

TOOLS FOR PROVING THAT GRAPHS ARE NOT GRACEFUL

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ABSTRACT. A graph with m edges is graceful if its vertices can be labeled with distinct integers in $[0, m]$ whose edge differences are exactly $1, \dots, m$. Graceful labelings have been constructed for sixty years, but there are few general methods for showing that a graph is *not* graceful. We start from a single generating-function identity that restates gracefulness, and derive three tools from it: residues at roots of unity, extending the classical parity condition; positivity on the unit circle, whose semidefinite form amounts exactly to the Goemans–Williamson max-cut relaxation tested against the minimum of the Dirichlet kernel; and the label interval itself, which forces the labels of a dense subgraph to have a large span. The consequences include new families of ungraceful graphs: every edge-transitive graph that contains K_6 , the Paley graphs $P(q)$ with $q \geq 37$, the Kneser graphs $K(n, k)$ with $n \geq 5.61k$, almost every d -regular graph with $d \geq 84$, and the polarity graphs ER_q with $q \geq 23$, which contain no 4-cycle. For the semidefinite form of positivity we also determine exactly what it can prove, and we show that a planar refinement is strictly stronger: it excludes the Kneser graph $K(11, 2)$ with any seven edges deleted.

About this article. The text of this article was written by Claude Opus 5.5, an AI model made by Anthropic, at the direction of Boon Suan Ho, who chose the questions and steered the work. Version 2 incorporates corrections and new results from a revision by GPT-6 Pro (OpenAI), and some of its passages are adapted from that revision. Results marked * were found by Claude, and results marked † by GPT-6 Pro. The author has not checked every proof in detail. Section 6 gives the details.

1. INTRODUCTION

A graph G with m edges is *graceful* if there is an injective map $f: V(G) \rightarrow \{0, 1, \dots, m\}$ such that the edge differences $|f(u) - f(v)|$, $uv \in E(G)$, are exactly $1, 2, \dots, m$. These labelings were introduced by Rosa [51, 50] under the name β -valuations, and named by Golomb [28]. Gallian’s dynamic survey [25] lists thousands of papers, almost all of which *construct* graceful labelings. Figure 1 shows a small example, and a connection that runs through this paper: on a complete subgraph, a graceful labeling is a *Golomb ruler*, a set of marks whose pairwise distances are all different.

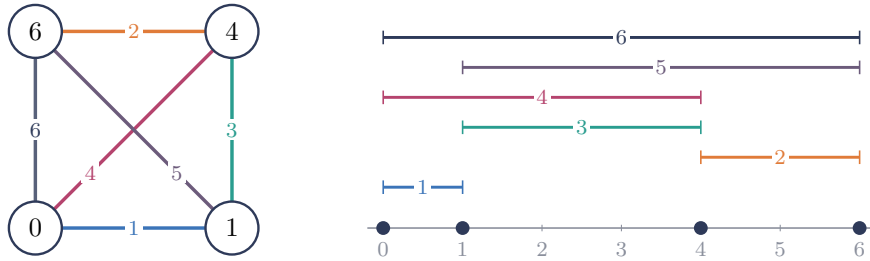


FIGURE 1. A graceful labeling of K_4 : the six edges have differences $1, \dots, 6$. Since K_4 is complete, this says that the marks $0, 1, 4, 6$ measure every distance from 1 to 6 exactly once. Such a ruler is a *perfect Golomb ruler*, and none exists with five or more marks.

family	not graceful when	source
<i>Tool I: parity and residues</i>		
graphs with all degrees even	$m \equiv 1, 2 \pmod{4}$	Bosák [51]; Cor. 3.3
friendship graphs $K_3^{(t)}$	$t \equiv 2, 3 \pmod{4}$	Bosák [51]; Ex. 3.4
windmills $K_5^{(t)}$	t odd, or $t = 2, 4$	[51, 3, 35, 29]; Rem. 4.7
decompositions into an odd number of K_5 's	always	Cor. 4.8
graphs split into two or three K_4 's	any overlap	[3]; Prop. 3.5
$K_s \vee \overline{K}_{2t}$	$s \equiv 8, 10 \pmod{16}$	Cor. 3.8 [†]
$K_{8k+6} \vee \overline{K}_{2t}$	$k + t$ odd	Cor. 3.9 [†]
$K_{1,1,1,1,a,b}$	$a \equiv b \equiv 2 \pmod{4}$	Cor. 3.10*
<i>Tool II: positivity, with clique and density certificates</i>		
complete graphs K_n	$n \geq 5$	Golomb [28]; Cor. 4.4
windmills $K_n^{(t)}$; unions tK_n	$n \geq 6$	[33, 34]; Cor. 4.4
barbells B_n	$n \geq 6$	Cor. 4.4*
$K_n \square P_r$, $K_n \square C_r$	$n \geq 13$ (all r)	Li [30]; Cor. 4.4
graphs with $m \geq 0.4108 n^2$ edges	always	[49, 38]; Cor. 4.5
complete multipartite $K_{a_1, \dots, a_r}$	$\sum a_i^2 \leq \frac{\kappa}{1+\kappa} n^2 + 1$	Cor. 4.5
graphs decomposing into K_n 's	$n \geq 6$	[3]; Thm. 4.6
mixed or fractional decompositions into $K_{\geq 6}$'s	always	Thm. 4.6*
edge-transitive graphs containing K_6	always	Cor. 4.8*
t copies of K_n sharing a K_r	$n \geq 2.4r + 6$, all t	Cor. 4.9*
line graphs of linear hypergraphs H	$\delta(H) \geq 6$	Cor. 4.11*
triangular graphs $T(n)$	$n \geq 7$	Cor. 4.11*
rook's graphs $K_a \square K_b$	$a + b \geq 12$; $K_2 \square K_9$, $K_3 \square K_8$	Cor. 4.11*
gen. quadrangles $\text{GQ}(s, t)$; nets of order n	$s \geq 5$; $n \geq 6$	Cor. 4.11*
Hamming graphs $H(d, q)$	$q \geq 6$, every d	Cor. 4.11*
Johnson graphs $J(n, k)$, $k \leq n/2$	$n \geq k + 5$	Cor. 4.11*
Grassmann graphs $J_q(n, k)$	all but K_3, K_4	Cor. 4.11* [†]
q -Kneser graphs $K_q(n, k)$, $k \geq 2$	all but $K_2(4, 2)$ (open)	Cor. 4.11 [†]
<i>Tool II: positivity, through eigenvalues</i>		
Kneser graphs $K(n, k)$	$n \geq 5.61 k$	Cor. 4.11*
Paley graphs $P(q)$	$q \geq 37$	Cor. 4.12*
graphs with $\lambda_{\min} \geq -2$	avg. degree ≥ 9.21	Cor. 4.14*
non-bipartite Ramanujan graphs	degree ≥ 84	Cor. 4.14*
random d -regular graphs	$d \geq 84$, a.a.s.	Cor. 4.13*
polarity graphs ER_q (no C_4)	$q \geq 23$	Cor. 4.14* [†]
<i>Tool II: the planar refinement</i>		
$K(11, 2)$ minus ≤ 7 edges; blowups, unions	always	Cor. 4.19 [†]
<i>Tool III: the interval</i>		
K_r plus $\lfloor r^2/4 \rfloor$ leaves at one vertex	$r \geq 46$	Cor. 5.2 [†]
t edge-disjoint K_{20} 's plus $\leq 59t + 4$ edges	always	Cor. 5.5 [†]

TABLE 1. Ungraceful families obtained in this paper. In the last column, a citation gives the first proof we know of, and a reference to a numbered result gives the proof here. The markers * and [†] indicate results that we believe are new (a row with both was found partly by each); see Section 6. Here $\kappa \approx 0.2172$ is Leech's constant (Lemma 2.1); notation is explained at the end of Section 1.

For a single graph the two directions are not symmetric. A labeling is a certificate of gracefulness that anyone can check, but in general there is no short certificate of the opposite, and an exhaustive search must rule out every injection of n vertices into $m + 1$ labels. For a whole family the difficulty can lie on either side: the conjecture of Ringel and Kotzig that every tree is graceful [25] has resisted both proof and counterexample for sixty years. For the negative direction, the classical tools are Bosák's parity condition, published by Rosa [51, 50], and Golomb's related partition condition [28], together with case-by-case

arguments and computer searches. We derive three tools, each applicable to every graph, from a single identity, and apply them to the families in Table 1.

All three start from the same observation. If the edge differences of a labeling are exactly $1, \dots, m$, then the generating function of the differences is known in advance:

$$\sum_{uv \in E} (z^{f(u)-f(v)} + z^{f(v)-f(u)}) = \sum_{0 < |k| \leq m} z^k.$$

The left side depends on the graph and the labeling; the right side does not. The identity sees only differences along edges, so it does not record that the labels are *distinct integers* in $[0, m]$: on a path with three edges, the labels 0, 1, 3, 0 satisfy it too. Tool I evaluates it at roots of unity and Tool II on the unit circle. Tools I and III also use part of this missing information (the capacities of residue classes, and the bounded span of the labels); Tool II needs only the identity itself, combined with positivity:

- I. Parity and residues** (Section 3). At $z = -1$, and more generally at roots of unity, the identity counts edges by the residues of their endpoints' labels; the extra information is that each residue class of $[0, m]$ holds only so many labels. This gives the classical conditions of Bosák and Golomb, an iterated form that is strictly stronger (Proposition 3.5), and integer equations for the residues of labels.
- II. Positivity on the unit circle** (Section 4). At $z = e^{it}$ the right side is $D_m(t) - 1$, where D_m is the Dirichlet kernel, whose minimum is below $-\kappa(2m+1)$. On the left, replacing each label x by the point e^{itx} turns the sum into a quadratic form in these points, and positive semidefinite matrices bound this form from below; the bound is strong for graphs close to unions of cliques and weak for graphs with a large cut. The resulting criterion (Theorem 4.1) has a spectral form, a combinatorial form (*a graph whose edges decompose, even fractionally, into cliques of order at least 6 is not graceful*), and an exact description as a max-cut relaxation.
- III. The interval** (Section 5). Labels are distinct integers in $[0, m]$, so a dense subgraph, which needs many distinct differences, forces a long label span. It excludes graphs that neither iterated parity nor positivity excludes (Proposition 5.4).

For positivity we determine exactly what its semidefinite form can prove (Proposition 4.15), and show that a planar refinement proves strictly more (Section 4.7); for parity we give graphs it does not exclude (Section 3.5), and we show that the interval excludes graphs that iterated parity and positivity both miss (Proposition 5.4); open cases that no tool reaches are collected in Section 7. Table 1 lists the families covered, grouped by tool. A few examples: every graph whose edges split into cliques with at least six vertices is not graceful (for cliques of one order this is, in the language of difference sets, a theorem of Bermond, Kotzig and Turgeon [3]), and neither is any edge-transitive graph that contains K_6 ; nor almost any d -regular graph with $d \geq 84$; nor the polarity graphs ER_q for $q \geq 23$, which contain no 4-cycle. Section 6 records the history and the origin of each result, and Section 7 collects open problems, among them the Beutner–Harborth conjecture on complete multipartite graphs.

How to read this paper. We assume only basic linear algebra (eigenvalues, positive semidefinite matrices), a little elementary number theory, and standard graph theory. After Section 2, the three tools can be read independently (except Proposition 5.4, which compares them). Each tool comes with a guiding example: cycles and K_9 for parity, complete graphs for positivity, and a clique with pendant leaves for the interval. Section 2.1 is background and can be skipped.

Notation. Graphs are finite and simple; G has n vertices and m edges, and we assume $m \geq 1$ (a graph with no edges is graceful exactly when it has at most one vertex). A labeling is a map $f: V(G) \rightarrow \{0, \dots, m\}$, and $d_{uv} = |f(u) - f(v)|$ is the difference on the edge uv . We write K_n , C_n and P_n for the complete graph, the cycle and the path on n vertices, $K_{a_1, \dots, a_r}$ for the complete multipartite graph with parts of sizes $a_1, \dots, a_r$, and

$\overline{K}_N$ for N isolated vertices. $G \vee H$ is the join (every vertex of G joined to every vertex of H), $G \square H$ the Cartesian product, tK_n the disjoint union of t copies of K_n , and $K_n^{(t)}$ the windmill of t copies of K_n sharing one vertex. A is the adjacency matrix, J the all-ones matrix, $\lambda_{\min}$ the smallest eigenvalue of A , and $M \succeq 0$ means that M is positive semidefinite. The Dirichlet kernel D_m , the trial frequency t^* , the number $\beta_m = 1 - \min_t D_m(t)$, and the constant $\kappa = 0.2172\dots$ (Leech's constant, minus the minimum of $\sin x/x$) are defined in Lemma 2.1. Other families of graphs are defined where they first appear.

2. THE MASTER IDENTITY

Let f be a graceful labeling of G , and for each edge $e = uv$ let $d_e = |f(u) - f(v)|$. Since $\{d_e\} = \{1, \dots, m\}$, we have, as Laurent polynomials in z (each edge contributes its difference with both signs),

$$(1) \quad \sum_{uv \in E} (z^{f(u)-f(v)} + z^{f(v)-f(u)}) = \sum_{0 < |k| \leq m} z^k.$$

The identity is elementary: it restates the definition as a generating function. For a complete graph it reads $A(z)A(z^{-1}) = n + \sum_{0 < |k| \leq m} z^k$ with $A(z) = \sum_v z^{f(v)}$, the familiar equation for Golomb rulers and, in cyclic form, for difference sets [6]; Rédei and Rényi [49] and Leech [38] used it on the unit circle, and Clark, Curtis, Gnanang and Hogben [18] relate graceful labelings to matrices of roots of unity. For the labeling of K_4 in Figure 1 both sides are $z^{-6} + \dots + z^{-1} + z + \dots + z^6$. On the unit circle the right-hand side is $D_m(t) - 1$, where D_m is the Dirichlet kernel.

Lemma 2.1 (Dirichlet kernel). *For $t \notin 2\pi\mathbb{Z}$,*

$$D_m(t) := 1 + 2 \sum_{k=1}^m \cos kt = \frac{\sin((m + \frac{1}{2})t)}{\sin(t/2)}.$$

Let $\alpha \approx 4.4934$ be the root of $\tan \alpha = \alpha$ in $(\pi, \frac{3}{2}\pi)$, and put

$$\kappa = -\min_{x>0} \frac{\sin x}{x} = -\frac{\sin \alpha}{\alpha} = 0.21723362821\dots, \quad \beta_m = 1 - \min_t D_m(t).$$

Then at the trial frequency $t^* = 2\alpha/(2m+1)$, which lies in the first interval where $D_m < 0$, we have $D_m(t^*) < -\kappa(2m+1)$, so $\beta_m > 1 + \kappa(2m+1)$.

Proof. Multiply by $\sin(t/2)$ and use $2 \sin(t/2) \cos kt = \sin((k + \frac{1}{2})t) - \sin((k - \frac{1}{2})t)$; the sum telescopes. At $t = 2\alpha/(2m+1)$ the numerator is $\sin \alpha < 0$, and the denominator is $\sin(\alpha/(2m+1)) < \alpha/(2m+1)$, so the value is below $-\kappa(2m+1)$. That α gives the global minimum of $\sin x/x$: the function is nonnegative on $[0, \pi]$, has its unique minimum on $[\pi, 2\pi]$ at α , and for $x \geq 2\pi$ its absolute value is at most $1/(2\pi) < 2/(3\pi)$, while its value at $\frac{3}{2}\pi$ is $-2/(3\pi)$. $\square$

Where does κ come from? Put $x = (m + \frac{1}{2})t$. For large m and small t , $\sin(t/2) \approx t/2$, so $D_m(t) \approx (2m+1) \sin x/x$: the kernel is a stretched copy of $\sin x/x$, whose deepest point is $-\kappa$ at $x = \alpha$. In words, as $m \rightarrow \infty$ the m cosines $\cos kt$ can be made to average about $-\kappa \approx -0.217$, and no lower. For finite m the minimum lies below $-\kappa(2m+1)$; β_m records its exact value, and κ gives convenient uniform bounds (Lemma A.1 bounds β_m from above). Every bound in Tool II is stated in terms of κ .

Rédei and Rényi [49] used the point $3\pi/(2m+1)$, where the numerator is -1 ; it gives the constant $2/(3\pi) \approx 0.2122$ in place of κ . Leech [38] used the minimum of $\sin x/x$, which gives κ . Neither point is exactly the minimum of D_m for finite m : the critical points satisfy $\tan x = N \tan(x/N)$, where $N = 2m+1$ and $x = Nt/2$, and the first one tends to α as $m \rightarrow \infty$. Figure 2 shows the first dip.

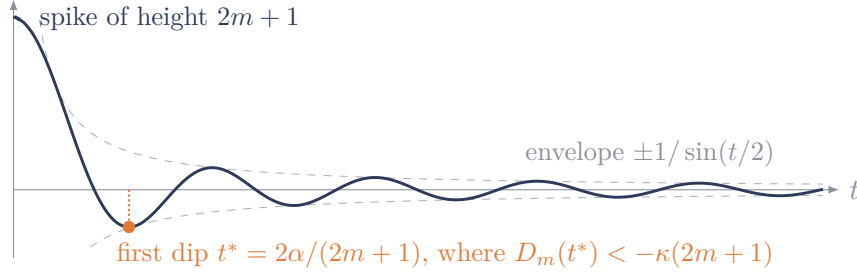


FIGURE 2. The Dirichlet kernel $D_m(t)$ for $m = 31$. Every positivity argument in Section 4 evaluates the master identity at, or near, the first dip.

2.1. A dictionary. Read modulo $2m + 1$, a graceful labeling becomes a ρ -labeling: an injective map $V \rightarrow \mathbb{Z}_{2m+1}$ whose edge differences, taken with both signs, are the nonzero residues, each once. Rosa showed that a ρ -labeling of G is the same thing as a cyclic decomposition of K_{2m+1} into copies of G [50]. A *difference family* is a collection of sets in a group whose differences cover every nonzero element exactly once; a graceful labeling is the integer, or “perfect”, form of a difference family whose base block is the graph G rather than a set. When G is complete the block is just a set, and we are back to difference sets. Write $A_S(z) = \sum_{v \in S} z^{f(v)}$ for a set S of vertices, $A = A_V$, and $\bar{A}(z) = A(z^{-1})$ (in this section, A is a polynomial, not the adjacency matrix). In the third row, “ X above Y ” means that every label in X exceeds every label in Y ; not every graceful labeling of $K_{a,b}$ has this form.

graph	left side of (1)	counterpart
K_n	$A\bar{A} - n$	Golomb ruler; cyclically, planar difference set [6]
clique partition $\{Q_i\}$	$\sum_i (A_{Q_i}\bar{A}_{Q_i} - Q_i)$	perfect system of difference sets [1, 3]
$K_{a,b}$, side X above Y	$A_X\bar{A}_Y + \bar{A}_X A_Y$	interval factorization $X - Y = [1, ab]$ [13]
$K_{n_1, \dots, n_r}$, parts P_i	$A\bar{A} - \sum_i A_{P_i}\bar{A}_{P_i}$	external difference family [47]
any G	a quadratic form in the $z^{f(v)}$	difference family with base block G [50]

Whether an argument from the theory of difference sets carries over to a graph depends on the algebraic form of this left side. Arguments that use only the list of differences and the range of the labels (parity, positivity, ordering) work in every row; these are the three tools of this paper, and the reason they apply to all graphs. Arguments that use multiplicative structure, such as multiplier theorems or the factorization of cyclotomic polynomials, need a norm $A\bar{A}$ or a product $A_X\bar{A}_Y$, and so reach only graphs built from cliques and bicliques. The second row is used in Section 4. The fourth row says that a graceful labeling of $K_{n_1, \dots, n_r}$, read modulo $2m + 1$, is an external difference family in $\mathbb{Z}_{2m+1}$. Paterson and Stinson [47] used graceful labelings to construct such families. In the other direction, $K_{4,4,4}$ has the ρ -labeling with parts $\{1, 3, 44, 57\}$, $\{8, 35, 55, 85\}$, $\{61, 65, 82, 86\}$ in $\mathbb{Z}_{97}$, and $K_{5,5,5}$ has one in $\mathbb{Z}_{151}$ (these are among the few known families of this kind; see [14]), although these graphs are not graceful (the Beutner–Harborth conjecture of Section 7 has been verified up to 23 vertices [25]); so that conjecture cannot be proved modulo $2m + 1$ alone.

3. TOOL I: PARITY AND RESIDUES

3.1. The classical conditions. At $z = -1$ the identity (1) becomes

$$\sum_{e \in E} (-1)^{d_e} = \sum_{k=1}^m (-1)^k,$$

which gives the classical conditions.

Corollary 3.1 (Golomb [28]). *If G is graceful, then $V(G)$ can be split into two sets (the odd and the even labels) with exactly $\lceil m/2 \rceil$ edges between them.*

Proof. An edge has an odd difference exactly when its ends have labels of different parity, and $\lceil m/2 \rceil$ of the numbers $1, \dots, m$ are odd. $\square$

Corollary 3.2 (Knuth's form of the parity condition [32]). *In a graceful labeling of a graph with $m \equiv 1$ or $2 \pmod{4}$ edges, the number of vertices that have odd degree and an odd label is odd.*

Proof. Since $|a - b| \equiv a + b \pmod{2}$, we have modulo 2 $\binom{m+1}{2} = \sum_e d_e \equiv \sum_e (f(u) + f(v)) = \sum_v \deg(v)f(v)$, and the right-hand side is congruent to the number of vertices with $\deg(v)$ and $f(v)$ both odd. Finally $\binom{m+1}{2}$ is odd exactly when $m \equiv 1, 2 \pmod{4}$. $\square$

Corollary 3.3 (Bosák's parity condition [51, 50]). *If every vertex of a graceful graph has even degree, then $m \equiv 0$ or $3 \pmod{4}$.*

Proof. If all degrees are even, no vertex has odd degree, so Corollary 3.2 rules out $m \equiv 1, 2 \pmod{4}$. $\square$

For example, the cycle C_n with $n \equiv 1, 2 \pmod{4}$ and the complete graph K_n with $n \equiv 5$ or $7 \pmod{8}$ are not graceful.

Example 3.4 (windmills). Let $K_n^{(t)}$ be the windmill formed by t copies of K_n sharing one vertex. For odd n all degrees are even, and $m = t\binom{n}{2}$. Hence the friendship graph $K_3^{(t)}$ is not graceful when $t \equiv 2, 3 \pmod{4}$; Skolem's construction [55] shows that it is graceful when $t \equiv 0, 1 \pmod{4}$. Likewise $K_5^{(t)}$ is not graceful for odd t , since then $m = 10t \equiv 2 \pmod{4}$. Their full status is recorded in Remark 4.7.

3.2. Iterating the parity condition. Golomb's condition is stronger than Bosák's, because it asks for a cut of an exact size and not merely of a given parity. For example, K_8 has odd degrees, so Bosák's condition says nothing, but the cuts of K_8 have 7, 12, 15 or 16 edges, never the required 14. The condition can also be applied again inside the two sides of the cut. Write $v_2(d)$ for the exponent of 2 in d .

Proposition 3.5 (iterated parity).^{*} *Let f be a graceful labeling of G . For $j \geq 0$ let G_j be the spanning subgraph formed by the edges whose difference is divisible by 2^j , and put $m_j = \lfloor m/2^j \rfloor$. Then G_j has exactly m_j edges, and the edges with $v_2(d_e) = j$ form a cut of G_j with exactly $\lceil m_j/2 \rceil$ edges. Removing this cut from G_j leaves G_{j+1} . In particular, if every vertex has even degree in G_j , then $m_j \equiv 0$ or $3 \pmod{4}$.*

Proof. The differences on the edges of G_j are $2^j, 2 \cdot 2^j, \dots, m_j 2^j$. If uv is an edge of G_j , then $f(u) \equiv f(v) \pmod{2^j}$, so $d_{uv} = 2^j |g(u) - g(v)|$ with $g(w) = \lfloor f(w)/2^j \rfloor$. Hence $v_2(d_{uv}) = j$ exactly when $g(u)$ and $g(v)$ have different parities, so these edges form the cut of G_j between the vertices with g even and those with g odd. Their number is the number of odd integers in $[1, m_j]$. For the last sentence: if all degrees of G_j are even, every cut of G_j has an even number of edges, and $\lceil m_j/2 \rceil$ is even exactly when $m_j \equiv 0, 3 \pmod{4}$. $\square$

The case $j = 0$ is Golomb's condition, and the last sentence with $j = 0$ is Bosák's. The higher levels give conditions that neither of these gives: Example 3.6 excludes K_9 , which passes both. We say that G passes the iterated parity test if some map $g: V \rightarrow \mathbb{Z}$ has, for every j , exactly $\lceil m_j/2 \rceil$ edges with $v_2(g(u) - g(v)) = j$. Since G_j depends on f , excluding a graph means checking every G_j that is consistent with the earlier levels, as in the next example. The case $j = 1$ appears, for perfect systems of 4-sets, in Bermond, Kotzig and Turgeon [3]: halving the even differences leaves a system of 3-sets that violates Bosák's condition.

Example 3.6 (K_9). Here $m = 36$; Figure 3 shows the argument. At level 0 we need a cut with 18 edges. The cuts of K_9 are the graphs $K_{a,9-a}$, with $a(9-a)$ edges, so $a = 3$ and $G_1 = K_3 \sqcup K_6$, which indeed has 18 edges. At level 1 we need a cut of G_1 with 9 edges.

A triangle contributes 0 or 2 cut edges, and a K_6 contributes 0, 5, 8 or 9. So the triangle is not cut, and the K_6 is cut as $K_{3,3}$. Then G_2 consists of three disjoint triangles. It has $9 = \lfloor 36/4 \rfloor$ edges and all its degrees are even, but $9 \equiv 1 \pmod{4}$. Hence K_9 is not graceful.

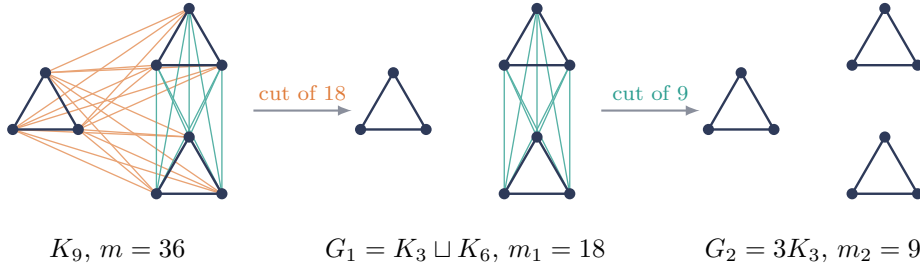


FIGURE 3. Iterated parity for K_9 (Example 3.6). The odd differences must form a cut with 18 edges, which forces the split $3 + 6$ (orange). The differences $\equiv 2 \pmod{4}$ must then form a cut of G_1 with 9 edges, which forces the split of the K_6 (green). What is left, three triangles, needs a cut with 5 edges at the next level, but every cut of a union of triangles is even.

For complete graphs the test is easy to run, since every G_j is a disjoint union of cliques. For $4 \leq n \leq 30$ it rules out every K_n except $n = 4, 6, 11, 18, 25, 27$. Golomb's condition alone, which requires n or $n - 2$ to be a perfect square [28], also leaves $n = 9$ and $n = 16$.

3.3. Other roots of unity. Fix $q \geq 2$ and reduce the labels modulo q . The identity then says: for each residue j modulo q , the number of ordered pairs (u, v) with $uv \in E$ and $f(u) - f(v) \equiv j \pmod{q}$ equals the number of integers $k \in [-m, m] \setminus \{0\}$ with $k \equiv j \pmod{q}$. Algebraically this is the identity (1) read modulo $z^q - 1$, coefficient by coefficient; equivalently, it holds at *every* q th root of unity, including those of lower order. (Evaluating at a single primitive root loses information when q is composite, so we work with the counts themselves.) For $q = 2$ the counts are exactly Golomb's condition.

A labeling must also respect the *capacities* of the interval: at most $\lfloor (m - j)/q \rfloor + 1$ vertices can have labels congruent to j , for $0 \leq j < q$. Applied to the residue map $v \mapsto f(v) \bmod q$ itself, together with the capacities, the conditions at any $q \geq 2m + 1$ are equivalent to gracefulness, since then a residue determines the label. Without the capacities this fails. For K_6 ($m = 15$), the residues $\{0, 1, 3, 8, 12, 18\}$ modulo 31 have every nonzero difference exactly once, so they satisfy the full residue counts at $q = 2m + 1$, although K_6 is not graceful: the label 18 lies outside $[0, 15]$. (This is a cyclic, not a perfect, difference set; compare Remark 4.7.) Distinct residues with the full counts at $q = 2m + 1$ form a ρ -labeling (Section 2.1), which Knuth calls a *rainbow graceful* labeling. By Singer's theorem [6], K_{n+1} has one whenever n is a prime power; K_7 , K_{11} and K_{13} do not [32, §7.2.3].

For a graph with a natural vertex partition, such as a complete multipartite graph, the conditions become quadratic Diophantine equations in the numbers $n_i(j)$ of labels in part i congruent to j . Other moduli exclude graphs that parity does not. Complete graphs are a convenient test: all K_n with $n \geq 5$ are excluded anyway, so the question is only what a fixed modulus can see.

Corollary 3.7 (a modulus-3 obstruction).[†] *The residue equations modulo 3 exclude K_n whenever $n \equiv 6, 10, 11 \pmod{12}$.*

Proof. Let x_0, x_1, x_2 count the labels in the three residue classes and put $r = m \bmod 3$, where $m = \binom{n}{2}$. Counting differences divisible by 3 gives $\sum_i x_i(x_i - 1) = 2\lfloor m/3 \rfloor$, with $\sum_i x_i = n$. Consequently, for $u = x_0 - x_2$ and $v = x_1 - x_2$,

$$u^2 - uv + v^2 = \frac{3\sum_i x_i^2 - n^2}{2} = n - r.$$

The left side is never $2 \pmod{4}$, while the stated congruence classes give $n - r \equiv 2 \pmod{4}$. $\square$

This excludes K_6 , K_{11} and K_{18} , which survive iterated parity, without a computer search.

3.4. A parity family of split graphs. Let $G = K_s \vee \overline{K}_N$: a clique on s vertices, joined to every vertex of an independent set of size N . Let S and Y be the sums of $(-1)^{f(v)}$ over the clique and over the independent set, and let $\varepsilon = m \pmod{2}$. At $z = -1$, the identity (1) reads $\frac{1}{2}(S^2 - s) + SY = -\varepsilon$, that is,

$$(2) \quad S^2 + 2SY = s - 2\varepsilon.$$

Corollary 3.8 (a parity family).[†] *If $s \equiv 8$ or $10 \pmod{16}$, then $K_s \vee \overline{K}_{2t}$ is not graceful for any $t \geq 1$.*

Proof. Since sN is even, $m \equiv \binom{s}{2} \pmod{2}$, so $\varepsilon = 0$ when $s \equiv 8$ and $\varepsilon = 1$ when $s \equiv 10 \pmod{16}$. Write $s = 4k$ or $4k + 2$; in both cases the right side of (2) is $4k$, and S is even. Put $S = 2h$, with $h > 0$ after a sign change. Then $h(h + Y) = k$. Here $k \equiv 2 \pmod{4}$, so in every factorization $k = h(h + Y)$ one factor is odd and the other even, and Y is odd. But Y is a sum of $N = 2t$ signs, so it is even. $\square$

For $K_{1,1,1,1,N} = K_4 \vee \overline{K}_N$, (2) gives $S = \pm 2$ and $Y = 0$: N is even, and exactly three clique labels share a parity. (For odd N this is already Bosák's condition: all degrees are even and $m = 4N + 6 \equiv 2 \pmod{4}$.) The clique vertices have odd degree in Corollary 3.8, so Bosák's condition does not apply there.

Evaluating also at $z = i$ gives two more families. They are the Gaussian counterpart of Corollary 3.7, whose form $u^2 - uv + v^2$ is the norm of $u + v\omega$ ($\omega = e^{2\pi i/3}$), the value at $z = \omega$ of $\sum_w z^{f(w)}$.

Corollary 3.9 (a Gaussian-integer obstruction).[†] *Let $k \geq 0$ and $t \geq 1$ with $k + t$ odd. Then $K_{8k+6} \vee \overline{K}_{2t}$ is not graceful. If k is even, the same holds after deleting any one edge of the clique.*

Proof. Put $s = 8k + 6$ and $N = 2t$; then $m = \binom{s}{2} + sN \equiv 3 \pmod{4}$. Let $C(z)$ and $X(z)$ be the sums of $z^{f(w)}$ over the clique and over the independent set. For $|z| = 1$ the identity (1) reads

$$|C(z)|^2 - s + 2 \operatorname{Re}(C(z)\overline{X(z)}) = \sum_{0 < |j| \leq m} z^j.$$

At $z = -1$, with $C(-1) = 2p$ and $X(-1) = 2q$, this is (2): $p^2 + 2pq = 2k + 1$. So p is odd, which makes the numbers $4k + 3 \pm p$ of even and of odd clique labels both even, and $q \equiv k \pmod{2}$. Hence $C(i) = 2u + 2vi$ with $u, v \in \mathbb{Z}$. Write $X(i) = x + yi$; then x and y have the parities of the numbers $t \pm q$ of even and of odd labels in the independent set, so both are odd, since $t + q \equiv t + k$ is odd. At $z = i$ the right side is -2 , because $m \equiv 3 \pmod{4}$, and the identity becomes

$$u^2 + v^2 + ux + vy = 2k + 1.$$

But $u(u + x)$ and $v(v + y)$ are even, since x and y are odd.

If an edge ab of the clique is deleted, then m drops by one and the left side gains the term $-2 \operatorname{Re} z^{f(a)-f(b)}$; since $m - 1 \equiv 2 \pmod{4}$, the right side becomes 0 at $z = -1$ and stays -2 at $z = i$. At $z = -1$, if $f(a)$ and $f(b)$ had the same parity, we would get $p^2 + 2pq = 2k + 2$, so $p = 2p'$ and $2p'(p' + q) = k + 1$, which is odd. So $f(a) - f(b)$ is odd, the extra term vanishes at $z = i$, and the same equations as before result. $\square$

The identity at the fourth roots of unity amounts to the residue counts modulo 4, which are levels 0 and 1 of Proposition 3.5; so the same family also follows from that proposition: the parities forced at $z = -1$ make every degree of G_1 even, while $m_1 \equiv 1 \pmod{4}$. Neither

argument reaches $K_6 \vee \overline{K}_4$, which is not graceful by an exhaustive search: it passes the iterated parity test, and its residue equations, with capacities, are solvable for every $q \leq 16$.

The same two evaluations exclude a relative of the graceful graphs $K_{1,1,a,b} = K_2 \vee K_{a,b}$ of the Beutner–Harborth conjecture (Section 7), with four singleton parts instead of two.

Corollary 3.10 (four singletons and two parts).^{*} *If $a \equiv b \equiv 2 \pmod{4}$, then $K_{1,1,1,1,a,b}$, that is, $K_4 \vee K_{a,b}$, is not graceful.*

Proof. Here $m = 6 + 4(a + b) + ab \equiv 2 \pmod{8}$. Let $T(z)$, $P(z)$, $Q(z)$ and $C(z)$ be the sums of $z^{f(w)}$ over all vertices, over the parts of sizes a and b , and over the four singletons. The non-edges are the pairs inside the two large parts, so for $|z| = 1$ the identity (1) reads

$$|T(z)|^2 - |P(z)|^2 - |Q(z)|^2 - 4 = \sum_{0 < |j| \leq m} z^j.$$

At $z = -1$ the right side is 0, and T , P , Q are even there; with $T(-1) = 2\tau$, $P(-1) = 2\alpha$ and $Q(-1) = 2\beta$ we get $\tau^2 = 1 + \alpha^2 + \beta^2$. Modulo 4 this forces α and β even and τ odd. Hence each large part has an odd number of even labels and an odd number of odd labels, and $C(-1) \equiv 2 \pmod{4}$, so the singletons split $3 + 1$ by parity. Consequently $P(i)$, $Q(i)$, $C(i)$, and therefore $T(i)$, all have odd real and imaginary parts, and such a Gaussian integer has squared modulus $\equiv 2 \pmod{8}$. At $z = i$ the right side is -2 , because $m \equiv 2 \pmod{4}$, so $|T(i)|^2 - |P(i)|^2 - |Q(i)|^2 = 2$; but the left side is $\equiv 2 - 2 - 2 \equiv 6 \pmod{8}$. $\square$

As with Corollary 3.9, level 1 of Proposition 3.5 gives the same family: the parities forced at $z = -1$ make every degree of G_1 even, while $m_1 = m/2 \equiv 1 \pmod{4}$. Golomb’s condition alone does not exclude these graphs.

3.5. Limits of parity. The residue equations work when few cut sizes are possible, as in a complete graph, and can fail when G is built from many interchangeable pieces whose contributions add up to the required cut sizes. In $K_5^{(t)}$ each blade contributes 0, 4 or 6 edges to Golomb’s cut, and these add up to the required $5t$ for every even t . This is consistent with Remark 4.7: some of these windmills are graceful. For the complete multipartite graphs $K_{3,3,3}$, $K_{2,2,2,2,2}$, $K_{4,4,4}$ and $K_{3,4,5}$, all of which are ungraceful by the verification of Section 7, the residue equations are solvable for every $q \leq 12$, even together with the capacities.

4. TOOL II: POSITIVITY ON THE UNIT CIRCLE

4.1. A warm-up: complete graphs. Replace each label x by the complex number e^{itx} of modulus 1, for a real parameter t (the *frequency*) to be chosen. For a labeling f we call the vector $z \in \mathbb{C}^V$ with entries $z_v = e^{itf(v)}$ the *phase vector* of f at frequency t ; its entries are points of the unit circle. Let f be a graceful labeling of K_n . Then

$$0 \leq \left| \sum_v z_v \right|^2 = \sum_{u,v} e^{it(f(u)-f(v))} = n + \sum_{u \neq v} e^{it(f(u)-f(v))} = n + D_m(t) - 1,$$

because the differences over ordered pairs $u \neq v$ are $\pm 1, \dots, \pm m$, each once. At the first dip $D_m(t) < -\kappa(2m + 1)$, so $n > 1 + \kappa(2m + 1) = 1 + \kappa(n^2 - n + 1)$, which is false for $n \geq 5$. So K_n is not graceful for $n \geq 5$, which is Golomb’s theorem; this is essentially Rédei and Rényi’s argument. Figure 4 shows the case $n = 4$, and the failure for $n = 5$.

4.2. The main theorem. For a general graph, $|\sum z_v|^2 = z^* J z$, with J the all-ones matrix, involves all pairs of vertices, while the identity only controls pairs that are edges. For K_n we have $J = A + I$, where A is the adjacency matrix. The remedy is to replace $A + I$ by any positive semidefinite matrix that agrees with A off the diagonal. For a graceful labeling, the phase vector $z_v = e^{itf(v)}$ satisfies

$$(3) \quad z^* A z = \sum_{uv \in E} 2 \cos(td_{uv}) = D_m(t) - 1,$$

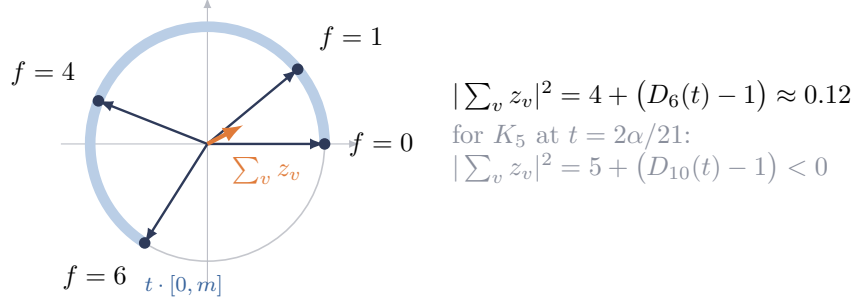


FIGURE 4. The idea of Tool II. The labels 0, 1, 4, 6 of Fig. 1 become the points $z_v = e^{itf(v)}$, here at the first dip $t = 2\alpha/13$. Because the differences are $1, \dots, 6$, their sum satisfies $|\sum z_v|^2 = n + D_m(t) - 1 \approx 0.12$. Five labels with ten distinct differences would need, at their own first dip $t = 2\alpha/21$, $|\sum z_v|^2 = 5 + D_{10}(t) - 1 \approx -0.60$, which is impossible. This proves that K_5 is not graceful.

which is below $-\kappa(2m+1)$ at the first dip. So we want matrices $A + \text{Diag}(y)$ that are positive semidefinite with small trace $\sum_v y_v$. Define the *diagonal completion number* and the *planar energy*

$$\mu(G) = \min \left\{ \sum_v y_v : A + \text{Diag}(y) \succeq 0 \right\}, \quad \gamma_2(G) = \max \{ -z^* A z : z \in \mathbb{C}^V, |z_v| = 1 \}.$$

We call $-z^* A z$ the *energy* of z . A complex number of modulus 1 is a unit vector in the plane, so γ_2 maximizes the energy over unit vectors in $\mathbb{R}^2$; we will see in Section 4.6 that μ is the same maximum over unit vectors in any dimension.

Theorem 4.1 (positivity). *For every graph, $\gamma_2(G) \leq \mu(G)$; if G is graceful, then $\beta_m \leq \gamma_2(G)$. In particular G is not graceful if $\mu(G) \leq 1 + \kappa(2m+1)$, or more generally if $\gamma_2(G) \leq 1 + \kappa(2m+1)$.*

Proof. By (3), the phase vector of a graceful labeling at frequency t has $-z^* A z = 1 - D_m(t)$, and maximizing over t gives $\gamma_2(G) \geq \beta_m$. If $A + \text{Diag}(y) \succeq 0$ and $|z_v| = 1$, then $0 \leq z^*(A + \text{Diag}(y))z = z^* A z + \sum_v y_v$, so $\gamma_2(G) \leq \mu(G)$. Finally $\beta_m > 1 + \kappa(2m+1)$ by Lemma 2.1. $\square$

Intuitively, $\mu(G)$ is small when G looks like a union of cliques, and large when G has a big cut; Section 4.6 makes this exact. For K_n , the matrix $J = A + I$ gives $\mu(K_n) \leq n$, and the theorem reduces to the warm-up.

The parameter $\mu(G)$ is the value of a semidefinite program, and every feasible y certifies the upper bound $\mu(G) \leq \sum_v y_v$. It is convenient to allow certificates whose entries on the edges lie in $[0, 1]$; this gains nothing over μ , but simplifies the constructions below.

Lemma 4.2 (certificates). *Let $M \succeq 0$ be indexed by $V(G)$, with $M_{uv} = 0$ for distinct non-adjacent u, v and $0 \leq M_{uv} \leq 1$ for every edge. Then $\mu(G) \leq \text{tr } M + 2 \text{def}(M)$, where $\text{def}(M) = \sum_{uv \in E} (1 - M_{uv})$.*

Proof. $M + \sum_{uv \in E} (1 - M_{uv})(e_u + e_v)(e_u + e_v)^T$ is positive semidefinite, equals A off the diagonal, and has trace $\text{tr } M + 2 \text{def}(M)$. $\square$

4.3. First consequences. Three choices of certificate give the three main corollaries.

Corollary 4.3 (spectral form).^{*} *If G has n vertices, m edges and smallest adjacency eigenvalue $\lambda_{\min}$, and $n|\lambda_{\min}| \leq 1 + \kappa(2m+1)$, then G is not graceful. In particular, a k -regular graph with $\lambda_{\min} \geq -\kappa k \approx -0.2172k$ is not graceful.*

Proof. The matrix $A + |\lambda_{\min}|I$ is positive semidefinite, so $\mu(G) \leq n|\lambda_{\min}|$. For a k -regular graph $m = nk/2$, and $n|\lambda_{\min}| \leq \kappa nk < 1 + \kappa(2m+1)$. $\square$

Corollary 4.4 (clique packings). *Suppose G contains edge-disjoint cliques $Q_1, \dots, Q_k$ with $\sum_i |Q_i| = \sigma$, and e_0 edges of G lie in none of them. If $\sigma + 2e_0 \leq 1 + \kappa(2m + 1)$, then G is not graceful. Consequently:*

- (a) K_n is not graceful for $n \geq 5$ (Golomb's theorem);
- (b) the windmill $K_n^{(t)}$ and the disjoint union tK_n are not graceful for $n \geq 6$;
- (c) the barbell B_n (two copies of K_n joined by an edge) is not graceful for $n \geq 6$;
- (d) $K_n \square P_r$ and $K_n \square C_r$ are not graceful for $n \geq 13$, for every r ($r \geq 3$ for the cycle).

Proof. Apply Lemma 4.2 to $M = \sum_i J_{Q_i}$, where J_Q is the all-ones matrix on Q : then $\text{tr } M = \sigma$ and $\text{def}(M) = e_0$. For (a), $\sigma = n$, $e_0 = 0$, and $n \leq 1 + \kappa(n^2 - n + 1)$ for $n \geq 5$. For (b), $\sigma = tn$, $e_0 = 0$, and $tn \leq \kappa tn(n - 1)$ once $n - 1 \geq 1/\kappa \approx 4.60$. For (c), $\sigma = 2n$, $e_0 = 1$, and $2n + 2 \leq 1 + \kappa(2n^2 - 2n + 3)$ for $n \geq 6$. For (d), with r cliques of size n joined by matchings, $\sigma + 2e_0 = 3rn$ and $2m = rn(n + 1)$ for the cycle, and $\sigma + 2e_0 = n(3r - 2)$ and $2m = rn(n + 1) - 2n$ for the path; both inequalities hold once $\kappa(n + 1) \geq 3$, that is, $n \geq 13$. $\square$

Corollary 4.5 (dense graphs). *A graph with n vertices and $m \geq (n^2 - 1 - \kappa)/(2 + 2\kappa) \approx 0.4108n^2$ edges is not graceful. In particular $K_{a_1, \dots, a_r}$ is not graceful when $\sum a_i^2 \leq \frac{\kappa}{1+\kappa}n^2 + 1 \approx 0.1785n^2 + 1$, which includes every complete multipartite graph with $r \geq 6$ parts, all of the same size.*

Proof. Take $M = J + L(\overline{G})$, where $L(\overline{G})$ is the Laplacian of the complement. It is positive semidefinite, it is 1 on the edges of G and 0 on its non-edges, and $\text{tr } M = n + 2\left(\binom{n}{2} - m\right) = n^2 - 2m$. The inequality $n^2 - 2m \leq 1 + \kappa(2m + 1)$ is the stated bound. For $K_{a_1, \dots, a_r}$, $2m = n^2 - \sum a_i^2$. $\square$

Corollary 4.5 is the Rédei–Rényi bound for difference bases [49], with Leech's constant [38], translated into graph language. A graceful labeling of an n -vertex graph gives n integers in $[0, m]$ whose differences cover $1, \dots, m$; conversely, any such set is the label set of *some* graceful graph. So the maximum number of edges of a graceful graph on n vertices is a difference-basis problem (OEIS [A004137](#)). For complete multipartite graphs with no part larger than $n/2$, the bound on $\sum a_i^2$ is exactly what positivity can prove, up to the gap between $1 + \kappa(2m + 1)$ and β_m , even with planar vectors (Proposition 4.15(c) below); for example $\mu(K_{3,3,3}) = 27$ and $\mu(K_{2,2,2,2,2}) = 20$, both too large.

4.4. Clique decompositions. Choosing M to be a weighted sum of all-ones matrices on cliques turns Theorem 4.1 into a purely combinatorial criterion.

Theorem 4.6 (fractional clique decompositions).^{*} *Let $\mathcal{Q}$ be a family of cliques of G with weights $w_Q \geq 0$, such that every edge of G lies in cliques of total weight exactly 1. If $\sum_Q w_Q |Q| \leq 1 + \kappa(2m + 1)$, then G is not graceful. This holds in particular when*

$$\sum_{Q \in \mathcal{Q}} w_Q |Q| \left(|Q| - 1 - \frac{1}{\kappa} \right) \geq 0,$$

for instance whenever every clique of positive weight has at least 6 vertices.

Proof. Let $M = \sum_Q w_Q J_Q$. It is positive semidefinite, $M_{uv} = 1$ on edges and 0 on non-edges, and $\text{tr } M = \sum_Q w_Q |Q|$, so Theorem 4.1 applies. Counting edges with weights gives $2m = \sum_Q w_Q |Q|(|Q| - 1)$, so the displayed inequality says $\text{tr } M \leq 2\kappa m$, which is less than $1 + \kappa(2m + 1)$. If $|Q| \geq 6$, then $|Q| - 1 \geq 5 > 1/\kappa \approx 4.60$. $\square$

For $K_n \square P_r$ and $K_n \square C_r$, whose columns are cliques, the clique-by-clique argument of Corollary 4.4(d) is due to Li [30]. The unweighted case has an older source, in the language of difference sets. If the edges of G split into cliques $Q_1, \dots, Q_t$ and f is a graceful labeling, then the label sets $f(Q_i)$, each translated to start at 0, have positive differences exactly

$1, \dots, m$, whatever the overlaps between the cliques: they form a *perfect system of difference sets* [1]. Bermond, Kotzig and Turgeon [3] proved that no perfect system has all of its sets of one size $n \geq 6$, which is the theorem for decompositions into cliques of one order. Their proof is an elementary rearrangement argument; it does not reach systems of mixed sizes, which the Dirichlet kernel handles (for example sets of sizes 6 and 4, or 6, 5 and 5, by the first inequality of the theorem). Ma and Yi [42] have recently reproved and strengthened their result, for families of rulers that all have the same number of marks, by the one-frequency argument used here, with the same constant. Bermond, Kotzig and Turgeon state the reduction from clique decompositions to perfect systems for windmills, and Gallian's survey [25] states it in general, but as far as we know it has been applied only to windmills and disjoint unions. The mixed and weighted forms, the eigenvalue criterion (Corollary 4.3), and the families below appear to be new.

The same reduction turns every nonexistence theorem for perfect systems into a theorem about graphs. For instance, Bermond, Kotzig and Turgeon showed that no perfect system consists of two or three sets of size 4, or of two sets of size 5 [3]. So a graph whose edges split into two or three copies of K_4 , or into two copies of K_5 , is not graceful, however the cliques meet; three K_4 's glued pairwise at three distinct vertices is an example that is neither a windmill nor a disjoint union. Their argument for K_4 's is the iterated parity of Proposition 3.5. With two or three K_4 's, $m = 12$ or 18 , and the cut of odd differences must have 6 or 9 edges. Each K_4 contributes 0, 3 or 4 cut edges, so every K_4 is cut as $K_{1,3}$, and G_1 consists of two or three edge-disjoint triangles. Then all degrees of G_1 are even, while $m_1 = 6$ or 9 , which violates the last sentence of the proposition.

Gallian's survey lists the K_4 -snakes, chains of t copies of K_4 in which consecutive copies share a vertex, as open. They are not graceful for $t = 2, 3$, by the reduction. Dore-Hall reported in 2018 that the snake with $t = 3$ is not graceful and that those with $t = 4, 5$ are [20]; one labeling for $t = 4$ has blocks $\{2, 12, 17, 20\}$, $\{0, 11, 20, 24\}$, $\{0, 6, 22, 23\}$, $\{1, 3, 15, 22\}$. A computer search by the author found one for $t = 6$, with blocks $\{0, 14, 18, 34\}$, $\{0, 1, 13, 36\}$, $\{1, 7, 22, 33\}$, $\{2, 9, 31, 33\}$, $\{2, 5, 30, 35\}$, $\{3, 11, 20, 30\}$, and we expect the snakes to be graceful for every $t \geq 4$.

The number 6 is the threshold of this criterion for decompositions into cliques of one size: $5 - 1 < 1/\kappa$. Graphs with a few K_5 's among larger cliques can still be excluded by the first inequality of the theorem, but no statement covering all K_5 -decompositions is possible by this method, and indeed some of them are graceful, as the next remark shows.

Remark 4.7 (the status of windmills). Graceful labelings of windmills come from *perfect difference families*: t sets of n integers, each containing 0, whose positive differences are exactly $1, \dots, t \binom{n}{2}$. Such a family gives a graceful labeling of $K_n^{(t)}$: identify the t zeros as the hub. A nonzero mark cannot occur in two sets, since it would repeat a difference from 0, so the labels are distinct, and the edge differences are the given ones. For $n = 4$, Ge, Miao and Sun [26] constructed these families for $4 \leq t \leq 1000$, and Liu, Feng, Wang and Zhang constructed them for every $t \geq 4$ [40], settling Bermond's conjecture: together with [3], $K_4^{(t)}$ is graceful exactly when $t \neq 2, 3$. For $n \geq 6$ the windmills are never graceful (Corollary 4.4(b)).

For $n = 5$ some windmills are graceful and some are not. $K_5^{(2)}$ is not graceful [3], and neither is $K_5^{(4)}$: Kotzig and Turgeon showed by exhaustive elimination that no perfect family exists for $t = 4$ [35], and an exhaustive search by B. D. McKay found no graceful labeling of $K_5^{(4)}$ at all [29]. On the other hand, perfect families with blocks of size 5 exist for $t = 6, 8, 10$, and a recursive construction produces infinitely many more, for example $t = 732$ and $t = 974$ [43]. So $K_5^{(t)}$ is graceful for infinitely many even t , and the status of the remaining even $t \geq 12$ is open. Gallian's survey [25] (28th edition, Section 2.4) records

$K_5^{(3)}$ and $K_5^{(4)}$ as graceful, and its summary table also lists $K_5^{(5)}$; as just explained, none of them is.

Corollary 4.8 (clique-decomposable graphs).*

- (a) *A graph whose edge set is a disjoint union of cliques, each with at least 6 vertices, is not graceful. In particular, every K_6 -decomposable graph is not graceful.*
- (b) *Every edge-transitive graph that contains a K_6 is not graceful.*
- (c) (Parity companion.) *A graph whose edges decompose into an odd number of copies of K_5 , or into T triangles with $T \equiv 2, 3 \pmod{4}$, is not graceful.*

Proof. For (a), give each clique weight 1. For (b), let $c \geq 1$ be the number of copies of K_6 containing an edge; by edge-transitivity this does not depend on the edge. Give every copy of K_6 weight $1/c$. For (c), let r_v be the number of cliques through v . For K_5 's the degrees are $4r_v$ and $m = 10T \equiv 2 \pmod{4}$ when T is odd, so Corollary 3.3 applies; for triangles, degrees are $2r_v$ and $m = 3T \equiv 1, 2 \pmod{4}$ exactly when $T \equiv 3, 2 \pmod{4}$. $\square$

Cliques that overlap in more than one vertex need a correction on the overlap. The simplest case is a family introduced by Koh, Rogers, Teo and Yap [33]: let $B(n, r, t)$ be the graph formed by t copies of K_n that share a common K_r . Thus $B(n, 1, t) = K_n^{(t)}$ and $B(n, r, 1) = K_n$. Bermond [4] asked which $B(n, r, t)$ are graceful, and Bermond and Farhi [2] showed that they are not when n is sufficiently large compared with r . Positivity gives an explicit linear threshold, uniform in t .

Corollary 4.9 (cliques sharing a clique).* *Let $C = 1 + 1/\kappa \approx 5.603$. If $n(n - C) \geq C r(r - 1)$, then $B(n, r, t)$ is not graceful for any $t \geq 1$. In particular this holds when $n \geq 2.4r + 6$, and for $n \geq 6, 8, 10, 12, 14$ when $r = 1, 2, 3, 4, 5$.*

Proof. Let R be the common K_r , and $Q_1, \dots, Q_t$ the copies of K_n . Put $M = \sum_j J_{Q_j} - (t - 1)J_R + (t - 1)r I_R$. Off the diagonal it is the adjacency matrix. With σ_R and σ_j the sums of a vector x over R and over $Q_j \setminus R$, and using $r \sum_{v \in R} x_v^2 \geq \sigma_R^2$,

$$x^T M x \geq \sum_j (\sigma_R + \sigma_j)^2 - (t - 1)\sigma_R^2 + (t - 1)\sigma_R^2 \geq 0.$$

Also $\text{tr } M = tn + (t - 1)r(r - 1)$ and $2m + 1 = 1 + r(r - 1) + t(n(n - 1) - r(r - 1))$. The hypothesis is equivalent to $n + r(r - 1) \leq \kappa(n(n - 1) - r(r - 1))$; multiplying by t and subtracting $r(r - 1)$ gives $\text{tr } M < 1 + \kappa(2m + 1)$. If $n \geq \sqrt{C}r + C$, then $n(n - C) \geq (n - C)^2 \geq Cr^2$, and $\sqrt{C} < 2.4$. $\square$

In fact γ_2 and μ of $B(n, r, t)$ can be computed exactly.

Proposition 4.10 (exact values for $B(n, r, t)$).[†] *Let $n > r \geq 1$, $t \geq 1$, $c = n - r$ and $v = r + tc$ (the number of vertices). Then*

$$\gamma_2(B(n, r, t)) = \mu(B(n, r, t)) = v + \begin{cases} (t - 1)r^2, & r \leq c, \\ -r^2 + 2tcr - tc^2, & c \leq r \leq tc, \\ t(t - 1)c^2, & r \geq tc. \end{cases}$$

Proof. By Proposition 4.15(a) below, $\mu(G)$ is the maximum of $-\sum_{uv \in E} 2\langle x_u, x_v \rangle$ over unit vectors x_v in any dimension, and γ_2 is the same maximum in the plane. Let S be the sum of the x_v over the common K_r and S_j the sum over the rest of the j th copy, with $s = \|S\| \leq r$ and $\|S_j\| \leq c$. The quantity to maximize is

$$v + (t - 1)s^2 - \sum_j \|S + S_j\|^2 \leq v + (t - 1)s^2 - t(s - c)_+^2,$$

and the right side is largest at $s = \min\{r, tc\}$, which gives the displayed formula. It is attained in the plane: choose unit complex numbers on the common K_r whose sum has

length s , and on each copy unit complex numbers whose sum points the opposite way with length $\min\{s, c\}$. $\square$

For $n \geq 2r$ this reduces to $\gamma_2 = \mu = tn + (t-1)r(r-1)$, so the certificate in the proof of Corollary 4.9 is optimal, and better thresholds for $B(n, r, t)$ will need either the exact test $\mu < \beta_m$ or information beyond positivity. For $r = 2$ the corollary extends the known fact that $B(n, 2, 2)$ is not graceful for $n > 5$ [2, 25] to all t , from $n = 8$ on.

Many classical graphs come with a natural partition of their edges into cliques, and then Theorem 4.6 applies directly. The *Hamming graph* $H(d, q)$ has the words of length d over a q -letter alphabet as vertices, adjacent when they differ in one coordinate. The *Johnson graph* $J(n, k)$ has the k -subsets of an n -set, adjacent when they share $k-1$ elements, and the *Kneser graph* $K(n, k)$ the same vertices, adjacent when disjoint. The *Grassmann graph* $J_q(n, k)$ and the *q -Kneser graph* $K_q(n, k)$ are their analogues for k -dimensional subspaces of $\mathbb{F}_q^n$: adjacent when they meet in dimension $k-1$, respectively only in 0. A *partial linear space* is a set of points with a family of lines such that two points lie on at most one line; its *collinearity graph* joins two points on a common line. Generalized quadrangles $\text{GQ}(s, t)$ have $s+1$ points on each line, nets of order n have n (Latin square graphs are the collinearity graphs of 3-nets), and the finite classical polar spaces over $\mathbb{F}_q$ have $q+1$.

Corollary 4.11 (line graphs, geometries and subspace graphs).^{*} *The following graphs are not graceful:*

- (a) *the line graph $L(H)$ of any graph H , or of any linear hypergraph H (two edges sharing at most one vertex), with $\delta(H) \geq 6$. More generally $L(H)$ is not graceful whenever $\sum_v d_v(d_v - 1 - \frac{1}{\kappa}) \geq 0$, the sum running over the vertices of H with $d_v \geq 2$. This includes the triangular graphs $T(n) = L(K_n)$ with $n \geq 7$ and the rook's graphs $K_a \square K_b = L(K_{a,b})$ with $a+b \geq 12$, and also, by the first inequality of Theorem 4.6, $K_2 \square K_9$ and $K_3 \square K_8$;*
- (b) *the collinearity graph of any partial linear space (any two points lie on at most one line) in which every line has at least 6 points. This includes generalized quadrangles $\text{GQ}(s, t)$ with $s \geq 5$, nets (in particular Latin square graphs) of order $n \geq 6$, and polar space graphs over $\mathbb{F}_q$ with $q \geq 5$. It also includes the block intersection graphs of Steiner systems $S(2, k, v)$ with replication number $(v-1)/(k-1) \geq 6$;*
- (c) *the Hamming graphs $H(d, q) = K_q^{\square d}$ with $q \geq 6$, in every dimension d ;*
- (d) *the Johnson graphs $J(n, k)$ ($k \leq n/2$) with $n \geq k+5$; in particular every $J(n, k)$ with $k \geq 5$;*
- (e) *every Grassmann graph $J_q(n, k)$, $1 \leq k \leq n-1$, except $J_2(2, 1) = K_3$ and $J_3(2, 1) = K_4$, which are graceful;*
- (f) *the Kneser graphs $K(n, k)$ with $n \geq (1 + 1/\kappa)k \approx 5.603k$;*
- (g) *the q -Kneser graphs $K_q(n, k)$ with $k \geq 2$, $n \geq 2k$, except possibly $K_2(4, 2)$.*

Proof. Items (a)–(d) and (e) for $k \leq n-2$ follow from Theorem 4.6 with a clique partition of the edges.

- (a) For each vertex v of H , the edges through v form a clique of size d_v in $L(H)$. By linearity each edge of $L(H)$ lies in exactly one of these cliques.
- (b) The lines are cliques, and they partition the edges. For a Steiner system, the blocks through a point form such a clique.
- (c) The words that agree outside one fixed coordinate form a clique of size q , and these cliques partition the edges.
- (d) The k -sets containing a fixed $(k-1)$ -set form a clique of size $n-k+1$, and an edge $\{A, B\}$ lies only in the clique of $A \cap B$.
- (e) If $k \leq n-2$, the k -spaces containing a fixed $(k-1)$ -space form a clique of size $(q^{n-k+1} - 1)/(q - 1) \geq q^2 + q + 1 \geq 7$, and these cliques partition the edges. If

$k = n - 1$, the graph is complete on $(q^n - 1)/(q - 1)$ vertices, which is at least 5 except in the two stated cases.

- (f) $K(n, k)$ is $\binom{n-k}{k}$ -regular with $\lambda_{\min} = -\binom{n-k-1}{k-1}$, so $|\lambda_{\min}|/\text{degree} = k/(n - k)$ [11]; apply Corollary 4.3.
- (g) $\text{GL}(n, q)$ acts transitively on the edges. Inside $\mathbb{F}_{q^k}^2$, viewed as a $2k$ -dimensional space over $\mathbb{F}_q$, the $q^k + 1$ subspaces $\{(x, ax)\}$ ($a \in \mathbb{F}_{q^k}$) and $\{(0, x)\}$ pairwise meet only in 0, so the graph contains K_{q^k+1} ; when $q^k \geq 5$, Corollary 4.8(b) applies. The only remaining case with $k \geq 2$ is $q = k = 2$. For $n \geq 5$, take a 2-dimensional subspace $W \subset \mathbb{F}_8$; in $W \oplus \mathbb{F}_8$ the eight subspaces $\{(x, ax) : x \in W\}$, $a \in \mathbb{F}_8$, together with any 2-dimensional subspace of $0 \oplus \mathbb{F}_8$, form a K_9 .

For (a) with H a graph, (c) and (d), Corollary 4.3 gives the same bounds, since the smallest eigenvalues are at least -2 , $-d$ and $-k$ respectively [11]. $\square$

4.5. Spectral families. Corollary 4.3 also reaches graphs without large cliques.

Corollary 4.12 (Paley graphs).^{*} *The Paley graph $P(q)$ (vertices $\mathbb{F}_q$ with $q \equiv 1 \pmod{4}$), adjacent when their difference is a nonzero square) is not graceful for $q \geq 37$.*

Proof. $P(q)$ is $\frac{q-1}{2}$ -regular with $\lambda_{\min} = -(1 + \sqrt{q})/2$ [11]. The ratio $(1 + \sqrt{q})/(q - 1)$ is decreasing, and about $0.197 < \kappa$ at $q = 37$. $\square$

$P(q)$ has even degrees and $m = q(q - 1)/4$ edges, so Bosák's condition already excludes $q \equiv 5, 9 \pmod{16}$, including $q = 25$ and $q = 37$. The new cases are $q \equiv 1, 13 \pmod{16}$, beginning with $q = 49$ and $q = 61$. The smaller graph $P(13)$ is graceful: in the cyclic order $0, 1, \dots, 12$ of its vertices, nickgard [46] gives the labels $0, 1, 5, 39, 23, 34, 13, 3, 21, 7, 38, 33, 24$. For comparison, the hypercubes $H(d, 2)$ and the Petersen graph $K(5, 2)$ are graceful [25]; their ratios $|\lambda_{\min}|/\text{degree}$ are 1 and $2/3$, far above the shaded region of Fig. 5.

Corollary 4.13 (random regular graphs).^{*} *For each fixed $d \geq 84$, a uniformly random d -regular graph on n vertices (nd even) is not graceful with probability tending to 1 as $n \rightarrow \infty$.*

Proof. By Friedman's theorem [24], in the form for the uniform model proved by Bordenave [9], for every $\epsilon > 0$ all eigenvalues other than d have absolute value at most $2\sqrt{d-1} + \epsilon$ with probability tending to 1. For $d \geq 84$, $2\sqrt{d-1} < \kappa d$, and we can choose ϵ to fit in between. $\square$

Counting shows that almost all graphs are not graceful [25], but it does not reach graphs of fixed degree: there are exactly $m!$ graphs on the vertex set $\{0, \dots, m\}$ whose m edges have the differences $1, \dots, m$ [54] (choose the edge of each difference k in one of $m - k + 1$ positions; see also [32, §7.2.2.3, Theorem S]), so at most $m!$ graceful graphs with m edges and no isolated vertices, and for fixed degree d this exceeds the number of d -regular graphs with m edges, up to isomorphism.

Corollary 4.14 (more spectral families).^{*} *The following graphs are not graceful:*

- (a) every graph with smallest eigenvalue at least -2 and average degree at least $2/\kappa \approx 9.21$;
- (b) every non-bipartite Ramanujan graph of degree $d \geq 84$, that is, every d -regular graph whose eigenvalues other than d have absolute value at most $2\sqrt{d-1}$; for example, the non-bipartite graphs of Lubotzky, Phillips and Sarnak [41] and of Morgenstern [45] of degree at least 84;
- (c) the polarity graph ER_q [21, 12] for every prime power $q \geq 23$. Its vertices are the points of the projective plane $\text{PG}(2, q)$, with $x \sim y$ when $x \neq y$ and $x^\top y = 0$. These graphs contain no 4-cycle.

Proof. (a) Here $n|\lambda_{\min}| \leq 2n \leq \kappa \cdot 2m$. (b) Here $\lambda_{\min} \geq -2\sqrt{d-1}$, and $2\sqrt{d-1} \leq \kappa d$ for $d \geq 84$. (c) Let P be the diagonal indicator of the $q+1$ absolute points ($x^\top x = 0$), so that $A' = A + P$ is the looped matrix. Two distinct points have exactly one common neighbour in A' , so $A'^2 = J + qI$, and the eigenvalues of A' are $q+1$ and $\pm\sqrt{q}$. Hence $A + P + \sqrt{q}I \geq 0$, with trace $n\sqrt{q} + q + 1$, where $n = q^2 + q + 1$ and $2m = (q+1)(n-1)$. The function $\kappa(q+1) - \sqrt{q}$ is increasing for $q \geq 23$ and exceeds 0.41 there, so $\kappa(2m+1) + 1 - (n\sqrt{q} + q + 1) \geq n(\kappa(q+1) - \sqrt{q}) - q - \kappa q > 0.41q^2 - 1.22q > 0$. $\square$

The graphs in (a) are classified: by the theorem of Cameron, Goethals, Seidel and Shult [15], each connected component is a generalized line graph or one of finitely many exceptional graphs coming from the root system E_8 . In (c) the certificate has a non-constant diagonal: adding 1 only at the $q+1$ absolute points gives trace $n\sqrt{q} + q + 1$, less than the trace $n(\sqrt{q} + 1)$ of a scalar certificate.

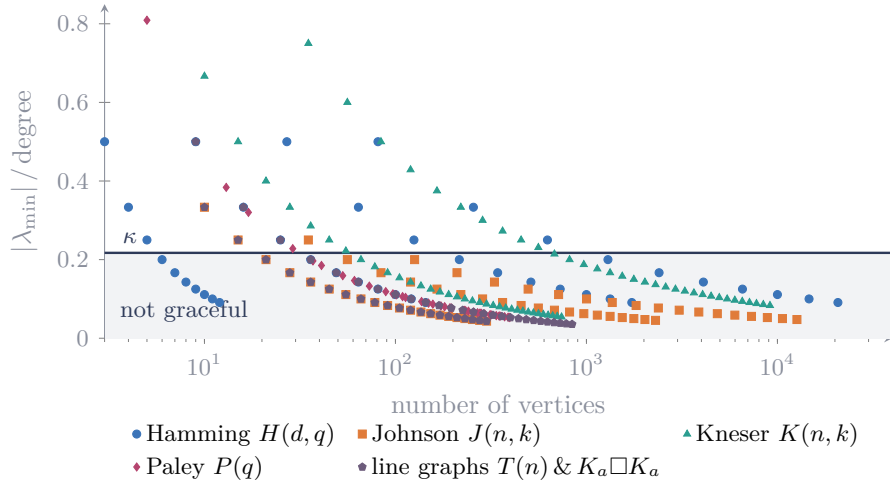


FIGURE 5. Members of several regular families plotted by size and by $|\lambda_{\min}|/\text{degree}$. Corollary 4.3 makes every regular graph in the shaded region ungraceful.

4.6. The optimal semidefinite certificate. The corollaries above each use one convenient certificate. The best possible certificate has an exact description, because μ is the dual of a standard semidefinite program. Let $\text{SDP}_{\text{MC}}(G) = \max\{\sum_{uv \in E} \frac{1}{2}(1 - X_{uv}) : X \succeq 0, X_{vv} = 1\}$ be the Goemans–Williamson relaxation of maximum cut [27]: a cut is an assignment of signs $x_v = \pm 1$, the edge uv is cut when $\frac{1}{2}(1 - x_u x_v) = 1$, and the relaxation replaces the signs by unit vectors and X by their Gram matrix.

Proposition 4.15 (exact form).[†]

- (a) $\mu(G) = \max\{-\langle A, X \rangle : X \succeq 0, X_{vv} = 1\} = 4\text{SDP}_{\text{MC}}(G) - 2m$. In particular $\mu(G) \geq 4\text{MaxCut}(G) - 2m$.
- (b) If G is vertex-transitive, then $\mu(G) = n|\lambda_{\min}|$.
- (c) If $G = K_{a_1, \dots, a_r}$ with $n = \sum a_i$, then $\gamma_2(G) = \mu(G) = \sum_i a_i^2 - (2\max_i a_i - n)_+^2$.

Proof. Parts (a) and (b) are standard [19, 44]; part (c) is new. For (a), the semidefinite dual of $\max\{-\langle A, X \rangle : X \succeq 0, X_{vv} = 1\}$ is $\min\{\sum_v y_v : A + \text{Diag}(y) \succeq 0\} = \mu(G)$; both programs are strictly feasible (y large; $X = I$), so the optima are equal and attained, and $\sum_{uv \in E} \frac{1}{2}(1 - X_{uv}) = \frac{m}{2} - \frac{1}{4}\langle A, X \rangle$ gives the second form. For (b), average a feasible y over the automorphism group: the diagonal becomes constant with the same sum, and $A + cI \succeq 0$ exactly when $c \geq -\lambda_{\min}$. For (c), let S_i be the sum of the vectors of X 's Gram factorization over part i . Then $-\langle A, X \rangle = \sum_i \|S_i\|^2 - \|\sum_i S_i\|^2$, and $\|S_i\| \leq a_i$. The expression is affine in each S_i separately (its own square cancels), so the maximum has $\|S_i\| = a_i$, realized by

equal vectors within each part. The smallest possible length of a sum of vectors of lengths a_i is $(2 \max a_i - n)_+$, attained in the plane; so the planar maximum equals the semidefinite one. $\square$

So the best that a semidefinite certificate can do, with any diagonal and any frequency, is this: *G is not graceful if $\text{SDP}_{\text{MC}}(G) < (2m + \beta_m)/4$.* Since $\beta_m > 1 + \kappa(2m + 1)$, this threshold exceeds $\frac{1}{2}(1 + \kappa)m \approx 0.6086m$. Graphs whose maximum cut is at least $(2m + \beta_m)/4$ are out of reach even of the planar bound, because a cut, as a vector of signs ± 1 , is a vector of unit complex numbers; for a bipartite graph $\text{SDP}_{\text{MC}}(G) = m$, the largest possible value. This statement concerns μ , not all of Theorem 4.1: a bound on γ_2 can be strictly stronger, as Section 4.7 shows.

Three simple rules transfer certificates from one graph to another; we use them in Section 4.7.

Lemma 4.16 (blocks and edits).[†] *Let η be μ or γ_2 .*

- (a) *If subgraphs F_i with weights $w_i \geq 0$ cover every edge of G with total weight exactly 1, then $\eta(G) \leq \sum_i w_i \eta(F_i)$.*
- (b) *If G arises from H by adding a edges and deleting b edges, then $|\eta(G) - \eta(H)| \leq 2(a + b)$.*
- (c) *For the independent blowup $F[t]$, obtained by replacing every vertex by t independent twins and every edge by a complete bipartite graph, $\eta(F[t]) = t^2 \eta(F)$.*

Proof. For μ , add certificates for (a); for (b), an added edge uv costs $(e_u + e_v)(e_u + e_v)^\top$ and a deleted one $(e_u - e_v)(e_u - e_v)^\top$, each of trace 2. For γ_2 , the energy of a fixed vector z with $|z_v| = 1$ splits over the blocks, and each changed edge changes it by at most 2. For (c), sum the vectors in each twin class to get vectors S_v of length at most t . The energy is affine in each S_v separately, so its maximum has $\|S_v\| = t$, realized by equal vectors within each class; this works in the plane and in any dimension. $\square$

Three further remarks bound what refinements of positivity can achieve.

Averaging frequencies. One could replace the function $x \mapsto e^{itx}$ by any positive definite function φ on $\mathbb{Z}$ with $\varphi(0) = 1$, since the matrix $(\varphi(f(u) - f(v)))_{u,v}$ is then positive semidefinite. By the Herglotz–Bochner theorem [52], $\varphi(x) = \int e^{itx} d\nu(t)$ for a probability measure ν , and the identity becomes $\sum_{uv \in E} 2 \operatorname{Re} \varphi(d_{uv}) = \int (D_m(t) - 1) d\nu(t)$, an average of the single-frequency case; it can never do better than the best single frequency. Bernshteyn and Tait [5] beat Leech’s constant for difference bases only by using several Fourier coefficients of one measure jointly. A similar gain for graphs would need constraints that tie different frequencies together.

Abelian Cayley graphs. For a Cayley graph of an abelian group, the eigenvector for $\lambda_{\min}$ can be taken to be a character, all of whose entries have modulus 1, so $\gamma_2 = \mu = n|\lambda_{\min}|$. This settles the unrestricted planar optimization for Paley, Hamming and rook’s graphs and for K_n , but not refinements that remember that the entries of an actual phase vector lie on a finite grid and, at the first dip, on an arc of length about $\alpha < 2\pi$.

Adding parity. Knowing which edges have even differences, one might give them weight 1 and give the odd ones weight $c \in [-1, 1]$. The right side then becomes $\frac{1+c}{2} D_m(t) + \frac{1-c}{2} D_m(t+\pi) - 1$, a convex combination of two translates of $D_m - 1$, whose minimum is no lower. So the kernel side gains nothing; any gain would have to come from the graph side, from certificates for the reweighted matrix, which depends on the unknown cut. We have not found a way to exploit this.

4.7. The planar refinement. The vectors $z_v = e^{itf(v)}$ are complex numbers of modulus 1, while $\mu(G)$ allows unit vectors in any dimension, and Theorem 4.1 already records the stronger inequality $\gamma_2(G) \geq \beta_m$. The following lemma is one way to use it: it bounds the projection onto the least eigenspace of a vector whose entries have modulus 1.

Lemma 4.17 (projection onto the least eigenspace).[†] *Let $\theta > \lambda_{\min}$ be the second smallest distinct adjacency eigenvalue of G , let $U \subseteq \mathbb{C}^V$ be the eigenspace of $\lambda_{\min}$, and put $\Lambda(U) = \max\{\|w\|_1 : w \in U, \|w\|_2 = 1\}$. Then*

$$\gamma_2(G) \leq (\theta - \lambda_{\min})\Lambda(U)^2 - \theta n.$$

Proof. Let P be the orthogonal projection onto U . If $|z_v| = 1$ for all v , then $\|z\|_2^2 = n$ and $\|Pz\|_2 = \max_{w \in U, \|w\|_2=1} |w^*z| \leq \Lambda(U)$. The spectral decomposition gives $z^*Az \geq \lambda_{\min}\|Pz\|_2^2 + \theta(n - \|Pz\|_2^2)$. $\square$

Theorem 4.18 (a planar bound for $K(11, 2)$).[†] *Let $H = K(11, 2)$, the Kneser graph on the 55 pairs from an 11-set. It is 36-regular with 990 edges, $\mu(H) = 440$, and*

$$\gamma_2(H) \leq \frac{26409}{64} = 412.640625.$$

The bottom eigenspace of $K(11, 2)$ consists of the vectors whose entry at a pair $\{i, j\}$ is $a_i + a_j$, with $\sum_i a_i = 0$. The proof bounds $\sum_{i < j} |a_i + a_j|$ by $173/8$ when $\sum_i a_i = 0$ and $\sum_i |a_i|^2 = 1$, using the relation between the largest $|a_i + a_j|$ and the fourth moment $\sum_i |a_i|^4$; then $\Lambda(U) \leq 173/24$, and Lemma 4.17 with $\lambda_{\min} = -8$ and $\theta = 1$ gives the theorem. The details are in Appendix B.

The max-cut of $K(n, 2)$ and its eigenvalue bound are classical [48, 44], and the gap between rank-constrained and unconstrained semidefinite programs is studied in [10]; we know of no earlier bound on the planar value of a Kneser graph. Theorem 4.18 gives an explicit margin, and hence the following family.

Corollary 4.19 (planar families).[†] *Let H_b be $K(11, 2)$ with any $b \in \{0, \dots, 7\}$ edges deleted. Every H_b is not graceful, and neither is any graph whose edges have a fractional decomposition into such graphs, or any independent blowup of such a graph. On every H_b and each of its independent blowups, every semidefinite certificate fails, at every frequency.*

Proof. Theorem 4.18 and Lemma 4.16(b) give $\gamma_2(H_b) \leq 26409/64 + 2b$. Since κ is the maximum of $-\sin x/x$, and $0 < \frac{3}{2}\pi - \frac{9}{2} < \frac{3}{14}$, the inequality $\cos x > 1 - x^2/2$ gives $\kappa \geq \frac{2}{9} \cos(\frac{3}{2}\pi - \frac{9}{2}) > \frac{383}{1764}$. Hence

$$2\kappa(990 - b) - \gamma_2(H_b) > \frac{383}{882}(990 - b) - \frac{26409}{64} - 2b \geq \frac{6143}{28224} > 0,$$

so $\gamma_2(H_b) < 2\kappa|E(H_b)| < 1 + \kappa(2m + 1)$. These inequalities add over a fractional decomposition, by Lemma 4.16(a), and both sides scale by t^2 under blowups, by Lemma 4.16(c); Theorem 4.1 applies in each case.

For the semidefinite comparison, let B be the incidence matrix of K_{11} and put $X_0 = \frac{11}{18}B^T(I - \frac{1}{11}J)B$. It is positive semidefinite with unit diagonal and has entry $-2/9$ on each edge of H ; it stays feasible after edges are deleted, so $\mu(H_b) \geq 440 - \frac{4}{9}b \geq \frac{3932}{9} > 436$. Lemma A.1 gives $\beta_{990-b} \leq 1 + 525 \cdot 1981/2399 < 436$ for $0 \leq b \leq 7$, and the same comparison holds after a blowup by t , since μ scales by t^2 while $\beta_{t^2m}/t^2 \leq (1 + \frac{525}{2399})/t^2 + \frac{1050m}{2399} < 436$. $\square$

The same margin also allows edge additions: a graph obtained from $K(11, 2)$ by adding a edges and deleting b edges is not graceful whenever $44192a + 68704b \leq 487070$, for instance after any eleven additions. The graph $K(11, 2)$ itself is excluded by Bosák's condition (36 is even and $990 \equiv 2 \pmod{4}$). For $1 \leq b \leq 7$ it excludes H_b only when the deleted edges form an even subgraph (every vertex meeting an even number of them) and $990 - b \equiv 1, 2 \pmod{4}$, that is, only for $b = 4, 5$ (for example, a deleted 4-cycle or 5-cycle). It says nothing about H_1 and H_2 , which have vertices of odd degree, nor about H_3 , H_6 and H_7 , whose numbers of edges are $\equiv 0, 3 \pmod{4}$.

5. TOOL III: THE INTERVAL

Tool III uses the bounded span of the labels, which are distinct integers in $[0, m]$. This forces the labels of a dense subgraph to span a long interval: a clique K_r has $\binom{r}{2}$ distinct differences among its labels, so those labels must spread over an interval of length at least $\binom{r}{2} \approx r^2/2$, and in fact of length about r^2 (Theorem 5.1 below); a graph with fewer than about r^2 edges cannot provide that interval. For a graph H , let $\ell(H)$ be the smallest span $\max f - \min f$ of an injective integer labeling of H whose edge differences are all distinct (they need not be $1, \dots, |E(H)|$). Then

$$(4) \quad H \subseteq G, G \text{ graceful} \implies \ell(H) \leq |E(G)|.$$

For $H = K_r$, $\ell(H)$ is the length of the shortest r -mark Golomb ruler. Ungracefulness of H says nothing about its supergraphs, but a lower bound on $\ell(H)$ excludes every graph with fewer than $\ell(H)$ edges that contains H .

Theorem 5.1 (dense core).[†] Suppose G has m edges and contains a subgraph H on r vertices that misses at most p of the $\binom{r}{2}$ pairs. For an integer $1 \leq h \leq r-1$, put $T = hr - \frac{h(h+1)}{2} - p$. If $T > 0$ and G is graceful, then

$$m \geq \frac{T(T+1) + 2p}{h(h+1)} + \frac{2(h-1)}{3}.$$

If $H = K_r$ and $r \geq 2h$, the bound improves to $m \geq \frac{T(T+1)}{h(h+1)} + \frac{h^2-1}{3}$.

Proof. Order the labels on H as $a_1 < \dots < a_r$, all in $[0, m]$, and put $S_h = \sum_{j=1}^h \sum_{i=1}^{r-j} (a_{i+j} - a_i)$. There are $P = hr - h(h+1)/2$ pairs whose indices differ by at most h ; let $p' \leq p$ of them be non-edges. The other $P - p'$ have distinct positive differences, and each non-edge has length at least one, so $S_h \geq \frac{(P-p')(P-p'+1)}{2} + p' \geq \frac{T(T+1)}{2} + p$. For fixed j , the inner sum splits into j telescoping chains by residue of i modulo j (Fig. 6); since the labels are distinct integers,

$$\sum_{i=1}^{r-j} (a_{i+j} - a_i) = \sum_{i=r-j+1}^r a_i - \sum_{i=1}^j a_i \leq j(a_r - a_1) - j(j-1) \leq jm - j(j-1).$$

Summing over $j \leq h$ gives $S_h \leq \frac{h(h+1)}{2}m - \frac{(h-1)h(h+1)}{3}$, which proves the first bound.

For a clique, write $S_h = \sum_i g_i w_i$ with gaps $g_i = a_{i+1} - a_i$, where w_i is the number of chosen pairs whose interval contains the i th gap. Interior gaps have weight $W = h(h+1)/2$; the first $h-1$ gaps and the last $h-1$ have deficits $W - w_i = \binom{h-i+1}{2}$, $1 \leq i \leq h-1$, each value occurring twice. These $2(h-1)$ gaps are distinct edge differences, so $\sum_i (W - w_i)g_i \geq \sum_{j=1}^{h-1} (4j-1)\binom{h-j+1}{2} = \frac{h(h-1)(h+1)^2}{6}$, by giving the smallest gaps to the largest deficits. Thus $S_h \leq Wm - \frac{h(h-1)(h+1)^2}{6}$, and dividing by W gives the extra $\frac{h^2-1}{3}$. $\square$

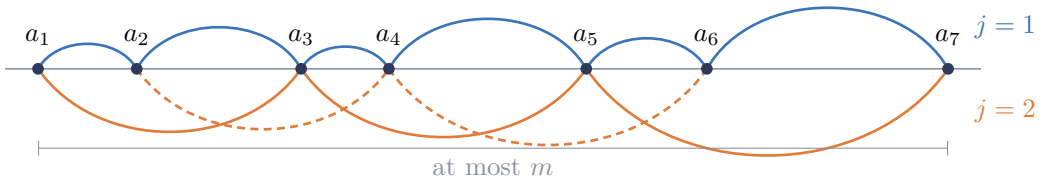


FIGURE 6. The counting behind Tool III, for $r = 7$ and $h = 2$. The arcs join labels whose indices differ by at most h ; at least T of them are edges, whose lengths are distinct and so add up to at least $1 + 2 + \dots + T$. But for each j the arcs of index distance j form j chains (solid and dashed below), each lying inside $[a_1, a_r]$, so their total length is at most $j m$.

Taking h of order $\sqrt{r}$ gives $\ell(K_r) \geq r^2 - O(r^{3/2})$, the classical Erdős–Turán–Lindström estimate [22, 39]; much sharper bounds are known for Golomb rulers [16]. That the clique labels of a graceful graph form a Golomb ruler is folklore (see [25]), and the asymptotic form of Theorem 5.1 for dense subgraphs is due to Ferrara, Kohayakawa and Rödl [23] and to Bollobás and Pikhurko [8], in the language of integer sets with prescribed distinct differences. What Theorem 5.1 adds is an explicit inequality, usable for specific r , and its application to gracefulness. So a graph containing K_r with at most $(1 - \varepsilon)r^2$ edges is not graceful once r is large, which allows almost $r^2/2$ edges beyond the clique’s own. The bound depends only on m , so the extra edges may be placed arbitrarily. In the next corollary they are pendant leaves, which create a large cut and absorb the parity constraints; neither positivity nor iterated parity excludes these graphs (Proposition 5.4).

Corollary 5.2 (cliques with many leaves).[†]

- (a) For $r \geq 46$, the graph obtained from K_r by attaching $\lfloor r^2/4 \rfloor$ pendant leaves to one vertex is not graceful.
- (b) For even $r \geq 102$, the graph obtained from K_r by deleting a perfect matching and attaching $r^2/4$ pendant leaves to one vertex is not graceful.

Proof. (a) Here $m \leq \frac{3}{4}r^2 - \frac{r}{2}$. Take $h = 6$, $T = 6r - 21$: $\frac{(6r-21)(6r-20)}{42} + \frac{35}{3} - (\frac{3}{4}r^2 - \frac{r}{2}) = \frac{9r^2 - 450r + 1820}{84} > 0$ for $r \geq 46$. (b) Here $p = r/2$ and $m = \frac{3}{4}r^2 - r$. Take $h = 13$, $T = (25r - 182)/2$: $\frac{T(T+1)}{182} - m = \frac{79r^2 - 8322r + 32760}{728} > 0$ for $r \geq 102$ (dropping the other, nonnegative terms of the bound). $\square$

Lemma 5.3 (leaves absorb the parity constraints).[†] Let H be a graph on r vertices, let v be a vertex of H , and let G be obtained by attaching leaves to v . If G has $m \geq r^2/2$ edges, then G passes the iterated parity test.

Proof. Give the vertices of H the auxiliary labels $0, \dots, r-1$, with 0 at v . For $q = 2^j$, let c_j be the number of edges of H whose difference has 2-adic valuation j . The q residue classes of $\{0, \dots, r-1\}$ have sizes $r_a \in \{k, k+1\}$, where $r = kq + s$ with $0 \leq s < q$, so $c_j \leq \sum_a \lfloor r_a^2/4 \rfloor \leq \lfloor r^2/(4q) \rfloor$: a direct computation shows that $r^2/(4q) - \sum_a \lfloor r_a^2/4 \rfloor$ equals $s^2/(4q)$ when k is even and $(q-s)^2/(4q)$ when k is odd. The test at level j requires $o_j = \lceil \lfloor m/q \rfloor / 2 \rceil \geq \lfloor m/(2q) \rfloor \geq c_j$ edges of valuation j . Give $o_j - c_j$ of the leaves labels of valuation j (the labels need not be distinct); this uses all the leaves, since $\sum_j o_j = m$ and $\sum_j c_j = |E(H)|$. The cuts defined by the successive binary digits then have exactly the required sizes. $\square$

The auxiliary labels need not lie in $[0, m]$, so the lemma concerns the parity test, not the residue tests with capacities.

Proposition 5.4 (what parity and positivity miss).[†] Neither iterated parity nor positivity can prove Corollary 5.2: both families pass the iterated parity test, and satisfy $\mu(G) \geq \gamma_2(G) > \beta_m$.

Proof. Both families have $m \geq r^2/2$, so Lemma 5.3 applies. Split K_r into two halves, and put the leaves on the side opposite to the vertex that carries them. In (a) this cut has $B = 2\lfloor r^2/4 \rfloor \geq 2m/3$ edges, and in (b) at least $r^2/2 - r/2 \geq 2m/3$. Its ± 1 vector has energy $4B - 2m$, so $\gamma_2(G) \geq 4B - 2m \geq 2m/3$. On the other hand, $D_m(t) < 0$ forces $t \geq 2\pi/(2m+1)$ on $[0, \pi]$, so $\beta_m \leq 1 + \csc \frac{\pi}{2m+1} \leq 1 + \frac{2m+1}{\pi} + \frac{\pi}{2m+1} < \frac{2m}{3}$ for $m \geq 100$, which holds in both families. $\square$

The same counting applies to several cliques at once.

Corollary 5.5 (the interval for clique packings).[†] Suppose G has m edges and contains t edge-disjoint copies of K_r . For an integer $1 \leq h \leq r-1$ put $T = hr - \frac{h(h+1)}{2}$. If G is

graceful, then

$$m \geq \frac{T(tT + 1)}{h(h + 1)},$$

and if $r \geq 2h$, then

$$m \geq \frac{T(tT + 1)}{h(h + 1)} + \frac{(h - 1)(th + 1)}{3}.$$

Proof. Run the proof of Theorem 5.1 in each copy. The tT chosen pairs are all edges, of the same copy or of different ones, so their differences are distinct: their sum is at least $tT(tT + 1)/2$ and at most tWm , with $W = h(h + 1)/2$. For the refinement, the deficits $\binom{h-j+1}{2}$, $1 \leq j \leq h - 1$, occur twice in each copy, and the corresponding $2t(h - 1)$ boundary gaps are distinct edge differences. Giving the smallest gaps to the largest deficits, the total deficit is at least $\sum_{j=1}^{h-1} t(4tj - 2t + 1)\binom{h-j+1}{2} = \frac{th(h-1)(h+1)(th+1)}{6}$. Subtract this from tWm and divide by tW . $\square$

For example, $r = 20$ and $h = 4$ give $m \geq 249t + \frac{9}{2}$: t edge-disjoint copies of K_{20} together with at most $59t + 4$ further edges, arranged in any way, never form a graceful graph. Shifting the labels of each copy to start at 0 turns the copies into a *difference triangle set* [17]: t sets of r integers, each containing 0, with all differences within the sets distinct; its *scope* is the largest element. Conversely, a windmill with pendant edges at its hub realizes any difference triangle set, one pendant edge for each missing difference. So the least number of edges of a graceful graph containing t edge-disjoint copies of K_r is exactly the least scope of a difference triangle set with t rows of r marks, and the first bound of Corollary 5.5 is the classical counting bound for that problem [17], translated into graph language. Per copy of K_r , positivity (Corollary 4.4) forces about $r(r - 2)/(2 - 2\kappa) \approx 0.64r^2$ edges, and Corollary 5.5 forces $r^2 - O(r^{3/2})$. For large t , positivity is the stronger of the two for $6 \leq r \leq 11$, and the interval from $r = 12$ on: with $h = 3$ the interval gives $\frac{3}{4}(r - 2)^2 + 2$ per copy, which is 77 at $r = 12$, against 76.65 from positivity. At $r = 6$ positivity gives $15.33t$ edges, more than Shearer's linear-programming bound $(91t + 6)/6$ for difference triangle sets [53]. Ma and Yi [42] obtain the same positivity constant for difference triangle sets; their counting bound, which lacks the boundary-gap term, overtakes positivity only from $r = 14$.

6. HISTORY AND PROVENANCE

Classical background.

- Graceful labelings (β -valuations) are due to Rosa [51, 50]. The parity condition (Corollary 3.3) is due to J. Bosák; Rosa credits it to him in his 1965 thesis [51, p. 17], and published it in [50]. The refinement in Corollary 3.2 appears in Section 7.2.2.3 of Knuth's *The Art of Computer Programming*, Volume 4, Fascicle 7 [32]. The application of the parity condition to the windmills $K_5^{(t)}$ comes from a MathOverflow answer by the author [29].
- The name, the partition condition, and the theorem that K_n is graceful only for $n \leq 4$ are due to Golomb [28].
- The analytic step of Tool II, evaluating the identity at the first dip of the Dirichlet kernel, is due to Rédei and Rényi [49], with Leech's constant κ [38]; Bernshteyn and Tait improved the difference-basis constant using coupled Fourier information [5]. They worked on difference bases, not graphs. The max-cut relaxation is Goemans and Williamson's [27].
- The ordered-difference argument of Tool III goes back to Lindström's bound for Sidon sets [39].
- Windmills $K_n^{(t)}$ with $n \geq 6$ were shown ungraceful by Koh, Rogers, Teo and Yap [33], and disjoint unions tK_n ($t \geq 2$) by Kotzig and Turgeon [34]. Bermond, Kotzig and

Turgeon [3] proved that no perfect system of difference sets has all its sets of one size $n \geq 6$, which is Theorem 4.6 for cliques of one order; they also used the case $j = 1$ of Proposition 3.5 and excluded $K_5^{(2)}$. Kotzig and Turgeon excluded perfect families for $K_5^{(4)}$ [35], and Mathon recorded perfect families of size-5 blocks for $t = 6, 8, 10$ and a recursive construction [43].

- The complete multipartite conjecture is Beutner and Harborth's [7]. The graphs $B(n, r, t)$ are due to Koh, Rogers, Teo and Yap [33], the question of their gracefulness to Bermond [4], and the non-explicit bound to Bermond and Farhi [2].

Recent work. In 2026 Shisheng Li applied the Rédei–Rényi argument to graphs built from cliques, with one sum per clique. He proved that $K_n \square P_4$ and $K_n \square P_5$ are ungraceful for $n \geq 11$ and $n \geq 12$, and that $K_n \square P_r$ and $K_n \square C_r$ are ungraceful for $n \geq 13$; this was reported by Knuth [30]. Earlier, Knuth's note [31], building on searches by Smith and Puget, gave computer proofs that $K_n \square P_2$, $K_n \square P_3$ and $K_n \square C_3$ are not graceful for $n \geq 6$, $n \geq 7$ and $n \geq 6$, respectively; $K_5 \square P_2$ and $K_6 \square P_3$ each have a unique graceful labeling up to symmetry. Ma and Yi [42] have used the same Dirichlet-kernel obstruction for families of Golomb rulers. Liu, Feng, Wang and Zhang settled the four-clique windmills [40].

Results new here, and how they were found. Results marked $\star$ or $\dagger$ are new as far as we know; several are short deductions, and we would be glad to hear of earlier appearances. The text was written by Claude Opus 5.5 (Anthropic), at the direction of the author; some passages of version 2 are adapted from a revision by GPT-6 Pro. The barbell theorem (Corollary 4.4(c)) comes from an earlier note written in the same collaboration, which started this project. A targeted search for prior statements of the main results found the antecedents cited in the text: Li's clique argument and the theorem of Bermond, Kotzig and Turgeon on perfect systems for Tool II, the asymptotic dense-subgraph bounds of [23, 8] for Tool III, and classical max-cut and rank-constrained semidefinite work for the planar refinement; several older proceedings papers (notably [4, 2]) could not be checked in full. Results marked $\star$ were found by Claude in this collaboration. Results marked $\dagger$ were found by GPT-6 Pro (OpenAI), most of them in a referee report on an earlier draft, which also corrected the status of the windmills $K_5^{(t)}$ and suggested using κ throughout. The author has not checked every proof in detail. The finite and numerical claims of the paper are checked by Python scripts at <https://github.com/boonsuan/tools>, in exact rational arithmetic wherever a proof uses them; their README lists which script checks which result. The search mentioned in Section 7 is author-reported.

Version 2 incorporates a revision by GPT-6 Pro. It corrected four points of version 1: the description of de Bruijn's tiles, the status of $P(13)$, which is graceful [46], the credit for the K_4 -snakes [20], and the scope of the statement that the max-cut relaxation describes what positivity can prove, which applies to μ and not to γ_2 . It also contributed the results marked $\dagger$ that are new in this version: the modulus-3 family, the exact energies of $B(n, r, t)$, the projection lemma, Theorem 4.18 and Corollary 4.19 for $K(11, 2)$, the boundary-gap terms in Theorem 5.1 and Corollary 5.5, and the leaf lemma. Corollary 3.9 was found by GPT-6 Pro later, after the author suggested evaluating the identity at other points; Claude then found Corollary 3.10 by a computer search of the residue equations modulo 3, 4, 5 and 8 for complete multipartite graphs. The verification scripts were written by Claude for version 2, partly adapted from scripts in that revision.

7. OPEN PROBLEMS

- (1) *The Beutner–Harborth conjecture.* Beutner and Harborth [7] gave graceful labelings of $K_{1,m,n}$, $K_{2,m,n}$ and $K_{1,1,m,n}$, and conjectured that these, together with $K_{m,n}$, are the only graceful complete multipartite graphs; it has been verified for up to 23 vertices [25] (and, by our own search, 26). The tools here settle every complete

multipartite graph with $\sum a_i^2 \leq \frac{\kappa}{1+\kappa}n^2 + 1$, and by Proposition 4.15(c) that is exactly the reach of positivity when no part exceeds $n/2$; for example $\mu(K_{3,3,3}) = 27$, and $\mu(K_{2,2,2,2,2}) = 20 > \beta_{40} \approx 18.6$. Residues alone also fail for $K_{3,3,3}$, $K_{2,2,2,2,2}$, $K_{4,4,4}$ and $K_{3,4,5}$ (Section 3.5). Since $K_{4,4,4}$ and $K_{5,5,5}$ have ρ -labelings (Section 2.1), a proof must use that the labels lie in $[0, m]$, not only their residues modulo $2m + 1$, and Tool III as stated does not exclude these graphs.

- (2) Prove that $K_{1,1,1,1,N} = K_4 \vee \overline{K}_N$, the simplest open case of the Beutner–Harborth conjecture, is not graceful for any $N \geq 1$. Parity forces N to be even (Section 3), and positivity cannot help, since $\mu = \gamma_2 = 8N - 12$ for $N \geq 4$ (Proposition 4.15(c)). The interval does constrain a labeling. Normalize so that 0 is a clique label, let C be the clique labels and $c = \max C$, and suppose that the largest label m lies in the independent part X . A difference $d > c$ can only come from a pair $u < u + d$ with $u \in C$ and $u + d \in f(X)$, and exactly one such pair exists. So every integer in $[0, m - c]$ is uniquely of the form $x' + u$ with $x' \in m - f(X)$ and $u \in C$: translates of C tile an initial segment of length $m - c$.
- (3) Determine which even $t \geq 12$ make $K_5^{(t)}$ graceful. Are the K_4 -snakes graceful for every $t \geq 4$?
- (4) Find the true thresholds in Table 1. For example, are the Paley graphs $P(17)$ and $P(29)$ graceful? Are $K_4 \square K_7$ and $K_5 \square K_6$? (Knuth found by exhaustive search that $K_4 \square K_4$ is graceful, while $K_4 \square K_5$ and $K_5 \square K_5$ are not [32, §7.2.2.3, exercise 88].) Is the q -Kneser graph $K_2(4, 2)$? The graphs $K_n \square P_r$ are ungraceful for $n \geq 6$ when $r = 2$, $n \geq 7$ when $r = 3$, $n \geq 11$ when $r = 4$, $n \geq 12$ when $r = 5$, and $n \geq 13$ for every r (Section 6); what are the thresholds for $r \geq 4$?
- (5) Can a *fixed* modulus, with capacities, exclude an infinite subfamily that the other tools miss? Which complete graphs pass the iterated parity test at every level?
- (6) Find more graphs with $\gamma_2(G) \leq 1 + \kappa(2m + 1) < \mu(G)$, other than the graphs H_b of Corollary 4.19, and a general method for proving such bounds.
- (7) Find certificates that couple residues, the interval and positivity and exclude an infinite family beyond the present tools. A natural setting is the moment hierarchy [36, 37] in the indicators $x_{v,a}$ of “vertex v has label a ”: its first level already implies the residue counts and the capacities, and is infeasible whenever $\mu(G) < \beta_m$; and level $\lceil (n + 2)/2 \rceil$ is feasible exactly when G is graceful (apply the relaxation to the indicators $\prod_v x_{v,f(v)}$ of complete assignments, whose values sum to 1). For a join $H \vee \overline{K}_N$, fixing the labels of H reduces the question to an exact-cover linear program, and infeasibility of that program for every labeling of H already makes level $|V(H)| + 2$ infeasible, independently of N . Can such certificates be found uniformly in N , for instance for $K_4 \vee \overline{K}_N$?

APPENDIX A. AN UPPER BOUND FOR β_m

Lemma A.1 (an upper bound for β_m).[†] For $m \geq 1$, put $N = 2m + 1$ and $\text{sinc } x = \sin x / x$. Then

$$\beta_m \leq 1 + \frac{\kappa N}{\text{sinc}(3\pi/(2N))}.$$

In particular, $\beta_m \leq 1 + \frac{525(2m+1)}{2399}$ for $m \geq 46$.

Proof. By symmetry restrict to $0 < t \leq \pi$, and put $x = t/2$ and $x_0 = 3\pi/(2N)$. At x_0 we have $D_m(2x_0) = -1/\sin x_0$, while for $x \geq x_0$, $D_m(2x) = \sin(Nx)/\sin x \geq -1/\sin x \geq -1/\sin x_0$. So the minimum occurs at some $x \leq x_0$, where

$$D_m(2x) = N \frac{\sin(Nx)}{Nx} \cdot \frac{x}{\sin x} \geq -\kappa N \frac{x_0}{\sin x_0},$$

since $\sin(Nx)/(Nx) \geq -\kappa$ and $x/\sin x$ increases on $(0, \frac{\pi}{2})$.

For the rational bound, first $\kappa < 1/\sqrt{21}$. Indeed, with $\delta = \frac{3}{2}\pi - \alpha$, $1/\alpha = \tan \delta > \delta$ gives $\alpha + 1/\alpha > \frac{3}{2}\pi > \sqrt{20} + 1/\sqrt{20}$ (because $\pi > \frac{157}{50} > \frac{7}{\sqrt{5}}$), so $\alpha > \sqrt{20}$, and $\kappa = 1/\sqrt{1 + \alpha^2}$. If $N \geq 93$, then $x_0 < 1/19$, so $\text{sinc } x_0 > 1 - x_0^2/6$ and

$$\frac{\kappa}{\text{sinc } x_0} < \frac{1}{\sqrt{21}(1 - 1/(6 \cdot 19^2))} = \frac{2166}{2165\sqrt{21}} < \frac{525}{2399}.$$

□

APPENDIX B. PROOF OF THEOREM 4.18

Proof. Let B be the 11×55 incidence matrix of K_{11} . Then $BB^T = 9I + J$ and $A_H = J + I - B^T B$, so A_H has eigenvalues 36, -8 and 1 on $\langle \mathbf{1} \rangle$, $U = \{B^T a : \sum_i a_i = 0\}$ and $\ker B$, of dimensions 1, 10, 44. By Proposition 4.15(b), $\mu(H) = 55 \cdot 8 = 440$.

Since $\|B^T a\|^2 = 9\|a\|^2$ for $a \perp \mathbf{1}$, the unit vectors of U are $B^T a/3$ with $\sum_i a_i = 0$ and $\sum_i |a_i|^2 = 1$, with coordinates $(a_i + a_j)/3$; so it suffices to show that $\sum_{i < j} |a_i + a_j| \leq 173/8$ for such a .

Put $x_{ij} = |a_i + a_j|^2$ and $Q = \sum_i |a_i|^4$. Expanding $|a_i + a_j|^2 = |a_i|^2 + |a_j|^2 + 2\text{Re}(a_i \bar{a}_j)$ gives

$$(5) \quad \sum_{i < j} x_{ij} = 9, \quad \sum_{i < j} x_{ij}^2 = 3Q + 2 + \left| \sum_i a_i^2 \right|^2 \geq 3Q + 2.$$

Let $h = \max\{2/\sqrt{11}, \max_{i < j} |a_i + a_j|\}$. Then

$$(6) \quad Q \geq \frac{1}{11} + \frac{(11h^2 - 4)^2}{792}.$$

If $h = 2/\sqrt{11}$ this is Cauchy-Schwarz. Otherwise take a pair attaining h and put $\sigma = |a_i|^2 + |a_j|^2 \geq h^2/2 \geq 2/11$; Cauchy-Schwarz on that pair and on the other nine entries gives $Q \geq \sigma^2/2 + (1 - \sigma)^2/9$, which increases for $\sigma \geq 2/11$, and $\sigma = h^2/2$ gives (6).

For $\rho > 0$ the identity $\sqrt{x} = \sqrt{\rho} + \frac{x-\rho}{2\sqrt{\rho}} - \frac{(x-\rho)^2}{2\sqrt{\rho}(\sqrt{x}+\sqrt{\rho})^2}$ and $\sqrt{x_{ij}} \leq h$ give, with $\rho = 9/64$ and (5),

$$\sum_{i < j} \sqrt{x_{ij}} \leq \frac{357}{16} - \frac{4Q + 2279/3072}{(h + 3/8)^2} \leq \frac{173}{8}.$$

For the last inequality: by (6), the numerator minus $\frac{11}{16}(h + 3/8)^2$ is at least

$$\frac{11}{18}h^4 - \frac{163}{144}h^2 - \frac{33}{64}h + \frac{5021}{4608} \geq \frac{2816h^4 - 6404h^2 + 3833}{4608} > 0,$$

using $2h \leq h^2 + 1$; the quadratic in h^2 has positive leading coefficient and discriminant -2163696 .

Consequently $\|Pz\|^2 \leq (173/24)^2$ when all $|z_v| = 1$. If $p = \|Pz\|^2$ and s is the squared length of the component of z along $\mathbf{1}$, then

$$-z^* A_H z = 9p - 55 - 35s \leq 9p - 55 \leq \left(\frac{173}{8}\right)^2 - 55 = \frac{26409}{64}.$$

□

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