

A graph with three card types and five vertex orbits

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Abstract

We prove that the answer to part (b) of Problem 20.71 of the Kourovka Notebook is yes: there is a connected graph on 63 vertices with exactly three isomorphism types of vertex-deleted subgraphs whose automorphism group has five orbits on the vertices. The graph is a Cayley graph of a Suzuki 2-group of order 64 with one vertex removed.

Keywords: vertex-deleted subgraph, pseudosimilar vertices, Cayley graph, Suzuki 2-group, divisible design graph

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1. Introduction

All graphs are finite and simple. For a vertex v of a graph G , the *card* $G - v$ is the subgraph induced on $V(G) \setminus \{v\}$. Let $k(G)$ be the number of isomorphism classes of cards of G and $r(G)$ the number of orbits of $\text{Aut}(G)$ on $V(G)$.³ An automorphism mapping u to v restricts to an isomorphism

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³Counting the orbits of a finite group on a finite configuration is the same task as counting the chamber types of a Voronoi domain under a Weyl group, which in [5] is done for $W(D_4)$ of order 192; here the group has order 64 and acts on 63 points, and the five orbits are told apart by invariants of the graph rather than of the geometry.

$G - u \rightarrow G - v$, so $k(G) \leq r(G)$. Vertices u, v in different orbits with $G - u \cong G - v$ are called *pseudosimilar*; they exist exactly when $k(G) < r(G)$.

Problem 20.71 of the Kourovka Notebook [1], posed by J. Lauri and communicated by V. Pannone, reads: “For a connected graph G with k isomorphism types of cards, can $\text{Aut}(G)$ have more than k orbits on the vertex set $V(G)$ a) for $k = 2$? b) for $k = 3$? c) for $k = 4$?” The comment to the problem records that $r(G) = 1$ when $k(G) = 1$, and that connected graphs with $(k, r) = (5, 6)$ and $(6, 7)$ are known. Prajapati [2] answered (c) with a connected graph on 9 vertices having $k = 4$ and $r = 5$, and found by exhaustive search that no connected graph on at most 12 vertices answers (a) or (b). We answer (b).

Theorem 1.1. *There is a connected graph X on 63 vertices with $k(X) = 3$ and $r(X) = 5$.*

The graph X is obtained from a Cayley graph Y of a Suzuki 2-group H of order 64 [3] by deleting the identity.⁴ Left multiplications of H make $X - x$ isomorphic to $X - x^{-1}$, and inversion of H exchanges two pairs of $\text{Aut}(X)$ -orbits (Section 4). The proof that these orbits are distinct rests on the fact that Y is a divisible design graph [4]: two distinct vertices have two common neighbours if they lie in the same coset of the centre of H and three otherwise (Section 2). The cosets carry a Fano plane defined by the graph, which separates the orbits (Section 3).⁵ The whole proof is formalized in Lean 4 and checked by its kernel (Section 5). Part (a) remains open.

2. The group and the graph

Let $\mathbb{F} = \mathbb{F}_2[t]/(t^3 + t + 1)$ be the field with 8 elements and $\mathbb{F}^\times = \mathbb{F} \setminus \{0\}$. Cubing is a bijection of $\mathbb{F}^\times$, since $\gcd(3, 7) = 1$. Let $\text{Tr}(\beta) = \beta + \beta^2 + \beta^4$ be the trace to $\mathbb{F}_2$, so $\text{Tr}(1) = 1$. Put

$$R_0 = \{t, t^2, t^4\}, \quad R_1 = \{t + 1, t^2 + 1, t^4 + 1\},$$

⁴A Cayley graph carries the word metric of its group with respect to the chosen generators. In the present finite 2-group every element is within distance two of the identity; the opposite extreme, where the word metric is negatively curved at large scale, is the setting of hyperbolic groups treated in [6].

⁵The plane is read off from the pattern of common neighbours between cosets, in the way that the intersection pattern of a family of sets is recorded by its nerve; the nerve and its iterated versions are studied in [7].

the roots of $x^3 + x + 1$ and of $x^3 + x^2 + 1$; then $\mathbb{F} = \{0, 1\} \cup R_0 \cup R_1$, Tr vanishes on R_0 and equals 1 on R_1 , and $\sum_{x \in R_0} x = 0$, $\sum_{x \in R_1} x = 1$. For $x \in R_0$ we have $x^3 = x + 1$ and $x^4 = x^2 + x$; for $x \in R_1$ we have $x^3 = x^2 + 1$ and $x^4 = x^2 + x + 1$.

Let H be the set $\mathbb{F} \times \mathbb{F}$ with the product

$$(a, b)(c, d) = (a + c, b + d + ac^2).$$

This is associative, since both bracketings of $(a, b)(c, d)(e, f)$ give $(a + c + e, b + d + f + ac^2 + ae^2 + ce^2)$. The identity is $e = (0, 0)$ and $(a, b)^{-1} = (a, b + a^3)$. The subgroup $Z = \{(0, b) : b \in \mathbb{F}\}$ is central, and its cosets are the sets $A_a = \{(a, b) : b \in \mathbb{F}\}$, $a \in \mathbb{F}$. For $c \in \mathbb{F}$ put $z_c = (0, c^3) \in Z$. Let

$$S = \{(a, 0), (a, a^3) : a \in \mathbb{F}^\times\}.$$

Then $|S| = 14$ and $S = S^{-1}$, as $(a, 0)^{-1} = (a, a^3)$. Let Y be the Cayley graph with vertex set H in which $g \sim h$ if and only if $g^{-1}h \in S$, and let $X = Y - e$. Then Y is 14-regular, and X has 63 vertices.

Lemma 2.1. $(a, b) \sim (c, d)$ in Y if and only if $a \neq c$ and $b + d \in \{a(a + c)^2, c(a + c)^2\}$.

Proof. $(a, b)^{-1}(c, d) = (a, b + a^3)(c, d) = (a + c, b + d + a^3 + ac^2) = (a + c, b + d + a(a + c)^2)$. This lies in S if and only if $a + c \neq 0$ and $b + d + a(a + c)^2 \in \{0, (a + c)^3\}$, and $(a + c)^3 + a(a + c)^2 = c(a + c)^2$. $\square$

Lemma 2.2. For $c \in \mathbb{F}^\times$ the maps $\mu_c(a, b) = (ca, c^3b)$ and $\varphi(a, b) = (a^2, b^2)$ are automorphisms of H with $\mu_c(S) = \varphi(S) = S$. Hence they are automorphisms of Y fixing e , and they restrict to automorphisms of X .

Proof. μ_c is a bijection and $\mu_c((a, b)(a', b')) = (c(a + a'), c^3(b + b' + aa'^2)) = (ca, c^3b)(ca', c^3b')$, because $ca(ca')^2 = c^3aa'^2$. The map φ is the Frobenius automorphism in both coordinates. Both map $(a, 0)$ to an element $(a', 0)$ and (a, a^3) to an element (a', a'^3) . $\square$

For $x \in H$ let $\lambda(x)$ be the number of pairs $(s, s') \in S \times S$ with $ss' = x$. If $g \neq h$, the common neighbours of g and h in Y are the vertices $gs = hs'$ with $s, s' \in S$, so their number is $\lambda(g^{-1}h)$, using $S = S^{-1}$.

Lemma 2.3. $\lambda(x) = 2$ for $x \in Z \setminus \{e\}$ and $\lambda(x) = 3$ for $x \notin Z$. Hence two distinct vertices of Y have exactly two common neighbours if they lie in the same coset of Z , and exactly three otherwise.

Proof. Write $s = (u, \delta u^3)$ and $s' = (u', \delta' u'^3)$ with $u, u' \in \mathbb{F}^\times$ and $\delta, \delta' \in \{0, 1\}$. Then $ss' = (u + u', \delta u^3 + \delta' u'^3 + uu'^2)$.

Let $x = (0, b)$ with $b \neq 0$. Then $u' = u$ and $b = (\delta + \delta' + 1)u^3$, which forces $\delta = \delta'$ and $u^3 = b$. There is one such u , and two choices of δ , so $\lambda(x) = 2$.

Let $x = (a, b)$ with $a \neq 0$. Since μ_a is an automorphism of H preserving S , $\lambda(a, b) = \lambda(\mu_a(1, a^{-3}b)) = \lambda(1, a^{-3}b)$, so we may take $a = 1$. Then $u' = 1 + u$ with $u \in \mathbb{F}^\circ = \mathbb{F} \setminus \{0, 1\}$, and $ss' = (1, f_{\delta\delta'}(u))$, where

$$f_{00}(u) = u + u^3, \quad f_{10}(u) = u, \quad f_{01}(u) = 1 + u^2, \quad f_{11}(u) = 1 + u^2 + u^3.$$

The maps f_{10} and f_{01} are bijections of $\mathbb{F}^\circ$. On R_0 we have $f_{00} = 1$ and $f_{11}(u) = u^2 + u = u^4$; on R_1 we have $f_{11} = 0$ and $f_{00}(u) = u^2 + u + 1 = u^4$. Since $u \mapsto u^4$ permutes R_0 and R_1 , every $b \in R_0$ is a value of f_{11} exactly once and of f_{00} never, and every $b \in R_1$ is a value of f_{00} exactly once and of f_{11} never. Hence each $b \in \mathbb{F}$ has exactly three preimages under the four maps. The value 0 is taken only by f_{11} , on the three elements of R_1 , and the value 1 only by f_{00} , on the three elements of R_0 . So $\lambda(1, b) = 3$ for every b . $\square$

Corollary 2.4. *X is connected, of diameter 2.*

Proof. Two distinct non-adjacent vertices of X have at least two common neighbours in Y by Lemma 2.3, so at least one in X . $\square$

3. The orbits of $\text{Aut}(X)$

Every element of $H \setminus Z$ is $(a, a^3\beta)$ with $a \neq 0$ and $\beta = a^{-3}b \in \mathbb{F}$. Define

$$\begin{aligned} P &= Z \setminus \{e\}, & O_0 &= \{(a, 0) : a \neq 0\}, & O_1 &= \{(a, a^3) : a \neq 0\}, \\ O_2 &= \{(a, a^3\beta) : a \neq 0, \beta \in R_0\}, & O_3 &= \{(a, a^3\beta) : a \neq 0, \beta \in R_1\}. \end{aligned}$$

They partition $V(X)$, have sizes 7, 7, 7, 21, 21, and $S = O_0 \cup O_1$; see Figure 1.

Lemma 3.1. *The group K generated by the maps μ_c and φ acts on each of P, O_0, O_1, O_2, O_3 transitively.*

Proof. $\mu_c(0, b) = (0, c^3b)$, and cubing is onto $\mathbb{F}^\times$. Further $\mu_c(a, a^3\beta) = (ca, (ca)^3\beta)$ and $\varphi(a, a^3\beta) = (a^2, (a^2)^3\beta^2)$, while $\{\beta, \beta^2, \beta^4\}$ equals R_0 for $\beta \in R_0$ and R_1 for $\beta \in R_1$. $\square$

Lemma 3.2. *Every automorphism of X extends to an automorphism of Y fixing e and preserving each of the sets P, S and $O_2 \cup O_3$. Every automorphism of Y permutes the cosets of Z .*

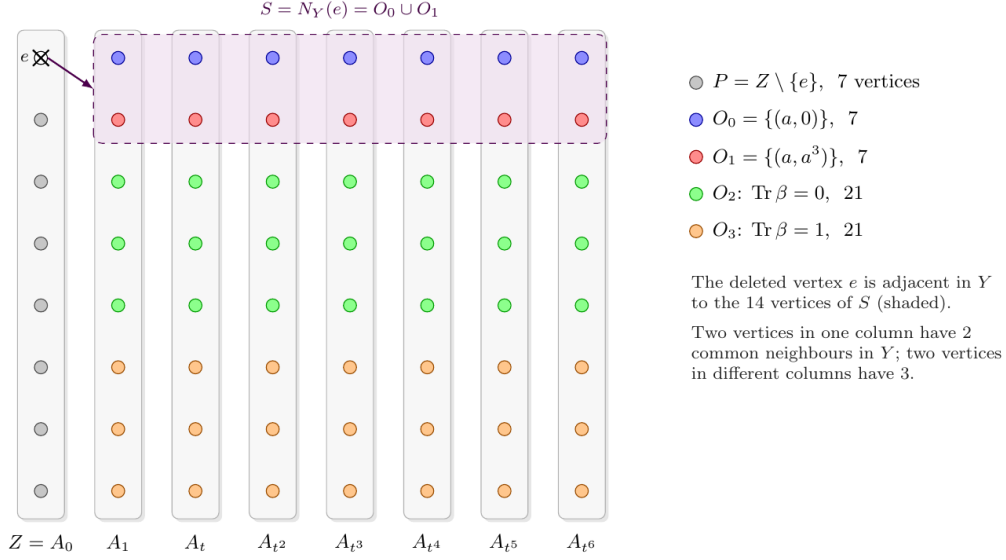


Figure 1: The vertices of Y arranged by the cosets of the centre Z . Deleting e gives X ; the fourteen neighbours of e form $S = O_0 \cup O_1$ and have degree 13 in X . Each coset other than Z meets O_0 and O_1 once and O_2 and O_3 three times.

Proof. In X the vertices of $S = N_Y(e)$ have degree 13 and all others degree 14. So an automorphism σ of X maps S onto S , and setting $\sigma(e) = e$ gives an automorphism of Y . By Lemma 2.3 the cosets of Z are the classes of the relation “ $g = h$, or g and h have exactly two common neighbours”, so every automorphism of Y permutes them; one fixing e maps Z onto Z , hence $P = Z \setminus \{e\}$ onto P , and it maps the complement $O_2 \cup O_3$ of $\{e\} \cup P \cup S$ onto itself. $\square$

Lemma 3.3. *Let $a \neq c$. Each $u = (a, b) \in A_a$ has exactly two neighbours in A_c , namely $(c, b + a(a+c)^2)$ and $(c, b + c(a+c)^2)$, and $uz_{a+c} = (a, b + (a+c)^3)$ is the only other vertex of A_a with the same two neighbours in A_c .*

Proof. The neighbours come from Lemma 2.1. The two values $\alpha = a(a+c)^2$ and $\alpha' = c(a+c)^2$ are distinct with $\alpha + \alpha' = (a+c)^3$. The vertex (a, b') has the neighbour set $\{(c, b' + \alpha), (c, b' + \alpha')\}$, which equals that of u if and only if $b' \in \{b, b + \alpha + \alpha'\}$. $\square$

For distinct cosets A, C let $\tau_{A,C}: A \rightarrow A$ map u to the other vertex of A with the same neighbours in C . By Lemma 3.3, $\tau_{A_a, A_c}(u) = uz_{a+c}$, and

$\sigma\tau_{A,C}\sigma^{-1} = \tau_{\sigma(A),\sigma(C)}$ for every automorphism σ of Y , since the definition uses only adjacency and the coset partition. On $Z = A_0$ write $\tau_c = \tau_{Z,A_c}$, so $\tau_c(0, b) = (0, b + c^3)$. Call three distinct cosets $A_{c_1}, A_{c_2}, A_{c_3} \neq Z$ *aligned* if $\tau_{c_1}\tau_{c_2}\tau_{c_3}$ is the identity of Z , that is, if $c_1^3 + c_2^3 + c_3^3 = 0$. An automorphism of Y fixing e fixes Z , conjugates τ_c to the map τ attached to $\sigma(A_c)$, and so maps aligned triples to aligned triples.

Lemma 3.4. *Each $s \in S$ has exactly three neighbours in S , lying in three distinct cosets of Z . These cosets are aligned if $s \in O_0$ and not aligned if $s \in O_1$.*

Proof. Let $s = (a, 0)$ and $(c, d) \in S$ with $c \neq a$. For $d = 0$, Lemma 2.1 would need $0 \in \{a(a+c)^2, c(a+c)^2\}$, which is impossible. For $d = c^3$ it needs $c^3 = c(a+c)^2$, which gives $a = 0$, or $c^3 = a(a+c)^2$, which for $x = c/a$ reads $x^3 = x^2 + 1$, that is $x \in R_1$. So the neighbours of s in S are $(ax, (ax)^3)$, $x \in R_1$, in the cosets A_{ax} , and

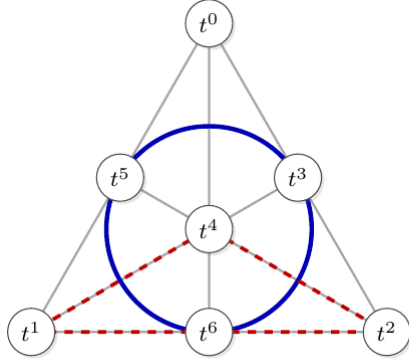
$$\sum_{x \in R_1} (ax)^3 = a^3 \sum_{x \in R_1} (x^2 + 1) = a^3 \left(\left(\sum_{x \in R_1} x \right)^2 + 1 \right) = 0.$$

Let $s = (a, a^3)$. For $(c, 0)$ the condition is $a^3 \in \{a(a+c)^2, c(a+c)^2\}$; the first gives $c = 0$, the second, with $x = c/a$, gives $x^3 + x = 1$, that is $x \in R_0$. For (c, c^3) it is $a^3 + c^3 \in \{a^3 + ac^2, a^2c + c^3\}$, which gives $c = a$ in both cases. So the neighbours of s in S are $(ay, 0)$, $y \in R_0$, in the cosets A_{ay} , and $\sum_{y \in R_0} (ay)^3 = a^3 \sum_{y \in R_0} (y + 1) = a^3 \neq 0$. $\square$

The aligned triples are the lines of a Fano plane on the seven cosets $A_c \neq Z$, since $c \mapsto c^3$ is a bijection of $\mathbb{F}^\times$ and the triples $\{x, y, x + y\}$ of nonzero elements of $\mathbb{F}$ are the lines of the Fano plane of the $\mathbb{F}_2$ -space $\mathbb{F}$. Figure 2 shows this plane and Lemma 3.4 for $a = 1$.

Lemma 3.5. *Each vertex of O_2 has one neighbour in O_0 and two in O_1 ; each vertex of O_3 has two neighbours in O_0 and one in O_1 .*

Proof. Let $v = (a, a^3\beta)$ with $\beta \in R_0 \cup R_1$, and write a vertex of S in a coset A_c , $c \notin \{0, a\}$, with $x = c/a \in \mathbb{F}^\circ$. Dividing the condition of Lemma 2.1 by a^3 , the vertex $(c, 0)$ is a neighbour of v if and only if $\beta \in \{(1+x)^2, x+x^3\}$, and (c, c^3) is a neighbour if and only if $\beta \in \{1+x^2+x^3, x\}$. In each pair the two values differ for $x \neq 1$, so the number of neighbours of v in O_0 is



- neighbours in S of $(1, 0) \in O_0$: cosets $A_{t^3}, A_{t^5}, A_{t^6}$, a line
- - - neighbours in S of $(1, 1) \in O_1$: cosets $A_{t^1}, A_{t^4}, A_{t^2}$, not a line

A point t^i is the coset A_{t^i} of Z . Three cosets $A_{c_1}, A_{c_2}, A_{c_3}$ lie on a line when $c_1^3 + c_2^3 + c_3^3 = 0$, that is, when the three translations τ_{c_j} of Z compose to the identity.

Figure 2: The seven cosets $A_c \neq Z$, $c = t^i$, as the points of a Fano plane whose lines are the aligned triples. The neighbours in S of $(1, 0) \in O_0$ lie in the cosets of a line, those of $(1, 1) \in O_1$ do not.

the number of solutions $x \in \mathbb{F}^\circ$ of $(1 + x)^2 = \beta$ plus that of $x + x^3 = \beta$, and similarly for O_1 .

The equations $(1 + x)^2 = \beta$ and $x + x^3 = \beta$ have one solution each in $\mathbb{F}^\circ$. By the proof of Lemma 2.3, $x + x^3$ equals 1 on R_0 and x^4 on R_1 , and $1 + x^2 + x^3$ equals x^4 on R_0 and 0 on R_1 . So $x + x^3 = \beta$ has one solution for $\beta \in R_1$ and none for $\beta \in R_0$, and $1 + x^2 + x^3 = \beta$ has one solution for $\beta \in R_0$ and none for $\beta \in R_1$. $\square$

Theorem 3.6. *The orbits of $\text{Aut}(X)$ on $V(X)$ are P , O_0 , O_1 , O_2 , O_3 . In particular $r(X) = 5$.*

Proof. By Lemmas 2.2 and 3.1 each of the five sets lies in one orbit. Let $\sigma \in \text{Aut}(X)$, extended to Y by Lemma 3.2. Then σ preserves P and S , permutes the cosets of Z , and maps aligned triples to aligned triples; as σ permutes the finite set of triples, it also maps triples that are not aligned to triples that are not aligned. It maps the three neighbours in S of a vertex $s \in S$ onto those of $\sigma(s)$, and their cosets onto the corresponding cosets. By Lemma 3.4, $\sigma(O_0) = O_0$ and $\sigma(O_1) = O_1$. Then σ preserves $O_2 \cup O_3$ and the numbers of neighbours in O_0 and in O_1 , so $\sigma(O_2) = O_2$ and $\sigma(O_3) = O_3$ by Lemma 3.5. $\square$

Remark 3.7. A complete computer search for automorphisms shows that Y has $1344 = 64 \cdot 21$ automorphisms and X has 21, so $\text{Aut}(X) = K$. This is not used in the proof.

4. The cards

Lemma 4.1. *For every vertex x of X , the cards $X - x$ and $X - x^{-1}$ are isomorphic.*

Proof. Left multiplication $g \mapsto x^{-1}g$ is an automorphism of Y mapping x to e and e to x^{-1} . It maps $V(Y) \setminus \{e, x\}$ onto $V(Y) \setminus \{e, x^{-1}\}$, so it restricts to an isomorphism $X - x = Y - \{e, x\} \rightarrow Y - \{e, x^{-1}\} = X - x^{-1}$. $\square$

Inversion fixes P pointwise, maps $(a, 0)$ to (a, a^3) , and maps $(a, a^3\beta)$ to $(a, a^3(\beta + 1))$ with $\text{Tr}(\beta + 1) = \text{Tr}(\beta) + 1$. So it exchanges O_0 with O_1 and O_2 with O_3 . In particular each $x \in O_0 \cup O_2$ is pseudosimilar to x^{-1} .

Theorem 4.2. *The cards of X fall into exactly three isomorphism classes: those at the vertices of P , of $O_0 \cup O_1$, and of $O_2 \cup O_3$. In particular $k(X) = 3$.*

Proof. Cards at vertices in one orbit are isomorphic, so Theorem 3.6 and Lemma 4.1 give at most these three classes. To separate them, count the vertices of degree 13 in $X - x = Y - \{e, x\}$. A vertex $u \neq e, x$ has degree $14 - [u \sim e] - [u \sim x]$ there, so these vertices are those of the symmetric difference of $N_Y(e)$ and $N_Y(x)$ other than e and x . Their number is $28 - 2\lambda(x) - 2[x \in S]$, which by Lemma 2.3 is 24 for $x \in P$, 20 for $x \in S = O_0 \cup O_1$, and 22 for $x \in O_2 \cup O_3$. $\square$

Theorem 1.1 follows from Corollary 2.4 and Theorems 3.6 and 4.2. Figure 3 summarizes the argument.

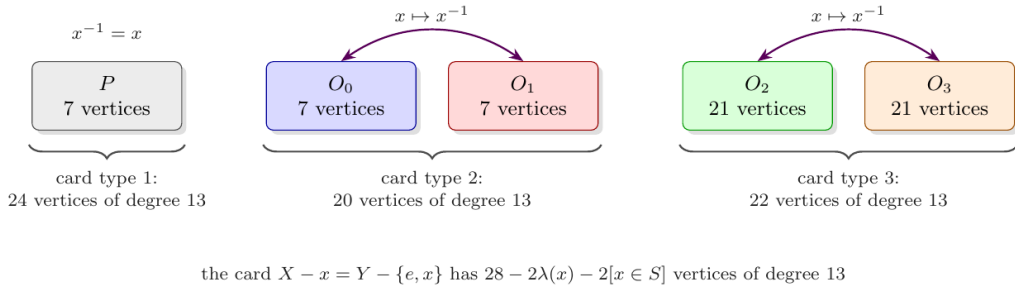


Figure 3: The five orbits of $\text{Aut}(X)$ and the three card types. Inversion in H fixes P pointwise and exchanges O_0 with O_1 and O_2 with O_3 , which gives the isomorphisms of Lemma 4.1; the number of vertices of degree 13 separates the three types.

5. Formal verification and computations

Theorem 1.1 is formalized in Lean 4 [8] with Mathlib [9], version 4.34.1, and the formal proof is checked by the Lean kernel. The formal statement says that there are a finite set V with 63 elements and a simple graph G on V such that G is connected, the relation “ $G - v$ and $G - w$ are isomorphic” on V has exactly three classes, and the relation “some automorphism of G maps v to w ” has exactly five classes. In the formal model the elements of $\mathbb{F}$ are coded by the integers $0, \dots, 7$ through their coordinates in the basis $1, t, t^2$, the vertex $k \in \{0, \dots, 62\}$ is the pair (a, b) with $8a + b = k + 1$, and adjacency is the formula of Lemma 2.1. The proof depends only on the axioms of Lean’s logic, namely propositional extensionality, choice and the soundness of quotients; it has no unproved step and adds no axiom.

The formal proof follows Sections 2–4 with one change of route. It gives the automorphisms μ_t and φ and the isomorphisms of Lemma 4.1 explicitly, and separates the five orbits by a single invariant in place of the Fano plane: for a vertex x , the number of pairs (y, z) with $x \sim y \sim z$, $z \neq x$, $z \not\sim x$, such that x and z have exactly two common neighbours and y and z have no common neighbour of degree 13. Every automorphism preserves this number, and it takes the values 6, 15, 21, 8, 4 on P, O_0, O_1, O_2, O_3 . The finitely many instances of each lemma are evaluated by the kernel itself.

Independent computations in exact integer arithmetic confirm that X has 63 vertices and 434 edges and is connected, that μ_t and φ are automorphisms with the orbits of Lemma 3.1, the values of the invariant above, Lemma 4.1 for every vertex, and the counts in the proof of Theorem 4.2.

Data and code availability

The L^AT_EX source of the paper, the Lean 4 formalization of Theorem 1.1 described in Section 5, and the Python, Julia, C, C++ and shell programs that recompute the graph, its orbits and its card types are archived at Zenodo under doi:10.5281/zenodo.23384197; the paper is available at <https://github.com/creelie/urban-chainsaw/blob/main/preprintKourovka2071b/Kourovka2071b.pdf>. Deep Bhattacharjee carried out the computations and the Lean, Julia, C, C++, Python and shell verifications, and the proofs and the text were drafted, with AI assistance (Claude, Anthropic); both authors approved the final version.

Ethical declarations

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References

- [1] Unsolved problems in group theory. The Kourovka notebook, no. 21, version 48 (2026). [arXiv:1401.0300](#), doi:10.48550/arXiv.1401.0300.
- [2] J. Prajapati, A graph with four card types and five vertex orbits: the $k = 4$ case of Kourovka Notebook Problem 20.71, Preprint, <https://kourovkanotebookorg.wordpress.com/wp-content/uploads/2026/10/kourovka-20-71-prajapati.pdf>; code at <https://github.com/infinityscroll/kourovka-20-71> (2026).
- [3] G. Higman, Suzuki 2-groups, Illinois J. Math. 7 (1963) 79–96. doi:10.1215/ijm/1255637483.
- [4] W. H. Haemers, H. Kharaghani, M. A. Meulenberg, Divisible design graphs, J. Combin. Theory Ser. A 118 (3) (2011) 978–992. doi:10.1016/j.jcta.2010.10.003.
- [5] D. Bhattacharjee, U. Bhattacharya, S. Bhattacharya, The sphere packing problem in dimension 4, Preprints.org (2026). doi:10.20944/preprints202606.0856.v11.
- [6] D. Bhattacharjee, Establishing equivariant class $[O]$ for hyperbolic groups, Asian Res. J. Math. 18 (11) (2022) 362–369. doi:10.9734/arjom/2022/v18i11615.
- [7] D. Bhattacharjee, U. Bhattacharya, Iterated nerve complexes for k -fold sensor coverage, Preprints.org (2026). doi:10.20944/preprints202606.1351.v2.
- [8] L. de Moura, S. Ullrich, The Lean 4 theorem prover and programming language, in: Automated Deduction – CADE 28, Vol. 12699 of Lecture Notes in Comput. Sci., Springer, Cham, 2021, pp. 625–635. doi:10.1007/978-3-030-79876-5_37.

- [9] The mathlib Community, The Lean mathematical library, in: Proceedings of the 9th ACM SIGPLAN International Conference on Certified Programs and Proofs, ACM, New York, 2020, pp. 367–381. doi: 10.1145/3372885.3373824.