

The $\mathfrak{sl}(2)$ weight system at $c = -3/32$ extends to a graph 4-invariant

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Abstract

A weight system is a function on chord diagrams that satisfies the four-term relations. The Lie algebra $\mathfrak{sl}(2)$ gives a weight system with values in the polynomial ring $\mathbb{Q}[c]$, and its value on a chord diagram depends only on the intersection graph of the diagram. Fomin, Karel, Pavutnitskiy and Usanov showed that this weight system does not come from a function on graphs, and that a specialisation at a value $c = c_0$ can come from one only for $c_0 = 0, 3/8, 1$ or $-3/32$. The first three values were settled before. They conjectured that the last one, $c = -3/32$, which is the eigenvalue of the Casimir element on the oscillator representation, also extends. We prove this. The extension is the function $\xi(G) = (-32)^{-n} S_G(-2)$, where G is a graph with n vertices and $S_G(N)$ is a signed count of the maps that send each vertex of G to a two-element subset of $\{1, \dots, N\}$: a map counts with the sign -1 for every edge whose two endpoints are sent to subsets meeting in exactly one element. The count is a polynomial in N , so its value at $N = -2$ makes sense. The four-term relation for ξ is proved by an involution on these maps. The identity with the weight system is proved by letting the chord diagram act on Verma modules, then on the oscillator representation, where a supertrace on the Weyl algebra turns the value into a sum over perfect matchings, and finally by reading the same sum as a trace in a Clifford algebra on N generators.

Keywords. weight system; intersection graph; four-term relation; Clifford algebra.

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1 Introduction

A weight system is a function on chord diagrams that satisfies the four-term relations. Weight systems are the symbols of finite type knot invariants [1, 2, 3].¹ A Lie algebra with an invariant form gives a weight system with values in the centre of its universal enveloping algebra. For $\mathfrak{sl}(2)$ with the Killing form the centre is the polynomial ring $\mathbb{Q}[c]$ in the Casimir element c , and Chmutov and Lando proved that the value $w_D \in \mathbb{Q}[c]$ of this weight system on a chord diagram D depends only on the intersection graph $\Gamma(D)$ of D [4]. Lando's four-term relations for graphs [5] are such that every function f on graphs satisfying them gives a weight system $D \mapsto f(\Gamma(D))$; such a function is called a 4-invariant. Lando asked whether w comes from a 4-invariant [4]. For graphs with at most 8 vertices this is true [7]; see [6] for a survey.

Fomichev, Karev, Pavutnitskiy and Usanov [8] answered the question in the negative. They found a linear combination of intersection graphs that vanishes modulo the four-term relations of graphs and on which w takes the value $c(c - \frac{3}{8})(c - 1)(c + \frac{3}{32})$. So the specialisation of w at a value $c = c_0$ can extend to a 4-invariant only if c_0 is one of $0, \frac{3}{8}, 1$ and $-\frac{3}{32}$. The first three values are the eigenvalues of the Casimir element on the irreducible representations of dimension 1, 2 and 3, and for them extensions are known [8, 9, 10].² The value $-\frac{3}{32}$ is the eigenvalue of the Casimir element on the oscillator representation, the representation of $\mathfrak{sl}(2)$ on polynomials in one variable by second order differential operators. The authors of [8] built an extension for graphs with at most 10 vertices and stated the following conjecture.

Conjecture 1.1 ([8, Section 4]). *The specialisation of the $\mathfrak{sl}(2)$ -weight system at $c = -\frac{3}{32}$ extends to a 4-invariant of graphs $\xi(G)$.*

We prove Conjecture 1.1, with an explicit formula for ξ . All graphs are finite and simple. For $N \in \mathbb{N}$ write $[N] = \{1, \dots, N\}$ and $\binom{[N]}{2}$ for the set of its 2-element subsets.

Definition 1.2. For a graph $G = (V, E)$ and $N \in \mathbb{N}$ put

$$S_G(N) = \sum_{p: V \rightarrow \binom{[N]}{2}} \prod_{xy \in E} (-1)^{|p(x) \cap p(y)|},$$

and for $k \in \mathbb{N}$ let $T_k(G)$ be the same sum restricted to the maps $p: V \rightarrow \binom{[k]}{2}$ with $\bigcup_{x \in V} p(x) = [k]$.

Theorem 1.3. *Let $G = (V, E)$ be a graph with n vertices.*

(a) $S_G(N) = \sum_{k=0}^{2n} \binom{N}{k} T_k(G)$ for all $N \in \mathbb{N}$; so S_G is a polynomial in N of degree at most $2n$.

(b) *The graph invariant*

$$\xi(G) = \frac{S_G(-2)}{(-32)^n} = \frac{1}{(-32)^n} \sum_{k=0}^{2n} (-1)^k (k+1) T_k(G)$$

satisfies the four-term relations of graphs.

¹A chord diagram with n chords is a circle with $2n$ marked points joined in pairs. A finite type invariant of degree n of knots, evaluated on a knot with n double points and extended by the Vassiliev skein relation, depends only on the chord diagram of the double points; the function obtained in this way satisfies the four-term relations, and by Kontsevich's theorem every function satisfying them arises so.

²At $c = 0$ the weight system is identically 1 on diagrams, since the Casimir element acts by 0 on the trivial representation. At $c = \frac{3}{8}$ the extension was conjectured in [9] and proved in [10]. At $c = 1$ it is given in [8].

(c) For every chord diagram D , $w_D(-\frac{3}{32}) = \xi(\Gamma(D))$.

Parts (b) and (c) are Conjecture 1.1. The two expressions for ξ agree by (a), because $\binom{-2}{k} = (-1)^k(k+1)$.

Example 1.4. For one vertex, $S_{K_1}(N) = \binom{N}{2}$ and $\xi(K_1) = 3/(-32) = -\frac{3}{32}$, the value of w on one chord. Since $S_{G \sqcup H} = S_G S_H$, ξ is multiplicative, as w is. For two adjacent vertices, the pairs $(p(x), p(y))$ with $|p(x) \cap p(y)| = 2, 1, 0$ number $\binom{N}{2}$, $\binom{N}{2} \cdot 2(N-2)$ and $\binom{N}{2} \binom{N-2}{2}$, so $S_{K_2}(N) = \binom{N}{2}(1 - 2(N-2) + \binom{N-2}{2})$ and $\xi(K_2) = 3 \cdot 19/1024 = \frac{57}{1024}$. This is $c(c - \frac{1}{2})$ at $c = -\frac{3}{32}$, the value of w on two crossing chords.³

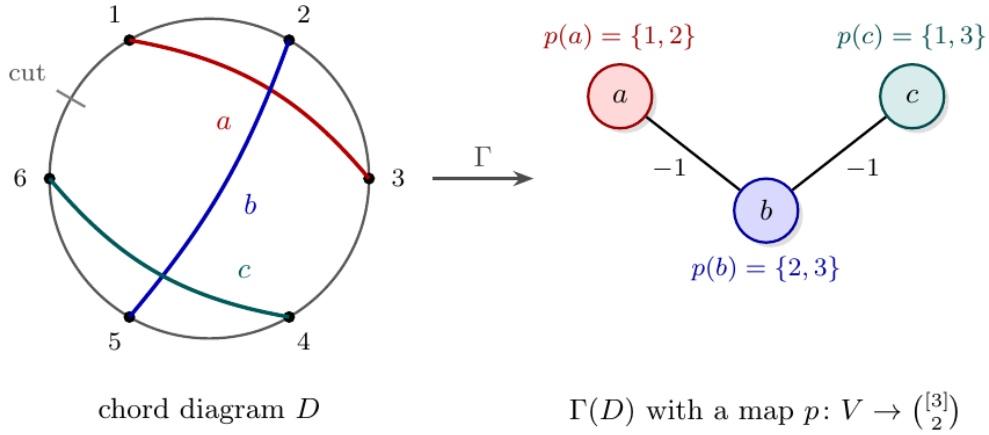


Figure 1: A chord diagram with three chords, cut between the points 6 and 1, and its intersection graph, a path. The map p sends each vertex to a 2-element subset of $[3]$; both edges meet in one element, so p contributes $(-1)(-1) = 1$ to $S_{\Gamma(D)}(3)$.

The proof of (b) is an involution on assignments (Section 6). For (c) we let the element $C_D \in U(\mathfrak{sl}(2))$ of a chord diagram act on the Verma module of highest weight μ ; it acts by a polynomial $\lambda_D(\mu)$, and $w_D(-\frac{3}{32}) = \lambda_D(-\frac{1}{2})$ (Section 2). The oscillator representation on $\mathbb{Q}[z]$ is the sum of the Verma modules of highest weights $-\frac{1}{2}$ and $-\frac{3}{2}$, so the image of C_D in the Weyl algebra is the scalar $\lambda_D(-\frac{1}{2})$. A supertrace on the Weyl algebra with a Wick formula writes this scalar as a sum over perfect matchings, in which every cycle contributes a factor -2 (Sections 3 and 4). With N in place of -2 the same sum is a trace in the Clifford algebra on N generators, and that trace equals $4^n S_\Gamma(N)$ (Section 5). Figure 2 shows the two halves of the argument.⁴

The combinatorial objects of the proof, chord diagrams with their legs, perfect matchings, their cycle and crossing counts, are finite and the proof uses nothing beyond linear algebra over $\mathbb{Q}$ and counting. Related constructions in which a finite complex or a group action is studied through counts of this kind appear in the authors' earlier work [13, 14, 15].

³On two crossing chords $C_D = C^2 - \frac{1}{2}C$ in the notation of Section 2; this is the standard computation for the $\mathfrak{sl}(2)$ weight system, see [3, Section 6.1], with the Killing form normalisation that gives the value c on one chord.

⁴The step from N to -2 is the point of the proof. The Clifford algebra on N generators is a quotient of the Weyl algebra's fermionic counterpart; the Weyl algebra itself behaves like a Clifford algebra on -2 generators, because in the Wick formula of Lemma 3.4(c) every closed cycle of contractions carries the factor -2 instead of N . This is the familiar sign that supertraces of bosonic systems carry negative dimensions. Nothing in the proof uses this heuristic; the identity is proved as an identity of polynomials in N and then evaluated at -2 .

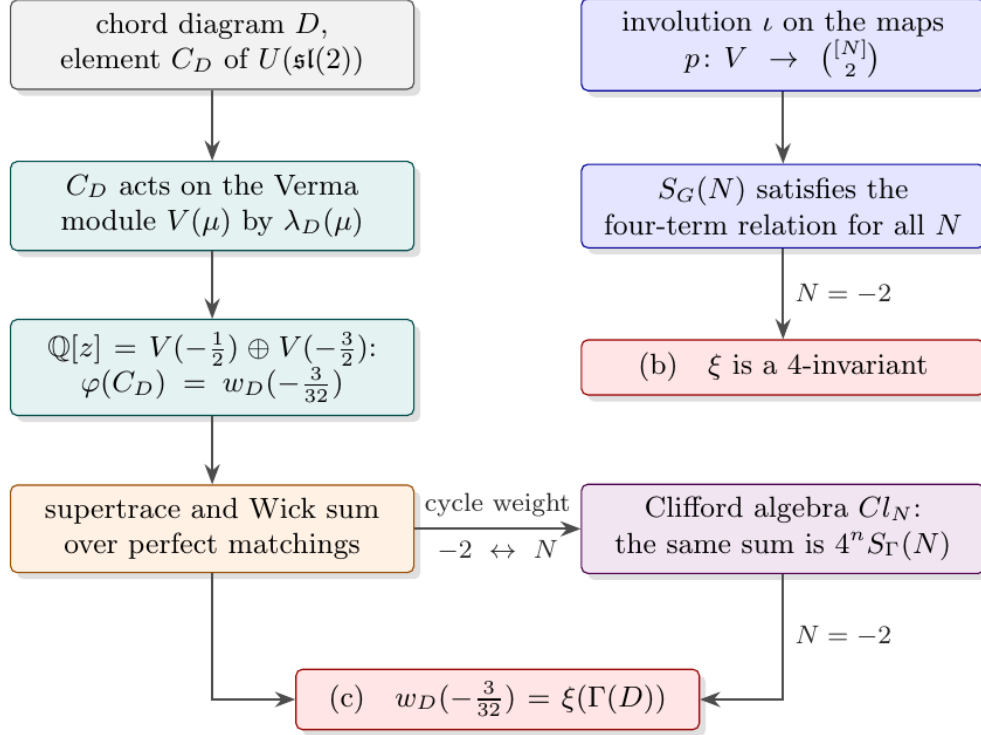


Figure 2: The proof of Theorem 1.3. The right column gives part (b), the left column and the bottom row part (c).

2 The weight system and Verma modules

2.1 Chord diagrams and the weight system

A chord diagram with n chords is a fixed-point-free involution σ of $[2n]$; its chords are the pairs $\{p, \sigma(p)\}$. It is a chord diagram on an oriented circle cut at a point that is not an endpoint; cutting at another point rotates $[2n]$ cyclically. Chords $\{p < q\}$ and $\{p' < q'\}$ cross if $p < p' < q < q'$ or $p' < p < q' < q$. The intersection graph $\Gamma(D)$ has the chords as vertices and the crossing pairs as edges; it does not depend on the cut.⁵

Let $\mathfrak{g} = \mathfrak{sl}(2)$ over $\mathbb{Q}$ with basis E, F, H , $[H, E] = 2E$, $[H, F] = -2F$, $[E, F] = H$, and let

$$t = \frac{1}{4} E \otimes F + \frac{1}{4} F \otimes E + \frac{1}{8} H \otimes H \in \mathfrak{g} \otimes \mathfrak{g}$$

be the Casimir tensor of the Killing form.⁶ It is symmetric and $\mathfrak{g}$ -invariant: $(\text{ad}_x \otimes 1 + 1 \otimes \text{ad}_x)t = 0$ for $x \in \mathfrak{g}$. For a chord diagram D let $t_D \in \mathfrak{g}^{\otimes 2n}$ be the tensor with a copy of t in the factors $p, \sigma(p)$ of every chord, and let $C_D \in U(\mathfrak{g})$ be the image of t_D under $u_1 \otimes \cdots \otimes u_{2n} \mapsto u_1 \cdots u_{2n}$. For one chord $C_D = C = \frac{1}{4}(EF + FE + \frac{1}{2}H^2)$.

For $d \geq 1$ let V_d be the irreducible representation of dimension d , with basis $e_0, \dots, e_{d-1}$ and

$$Ee_j = j(d-j)e_{j-1}, \quad Fe_j = e_{j+1}, \quad He_j = (d-1-2j)e_j,$$

⁵Two chords cross on the circle exactly when their endpoints alternate around the circle, and alternation is unchanged by a cyclic rotation of the labels.

⁶The Killing form of $\mathfrak{sl}(2)$ has $\kappa(E, F) = 4$ and $\kappa(H, H) = 8$, and t is the element dual to it: $t = \sum_i x_i \otimes x^i$ for a basis x_i and its dual basis x^i with respect to κ . Any other invariant form is a multiple of κ , and rescaling the form rescales w_D by a power of the scale; the four special values of [8] are stated for the Killing form.

where $e_{-1} = e_d = 0$. The Casimir element C acts on V_d by $c_d = (d^2 - 1)/8$.⁷ Put $W_d(D) = \text{tr}_{V_d}(C_D)/d$. Rotating the cut permutes the factors of t_D cyclically, so $W_d(D)$ depends only on the diagram on the circle.

By Proposition 2.3 below there is a unique $w_D \in \mathbb{Q}[c]$ with $w_D(c_d) = W_d(D)$ for all $d \geq 1$. It is the $\mathfrak{sl}(2)$ -weight system of [11, 8]: by Lemma 2.1 C_D lies in the centre of $U(\mathfrak{g})$, which is $\mathbb{Q}[C]$ [12], so $C_D = w(C)$ for a polynomial w , and taking traces on V_d gives $w(c_d) = W_d(D)$; the weight system of [8] is defined with the Killing form and takes the value c on one chord.

Lemma 2.1. C_D commutes with every $x \in \mathfrak{g}$.

Proof. ad_x is a derivation of $U(\mathfrak{g})$, so $[x, C_D]$ is the image of $\sum_{p=1}^{2n} (\text{ad}_x)_p t_D$, where $(\text{ad}_x)_p$ acts on the factor p . Grouping the two endpoints of every chord writes this sum as a sum over chords of tensors containing $(\text{ad}_x \otimes 1 + 1 \otimes \text{ad}_x)t = 0$. $\square$

2.2 Verma modules

Let $R = \mathbb{Q}[\mu]$ and let M be the free R -module with basis $v_0, v_1, \dots$ and

$$Ev_j = j(\mu - j + 1)v_{j-1}, \quad Fv_j = v_{j+1}, \quad Hv_j = (\mu - 2j)v_j. \quad (1)$$

These operators satisfy the relations of $\mathfrak{g}$:

$$(EF - FE)v_j = ((j+1)(\mu - j) - j(\mu - j + 1))v_j = (\mu - 2j)v_j = Hv_j,$$

and $(HE - EH)v_j = 2Ev_j$, $(HF - FH)v_j = -2Fv_j$, because H has eigenvalue $\mu - 2j + 2$ on v_{j-1} and $\mu - 2j - 2$ on v_{j+1} . So M is a $U(\mathfrak{g})$ -module on which $U(\mathfrak{g})$ acts R -linearly. For $\mu_0 \in \mathbb{Q}$, specialising $\mu = \mu_0$ gives the Verma module $V(\mu_0)$ of highest weight μ_0 .

Proposition 2.2. *There is $\lambda_D \in \mathbb{Q}[\mu]$ with $C_D m = \lambda_D m$ for all $m \in M$. Moreover*

(a) $W_d(D) = \lambda_D(d - 1)$ for every $d \geq 1$;

(b) $\lambda_D(\mu) = \lambda_D(-2 - \mu)$.

Proof. $H - \mu$ multiplies v_j by $-2j$, so its kernel is Rv_0 . As C_D commutes with H , $C_D v_0 = \lambda_D v_0$ for some $\lambda_D \in R$, and $C_D v_j = C_D F^j v_0 = F^j C_D v_0 = \lambda_D v_j$.

(a) In $V(d - 1)$, $Ev_d = d(d - 1 - d + 1)v_{d-1} = 0$, so $U_d = \text{span}\{v_j : j \geq d\}$ is a submodule. The quotient $V(d - 1)/U_d$, with basis the images of $v_0, \dots, v_{d-1}$, is V_d with the formulas above. C_D acts on $V(d - 1)$, hence on V_d , by $\lambda_D(d - 1)$, so $\text{tr}_{V_d} C_D = d \lambda_D(d - 1)$.

(b) The map $V(-1 - d) \rightarrow U_d$, $v_j \mapsto v_{d+j}$, is an isomorphism of $\mathfrak{g}$ -modules: it commutes with F ; both Hv_j in $V(-1 - d)$ and Hv_{d+j} in $V(d - 1)$ have eigenvalue $-1 - d - 2j$; and $Ev_j = j(-d - j)v_{j-1}$ in $V(-1 - d)$ while $Ev_{d+j} = (d + j)(-j)v_{d+j-1}$ in $V(d - 1)$. Hence $\lambda_D(-1 - d) = \lambda_D(d - 1)$ for all $d \geq 1$, and the polynomial $\lambda_D(\mu) - \lambda_D(-2 - \mu)$ has infinitely many roots.⁸ $\square$

Proposition 2.3. *There is a unique $w_D \in \mathbb{Q}[c]$ with $w_D(c_d) = W_d(D)$ for all $d \geq 1$, and $\deg w_D \leq n$. It satisfies $\lambda_D(\mu) = w_D(\mu(\mu + 2)/8)$; in particular*

$$w_D(-\frac{3}{32}) = \lambda_D(-\frac{1}{2}) = \lambda_D(-\frac{3}{2}).$$

⁷On the highest weight vector e_0 one has $Fe_0 = e_1$, $EF e_0 = (d - 1)e_0$, $FE e_0 = 0$ and $H^2 e_0 = (d - 1)^2 e_0$, so $C e_0 = \frac{1}{4}((d - 1) + \frac{1}{2}(d - 1)^2)e_0 = \frac{(d - 1)(d + 1)}{8}e_0$; since C is central and V_d is irreducible, C acts by this scalar everywhere.

⁸This is the symmetry of the Harish-Chandra isomorphism for $\mathfrak{sl}(2)$: the central character of $V(\mu)$ depends on μ only through $\mu(\mu + 2)$, see [12, Section 1.10]. The direct argument above avoids the general theory.

Proof. By Proposition 2.2(b), $\nu \mapsto \lambda_D(\nu - 1)$ is even, so $\lambda_D(\nu - 1) = g(\nu^2)$ for some $g \in \mathbb{Q}[x]$. Put $w_D(c) = g(8c + 1)$. Then $w_D(\mu(\mu + 2)/8) = g((\mu + 1)^2) = \lambda_D(\mu)$, and $w_D(c_d) = \lambda_D(d - 1) = W_d(D)$ by Proposition 2.2(a). Uniqueness holds because the c_d are pairwise distinct. Each term of $C_D v_0$ is obtained by applying $2n$ operators from (1); their coefficients have degree at most one in μ . So $\deg \lambda_D \leq 2n$. Translation by 1 preserves this degree bound, so $\lambda_D(\nu - 1) = g(\nu^2)$ gives $\deg g \leq n$, hence $\deg w_D \leq n$. In particular the values $W_d(D)$ for $1 \leq d \leq n + 1$ determine w_D by polynomial interpolation. Finally $\mu(\mu + 2)/8 = -\frac{3}{32}$ for $\mu = -\frac{1}{2}$ and for $\mu = -\frac{3}{2}$. $\square$

3 The oscillator representation and a supertrace

Let A be the Weyl algebra over $\mathbb{Q}$ generated by z and ∂ with $\partial z - z\partial = 1$. It acts on $\mathbb{Q}[z]$, with z acting by multiplication and ∂ by differentiation, and the monomials $z^i \partial^j$ ($i, j \geq 0$) span A . In A ,

$$z^i \partial^j z = z^{i+1} \partial^j + j z^i \partial^{j-1}, \quad \partial z^i \partial^j = z^i \partial^{j+1} + i z^{i-1} \partial^j. \quad (2)$$

Lemma 3.1. *The operators $z^i \partial^j$, $i, j \geq 0$, are linearly independent on $\mathbb{Q}[z]$. In particular the monomials $z^i \partial^j$ form a basis of A and the action of A on $\mathbb{Q}[z]$ is faithful.*

Proof. The relation $\partial z = z\partial + 1$ rewrites every word as a linear combination of these monomials. Suppose a finite sum $B = \sum_{i,j} a_{ij} z^i \partial^j$ acts by zero. On z^k , for all large enough integers k , the coefficient of z^{k+r} is

$$\sum_{i-j=r} a_{ij} k(k-1) \cdots (k-j+1).$$

It vanishes for infinitely many k , so it is the zero polynomial in k . For each fixed r the falling factorials in this sum have distinct degrees j and leading coefficient 1; hence every a_{ij} is zero. This also shows that the normal forms are unique, so the action of the abstract Weyl algebra is faithful. $\square$

Lemma 3.2. *The assignment $\varphi(E) = -\frac{1}{4}\partial^2$, $\varphi(F) = z^2$, $\varphi(H) = -\frac{1}{2}(z\partial + \partial z)$ extends to an algebra homomorphism $\varphi: U(\mathfrak{g}) \rightarrow A$. On $\mathbb{Q}[z]$, the maps $v_j \mapsto z^{2j}$ and $v_j \mapsto z^{2j+1}$ are isomorphisms of $\mathfrak{g}$ -modules $V(-\frac{1}{2}) \rightarrow \mathbb{Q}[z]_{\text{even}}$ and $V(-\frac{3}{2}) \rightarrow \mathbb{Q}[z]_{\text{odd}}$.*

Proof. On $\mathbb{Q}[z]$,

$$Ez^k = -\frac{1}{4}k(k-1)z^{k-2}, \quad Fz^k = z^{k+2}, \quad Hz^k = -(k + \frac{1}{2})z^k.$$

Hence $(EF - FE)z^k = -\frac{1}{4}((k+2)(k+1) - k(k-1))z^k = Hz^k$, and $(HE - EH)z^k = 2Ez^k$, $(HF - FH)z^k = -2Fz^k$, because H has eigenvalue $-(k - \frac{3}{2})$ on z^{k-2} and $-(k + \frac{5}{2})$ on z^{k+2} . The relations of $\mathfrak{g}$ hold on $\mathbb{Q}[z]$, hence in A by Lemma 3.1. For $k = 2j$ the formulas read $Ez^{2j} = j(-\frac{1}{2} - j + 1)z^{2j-2}$ and $H z^{2j} = (-\frac{1}{2} - 2j)z^{2j}$, which is (1) at $\mu = -\frac{1}{2}$; for $k = 2j + 1$ they read $Ez^{2j+1} = j(-\frac{3}{2} - j + 1)z^{2j-1}$ and $H z^{2j+1} = (-\frac{3}{2} - 2j)z^{2j+1}$, which is (1) at $\mu = -\frac{3}{2}$.⁹ $\square$

Proposition 3.3. $\varphi(C_D) = w_D(-\frac{3}{32}) \cdot 1$ in A .

⁹This is the oscillator, or metaplectic, representation of $\mathfrak{sl}(2)$: the even and odd polynomials are the two irreducible pieces, of lowest weights $\frac{1}{2}$ and $\frac{3}{2}$ in the usual convention, which here appear as highest weights $-\frac{1}{2}$ and $-\frac{3}{2}$ because $F = z^2$ raises the degree. The Casimir element acts on both pieces by $-\frac{3}{32}$, as the computation $\mu(\mu + 2)/8$ at $\mu = -\frac{1}{2}, -\frac{3}{2}$ shows.

Proof. By Proposition 2.2 and Lemma 3.2, $\varphi(C_D)$ acts on $\mathbb{Q}[z]_{\text{even}}$ by $\lambda_D(-\frac{1}{2})$ and on $\mathbb{Q}[z]_{\text{odd}}$ by $\lambda_D(-\frac{3}{2})$. Both equal $w_D(-\frac{3}{2})$ by Proposition 2.3, and Lemma 3.1 gives faithfulness. $\square$

Let $\tau: A \rightarrow \mathbb{Q}$ be the linear map with

$$\tau(z^i \partial^j) = \delta_{ij} (-1)^j \frac{j!}{2^j}.$$

Since $z^j \partial^j$ multiplies z^k by $k(k-1)\cdots(k-j+1)$ and $\sum_k k(k-1)\cdots(k-j+1)x^k = j! x^j (1-x)^{-j-1}$, the number $\tau(a)$ is twice the Abel sum $\lim_{x \rightarrow -1+} \sum_k a_{kk} x^k$ of the supertrace of a on $\mathbb{Q}[z]$, where $az^k = a_{kk} z^k + (\text{other powers})$.¹⁰ Only the following properties are used.

Lemma 3.4. (a) $\tau(1) = 1$.

(b) $\tau(xa) = -\tau(ax)$ for $x \in \{z, \partial\}$ and $a \in A$.

(c) For $u_1, \dots, u_m \in \{z, \partial\}$,

$$\tau(u_1 \cdots u_m) = \sum_M \prod_{\{a < b\} \in M} \frac{1}{2} [u_a, u_b],$$

the sum over the perfect matchings M of $[m]$; here $[z, \partial] = -1$, $[\partial, z] = 1$ and $[z, z] = [\partial, \partial] = 0$. For odd m both sides vanish.

Proof. (a) is the case $i = j = 0$. For (b) it is enough to take $a = z^i \partial^j$. By (2), $\tau(z^i \partial^j z) = \tau(z^{i+1} \partial^j) + j \tau(z^i \partial^{j-1})$. Both terms vanish unless $j = i + 1$, and then the sum is $(-1)^j \frac{j!}{2^j} + j(-1)^{j-1} \frac{(j-1)!}{2^{j-1}} = -(-1)^j \frac{j!}{2^j} = -\tau(z \cdot z^i \partial^j)$. In the same way $\tau(\partial z^i \partial^j) = \tau(z^i \partial^{j+1}) + i \tau(z^{i-1} \partial^j)$ vanishes unless $i = j + 1$, and then it equals $(-1)^{j+1} \frac{(j+1)!}{2^{j+1}} + (j+1)(-1)^j \frac{j!}{2^j} = (-1)^j \frac{(j+1)!}{2^{j+1}} = -\tau(z^i \partial^j \cdot \partial)$.

(c) Induction on m . For $m = 0$ use (a); for $m = 1$, $\tau(z) = \tau(\partial) = 0$. For $m \geq 2$, as the $[u_1, u_s]$ are scalars,

$$u_1 u_2 \cdots u_m = \sum_{s=2}^m [u_1, u_s] u_2 \cdots \widehat{u_s} \cdots u_m + u_2 \cdots u_m u_1.$$

Applying τ and (b) to the last term gives

$$\tau(u_1 \cdots u_m) = \frac{1}{2} \sum_{s=2}^m [u_1, u_s] \tau(u_2 \cdots \widehat{u_s} \cdots u_m),$$

which is the expansion of the right side of (c) according to the partner s of 1. $\square$

Corollary 3.5. $w_D(-\frac{3}{2}) = \tau(\varphi(C_D))$.

4 Expansion over matchings

4.1 Matchings

Let X be a finite totally ordered set and P, M perfect matchings of X . Two edges $\{a < b\}$ and $\{a' < b'\}$ cross if $a < a' < b < b'$ or $a' < a < b' < b$, and $\text{cr}(P)$ is the number of crossing pairs of edges of P . The multigraph $P \cup M$ is a disjoint union of cycles alternating between

¹⁰At $x = -1$ the series $\sum_k k(k-1)\cdots(k-j+1)x^k$ does not converge, which is why τ is defined directly by the formula and the Abel sum is only mentioned as the origin of the formula. The three properties in Lemma 3.4 are proved from the formula alone.

edges of P and of M (a common edge gives a cycle of length 2); $\text{cyc}(P, M)$ is their number. Orient every cycle; an edge oriented from a larger to a smaller element is a descent, and $\text{des}(P, M)$ is the number of descents. Reversing a cycle of length $2r$ turns δ descents into $2r - \delta$, so the parity of $\text{des}(P, M)$ does not depend on the orientations. Put

$$L_P(N) = \sum_M (-1)^{\text{cr}(M)} N^{\text{cyc}(P, M)} \in \mathbb{Z}[N],$$

the sum over all perfect matchings M of X .

Lemma 4.1. $\text{cr}(P) + \text{cr}(M) + \text{cyc}(P, M) + \text{des}(P, M)$ is even.

Proof. Let $Q(P, M)$ be the sum. Let $x < y$ be consecutive in X , and exchange their positions in the order, keeping P and M . If $\{x, y\} \notin P$, let $e \ni x$ and $f \ni y$ be the edges of P ; the exchange moves y to the other side of the endpoint x of e and keeps the side of the other endpoint of f , so it changes whether e and f cross, and no other pair. So $\text{cr}(P)$ changes by $1 - [\{x, y\} \in P]$ modulo 2, and likewise $\text{cr}(M)$. $\text{cyc}(P, M)$ does not change. A descent status changes only on edges joining x and y , so $\text{des}(P, M)$ changes by $[\{x, y\} \in P] + [\{x, y\} \in M]$. The total change is 2. Any two total orders of X are related by exchanges of consecutive elements, so Q modulo 2 is the same for all orders. Choose an order in which every cycle $x_1 x_2 \cdots x_{2r}$ (with $x_1 x_2 \in P$) occupies consecutive positions in this order. Then P and M have no crossings, and each cycle, oriented as listed, has exactly one descent, from x_{2r} to x_1 . Hence $Q \equiv 2 \text{cyc}(P, M) \equiv 0$. $\square$

4.2 The bosonic expansion

For a perfect matching P of X with k edges let

$$X_P = \sum_o \text{sgn}(o) \prod_{x \in X}^{\rightarrow} u_x^o \in A,$$

the sum over the 2^k orientations o of the edges of P , where $u_x^o = z$ if x is the tail of its edge and $u_x^o = \partial$ if it is the head, the product is taken in the order of X , and $\text{sgn}(o) = (-1)^{\text{number of descents of } o}$.

Proposition 4.2. $\tau(X_P) = 2^{-k} (-1)^{\text{cr}(P)} L_P(-2)$.

Proof. By Lemma 3.4(c),

$$\tau(X_P) = \sum_o \sum_M \text{sgn}(o) \prod_{\{a < b\} \in M} \frac{1}{2} [u_a^o, u_b^o].$$

A term vanishes unless every edge of M joins a head and a tail of o . Orient such an edge from the head to the tail; then $\frac{1}{2} [u_a^o, u_b^o]$ is $\frac{1}{2}$ if it is oriented from a to b ($u_a^o = \partial$, $u_b^o = z$) and $-\frac{1}{2}$ if it is a descent. As $\text{sgn}(o)$ is -1 to the number of descents among the edges of P , the term equals $2^{-k} (-1)^{\text{des}(P, M)}$. Every element of X is then entered by exactly one oriented edge and left by exactly one, so the cycles of $P \cup M$ are directed cycles. Conversely, each of the $2^{\text{cyc}(P, M)}$ orientations of the cycles of $P \cup M$ as directed cycles restricts on P to an orientation o for which every edge of M joins a head to a tail. Hence

$$\begin{aligned} \tau(X_P) &= 2^{-k} \sum_M 2^{\text{cyc}(P, M)} (-1)^{\text{des}(P, M)} \\ &= 2^{-k} (-1)^{\text{cr}(P)} \sum_M (-1)^{\text{cr}(M)} (-2)^{\text{cyc}(P, M)} \end{aligned}$$

by Lemma 4.1. $\square$

4.3 Legs

Let D be a chord diagram with n chords. Every point $p \in [2n]$ carries two legs $(p, 1) < (p, 2)$, and the set X_D of the $4n$ legs is ordered lexicographically. For a chord $c = \{p < q\}$ and $\beta \in \{0, 1\}$ let P_c^β be the pair of edges

$$P_c^0 = \{(p, 1)(q, 1), (p, 2)(q, 2)\}, \quad P_c^1 = \{(p, 1)(q, 2), (p, 2)(q, 1)\},$$

and for $\beta: \{\text{chords}\} \rightarrow \{0, 1\}$ let $P_\beta = \bigcup_c P_c^{\beta(c)}$, a perfect matching of X_D . Let $\pi = z \otimes \partial - \partial \otimes z \in A \otimes A$. Writing $\pi = \sum_r \varepsilon_r x_r \otimes y_r$, put $\pi_{13}\pi_{24} = \sum_{r,s} \varepsilon_r \varepsilon_s x_r x_s \otimes y_r y_s$ and $\pi_{14}\pi_{23} = \sum_{r,s} \varepsilon_r \varepsilon_s x_r x_s \otimes y_s y_r$.

Lemma 4.3. $(\varphi \otimes \varphi)(t) = -\frac{1}{32}(\pi_{13}\pi_{24} + \pi_{14}\pi_{23})$.

Proof. Both sides equal $-\frac{1}{32}(2z^2 \otimes \partial^2 + 2\partial^2 \otimes z^2 - (z\partial + \partial z) \otimes (z\partial + \partial z))$. On the left, $\frac{1}{4}\varphi(E) \otimes \varphi(F) + \frac{1}{4}\varphi(F) \otimes \varphi(E) = -\frac{1}{16}(\partial^2 \otimes z^2 + z^2 \otimes \partial^2)$ and $\frac{1}{8}\varphi(H) \otimes \varphi(H) = \frac{1}{32}(z\partial + \partial z)^{\otimes 2}$. On the right, the terms with $r = s$ give $2z^2 \otimes \partial^2 + 2\partial^2 \otimes z^2$ and the terms with $r \neq s$ give $-z\partial \otimes (\partial z + z\partial) - \partial z \otimes (z\partial + \partial z)$. $\square$

Corollary 4.4. $\varphi(C_D) = (-\frac{1}{32})^n \sum_\beta X_{P_\beta}$, the sum over all $\beta: \{\text{chords}\} \rightarrow \{0, 1\}$.

Proof. $\varphi(C_D)$ is the product, in the order of the points, of the factors of $(\varphi \otimes \varphi)(t)^{\otimes n}$ placed at the endpoints of the chords. Put the two letters of each factor $x_r x_s$ or $y_r y_s$ on the two legs of its point, in this order. By Lemma 4.3, a chord contributes $-\frac{1}{32}$ times the sum of π placed on the two edges of P_c^0 and π placed on the two edges of P_c^1 . A copy of π on an edge $\{a < b\}$ contributes z at a and ∂ at b with sign $+1$, or ∂ at a and z at b with sign -1 ; these are the two orientations of the edge with their signs in X_{P_β} . $\square$

5 Clifford algebras

For $N \geq 1$ let Cl_N be the $\mathbb{Q}$ -algebra generated by $e_1, \dots, e_N$ with $e_i e_j + e_j e_i = 2\delta_{ij}$. For $I = \{i_1 < \dots < i_r\} \subseteq [N]$ put $e_I = e_{i_1} \cdots e_{i_r}$; these form a basis, and $e_I e_J = \pm e_{I \Delta J}$. Let $\text{tr}: Cl_N \rightarrow \mathbb{Q}$ be linear with $\text{tr}(e_I) = \delta_{I\emptyset}$. Then $\text{tr}(ab) = \text{tr}(ba)$: for basis elements both sides vanish unless $I = J$.¹¹

Lemma 5.1. For $i_1, \dots, i_m \in [N]$,

$$\text{tr}(e_{i_1} \cdots e_{i_m}) = \sum_M (-1)^{\text{cr}(M)} \prod_{\{a,b\} \in M} \delta_{i_a i_b},$$

the sum over the perfect matchings M of $[m]$.

Proof. For odd m the product is $\pm e_I$ with $|I|$ odd, so both sides vanish. For even $m \geq 2$, moving e_{i_1} to the right end gives

$$e_{i_1} e_{i_2} \cdots e_{i_m} = \sum_{s=2}^m (-1)^s 2\delta_{i_1 i_s} e_{i_2} \cdots \widehat{e_{i_s}} \cdots e_{i_m} - e_{i_2} \cdots e_{i_m} e_{i_1}.$$

Taking traces and using $\text{tr}(e_{i_2} \cdots e_{i_m} e_{i_1}) = \text{tr}(e_{i_1} \cdots e_{i_m})$,

$$\text{tr}(e_{i_1} \cdots e_{i_m}) = \sum_{s=2}^m (-1)^s \delta_{i_1 i_s} \text{tr}(e_{i_2} \cdots \widehat{e_{i_s}} \cdots e_{i_m}).$$

¹¹For $I = J$ both products are $e_I e_I = \pm 1$ with the same sign, since $e_I e_I$ and $e_I e_I$ are the same word. For $I \neq J$ the products are $\pm e_{I \Delta J}$ with $I \Delta J \neq \emptyset$, so both traces vanish. The trace is 2^{-N} times the trace of the left regular representation.

The edge $\{1, s\}$ crosses exactly the edges with one endpoint in $\{2, \dots, s-1\}$; the elements of this set matched among themselves come in pairs, so the number of such edges is congruent to $s-2$ modulo 2. By induction both sides satisfy the same recursion. $\square$

Corollary 5.2. *For a perfect matching P of a totally ordered set $X = \{x_1 < \dots < x_{2k}\}$ and $N \geq 1$,*

$$\sum_{\kappa} \text{tr}(e_{\kappa(x_1)} \cdots e_{\kappa(x_{2k})}) = L_P(N),$$

the sum over the maps $\kappa: X \rightarrow [N]$ that are constant on every edge of P .

Proof. By Lemma 5.1 the left side is $\sum_M (-1)^{\text{cr}(M)}$ times the number of maps $X \rightarrow [N]$ constant on the edges of P and of M , which is $N^{\text{cyc}(P, M)}$. $\square$

For $I = \{i < j\} \in \binom{[N]}{2}$ write $b_I = e_i e_j$. Then

$$b_I b_J = (-1)^{|I \cap J|} b_J b_I, \quad b_I^2 = -1. \quad (3)$$

Proposition 5.3. *Let D be a chord diagram with n chords and $\Gamma = \Gamma(D)$. Then*

$$\sum_{\beta} (-1)^{\text{cr}(P_{\beta})} L_{P_{\beta}}(N) = 4^n S_{\Gamma}(N)$$

as polynomials in N .

Proof. Both sides are polynomials (Theorem 1.3(a) is proved in Section 6 on its own), so it is enough to take $N \geq 1$.

The two edges of P_c^{β} cross if and only if $\beta = 0$. If chords c and c' cross, every edge of P_c^{β} crosses every edge of $P_{c'}^{\beta'}$; otherwise none does. Hence

$$(-1)^{\text{cr}(P_{\beta})} = (-1)^n \prod_c (-1)^{\beta(c)}.$$

A map κ constant on the edges of P_{β} is given by a pair $(i_c, j_c) \in [N]^2$ for every chord $c = \{p < q\}$: i_c on the edge through $(p, 1)$ and j_c on the edge through $(p, 2)$. The legs at p then read $e_{i_c} e_{j_c}$, and the legs at q read $e_{i_c} e_{j_c}$ if $\beta(c) = 0$ and $e_{j_c} e_{i_c}$ if $\beta(c) = 1$. By Corollary 5.2, summing over $\beta(c)$ with the sign $(-1)^{\beta(c)}$ replaces the factor at q by $e_{i_c} e_{j_c} - e_{j_c} e_{i_c}$, which is 0 if $i_c = j_c$ and $2e_{i_c} e_{j_c}$ otherwise. Exchanging i_c and j_c changes the sign of both factors of c , so every 2-element set $s(c) = \{i_c, j_c\}$ arises twice with the same value. Therefore

$$\sum_{\beta} (-1)^{\text{cr}(P_{\beta})} L_{P_{\beta}}(N) = (-1)^n 4^n \sum_s \text{tr}(B_s), \quad B_s = \prod_{p=1}^{2n} b_{s(c_p)},$$

the sum over $s: \{\text{chords}\} \rightarrow \binom{[N]}{2}$, where c_p is the chord through p .

We show $\text{tr}(B_s) = (-1)^n \prod_{cc' \in E(\Gamma)} (-1)^{|s(c) \cap s(c')|}$ by induction on n . Let $c = \{p < q\}$. Move the factor at p to the right until it is next to the factor at q . By (3) it passes the factors at the points strictly between p and q ; a chord with both endpoints there contributes the square of a sign, and the chords with one endpoint there are the neighbours of c in Γ . Then $b_{s(c)}^2 = -1$, and the remaining word is $B_{s'}$ for the diagram $D \setminus c$, whose intersection graph is $\Gamma - c$, with $s' = s|_{\Gamma - c}$. Summing over s gives $\sum_s \text{tr}(B_s) = (-1)^n S_{\Gamma}(N)$. $\square$

6 Proof of Theorem 1.3

Proof of (a). Group the maps $p: V \rightarrow \binom{[N]}{2}$ by $U = \bigcup_x p(x)$. Relabelling U in increasing order by $||U||$ preserves all intersection sizes, so the maps with $|U| = k$ contribute $\binom{N}{k} T_k(G)$; and $T_k(G) = 0$ for $k > 2n$. $\square$

Proof of (c). By Corollaries 3.5 and 4.4, Proposition 4.2 with $k = 2n$, and Proposition 5.3,

$$\begin{aligned} w_D(-\tfrac{3}{32}) &= \left(-\tfrac{1}{32}\right)^n \sum_{\beta} \tau(X_{P_{\beta}}) = \left(-\tfrac{1}{32}\right)^n 4^{-n} \sum_{\beta} (-1)^{\text{cr}(P_{\beta})} L_{P_{\beta}}(-2) \\ &= \frac{S_{\Gamma}(-2)}{(-32)^n} = \xi(\Gamma). \end{aligned} \quad \square$$

For a graph G and distinct vertices a, b let G'_{ab} be G with the adjacency of a and b reversed, and $\tilde{G}_{ab}$ be G with the adjacency of a and y reversed for every neighbour $y \neq a$ of b . The four-term relation for a function f on graphs is

$$f(G) - f(G'_{ab}) = f(\tilde{G}_{ab}) - f(\tilde{G}'_{ab}), \quad \tilde{G}'_{ab} = (\tilde{G}_{ab})'_{ab},$$

as in Figure 3.¹²

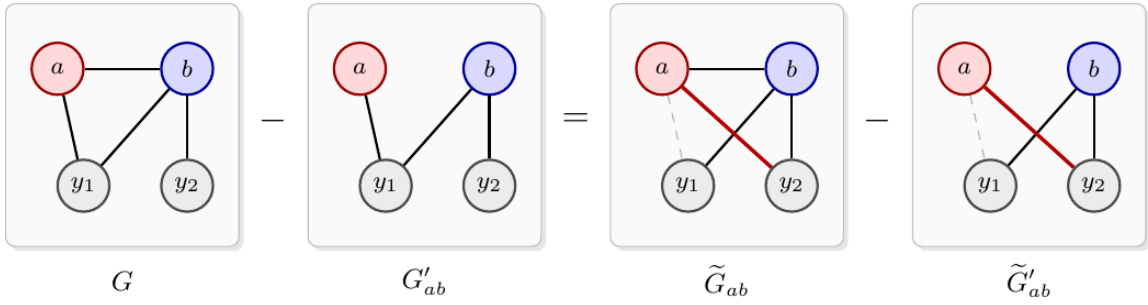


Figure 3: The four graphs of a four-term relation. Here b has the neighbours a, y_1, y_2 ; passing from G to $\tilde{G}_{ab}$ removes the edge ay_1 (dashed) and adds the edge ay_2 .

Proposition 6.1. *For every $N \in \mathbb{N}$, $G \mapsto S_G(N)$ satisfies the four-term relation.*

Proof. For $p: V \rightarrow \binom{[N]}{2}$ write

$$\epsilon(G, p) = \prod_{xy \in E(G)} (-1)^{|p(x) \cap p(y)|}.$$

Since $\epsilon(G'_{ab}, p) = \epsilon(G, p)(-1)^{|p(a) \cap p(b)|}$ and $|p(a) \cap p(b)| \in \{0, 1, 2\}$,

$$S_G(N) - S_{G'_{ab}}(N) = 2 \sum_{p \in \Pi} \epsilon(G, p), \quad \Pi = \{p : |p(a) \cap p(b)| = 1\}.$$

The construction $\tilde{G}_{ab}$ does not change the adjacency of a and b , so the same holds for $\tilde{G}_{ab}$. On Π define $\iota(p)$ by $\iota(p)(b) = p(a) \triangle p(b)$ and $\iota(p)(x) = p(x)$ for $x \neq b$. As $|p(a) \cap p(b)| = 1$,

¹²This is Lando's relation [5]. For intersection graphs it is the image of the four-term relation of chord diagrams: if the chords a and b share an endpoint region, then moving one endpoint of a past the endpoint of b reverses the adjacency of a with b , and moving it past the other endpoint of b reverses the adjacency of a with every chord that crosses b and not a , which is the passage to $\tilde{G}_{ab}$. So a function on graphs satisfying the relation gives a weight system.

$\iota(p)(b)$ is a 2-element set meeting $p(a)$ in one element, and ι is an involution of Π .¹³ Let Y be the set of neighbours $y \neq a$ of b in G , which is also the set of such neighbours in $\tilde{G}_{ab}$. For $y \in Y$, $|(p(a) \triangle p(b)) \cap p(y)| \equiv |p(a) \cap p(y)| + |p(b) \cap p(y)|$ modulo 2, and the edge ab , if present, keeps its sign because $|p(a) \cap \iota(p)(b)| = 1$. Hence

$$\begin{aligned}\epsilon(G, \iota(p)) &= \epsilon(G, p) \prod_{y \in Y} (-1)^{|p(a) \cap p(y)|}, \\ \epsilon(\tilde{G}_{ab}, q) &= \epsilon(G, q) \prod_{y \in Y} (-1)^{|q(a) \cap q(y)|},\end{aligned}$$

the second for every q , since $\tilde{G}_{ab}$ differs from G exactly in the pairs ay , $y \in Y$. With $q = \iota(p)$, which agrees with p at a and on Y , this gives $\epsilon(\tilde{G}_{ab}, \iota(p)) = \epsilon(G, p)$. Therefore $\sum_{p \in \Pi} \epsilon(\tilde{G}_{ab}, p) = \sum_{p \in \Pi} \epsilon(G, p)$, and the two differences agree. $\square$

Proof of (b). ξ depends only on the isomorphism class of G . The four graphs of a four-term relation have the same vertex set, so by Proposition 6.1 and (a) the four-term relation holds for the polynomials S_G , hence at $N = -2$, and after division by $(-32)^n$ for ξ . $\square$

Remark 6.2. Proposition 6.1 applied to the sets Π restricted to the maps with $\bigcup_x p(x) = [k]$, which ι preserves, shows that every T_k satisfies the four-term relation. By the proof of Proposition 5.3, for $N \geq 1$ the 4-invariant $S_\Gamma(N)$ is $(-4)^{-n}$ times the Clifford trace of the element obtained by placing the tensor $\sum_{i \neq j} e_i e_j \otimes 2e_i e_j$ on the chords as in the definition of C_D , and Theorem 1.3(c) identifies $w_D(-\frac{3}{32})$ with $(-32)^{-n}$ times its value at $N = -2$.

Remark 6.3. The formula for ξ is a finite formula for every finite graph, with no bound on the number of vertices, since $T_k(G) = 0$ for $k > 2|V(G)|$. The involution of Proposition 6.1 is defined on every assignment with $|p(a) \cap p(b)| = 1$, so the proof of (b) does not rest on a table of graphs. The identities of Sections 2 to 5 hold for every chord diagram. The evaluation at $N = -2$ in the proof of (c) is substitution into a polynomial; no limit or analytic continuation is involved. The finite-dimensional normalised traces $W_d(D)$ are the starting definition, and the Verma module argument proves that they interpolate to a unique polynomial before its value at the oscillator representation is taken. The theorem says nothing about the unspecialised weight system, which does not extend to graphs by [8].

Declarations

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¹³If $p(a) = \{u, v\}$ and $p(b) = \{v, w\}$ with u, v, w distinct, then $\iota(p)(b) = \{u, w\}$, and applying ι again gives $\{u, v\} \triangle \{u, w\} = \{v, w\} = p(b)$.

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