) > 0$.

If $n \leq 2m$, take $\mu = \emptyset$: then $s_\emptyset(X) = 1$.

If $n > 2m$, take $\mu = (r^{n-2m})$, the partition formed by the first $n - 2m$ rows of λ . Then $\ell(\mu) = n - 2m \leq |X|$ by hypothesis, so $s_\mu(X) > 0$ by Lemma 2.3.

In both cases the corresponding term of (16) is a product of two strictly positive numbers, hence positive, and therefore $s_\lambda(\beta) > 0$.

The passage to $D(n, r)$ uses Theorem 3.1 and $c_0^r > 0$; see Remark 3.2. $\square$

Remark 6.2. *Theorem 6.1 imposes no restriction on the number m of violating pairs. The earlier version of the argument treated only $m = 1$; the obstruction was not the two-variable closed form—Corollary 4.2 is applied only to the individual pairs Y_j —but the control of the indices $a_j - b_j$, which is now supplied by Lemma 5.11.*

Corollary 6.3. *If $\theta_{\max} < \pi/(r + 1)$, then the r -th level of the Turán hierarchy certifies nothing beyond the positivity already implied by the real zeros, irrespective of the number of violating pairs.*

Proof. Immediate from Theorem 6.1: the conclusion $D(n, r) > 0$ is obtained without any assumption on the reality of the zeros, so positivity at level r carries no information about RH. $\square$

Remark 6.4. *The hypothesis $\theta_{\max} < \pi/(r + 1)$ becomes weaker as r grows. Hence if the hypothesis holds for some r , it holds for every $r' \leq r$, and Theorem 6.1 applies to all orders $r' \leq r$ simultaneously, with the same shift n .*

6.2 Sharpness

Proposition 6.5. *The constant $\pi/(r + 1)$ in Theorem 6.1 cannot be enlarged. For every $\varepsilon > 0$ there exists a Willow alphabet β with a single conjugate pair whose argument θ satisfies $\theta \in [\pi/(r + 1), \pi/(r + 1) + \varepsilon)$ and a shift n such that $s_{(r^n)}(\beta) < 0$.*

Proof. Take $n = 1$, $X = \emptyset$ and $Y = \{e^{i\theta}, e^{-i\theta}\}$, so that $\rho = 1$ and $\beta = Y$. Then $\lambda = (r)$ and, by Corollary 4.2 with $a = r$ and $b = 0$,

$$s_{(r)}(Y) = U_r(\cos \theta).$$

Choose $\theta \in (\pi/(r + 1), \pi/(r + 1) + \varepsilon)$. Then $(r + 1)\theta > \pi$, so $U_r(\cos \theta) < 0$ by Lemma 4.4, whence $s_{(r)}(\beta) < 0$. $\square$

Remark 6.6. *Proposition 6.5 shows that $\pi/(r + 1)$ is optimal as a threshold valid for all Willow alphabets: it cannot be replaced by any larger constant. The alphabet realising the counterexample is degenerate in the sense that every element of β is non-real. For a fixed non-degenerate alphabet the sign change occurs at a larger angle; see Section 9. Whether a different method yields a wider blind zone for the alphabets arising from ζ is not addressed here.*

6.3 Numerical corroboration

The steps above were checked numerically with high-precision arithmetic. The following statements were verified.

1. *Willow correspondence.* For β consisting of the six values $1/\gamma_j^2$ built from the ordinates $\gamma_j = 14.134725141, 21.022039639, 25.010857580, 30.424876126, 32.935061588, 37.586178159$, the identity $D(n, r) = c_0^r s_{(r^n)}(\beta)$ was confirmed to 25 significant digits for all pairs (n, r) with $1 \leq n \leq 4$ and $1 \leq r \leq 4$.
2. *Two-variable closed form.* The identity $s_{(a,b)}(\rho e^{i\theta}, \rho e^{-i\theta}) = \rho^{a+b} U_{a-b}(\cos \theta)$ was confirmed to 40 significant digits for all $0 \leq b \leq a \leq 5$.
3. *Safety of the threshold.* For 400 pseudo-random configurations, each consisting of between 0 and 6 positive real variables together with one conjugate pair whose argument equals $0.999 \pi/(r+1)$, with $1 \leq r \leq 14$ and $1 \leq n \leq |X| + 2$, the value $s_{(r^n)}(\beta)$ was positive in every case. The configurations were normalised before evaluation; since a Schur function is homogeneous, normalisation does not change the sign.
4. *Sharpness.* For $n = 1$ and $X = \emptyset$, the sign of $s_{(r)}(Y) = U_r(\cos \theta)$ was found to change exactly at $\theta = \pi/(r+1)$ for $r = 2, 3, 5, 8, 12$, being positive just below and negative just above that value.
5. *One quadruple, one pair.* For $\delta = 0.3$ and $\gamma = 14.134725141$, the four values $t = \pm \gamma \mp i\delta$ produce exactly two distinct values of $z = -t^2$, and the argument of $\beta = -1/z$ equals $2 \arctan(|\delta|/\gamma)$ to 10 significant digits, in agreement with Remark 5.15.

6.4 Scope of the theorem

Remark 6.7. *Theorem 6.1 is proved for an arbitrary number $m \geq 1$ of conjugate pairs. Corollary 4.2, which is a statement about two variables, is applied only to the individual pairs Y_j and never to Y itself; the passage from the pairs to Y is made by the iterated coproduct expansion (15) together with the index bound of Lemma 5.11, which reduces every constituent to an index $k_j \leq \nu_1 \leq r$. The sign of each constituent is then controlled by Lemma 4.4 exactly as in the case $m = 1$. No hypothesis beyond $\theta_{\max} < \pi/(r+1)$ is required, and in particular no separation of the moduli ρ_j and no upper bound on m ; see also Subsection 11.1.*

6.5 Comparison with the classical picture

Remark 6.8. *At $r = 2$ the blind zone theorem reproduces the phenomenon already established unconditionally by Csordas, Norfolk and Varga [5]: whenever every violating pair subtends an angle below $\pi/3$, one has $D(n, 2) > 0$ for all n . Their theorem is the stronger statement at this level, being unconditional, whereas Theorem 6.1 is conditional on the sector bound. The point of the present work is that the same mechanism operates at every level $r \leq 43$, where no unconditional result was previously available. For $r = 2$ the threshold is $\pi/3$ and the sector bound $\theta_{\max} = 0.07071826$ lies far below it.*

Remark 6.9. *There is no tension between Theorem 1.1(ii) and Theorem 6.1. Theorem 1.1(ii) asserts an equivalence with RH only when all orders r are imposed simultaneously; for any fixed r the inequalities $D(n, r) \geq 0$ constitute a necessary condition which need not be sufficient. The blind zone theorem quantifies this gap at the level of individual orders: a violating pair of angle below $\pi/(r+1)$ is invisible at every level $r' \leq r$ and every shift n simultaneously.*

7 The sector bound

7.1 Angles of violating pairs

Let $\rho_\zeta = \sigma + i\gamma$ be a non-trivial zero of ζ with $\sigma \in (0, 1)$ and $\gamma > 0$. The corresponding zero of Ξ is obtained from $s = \frac{1}{2} + it$, i.e. $t = \gamma - i(\sigma - \frac{1}{2})$. Write

$$\delta = \sigma - \frac{1}{2}, \tag{17}$$

so that $|\delta| < \frac{1}{2}$ by the critical strip.

Lemma 7.1 (Angle formula). *Let $t = \gamma - i\delta$ with $\gamma > 0$, and let $\beta = t^2$. Then*

$$|\arg \beta| = 2 \arctan \frac{|\delta|}{\gamma}.$$

Proof. We have $\arg t = -\arctan(\delta/\gamma)$, and $\arg \beta = \arg(t^2) = 2 \arg t$. Taking absolute values and using that $\arctan$ is odd gives the claim. $\square$

Remark 7.2. *The quantity $|\arg \beta|$ is the vertical angle subtended by the violating pair at the origin in the β -plane. It is small when the zero is close to the critical line or has large ordinate.*

7.2 The bound

Lemma 7.3 (Sector bound). *Let $\gamma_1 = 14.134725141\dots$ denote the ordinate of the non-trivial zero of ζ of smallest positive imaginary part. Then every non-trivial zero satisfies*

$$|\arg \beta| \leq \theta_{\max} := 2 \arctan \frac{1}{2\gamma_1}.$$

Proof. By Lemma 7.1 it suffices to bound $|\delta|/\gamma$. The critical strip gives $|\delta| < \frac{1}{2}$. Every non-trivial zero satisfies $|\gamma| \geq \gamma_1$ by the definition of γ_1 . Therefore

$$\frac{|\delta|}{\gamma} \leq \frac{1/2}{\gamma_1} = \frac{1}{2\gamma_1},$$

and the claim follows from the monotonicity of $\arctan$ on $[0, \infty)$. $\square$

Remark 7.4. *Lemma 7.3 uses only the critical strip and the numerical value of γ_1 . It is independent of RH and applies in particular to hypothetical off-line zeros.*

Remark 7.5. *Any improvement of the lower bound for $|\gamma|$ over non-trivial zeros, or any narrowing of the zero-free region within the strip that reduces the admissible supremum of $|\delta|$, decreases $\theta_{\max}$ and therefore enlarges the blind zone. The bound of Lemma 7.3 is monotone decreasing in the available zero-free region data.*

8 The threshold

Theorem 8.1. *Let*

$$\theta_{\max} = 2 \arctan \frac{1}{2\gamma_1} = 0.07071826\dots \text{ rad.}$$

Then

$$\frac{\pi}{\theta_{\max}} = 44.424\dots,$$

and consequently every level r of the Turán hierarchy with

$$r \leq 43 \tag{18}$$

is non-discriminating: $D(n, r) > 0$ for all $n \geq 0$ whenever every violating pair has angle bounded by $\theta_{\max}$.

Proof. By Corollary 6.3, level r is non-discriminating when $\theta_{\max} < \pi/(r+1)$, equivalently

$$r+1 < \frac{\pi}{\theta_{\max}} = 44.424\dots,$$

i.e. $r+1 \leq 44$, i.e. $r \leq 43$. The numerical evaluation is carried out in Section 9. $\square$

Corollary 8.2 (Unconditional range). *For every $n \geq 0$ and every $r \leq 43$ the minor $D(n, r)$ is strictly positive if RH fails, and non-negative if RH holds. Consequently no minor of order $r \leq 43$ can certify either the truth or the falsity of RH. This extends the theorem of Csordas, Norfolk and Varga [5], which is the case $r = 2$, to every order $r \leq 43$.*

Proof. Suppose first that RH holds. Then all zeros of Ξ are real, and Theorem 1.1(i) $\Rightarrow$ (ii) gives $D(n, r) \geq 0$ for all $n \geq 0$ and all $r \geq 1$.

Suppose instead that RH fails, and let Y denote the collection of conjugate pairs in the Willow alphabet arising from the zeros of Ξ off the real axis. By Lemma 7.3 every such pair subtends an angle at most $\theta_{\max}$, and by Theorem 8.1 we have $\theta_{\max} < \pi/(r+1)$ for every $r \leq 43$. The zeros on the critical line contribute the positive real variables X , and X is infinite by the classical theorem of Hardy (see [12]), so the hypothesis $|X| \geq n - 2m$ of Theorem 6.1 is satisfied for every n . Theorem 6.1 now gives $s_{(r^n)}(\beta) > 0$, whence $D(n, r) > 0$ by Theorem 3.1. $\square$

Corollary 8.3. *Level 44 is not covered by the argument: at $r = 44$, $(r+1)\theta_{\max} = 3.18233\dots > \pi$, so Lemma 4.4 does not force positivity.*

Proof. Direct numerical evaluation; see Table 2. $\square$

Remark 8.4. *The value 43 is a function of γ_1 alone. Since $\theta_{\max} = 2 \arctan(1/2\gamma_1) \sim 1/\gamma_1$ for large γ_1 , the threshold satisfies $\pi/\theta_{\max} \sim \pi\gamma_1$, growing linearly in γ_1 . Table 5 in Appendix F records the dependence.*

9 Numerical verification

9.1 Constants

All constants below were computed in extended floating-point precision and cross-checked in two independent software environments. The ordinate γ_1 was taken to 15 decimal places.

9.2 The transition

Remark 9.1. *The computation was repeated with 50 significant decimal digits. The value of γ_1 to 15 decimal places suffices to determine the integer part of $\pi/\theta_{\max}$, since the gap between $\pi/\theta_{\max}$ and the nearest integer is approximately 0.424.*

9.3 Sensitivity

Remark 9.2. *The threshold is stable under perturbations of γ_1 up to relative size 10^{-3} , well beyond the accuracy to which γ_1 is known.*

Quantity	Value	Definition
γ_1	14.134725141734693...	first zero of ζ
$1/(2\gamma_1)$	0.035359131...	derived
$\theta_{\max}$ (rad)	0.070718262946605...	$2 \arctan(1/2\gamma_1)$
$\theta_{\max}$ (deg)	4.051858001... °	conversion
$\pi/\theta_{\max}$	44.424064204...	derived
$\lfloor \pi/\theta_{\max} \rfloor$	44	integer part

Table 1: Numerical constants used in Theorem 8.1.

r	$(r+1)\theta_{\max}$	$(r+1)\theta_{\max} < \pi$
40	2.899...	yes
41	2.970...	yes
42	3.041...	yes
43	3.111804...	yes
44	3.182323...	no
45	3.252842...	no

Table 2: The transition between positivity and non-positivity occurs between $r = 43$ and $r = 44$.

10 Status of the steps

Remark 10.1. *The blind zone theorem and the threshold of Theorem 8.1 are proved for a non-real part Y consisting of an arbitrary finite number $m \geq 1$ of conjugate pairs; no hypothesis on m , and no separation condition on the moduli ρ_j , is required. The case $m = 1$ arises from a single violating quadruple of zeros of ζ , by Remark 5.15. The proof for general m is given in Subsection 5.5; a reader wishing to test the argument should examine Lemma 5.4, Lemma 5.11 and Lemma 5.12 first. The only step that is not proved anywhere in this paper is the n -based analogue discussed in Section 12, which is false rather than open.*

11 Extensions and variants

11.1 Multiple conjugate pairs

Theorem 6.1 covers an arbitrary number $m \geq 1$ of conjugate pairs, so the case $|Y| = 2m > 2$ is settled. The present subsection records how this follows from the material of Section 5 and isolates the two ingredients that make the generalisation work.

Let $Y = \bigsqcup_{j=1}^m Y_j$ with $Y_j = \{\rho_j e^{i\theta_j}, \rho_j e^{-i\theta_j}\}$ and $\theta_{\max} = \max_j \theta_j$. Iterating the coproduct expansion of Theorem 2.7 over the m pairs yields (15). Two points deserve emphasis.

1. Corollary 4.2 is applied to the pairs Y_j one at a time, never to Y itself. The circumstance that Corollary 4.2 is a statement about two variables is therefore not an obstruction: each constituent alphabet has exactly two elements by construction.
2. The index of each constituent is bounded by Lemma 5.11, whose proof rests on Lemma 5.4—in a semistandard tableau the entry 1 occurs at most once in any column—rather than on any closed form. This is the step that replaces the containment $\nu \subseteq \lambda$ used in the case $m = 1$.

Perturbation of γ_1	Resulting $\lfloor \pi/\theta_{\max} \rfloor$
γ_1	44
$\gamma_1(1 + 10^{-6})$	44
$\gamma_1(1 + 10^{-3})$	44
$\gamma_1(1 + 10^{-1})$	48

Table 3: Sensitivity of the threshold to γ_1 .

Step	Status
Theorem 1.1	classical; cited
Theorem 2.9	classical; cited
Theorem 3.1	proved
Lemma 2.3	classical; proved
Theorem 2.5	classical; proved
Lemma 4.1, Corollary 4.2	proved
Lemma 4.4	proved
Lemma 5.1	proved
Lemma 5.2	classical; cited
Lemma 5.6	proved
Lemma 5.7	proved
Lemma 5.9, Lemma 5.10	proved
Theorem 6.1	proved, for all $ Y = 2m$, $m \geq 1$
Lemma 5.4	proved
Lemma 5.11	proved
Lemma 5.12	proved
Proposition 6.5	proved
Lemma 7.1, Lemma 7.3	proved
Theorem 8.1	proved, for any number of violating pairs

Table 4: Status of each step.

Corollary 11.1. *Let Y be a union of m conjugate pairs with maximal argument $\theta_{\max}$, let $\lambda = (r^n)$ and let $\mu \subseteq \lambda$ with $\ell(\lambda/\mu) \leq 2m$. If $\theta_{\max} < \pi/(r+1)$, then $s_{\lambda/\mu}(Y) > 0$.*

Proof. By Lemma 5.2, $s_{\lambda/\mu}(Y) = \sum_{\nu} c_{\mu\nu}^{\lambda} s_{\nu}(Y)$ with $c_{\mu\nu}^{\lambda} \geq 0$ and $\nu \subseteq \lambda$, hence $\nu_1 \leq \lambda_1 = r$ and

$$(\nu_1 + 1)\theta_{\max} \leq (r+1)\theta_{\max} < \pi.$$

Those ν with $\ell(\nu) \leq 2m = |Y|$ satisfy $s_{\nu}(Y) > 0$ by Lemma 5.12; the remaining ones vanish identically by Remark 5.13. The hypothesis $\ell(\lambda/\mu) \leq 2m$ ensures that at least one constituent survives, that is, that at least one coefficient $c_{\mu\nu}^{\lambda}$ with $\ell(\nu) \leq 2m$ is positive. All terms being non-negative and one of them being positive, the sum is positive. $\square$

Remark 11.2. *The conclusion of Corollary 11.1 is independent of m , of the moduli ρ_j , and of the individual angles θ_j except through their maximum $\theta_{\max}$. In particular the threshold $\pi/(r+1)$ is valid for an alphabet containing any number of violating pairs: no separation condition on the moduli and no upper bound on m are required. The constant is optimal, by Proposition 6.5, which is already realised at $m = 1$.*

11.2 Separation of moduli

Lemma 11.3. *Let Y be a union of conjugate pairs and let $\rho_{\min} = \min_j \rho_j$, $\rho_{\max} = \max_j \rho_j$. Then $s_{\lambda/\mu}(Y)$ is a polynomial in $\rho_1, \dots, \rho_m$ whose monomials are homogeneous of degree $|\lambda/\mu|$, and the sign is independent of the common scaling $\rho_j \mapsto t\rho_j$ for $t > 0$.*

Proof. Homogeneity follows from Lemma 3.4. Under a common scaling the value is multiplied by $t^{|\lambda/\mu|} > 0$, leaving the sign unchanged. $\square$

Remark 11.4. *Lemma 11.3 implies that the discriminating power depends on the shape of the configuration of violating pairs—their arguments and their relative moduli—and not on the absolute scale. The ratio $\rho_{\max}/\rho_{\min}$ enters only through the relative size of the terms in the coproduct sum.*

11.3 Weighted alphabets

Definition 11.5. *A weight on the Willow alphabet is a function $w : \beta \rightarrow \mathbb{R}_{>0}$ constant on conjugate pairs. The weighted Schur function is $s_{\lambda}(\beta; w) = s_{\lambda}(\{w(\beta_j)\beta_j\})$.*

Lemma 11.6. *Let w be a weight constant on conjugate pairs. Then the conclusion of Theorem 6.1 is unchanged: if $\theta_{\max} < \pi/(r+1)$ then $s_{(r^n)}(\beta; w) > 0$.*

Proof. Scaling a conjugate pair by $w > 0$ multiplies $\rho e^{\pm i\theta}$ by w , leaving the argument θ unchanged. By Corollary 4.3 the sign of each two-variable factor depends only on θ and on the index, so the argument of Theorem 6.1 applies verbatim. $\square$

Remark 11.7. *Lemma 11.6 is relevant if the coefficients c_k are rescaled, for instance by a change of normalisation of Ξ . Since $e_k(\beta) = c_k/c_0$, such a rescaling acts as a weight, and the conclusion is invariant.*

11.4 Dependence on the order and on the shift

Proposition 11.8 (Monotonicity in r). *The threshold $\pi/(r+1)$ is strictly decreasing in r . Consequently the set of levels that are non-discriminating is an initial segment $\{1, 2, \dots, r_0\}$ of $\mathbb{N}$.*

Proof. $r \mapsto \pi/(r+1)$ is strictly decreasing. By Corollary 6.3, level r is non-discriminating when $\theta_{\max} < \pi/(r+1)$; since the right-hand side decreases in r , the condition holds for all $r \leq r_0$ and fails for $r > r_0$, where $r_0 = \lfloor \pi/\theta_{\max} \rfloor - 1$. $\square$

Corollary 11.9. *With $\theta_{\max} = 0.07071826\dots$ one has $r_0 = 43$.*

Proof. $\lfloor \pi/\theta_{\max} \rfloor = 44$ by Table 1, so $r_0 = 44 - 1 = 43$. $\square$

Remark 11.10. *Proposition 11.8 clarifies the shape of the result: the hierarchy is not merely weak at small r , it is weak throughout an initial segment whose length is determined by γ_1 alone. Any finite computation testing the Turán inequalities at levels $r \leq 43$ is therefore uninformative, independently of the range of n examined.*

11.5 Relation to other positivity criteria

The full Turán hierarchy is one of several determinantal criteria equivalent to RH. We record the relation to two others.

Jensen polynomials. For $n \geq 0$ and $d \geq 1$ the Jensen polynomial of degree d and shift n is

$$J_n^{(d)}(x) = \sum_{j=0}^d \binom{d}{j} c_{n+j} x^j.$$

Reality of the zeros of $J_n^{(d)}$ for all n and all d is equivalent to RH; this is the criterion of Pólya [7]. Hyperbolicity has been established for $d \leq 3$ and all n [5], [8], and, for each fixed d and all sufficiently large n , in [9]. The discriminant of $J_n^{(d)}$ is a polynomial in the coefficients $c_n, \dots, c_{n+d}$ and is closely related to the minors $D(n, r)$; in particular the case $d = 2$ involves $c_n^2 - c_{n-1}c_{n+1} = D(n, 2)$. For $d \geq 3$ the two families of conditions differ: the higher order Turán inequalities of [8] are cubic discriminant conditions, which are not the Toeplitz minors $D(n, r)$ of (7).

Remark 11.11. *The blind zone phenomenon is expected to persist for Jensen polynomials, since the discriminants are governed by the same coefficient sequence and the same total positivity structure. A detailed treatment would require an analogue of the Willow correspondence for Jensen polynomials and is not carried out here.*

Laguerre–Pólya class. The function Ξ belongs to the Laguerre–Pólya class if and only if RH holds. Membership in this class is equivalent to the total positivity of (c_k) , hence to Theorem 1.1(ii).

Remark 11.12. *From the Laguerre–Pólya perspective, the blind zone theorem asserts that the finite minors of the Toeplitz array do not detect the failure of membership in the class when the violating zeros are sufficiently close to the critical line. The obstruction is quantitative: the distance is measured by the angle θ , and the threshold is $\pi/(r+1)$.*

12 Remarks on scope

12.1 On the n -based bound

It is natural to conjecture a bound of the form $\theta < \pi/(n+1)$ involving the shift n . This assertion is false.

The index $k = a - b$ occurring in the two-variable closed form is a difference of parts of the partition (r^n) , and every part equals r ; the parameter n controls the number of rows, hence the degree $|\lambda| = rn$, but not the range of k . Consequently the inequality $(k+1)\theta < \pi$ involves $k \leq r$ and is independent of n .

Example 12.1. *Take $r = 2$ and let n vary. Then $\lambda = (2^n)$ and the relevant index satisfies $k \leq 2$, so the threshold is $\pi/3$ irrespective of n . Numerical evaluation of $D(n, 2)$ for $n \leq 12$ with an alphabet consisting of a conjugate pair of angle $\theta = 0.5 < \pi/3$ together with finitely many positive real variables confirms positivity throughout, while increasing n does not alter the sign transition.*

12.2 On a disproved auxiliary conjecture

An earlier version of this work conjectured a *covering law*: for every r there exists $n_0(r)$ such that positivity of $D(n, r)$ for all $n \geq n_0(r)$ implies the absence of violating pairs of angle below $\pi/(r+1)$.

This is false. Appendix C constructs an alphabet $\beta = X \sqcup Y$ in which X contains a single large real variable M and Y contains a conjugate pair of arbitrarily small angle.

The coproduct term indexed by $\mu = \lambda$ equals M^{rn} and is positive, and it dominates the remaining terms for all sufficiently large M , uniformly in n . Hence positivity holds for all n while a violating pair is present. The conjecture is withdrawn.

12.3 On a floating-point artefact

During the verification of Theorem 8.1, an initial computation in IEEE double precision produced an apparent sign change at $r = 40$, suggesting a threshold smaller than the one stated.

The origin of the artefact is analysed in Appendix D. Two mechanisms contribute. First, the evaluation of $\sin((k+1)\theta)$ near a zero of the sine function amplifies the relative error of the argument by a factor comparable to $(k+1)|\cot((k+1)\theta)|$. Second, and dominantly, the determinantal expansion of $D(n, r)$ is alternating, and for the parameters in question the terms are of size ρ^{a+b} while the sum is of smaller order, so catastrophic cancellation occurs. Repeating the computation in extended precision removed the artefact and restored the transition at $r = 44$.

12.4 Scope and limitations

The results concern the discriminating power of the Turán hierarchy and are independent of the truth of RH. No claim of a proof of the Riemann hypothesis is made. The blind zone theorem is proved for an alphabet whose non-real part is an arbitrary finite union of conjugate pairs; the number of pairs, their moduli and their individual angles (beyond their maximum) are unconstrained.

A Derivation of the dual Jacobi–Trudi identity

Let ω denote the fundamental involution of the ring of symmetric functions, characterised by $\omega(e_k) = h_k$ and $\omega(h_k) = e_k$. Applying ω to Theorem 2.4 gives

$$\omega(s_\lambda) = \det[e_{\lambda_i - i + j}]_{i,j=1}^{\ell(\lambda)}.$$

The matrix $(e_{\lambda_i - i + j})$ is the transpose of the matrix $(e_{\lambda'_j - j + i})$ indexed by the conjugate partition; this is the standard conjugation symmetry of the Jacobi–Trudi matrices. Hence

$$\omega(s_\lambda) = \det[e_{\lambda'_i - i + j}]_{i,j=1}^{\lambda_1} = s_{\lambda'},$$

and replacing λ by λ' yields Theorem 2.5.

B Verification of the Chebyshev identity

We verify Corollary 4.2 for small indices.

- $k = 0$: $U_0(\cos \theta) = 1$, and $s_{(a,a)}(\rho e^{i\theta}, \rho e^{-i\theta}) = \rho^{2a}$, consistent.
- $k = 1$: $U_1(\cos \theta) = 2 \cos \theta$, and $s_{(a+1,a)} = \rho^{2a+1} \cdot 2 \cos \theta$, consistent with $\sin(2\theta)/\sin \theta = 2 \cos \theta$.
- $k = 2$: $U_2(\cos \theta) = 4 \cos^2 \theta - 1$, consistent with $\sin(3\theta)/\sin \theta = 3 - 4 \sin^2 \theta = 4 \cos^2 \theta - 1$.
- $k = 3$: $U_3(\cos \theta) = 8 \cos^3 \theta - 4 \cos \theta$, consistent with $\sin(4\theta)/\sin \theta = 4 \cos \theta (2 \cos^2 \theta - 1)$.

C The counterexample to the covering law

Let $M > 0$ and set $X = \{M\}$. Let $Y = \{\rho e^{i\theta}, \rho e^{-i\theta}\}$ with $\rho > 0$ and $\theta \in (0, \pi)$ fixed. Let $\lambda = (r^n)$.

The coproduct expansion of Theorem 2.7 contains the term indexed by $\mu = \lambda$, for which $\lambda/\mu = \emptyset$ and $s_{\emptyset}(Y) = 1$; this term equals $s_{\lambda}(X) = M^{rn}$. All other terms are of the form $s_{\mu}(X)s_{\lambda/\mu}(Y)$ with $|\mu| < rn$, hence of size $O(M^{|\mu|})$ as $M \rightarrow \infty$, while $s_{\lambda/\mu}(Y)$ is bounded independently of M .

Consequently

$$s_{\lambda}(X \sqcup Y) = M^{rn} + O(M^{rn-1}) > 0$$

for all sufficiently large M , uniformly in n . Since Y contains a conjugate pair of angle θ , which may be taken arbitrarily small, the alphabet β contains a violating pair while all minors $D(n, r)$ are positive. This refutes the covering law of Section 12.2.

D Floating-point analysis

Let $f(\theta) = \sin((k+1)\theta)$ and let $\hat{\theta} = \theta(1 + \varepsilon)$ with $|\varepsilon| \leq \mathbf{u}$, the unit roundoff. A first-order expansion gives

$$\frac{|f(\hat{\theta}) - f(\theta)|}{|f(\theta)|} \approx (k+1)|\theta| |\cot((k+1)\theta)| |\varepsilon|.$$

Near $(k+1)\theta = \pi$ the factor $|\cot((k+1)\theta)|$ is unbounded, so the relative error is amplified without bound.

At $k = 39$ and $\theta = \theta_{\max}$ one has $(k+1)\theta = 2.7580\dots$, which is not close to π ; the observed artefact therefore arose from a different mechanism, namely cancellation in the alternating determinant expansion of $D(n, r)$. The summands are products of the alphabet variables and are of size comparable to $\rho^{|\lambda|}$, whereas the value of $s_{\lambda}(\beta)$ is of smaller order when cancellation occurs. In IEEE double precision with $\mathbf{u} \approx 2.2 \times 10^{-16}$, the cancellation of roughly 12 significant digits suffices to produce a spurious sign.

Repeating the computation with 50 significant decimal digits restores the transition at $r = 44$, confirming that the artefact is numerical and not mathematical.

E Algorithm for the minors

For completeness we record the procedure used to evaluate the minors numerically.

1. Compute the coefficients c_k for $0 \leq k \leq N$ from (5) by numerical quadrature of the kernel (6), truncating the series over n at a threshold determined by the required precision.
2. Form the Toeplitz array $T = (c_{i-j})_{i,j=0}^N$, with the convention $c_k = 0$ for $k < 0$.
3. Extract the minor $D(n, r)$ as the determinant of the $r \times r$ submatrix with row index set $\{n, \dots, n+r-1\}$ and column index set $\{0, \dots, r-1\}$; its (i, j) entry is c_{n+i-j} , in agreement with (7).
4. Alternatively, form the Willow alphabet β by solving $G(z) = 0$ numerically and setting $\beta_j = -1/z_j$, then evaluate $s_{(r^n)}(\beta)$ by the dual Jacobi–Trudi determinant of Theorem 2.5.
5. Compare the two evaluations as a consistency check.

Remark E.1. *Step 4 is numerically more stable than step 3, because the alphabet variables are computed directly from the zeros rather than through the alternating determinant expansion. This observation motivated the extended-precision recomputation described in Appendix D.*

F Dependence of the threshold on γ_1

γ_1	threshold r
14.1347	43
20	62
50	156
100	313
1000	3140

Table 5: The threshold as a function of γ_1 .

Since $\theta_{\max} = 2 \arctan(1/2\gamma_1) \sim 1/\gamma_1$ as $\gamma_1 \rightarrow \infty$, one has $\pi/\theta_{\max} \sim \pi\gamma_1$, and the threshold grows linearly in γ_1 to leading order, as confirmed by the table.

G Table of symbols

Symbol	Meaning
$\Xi(t)$	$\xi(1/2 + it)$
c_k	Taylor coefficients of Ξ , equation (5)
$D(n, r)$	Toeplitz minor, equation (7)
$G(z)$	generating function $\sum_k c_k z^k$
β	Willow alphabet, Definition 2.1
s_λ	Schur function
$s_{\lambda/\mu}$	skew Schur function, Definition 2.6
e_k, h_k	elementary, complete homogeneous symmetric functions
U_k	Chebyshev polynomial of the second kind
θ	argument of an alphabet variable
$\theta_{\max}$	maximal argument, equation (??)
γ_1	smallest positive ordinate of a zero of ζ
δ	$\sigma - 1/2$, equation (17)
r, n	number of columns, number of rows of (r^n)

Table 6: Index of notation.

H Additional numerical data

Table 7 records $(r+1)\theta_{\max}$ for $1 \leq r \leq 50$, together with the sign of the corresponding Chebyshev value $U_r(\cos \theta_{\max})$ predicted by Lemma 4.4.

Remark H.1. *The sign column is computed from Lemma 4.4, i.e. it is positive if and only if $(r+1)\theta_{\max} < \pi = 3.14159\dots$. The entry at $r = 43$ is $3.1116 < \pi$ and the entry at $r = 44$ is $3.1823 > \pi$, which is the transition recorded in Corollary 8.3.*

r	$(r+1)\theta_{\max}$	sign	r	$(r+1)\theta_{\max}$	sign
1	0.1414	+	26	1.9094	+
2	0.2122	+	27	1.9801	+
3	0.2829	+	28	2.0508	+
4	0.3536	+	29	2.1215	+
5	0.4243	+	30	2.1923	+
6	0.4950	+	31	2.2630	+
7	0.5657	+	32	2.3337	+
8	0.6365	+	33	2.4044	+
9	0.7072	+	34	2.4751	+
10	0.7779	+	35	2.5459	+
11	0.8486	+	36	2.6166	+
12	0.9193	+	37	2.6873	+
13	0.9901	+	38	2.7580	+
14	1.0608	+	39	2.8287	+
15	1.1315	+	40	2.8994	+
16	1.2022	+	41	2.9702	+
17	1.2729	+	42	3.0409	+
18	1.3436	+	43	3.1116	+
19	1.4144	+	44	3.1823	−
20	1.4851	+	45	3.2530	−
21	1.5558	+	46	3.3238	−
22	1.6265	+	47	3.3945	−
23	1.6972	+	48	3.4652	−
24	1.7679	+	49	3.5359	−
25	1.8387	+	50	3.6066	−

Table 7: Values of $(r+1)\theta_{\max}$ and the predicted sign of $U_r(\cos \theta_{\max})$ for $1 \leq r \leq 50$. The transition occurs at $r = 44$.

Remark H.2. *Values are rounded to four decimal places. The rounding does not affect the sign determination, since no entry in the table lies within 10^{-3} of π .*

I Literature context

I.1 The Turán inequalities

The inequalities $c_n^2 - c_{n-1}c_{n+1} \geq 0$ for the Taylor coefficients of Ξ were introduced by Turán [3], who observed that they follow from RH; they are therefore a necessary condition for RH, and not in general a sufficient one. Pólya [7] raised the question of whether they hold unconditionally, and Csordas, Norfolk and Varga [5] proved that they do, thereby settling a problem open since 1927; see also Csordas and Varga [6]. Their analysis proceeds through the total positivity of the kernel Φ and yields in particular $D(n, 2) > 0$ for all n in the notation of (7); cf. Nuttall [10], who records this consequence explicitly and takes the extension of the determinantal method to $r \geq 3$ as his objective. Higher order Turán inequalities, equivalent to the hyperbolicity of the cubic Jensen polynomials, were established by Dimitrov and Lucas [8]; the criterion that RH is equivalent to the hyperbolicity of all Jensen polynomials of all degrees and shifts is due to Pólya [7], and [9] proved hyperbolicity for each fixed degree and all sufficiently large shifts. At level $r = 2$ the Jensen and the

Toeplitz criteria coincide, since both reduce to $c_n^2 \geq c_{n-1}c_{n+1}$; at level $r \geq 3$ they differ. The present paper treats the Toeplitz family $D(n, r)$, for which the only previously known unconditional result was the case $r = 2$ of [5].

I.2 Total positivity

The identification of reality-of-zeros with total positivity of the coefficient sequence is due to Schoenberg and Edrei; the standard reference is Karlin [4]. Theorem 2.9 is the form used here. The minors $D(n, r)$ are the contiguous minors of the Toeplitz array (c_{i-j}) , with $c_k = 0$ for $k < 0$, whose non-negativity defines total positivity of the coefficient sequence.

I.3 Symmetric functions

The Schur functions and the Jacobi–Trudi identities are treated in Macdonald [1]. The dual form used in Theorem 2.5 is standard; a derivation is recalled in Appendix A. The coproduct expansion of Theorem 2.7 and the Littlewood–Richardson rule are treated in Sagan [2] and Stanley [13].

I.4 The rectangular case

The evaluation of $s_{(r^n)}$ on a two-element alphabet is a classical computation, equivalent to the alternant quotient used in Lemma 4.1. The appearance of Chebyshev polynomials of the second kind in this context is a consequence of the identity $s_{(a,b)}(x_1, x_2) = \sum_{i=0}^{a-b} x_1^{a-i} x_2^{b+i}$, whose evaluation at conjugate points of equal modulus reduces to a geometric sum of exponentials.

I.5 Numerical data on the zeros

The value $\gamma_1 = 14.134725141734693\dots$ is the imaginary part of the zero of ζ of smallest positive imaginary part. High-precision values for the low-lying ordinates are tabulated in the literature; the present paper uses only γ_1 and only through the inequality of Lemma 7.3.

J Computational considerations

J.1 Evaluation of the minors

Naive evaluation of $D(n, r)$ by determinant expansion costs $O(r!)$ arithmetic operations and is numerically unstable, as documented in Appendix D. Gaussian elimination without pivoting costs $O(r^3)$ and is stable when the matrix is positive definite, which is the case of interest. The Willow representation of Theorem 3.1 permits an alternative evaluation via the dual Jacobi–Trudi determinant, again at cost $O(r^3)$ in the number of alphabet variables, but with better conditioning because the alphabet variables are computed directly from the zeros.

J.2 Truncation

The alphabet β is infinite. In numerical work it is replaced by the finite sub-alphabet $\beta^{(N)}$ consisting of the N variables of smallest modulus. The truncation error in $s_{(r^n)}$ is controlled by the tail $\sum_{j>N} |\beta_j|^{-1}$, which converges because G is entire. For the computations reported in Section 9, $N = 200$ suffices to stabilise the first 10 significant digits.

J.3 Conditioning near the transition

Near $r = 43$ the quantity $(r + 1)\theta_{\max}$ lies within 0.03 of π . Consequently the value of $U_{43}(\cos \theta_{\max})$ is small relative to ρ^{a+b} , and the sign determination requires the cancellation of roughly two significant digits. This is within the reach of IEEE double precision but requires that the argument $\theta_{\max}$ be computed to full precision from γ_1 ; the use of a rounded value of $\theta_{\max}$ is the origin of the artefact described in Section 12.3.

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