

EVERY ODD ANTIHOLE IS STRONGLY CHROMATIC-CHOOSABLE

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ABSTRACT. We prove the complete Fiujaali–McDonald conjecture on odd antiholes unconditionally: for every $\ell \geq 3$ the only assignments of $(\ell - 1)$ -element lists from which the odd antihole $\overline{C_{2\ell-1}}$ cannot be coloured are the constant ones. Consequently, an odd-hole-free k -critical graph of maximum degree Δ containing an odd antihole of length $2(\Delta - k) + 7$ is that antihole.

1. INTRODUCTION

All graphs are finite and simple. A *hole* is an induced cycle of length at least four and an *antihole* is an induced copy of the complement $\overline{C_t}$ of a cycle C_t with $t \geq 4$ ¹; holes and antiholes are *odd* when t is odd. By the Strong Perfect Graph Theorem of Chudnovsky, Robertson, Seymour and Thomas [7], a graph whose chromatic number exceeds its clique number contains an odd hole or an odd antihole. Following Fiujaali and McDonald [9], a graph G is *k-critical* if $\chi(G) = k$ and $\chi(G - v) < k$ for every vertex v .

Fiujaali and McDonald [9] study the odd antiholes in critical graphs of given maximum degree. They prove that every odd antihole in a k -critical graph of maximum degree Δ has length at most $2(\Delta - k) + 7$, and they ask whether the bound is attained only by the antihole itself.

Conjecture A ([9, Conjecture 3]). *Let G be an odd-hole-free k -critical graph with maximum degree Δ . If G contains an odd antihole of length $2(\Delta - k) + 7$, then $G = \overline{C_{2(\Delta-k)+7}}$.*

They prove Conjecture A when $\Delta \leq k$ and when $\Delta \geq 2k - 5$ [9, Theorem 4]; the case $\Delta = k$ is a theorem of Chen, Lan, Lin and Zhou [6] and of Galindo, McDonald and Shan [10]. The range $k < \Delta < 2k - 5$ remained open, and its first case is $(k, \Delta) = (7, 8)$.

The tool is list colouring. A *list assignment* L for a graph G gives each vertex v a finite set $L(v)$ of colours, and G is *L-colourable* if there is a proper colouring f of G with $f(v) \in L(v)$ for every v . An *m-assignment* is a list assignment in which every list has exactly m colours. Following Kaul and Mudrock [12], a graph G is *strongly k-chromatic-choosable* if $\chi(G) = k$ and every $(k - 1)$ -assignment L for which G is not L -colourable is constant, that is, $L(u) = L(v)$ for all vertices u, v .

2020 *Mathematics Subject Classification.* Primary 05C15; Secondary 05C17, 05C38.

Key words and phrases. list colouring, odd antihole, chromatic-choosability, critical graph, odd hole.

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¹The cycle C_t is the Cayley graph of the cyclic group of order t with respect to a generator and its inverse. Cayley graphs with their word metrics, for Gromov hyperbolic groups, are studied in [3].

Fiujlaali and McDonald prove that $\overline{C_{2\ell-1}}$ is L -colourable for every non-constant $(\ell - 1)$ -assignment whose lists lie in a common set of ℓ colours [9, Theorem 8], and they conjecture that the common set can be dropped.

Conjecture B ([9, Conjecture 9]). *For every integer $\ell \geq 3$, the odd antihole $\overline{C_{2\ell-1}}$ is strongly ℓ -chromatic-choosable.*

They show that Conjecture B for ℓ implies Conjecture A for every pair (k, Δ) with $\Delta - k = \ell - 4$ [9, Theorem 10]. The case $\ell = 3$ is the classical fact that an odd cycle with lists of size two can be coloured unless all lists are equal. We prove the conjecture for every ℓ .

Theorem 1.1. *For every integer $\ell \geq 3$ the odd antihole $\overline{C_{2\ell-1}}$ is strongly ℓ -chromatic-choosable. More precisely, $\chi(\overline{C_{2\ell-1}}) = \ell$, and if L is a list assignment for $\overline{C_{2\ell-1}}$ in which every list has at least $\ell - 1$ colours and $\overline{C_{2\ell-1}}$ is not L -colourable, then all lists are one and the same set of $\ell - 1$ colours.*

Corollary 1.2. *Conjecture A holds: if G is an odd-hole-free k -critical graph of maximum degree Δ and G contains an odd antihole of length $2(\Delta - k) + 7$, then G is this antihole.*

Theorem 1.1 is a sharp form of Ohba's conjecture for odd antiholes. Ohba's conjecture, proved by Noel, Reed and Wu [15], states that a k -chromatic graph with at most $2k + 1$ vertices is k -choosable [16]; $\overline{C_{2\ell-1}}$ has $2\ell - 1$ vertices and chromatic number ℓ , so lists of size ℓ always suffice. Theorem 1.1 says that lists of size $\ell - 1$ suffice as well, unless they are all equal.

Strong chromatic-choosability is a relative of the strongly critical graphs of Stiebitz, Tuza and Voigt [17]: a graph is *strongly k -critical* if it is k -critical in the sense that every proper subgraph has smaller chromatic number, and every $(k - 1)$ -assignment from which it cannot be coloured is constant. Complete graphs, odd cycles, two further families of graphs on $2k - 1$ vertices, and joins of strongly critical graphs with cliques are strongly critical [17], and Bernshteyn, Kaul, Mudrock and Sharma [5] add joins of critical graphs with large cliques. For $\ell \geq 4$ the antihole $\overline{C_{2\ell-1}}$ is not among these graphs, and it is not strongly critical: if $v_0v_1 \cdots v_{2\ell-2}v_0$ is the cycle, the vertices v_0 and v_3 have no common neighbour on it, so after the edge v_0v_3 is removed from $\overline{C_{2\ell-1}}$ every colour class still has at most two vertices, and the chromatic number is still ℓ .² Theorem 1.1 therefore gives an infinite family of strongly chromatic-choosable graphs that are not strongly critical; Kaul and Mudrock [12] and Bernshteyn, Kaul, Mudrock and Sharma [5] give other such graphs.

The proof. A colour class of $\overline{C_{2\ell-1}}$ is a clique of $C_{2\ell-1}$, so it is a single vertex or two consecutive vertices of the cycle. Removing two consecutive vertices from the cycle leaves a path, so the proof turns on colourings of complements of paths, and more generally of disjoint unions of paths with an odd number of vertices. On such a union we call the vertices in even positions of each path *outer*. If there are s outer vertices, they form a clique of size s in the complement, so constant lists of size $s - 1$ on the outer vertices cannot be completed to a colouring. Theorem 4.1 shows that, for lists of size at least $s - 1$, this is the only obstruction, whatever the

²Every graph in the two families of [17] on $2k - 1$ vertices has a vertex of degree $k - 1$, whereas $\overline{C_{2\ell-1}}$ is regular of degree $2\ell - 4 > \ell - 1$ for $\ell \geq 4$. The antihole is not a join, since its complement $C_{2\ell-1}$ is connected.

lists of the other vertices are. The proof is an induction on the number of vertices, driven by Hall's theorem and by the choosability of complete multipartite graphs with parts of size at most two (Lemma 2.2); the extremal case, a single path with $2k$ colours in all, is settled separately in Lemma 3.1. Theorem 5.2 then closes the cycle: if two consecutive vertices share a colour c , the path theorem applied to the rest of the cycle with c removed propagates c and the lists around the cycle in steps of two, and since the cycle is odd these steps reach every vertex.³ Figure 1 shows the order of the argument. Section 6 deduces Corollary 1.2 by the argument of Fiujaali and McDonald.

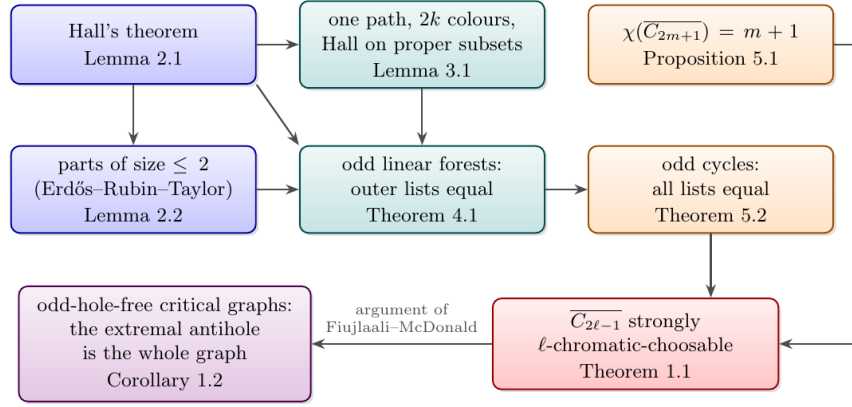


FIGURE 1. The order of the proof. Arrows point from a result to the results that use it.

Theorem 1.1, with all the lemmas of Sections 2–5 on which it rests, has been formalized in the Lean 4 proof assistant (Section 7).

2. COLOURINGS OF COMPLEMENTS

Throughout, H is a graph with vertex set V and L is a list assignment on V . An L -colouring of the complement $\overline{H}$ is a map f with $f(v) \in L(v)$ for every $v \in V$ such that two distinct vertices receive the same colour only if they are adjacent in H . For $W \subseteq V$ we write

$$L(W) = \bigcup_{w \in W} L(w), \quad \text{def}(W) = |W| - |L(W)|,$$

so that $\text{def}(\emptyset) = 0$. The complement of an induced subgraph $H[X]$ is the induced subgraph $\overline{H}[X]$, and we use this without comment. We use Hall's theorem [11] in the following form.⁴

³The passage from colourings of the paths obtained by cutting the cycle to a colouring of the whole cycle is a gluing step. Local-to-global gluing conditions of this kind, for signals on circuits laid out as networks, are formalised with sheaves in [4].

⁴Hall's condition $|L(W)| \geq |W|$ is a condition on the intersection pattern of the family of lists. Intersection patterns of families of sets, recorded in nerve complexes, are used in [2] to decide how many times a region is covered.

Lemma 2.1. *If $|L(W)| \geq |W|$ for every $W \subseteq V$, then there is an injective map f on V with $f(v) \in L(v)$ for every v . Such a map is an L -colouring of $\overline{H}$ for every graph H on V .*

The next lemma is the theorem of Erdős, Rubin and Taylor [8] that complete multipartite graphs whose parts have at most two vertices are chromatic-choosable. We include the short proof, since the same counting returns in Sections 3 and 5.

Lemma 2.2. *Let $q \geq 0$, and suppose that V is the union of at most q pairwise disjoint sets, each of which is a single vertex or the two ends of an edge of H . If $|L(v)| \geq q$ for every $v \in V$, then $\overline{H}$ is L -colourable.*

Proof. We use induction on q . If $q = 0$ then V is empty. Let $q \geq 1$. Suppose first that some part $\{x, y\}$ has a colour $c \in L(x) \cap L(y)$. Give c to x and to y , and remove c from the lists of the other vertices. The remaining vertices are covered by at most $q - 1$ parts and their lists keep at least $q - 1$ colours, so by induction they can be coloured. Since xy is an edge of H and no other vertex uses c , the result is an L -colouring of $\overline{H}$.

Otherwise the two lists of each part are disjoint, so each colour lies in the lists of at most one vertex of each part, and hence in at most q lists. For $W \subseteq V$, counting pairs (w, c) with $w \in W$ and $c \in L(w)$ gives

$$q|W| \leq \sum_{w \in W} |L(w)| \leq q|L(W)|,$$

so $|L(W)| \geq |W|$, and Lemma 2.1 applies. $\square$

3. THE KEY LEMMA

This section treats a single path with an odd number of vertices in the extremal situation that arises in the proof of Theorem 4.1.

Lemma 3.1. *Let $k \geq 1$, let $P = v_0 v_1 \cdots v_{2k}$ be a path with vertex set V , and let L be a list assignment with $|L(v)| \geq k$ for every $v \in V$, with $|L(V)| = 2k$, and with $|L(W)| \geq |W|$ for every $W \subsetneq V$. If $\overline{P}$ is not L -colourable, then $L(v_j) \cap L(v_{j+1}) = \emptyset$ for $0 \leq j < 2k$. Consequently every list has exactly k colours and $L(v_0) = L(v_2) = \cdots = L(v_{2k})$.*

Proof. Suppose that $\overline{P}$ is not L -colourable. Call a nonempty set $W \subsetneq V$ *tight* if $|L(W)| \leq |W|$; by hypothesis a tight set satisfies $|L(W)| = |W|$. A tight set is *minimal* if no proper nonempty subset of it is tight.

(a) *For every $x \in V$ there is a minimal tight set not containing x .* The set $V \setminus \{x\}$ is nonempty, and it is tight because $|L(V \setminus \{x\})| \leq |L(V)| = 2k = |V \setminus \{x\}|$. A tight subset of $V \setminus \{x\}$ of least size is minimal.

(b) *Matchings that avoid one vertex.* For $x \in V$, every subset of $V \setminus \{x\}$ is a proper subset of V , so Lemma 2.1 gives an injective map φ_x on $V \setminus \{x\}$ with $\varphi_x(v) \in L(v)$. If W is tight and $x \notin W$, then $\varphi_x(W) \subseteq L(W)$ and $|\varphi_x(W)| = |W| \geq |L(W)|$, so $\varphi_x(W) = L(W)$. By injectivity,

$$(1) \quad \varphi_x(v) \notin L(W) \quad \text{for every } v \in V \setminus (W \cup \{x\}).$$

(c) *A shared colour across the boundary of a minimal tight set gives a colouring.* Let W be a minimal tight set, and let $x \notin W$ and $y \in W$ be adjacent in P with a common colour $c \in L(x) \cap L(y)$. A nonempty set $Y \subseteq W \setminus \{y\}$ is a proper subset of

W , so it is not tight and $|L(Y)| \geq |Y| + 1$; hence $|L(Y) \setminus \{c\}| \geq |Y|$. By Lemma 2.1 there is an injective map g on $W \setminus \{y\}$ with $g(v) \in L(v) \setminus \{c\}$. Put

$$f(x) = f(y) = c, \quad f(v) = g(v) \quad (v \in W \setminus \{y\}), \quad f(v) = \varphi_x(v) \quad (v \notin W \cup \{x\}).$$

The colours $g(v)$ are distinct and lie in $L(W) \setminus \{c\}$; by (1) the colours $\varphi_x(v)$ with $v \notin W \cup \{x\}$ are distinct and lie outside $L(W)$, which contains c . So x and y are the only two vertices with a common colour, and they are adjacent in P . Thus f is an L -colouring of $\bar{P}$, which is impossible. Hence no such W , x , y exist.

(d) *Disjoint lists across the boundary give a colouring.* Let W be a tight set, let $y \in W$, and let $b, z \notin W$ be adjacent in P with $L(b) \cap L(y) = \emptyset$. The lists $L(b)$ and $L(y)$ are disjoint subsets of $L(V)$ with at least k colours each, so $L(b) \cup L(y) = L(V)$. By (1) with $x = b$ we have $\varphi_b(z) \notin L(W) \supseteq L(y)$, hence $\varphi_b(z) \in L(b)$. Colouring b with $\varphi_b(z)$ and every other vertex v with $\varphi_b(v)$ gives an L -colouring of $\bar{P}$ in which only b and z share a colour. This is impossible, so no such W , y , b , z exist.

(e) *Every edge of P meets every minimal tight set.* Suppose that W is a minimal tight set and $v_j, v_{j+1} \notin W$. Reversing the path if necessary, we may assume that W has a vertex v_h with $h < j$, and we choose h as large as possible, so that $v_{h+1}, \dots, v_{j+1} \notin W$. Put $y = v_h$ and $b = v_{h+1}$. If $L(b) \cap L(y) \neq \emptyset$, then b, y contradict (c). Otherwise $z = v_{h+2}$ satisfies $h + 2 \leq j + 1$, so $z \notin W$, and W , y , b , z contradict (d). See Figure 2.

(f) *Conclusion.* Let $0 \leq j < 2k$. By (a) there is a minimal tight set W with $v_j \notin W$, and by (e) $v_{j+1} \in W$. By (c), $L(v_j) \cap L(v_{j+1}) = \emptyset$. Two disjoint subsets of $L(V)$ with at least k colours each have exactly k colours each and are complements of each other in $L(V)$. Hence $L(v_{j+1}) = L(V) \setminus L(v_j)$ for every j , and $L(v_{j+2}) = L(V) \setminus L(v_{j+1}) = L(v_j)$. $\square$

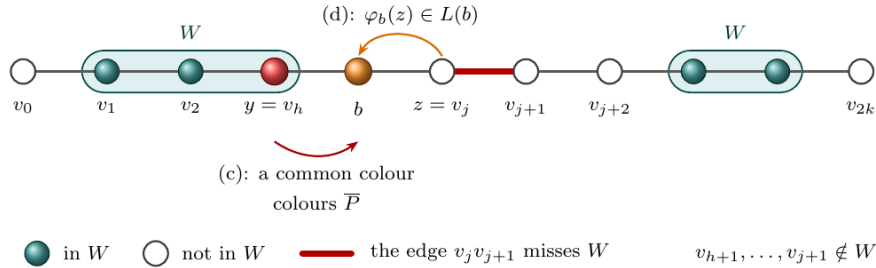


FIGURE 2. Step (e) of the proof of Lemma 3.1. The shaded vertices form a minimal tight set W and the edge $v_j v_{j+1}$ misses W . Either b and y share a colour, which gives a colouring by (c), or their lists are complementary, and then the matching φ_b gives z a colour of $L(b)$, which b can share with z by (d).

4. UNIONS OF ODD PATHS

An *odd linear forest* is a graph whose components are paths with an odd number of vertices. On a component $u_0 u_1 \dots u_{2r}$ the vertices $u_0, u_2, \dots, u_{2r}$ are *outer* and $u_1, u_3, \dots, u_{2r-1}$ are *inner*. If P has p components, s outer vertices and t inner

vertices, then $s = t + p$, so $t \leq s - 1$ when P is nonempty. Two outer vertices are never adjacent in P , so the outer vertices form a clique of size s in $\overline{P}$.

Theorem 4.1. *Let P be a nonempty odd linear forest with s outer vertices, and let L be a list assignment for P with $|L(v)| \geq s - 1$ for every vertex v . Then $\overline{P}$ is not L -colourable if and only if all outer vertices have the same list and this list has exactly $s - 1$ colours.*

Proof. If the outer vertices have a common list of $s - 1$ colours, they cannot be coloured, since they form a clique of size s in $\overline{P}$. We prove the converse by induction on the number of vertices. Let P , with vertex set V , s outer and t inner vertices, and L be such that $\overline{P}$ is not L -colourable, and suppose that the converse holds for all odd linear forests with fewer vertices. Assume, for a contradiction, that the outer lists are not one common list of $s - 1$ colours.

Step 1: $\text{def}(W) < \text{def}(V)$ for every $W \subsetneq V$. Let $X \subseteq V$ have the largest deficiency, and suppose that $X \neq V$. We first show that $\overline{P}[X]$ is L -colourable.

If some outer vertex w is not in X , assign to every inner vertex u an outer neighbour $\pi(u) \neq w$: on the component of w , the neighbour of u on the side away from w , and on every other component, the neighbour of u nearer to a fixed end of that component. Each outer vertex receives at most one inner vertex in this way, and w receives none (Figure 3). The sets $X \cap (\{o\} \cup \pi^{-1}(o))$, for outer vertices $o \neq w$, cover X ; there are at most $s - 1$ of them, and each is a single vertex or the two ends of an edge of P . Lemma 2.2 with $q = s - 1$ colours $\overline{P}[X]$.

If X contains every outer vertex, then $P[X]$ is an odd linear forest with the same outer vertices as P , since deleting an inner vertex splits its path into two paths with an odd number of vertices whose outer vertices keep their positions. As $|X| < |V|$, the induction hypothesis and our assumption on the outer lists show that $\overline{P}[X]$ is L -colourable.

Let f_X be an L -colouring of $\overline{P}[X]$, and put $L'(v) = L(v) \setminus L(X)$ for $v \in V \setminus X$. For $Z \subseteq V \setminus X$ the set $L(X \cup Z)$ is the disjoint union of $L(X)$ and $L'(Z)$, so $\text{def}(X \cup Z) \leq \text{def}(X)$ gives $|L'(Z)| \geq |Z|$. By Lemma 2.1 there is an injective map g on $V \setminus X$ with $g(v) \in L'(v)$. The colours of g are distinct and avoid $L(X)$, so f_X and g together form an L -colouring of $\overline{P}$, a contradiction. Hence V is the only set of largest deficiency, which is Step 1. Taking $W = \emptyset$ we obtain $\text{def}(V) \geq 1$, that is,

$$(2) \quad |L(V)| \leq |V| - 1 = s + t - 1.$$

Step 2: $t \geq 1$. If $t = 0$, then $|V| = s$ and every list is a subset of $L(V)$ with at least $s - 1 \geq |L(V)|$ colours, so every list equals $L(V)$ and has $s - 1$ colours, contrary to our assumption.

Step 3: any two lists meet. Suppose that $L(u) \cap L(v) = \emptyset$ for some $u \neq v$. Then $2(s - 1) \leq |L(V)| \leq s + t - 1$ by (2), so $t \geq s - 1$, and since $t = s - p \leq s - 1$ we get $t = s - 1$ and $p = 1$. Thus P is a single path $v_0 v_1 \cdots v_{2k}$ with $k = s - 1 = t \geq 1$, $|L(V)| = 2k$, and $\text{def}(V) = 1$. By Step 1, $\text{def}(W) \leq 0$, that is $|L(W)| \geq |W|$, for every $W \subsetneq V$. Every list has at least k colours, so Lemma 3.1 shows that $L(v_0) = L(v_2) = \cdots = L(v_{2k})$ with $k = s - 1$ colours, contrary to our assumption.

Step 4: the two ends of a path. By Step 2 some component $Q = u_0 u_1 \cdots u_{2r}$ has $r \geq 1$, so its ends $u_0 \neq u_{2r}$ are outer. By Step 3 there are colours $c \in L(u_0) \cap L(u_1)$ and $c' \in L(u_{2r}) \cap L(u_{2r-1})$.

Let $P' = P - \{u_0, u_1\}$. It is an odd linear forest whose outer vertices are those of P other than u_0 , so it has $s - 1 \geq 1$ of them. Put $L'(v) = L(v) \setminus \{c\}$, a list with at least $s - 2$ colours. If $\overline{P'}$ had an L' -colouring, giving c to u_0 and u_1 would extend it to an L -colouring of $\overline{P}$, since u_0u_1 is an edge of P . By the induction hypothesis the outer vertices of P' have one common L' -list of $s - 2$ colours. For an outer vertex $v \neq u_0$ the list $L(v)$ has at least $s - 1$ colours, so $c \in L(v)$ and $L(v) = L'(v) \cup \{c\}$. Hence

(3) $L(v) = S_0$ for every outer vertex $v \neq u_0$, where $|S_0| = s - 1$ and $c \in S_0$.

The same argument with u_{2r}, u_{2r-1} and c' shows that

(4) $L(v) = S_1$ for every outer vertex $v \neq u_{2r}$, where $|S_1| = s - 1$.

If $s \geq 3$, there is an outer vertex $z \notin \{u_0, u_{2r}\}$, and $L(u_0) = S_1 = L(z) = S_0 = L(u_{2r})$ by (4) and (3). If $s = 2$, the outer vertices are u_0 and u_{2r} ; the list $L(u_0) = S_1$ has one colour and contains c , and $L(u_{2r}) = S_0$ has one colour and contains c , so $L(u_0) = L(u_{2r}) = \{c\}$. In either case all outer vertices have one common list of $s - 1$ colours, contrary to our assumption. $\square$

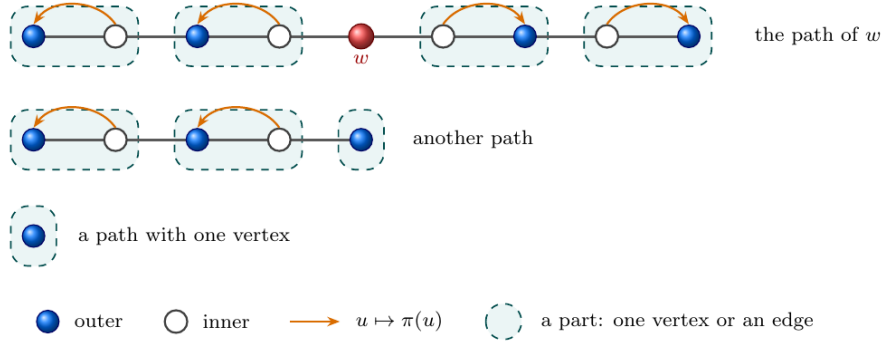


FIGURE 3. Step 1 of the proof of Theorem 4.1. Outer vertices are filled and inner vertices are hollow. Each inner vertex is sent to an outer neighbour other than w ; on the path of w it is sent away from w . The resulting parts (outlined) are single vertices or edges of P , and there are at most $s - 1$ of them.

5. ODD CYCLES

Let $m \geq 2$ and let $C = v_0v_1 \cdots v_{2m}v_0$ be the cycle of length $2m + 1$; indices are taken modulo $2m + 1$. The complement $\overline{C}$ is the odd antihole $\overline{C_{2m+1}}$.

Proposition 5.1. $\chi(\overline{C_{2m+1}}) = m + 1$ for every $m \geq 2$.

Proof. A colour class of $\overline{C}$ is a clique of C , and C has no triangles, so every class has at most two vertices and at least $m + 1$ colours are needed. The classes $\{v_{2i}, v_{2i+1}\}$ for $0 \leq i < m$ and $\{v_{2m}\}$ form a colouring with $m + 1$ colours. $\square$

Theorem 5.2. Let $m \geq 2$ and let L be a list assignment for $\overline{C}$ with $|L(v)| \geq m$ for every vertex v . If $\overline{C}$ is not L -colourable, then all lists are equal.

Proof. Suppose that $\overline{C}$ is not L -colourable.

Claim 1. *Some two consecutive lists meet.* Otherwise the vertices whose lists contain a given colour are pairwise non-consecutive, so they form an independent set of C and there are at most m of them. As in the proof of Lemma 2.2, $m|W| \leq \sum_{w \in W} |L(w)| \leq m|L(W)|$ for every set W of vertices, and Lemma 2.1 gives an L -colouring of $\overline{C}$.

Claim 2. *If $c \in L(v_j) \cap L(v_{j+1})$, then $c \in L(v_{j+r})$ and $L(v_{j+r}) = L(v_{j+2})$ for every even r with $2 \leq r \leq 2m$.* The graph $C - \{v_j, v_{j+1}\}$ is the path $v_{j+2}v_{j+3} \cdots v_{j+2m}$ with $2m - 1$ vertices, whose outer vertices are the m vertices v_{j+r} with r even, $2 \leq r \leq 2m$, and its complement is $\overline{C} - \{v_j, v_{j+1}\}$. Put $L'(v) = L(v) \setminus \{c\}$, a list with at least $m - 1$ colours. An L' -colouring of this complement, together with the colour c on v_j and v_{j+1} , would be an L -colouring of $\overline{C}$. So Theorem 4.1 with $s = m$ shows that the lists $L'(v_{j+r})$, r even, are one common set of $m - 1$ colours. Since $|L(v_{j+r})| \geq m$, each $L(v_{j+r})$ contains c and equals $L'(v_{j+r}) \cup \{c\}$. This proves Claim 2 (Figure 4).

Conclusion. By Claim 1 there are j and c with $c \in L(v_j) \cap L(v_{j+1})$. Write $L_r = L(v_{j+r})$. We apply Claim 2 four times, at the consecutive pairs starting at v_j, v_{j-1}, v_{j+1} and v_{j+2} .

- (i) At v_j : $c \in L_r$ and $L_r = L_2$ for even $r \in [2, 2m]$. In particular $c \in L_{2m} = L_{-1}$ and $c \in L_2$.
- (ii) At v_{j-1} , since $c \in L_{-1} \cap L_0$: $c \in L_r$ and $L_r = L_1$ for odd $r \in [1, 2m - 1]$. In particular $c \in L_3$, as $m \geq 2$.
- (iii) At v_{j+1} , since $c \in L_1 \cap L_2$, with $r = 2m$: $L_0 = L_{1+2m} = L_3$.
- (iv) At v_{j+2} , since $c \in L_2 \cap L_3$, with $r = 2m$: $L_1 = L_{2+2m} = L_4$.

By (iii) and (ii), $L_0 = L_3 = L_1$, and by (iv) and (i), $L_1 = L_4 = L_2$. So $L_0 = L_1 = L_2$, and by (i) and (ii) every list equals L_1 or L_2 . $\square$

Proof of Theorem 1.1. Put $m = \ell - 1 \geq 2$, so that $\overline{C_{2\ell-1}} = \overline{C}$. By Proposition 5.1, $\chi(\overline{C}) = \ell$. Let L be a list assignment in which every list has at least $\ell - 1 = m$ colours and suppose that $\overline{C}$ is not L -colourable. By Theorem 5.2 all lists are one set S . If $|S| \geq \ell$, a colouring of $\overline{C}$ with ℓ colours from S would be an L -colouring, so $|S| = \ell - 1$. In particular every $(\ell - 1)$ -assignment from which $\overline{C}$ cannot be coloured is constant, and $C_{2\ell-1}$ is strongly ℓ -chromatic-choosable. $\square$

6. ODD ANTIHOLES IN CRITICAL GRAPHS

We now deduce Corollary 1.2. The argument is that of Fiujaali and McDonald [9, Theorem 10], with Theorem 1.1 in place of Conjecture B; we give it in full so that the proof is complete.

Proof of Corollary 1.2. Let A be an odd antihole of length $2\ell - 1$ in G , where $\ell = \Delta - k + 4$; an odd antihole has length at least five, so $\ell \geq 3$. Suppose that $G \neq A$. Since A is an induced subgraph of G , there is a vertex $v \notin V(A)$. Since A is a subgraph of $G - v$, criticality gives $\ell = \chi(A) \leq \chi(G - v) \leq k - 1$. Put $q = k - \ell \geq 1$, so that

$$k = \ell + q \quad \text{and} \quad \Delta = 2\ell + q - 4.$$

By criticality $G - V(A)$ has a proper colouring φ with a set P of $k - 1 = \ell + q - 1$ colours. For $v \in V(A)$ let $L(v) = P \setminus \varphi(N(v) \setminus V(A))$. Each vertex of A has

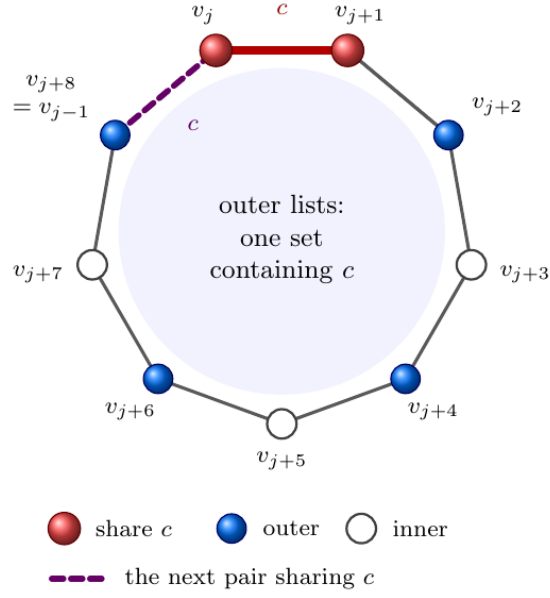


FIGURE 4. Claim 2 in the proof of Theorem 5.2 for $m = 4$. The consecutive vertices v_j and v_{j+1} share the colour c . The rest of the cycle is a path whose outer vertices $v_{j+2}, v_{j+4}, \dots, v_{j+2m}$ (filled) then all carry one list containing c . Since $v_{j+2m} = v_{j-1}$, the colour c passes to the pair $v_{j-1}v_j$, and the claim applies again.

$2\ell - 4$ neighbours in A , hence at most $\Delta - (2\ell - 4) = q$ neighbours outside A , and $|L(v)| \geq \ell - 1$. An L -colouring of A would extend φ to a proper colouring of G with $k - 1$ colours, so by Theorem 1.1 all the lists $L(v)$ are one set S of $\ell - 1$ colours.

Let $F = P \setminus S$, so $|F| = q$. Every $v \in V(A)$ has a neighbour outside A of each colour in F and at most q neighbours outside A . Hence v has exactly one outside neighbour $z_v(c)$ of each colour $c \in F$, and no outside neighbour with a colour in S .

Fix $c \in F$ and $a \in S$, and consider the components of the subgraph of $G - V(A)$ induced by the vertices of colours a and c . Suppose that one of these components, B , contains $z_x(c)$ but not $z_y(c)$ for some $x, y \in V(A)$. Exchanging a and c on B gives a proper colouring φ' of $G - V(A)$. The only outside neighbour of a vertex $v \in V(A)$ whose colour is a or c is $z_v(c)$, so the new lists are $(S \setminus \{a\}) \cup \{c\}$ when $z_v(c) \in B$ and S otherwise. These are $(\ell - 1)$ -sets and they are not all equal, so Theorem 1.1 colours A from them, and G is $(k - 1)$ -colourable, a contradiction. Hence all the vertices $z_v(c)$, $v \in V(A)$, lie in one component B .

Suppose next that $z_x(c) \neq z_y(c)$ for some $x, y \in V(A)$. The antihole A is connected, so we may take x and y adjacent. Let $T = t_0 t_1 \dots t_{2h}$ be a shortest path in B from $t_0 = z_x(c)$ to $t_{2h} = z_y(c)$; its vertices alternate between the colours c and a , so its length $2h$ is even and positive. A chord $t_i t_j$ of T would join vertices of different colours, so it would be an edge of B and would shorten T ; hence T is an induced path. The vertex x has no outside neighbour of colour a and exactly one of colour c , so t_0 is its only neighbour on T ; likewise t_{2h} is the only neighbour

of y on T . Then $xt_0t_1\cdots t_{2h}yx$ is an induced cycle of odd length $2h+3 \geq 5$, an odd hole in G , a contradiction. So for each $c \in F$ there is one vertex z_c adjacent to every vertex of A .

The set $Z = \{z_c : c \in F\}$ has q elements, and $N(v) \setminus V(A) = Z$ for every $v \in V(A)$. Every $z \in Z$ has $2\ell - 1$ neighbours in A , so

$$d_{G-V(A)}(z) \leq \Delta - (2\ell - 1) = q - 3,$$

and in particular $q \geq 3$. Let $R = G - (V(A) \cup Z)$, keep the colouring φ on R , and choose a set $Q \subseteq P$ of $q - 1$ colours. For $z \in Z$ let $M(z) = Q \setminus \varphi(N(z) \cap V(R))$. Then

$$|M(z)| \geq q - 1 - d_R(z) \geq q - 1 - (q - 3 - d_{G[Z]}(z)) = d_{G[Z]}(z) + 2,$$

so $G[Z]$ can be coloured greedily from the lists $M(z)$. This gives a proper colouring of $G - V(A)$ in which Z uses only colours of Q . Every vertex of A has outside neighbourhood Z , so the ℓ colours of $P \setminus Q$ are available at every vertex of A , and since $\chi(A) = \ell$ the colouring extends to G with $k - 1$ colours. This contradicts $\chi(G) = k$, so $G = A$. $\square$

7. FORMAL VERIFICATION

Theorem 1.1 has been formalized in Lean 4 [14] with the Mathlib library [13]. The formal statement takes the odd antihole on $2\ell - 1$ vertices to be the complement of the cycle graph of Mathlib, defines list colourings and strong chromatic-choosability as in Section 1 with colours from an arbitrary type, and asserts that for every $\ell \geq 3$ the antihole has chromatic number ℓ and that every assignment of $(\ell - 1)$ -element lists from which it cannot be coloured is constant. Its proof follows Sections 2–5: Lemmas 2.1–3.1, the converse direction of Theorem 4.1, Proposition 5.1 and Theorem 5.2 are all proved in the formalization, and the main theorem depends only on the standard axioms of Lean (propositional extensionality, choice and quotient soundness). Corollary 1.2 is not formalized; its proof is Section 6.

As an independent check, every assignment of lists of size $\ell - 1$ to $\overline{C_{2\ell-1}}$, up to renaming the colours, was examined for $\ell = 3$ (4,900 assignments) and $\ell = 4$ (7,897,749,769 assignments): the constant assignment is the only one from which the antihole cannot be coloured. No statement of this paper rests on this computation.

Data and code availability. The L^AT_EX source of the paper, the Lean 4 formalization of Theorem 1.1 and the programs of Section 7 are archived at Zenodo under doi:10.5281/zenodo.23393861 [1]; the paper is available at <https://github.com/creelie/odd-antihole/blob/main/preprintOddAntiholeConjecture/OddAntiholeConjecture.pdf>. Deep Bhattacharjee carried out the computations and the Lean, Julia, C, C++, Python and shell verifications, and the proofs and the text were drafted, with AI assistance (Claude, Anthropic); both authors approved the final version.

Ethical declarations. Funding: none. Competing interests: none.

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