

Endotactic permanence and quantitative recovery with vanishing reverse rates

Abstract

For a finite ordinary endotactic reaction network, measurable reaction rates in a fixed positive compact box admit a compact convex invariant absorbing set in every positive stoichiometric class. Global positive Carathéodory solutions are unique, and entry times are uniform over compact initial sets and all admitted rate histories. The argument combines the finite affine construction of OpenAI with an endotactic witness estimate that retains the reaction projection lengths. We then quantify what permanence permits as reverse rates shrink. For autonomous weakly reversible networks with $k_e = \bar{k}_e \varepsilon^{a_e}$, eventual reciprocal concentration bounds depend on the exponent spread $\Delta = a_+ - a_-$ through $\varepsilon^{\Delta/(2H)}$, where $H > 0$ is a network-dependent geometric constant chosen before the individual rates. In an explicit three-species system with small positive rate parameters p and h , prepared concentrations have scales ph^3 , ph^2 , ph , and the worst-case prepared recovery time lies between $(400/1313)(p^2h^7)^{-1}$ and $(31/10)(p^2h^7)^{-1}$. A weighted inventory gives necessary restoration and depletion times from arbitrary positive seeds. Further results treat common clocks, reverse-exponent families and a conventional stochastic lower comparison. Two registered Lean developments support the permanence theorem, continuous dependence and selected deterministic quantitative results. Their precise coverage is separated from conventional extensions and numerical illustrations. Permanence at fixed positive rates supplies neither a uniform guarantee at zero reverse rate nor a global sharp-corridor recovery deadline.

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1 Introduction

Permanence asks for an eventual compact region inside a positive stoichiometric class. For mass-action systems this question separates the direction of the reaction vectors from the sizes and time dependence of their coefficients. Craciun, Nazarov and Pantea formulated the general endotactic permanence problem; Johnston, Pantea and Donnell state it as Conjecture 1.3 and give the ordinary-endotactic witness condition in Definition 3.1 [7, 14]. Strong endotacticity and weak reversibility provide important special cases [3, 6, 12].

This paper develops the qualitative theorem and its reverse-rate consequences in one account. Part I proves permanence for ordinary endotactic networks under arbitrary measurable histories within a fixed positive rate box. The compact convex target is chosen before the initial compact set and history; the absorption deadline is chosen before the initial point and history. Existence and uniqueness are conclusions. The proof is for nonnegative real complexes. A registered formal statement also covers arbitrary signed real complexes on the positive orthant.

Part II asks what survives quantitatively when some coefficients approach zero. It treats autonomous weakly reversible networks with natural-number complexes and rates $k_e = \bar{k}_e \varepsilon^{a_e}$, with prefactors in a fixed positive box. For each fixed $\varepsilon > 0$ the rates remain positive; the quantitative theorem tracks the deterioration of concentration bounds as $\varepsilon \downarrow 0$. This is a parameter family, rather than a history whose coefficients vanish in time. Uniform permanence over one fixed rate box does not supply a bound uniform over these degenerating boxes.

The common geometric input and the new estimates. Both arguments use the attributed finite affine construction of OpenAI [17–19]. It supplies a concave, piecewise-affine barrier $F_h(x) = \min_{\ell \in I} \{r_\ell \cdot x + b_\ell(h)\}$. The offsets are chosen so that a slope active at $x = h^q$ approximates the logarithmic concentration profile $q = \log x / \log h$, even for a discontinuous pointwise tolerance. Thus the directions of the affine pieces encode which concentrations are near the small- or large-coordinate faces of a positive box. For ordinary endotacticity, a single forward witness below all adverse sources gives a rate-box drift margin after the projection lengths are retained. For weak reversibility, a return-path estimate across source-level cuts exposes the ratio of the largest and smallest rates. Convex plateaux and dyadic scale descent then supply absorption.

What the quantitative examples establish. The measurable-rate example in Section 7 is neither weakly reversible nor strongly endotactic; its staged scalar clocks give a history-uniform operating deadline 347/16. The reverse-rate example in Section 10 is a different, weakly reversible three-species network. Its sharp concentration scales and recovery upper bound require a declared prepared basin. An inventory preserved by the fast reactions supplies necessary recovery times from arbitrary positive seeds, but does not prove that every seed reaches the sharp corridor. The stochastic comparison supplies a lower bound on recovery duration, not a success probability or a stochastic upper deadline.

Evidence and organization. The conventional proofs are given here in full at their stated scope. Section 15 identifies the two registered formalization records, their selected statements and the remaining conventional or numerical components. The registered interfaces support the real-complex permanence theorem and a state-dependent inventory comparison. The stochastic and optimization arguments retain their conventional proofs, and declared numerical trajectories provide illustrations. Source and proof snapshots, historical receipts and computational files are preserved separately from the manuscript-only TeX archive. No proof development or numerical campaign was rerun to prepare this edition.

Part I contains the model, drift argument, solution construction and measurable-rate example. Part II gives the rate-spread theorem, the coupled-network certificate, recovery clocks,

stochastic comparison and dimensional limitations. The final evidence section and Appendix A make the computational and formal boundaries explicit. Throughout, E indexes reactions, V denotes the comparison directions, q the logarithmic exponent vector, and $\ell \in I$ an affine label. We use P_c for a positive stoichiometric class, C_h for an affine plateau, D_* for the largest source degree and h_0 for a dyadic base. The scalar $p = uh$ is reserved for the rate parameters of the second example; the two examples have distinct reaction lists and assumptions. Part II reuses the geometric and absorption machinery proved in Part I.

Part I

Uniform permanence under measurable rates

2 Setting and main theorem

Let E be a finite index set of reactions. For $e \in E$ let $y_e, y'_e \in \mathbb{R}_{\geq 0}^d$ be its source and target complex and $\nu_e = y'_e - y_e$ its reaction vector. Parallel reactions, self-loops ($\nu_e = 0$) and the empty family are permitted. Put

$$S = \text{span}_{\mathbb{R}}\{\nu_e : e \in E\}, \quad P_c = (c + S) \cap \mathbb{R}_{>0}^d \quad (c \in \mathbb{R}_{>0}^d).$$

For $x \in \mathbb{R}_{>0}^d$ and $y \in \mathbb{R}_{\geq 0}^d$ write $x^y = \exp(\sum_i y_i \log x_i)$, the usual monomial when y has natural coordinates.

Definition 2.1 (Ordinary endotacticity). The network is *endotactic* if for every $r \in \mathbb{R}^d$ and every $e \in E$,

$$r \cdot \nu_e < 0 \implies \exists j \in E : r \cdot \nu_j > 0 \text{ and } r \cdot y_j < r \cdot y_e. \quad (1)$$

All directions $r \in \mathbb{R}^d$ are tested, not only nonnegative ones (the latter would give the weaker “lower endotactic” condition of [20]). Nothing is required of neutral reactions ($r \cdot \nu_e = 0$), and the witness need not start at a globally lowest source; requiring that defines strong endotacticity [12, 14].

Fix numbers $0 < \underline{k}_e \leq \bar{k}_e < \infty$ and put

$$\mathcal{B} = \prod_{e \in E} [\underline{k}_e, \bar{k}_e], \quad f_k(x) = \sum_{e \in E} k_e x^{y_e} \nu_e \quad (k \in \mathcal{B}, x \in \mathbb{R}_{>0}^d).$$

An *admissible history* is a function $k : [0, \infty) \rightarrow \mathcal{B}$ each of whose coordinates is Lebesgue measurable. A *solution* from $x_0 \in \mathbb{R}_{>0}^d$ under k is a function $x : [0, \infty) \rightarrow \mathbb{R}_{>0}^d$ that is absolutely continuous on every compact interval, satisfies $x(0) = x_0$, and satisfies $x'(t) = f_{k(t)}(x(t))$ for almost every t . Modifying k on a null set changes neither the class of admissible histories nor the solutions, so histories that are merely essentially bounded in $\mathcal{B}$ are covered after a modification on one common null set (E is finite).

Theorem 2.2 (Uniform endotactic permanence). *Assume the network is endotactic. For every rate box $\mathcal{B}$ and every $c \in \mathbb{R}_{>0}^d$ there is a nonempty compact convex set $K \subset P_c$ such that:*

1. *every $x_0 \in P_c$ and every admissible history have a unique global solution, and it stays in P_c ; uniqueness is among all solutions in the above sense, with no a priori boundedness assumption;*
2. *K is invariant: a solution starting in K remains in K for all $t \geq 0$, under every admissible history;*

3. for every compact $Q \subset P_c$ there is $T_Q \geq 0$ such that every solution from every $x_0 \in Q$ under every admissible history lies in K for every $t \geq T_Q$.

The set K is chosen before Q , and T_Q before x_0 and the history.

Compact containment in P_c gives one positive lower bound and one finite upper bound for every coordinate, valid after T_Q for all histories; these constants depend on the network, $\mathcal{B}$ and c , not on the history, and need not be uniform across classes or boxes. If $d = 0$ the state space is a point and there is nothing to prove; if $S = \{0\}$ (empty family or only self-loops) then $P_c = \{c\}$ and $K = \{c\}$ works with $T_Q = 0$. The proof for $d \geq 1$ occupies Sections 3–5. The argument handles a proper stoichiometric subspace without modification; the class enters through the neutral normals discussed in Section 4.

3 The finite affine geometric input

For $0 < h < 1$ put $B_h = [h, h^{-1}]^d$. Every $x \in B_h$ can be written uniquely as $x = h^q$ coordinatewise, with $q = \log x / \log h \in [-1, 1]^d$. Let $V \subset \mathbb{R}^d$ consist of the coordinate vectors e_i and all nonzero ordered differences of complexes occurring as sources or targets. This common comparison set includes differences between different connected components and will also be used for the cut estimates in Part II. For $r \in [-1, 1]^d$ define the tolerance

$$\vartheta(r) = \min\left(\{1/2\} \cup \left\{\frac{|r \cdot v|}{2\|v\|_1} : v \in V, r \cdot v \neq 0\right\}\right). \quad (2)$$

Zero scalar products are omitted, so $\vartheta(r) > 0$ for every r and $\vartheta(0) = 1/2$; no continuity of ϑ is assumed or true.

Lemma 3.1 (Affine approximation with discontinuous tolerance). *There are a nonempty finite label set I , slopes $r_\ell \in [-1, 1]^d$ ($\ell \in I$) independent of h , offsets $b_\ell(h) \in \mathbb{R}$ defined for small $h > 0$, and $h_g \in (0, 1)$ such that, whenever $0 < h < h_g$, $q \in [-1, 1]^d$, and ℓ attains the minimum in*

$$F_h(h^q) = \min_{m \in I} \{r_m \cdot h^q + b_m(h)\},$$

one has $\|q - r_\ell\|_\infty < \vartheta(r_\ell)$.

This is the uniform form, Corollary 2.2, of Lemma 2.1 in [17] (reproved in [19, Section 2]) applied with $m = d$, $[a, b] = [-1, 1]$ and the tolerance (2). The lemma is graph independent: it concerns only the finite set V of comparison vectors. Its sequential statement covers every fixed label, including ties; the uniform statement follows because a failure sequence $h_\nu \rightarrow 0$, $q_\nu \in [-1, 1]^d$ with a violating active label would, by finiteness and compactness, have a subsequence with one fixed label and convergent exponents, contradicting the sequential bound for that label, the tolerance being evaluated at the fixed slope throughout. The family, hence its finite set of slopes, is fixed before the threshold h_g . The Lean port of this construction is discussed in Section 15.

Fix the family and define the positive separation

$$H = \min(\{1\} \cup \{|r_\ell \cdot v|/2 : \ell \in I, v \in V, r_\ell \cdot v \neq 0\}) > 0. \quad (3)$$

Let $x = h^q \in B_h$ with $0 < h < h_g$, let ℓ be active at x and write $r = r_\ell$. For $v \in V$ with $r \cdot v \neq 0$,

$$|(q - r) \cdot v| \leq \|q - r\|_\infty \|v\|_1 < \vartheta(r) \|v\|_1 \leq |r \cdot v|/2,$$

and hence

$$\text{sign}(q \cdot v) = \text{sign}(r \cdot v), \quad |q \cdot v| > |r \cdot v|/2 \geq H. \quad (4)$$

Since $x^y = h^{q \cdot y}$ and $0 < h < 1$, this gives the *complex-ordering estimate*

$$r \cdot y < r \cdot z \implies x^z \leq h^H x^y \quad (x = h^q \in B_h, r = r_\ell \text{ active at } x), \quad (5)$$

and the *coordinate-sign estimate*: if $r_i \neq 0$ then q_i has the sign of r_i and $|q_i| > H$. Here y, z are any complexes of the network. The positive gap H belongs to the fixed finite slope family; exact zero comparisons are omitted from its definition.

4 Endotactic drift and the convex plateau

Call a slope r *neutral* if $r \cdot \nu_e = 0$ for every e , that is, $r \perp S$.

Lemma 4.1 (One witness below all adverse sources). *Let r be nonneutral. There is $j \in E$ with $r \cdot \nu_j > 0$ and $r \cdot y_j < r \cdot y_e$ for every e with $r \cdot \nu_e < 0$.*

Proof. If adverse reactions exist, pick e_- minimizing $r \cdot y_e$ among them. Condition (1) gives j with $r \cdot \nu_j > 0$ and $r \cdot y_j < r \cdot y_{e_-} \leq r \cdot y_e$ for every adverse e . If no adverse reaction exists, nonneutrality gives some j with $r \cdot \nu_j > 0$, and there is no inequality to check. $\square$

For each nonneutral slope $r = r_\ell$ of the fixed family choose such a witness $j = j(\ell)$ and set

$$\alpha_r = \underline{k}_j(r \cdot \nu_j) > 0, \quad \beta_r = \sum_{e: r \cdot \nu_e < 0} \bar{k}_e |r \cdot \nu_e| \geq 0.$$

Let $k \in \mathcal{B}$, $0 < h < h_g$, $x = h^q \in B_h$, and let ℓ be active at x . Keeping the witness term, discarding the other nonnegative terms, bounding the coefficients by the box, and applying (5) to every adverse source gives

$$r \cdot f_k(x) \geq \alpha_r x^{y_j} - \sum_{r \cdot \nu_e < 0} \bar{k}_e |r \cdot \nu_e| x^{y_e} \geq x^{y_j} (\alpha_r - h^H \beta_r). \quad (6)$$

The coefficients retain the projection lengths $|r \cdot \nu_e|$ of each reaction. Since the family is finite, there is $h_* \in (0, h_g)$ such that $h^H \beta_r \leq \alpha_r/2$ for all nonneutral slopes whenever $0 < h < h_*$. Let $D_* = \max(\{0\} \cup \{\sum_i y_{e,i} : e \in E\})$. Sources are nonnegative, so $x \in B_h$ implies $x^y \geq h^{D_*}$ for every source y . If nonneutral slopes exist let $\alpha_* > 0$ be the least of their α_r . Then for $0 < h < h_*$, every $k \in \mathcal{B}$, and every $x \in B_h$,

$$r_\ell \cdot f_k(x) \geq \delta_h := \frac{1}{2} \alpha_* h^{D_*} > 0 \quad \text{for every nonneutral label } \ell \text{ active at } x. \quad (7)$$

A neutral slope has $r \cdot f_k(x) = 0$ identically. If every slope is neutral the strict statements below are vacuous and any $\delta_h > 0$ may be used. All choices so far depend on the network and $\mathcal{B}$ only.

The plateau at a class representative. Now fix $c \in \mathbb{R}_{>0}^d$. Choose $h_c \in (0, h_g)$, depending on c , so that $\|\log c / \log h\|_\infty < H \leq 1$ whenever $0 < h \leq h_c$ (hence $c \in \text{int } B_h$), and reduce the drift threshold h_* to be at most h_c . If a label ℓ is active at c and $r_{\ell,i} \neq 0$ for some i , the coordinate-sign estimate would give $|\log c_i / \log h| > H$; hence every slope active at c is zero. Pick such a label and put $M_h = F_h(c)$. Its branch is the constant M_h , so $F_h \leq M_h$ on all of $\mathbb{R}^d$, and every other constant branch is $\geq M_h$. Define the *plateau*

$$C_h = \{x \in \mathbb{R}^d : F_h(x) = M_h\} = \bigcap_{\ell \in I} \{x : r_\ell \cdot x + b_\ell(h) \geq M_h\}, \quad (8)$$

a closed convex polyhedron containing c .

Lemma 4.2 (Compact plateau and classwise strict drift). *For every $0 < h \leq h_c$,*

$$C_h \subset \text{int } B_{\sqrt{h}}, \quad C_h \text{ is compact.}$$

If in addition $h < h_*$ and $x \in (P_c \cap B_h) \setminus C_h$, every label active at x is nonneutral, so $r_\ell \cdot f_k(x) \geq \delta_h$ for every active ℓ and every $k \in \mathcal{B}$.

Proof. Let $x = h^q \in C_h \cap B_h$. The chosen zero-slope branch equals $M_h = F_h(x)$, so it is active at x , and Lemma 3.1 with $\vartheta(0) = 1/2$ gives $|q_i| < 1/2$, that is $\sqrt{h} < x_i < h^{-1/2}$: $C_h \cap B_h \subset \text{int } B_{\sqrt{h}}$. Suppose some $z \in C_h$ lay outside B_h . Since $c \in \text{int } B_h$ and C_h is convex, the segment from c to z lies in C_h ; its first point on the boundary of B_h would belong to $C_h \cap B_h$ yet not to $\text{int } B_{\sqrt{h}}$, a contradiction. Hence $C_h \subset \text{int } B_{\sqrt{h}}$; it is closed and bounded, hence compact.

If r_ℓ is neutral then $r_\ell \cdot x = r_\ell \cdot c$ on $c + S$, so the branch ℓ is constant on the class with value $\geq F_h(c) = M_h$. At $x \in P_c$ with $F_h(x) < M_h$ this branch is not minimal, so ℓ is not active. The remaining active labels are nonneutral and (7) applies. $\square$

The classwise argument uses neutrality with respect to the whole subspace S ; individual reactions may have zero projection even for a nonneutral slope.

5 Solutions, trapping, and uniform absorption

5.1 Carathéodory solutions for measurable histories

On any compact positive box the finitely many monomials x^{y_e} and their state derivatives $\partial_i x^y = y_i x^y / x_i$ are bounded. Hence $(t, x) \mapsto f_{k(t)}(x)$ is measurable in t , smooth in x on $\mathbb{R}_{>0}^d$, and bounded and Lipschitz in x on each compact positive box uniformly over all histories with values in $\mathcal{B}$. These are the standard Carathéodory conditions [5, Chapter 2], [10, §1], and we record the argument in the form used by the formal proof. Fix $x_0 \in \mathbb{R}_{>0}^d$, $s \geq 0$, a closed ball of radius ρ about x_0 inside $\mathbb{R}_{>0}^d$, a bound A for $\|f_k\|$ and a Lipschitz constant L for f_k on that ball, uniform in $k \in \mathcal{B}$. For $\eta > 0$ with $A\eta \leq \rho$ and $L\eta < 1$, the Picard map

$$(\mathcal{T}u)(t) = x_0 + \int_s^t f_{k(v)}(u(v)) dv \quad (t \in [s, s + \eta])$$

sends continuous paths in the ball to continuous paths in the ball and is a contraction; its fixed point is an integral solution, hence absolutely continuous and a solution of the equation almost everywhere, and any two solutions agree on $[s, s + \eta]$ by the same estimate. Repeating the construction yields a unique maximal positive solution, and all its integral increments lie in the closed subspace S , so it stays in the affine class $c + S$ as long as it exists.

Let x be a solution segment in B_h and put $g(t) = F_h(x(t))$. The minimum of finitely many affine functions is Lipschitz with constant at most d in the maximum norm, so g is absolutely continuous on every compact interval. At almost every t the curve, g , and each branch composition $t \mapsto r_\ell \cdot x(t) + b_\ell(h)$ are differentiable and the equation holds. If ℓ is active at such a t , the nonnegative differentiable function (branch minus envelope) vanishes at t , so its derivative there is zero and $g'(t) = r_\ell \cdot f_{k(t)}(x(t))$. This handles ties among active branches without selecting one. Hence, while $x(t) \in P_c \cap B_h$,

$$g'(t) \geq 0 \quad \text{a.e.}, \quad g'(t) \geq \delta_h \quad \text{a.e. on } \{t : g(t) < M_h\}, \quad (9)$$

by Lemma 4.2 and the neutral-slope identity. These are almost-everywhere statements; nothing is assumed at switching times of the active set.

5.2 Trapping and global existence

Let $x_0 \in C_h \cap P_c$. Then $x_0 \in \text{int } B_h$. While the maximal solution stays in B_h , (9) and the fundamental theorem for absolutely continuous functions make g nondecreasing; since $g(0) = M_h$ and $g \leq M_h$, the solution stays in C_h . A first exit from $\text{int } B_h$ would be a point of C_h on the boundary of B_h , contradicting Lemma 4.2. So the maximal solution stays in the compact positive set $C_h \cap (c + S)$.

If its maximal interval of existence ended at a finite b , the field bound A on a positive box containing C_h would give $\|x(t) - x(s)\| \leq A|t - s|$, so $x(t)$ would converge as $t \uparrow b$ to a point x_b of that compact set; local existence from x_b under the shifted history and concatenation of the integral equations would extend the solution beyond b . Hence the solution is global and positive. The same argument shows that $C_h \cap P_c$ is invariant for every admissible history, and uniqueness among all solutions follows from local Lipschitz uniqueness on overlapping compact intervals. For natural-number complexes the field is polynomial on all of $\mathbb{R}^d$. The registered real-complex uniqueness statement concerns positive competitors; no extension of the real-power field outside $\mathbb{R}_{>0}^d$ is used.

5.3 Finite entry from a common drift margin

The trapping and entry arguments use only compact plateau geometry, zero drift for neutral slopes and a common positive drift margin for active nonneutral slopes. Thus they apply with any such margin δ_h ; the endotactic estimate (7) is one choice, and Part II will supply another.

Suppose a global positive solution stays in $P_c \cap B_h$ for all $t \geq a$. Write $g(t) = F_h(x(t))$ and set

$$b = a + \frac{M_h - F_h(x(a))}{\delta_h}. \quad (10)$$

If $g(a) = M_h$, invariance applies immediately. Otherwise, until the first entry into C_h , integration of $g' \geq \delta_h$ gives $g(t) \geq g(a) + \delta_h(t - a)$. Continuity at b then gives $g(b) \geq M_h$ if entry has not occurred earlier. Since $F_h \leq M_h$, entry occurs by b and invariance gives residence thereafter. With a known lower bound $A_h \leq F_h(x(a))$ we may, in particular, use the convenient deadline

$$a + \frac{M_h - A_h}{\delta_h} + 1. \quad (11)$$

5.4 The dyadic scale iteration

Choose one $h_0 \in (0, 1)$ below all thresholds above; it depends only on the network, $\mathcal{B}$ and c . Define

$$h_n = h_0^{2^n}, \quad C_n = C_{h_n}, \quad F_n = F_{h_n}, \quad M_n = F_n(c) \quad (n \geq 0).$$

These scales decrease to 0 and satisfy $\sqrt{h_{n+1}} = h_n$, so Lemma 4.2 gives

$$C_{n+1} \subset \text{int } B_{h_n}. \quad (12)$$

The plateaux need not be nested; only (12) is used. By compactness $A_n = \min_{x \in C_{n+1}} F_n(x)$ exists, and $A_n \leq M_n$ because $c \in C_{n+1}$. A solution trapped in $C_{n+1} \cap P_c$ from time a on stays in B_{h_n} , so by (11) it enters, and then stays in, C_n by time $a + D_n$ where

$$D_n = \frac{M_n - A_n}{\delta_{h_n}} + 1, \quad T_N = \sum_{n=0}^{N-1} D_n. \quad (13)$$

By induction, a solution trapped in $C_N \cap P_c$ from time 0 lies in C_0 for all $t \geq T_N$, under every admissible history.

Proof of Theorem 2.2. Let $x_0 \in P_c$. The numbers $|\log x_{0,i}|$ are finite and $|\log h_N| \rightarrow \infty$, so there is N with $\|\log x_0 / \log h_N\|_\infty < H$; by the coordinate-sign estimate every slope active at x_0 at scale h_N is zero, so $F_N(x_0)$ equals a constant branch, which is $\geq M_N$, while $F_N \leq M_N$; thus $x_0 \in C_N$. Trapping gives the unique global positive solution in $C_N \cap P_c$, and the iteration places it in C_0 by time T_N . Put

$$K = C_0 \cap (c + S).$$

It contains c , is compact and convex, and lies in $\mathbb{R}_{>0}^d$ by Lemma 4.2, so $K \subset P_c$; it is invariant because $C_0 \cap P_c$ and $c + S$ are. This proves (1) and (2) and the absorption of each individual initial state.

For (3) let $Q \subset P_c$ be compact. The coordinate logarithms are continuous on Q , so $|\log x_i| \leq A_Q$ on Q for some $A_Q \geq 0$, and one index N with $A_Q / |\log h_N| < H$ places all of Q in C_N . The deadline $T_Q = T_N$ then works for every $x_0 \in Q$ and every admissible history. This common starting plateau is what gives uniformity; the pointwise existence of entry times together with compactness of Q would not. $\square$

Formula (13) is a rigorous bound in terms of the barrier data (M_n, A_n, δ_{h_n}) . The affine construction does not make these data efficiently computable, and Section 7 shows that direct scalar structure, when available, yields far better constants.

5.5 Nonnegative real complexes

The argument above requires regularity only on compact positive boxes. The polynomial extension available for natural complexes is not needed for the real-complex conclusion. For $y_e, y'_e \in \mathbb{R}_{\geq 0}^d$ the monomials x^y are smooth on $\mathbb{R}_{>0}^d$, bounded with bounded derivatives on compact positive boxes, and satisfy $x^y \geq h^{D^*}$ on B_h because the exponents are nonnegative; the comparison set V is still finite; and Lemma 3.1 concerns exponent vectors $q \in [-1, 1]^d$ only. Hence Theorem 2.2 holds verbatim for nonnegative real complexes, with uniqueness understood among solutions in $\mathbb{R}_{>0}^d$, the domain where the real-power field is defined. The argument gives this extension directly. The registered statement also covers real complexes; Section 15 records its exact scope.

6 Immediate consequences

Proposition 6.1 (Uniform continuous dependence). *Let $C \subset \mathbb{R}_{>0}^d$ be compact. There is $L \geq 0$ depending on the network, $\mathcal{B}$ and C only, such that any two solutions x, y under the same admissible history that both remain in C satisfy*

$$\|x(t) - y(t)\| \leq \|x(0) - y(0)\| e^{Lt} \quad (t \geq 0). \quad (14)$$

In particular this applies on the invariant set K of Theorem 2.2.

Proof. Let L be a Lipschitz constant of f_k on C , uniform over $k \in \mathcal{B}$. The two integral equations give $q(t) \leq q(0) + L \int_0^t q(s) ds$ for the continuous function $q(t) = \|x(t) - y(t)\|$. Put $U(t) = q(0) + L \int_0^t q(s) ds$; then $q \leq U$, U is C^1 with $U' = Lq \leq LU$, and $(e^{-Lt}U)' \leq 0$ gives $q \leq U \leq q(0)e^{Lt}$. Only the auxiliary primitive U is differentiated pointwise; the measurable field enters in integral form. $\square$

Corollary 6.2 (Periodic operation). *Let $\tau > 0$ and let an admissible history satisfy $k(t + \tau) = k(t)$ for almost every $t \geq 0$. Then there is a positive solution with $x(t + \tau) = x(t)$ for all $t \geq 0$, taking all its values in K .*

Proof. The time- τ map $\Phi_\tau(z) = x_z(\tau)$ sends K to itself by invariance and is continuous by Proposition 6.1. Brouwer's fixed-point theorem, applied in the affine hull of the nonempty compact convex set K , gives $z \in K$ with $\Phi_\tau(z) = z$. The shifted curve $t \mapsto x_z(t + \tau)$ is absolutely

continuous on compact intervals, starts at z , and satisfies the equation almost everywhere because translation preserves null sets and the coefficients agree almost everywhere; by uniqueness it equals x_z . $\square$

Corollary 6.3 (Autonomous equilibria). *For every constant $k \in \mathcal{B}$, every positive stoichiometric class of an endotactic network contains a positive equilibrium of f_k .*

Proof. Write $f = f_k$ and Φ_s for its flow. For $\tau_n = 1/(n+1)$, Brouwer gives $z_n \in K$ with $\Phi_{\tau_n}(z_n) = z_n$; pass to a subsequence $z_n \rightarrow z_* \in K$. With A a bound for $\|f\|$ on K , $\|\Phi_s(z_n) - z_n\| \leq As$ for $0 \leq s \leq \tau_n$, so these short arcs converge uniformly to z_* , and uniform continuity of f on K gives

$$0 = \frac{\Phi_{\tau_n}(z_n) - z_n}{\tau_n} = \frac{1}{\tau_n} \int_0^{\tau_n} f(\Phi_s(z_n)) ds \rightarrow f(z_*).$$

$\square$

A fixed point of a single time- τ map could lie on a periodic orbit; the shrinking times are needed. The permanence-plus-Brouwer route to equilibria has a two-species antecedent in [7, Remark 6.2]; here it is available at the general endotactic scope.

Corollary 6.4 (Bounded multiplicative modulation). *If $k_e(t) = \bar{k}_e \theta_e(t)$ with $\bar{k}_e > 0$ and measurable $0 < \underline{\theta}_e \leq \theta_e(t) \leq \bar{\theta}_e < \infty$, then Theorem 2.2 applies with $\underline{k}_e = \bar{k}_e \underline{\theta}_e$ and $\bar{k}_e = \bar{k}_e \bar{\theta}_e$. No slow-variation, switching-rate or independence assumption is involved.*

Proposition 6.5 (State feedback with local hypotheses). *Let $k : [0, \infty) \times \mathbb{R}_{>0}^d \rightarrow \mathcal{B}$ be jointly measurable and such that the closed-loop field $(t, x) \mapsto f_{k(t,x)}(x)$ is bounded and Lipschitz in x on every compact positive box, uniformly in t . Then every positive initial state has a unique global positive closed-loop solution, K is invariant for it, and the deadline T_Q of Theorem 2.2 applies to closed-loop solutions from Q .*

Proof. The closed-loop field satisfies the Carathéodory conditions, so a local solution exists and is unique. Along it, the evaluated history $t \mapsto k(t, x(t))$ is measurable with values in $\mathcal{B}$, so the trapping argument of Section 5.2 applies to the local solution and confines it to a compact positive plateau; the field bound then gives a limit at any finite maximal time and local existence continues the solution. Only after global existence is established is the evaluated history a global admissible history, to which Theorem 2.2 applies; its bounds and the starting plateau do not depend on the feedback law. $\square$

7 A quantitative three-species example

Consider the nondimensional system

$$\dot{a} = c(k_1 - k_2 a^2), \quad \dot{b} = ab(k_3 - k_4 b^2), \quad \dot{c} = bc(k_5 - k_6 c^2), \quad \frac{1}{2} \leq k_i(t) \leq 2, \quad (15)$$

with six independent measurable coefficients. It is the mass-action system of the six reactions of Figure 1.

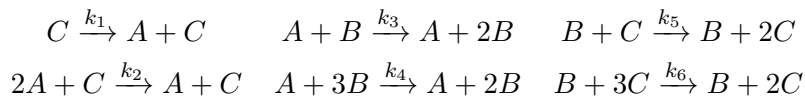


Figure 1: Three inward reaction pairs with terminal targets. Each column has sources u and $u + 2v$ and common target $u + v$, with $v = e_A, e_B, e_C$ respectively. For the direction $r = (0, 1, 0)$ the globally lowest sources are the neutral ones in the first column.

7.1 Structural classification

For a pair $u \rightarrow u + v$, $u + 2v \rightarrow u + v$ and a direction r : if $r \cdot v > 0$ the first reaction is a forward witness with source u below the adverse source $u + 2v$; if $r \cdot v < 0$ the roles are reversed; if $r \cdot v = 0$ neither reaction is adverse. Hence each pair, and their union, is endotactic in every real direction. The reaction vectors are $\pm e_A, \pm e_B, \pm e_C$, so $S = \mathbb{R}^3$ and the positive class is the whole open orthant. At $r = (0, 1, 0)$ the globally lowest sources C and $2A + C$ are neutral while the forward witness $A + B$ starts at level 1, so the network is not strongly endotactic. Each target is terminal, so the network is not weakly reversible. It has nine complexes, three linkage classes, rank 3 and deficiency $9 - 3 - 3 = 3$; the displayed reaction graph is acyclic and cannot be complex balanced at positive rates, since a terminal target receives positive influx and has no outflux (this concerns the displayed realization; dynamically equivalent realizations are not excluded here). Full rank excludes any nonzero linear conservation law.

7.2 Invariant boxes

For a positive initial state put

$$m = \min(a_0, b_0, c_0, 1/2), \quad M = \max(a_0, b_0, c_0, 2). \quad (16)$$

The box $[m, M]^3$ is invariant for every admissible history: on a lower face of a coordinate the bracket in its equation is at least $1/2 - 2m^2 \geq 0$, on an upper face at most $2 - M^2/2 \leq 0$, and the remaining factors are positive. To make this a rigorous almost-everywhere argument even with weak face signs, clamp the state to the box, construct the bounded Lipschitz integral solution of the clamped field, and note that the positive part of the distance of each coordinate to its interval is absolutely continuous with nonpositive derivative a.e. where positive and zero derivative a.e. where zero; starting from zero it stays zero, so the clamped solution solves (15) and uniqueness proves invariance. The same face signs, now strict ($9/50 > 0$ and $-9/8 < 0$), make the operating box $[2/5, 5/2]^3$ invariant. Enlarging the operating box beyond $[1/2, 2]^3$ matters: with $k_1 = 1/2$, $k_2 = 2$ and $a_0 < 1/2$ the first coordinate approaches $1/2$ from below without reaching it in finite time.

7.3 Clocks and scalar comparisons

Assume $(a_0, b_0, c_0) \in [1/10, 10]^3$; by invariance all three coordinates stay in $[1/10, 10]$. Each coordinate has an *operational clock*

$$\sigma_a(t) = \int_0^t c(s) ds, \quad \sigma_b(t) = \int_0^t a(s) ds, \quad \sigma_c(t) = \int_0^t b(s) ds,$$

an absolutely continuous nondecreasing function whose derivative is the prefactor of the corresponding equation. The startup box bounds every clock speed below by $1/10$: $\sigma(t) \geq t/10$ for each clock.

The first coordinate (conventional). Since σ_a is locally bi-Lipschitz with $\sigma'_a \geq 1/10$, the substitution $t \mapsto \sigma_a$ is valid for absolutely continuous equations and preserves the almost-everywhere statements. In the clock $\sigma = \sigma_a$,

$$\frac{1}{2} - 2a^2 \leq \frac{da}{d\sigma} \leq 2 - \frac{1}{2}a^2 \quad \text{a.e.}$$

The Riccati comparison equations $a' = \frac{1}{2} - 2a^2$ from $a(0) = 1/10$ and $a' = 2 - \frac{1}{2}a^2$ from $a(0) = 10$ have the solutions $\frac{1}{2} \tanh(\sigma + a \tanh \frac{1}{5})$ and $2 \coth(\sigma + a \tanh \frac{1}{5})$, and the positive-part argument with a local Lipschitz bound (valid for measurable coefficients) gives

$$\frac{1}{2} \tanh(\sigma + a \tanh \frac{1}{5}) \leq a \leq 2 \coth(\sigma + a \tanh \frac{1}{5}). \quad (17)$$

Both $a \geq 2/5$ and $a \leq 5/2$ hold once $\sigma \geq \operatorname{atanh} \frac{4}{5} - \operatorname{atanh} \frac{1}{5} = \frac{1}{2} \log 6$, hence for all

$$t \geq T_a := 5 \log 6 < 9, \quad (18)$$

and they persist because the comparison envelopes move inward. The rounding $5 \log 6 < 9$ is the exact inequality $e^{9/5} > \sum_{j=0}^6 (9/5)^j / j! = 7542629/1250000 > 6$.

The second and third coordinates (clocked linear comparison). Let $u \in \{b, c\}$ with clock $\sigma \in \{\sigma_b, \sigma_c\}$ and coefficients $(k_+, k_-) \in \{(k_3, k_4), (k_5, k_6)\}$, so that $u' = \sigma' u (k_+ - k_- u^2)$ almost everywhere. Because u is confined to $[1/10, 10]$ and $v \mapsto v^{-2}$ is Lipschitz there, $z = u^{-2}$ is absolutely continuous, and by the chain rule, for almost every t ,

$$z' = -2u^{-3}u' = \sigma'(-2k_+z + 2k_-), \quad \text{so} \quad \sigma'(1 - 4z) \leq z' \leq \sigma'(4 - z), \quad (19)$$

using $k_+ \geq 1/2$, $k_- \leq 2$, $z > 0$ and $\sigma' \geq 0$. The relevant comparison is stated in the clock directly, without changing the time variable.

Lemma 7.1 (Clocked linear comparison). *Let $z, \sigma : [0, T] \rightarrow \mathbb{R}$ be absolutely continuous, $\beta \in \mathbb{R}$, and suppose $z'(t) \leq \sigma'(t)(\beta - z(t))$ for almost every $t \in [0, T]$. Then*

$$z(T) \leq \beta + (z(0) - \beta)e^{-(\sigma(T) - \sigma(0))}.$$

The symmetric statement with $\geq$ holds for a lower bound.

Proof. σ is continuous, hence bounded by some M on $[0, T]$, and $\exp$ is Lipschitz on $(-\infty, M]$; so $w = e^\sigma(z - \beta)$ is absolutely continuous as a product of absolutely continuous functions. Almost everywhere $w' = e^\sigma(\sigma'(z - \beta) + z') \leq 0$. Integrating, $w(T) \leq w(0)$, which is the claim after division by $e^{\sigma(T)} > 0$. $\square$

This lemma, including the composition step and the integration, is verified in Lean (`ac_clocked_linear_upper`; the constant-speed case is `ac_linear_upper`). Applied to (19) with $z(0) = u_0^{-2} \in [1/100, 100]$:

$$\frac{1}{4} - \frac{6}{25}e^{-4\sigma(t)} \leq z(t) \leq 4 + 96e^{-\sigma(t)} \quad (t \geq 0). \quad (20)$$

The upper bound on z is a *lower* bound on u : $u \geq 2/5$ as soon as $\sigma(t) \geq \log(128/3)$, and persistently so because σ is nondecreasing. The lower bound gives $u \leq 5/2$ as soon as $\sigma(t) \geq \frac{1}{4} \log(8/3)$.

7.4 Two certified deadlines

With the uniform clock floor $\sigma(t) \geq t/10$ alone, (20) gives $u \geq 2/5$ for $t \geq T_* := 10 \log(128/3)$ and $u \leq 5/2$ for $t \geq \frac{5}{2} \log(8/3)$; together with (18) this is the one-stage certificate.

Proposition 7.2 (One-stage certificate). *Every solution of (15) from $[1/10, 10]^3$ lies in $[2/5, 5/2]^3$ for every $t \geq T_* = 10 \log(128/3)$, and $T_* < 38$.*

Proof. All deadlines in the preceding paragraph are at most T_* since $5 \log 6 < T_*$ and $\frac{5}{2} \log(8/3) < T_*$. The rounding is the exact rational inequality

$$e^{19/5} \geq \sum_{j=0}^9 \frac{(19/5)^j}{j!} = \frac{1968639501739}{44296875000} > \frac{128}{3}, \quad (21)$$

so $\log(128/3) < 19/5$ and $T_* < 38$. $\square$

The one-stage bound of Proposition 7.2 wastes the fact that the first coordinate recovers early. Once $a \geq 2/5$, the clock σ_b runs at speed at least $2/5$, and once $b \geq 2/5$, so does σ_c . Feeding these floors back in sequence gives a staged certificate; the order matters, since assuming all three floors from the start would be circular.

Theorem 7.3 (Staged certificate). *Every solution of (15) from $[1/10, 10]^3$ under every admissible history satisfies*

$$a(t) \in [\frac{2}{5}, \frac{5}{2}] \quad (t \geq 9), \quad b(t) \in [\frac{2}{5}, \frac{5}{2}] \quad (t \geq \frac{65}{4}), \quad c(t) \in [\frac{2}{5}, \frac{5}{2}] \quad (t \geq \frac{347}{16}).$$

In particular every such solution lies in $[0.4, 2.5]^3$ for all $t \geq 347/16 = 21.6875$, hence for all $t \geq 22$.

Proof. Upper bounds. By (18), $a \leq 5/2$ for $t \geq 9$. For $u \in \{b, c\}$ the lower half of (20) with $\sigma \geq t/10$ gives $z \geq 4/25$, i.e. $u \leq 5/2$, for $t \geq \frac{5}{2} \log(8/3)$, and $\frac{5}{2} \log(8/3) < \frac{5}{2} < 9$ since $8/3 < e$. So all upper bounds hold from $t = 9$ on, and persist.

Stage a. By (18), $a(t) \geq 2/5$ for all $t \geq 9$; also $a \geq 1/10$ for all $t \geq 0$.

Stage b. The clock σ_b has $\sigma'_b = a \geq 1/10$ a.e. on $[0, 9]$ and $\sigma'_b \geq 2/5$ a.e. on $[9, \infty)$, so for $t \geq 9$

$$\sigma_b(t) \geq \frac{9}{10} + \frac{2}{5}(t - 9),$$

and the right side equals $19/5$ at $t = 65/4$. By (20) and (21), $z_b(t) \leq 4 + 96e^{-19/5} < 4 + 96 \cdot \frac{3}{128} = \frac{25}{4}$ for $t \geq 65/4$, that is $b(t) > 2/5$ for all $t \geq 65/4$.

Stage c. Now $\sigma'_c = b \geq 1/10$ a.e. on $[0, 65/4]$ and $\sigma'_c \geq 2/5$ a.e. on $[65/4, \infty)$, so for $t \geq 65/4$

$$\sigma_c(t) \geq \frac{65}{40} + \frac{2}{5}\left(t - \frac{65}{4}\right),$$

which equals $19/5$ at $t = 347/16$. The same comparison gives $c(t) > 2/5$ for all $t \geq 347/16$. Finally $347/16 < 22$. $\square$

The two clock identities $\frac{9}{10} + \frac{2}{5}(\frac{65}{4} - 9) = \frac{19}{5}$ and $\frac{65}{40} + \frac{2}{5}(\frac{347}{16} - \frac{65}{4}) = \frac{19}{5}$ are exact. Keeping the exact stage times instead of rounding gives $T_a = 5 \log 6$, $T_b = \frac{3}{4}T_a + \frac{5}{2} \log \frac{128}{3}$ and $T_c = \frac{3}{4}T_b + \frac{5}{2} \log \frac{128}{3}$, approximately 8.959, 16.103 and 21.461; neither this nor $347/16$ is claimed optimal. Table 1 summarizes the deadlines.

Requirement	Deadline (exact)	Decimal	Rounded, certified
$a \in [2/5, 5/2]$	$5 \log 6$	8.95880	9
$b, c \leq 5/2$	$\frac{5}{2} \log(8/3)$	2.45207	9
$b, c \geq 2/5$, one stage	$10 \log(128/3)$	37.53418	38
$b \geq 2/5$, staged	$\frac{3}{4} 5 \log 6 + \frac{5}{2} \log(128/3)$	16.10264	$65/4 = 16.25$
$c \geq 2/5$, staged	$\frac{3}{4} T_b + \frac{5}{2} \log(128/3)$	21.46053	$347/16 = 21.6875$

Table 1: Sufficient recovery deadlines in nondimensional time, uniform over all measurable histories in $[1/2, 2]^6$ and all startups in $[1/10, 10]^3$. The decimal column only evaluates the exact formulas; the certified column uses the rational roundings proved in Lean.

Figure 2 shows illustrative trajectories under three declared rate schedules; every sampled state after $347/16$ lies in the operating box, as Theorem 7.3 requires, and the pictures show how far from tight the deadline is for these particular histories.

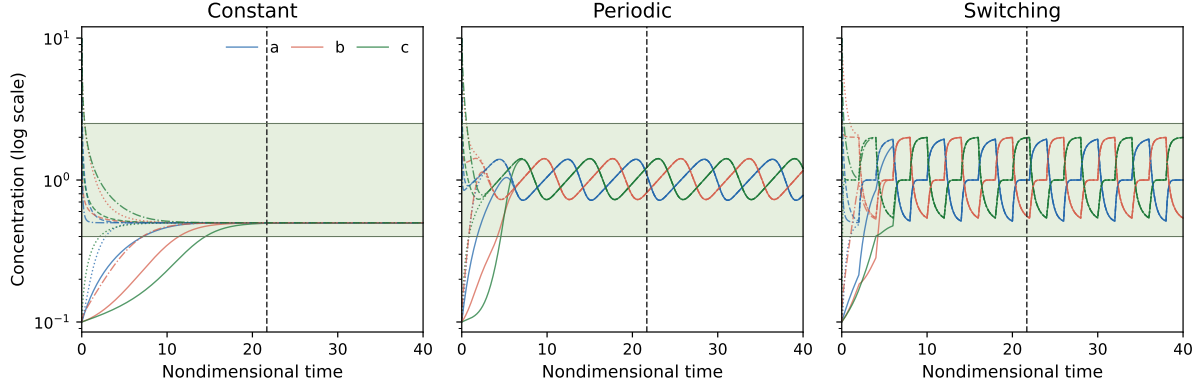


Figure 2: Illustrative trajectories under three declared schedules. The shaded strip is the proved operating interval $[0.4, 2.5]$ for each coordinate and the dashed line is the staged deadline $347/16$. Colours distinguish a, b, c ; line styles distinguish the four startups $(0.1, 0.1, 0.1)$ (solid), $(10, 10, 10)$ (dashed), $(0.1, 10, 0.1)$ (dotted) and $(10, 0.1, 10)$ (dash-dotted). Constant rates are $(0.5, 2, 0.5, 2, 0.5, 2)$; periodic rates are $k_i(t) = 1.25 + 0.75 \sin(2\pi t/8 + (i-1)\pi/3)$; switching rates are 0.5 when $(\lfloor t/2 \rfloor + i - 1) \bmod 3 = 0$ and 2 otherwise, integrated segment by segment at the exact switching times (SciPy LSODA, relative tolerance 10^{-9} , absolute tolerance 10^{-11}).

7.5 Dimensions and physical reading

Let $A = X_*a$, $B = X_*b$, $C = X_*c$ and $t_{\text{phys}} = t_*t$ for declared positive reference scales. The corresponding physical effective model is

$$\dot{A} = C(\kappa_1 - \kappa_2 A^2), \quad \dot{B} = AB(\kappa_3 - \kappa_4 B^2), \quad \dot{C} = BC(\kappa_5 - \kappa_6 C^2),$$

with $k_1 = t_*\kappa_1$, $k_2 = t_*X_*^2\kappa_2$, $(k_3, k_5) = t_*X_*(\kappa_3, \kappa_5)$ and $(k_4, k_6) = t_*X_*^3(\kappa_4, \kappa_6)$. The κ_i carry different units (κ_1 : time^{-1} ; κ_2 : $\text{concentration}^{-2}\text{time}^{-1}$; κ_3, κ_5 : $\text{concentration}^{-1}\text{time}^{-1}$; κ_4, κ_6 : $\text{concentration}^{-3}\text{time}^{-1}$), so the common dimensionless box $[1/2, 2]^6$ is an assumption on scaled coefficients, not a statement that unlike physical constants share a numerical range; whether useful X_* , t_* exist for given measured ranges must be checked case by case. The physical startup and operating boxes are $[0.1X_*, 10X_*]^3$ and $[0.4X_*, 2.5X_*]^3$, and the deadline is $22t_*$. Calibrating t_* requires physical rate data.

The reduced full-rank system is an open, driven effective model. One mass-balanced bookkeeping realization replaces the forward reactions by $F_A + C \rightarrow A + C$, $F_B + A + B \rightarrow A + 2B$, $F_C + B + C \rightarrow B + 2C$ and the backward ones by $2A + C \rightarrow A + C + W_A$, $A + 3B \rightarrow A + 2B + W_B$, $B + 3C \rightarrow B + 2C + W_C$, with maintained feed reservoirs $F_\bullet$ and removed waste $W_\bullet$ absorbed into the effective coefficients. This realization supplies open-system bookkeeping for effective co-operative kinetics; its physical use requires specifying the maintained reservoirs and the detailed reaction mechanisms.

7.6 Limits of the fixed-box hypothesis and transition to Part II

The positive rate box fixes both a lower and an upper bound. For the endotactic pair $A \rightleftharpoons 2A$ with $k_+(t) = k_-(t) = e^{-t}$, the operational clock has finite total length and the logistic solution satisfies $a(t) \leq e a_0$; arbitrarily small startups therefore never reach a common positive floor. For $0 \rightleftharpoons A$ with birth rate $1 + t$ and death rate 1, the equation $a' = 1 + t - a$ has unbounded solutions.

Part II keeps the rates autonomous and strictly positive for each $\varepsilon > 0$, while allowing their lower bounds to degenerate across a parameter family. The resulting questions are quantitative: how the absorbing concentration scales change, and how long an inventory can take to recover. Both use the shared geometry and absorption argument.

Part II

Concentration scales and recovery with vanishing reverse rates

8 Setting and main results

8.1 Autonomous weakly reversible networks

We use the reaction indices E , vectors ν_e , field f_k , stoichiometric subspace S , classes P_c and boxes B_h of Sections 2–3. In this part complexes lie in $\mathbb{N}^d$, $d \geq 1$, and the rates are constant in time; their prefactors may depend on ε . Thus the positive Carathéodory solutions of Part I are C^1 solutions here.

A network is *weakly reversible* when every reaction lies on a directed cycle. Such a network is endotactic: if a reaction has negative projection in a direction r , a return path crosses upward from below its source level, providing the witness in Definition 2.1. Consequently Theorem 2.2, applied to a singleton rate box, supplies global positive solutions for every positive constant rate vector. The task below is to quantify their eventual containment as that rate vector degenerates.

8.2 The general theorem

Theorem 8.1 (General rate-dependent containment). *For every finite weakly reversible network with $d \geq 1$ species there is $H > 0$, depending on the network alone, with the following property. Fix a positive class representative c and prefactor bounds $0 < \kappa_- \leq \kappa_+$. There is $\alpha \in (0, 1)$, depending on the network, c , $\kappa_{\pm}$ only, such that for all*

$$0 \leq a_- \leq a_+, \quad a_e \in [a_-, a_+], \quad 0 < \varepsilon \leq 1, \quad \bar{k}_e \in [\kappa_-, \kappa_+],$$

the rates $k_e = \bar{k}_e \varepsilon^{a_e}$ admit a global forward solution from every $x_0 \in P_c$, and every global forward solution from x_0 has a finite $T \geq 0$ such that for all $t \geq T$ and every coordinate i ,

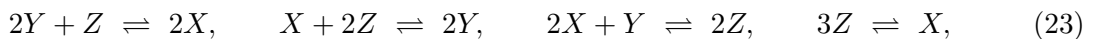
$$\sqrt{\alpha} \varepsilon^{\Delta/(2H)} \leq x_i(t) \leq \alpha^{-1/2} \varepsilon^{-\Delta/(2H)}, \quad \Delta = a_+ - a_-. \quad (22)$$

The entry time T may depend on x_0 and on the rate data.

The bound depends on the *spread* of the exponents, not on their common level. Multiplying every rate by ε^a changes the clock but not the phase portrait (Corollary 11.1), so a common offset cannot affect concentration geometry; it does affect physical time. An empty reaction set gives constant solutions and singleton classes and is covered trivially. The constants H and α are existence constants: H is a finite minimum over the slope data of the affine construction, and α is chosen from the class threshold and rate-margin inequality; Section 9.5 discusses what blocks their numerical extraction.

8.3 The coupled network and the prepared basin

Consider the four reversible pairs



with forward rate constants a_1, a_2, a_3, K and reverse rate constants $b_1 p^2, b_2 p^2, b_3 p h^6, b_4 p^2$, where $p = uh$. The admitted parameter rectangle is

$$0 < h \leq \frac{1}{600}, \quad 0 < u \leq \frac{1}{20}, \quad a_1, a_2, a_3, b_1, b_2, b_3, b_4 \in \left[\frac{99}{100}, \frac{101}{100}\right], \quad K \in \left[\frac{99}{25}, \frac{101}{25}\right]. \quad (24)$$

The net currents and the literal mass-action field are

$$\begin{aligned} J_1 &= a_1 y^2 z - b_1 p^2 x^2, & J_2 &= a_2 x z^2 - b_2 p^2 y^2, \\ J_3 &= a_3 x^2 y - b_3 p h^6 z^2, & J_4 &= K z^3 - b_4 p^2 x, \\ \dot{x} &= 2J_1 - J_2 - 2J_3 + J_4, & \dot{y} &= -2J_1 + 2J_2 - J_3, & \dot{z} &= -J_1 - 2J_2 + 2J_3 - 3J_4. \end{aligned} \quad (25)$$

For positive states define the inventory, the inventory coordinate and two shape coordinates

$$\begin{aligned} W &= 6x + 5y + 2z, & Q &= \frac{W}{2ph}, & A &= \frac{x}{ph^3 Q^3}, & B &= \frac{y}{ph^2 Q^2}, \\ C &= 1 - 3h^2 A Q^2 - \frac{5}{2} h B Q, \end{aligned} \quad (26)$$

so that $x = ph^3 A Q^3$, $y = ph^2 B Q^2$ and $z = ph Q C$.

Why these coordinates and boxes? The vector $(6, 5, 2)$ annihilates the reaction vectors of pairs 1, 2, 4, so these fast reactions preserve W ; only pair 3 changes it. In a Z -dominated state, $z \approx W/2 = phQ$. Balancing the leading fast X terms gives

$$K z^3 \approx b_4 p^2 x \implies x \approx \frac{K}{b_4} ph^3 Q^3.$$

The leading Y balance is $J_1 \approx J_2$, or $a_1 y^2 z \approx a_2 x z^2$, and therefore

$$y \approx \sqrt{\frac{a_2 K}{a_1 b_4}} ph^2 Q^2.$$

At nominal prefactors these shape coordinates are centered at $A = 4$, $B = 2$. The positive inventory input is $13b_3 ph^6 z^2$, whose scale after division by $2ph$ is $p^2 h^7 Q^2$; this identifies the slow clock $\lambda = p^2 h^7$. At the same leading balance, the slow residual is $1 - 32Q^6$, with zero $Q = 2^{-5/6} \in (1/2, 3/4)$. The inner intervals below surround these balances, while the wider outer box provides a preparation margin. This scaling argument motivates the choices; Section 10 verifies them for the full coupled field by exact differential identities and interval inequalities. The prepared outer basin and the inner target are the preimages under (A, B, Q) of

$$\mathcal{O} = [3, 5] \times [1, 3] \times [\tfrac{1}{4}, 1], \quad \mathcal{I} = [\tfrac{7}{2}, \tfrac{9}{2}] \times [\tfrac{3}{2}, \tfrac{5}{2}] \times [\tfrac{1}{2}, \tfrac{3}{4}]. \quad (27)$$

Membership in $\mathcal{O}$ is a condition on the initial state only; no future behaviour is assumed.

Theorem 8.2 (Prepared-basin concentration scales and recovery). *Assume (24) and write $\lambda = p^2 h^7$. Every positive initial state with $(A(0), B(0), Q(0)) \in \mathcal{O}$ admits a global positive solution of (25). Every such solution remains in the preimage of $\mathcal{O}$ and satisfies, for all $t \geq 0$,*

$$\tfrac{3}{64} ph^3 \leq x(t) \leq 5 ph^3, \quad \tfrac{1}{16} ph^2 \leq y(t) \leq 3 ph^2, \quad \tfrac{1}{8} ph \leq z(t) \leq ph. \quad (28)$$

It lies in the preimage of $\mathcal{I}$ for every $t \geq (31/10)\lambda^{-1}$. For the prepared initial state

$$x_0 = \frac{ph^3}{16}, \quad y_0 = \frac{ph^2}{8}, \quad z_0 = \frac{ph}{4} \left(1 - \frac{5}{4}h - \frac{3}{4}h^2\right), \quad (29)$$

which has $(A, B, Q)(0) = (4, 2, \frac{1}{4})$, every time $T \geq 0$ at which the solution is in the target satisfies

$$T \geq \frac{400}{1313} \lambda^{-1} \approx 0.3046 \lambda^{-1}. \quad (30)$$

Thus the worst-case prepared recovery time is pinned between $0.3046 \lambda^{-1}$ and $3.1 \lambda^{-1}$, a ratio of $40703/4000 \approx 10.18$, uniformly over (24). The three concentration orders in (28) hold along every prepared trajectory with the displayed parameter-independent constants. The witness trajectory supplies the matching recovery order.

Remark 8.3 (Reciprocal inventory comparison). The constants $400/1313$ and $31/10$ result from comparing $1/Q$: its derivative cancels the factor Q^2 in $\dot{Q}$ (Section 10.5). The resulting certified window can be compared with the sampled entry near $0.358 \lambda^{-1}$ in Figure 3.

9 Proof of the general theorem

9.1 The common affine geometry

Use the family I , slopes r_ℓ , offsets $b_\ell(h)$, tolerance ϑ and threshold h_g from Lemma 3.1. The comparison set V already contains every complex difference, so the gap $H \in (0, 1]$ in (3) is fixed by the network before its rates are chosen. For an active label at $x = h^q \in B_h$, the sign and ordering estimates (4)–(5) apply. Weak reversibility supplies the additional return paths needed to turn these geometric estimates into an explicit margin in terms of the rate spread.

9.2 The quantitative cut-drift lemma

Use the maximum source degree D_* defined in Section 4. For $x \in B_h$ every source monomial satisfies $x^{y_e} \geq h^{D_*}$. Let $k_0 > 0$ be a lower bound for all rates and assume the *rate margin*

$$h^H \sum_e k_e \leq \frac{k_0}{2}. \quad (31)$$

Lemma 9.1 (Quantitative cut drift). *Let $0 < h < h_g$, $x = h^q \in B_h$, and let ℓ be active at x with slope $r = r_\ell$. Then $r \cdot f_k(x) \geq 0$. If moreover $r \notin S^\perp$, then*

$$r \cdot f_k(x) \geq H k_0 h^{D_*}. \quad (32)$$

Proof. Order the complexes by the level $r \cdot y$ and consider the finitely many distinct levels $L_1 < \dots < L_m$. For two complexes y, z with $r \cdot z > r \cdot y$ the difference $z - y \in V$ has $r \cdot (z - y) \geq 2H$ by (3), so by (4) $q \cdot (z - y) > H$ and

$$x^z = h^{q \cdot z} = h^{q \cdot (z - y)} x^y \leq h^H x^y.$$

Fix a cut between consecutive levels $L_j < L_{j+1}$. Each reaction crossing the cut downward (source above, product below) has, by weak reversibility, a directed return path, which must cross the cut upward somewhere; hence if any reaction crosses the cut downward, some reaction e^* crosses it upward. Let $w = y_{e^*}$, which lies at or below L_j . Every downward source v lies at or above L_{j+1} , so $x^v \leq h^H x^w$. The total downward flux through the cut is therefore at most $h^H (\sum_e k_e) x^w \leq (k_0/2) x^w$ by (31), while the upward flux is at least $k_{e^*} x^w \geq k_0 x^w$. The net upward flux across a crossed cut is thus at least $(k_0/2) x^w \geq (k_0/2) h^{D_*}$; a cut with upward crossings only has positive net flux, and a cut with no crossing has zero net flux.

Writing $r \cdot f_k(x) = \sum_e k_e x^{y_e} (r \cdot y'_e - r \cdot y_e)$ and telescoping each level difference over the cuts it spans gives

$$r \cdot f_k(x) = \sum_{j=1}^{m-1} (L_{j+1} - L_j) \cdot (\text{net upward flux across cut } j) \geq 0.$$

Each gap $L_{j+1} - L_j$ is at least $2H$ by (3). If $r \notin S^\perp$, some reaction has $r \cdot y'_e \neq r \cdot y_e$, so at least one cut is crossed, and the sum is at least $2H \cdot (k_0/2) h^{D_*}$. If $r \in S^\perp$ all level differences vanish and the work is zero. $\square$

9.3 Applying the shared plateau and entry argument

For the chosen representative c , take the class threshold $h_c < h_g$ as in Section 4, so that

$$\|\log c / \log h\|_\infty < H \leq 1 \quad (0 < h \leq h_c). \quad (33)$$

In particular $c \in \text{int } B_h$. Write $M_h = F_h(c)$ and use the same plateau $C_h = \{x : F_h(x) = M_h\}$ as in (8).

Lemma 4.2 gives compact convex containment $C_h \subset \text{int } B_{\sqrt{h}}$ directly from the common affine family.

Lemma 9.2 (Invariance and entry at one scale). *Let $0 < h \leq h_c$, let k satisfy (31) with lower bound k_0 , and let x be a global forward solution in P_c .*

- (a) *If $x(a) \in C_h$ then $x(t) \in C_h$ for all $t \geq a$.*
- (b) *If $x(t) \in B_h$ for all $t \geq a$, then $x(t) \in C_h$ for every*

$$t \geq a + \frac{M_h - F_h(x(a))}{Hk_0h^{D^*}}.$$

Proof. The neutral-branch argument of Lemma 4.2 excludes active slopes in $S^\perp$ off C_h . Lemma 9.1 therefore supplies the common margin $\delta_h = Hk_0h^{D^*}$. Apply the trapping argument of Section 5.2 and the height-deficit deadline (10); autonomous C^1 solutions satisfy the same absolutely continuous envelope calculation. $\square$

9.4 Rate margin and the dyadic scale chain

Let $m \geq 1$ be the number of reactions, $k_e = \bar{k}_e \varepsilon^{a_e}$, $\Delta = a_+ - a_-$. Then $k_e \geq \kappa_- \varepsilon^{a_+} =: k_0$ and $\sum_e k_e \leq m\kappa_+ \varepsilon^{a_-}$. Choose

$$0 < \alpha < h_c \quad \text{with} \quad \alpha^H m \kappa_+ \leq \frac{\kappa_-}{2}, \quad (34)$$

and put $h_* = \alpha \varepsilon^{\Delta/H} \leq \alpha$. For every $0 < h \leq h_*$,

$$h^H \sum_e k_e \leq \alpha^H \varepsilon^\Delta m \kappa_+ \varepsilon^{a_-} \leq \frac{\kappa_-}{2} \varepsilon^{a_+} = \frac{k_0}{2},$$

which is (31). Note that α depends on the network (through H, m, h_c), on c (through h_c) and on $\kappa_\pm$, and on nothing else.

Proof of Theorem 8.1. Theorem 2.2 gives global positive solutions for these fixed rates, and every solution remains in P_c . At every scale $h \leq h_*$ the preceding rate-margin estimate and Lemma 9.2 give invariance and finite entry. Apply the dyadic descent of Section 5.4 with base $h_0 = h_*$: $h_n = h_0^{2^n}$ and $C_{h_{n+1}} \subset \text{int } B_{h_n}$. For any initial state, $\|\log x_0 / \log h_n\|_\infty < H$ for sufficiently large n , so the zero-slope argument places it in C_{h_n} . The finite chain of entry estimates then gives eventual residence in $C_{h_*} \subset \text{int } B_{\sqrt{h_*}}$. Since $\sqrt{h_*} = \sqrt{\alpha} \varepsilon^{\Delta/(2H)}$, this is (22). The finite number of steps and their deadlines may depend on the seed and rate data. $\square$

Remark 9.3 (Explicit single-scale deadline). Slopes have coordinates in $[-1, 1]$, so F_h is 1-Lipschitz in ℓ^1 and $0 \leq M_h - F_h(x) \leq \|c - x\|_1 \leq 2d/h$ on B_h . Lemma 9.2(b) therefore gives the explicit deadline $a + 2d\varepsilon^{-a_+}h^{-(D^*+1)}/(H\kappa_-)$ for entering C_h from a box-contained trajectory. Summing over the chain gives an explicit, seed-dependent total deadline. The formal statement keeps the sharper height-deficit form.

9.5 On the effectiveness of H and α

Given an explicit affine family, the remaining choices are finite: H is the minimum in (3), h_c satisfies (33), and α satisfies (34). The construction used here supplies the family existentially. Its induction on dimension chooses layer thresholds through limiting comparisons of offsets known to be $o(h^{-1})$. Extracting a network-specific numerical H therefore requires making those choices and the uniform activity threshold explicit. This is the constructive step left open by the present proof.

The inventory clocks of Theorem 10.6 and Section 11.4 instead use directly available data: a left null vector of the fast stoichiometry, the slow rate constants and regional bounds on the slow source monomials. Corollary 11.8 also applies these clocks after entry into the general reciprocal box. Thus geometric containment and inventory replenishment have separate quantitative inputs.

10 The coupled network

10.1 Stoichiometry and the neutral inventory

The four forward reaction vectors of (23) are the columns of

$$\nu = \begin{pmatrix} 2 & -1 & -2 & 1 \\ -2 & 2 & -1 & 0 \\ -1 & -2 & 2 & -3 \end{pmatrix}, \quad (35)$$

and the first three columns have determinant -13 . Hence $S = \mathbb{R}^3$: every positive stoichiometric class is the whole open orthant, and there is no nonzero linear conservation law. In particular no positive mass assignment makes (23) a closed system; it is an effective open network. Nevertheless $\ell = (6, 5, 2)$ annihilates columns 1, 2 and 4, and the full field satisfies the exact identity

$$\dot{W} = -13 J_3, \quad W = 6x + 5y + 2z, \quad (36)$$

for every choice of prefactors. Only the reaction pair $2X + Y \rightleftharpoons 2Z$ changes W , and its reverse direction, with rate constant $b_3 p h^6$, is the only process that increases it. This is the structural source of slow recovery.

10.2 Normalised coordinates and three clocks

For reference, $p = uh$, $W = 6x + 5y + 2z$ and $\lambda = p^2 h^7$. The coordinate definitions and characteristic inverse speeds are

Coordinate	Definition	Characteristic time
A	$x/(ph^3Q^3)$	p^{-2}
B	$y/(ph^2Q^2)$	$(p^2h^3)^{-1}$
Q	$W/(2ph)$	$\lambda^{-1} = (p^2h^7)^{-1}$

The exact equations below retain the additional powers of Q in each derivative. On the preimage of $\mathcal{O}$ under (26), the admitted rectangle gives

$$\frac{1}{2} \leq 1 - 15h^2 - \frac{15}{2}h \leq C \leq 1, \quad (37)$$

so the map $(A, B, Q) \mapsto (x, y, z) = (ph^3AQ^3, ph^2BQ^2, phQC)$ is positive and nonsingular there. Differentiating the full literal field (25), including the derivatives of Q inside A and B , gives exact equations with three separated characteristic speeds,

$$\dot{A} = p^2 R_A, \quad \dot{B} = p^2 h^3 Q^3 R_B, \quad \dot{Q} = \frac{13}{2} p^2 h^7 Q^2 R_Q, \quad (38)$$

where, with $G = a_3 A^2 B Q^6 - b_3 C^2$,

$$\begin{aligned} R_A = & KC^3 - b_4 A + h^2 Q^2 (2a_1 B^2 C - a_2 A C^2) - 2b_1 u h^4 A^2 Q^3 + b_2 u h^2 B^2 Q \\ & - 2a_3 h^5 A^2 B Q^5 + \frac{2b_3 h^5 C^2}{Q} + \frac{39}{2} A h^7 Q G, \end{aligned} \quad (39)$$

$$R_B = 2(a_2 A C^2 - a_1 B^2 C) + 2b_1 u h^2 A^2 Q - \frac{2b_2 u B^2}{Q} - a_3 h^3 A^2 B Q^3 + \frac{b_3 h^3 C^2}{Q^3} + \frac{13 B h^4 G}{Q^2}, \quad (40)$$

$$R_Q = b_3 C^2 - a_3 A^2 B Q^6. \quad (41)$$

Equations (38) express the full coupled mass-action system exactly in the new coordinates. The transformation separates characteristic clocks p^2 , $p^2 h^3$, $p^2 h^7$ and exposes useful signs; on the reference path $h = t$, $u = t^5$, $\varepsilon = t^{12}$ these are t^{12} , t^{15} , t^{19} .

10.3 The corridor certificate

Lemma 10.1 (Continuous corridor certificate). *Under (24) the following inequalities hold for all (A, B, Q) in the indicated sets, simultaneously at face intersections and uniformly over the parameter and prefactor boxes.*

- (a) Outer faces. $R_A > 0$ on $\{A = 3\} \cap \mathcal{O}$ and $R_A < 0$ on $\{A = 5\} \cap \mathcal{O}$; $R_B > 0$ on $\{B = 1\} \cap \mathcal{O}$ and $R_B < 0$ on $\{B = 3\} \cap \mathcal{O}$; $R_Q > 0$ on $\{Q = \frac{1}{4}\} \cap \mathcal{O}$ and $R_Q < 0$ on $\{Q = 1\} \cap \mathcal{O}$.
- (b) Inner faces. *The same six strict sign statements on the faces of $\mathcal{I}$.*
- (c) Entry slabs with margins.

slab (other coordinates)	lower slab	upper slab
A (B, Q in outer ranges)	$R_A \geq \frac{1}{4}$ for $3 \leq A \leq \frac{7}{2}$	$R_A \leq -\frac{1}{4}$ for $\frac{9}{2} \leq A \leq 5$
B (A inner, Q outer)	$R_B \geq 1$ for $1 \leq B \leq \frac{3}{2}$	$R_B \leq -1$ for $\frac{5}{2} \leq B \leq 3$
Q (A, B inner)	$R_Q \geq \frac{1}{10}$ for $\frac{1}{4} \leq Q \leq \frac{1}{2}$	$R_Q \leq -\frac{1}{10}$ for $\frac{3}{4} \leq Q \leq 1$

Proof. Each of the eighteen statements is an inclusion of the range of a rational function of thirteen variables (h, u, Q, A, B and the eight prefactors) over a rational box in a rational interval with the required sign or margin. We verify it by interval arithmetic [16, 22]: constants and variable boxes initialise an induction over the expression tree; addition adds endpoints, negation swaps them, multiplication takes the extreme endpoint products, nonnegative integer powers and reciprocals of positive intervals preserve inclusion. Every reciprocal in (39)–(40) has base $Q \geq \frac{1}{4}$, and every power base is nonnegative on the boxes. The checker is a small expression language whose evaluation is proved sound for every real point of the box; its rational endpoint computations are verified by kernel arithmetic, and separately proved polynomial identities equate the checked expressions with (39)–(41). Appendix A lists the eighteen boxes and the certified enclosures. The inclusions establish the face signs and slab margins on the complete declared boxes. $\square$

10.4 Invariance, global existence and the concentration bounds

Proposition 10.2. *Assume (24). Every positive initial state in the preimage of $\mathcal{O}$ has a global positive solution of (25); every such solution stays in the preimage of $\mathcal{O}$ for all $t \geq 0$ and satisfies (28).*

Proof. On any time interval on which a positive solution exists, (A, B, Q) is C^1 and obeys (38). Consider the six margins $A - 3, 5 - A, B - 1, 3 - B, Q - \frac{1}{4}, 1 - Q$. At a time when all are nonnegative and one of them vanishes, Lemma 10.1(a) gives that margin a strictly positive derivative (using $p^2 > 0, Q^3 > 0, Q^2 > 0$); corner ties are included since each face statement holds on the closed face. A standard first-exit argument for closed boxes then shows that no margin can become negative, so the solution remains in the preimage of $\mathcal{O}$. There, (37) and $A \in [3, 5], B \in [1, 3], Q \in [\frac{1}{4}, 1]$ give (28), a compact subset of the open orthant. Local existence for the polynomial field together with this a priori compact bound yields continuation to all $t \geq 0$ by the usual argument; the same reasoning applies to every solution with the stated initial data. $\square$

10.5 Sequential capture by reciprocal comparison

The following comparison gives a deadline for permanent residence in the target interval.

Lemma 10.3 (Slab entry). *Let $f : [s, t] \rightarrow \mathbb{R}$ be C^1 with $f \geq L$ on $[s, t]$ and let $\delta > 0$. Suppose $f'(\tau) \geq \delta$ whenever $f(\tau) \leq l$. If $t - s \geq (l - L)/\delta$ then $f(t) \geq l$. Symmetrically, if $f \leq U$, $f' \leq -\delta$ whenever $f \geq l$, and $t - s \geq (U - l)/\delta$, then $f(t) \leq l$.*

Proof. If $f(t) < l$, then $f \leq l$ on all of $[s, t]$: otherwise take the last time $\tau_1 < t$ with $f(\tau_1) \geq l$; just after τ_1 we have $f < l$ and hence $f' \geq \delta > 0$, so f cannot drop below $f(\tau_1) \geq l$, a contradiction. Then $f' \geq \delta$ on $[s, t]$ and $f(t) \geq f(s) + \delta(t - s) \geq L + (l - L) = l$, again a contradiction. The second statement follows by replacing f with $-f$. $\square$

Proposition 10.4 (Entry deadline). *Assume (24), let $\lambda = p^2 h^7$, and let (x, y, z) be a solution from the preimage of $\mathcal{O}$. Put*

$$T_A = \frac{2}{p^2}, \quad T_B = T_A + \frac{32}{p^2 h^3}, \quad T_Q = T_B + \frac{40}{13\lambda}. \quad (42)$$

Then $A(t) \in [\frac{7}{2}, \frac{9}{2}]$ for $t \geq T_A$, $B(t) \in [\frac{3}{2}, \frac{5}{2}]$ for $t \geq T_B$, $Q(t) \in [\frac{1}{2}, \frac{3}{4}]$ for $t \geq T_Q$, and

$$T_Q = \frac{2h^7 + 32h^4 + 40/13}{\lambda} \leq \frac{31}{10} \lambda^{-1} \quad (0 < h \leq \frac{1}{600}). \quad (43)$$

Proof. Throughout, Proposition 10.2 keeps $(A, B, Q) \in \mathcal{O}$.

Phase A. On $[0, T_A]$ apply Lemma 10.3 to $f = A$ with $L = 3$, $l = \frac{7}{2}$, $\delta = p^2/4$: whenever $A \leq \frac{7}{2}$, the entry slab of Lemma 10.1(c) gives $\dot{A} = p^2 R_A \geq p^2/4$, and $(l - L)/\delta = 2/p^2 = T_A$. Hence $A(T_A) \geq \frac{7}{2}$, and since the same inequality holds at later times whenever $A \leq \frac{7}{2}$, the first-exit argument of Lemma 10.3 keeps $A \geq \frac{7}{2}$ for all $t \geq T_A$. The upper slab gives $A \leq \frac{9}{2}$ in the same way.

Phase B. On $[T_A, T_B]$, A is inner, so whenever $B \leq \frac{3}{2}$ we have $R_B \geq 1$ and $\dot{B} = p^2 h^3 Q^3 R_B \geq p^2 h^3/64$ since $Q \geq \frac{1}{4}$. With $L = 1$, $l = \frac{3}{2}$, $\delta = p^2 h^3/64$ the deadline is $32/(p^2 h^3)$. The upper slab is symmetric.

Phase Q by reciprocal comparison. On $[T_B, T_Q]$, A and B are inner. Put $f = 1/Q$, which is C^1 with $f' = -\dot{Q}/Q^2 = -\frac{13}{2}\lambda R_Q$ by (38); the factor Q^2 cancels exactly. Whenever $Q \leq \frac{1}{2}$, i.e. $f \geq 2$, the slab gives $R_Q \geq \frac{1}{10}$ and hence $f' \leq -\frac{13}{20}\lambda$. Since $Q \geq \frac{1}{4}$ we have $f \leq 4$. Lemma 10.3 (second form) with $U = 4$, $l = 2$, $\delta = \frac{13}{20}\lambda$ gives $f \leq 2$, i.e. $Q \geq \frac{1}{2}$, after $(4 - 2)/\delta = 40/(13\lambda)$. Whenever $Q \geq \frac{3}{4}$, i.e. $f \leq \frac{4}{3}$, we have $R_Q \leq -\frac{1}{10}$ and $f' \geq \frac{13}{20}\lambda$; with $f \geq 1$ the first form gives $f \geq \frac{4}{3}$, i.e. $Q \leq \frac{3}{4}$, after $(\frac{4}{3} - 1)/\delta = 20/(39\lambda) \leq 40/(13\lambda)$. Residence thereafter follows as in phase A.

Finally $T_Q \lambda = 2h^7 + 32h^4 + 40/13$, and for $h \leq \frac{1}{600}$ the first two terms are below $3 \cdot 10^{-10}$ while $40/13 < 3.0770$, so $T_Q \leq 3.1\lambda^{-1}$. $\square$

Comparing $1/Q$ retains the exact cancellation of Q^2 . A direct comparison of Q using $Q^2 \geq 1/16$ on the slab would lose a factor of 16 in the Q -phase estimate.

10.6 The witness

Proposition 10.5 (Witness lower bound). *Assume (24). Let (x, y, z) be any solution from a prepared state with $Q(0) = \frac{1}{4}$ (for instance (29)). If $Q(T) \geq \frac{1}{2}$ for some $T \geq 0$, then $T \geq \frac{400}{1313}\lambda^{-1}$.*

Proof. On the preimage of $\mathcal{O}$ the forward term of R_Q is nonnegative and $b_3 C^2 \leq \frac{101}{100}$ by (37), so

$$\dot{Q} = \frac{13}{2}\lambda Q^2 R_Q \leq \frac{13}{2} \cdot \frac{101}{100} \lambda Q^2 = \frac{1313}{200} \lambda Q^2.$$

Hence $f = 1/Q$ satisfies $f' = -\dot{Q}/Q^2 \geq -\frac{1313}{200}\lambda$ for all $t \geq 0$. Integrating from $f(0) = 4$ gives $f(T) \geq 4 - \frac{1313}{200}\lambda T$. Target membership gives $Q(T) \geq \frac{1}{2}$, i.e. $f(T) \leq 2$, so $\frac{1313}{200}\lambda T \geq 2$ and $T \geq \frac{400}{1313}\lambda^{-1}$. The differential inequality holds throughout the trajectory, so the bound applies at every target time. $\square$

Proof of Theorem 8.2. Global existence, outer invariance and (28) are Proposition 10.2; the deadline is Proposition 10.4 together with (43); the state (29) is positive, has $(A, B, Q)(0) = (4, 2, \frac{1}{4}) \in \mathcal{O}$, hence has a global solution, and Proposition 10.5 applies to it since the target requires $Q \geq \frac{1}{2}$. $\square$

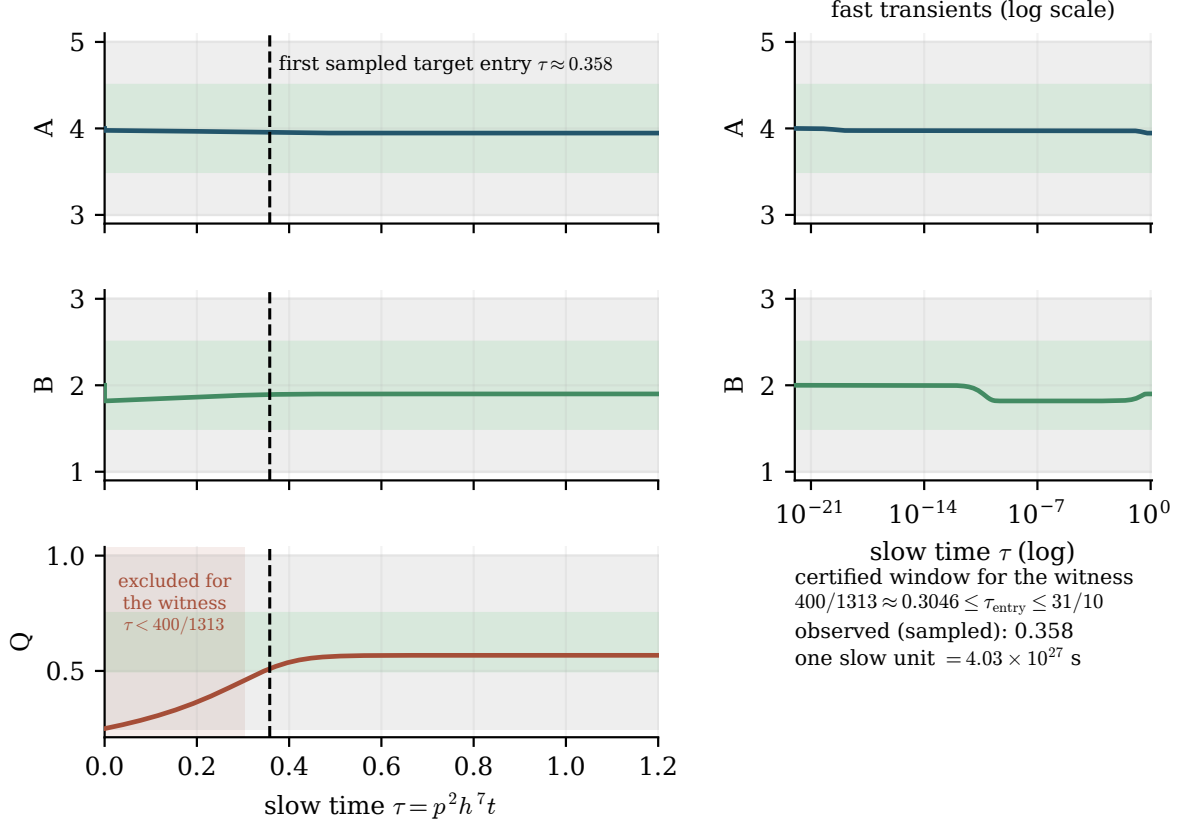


Figure 3: Numerical illustration of the complete system (25) at $h = 1/600$, $u = 1/20$, nominal prefactors, from the witness (29), plotted against the exact slow time $\tau = p^2 h^7 t$ (left) and on a logarithmic time axis for the fast coordinates (right). Grey bands are the outer intervals of $\mathcal{O}$, green bands the target intervals of $\mathcal{I}$. The dashed vertical line marks the first sampled time at which all three coordinates lie in $\mathcal{I}$, $\tau \approx 0.358$; the shaded region $\tau < 400/1313$ is excluded for the witness by Proposition 10.5, and the certified deadline $\tau = 31/10$ lies beyond the plotted window. One slow unit is 4.03×10^{27} seconds in the calibration of Section 13.

10.7 A numerical illustration

Figure 3 shows the solution from (29) at the corner $h = \frac{1}{600}$, $u = \frac{1}{20}$ with nominal prefactors, integrated with the stiff Radau method [13] on the complete transformed system (38) in the exact slow time $\tau = \lambda t$ (solver parameters are recorded in the reproducibility material). The first sampled time at which all three coordinates are inside $\mathcal{I}$ is $\tau \approx 0.358$, inside the certified window $[400/1313, 31/10] \approx [0.3046, 3.1]$. The fast coordinates A, B settle on the p^2 and $p^2 h^3$ clocks, invisible on the slow axis and shown on a logarithmic inset; the slow coordinate Q rises monotonically here, although no such monotonicity is used in the proofs.

10.8 Global inventory clocks for arbitrary seeds

Theorem 8.2 assumes a prepared seed. The inventory identity, however, carries information on the whole positive orthant, because positivity alone gives

$$z \leq \frac{1}{2}W, \quad x \leq \frac{1}{6}W, \quad y \leq \frac{1}{5}W \quad \text{for every positive state.} \quad (44)$$

Dropping the nonnegative term of the appropriate sign in $\dot{W} = -13J_3 = -13a_3x^2y + 13b_3ph^6z^2$ and inserting (44) gives the two-sided differential inequality

$$-\frac{13}{180}a_3W^3 \leq \dot{W} \leq \frac{13}{4}b_3ph^6W^2, \quad (45)$$

valid along every positive solution at every time.

Theorem 10.6 (Global inventory clocks). *Assume (24). Every positive initial state has a global positive solution of (25), and every global positive solution satisfies, for all $T \geq 0$,*

$$T \geq \frac{4}{13b_3ph^6} \left(\frac{1}{W(0)} - \frac{1}{W(T)} \right) \quad (\text{restoration clock}), \quad (46)$$

$$T \geq \frac{90}{13a_3} \left(\frac{1}{W(T)^2} - \frac{1}{W(0)^2} \right) \quad (\text{depletion clock}). \quad (47)$$

In particular, any positive solution that raises its inventory from W_0 to $W_1 > W_0$ needs time at least $\frac{4}{13b_3ph^6}(W_0^{-1} - W_1^{-1})$, and any solution that lowers it from W_0 to $W_1 < W_0$ needs at least $\frac{90}{13a_3}(W_1^{-2} - W_0^{-2})$, whatever the shape coordinates A, B do.

Proof. Global existence from every positive seed follows from Theorem 2.2 applied to the weakly reversible network (23) with fixed positive rates. For (46), $f = 1/W$ is C^1 with $f' = -\dot{W}/W^2 \geq -\frac{13}{4}b_3ph^6$ by (45); integrating, $f(0) - f(T) \leq \frac{13}{4}b_3ph^6 T$. For (47), $g = 1/W^2$ has $g' = -2\dot{W}/W^3 \leq \frac{13}{90}a_3$, so $g(T) - g(0) \leq \frac{13}{90}a_3T$. Neither bound uses monotonicity of W , so both hold at every time, not only at first passage. $\square$

The restoration clock with $W(0) = \frac{ph}{2}$ and $W(T) \geq ph$ gives $T \geq \frac{4}{13b_3}\lambda^{-1} \geq \frac{400}{1313}\lambda^{-1}$. Proposition 10.5 is the special case of Theorem 10.6 at the witness, and the prepared basin was never needed for the lower bound. The depletion clock quantifies the opposite transient: from a macroscopic seed such as $x = y = z = 1$ ($W = 13$), reaching the corridor inventory $W \leq ph$ needs at least $\frac{90}{13a_3}((ph)^{-2} - 13^{-2}) \approx 6.9(ph)^{-2}$ time units at nominal a_3 , which is h^5 times the restoration scale λ^{-1} : collapse of a surplus is governed by the forward reaction $2X + Y \rightarrow 2Z$ and is at least $(ph)^{-2}$ slow, while rebuilding a deficit is $(p^2h^7)^{-1}$ slow. What the theorem does not give is an upper bound on the time for an arbitrary seed to enter the sharp corridor; see Section 14.

11 Corollaries and the inventory mechanism

11.1 Common clocks

Corollary 11.1 (Exact common-clock transport). *For autonomous mass action and $b > 0$, $\tilde{x}$ is a global forward solution for rates $\tilde{k}$ if and only if $x(t) = \tilde{x}(bt)$ is a global forward solution for rates $k_e = b\tilde{k}_e$ from the same initial state. Phase-space paths, invariant sets and residence statements correspond exactly; all times are divided by b .*

Proof. $f_{b\tilde{k}} = bf_{\tilde{k}}$, and the chain rule gives $\dot{x}(t) = b\dot{\tilde{x}}(bt) = bf_{\tilde{k}}(\tilde{x}(bt)) = f_{b\tilde{k}}(x(t))$; the converse uses b^{-1} . $\square$

This is why (22) depends only on the spread Δ : a common factor ε^{a-} leaves concentration geometry unchanged and multiplies all times by ε^{-a-} . Different exponent assignments with the same spread can nevertheless have different sharp species scales, as the coupled network shows.

11.2 A continuous reverse-exponent family

Let $\frac{1}{2} < \gamma < \frac{7}{2}$ and set

$$p = \varepsilon^{1/2}, \quad h = \varepsilon^{(2\gamma-1)/12}, \quad u = \varepsilon^{(7-2\gamma)/12}, \quad (48)$$

so that $p = uh$, $p^2 = \varepsilon$ and $ph^6 = \varepsilon^\gamma$: the reverse rates on pairs 1, 2, 4 are proportional to ε and pair 3 has reverse exponent γ .

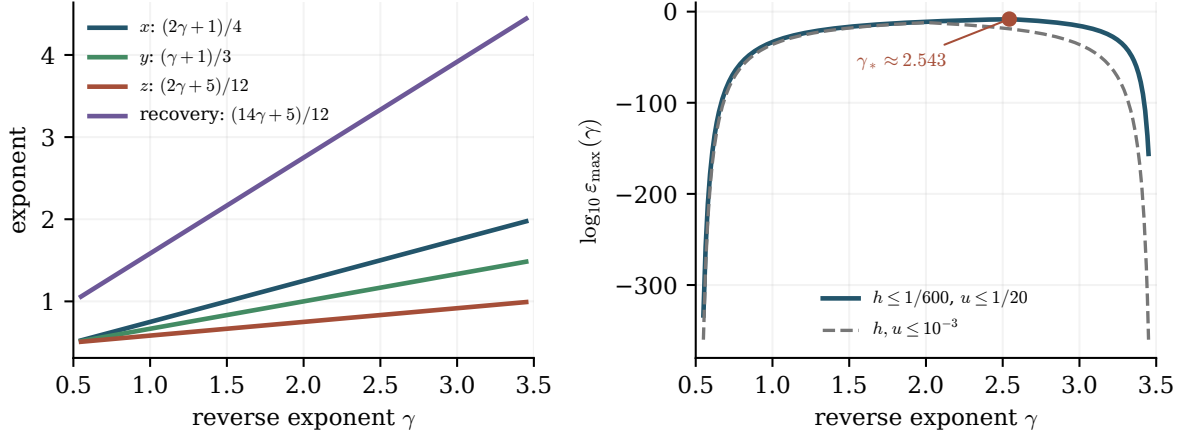


Figure 4: Left: exponent laws (50) and the recovery exponent along the family (48). Right: the certified cutoff $\log_{10} \varepsilon_{\max}(\gamma)$ for the admitted rectangle (solid) and the symmetric reference sub-rectangle (dashed). The marked point is the corner γ_* of Proposition 11.3, where the two branches of (49) balance and $\varepsilon_{\max}$ attains its largest value $1/144,000,000$.

Corollary 11.2 (Certified exponent family). *Define*

$$\varepsilon_{\max}(\gamma) = \min \{600^{-12/(2\gamma-1)}, 20^{-12/(7-2\gamma)}\}. \quad (49)$$

For $0 < \varepsilon \leq \varepsilon_{\max}(\gamma)$ and all admitted prefactors, Theorem 8.2 applies to (48). Along every prepared trajectory,

$$x \asymp \varepsilon^{(2\gamma+1)/4}, \quad y \asymp \varepsilon^{(\gamma+1)/3}, \quad z \asymp \varepsilon^{(2\gamma+5)/12}, \quad (50)$$

with the explicit constants of (28); every prepared start recovers by $\frac{31}{10}\varepsilon^{-(14\gamma+5)/12}$ and the witness needs at least $\frac{400}{1313}\varepsilon^{-(14\gamma+5)/12}$.

Proof. $h \leq \frac{1}{600}$ and $u \leq \frac{1}{20}$ are exactly the two real-power inequalities in (49), and $ph^3 = \varepsilon^{(2\gamma+1)/4}$, $ph^2 = \varepsilon^{(\gamma+1)/3}$, $ph = \varepsilon^{(2\gamma+5)/12}$, $p^2h^7 = \varepsilon^{(14\gamma+5)/12}$ by exponent arithmetic. Substitute into Theorem 8.2. $\square$

The symmetric reference sub-rectangle $h, u \leq 10^{-3}$ corresponds to the cutoff

$$\varepsilon \leq 10^{-36/\min\{2\gamma-1, 7-2\gamma\}}, \quad (51)$$

at $\gamma = 1$ it gives 10^{-36} , while (49) gives $600^{-12} \approx 4.59 \times 10^{-34}$. At $\gamma = 2$ the concentration exponents are $\frac{5}{4}, 1, \frac{3}{4}$ and the recovery exponent $\frac{11}{4}$. The endpoints $\gamma = \frac{1}{2}$ and $\frac{7}{2}$ are not covered: there one of h, u stops tending to zero. Figure 4 displays the exponent laws and the cutoffs.

Proposition 11.3 (Best corner of the certificate). *Over the rectangle (24), $p = uh \leq \frac{1}{12000}$, so the baseline reverse parameter satisfies $\varepsilon = p^2 \leq \frac{1}{144,000,000} \approx 6.94 \times 10^{-9}$. The family (48) reaches this corner exactly when*

$$\gamma_* = \frac{1}{2} + \frac{3 \ln 600}{\ln 12000} \approx 2.5432, \quad (52)$$

which maximises $\varepsilon_{\max}(\gamma)$. For fixed $p \leq \frac{1}{12000}$ the admissible h range is $[20p, \frac{1}{600}]$, and $\lambda = p^2h^7$ is increasing in h and in p , so the fastest certified recovery scale over the whole rectangle is attained at $h = \frac{1}{600}$, $u = \frac{1}{20}$:

$$\lambda^{-1} = 12000^2 \cdot 600^7 = 4.0310784 \times 10^{27}. \quad (53)$$

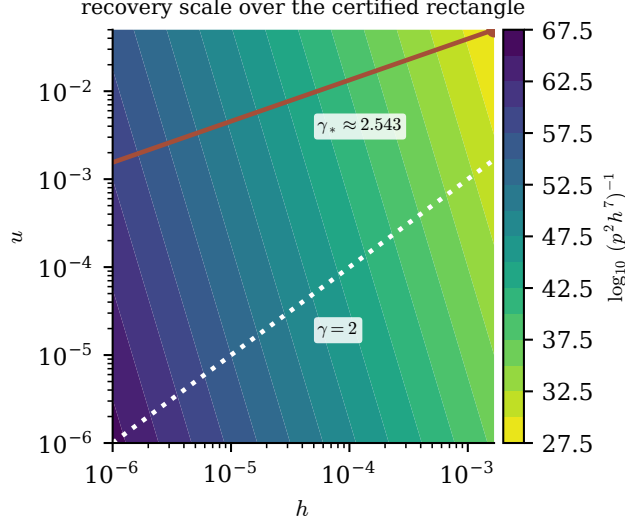


Figure 5: The certified recovery scale $\log_{10}(p^2 h^7)^{-1}$ over the admitted rectangle (logarithmic axes). Dotted: the family path $\gamma = 2$. Solid: the path $\gamma_* \approx 2.543$, which passes through the corner $(h, u) = (1/600, 1/20)$ (dot), the point of fastest certified recovery.

Proof. The two branches of (49) are respectively increasing and decreasing in γ and balance when $(7 - 2\gamma) \ln 600 = (2\gamma - 1) \ln 20$, i.e. at (52); substituting $h = \frac{1}{600}$, $p = \frac{1}{12000}$ into (48) gives the same value. The remaining statements are monotonicity of $p^2 h^7$ in each variable. $\square$

Figure 5 shows $\log_{10} \lambda^{-1}$ over the rectangle together with two family paths. The corner minimises the certified recovery scale over this rectangle. Enlarging the rectangle requires new face and entry estimates (Section 14).

11.3 Scale identities and necessary reaction extent

Corollary 11.4 (Preparation–recovery trade-off). *With the preparation scales $s_x = ph^3$, $s_y = ph^2$, $s_z = ph$,*

$$s_y^2 = s_x s_z, \quad \lambda^{-1} = \frac{s_y}{s_x^3} = \frac{s_z^3}{s_y^5}. \quad (54)$$

Hence the witness lower bound and the universal prepared deadline are $\frac{400}{1313} s_y/s_x^3$ and $\frac{31}{10} s_y/s_x^3$.

Proof. Substitute and cancel positive powers of p and h . $\square$

These dimensionless preparation-scale identities explain why extreme separation of concentration scales in this family is accompanied by slow recovery.

Corollary 11.5 (Necessary cumulative reverse extent). *For any solution from (29) and any $T \geq 0$ at which it is in the target,*

$$-\int_0^T J_3(t) dt = \frac{W(T) - W(0)}{13} \geq \frac{ph}{26}, \quad \int_0^T b_3 p h^6 z(t)^2 dt \geq \frac{ph}{26}. \quad (55)$$

Proof. Integrate (36); $W = 2phQ$ rises from $\frac{ph}{2}$ to at least ph . Since $-J_3 = b_3 p h^6 z^2 - a_3 x^2 y \leq b_3 p h^6 z^2$, the second bound follows. $\square$

11.4 A general inventory-bottleneck principle

The witness argument is an instance of a general principle that uses stoichiometry and kinetics rather than the reaction graph alone. Write a mass-action system as $\dot{x} = \sum_e \nu_e v_e(x)$ with $v_e \geq 0$, each direction of a reversible pair counted separately.

Proposition 11.6 (Inventory bottleneck, constant capacity). *Partition the reactions into a fast set F and a remainder R , and let $\ell \in \mathbb{R}^d$ satisfy $\ell \cdot \nu_e = 0$ for all $e \in F$. Put $U = \ell \cdot x$. Suppose that, up to the target time T , the trajectory stays in a region $\mathcal{D}$ on which*

$$\sum_{e \in R} (\ell \cdot \nu_e)_+ v_e(x) \leq M.$$

If reaching the target requires $U(T) \geq U(0) + \Delta U$ with $\Delta U > 0$, then $T \geq \Delta U/M$ when $M > 0$, and the target is unreachable along such a trajectory when $M = 0$.

Proof. $\dot{U} = \sum_{e \in R} (\ell \cdot \nu_e) v_e \leq \sum_{e \in R} (\ell \cdot \nu_e)_+ v_e \leq M$ on $\mathcal{D}$; integrate. $\square$

Proposition 11.7 (Inventory bottleneck, state-dependent capacity). *In the setting of Proposition 11.6 suppose instead that $\dot{U} \leq g(U)$ on $\mathcal{D}$ for a continuous $g > 0$ on an interval containing $[U(0), U_{\text{target}}]$ and all values taken by U before T . Then*

$$T \geq \int_{U(0)}^{U_{\text{target}}} \frac{ds}{g(s)}. \quad (56)$$

Proof. Let $\Phi' = 1/g$; then $(\Phi \circ U)' = \dot{U}/g(U) \leq 1$, so $\Phi(U(T)) - \Phi(U(0)) \leq T$, and Φ is increasing. $\square$

Corollary 11.8 (Explicit ε -scaling). *In the setting of Proposition 11.6, suppose every reaction $e \in R$ with $\ell \cdot \nu_e > 0$ has rate $k_e = \bar{k}_e \varepsilon^{a_e}$ with $a_e \geq a$ and $\bar{k}_e \leq \kappa_+$, and that $x^{y_e} \leq m_e$ on $\mathcal{D}$. Then any target requiring U to rise by $\Delta U > 0$ satisfies*

$$T \geq \frac{\Delta U}{\kappa_+ \sum_{e \in R, \ell \cdot \nu_e > 0} (\ell \cdot \nu_e) m_e} \varepsilon^{-a} = \Omega(\varepsilon^{-a}), \quad (57)$$

uniformly over the admitted prefactors. In the general setting of Theorem 8.1, after the entry time the trajectory lies in the box (22), so $m_e = (\alpha^{-1/2} \varepsilon^{-\Delta/(2H)})^{|y_e|}$ is admissible and (57) holds from that time on.

Proof. $\dot{U} \leq \sum (\ell \cdot \nu_e)_+ k_e x^{y_e} \leq \varepsilon^a \kappa_+ \sum (\ell \cdot \nu_e)_+ m_e =: C \varepsilon^a$ on $\mathcal{D}$, and Proposition 11.6 with $M = C \varepsilon^a$ gives $T \geq (\Delta U/C) \varepsilon^{-a}$. $\square$

This is the universal form of the slow-channel obstruction: it uses a left null vector of the fast reaction vectors, the slow rate scalings and a region bound. It is one-sided; a matching upper bound requires network-specific progress estimates, which is what Theorem 8.2 supplies for (23). For the coupled network take $\ell = (6, 5, 2)$, F the six directed reactions of pairs 1, 2, 4, and R the two directions of pair 3. The only positive contribution to $\dot{W}$ is $13 b_3 p h^6 z^2$, and on the preimage of $\mathcal{O}$ we have $z = p h Q C \leq p h Q = W/2$, so $\dot{W} \leq g(W) := \frac{13}{4} b_3 p h^6 W^2$. For the witness W must rise from $\frac{ph}{2}$ to ph , and (56) gives

$$T \geq \frac{4}{13 b_3 p h^6} \left(\frac{2}{ph} - \frac{1}{ph} \right) = \frac{4}{13 b_3} \lambda^{-1} \geq \frac{400}{1313} \lambda^{-1},$$

which is Proposition 10.5 and, since (44) needs no basin, Theorem 10.6: the reciprocal comparison is the state-dependent capacity bound for $g(W) \propto W^2$. The constant-capacity version with $z \leq ph$ gives the weaker $T \geq \lambda^{-1}/(26 \cdot \frac{101}{100})$, and Corollary 11.8 with $a = \gamma$ along the family (48) gives the recovery exponent up to the region-dependent prefactor. A matching upper deadline

additionally uses forward progress ($R_Q \geq \frac{1}{10}$ on the slab) and invariance of the comparison region.

Fast-subsystem conservation laws as slow variables have an established literature; see [8] for approximate conservation laws and reduction, and [9] for the geometric theory. The propositions above are elementary; their value is that they turn such an inventory into an explicit, regionally justified lower bound on recovery time. The productive search pattern for related mechanisms is: candidate fast subnetwork $\rightarrow$ exact left null vector $\ell \rightarrow$ regional bound on the increasing slow fluxes $\rightarrow$ lower time bound $\rightarrow$ a separate progress mechanism for an upper bound. The reaction graph alone does not determine ℓ , the flux bound or a useful region.

12 Stochastic remarks

Consider the standard stochastic mass-action model of (23), a continuous-time Markov chain on molecule counts $n = (n_X, n_Y, n_Z) \in \mathbb{N}^3$ with combinatorial propensities [1, 11, 15], in which every one of the eight directed reactions has a positive rate constant.

Proposition 12.1 (No jump into an absorbing state). *(a) For any reversible network with all rate constants positive, no reaction event can lead from a state in which some reaction is enabled to a state in which no reaction is enabled.*

(b) For (23) the absorbing count states are exactly $(0, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ and $(0, 1, 1)$, and none of them is reachable from any other state.

Proof. (a) A jump consuming the complex y and producing y' leaves a state containing at least y' molecules of the product complex, so the reverse reaction $y' \rightarrow y$, which has positive rate, is enabled at the destination. (b) $n_X \geq 1$ enables $X \rightarrow 3Z$; $n_Y \geq 2$ enables $2Y \rightarrow X + 2Z$; $n_Z \geq 2$ enables $2Z \rightarrow 2X + Y$. If none holds then $n_X = 0$, $n_Y \leq 1$, $n_Z \leq 1$, and inspection shows that none of the eight reactions is enabled. Unreachability is (a). $\square$

Individual species counts can reach zero and return: for instance, $n_X = 0$ is left by $3Z \rightarrow X$. Species absence, loss of a functional threshold and absorption are therefore distinct events. Recurrence and stationary distributions for this four-linkage-class network require a separate analysis; the single-linkage-class results of [2] provide related context.

The deterministic identity (36) has an exact discrete counterpart. Let $N_3^+(t)$ and $N_3^-(t)$ count the forward and reverse events of pair 3 and put $W_n = 6n_X + 5n_Y + 2n_Z$. Before any explosion,

$$W_n(t) = W_n(0) + 13(N_3^-(t) - N_3^+(t)), \quad (58)$$

so $W_n \bmod 13$ is invariant, and a target requiring W_n to rise by at least L requires at least $m = \lceil L/13 \rceil$ reverse events of pair 3.

The reverse propensity of pair 3 is $\lambda_3^-(n) = \kappa n_Z(n_Z - 1)$ with $\kappa = b_3 p h^6 / (V N_A)$ in the units of Section 13, and $2n_Z \leq W_n$ for every count state. Hence $\lambda_3^- \leq \frac{\kappa}{4} W_n^2$ everywhere, which is the discrete form of (45). This turns the inventory bottleneck into a pathwise statement with no localisation.

Proposition 12.2 (Pure-birth comparison). *Let $W_0 = W_n(0)$ and let T be any random time at which $W_n(T) \geq W_0 + 13m$ for an integer $m \geq 1$ (with $T = \infty$ if this never happens before explosion). Put $g_k = \frac{\kappa}{4}(W_0 + 13k)^2$. Then there are i.i.d. unit exponential random variables $E_0, \dots, E_{m-1}$ with*

$$T \geq \sum_{k=0}^{m-1} \frac{E_k}{g_k} \quad \text{pathwise.} \quad (59)$$

Consequently

$$\mathbb{E}[T] \geq \sum_{k=0}^{m-1} \frac{1}{g_k} \geq \frac{4}{13\kappa} \left(\frac{1}{W_0} - \frac{1}{W_0 + 13m} \right), \quad \Pr\{T \leq t\} \leq e^{\theta t} \prod_{k=0}^{m-1} \left(1 + \frac{\theta}{g_k} \right)^{-1} \quad (\theta > 0), \quad (60)$$

and the lower bound in (59) has variance $\sum_k g_k^{-2}$.

Proof. By the random time-change representation [1, 15], $N_3^-(t) = Y(\int_0^t \lambda_3^-(n(s)) ds)$ for a unit-rate Poisson process Y , up to explosion. Let σ_k be the time of the k -th reverse pair-3 event and E_k the k -th interarrival time of Y , so the E_k are i.i.d. unit exponentials. Forward pair-3 events lower W_n and all other events preserve it, so $W_n(s) \leq W_0 + 13N_3^-(s)$ by (58); on $(\sigma_k, \sigma_{k+1}]$ we have $N_3^- = k$ and therefore $\lambda_3^- \leq \frac{\kappa}{4}(W_0 + 13k)^2 = g_k$. Hence $E_k = \int_{\sigma_k}^{\sigma_{k+1}} \lambda_3^- \leq g_k(\sigma_{k+1} - \sigma_k)$. Reaching $W_n \geq W_0 + 13m$ requires $N_3^- \geq m$, so $T \geq \sigma_m = \sum_{k < m} (\sigma_{k+1} - \sigma_k) \geq \sum_{k < m} E_k / g_k$. The expectation, the integral comparison $\sum_{k < m} (W_0 + 13k)^{-2} \geq \int_0^m (W_0 + 13s)^{-2} ds$, the variance and the Chernoff bound $\Pr\{S \leq t\} \leq e^{\theta t} \mathbb{E}e^{-\theta S}$ follow from independence. $\square$

Writing $W_n = WVN_A$ and $\kappa = b_3ph^6/(VN_A)$, the mean bound in (60) is exactly the deterministic restoration clock (46), up to the discreteness correction $W_0 + 13m$ versus W_1 : the deterministic clock is the expectation of a pathwise stochastic lower bound. At the corner of Section 13 ($V = 1\text{ L}$), $W_0 = 4.18 \times 10^{16}$ counts, $m = 3.2 \times 10^{15}$, and the bound has mean $1.228 \times 10^{27}\text{ s}$ with relative standard deviation 1.9×10^{-8} ; the stochastic recovery time is at least this, essentially deterministically. At the reference endpoint $\varepsilon = 10^{-36}$, where the deterministic X concentration corresponds to 0.003 molecules, the inventory sits almost entirely in Z : $W_0 \approx 301$ counts, $m = 24$, mean bound $3.2 \times 10^{56}\text{ s}$ with relative standard deviation 0.22. The bottleneck therefore survives the breakdown of the deterministic description; what changes at low counts is the width of the distribution, not the order of the delay. The comparison is global up to explosion and uses only the inventory itself.

For completeness, a localised Poisson bound also holds: if the reverse propensity is at most Λ on a finite set $\mathcal{D}$ of count states, then $\Pr\{T_{\text{target}} \leq t, T_{\text{target}} \leq \tau_{\text{exit}}\} \leq \Pr\{\text{Poisson}(\Lambda t) \geq m\}$. It is weaker than Proposition 12.2: at the corner, bounding n_Z by its outer-box maximum gives $m/\Lambda \approx 1.5 \times 10^{26}\text{ s}$, a factor eight below the birth-chain mean, because it ignores that the propensity is smallest exactly when the deficit is largest.

13 Dimensional calibration and operational limitations

Fix a concentration unit C_0 , a time unit τ_0 and a volume V , independently of the parameters. If $\hat{k}$ is a dimensionless rate constant for a source of molecularity $|y|$, then

$$k_{\text{dim}} = \frac{\hat{k}C_0^{1-|y|}}{\tau_0}, \quad c_i = C_0x_i, \quad t_{\text{dim}} = \tau_0t, \quad N_i = c_iVN_A, \quad (61)$$

with $N_A = 6.02214076 \times 10^{23}\text{ mol}^{-1}$ exactly [4]. All forward sources in (23) are cubic; the reverse sources of pairs 1–3 are quadratic and that of pair 4 is linear. Take $C_0 = 1\text{ mol L}^{-1}$, $\tau_0 = 1\text{ s}$, $V = 1\text{ L}$ and the corner $h = \frac{1}{600}$, $u = \frac{1}{20}$ ($p = \frac{1}{12000}$) with nominal prefactors. The forward constants are 1, 1, 1, 4 $\text{L}^2\text{mol}^{-2}\text{s}^{-1}$; the reverse constants are $6.944 \times 10^{-9}\text{ L mol}^{-1}\text{s}^{-1}$ for pairs 1 and 2, $1.786 \times 10^{-21}\text{ L mol}^{-1}\text{s}^{-1}$ for pair 3 and $6.944 \times 10^{-9}\text{ s}^{-1}$ for pair 4. The certified all-time bounds, rounded outward, are

species	lower bound (mol/L)	upper bound (mol/L)	upper count in 1 L
X	1.8084×10^{-14}	1.9291×10^{-12}	1.1617×10^{12}
Y	1.4467×10^{-11}	6.9445×10^{-10}	4.1821×10^{14}
Z	1.7361×10^{-8}	1.3889×10^{-7}	8.3641×10^{16}

and the certified recovery window is

$$1.2281 \times 10^{27} \text{ s} \leq T_{\text{witness}}, \quad T_{\text{all}} \leq 1.2497 \times 10^{28} \text{ s}.$$

Copy numbers are macroscopic, yet the certified recovery time exceeds the age of the universe by nine orders of magnitude.

Other admissible points have the opposite limitation. On the reference path $\gamma = 1$ at the symmetric reference endpoint $\varepsilon = 10^{-36}$, the upper bound for X is 5×10^{-27} mol/L, fewer than 0.004 expected molecules in one litre, with $3.05 \times 10^{56} \text{ s} \leq T_{\text{witness}}$ and $T_{\text{all}} \leq 3.1 \times 10^{57} \text{ s}$; at the admitted-rectangle endpoint $\varepsilon = 600^{-12}$ the X upper count is still below 0.30, with $1.86 \times 10^{52} \text{ s} \leq T_{\text{witness}}$ and $T_{\text{all}} \leq 1.89 \times 10^{53} \text{ s}$. These expected counts lie below one molecule, outside the macroscopic regime of a deterministic concentration description.

There is also a thermodynamic distinction. With r_i the forward-to-reverse ratio of pair i on the reference path, the rate ratio around the independent cycle formed by pairs 1, 2 and 4 is

$$\frac{r_1 r_2}{r_4} = \frac{1}{4\varepsilon}, \quad (62)$$

whose logarithm is about 81.5 at $\varepsilon = 10^{-36}$ and 17.4 at the corner. Graph reversibility therefore does not imply equilibrium thermodynamic reversibility; reading such a cycle logarithm as an affinity or chemical work [21] requires a thermodynamically specified reservoir realisation that this three-species effective model does not supply. For practitioners the model-specific conclusion is: adding tiny reverse reactions can restore mathematical permanence while the certified concentrations or recovery times remain operationally unacceptable. The resulting diagnostic is to identify nearly conserved inventories, bound their replenishment capacity, compare with the operating horizon and check copy numbers.

14 Discussion and open problems

Weak reversibility protects against concentration loss quantitatively as rates degenerate, but concentration geometry, preparation and recovery obey different laws. The general bound is governed by the exponent spread and an existence constant H . In the coupled network the preparation scales obey exact algebraic relations, the slow reverse reaction must supply a definite cumulative inventory restoration, and the state-dependent capacity bound reproduces the certified witness constant exactly. These facts explain the recovery exponent $(14\gamma + 5)/12$ without any convergence or equilibrium argument.

- (O1) *Whole-class capture.* Theorem 8.2 is restricted to the prepared basin. Theorem 10.6 removes the basin hypothesis from the *lower* bounds: from any positive seed, restoring a deficit costs at least the restoration clock and shedding a surplus at least the depletion clock. The general theorem absorbs every positive seed into a reciprocal box, but neither result gives a seed-independent deadline for entering the sharp corridor. A capture theorem would need to control A and B before corridor entry; the inventory identity alone does not, since $\dot{Q}$ depends on $A^2 B Q^6$ and the identity gives no bound on A, B .
- (O2) *Limiting scaled entry time.* Matching powers do not give a limit for λT_{entry} as $h, u \rightarrow 0$. A promising route is a reduced slow equation after fast-layer control, with a quantified residual.
- (O3) *Optimal constants.* The certified window $[0.3046, 3.1]$ leaves room for sharper constants. Subdividing the entry slabs would sharpen the A and B phases, which are negligible here, and the Q phase through a better bound on R_Q along the actual corridor.
- (O4) *Larger rectangles.* At $h \leq \frac{1}{500}$ with $u \leq 10^{-3}$ only the sufficient lower A entry margin falls slightly below $\frac{1}{4}$ (a failure of the estimate, not a counterexample), and at $u \leq \frac{1}{10}$

the term $-2b_2uB^2/Q$ pushes the lower B entry margin below 1. There is also a genuine adverse face: at $h = \frac{1}{100}$, $u = \frac{1}{20}$, $(A, B, Q) = (\frac{7}{2}, \frac{5}{2}, \frac{3}{4})$, $a_i = b_1 = b_2 = b_3 = 1$, $K = \frac{99}{25}$, $b_4 = \frac{101}{100}$ the exact value of R_A is negative, so the inner box of (27) is not invariant there; the mechanism is $KC^3 - b_4A$, where inventory stored in Y reduces C and weakens the cubic restoring source. Enlarging h substantially requires different boxes, not merely more interval subdivision.

- (O5) *Effective affine data and network-specific clocks.* The barrier delays of Part I depend on the finite affine data. Section 9.5 identifies the limiting choices that prevent their numerical extraction in the construction used here. Explicit slopes and activity thresholds, a worked H for a small network, or bounds in terms of the complexes would make Theorem 8.1 numerically usable. Direct network structure can also yield sharper recovery constants, as the staged scalar example shows.
- (O6) *A general staged-recovery principle.* In Section 7, a recovered component accelerates the guaranteed recovery of the components it drives. A general result should specify the dependency order and the clock inequalities needed to propagate these improved bounds through a network.
- (O7) *Periodic entrainment.* Corollary 6.2 gives periodic operation under periodic rate histories. Conditions for stability, uniqueness and entrainment of that periodic regime, including state-dependent feedback under the hypotheses of Section 6, remain to be determined.
- (O8) *Stochastic viability.* Proposition 12.2 bounds the stochastic recovery time from below, pathwise, with no localisation. Matching stochastic upper bounds, positive recurrence, the time spent below functional thresholds and the full distribution of inventory events for this four-linkage-class network are open.

15 Formal evidence and reproducibility

The conventional arguments above are supported by two separate registered Lean developments. They retain their own module names, dependencies and proof interfaces; consolidation of the exposition does not create a new compared formal theorem. Both registered projects pin Lean v4.35.0-rc2 and Mathlib revision 065356127b1dc0016f66b7283ce0ce2c4055aa55. Their permitted axioms are `propext`, `Classical.choice` and `Quot.sound`. References to earlier Lean 4.30.0 research receipts identify historical checks rather than the environment of these registrations.

15.1 Registered endotactic permanence

PALOMAR-2026-10-09-000015, version 1 records the source commit [revision c72c94526456](#). The selected declarations, in namespace `EndotacticSubmission`, are `uniform_permanence` and `continuous_dependence`.

The compared permanence statement allows arbitrary signed real source and target coordinates and evaluates the exponential-log monomials on positive states. It tests ordinary endotacticity in every real direction, admits almost-everywhere measurable histories bounded almost everywhere in a fixed positive rate box, and selects the compact convex invariant target before the initial compact set and history. The common entry deadline precedes the initial point and history. A global locally absolutely continuous solution with the almost-everywhere ODE is constructed, and uniqueness is among positive Carathéodory competitors. The statement thus supports the nonnegative real-complex scope of Theorem 2.2, as well as the compact-initial uniformity. The continuous-dependence statement supports Proposition 6.1 for a common history on a compact positive region.

The periodic-forcing and equilibrium corollaries and the state-feedback extension retain the conventional proofs given here. For the first three-species example, historical Lean receipts establish the clocked comparison, inverse-square transformation, two-stage clock estimates and rational roundings. The reaction classification, invariant-box checks, clock identification and Riccati comparison retain their conventional status. Registration of the general theorem is not a claim that all example-specific arguments were selected for independent comparison.

Scalar-clock evidence for the first example. The steps of the proof of Theorem 7.3 after the clocks and the inverse-square transformation are formalized as one Lean theorem, `staged_coordinate_recovery` (Section 15): given an absolutely continuous coordinate u with values in $[1/10, 10]$, an absolutely continuous clock σ with $\sigma(0) = 0$ and $\sigma' \geq 0$, coefficients with $k_+ \geq 1/2$, $k_- \leq 2$, the equation $u' = \sigma'u(k_+ - k_-u^2)$ almost everywhere, clock floors $\sigma' \geq 1/10$ on $[0, t_1]$ and $\sigma' \geq 2/5$ on $[t_1, \infty)$, and the arithmetic condition $t_1/10 + \frac{2}{5}(t_2 - t_1) \geq 19/5$, it concludes $u(t) \geq 2/5$ for all $t \geq t_2$. Its instances `staged_recovery_second` ($t_1 = 9$, $t_2 = 65/4$) and `staged_recovery_third` ($t_1 = 65/4$, $t_2 = 347/16$) are the two stages, and `a_stage_deadline_lt_nine`, `recovery_deadline_lt_38` and `staged_deadline_lt_22` are the roundings. The conventional parts are the invariance of the startup box, the existence of the clocks with the stated almost-everywhere derivatives (the integral of a bounded measurable function), the identification $\sigma'_b = a$ with the floors from stage a , and the Riccati comparison (17).

15.2 Registered reverse-rate bounds and recovery

PALOMAR-2026-10-11-000020, version 1 records source commit [revision e54242294603](#) and eighteen selected declarations. The principal correspondences are listed below. Except where a different namespace is printed, declaration names are in `ReverseRateLimits`.

Table 2: Selected reverse-rate statements and their manuscript counterparts.

Manuscript conclusion	Selected declaration or coverage
Theorem 8.1: network-first geometric gap, rate-dependent reciprocal containment and global solutions	<code>general_quantitative_root</code>
Theorem 8.2: prepared-basin existence, concentration bounds and residence by $(31/10)\lambda^{-1}$	<code>SharpRecovery.prepared_recovery_sharp</code>
The designated witness and its $(400/1313)\lambda^{-1}$ necessary target time	<code>SharpRecovery.witness_sharp</code>
Theorem 10.6: arbitrary positive seeds, global solutions and necessary inventory restoration/depletion times	<code>GlobalInventory.global_inventory_clocks</code>
Reverse-exponent family, both concentration and worst-case prepared recovery orders	<code>SharpRecovery.family_root_sharp</code> ; exact scale identities and cutoff comparisons are selected separately
Common-clock transport, preparation identities and cumulative reverse-reaction requirements	<code>CommonClock.rescale_iff</code> , <code>RecoveryCorollaries.scale_tradeoff</code> , <code>ReactionExtent.restoration_extent</code>
Constant-capacity inventory comparison, including zero capacity	<code>SharpRecovery.inventory_bottleneck</code> , <code>SharpRecovery.inventory_unreachable</code>

Table 2 (continued)

Manuscript conclusion	Selected declaration or coverage
Abstract inverse-power clock and absence of a linear conservation law	<code>GlobalInventory.inventory_bottleneck_scaling</code> , <code>RecoveryCorollaries.no_linear_inventory</code>
Admitted parameters and family admission with arbitrary admitted prefactors	<code>QuantitativeRegistry.admitted_parameters</code> , <code>QuantitativeRegistry.family_parameters</code>
Open-range version of the state-dependent capacity comparison in Proposition 11.7	<code>QuantitativeRegistry.state_capacity_clock</code> : positive continuous capacity on an open convex range containing the trajectory and target; no inventory monotonicity assumption

The exact eighteen statement names and complete independent Challenge definitions are preserved with the source archives. The registered endpoints include the exact exponent identities, original cutoff comparison and cutoff decimal identity in addition to the grouped rows above. Definitions of literal trajectories do not assume the corridor, residence deadline or witness conclusion. The interval expression checker and the polynomial identifications used for the eighteen inclusions in Appendix A are contained in the proof closure.

The stochastic absorbing-state correction and birth-process comparison, the parameter-corner optimization, and the reaction-flux bookkeeping used to instantiate the abstract inventory estimate remain conventional arguments. Proposition 11.7 is proved conventionally at its displayed interval scope; the selected formal version explicitly assumes an open convex positive-capacity range. The saved trajectory is a numerical illustration and does not certify its sampled hitting time. No stochastic success probability, upper recovery deadline or whole-class sharp-corridor capture theorem is inferred.

15.3 Sources, historical verification and computational material

The registered source archives include independent Challenge statements, completed Solution entry points, the supporting proof modules, dependency pins, licenses and upstream attribution. The affine construction is attributed to the pinned OpenAI development [17–19]; the port preserves the mathematical dependencies rather than replacing them by additional axioms. The public registration records describe their historical verification. Saved local receipts additionally report independent comparison and six configured kernel checks. Neither those historical runs nor the public registry status is presented as a fresh proof replay for this manuscript edition.

The exact registered source archives and their registration records are preserved in [formalization.zip](#). Figure producers, exact certificate data, saved trajectories, parameter-corner files and historical evidence are in [computational-companion.zip](#). The manuscript, TeX source and both archives are publicly available together at [doi:10.5281/zenodo.23442569](https://doi.org/10.5281/zenodo.23442569). The immutable formalization repositories are also linked through the two Palomar records above.

All manuscript figures are carried over from the source papers. Their saved solver settings and numerical status are retained: declared simulations illustrate the analytic statements, while rational interval inclusions and formal statement comparisons are distinct evidence channels. Preparation of this edition checked source identities, bibliographic records, TeX compilation and rendered pages. No Lean, Lake, independent kernel, research numerical solver or certificate producer was executed.

Acknowledgements. The affine permanence construction and its Lean formalisation are due to OpenAI [17–19] and are used here as an attributed input at a pinned revision.

Use of AI tools. Generative AI was used extensively for mathematical development, manuscript drafting, computations and Lean formalization under human direction.

A The certified interval inclusions

Each row of Table 3 is one proved inclusion: over the rational box on (Q, A, B) shown, with $h \in (0, \frac{1}{600}]$, $u \in (0, \frac{1}{20}]$ and the prefactor intervals of (24), the indicated normalised derivative lies in the enclosure. The sign or margin required by Lemma 10.1 is read off the relevant endpoint; the enclosures are conservative, so the actual margins are larger than the required ones (for instance the lower A entry slab certifies $R_A \geq 0.278 > \frac{1}{4}$, the lower B entry slab $R_B \geq 1.389 > 1$ and the lower Q entry slab $R_Q \geq 0.1808 > \frac{1}{10}$). The checker operates on a rational expression tree with atoms for the thirteen variables; the interval evaluation is proved sound for every real point of the box, the endpoint arithmetic is verified by Lean’s kernel, and the identification of the expression trees with (39)–(41) is a separately proved polynomial identity.

Table 3: The eighteen certified inclusions (enclosures rounded outward to four digits).

name	function	Q	A	B	certified enclosure
outer A lower	R_A	$[1/4, 1]$	3	$[1, 3]$	$[0.7830, 1.0701]$
outer A upper	R_A	$[1/4, 1]$	5	$[1, 3]$	$[-1.2372, -0.9099]$
outer B lower	R_B	$[1/4, 1]$	$[3, 5]$	1	$[3.4661, 8.1284]$
outer B upper	R_B	$[1/4, 1]$	$[3, 5]$	3	$[-16.025, -7.4965]$
outer Q lower	R_Q	$1/4$	$[3, 5]$	$[1, 3]$	$[0.9653, 1.0079]$
outer Q upper	R_Q	1	$[3, 5]$	$[1, 3]$	$[-74.785, -7.9000]$
inner A lower	R_A	$[1/2, 3/4]$	$7/2$	$[3/2, 5/2]$	$[0.3327, 0.5751]$
inner A upper	R_A	$[1/2, 3/4]$	$9/2$	$[3/2, 5/2]$	$[-0.6774, -0.4149]$
inner B lower	R_B	$[1/2, 3/4]$	$[7/2, 9/2]$	$3/2$	$[1.8653, 4.6560]$
inner B upper	R_B	$[1/2, 3/4]$	$[7/2, 9/2]$	$5/2$	$[-7.0657, -3.1880]$
inner Q lower	R_Q	$1/2$	$[7/2, 9/2]$	$[3/2, 5/2]$	$[0.1807, 0.7258]$
inner Q upper	R_Q	$3/4$	$[7/2, 9/2]$	$[3/2, 5/2]$	$[-8.1258, -2.2276]$
entry A lower	R_A	$[1/4, 1]$	$[3, 7/2]$	$[1, 3]$	$[0.2780, 1.0701]$
entry A upper	R_A	$[1/4, 1]$	$[9/2, 5]$	$[1, 3]$	$[-1.2372, -0.4149]$
entry B lower	R_B	$[1/4, 1]$	$[7/2, 9/2]$	$[1, 3/2]$	$[1.3891, 7.1225]$
entry B upper	R_B	$[1/4, 1]$	$[7/2, 9/2]$	$[5/2, 3]$	$[-15.059, -3.1298]$
entry Q lower	R_Q	$[1/4, 1/2]$	$[7/2, 9/2]$	$[3/2, 5/2]$	$[0.1807, 1.0056]$
entry Q upper	R_Q	$[3/4, 1]$	$[7/2, 9/2]$	$[3/2, 5/2]$	$[-50.162, -2.2276]$

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