

The $\mathfrak{sl}(2)$ weight system at $c = -3/32$ extends to a graph 4-invariant

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Abstract

We prove unconditionally the conjecture of Fomichev, Karev, Pavutnitskiy and Usanov that the specialization of the $\mathfrak{sl}(2)$ weight system at $c = -3/32$ extends to a 4-invariant of graphs. The extension is $\xi(G) = (-32)^{-n} S_G(-2)$, where $S_G(N)$ counts with signs the assignments of 2-element subsets of $\{1, \dots, N\}$ to the n vertices of G .

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1. Introduction

A weight system is a function on chord diagrams that satisfies the four-term relations; weight systems are the symbols of finite type knot invariants [1, 2, 3].³ A metrized Lie algebra gives a weight system with values in the

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³By Thurston's theorem the complement of a knot that is neither a torus knot nor a satellite knot carries a hyperbolic metric, so its fundamental group is a Gromov hyperbolic group; the equivariant classes of such groups under Mostow–Prasad rigidity are the subject

centre of its universal enveloping algebra. For $\mathfrak{sl}(2)$ with the Killing form the centre is the polynomial ring $\mathbb{Q}[c]$ in the Casimir element c , and the value $w_D \in \mathbb{Q}[c]$ of the weight system on a chord diagram D depends only on the intersection graph $\Gamma(D)$ [4].⁴ Lando's four-term relations for graphs [5] are such that every function f on graphs satisfying them gives a weight system $D \mapsto f(\Gamma(D))$; such f are called 4-invariants. Lando asked whether w is obtained in this way [4]; see [6] for a survey and [7] for graphs with at most 8 vertices.

Fomichev, Karev, Pavutnitskiy and Usanov [8] answered the question negatively. They found a linear combination of intersection graphs that vanishes modulo the four-term relations of graphs and on which w takes the value $c(c - \frac{3}{8})(c - 1)(c + \frac{3}{32})$. Hence the specialization of w at $c = c_0$ can extend to a 4-invariant only if $c_0 \in \{0, \frac{3}{8}, 1, -\frac{3}{32}\}$. For $c_0 = 0, \frac{3}{8}, 1$, the eigenvalues of the Casimir element on the irreducible representations of dimension 1, 2, 3, extensions are known [8, 9, 10]. The value $-\frac{3}{32}$ is the eigenvalue of the Casimir element on the oscillator representation. The authors of [8] constructed an extension for graphs with at most 10 vertices and stated:

Conjecture 1.1 ([8, Section 4]). *The specialization of the $\mathfrak{sl}(2)$ -weight system at $c = -\frac{3}{32}$ extends to a 4-invariant of graphs $\xi(G)$.*

We prove Conjecture 1.1 with an explicit formula. All graphs are finite and simple. For $N \in \mathbb{N}$ write $[N] = \{1, \dots, N\}$ and $\binom{[N]}{2}$ for the set of its 2-element subsets.

Definition 1.2. For a graph $G = (V, E)$ and $N \in \mathbb{N}$ put

$$S_G(N) = \sum_{p: V \rightarrow \binom{[N]}{2}} \prod_{xy \in E} (-1)^{|p(x) \cap p(y)|},$$

and for $k \in \mathbb{N}$ let $T_k(G)$ be the same sum restricted to the maps $p: V \rightarrow \binom{[k]}{2}$ with $\bigcup_{x \in V} p(x) = [k]$.

Theorem 1.3. *Let $G = (V, E)$ be a graph with n vertices.*

of [15].

⁴The intersection graph records which pairs of chords of D cross. Intersection patterns of families of sets, recorded in their nerve complexes, are used in [16] to decide how many times a region is covered.

- (a) $S_G(N) = \sum_{k=0}^{2n} \binom{N}{k} T_k(G)$ for all $N \in \mathbb{N}$; thus S_G is a polynomial in N of degree at most $2n$.
- (b) The graph invariant

$$\xi(G) = \frac{S_G(-2)}{(-32)^n} = \frac{1}{(-32)^n} \sum_{k=0}^{2n} (-1)^k (k+1) T_k(G)$$

satisfies the four-term relations of graphs.

- (c) For every chord diagram D , $w_D(-\frac{3}{32}) = \xi(\Gamma(D))$.

Parts (b) and (c) are Conjecture 1.1. The two expressions for ξ agree by (a), since $\binom{-2}{k} = (-1)^k (k+1)$.

Example 1.4. For one vertex, $S_{K_1}(N) = \binom{N}{2}$ and $\xi(K_1) = 3/(-32) = -\frac{3}{32}$, the value of w on one chord. Since $S_{G \sqcup H} = S_G S_H$, ξ is multiplicative, as w is. For two adjacent vertices, the pairs $(p(x), p(y))$ with $|p(x) \cap p(y)| = 2, 1, 0$ number $\binom{N}{2}$, $\binom{N}{2} \cdot 2(N-2)$ and $\binom{N}{2} \binom{N-2}{2}$, so $S_{K_2}(N) = \binom{N}{2} (1 - 2(N-2) + \binom{N-2}{2})$ and $\xi(K_2) = 3 \cdot 19/1024 = \frac{57}{1024}$. This is $c(c - \frac{1}{2})$ at $c = -\frac{3}{32}$, the value of w on two crossing chords.

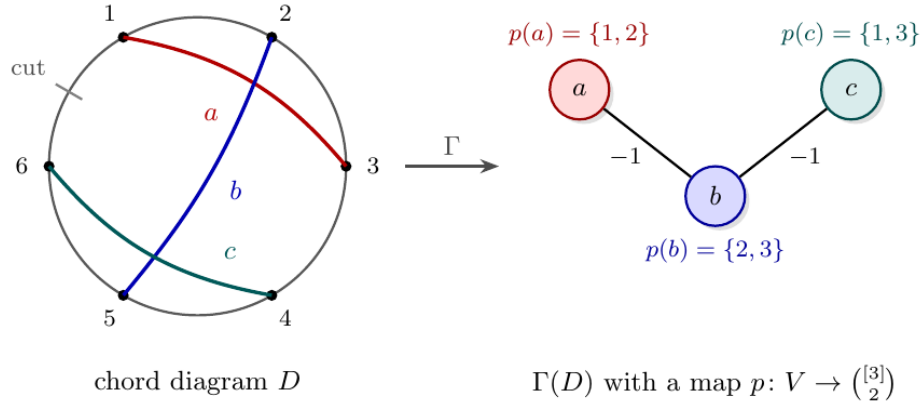


Figure 1: A chord diagram with three chords, cut between the points 6 and 1, and its intersection graph, a path. The map p sends each vertex to a 2-element subset of $[3]$; both edges meet in one element, so p contributes $(-1)(-1) = 1$ to $S_{\Gamma(D)}(3)$.

The proof of (b) is an involution on assignments (Section 6). For (c) we let the element $C_D \in U(\mathfrak{sl}(2))$ of a chord diagram act on the Verma module

of highest weight μ ; it acts by a polynomial $\lambda_D(\mu)$, and $w_D(-\frac{3}{32}) = \lambda_D(-\frac{1}{2})$ (Section 2). The oscillator representation on $\mathbb{Q}[z]$ is the sum of the Verma modules of highest weights $-\frac{1}{2}$ and $-\frac{3}{2}$, so the image of C_D in the Weyl algebra is the scalar $\lambda_D(-\frac{1}{2})$. A supertrace on the Weyl algebra with a Wick formula writes this scalar as a sum over perfect matchings, in which every cycle contributes a factor -2 (Sections 3 and 4). With N in place of -2 the same sum is a trace in the Clifford algebra on N generators, and that trace equals $4^n S_\Gamma(N)$ (Section 5). Figure 2 shows the two halves of the argument. The proof is formalized in Lean 4 (Section 7).

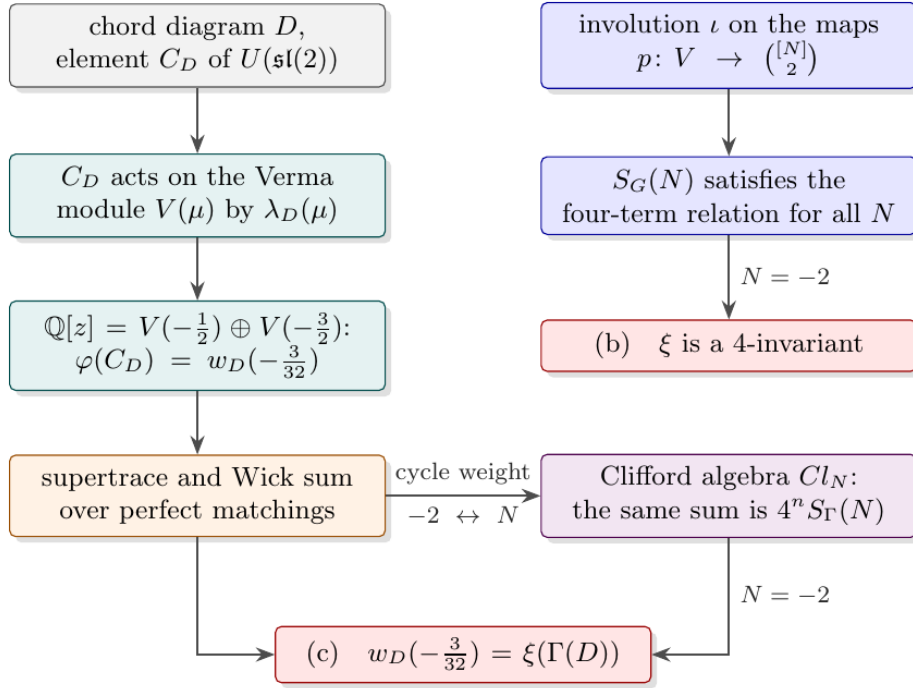


Figure 2: The proof of Theorem 1.3. The right column gives part (b), the left column and the bottom row part (c).

2. The weight system and Verma modules

2.1. Chord diagrams and the weight system

A chord diagram with n chords is a fixed-point-free involution σ of $[2n]$; its chords are the pairs $\{p, \sigma(p)\}$. It is a chord diagram on an oriented circle

cut at a point that is not an endpoint; cutting at another point rotates $[2n]$ cyclically. Chords $\{p < q\}$ and $\{p' < q'\}$ cross if $p < p' < q < q'$ or $p' < p < q' < q$. The intersection graph $\Gamma(D)$ has the chords as vertices and the crossing pairs as edges; it does not depend on the cut.

Let $\mathfrak{g} = \mathfrak{sl}(2)$ over $\mathbb{Q}$ with basis E, F, H , $[H, E] = 2E$, $[H, F] = -2F$, $[E, F] = H$, and let

$$t = \frac{1}{4} E \otimes F + \frac{1}{4} F \otimes E + \frac{1}{8} H \otimes H \in \mathfrak{g} \otimes \mathfrak{g}$$

be the Casimir tensor of the Killing form. It is symmetric and $\mathfrak{g}$ -invariant: $(\text{ad}_x \otimes 1 + 1 \otimes \text{ad}_x)t = 0$ for $x \in \mathfrak{g}$. For a chord diagram D let $t_D \in \mathfrak{g}^{\otimes 2n}$ be the tensor with a copy of t in the factors $p, \sigma(p)$ of every chord, and let $C_D \in U(\mathfrak{g})$ be the image of t_D under $u_1 \otimes \cdots \otimes u_{2n} \mapsto u_1 \cdots u_{2n}$. For one chord $C_D = C = \frac{1}{4}(EF + FE + \frac{1}{2}H^2)$.

For $d \geq 1$ let V_d be the irreducible representation of dimension d , with basis $e_0, \dots, e_{d-1}$ and

$$Ee_j = j(d-j)e_{j-1}, \quad Fe_j = e_{j+1}, \quad He_j = (d-1-2j)e_j,$$

where $e_{-1} = e_d = 0$. The Casimir element C acts on V_d by $c_d = (d^2 - 1)/8$. Put $W_d(D) = \text{tr}_{V_d}(C_D)/d$. Rotating the cut permutes the factors of t_D cyclically, so $W_d(D)$ depends only on the diagram on the circle.

By Proposition 2.3 below there is a unique $w_D \in \mathbb{Q}[c]$ with $w_D(c_d) = W_d(D)$ for all $d \geq 1$. It is the $\mathfrak{sl}(2)$ -weight system of [11, 8]: by Lemma 2.1 C_D lies in the centre of $U(\mathfrak{g})$, which is $\mathbb{Q}[C]$ [12], so $C_D = w(C)$ for a polynomial w , and taking traces on V_d gives $w(c_d) = W_d(D)$; the weight system of [8] is defined with the Killing form and takes the value c on one chord.

Lemma 2.1. C_D commutes with every $x \in \mathfrak{g}$.

Proof. ad_x is a derivation of $U(\mathfrak{g})$, so $[x, C_D]$ is the image of $\sum_{p=1}^{2n} (\text{ad}_x)_p t_D$, where $(\text{ad}_x)_p$ acts on the factor p . Grouping the two endpoints of every chord writes this sum as a sum over chords of tensors containing $(\text{ad}_x \otimes 1 + 1 \otimes \text{ad}_x)t = 0$. $\square$

2.2. Verma modules

Let $R = \mathbb{Q}[\mu]$ and let M be the free R -module with basis $v_0, v_1, \dots$ and

$$Ev_j = j(\mu - j + 1)v_{j-1}, \quad Fv_j = v_{j+1}, \quad Hv_j = (\mu - 2j)v_j. \quad (1)$$

These operators satisfy the relations of $\mathfrak{g}$:

$$(EF - FE)v_j = ((j+1)(\mu - j) - j(\mu - j + 1))v_j = (\mu - 2j)v_j = Hv_j,$$

and $(HE - EH)v_j = 2Ev_j$, $(HF - FH)v_j = -2Fv_j$, because H has eigenvalue $\mu - 2j + 2$ on v_{j-1} and $\mu - 2j - 2$ on v_{j+1} . Thus M is a $U(\mathfrak{g})$ -module on which $U(\mathfrak{g})$ acts R -linearly. For $\mu_0 \in \mathbb{Q}$, specializing $\mu = \mu_0$ gives the Verma module $V(\mu_0)$ of highest weight μ_0 .

Proposition 2.2. *There is $\lambda_D \in \mathbb{Q}[\mu]$ with $C_D m = \lambda_D m$ for all $m \in M$. Moreover*

- (a) $W_d(D) = \lambda_D(d - 1)$ for every $d \geq 1$;
- (b) $\lambda_D(\mu) = \lambda_D(-2 - \mu)$.

Proof. $H - \mu$ multiplies v_j by $-2j$, so its kernel is Rv_0 . As C_D commutes with H , $C_D v_0 = \lambda_D v_0$ for some $\lambda_D \in R$, and $C_D v_j = C_D F^j v_0 = F^j C_D v_0 = \lambda_D v_j$.

(a) In $V(d-1)$, $Ev_d = d(d-1-d+1)v_{d-1} = 0$, so $U_d = \text{span}\{v_j : j \geq d\}$ is a submodule. The quotient $V(d-1)/U_d$, with basis the images of $v_0, \dots, v_{d-1}$, is V_d with the formulas above. C_D acts on $V(d-1)$, hence on V_d , by $\lambda_D(d-1)$, so $\text{tr}_{V_d} C_D = d \lambda_D(d-1)$.

(b) The map $V(-1-d) \rightarrow U_d$, $v_j \mapsto v_{d+j}$, is an isomorphism of $\mathfrak{g}$ -modules: it commutes with F ; both Hv_j in $V(-1-d)$ and Hv_{d+j} in $V(d-1)$ have eigenvalue $-1-d-2j$; and $Ev_j = j(-d-j)v_{j-1}$ in $V(-1-d)$ while $Ev_{d+j} = (d+j)(-j)v_{d+j-1}$ in $V(d-1)$. Hence $\lambda_D(-1-d) = \lambda_D(d-1)$ for all $d \geq 1$, and the polynomial $\lambda_D(\mu) - \lambda_D(-2-\mu)$ has infinitely many roots. $\square$

Proposition 2.3. *There is a unique $w_D \in \mathbb{Q}[c]$ with $w_D(c_d) = W_d(D)$ for all $d \geq 1$. It satisfies $\lambda_D(\mu) = w_D(\mu(\mu+2)/8)$; in particular*

$$w_D(-\frac{3}{32}) = \lambda_D(-\frac{1}{2}) = \lambda_D(-\frac{3}{2}).$$

Proof. By Proposition 2.2(b), $\nu \mapsto \lambda_D(\nu-1)$ is even, so $\lambda_D(\nu-1) = g(\nu^2)$ for some $g \in \mathbb{Q}[x]$. Put $w_D(c) = g(8c+1)$. Then $w_D(\mu(\mu+2)/8) = g((\mu+1)^2) = \lambda_D(\mu)$, and $w_D(c_d) = \lambda_D(d-1) = W_d(D)$ by Proposition 2.2(a). Uniqueness holds because the c_d are pairwise distinct. Finally $\mu(\mu+2)/8 = -\frac{3}{32}$ for $\mu = -\frac{1}{2}$ and for $\mu = -\frac{3}{2}$. $\square$

3. The oscillator representation and a supertrace

Let A be the Weyl algebra over $\mathbb{Q}$ generated by z and ∂ with $\partial z - z\partial = 1$.⁵ It acts faithfully on $\mathbb{Q}[z]$, with z acting by multiplication and ∂ by differentiation, and the monomials $z^i \partial^j$ ($i, j \geq 0$) form a basis of A . In A ,

$$z^i \partial^j z = z^{i+1} \partial^j + j z^i \partial^{j-1}, \quad \partial z^i \partial^j = z^i \partial^{j+1} + i z^{i-1} \partial^j. \quad (2)$$

Lemma 3.1. *The assignment $\varphi(E) = -\frac{1}{4}\partial^2$, $\varphi(F) = z^2$, $\varphi(H) = -\frac{1}{2}(z\partial + \partial z)$ extends to an algebra homomorphism $\varphi: U(\mathfrak{g}) \rightarrow A$. On $\mathbb{Q}[z]$, the maps $v_j \mapsto z^{2j}$ and $v_j \mapsto z^{2j+1}$ are isomorphisms of $\mathfrak{g}$ -modules $V(-\frac{1}{2}) \rightarrow \mathbb{Q}[z]_{\text{even}}$ and $V(-\frac{3}{2}) \rightarrow \mathbb{Q}[z]_{\text{odd}}$.*

Proof. On $\mathbb{Q}[z]$,

$$Ez^k = -\frac{1}{4}k(k-1)z^{k-2}, \quad Fz^k = z^{k+2}, \quad Hz^k = -(k + \frac{1}{2})z^k.$$

Hence $(EF - FE)z^k = -\frac{1}{4}((k+2)(k+1) - k(k-1))z^k = Hz^k$, and $(HE - EH)z^k = 2Ez^k$, $(HF - FH)z^k = -2Fz^k$, because H has eigenvalue $-(k - \frac{3}{2})$ on z^{k-2} and $-(k + \frac{5}{2})$ on z^{k+2} . The relations of $\mathfrak{g}$ hold on $\mathbb{Q}[z]$, hence in A . For $k = 2j$ the formulas read $Ez^{2j} = j(-\frac{1}{2} - j + 1)z^{2j-2}$ and $H z^{2j} = (-\frac{1}{2} - 2j)z^{2j}$, which is (1) at $\mu = -\frac{1}{2}$; for $k = 2j + 1$ they read $Ez^{2j+1} = j(-\frac{3}{2} - j + 1)z^{2j-1}$ and $H z^{2j+1} = (-\frac{3}{2} - 2j)z^{2j+1}$, which is (1) at $\mu = -\frac{3}{2}$. $\square$

Proposition 3.2. $\varphi(C_D) = w_D(-\frac{3}{32}) \cdot 1$ in A .

Proof. By Proposition 2.2 and Lemma 3.1, $\varphi(C_D)$ acts on $\mathbb{Q}[z]_{\text{even}}$ by $\lambda_D(-\frac{1}{2})$ and on $\mathbb{Q}[z]_{\text{odd}}$ by $\lambda_D(-\frac{3}{2})$. Both equal $w_D(-\frac{3}{32})$ by Proposition 2.3, and A acts faithfully. $\square$

Let $\tau: A \rightarrow \mathbb{Q}$ be the linear map with

$$\tau(z^i \partial^j) = \delta_{ij} (-1)^j \frac{j!}{2^j}.$$

Since $z^j \partial^j$ multiplies z^k by $k(k-1) \cdots (k-j+1)$ and $\sum_k k(k-1) \cdots (k-j+1)x^k = j! x^j (1-x)^{-j-1}$, $\tau(a)$ is twice the Abel sum $\lim_{x \rightarrow -1^+} \sum_k a_{kk} x^k$ of the supertrace of a on $\mathbb{Q}[z]$, where $az^k = a_{kk} z^k + (\text{other powers})$. Only the following properties are used.

⁵The relation $\partial z - z\partial = 1$ is the simplest instance of non-commutativity in an algebra over a ring; non-commutativity under permutation representations of a parametrized ring is studied in [17].

Lemma 3.3. (a) $\tau(1) = 1$.

(b) $\tau(xa) = -\tau(ax)$ for $x \in \{z, \partial\}$ and $a \in A$.

(c) For $u_1, \dots, u_m \in \{z, \partial\}$,

$$\tau(u_1 \cdots u_m) = \sum_M \prod_{\{a < b\} \in M} \frac{1}{2} [u_a, u_b],$$

the sum over the perfect matchings M of $[m]$; here $[z, \partial] = -1$, $[\partial, z] = 1$ and $[z, z] = [\partial, \partial] = 0$. For odd m both sides vanish.

Proof. (a) is the case $i = j = 0$. For (b) it suffices to take $a = z^i \partial^j$. By (2), $\tau(z^i \partial^j z) = \tau(z^{i+1} \partial^j) + j \tau(z^i \partial^{j-1})$. Both terms vanish unless $j = i + 1$, and then the sum is $(-1)^j \frac{j!}{2^j} + j(-1)^{j-1} \frac{(j-1)!}{2^{j-1}} = -(-1)^j \frac{j!}{2^j} = -\tau(z \cdot z^i \partial^j)$. Likewise $\tau(\partial z^i \partial^j) = \tau(z^i \partial^{j+1}) + i \tau(z^{i-1} \partial^j)$ vanishes unless $i = j + 1$, and then it equals $(-1)^{j+1} \frac{(j+1)!}{2^{j+1}} + (j+1)(-1)^j \frac{j!}{2^j} = (-1)^j \frac{(j+1)!}{2^{j+1}} = -\tau(z^i \partial^j \cdot \partial)$.

(c) Induction on m . For $m = 0$ use (a); for $m = 1$, $\tau(z) = \tau(\partial) = 0$. For $m \geq 2$, as the $[u_1, u_s]$ are scalars,

$$u_1 u_2 \cdots u_m = \sum_{s=2}^m [u_1, u_s] u_2 \cdots \widehat{u_s} \cdots u_m + u_2 \cdots u_m u_1.$$

Applying τ and (b) to the last term gives

$$\tau(u_1 \cdots u_m) = \frac{1}{2} \sum_{s=2}^m [u_1, u_s] \tau(u_2 \cdots \widehat{u_s} \cdots u_m),$$

which is the expansion of the right side of (c) according to the partner s of 1. $\square$

Corollary 3.4. $w_D(-\frac{3}{32}) = \tau(\varphi(C_D))$.

4. Expansion over matchings

4.1. Matchings

Let X be a finite totally ordered set and P, M perfect matchings of X . Two edges $\{a < b\}$ and $\{a' < b'\}$ cross if $a < a' < b < b'$ or $a' < a < b' < b$, and $\text{cr}(P)$ is the number of crossing pairs of edges of P . The multigraph $P \cup M$ is a disjoint union of cycles alternating between edges of P and of M (a common edge gives a cycle of length 2); $\text{cyc}(P, M)$ is their number. Orient every cycle; an edge oriented from a larger to a smaller element is

a descent, and $\text{des}(P, M)$ is the number of descents. Reversing a cycle of length $2r$ turns δ descents into $2r - \delta$, so the parity of $\text{des}(P, M)$ does not depend on the orientations. Put

$$L_P(N) = \sum_M (-1)^{\text{cr}(M)} N^{\text{cyc}(P, M)} \in \mathbb{Z}[N],$$

the sum over all perfect matchings M of X .

Lemma 4.1. $\text{cr}(P) + \text{cr}(M) + \text{cyc}(P, M) + \text{des}(P, M)$ is even.

Proof. Let $Q(P, M)$ be the sum. Let $x < y$ be consecutive in X , and exchange their positions in the order, keeping P and M . If $\{x, y\} \notin P$, let $e \ni x$ and $f \ni y$ be the edges of P ; the exchange moves y to the other side of the endpoint x of e and keeps the side of the other endpoint of f , so it changes whether e and f cross, and no other pair. Thus $\text{cr}(P)$ changes by $1 - [\{x, y\} \in P]$ modulo 2, and likewise $\text{cr}(M)$. $\text{cyc}(P, M)$ does not change. A descent status changes only on edges joining x and y , so $\text{des}(P, M)$ changes by $[\{x, y\} \in P] + [\{x, y\} \in M]$. The total change is 2. Any two total orders of X are related by exchanges of consecutive elements, so Q modulo 2 is the same for all orders. Choose an order in which every cycle $x_1 x_2 \cdots x_{2r}$ (with $x_1 x_2 \in P$) occupies consecutive positions in this order. Then P and M have no crossings, and each cycle, oriented as listed, has exactly one descent, from x_{2r} to x_1 . Hence $Q \equiv 2 \text{cyc}(P, M) \equiv 0$. $\square$

4.2. The bosonic expansion

For a perfect matching P of X with k edges let

$$X_P = \sum_o \text{sgn}(o) \prod_{x \in X}^{\rightarrow} u_x^o \in A,$$

the sum over the 2^k orientations o of the edges of P , where $u_x^o = z$ if x is the tail of its edge and $u_x^o = \partial$ if it is the head, the product is taken in the order of X , and $\text{sgn}(o) = (-1)^{\text{number of descents of } o}$.

Proposition 4.2. $\tau(X_P) = 2^{-k} (-1)^{\text{cr}(P)} L_P(-2)$.

Proof. By Lemma 3.3(c),

$$\tau(X_P) = \sum_o \sum_M \text{sgn}(o) \prod_{\{a < b\} \in M} \frac{1}{2} [u_a^o, u_b^o].$$

A term vanishes unless every edge of M joins a head and a tail of o . Orient such an edge from the head to the tail; then $\frac{1}{2}[u_a^o, u_b^o]$ is $\frac{1}{2}$ if it is oriented from a to b ($u_a^o = \partial$, $u_b^o = z$) and $-\frac{1}{2}$ if it is a descent. As $\text{sgn}(o)$ is -1 to the number of descents among the edges of P , the term equals $2^{-k}(-1)^{\text{des}(P,M)}$. Every element of X is then entered by exactly one oriented edge and left by exactly one, so the cycles of $P \cup M$ are directed cycles. Conversely, each of the $2^{\text{cyc}(P,M)}$ orientations of the cycles of $P \cup M$ as directed cycles restricts on P to an orientation o for which every edge of M joins a head to a tail. Hence

$$\begin{aligned}\tau(X_P) &= 2^{-k} \sum_M 2^{\text{cyc}(P,M)} (-1)^{\text{des}(P,M)} \\ &= 2^{-k} (-1)^{\text{cr}(P)} \sum_M (-1)^{\text{cr}(M)} (-2)^{\text{cyc}(P,M)}\end{aligned}$$

by Lemma 4.1. □

4.3. Legs

Let D be a chord diagram with n chords. Every point $p \in [2n]$ carries two legs $(p, 1) < (p, 2)$, and the set X_D of the $4n$ legs is ordered lexicographically. For a chord $c = \{p < q\}$ and $\beta \in \{0, 1\}$ let P_c^β be the pair of edges

$$P_c^0 = \{(p, 1)(q, 1), (p, 2)(q, 2)\}, \quad P_c^1 = \{(p, 1)(q, 2), (p, 2)(q, 1)\},$$

and for $\beta: \{\text{chords}\} \rightarrow \{0, 1\}$ let $P_\beta = \bigcup_c P_c^{\beta(c)}$, a perfect matching of X_D . Let $\pi = z \otimes \partial - \partial \otimes z \in A \otimes A$. Writing $\pi = \sum_r \varepsilon_r x_r \otimes y_r$, put $\pi_{13}\pi_{24} = \sum_{r,s} \varepsilon_r \varepsilon_s x_r x_s \otimes y_r y_s$ and $\pi_{14}\pi_{23} = \sum_{r,s} \varepsilon_r \varepsilon_s x_r x_s \otimes y_s y_r$.

Lemma 4.3. $(\varphi \otimes \varphi)(t) = -\frac{1}{32}(\pi_{13}\pi_{24} + \pi_{14}\pi_{23})$.

Proof. Both sides equal $-\frac{1}{32}(2z^2 \otimes \partial^2 + 2\partial^2 \otimes z^2 - (z\partial + \partial z) \otimes (z\partial + \partial z))$. On the left, $\frac{1}{4}\varphi(E) \otimes \varphi(F) + \frac{1}{4}\varphi(F) \otimes \varphi(E) = -\frac{1}{16}(\partial^2 \otimes z^2 + z^2 \otimes \partial^2)$ and $\frac{1}{8}\varphi(H) \otimes \varphi(H) = \frac{1}{32}(z\partial + \partial z)^{\otimes 2}$. On the right, the terms with $r = s$ give $2z^2 \otimes \partial^2 + 2\partial^2 \otimes z^2$ and the terms with $r \neq s$ give $-z\partial \otimes (\partial z + z\partial) - \partial z \otimes (z\partial + \partial z)$. □

Corollary 4.4. $\varphi(C_D) = \left(-\frac{1}{32}\right)^n \sum_\beta X_{P_\beta}$, the sum over all $\beta: \{\text{chords}\} \rightarrow \{0, 1\}$.

Proof. $\varphi(C_D)$ is the product, in the order of the points, of the factors of $(\varphi \otimes \varphi)(t)^{\otimes n}$ placed at the endpoints of the chords. Put the two letters of each factor $x_r x_s$ or $y_r y_s$ on the two legs of its point, in this order. By Lemma 4.3, a chord contributes $-\frac{1}{32}$ times the sum of π placed on the two edges of P_c^0 and π placed on the two edges of P_c^1 . A copy of π on an edge $\{a < b\}$ contributes z at a and ∂ at b with sign $+1$, or ∂ at a and z at b with sign -1 ; these are the two orientations of the edge with their signs in X_{P_β} . $\square$

5. Clifford algebras

For $N \geq 1$ let Cl_N be the $\mathbb{Q}$ -algebra generated by $e_1, \dots, e_N$ with $e_i e_j + e_j e_i = 2\delta_{ij}$. For $I = \{i_1 < \dots < i_r\} \subseteq [N]$ put $e_I = e_{i_1} \cdots e_{i_r}$; these form a basis, and $e_I e_J = \pm e_{I \Delta J}$. Let $\text{tr}: Cl_N \rightarrow \mathbb{Q}$ be linear with $\text{tr}(e_I) = \delta_{I\emptyset}$. Then $\text{tr}(ab) = \text{tr}(ba)$: for basis elements both sides vanish unless $I = J$.

Lemma 5.1. *For $i_1, \dots, i_m \in [N]$,*

$$\text{tr}(e_{i_1} \cdots e_{i_m}) = \sum_M (-1)^{\text{cr}(M)} \prod_{\{a,b\} \in M} \delta_{i_a i_b},$$

the sum over the perfect matchings M of $[m]$.

Proof. For odd m the product is $\pm e_I$ with $|I|$ odd, so both sides vanish. For even $m \geq 2$, moving e_{i_1} to the right end gives

$$e_{i_1} e_{i_2} \cdots e_{i_m} = \sum_{s=2}^m (-1)^s 2\delta_{i_1 i_s} e_{i_2} \cdots \widehat{e_{i_s}} \cdots e_{i_m} - e_{i_2} \cdots e_{i_m} e_{i_1}.$$

Taking traces and using $\text{tr}(e_{i_2} \cdots e_{i_m} e_{i_1}) = \text{tr}(e_{i_1} \cdots e_{i_m})$,

$$\text{tr}(e_{i_1} \cdots e_{i_m}) = \sum_{s=2}^m (-1)^s \delta_{i_1 i_s} \text{tr}(e_{i_2} \cdots \widehat{e_{i_s}} \cdots e_{i_m}).$$

The edge $\{1, s\}$ crosses exactly the edges with one endpoint in $\{2, \dots, s-1\}$; the elements of this set matched among themselves come in pairs, so the number of such edges is congruent to $s-2$. By induction both sides satisfy the same recursion. $\square$

Corollary 5.2. *For a perfect matching P of a totally ordered set $X = \{x_1 < \dots < x_{2k}\}$ and $N \geq 1$,*

$$\sum_{\kappa} \text{tr}(e_{\kappa(x_1)} \cdots e_{\kappa(x_{2k})}) = L_P(N),$$

the sum over the maps $\kappa: X \rightarrow [N]$ that are constant on every edge of P .

Proof. By Lemma 5.1 the left side is $\sum_M (-1)^{\text{cr}(M)}$ times the number of maps $X \rightarrow [N]$ constant on the edges of P and of M , which is $N^{\text{cyc}(P,M)}$. $\square$

For $I = \{i < j\} \in \binom{[N]}{2}$ write $b_I = e_i e_j$. Then

$$b_I b_J = (-1)^{|I \cap J|} b_J b_I, \quad b_I^2 = -1. \quad (3)$$

Proposition 5.3. *Let D be a chord diagram with n chords and $\Gamma = \Gamma(D)$. Then*

$$\sum_{\beta} (-1)^{\text{cr}(P_{\beta})} L_{P_{\beta}}(N) = 4^n S_{\Gamma}(N)$$

as polynomials in N .

Proof. Both sides are polynomials (Theorem 1.3(a) is proved in Section 6 independently), so it suffices to take $N \geq 1$.

The two edges of P_c^{β} cross if and only if $\beta = 0$. If chords c and c' cross, every edge of P_c^{β} crosses every edge of $P_{c'}^{\beta'}$; otherwise none does. Hence

$$(-1)^{\text{cr}(P_{\beta})} = (-1)^n \prod_c (-1)^{\beta(c)}.$$

A map κ constant on the edges of P_{β} is given by a pair $(i_c, j_c) \in [N]^2$ for every chord $c = \{p < q\}$: i_c on the edge through $(p, 1)$ and j_c on the edge through $(p, 2)$. The legs at p then read $e_{i_c} e_{j_c}$, and the legs at q read $e_{i_c} e_{j_c}$ if $\beta(c) = 0$ and $e_{j_c} e_{i_c}$ if $\beta(c) = 1$. By Corollary 5.2, summing over $\beta(c)$ with the sign $(-1)^{\beta(c)}$ replaces the factor at q by $e_{i_c} e_{j_c} - e_{j_c} e_{i_c}$, which is 0 if $i_c = j_c$ and $2e_{i_c} e_{j_c}$ otherwise. Exchanging i_c and j_c changes the sign of both factors of c , so every 2-element set $s(c) = \{i_c, j_c\}$ arises twice with the same value. Therefore

$$\sum_{\beta} (-1)^{\text{cr}(P_{\beta})} L_{P_{\beta}}(N) = (-1)^n 4^n \sum_s \text{tr}(B_s), \quad B_s = \prod_{p=1}^{2n} b_{s(c_p)},$$

the sum over $s: \{\text{chords}\} \rightarrow \binom{[N]}{2}$, where c_p is the chord through p .

We show $\text{tr}(B_s) = (-1)^n \prod_{cc' \in E(\Gamma)} (-1)^{|s(c) \cap s(c')|}$ by induction on n . Let $c = \{p < q\}$. Move the factor at p to the right until it is next to the factor at q . By (3) it passes the factors at the points strictly between p and q ; a chord with both endpoints there contributes the square of a sign, and the chords with one endpoint there are the neighbours of c in Γ . Then $b_{s(c)}^2 = -1$, and the remaining word is $B_{s'}$ for the diagram $D \setminus c$, whose intersection graph is $\Gamma - c$, with $s' = s|_{\Gamma - c}$. Summing over s gives $\sum_s \text{tr}(B_s) = (-1)^n S_\Gamma(N)$. $\square$

6. Proof of Theorem 1.3

Proof of (a). Group the maps $p: V \rightarrow \binom{[N]}{2}$ by $U = \bigcup_x p(x)$. Relabelling U in increasing order by $[|U|]$ preserves all intersection sizes, so the maps with $|U| = k$ contribute $\binom{N}{k} T_k(G)$; and $T_k(G) = 0$ for $k > 2n$. $\square$

Proof of (c). By Corollaries 3.4 and 4.4, Proposition 4.2 with $k = 2n$, and Proposition 5.3,

$$\begin{aligned} w_D(-\tfrac{3}{32}) &= \left(-\tfrac{1}{32}\right)^n \sum_{\beta} \tau(X_{P_\beta}) = \left(-\tfrac{1}{32}\right)^n 4^{-n} \sum_{\beta} (-1)^{\text{cr}(P_\beta)} L_{P_\beta}(-2) \\ &= \frac{S_\Gamma(-2)}{(-32)^n} = \xi(\Gamma). \end{aligned} \quad \square$$

For a graph G and distinct vertices a, b let G'_{ab} be G with the adjacency of a and b reversed, and $\tilde{G}_{ab}$ be G with the adjacency of a and y reversed for every neighbour $y \neq a$ of b . The four-term relation for a function f on graphs is

$$f(G) - f(G'_{ab}) = f(\tilde{G}_{ab}) - f(\tilde{G}'_{ab}), \quad \tilde{G}'_{ab} = (\tilde{G}_{ab})'_{ab},$$

as in Figure 3.

Proposition 6.1. *For every $N \in \mathbb{N}$, $G \mapsto S_G(N)$ satisfies the four-term relation.*

Proof. For $p: V \rightarrow \binom{[N]}{2}$ write

$$\epsilon(G, p) = \prod_{xy \in E(G)} (-1)^{|p(x) \cap p(y)|}.$$

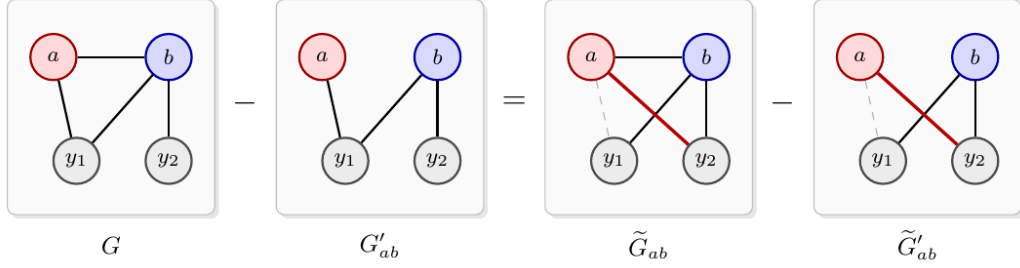


Figure 3: The four graphs of a four-term relation. Here b has the neighbours a, y_1, y_2 ; passing from G to $\tilde{G}_{ab}$ removes the edge ay_1 (dashed) and adds the edge ay_2 .

Since $\epsilon(G'_{ab}, p) = \epsilon(G, p)(-1)^{|p(a) \cap p(b)|}$ and $|p(a) \cap p(b)| \in \{0, 1, 2\}$,

$$S_G(N) - S_{G'_{ab}}(N) = 2 \sum_{p \in \Pi} \epsilon(G, p), \quad \Pi = \{p : |p(a) \cap p(b)| = 1\}.$$

The construction $\tilde{G}_{ab}$ does not change the adjacency of a and b , so the same holds for $\tilde{G}'_{ab}$. On Π define $\iota(p)$ by $\iota(p)(b) = p(a) \Delta p(b)$ and $\iota(p)(x) = p(x)$ for $x \neq b$. As $|p(a) \cap p(b)| = 1$, $\iota(p)(b)$ is a 2-element set meeting $p(a)$ in one element, and ι is an involution of Π . Let Y be the set of neighbours $y \neq a$ of b in G , which is also the set of such neighbours in $\tilde{G}_{ab}$. For $y \in Y$, $|(p(a) \Delta p(b)) \cap p(y)| \equiv |p(a) \cap p(y)| + |p(b) \cap p(y)|$, and the edge ab , if present, keeps its sign because $|p(a) \cap \iota(p)(b)| = 1$. Hence

$$\begin{aligned} \epsilon(G, \iota(p)) &= \epsilon(G, p) \prod_{y \in Y} (-1)^{|p(a) \cap p(y)|}, \\ \epsilon(\tilde{G}_{ab}, q) &= \epsilon(G, q) \prod_{y \in Y} (-1)^{|q(a) \cap q(y)|}, \end{aligned}$$

the second for every q since $\tilde{G}_{ab}$ differs from G exactly in the pairs ay , $y \in Y$. With $q = \iota(p)$, which agrees with p at a and on Y , this gives $\epsilon(\tilde{G}_{ab}, \iota(p)) = \epsilon(G, p)$. Therefore $\sum_{p \in \Pi} \epsilon(\tilde{G}_{ab}, p) = \sum_{p \in \Pi} \epsilon(G, p)$, and the two differences agree. $\square$

Proof of (b). ξ depends only on the isomorphism class of G . The four graphs of a four-term relation have the same vertex set, so by Proposition 6.1 and (a) the four-term relation holds for the polynomials S_G , hence at $N = -2$, and after division by $(-32)^n$ for ξ . $\square$

Remark 6.2. Proposition 6.1 applied to the sets Π restricted to the maps with $\bigcup_x p(x) = [k]$, which ι preserves, shows that every T_k satisfies the four-term relation. By the proof of Proposition 5.3, for $N \geq 1$ the 4-invariant $S_\Gamma(N)$ is $(-4)^{-n}$ times the Clifford trace of the element obtained by placing the tensor $\sum_{i \neq j} e_i e_j \otimes 2e_i e_j$ on the chords as in the definition of C_D , and Theorem 1.3(c) identifies $w_D(-\frac{3}{32})$ with $(-32)^{-n}$ times its value at $N = -2$.

7. Formal verification

Parts (b) and (c) of Theorem 1.3 are formalized in Lean 4 [13] with Mathlib [14], version 4.34.1, and the formal proof is checked by the Lean kernel. The formal statement has two parts. The first says that ξ is invariant under isomorphisms of graphs and satisfies the four-term relation for every graph and every pair of distinct vertices a, b . The second says that for every chord diagram D there is a polynomial $P \in \mathbb{Q}[c]$ with $P(c_d) = W_d(D)$ for all $d \geq 1$, and that every such polynomial, that is w_D , takes the value $\xi(\Gamma(D))$ at $c = -\frac{3}{32}$. A chord diagram is a fixed-point-free involution of $\{0, \dots, 2n-1\}$, the trace $W_d(D)$ is computed from the matrices of E, F, H on V_d in the basis of Section 2, and $\Gamma(D)$, G'_{ab} , $\tilde{G}_{ab}$, T_k and ξ are as in Sections 1 and 6. The proof depends only on the axioms of Lean's logic, namely propositional extensionality, choice and the soundness of quotients; it has no unproved step and adds no axiom.

The formal proof follows Sections 2–6 with three changes of route. Proposition 2.2(b) is derived from an identity between the coefficients of words acting on the Verma modules of highest weights μ and $-2 - \mu$, which is the isomorphism $V(-1-d) \cong U_d$ in coordinates. The identity $W_d(D) = \lambda_D(d-1)$ is proved by expanding matrix entries of words over paths. The sums over matchings are encoded by a loop polynomial of a word in which every label occurs twice, defined by contracting the first label; the bosonic expansion (Proposition 4.2) and the Clifford colour sum (Corollary 5.2, with the Clifford trace realized as (-1) to the number of inversions of a word in which every letter occurs an even number of times) are shown to satisfy the same recursion, which replaces Lemma 4.1.

Independent computations confirm the theorem on small cases. In exact rational arithmetic, $w_D(-\frac{3}{32})$, obtained by interpolating $W_d(D)$ for $d \leq n+1$, agrees with $\xi(\Gamma(D))$ on all chord diagrams with at most 5 chords and on random larger ones, the four-term relation holds on all graphs with at most 4 vertices, and each identity of Sections 3–5 holds on small cases. Modulo the

prime $2^{61} - 1$, the two sides agree on all 11465 chord diagrams with at most 6 chords and on 400 random diagrams with 7 and 8 chords, the four-term relation holds on all graphs with at most 5 vertices, and ξ reproduces the values of the extension tabulated in [8] on 63 graphs with 9 and 10 vertices.

Data and code availability

The $\text{\LaTeX}$ source of the paper, the Lean 4 formalization of Theorem 1.3(b) and (c) and the programs of Section 7 are archived at Zenodo under doi:10.5281/zenodo.23384025 [18]; the paper is available at <https://github.com/creelie/automatic-memory/blob/main/preprintSl2WeightSystemAtMinus30ver32/Sl2WeightSystemAtMinus30ver32.pdf>. Deep Bhattacharjee carried out the computations and the Lean, Julia, C, C++, Python and shell verifications, and the proofs and the text were drafted, with AI assistance (Claude, Anthropic); both authors approved the final version.

Ethical declarations

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