

DENSE ALGEBRAIC LOCI OF HODGE CLASSES

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ABSTRACT. *We prove unconditionally that the Hodge conjecture is equivalent to two statements about abelian varieties, that the locus on which either statement holds is dense in every family and is either the whole family or a meagre set, that the real multiplications by $\sqrt{6}$ and by the cubic fields $\mathbb{Q}(\zeta_7)^+$ and $\mathbb{Q}(\zeta_9)^+$ are algebraic on explicit families of K3 surfaces, the cubic ones on the surfaces of Mumford type whose quaternion algebra is ramified at two real places, and that the Hodge conjecture holds on every power and every Hilbert scheme of points of the K3 surfaces of the one-dimensional Fricke families of levels 5 and 7, which carry complex multiplication by $\mathbb{Q}(\sqrt{-5})$ and $\mathbb{Q}(\sqrt{-7})$. For each of the two statements we build the object it needs next, a complex on a product of two Weil fourfolds and a class on an abelian fourteenfold, prove what it does, and prove at which step it stops; among the steps, every convolution of the Weil line bundles on E_0^8 whose shifts lie in seven consecutive integers fails the polarised criterion in eighteen of the twenty-eight arrangements of the shifts, an explicit curve of genus four makes the Prym map of cyclic triple covers étale onto the Weil domain of $\mathbb{Q}(\sqrt{-3})$, and the conjecture for a class is equivalent to the vanishing of the mass gap of a multiple of it plus a complete intersection, which Kollár's hypersurfaces show cannot be dropped for integral classes. Both statements remain open.*

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1. INTRODUCTION

The Hodge conjecture asks that every rational cohomology class of type (p, p) on a smooth complex projective variety be a rational combination of classes of algebraic subvarieties.¹ Hodge raised it in

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¹An earlier preprint of the authors, *Relative secant cycles and Hodge classes*, Preprints.org (2026), doi:10.20944/preprints202602.0462.v5 [9], claimed a proof of the Hodge conjecture that does not hold, and the present paper replaces it. The secant variety $\text{Sec}^n(A^\vee)$ contains $A^\vee$, so the cycle $\Sigma \cap A^\vee$ of its Definition 3 is $A^\vee$ itself, whose Fourier–Mukai transform is a skyscraper sheaf with no component in degree $2n$, contrary to its Lemma 4. The Weil classes span a 2-dimensional space on which $K^\times$ acts by $\tau \mapsto \tau^{2n}$, while η^n has character $N(\tau)^n$, contrary to its Propositions 1 and 2.

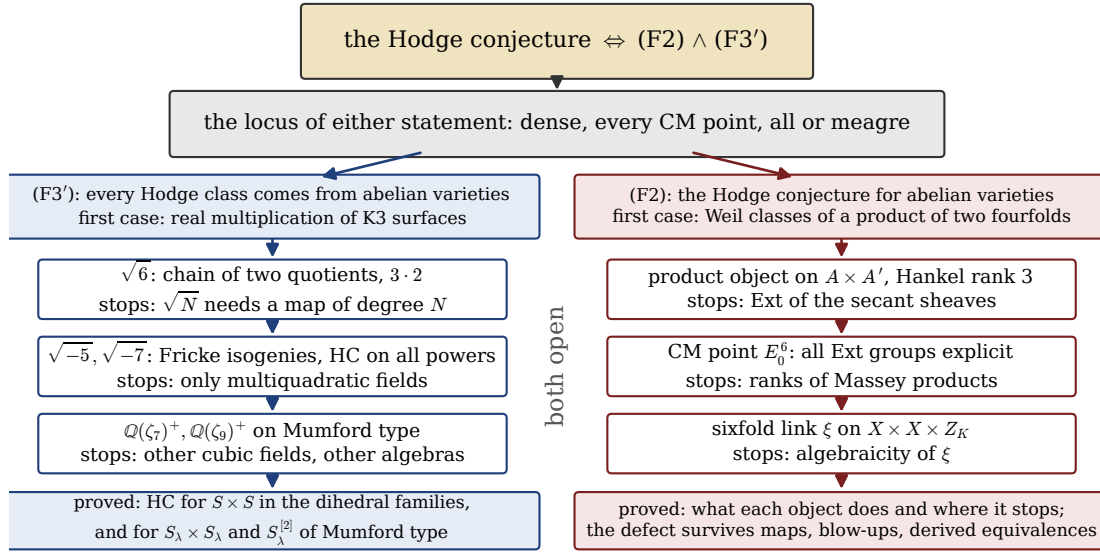


FIGURE 1. The plan of the paper. The conjecture is equivalent to (F2) and (F3') (Theorem 1.1, Proposition 3.9); the locus of either statement is dense, contains every CM point and is all or meagre (Section 4). For (F3') the first class is the real multiplication of K3 surfaces: the chain of quotients for $\sqrt{6}$ (Theorem 5.5), the Fricke isogenies, complex multiplications by $\mathbb{Q}(\sqrt{-5})$ and $\mathbb{Q}(\sqrt{-7})$ that settle every power (Theorem 5.9), the cyclic cubic fields on the surfaces of Mumford type (Theorem 5.14), and the steps at which each stops (Propositions 5.11 and 5.15). For (F2) the first class is the Weil classes of a product of two fourfolds: the product object (Theorem 6.3, stopping at Proposition 6.5), the point of complex multiplication (Lemma 6.8, Proposition 6.11, stopping at Section 6.5) and the sixfold link (Theorem 6.45, stopping at Section 6.13); the defect survives the operations of Proposition 8.2.

1950 [37]; Lefschetz had proved it for divisors [43]. It is known in a scattered family of cases and open in general. This paper does three things. It proves that the conjecture is equivalent to two statements about abelian varieties. It proves that the locus on which either statement is known has the same shape in every family, dense and either everything or meagre, so that each statement asks that algebraicity propagate from a dense set of special points to the whole family. And it builds, at the first class each statement has to reach, an explicit object, proves what that object does, and names the step at which it stops. The conjecture does not close here. What the paper adds is a complete account of where the two remaining statements live, two new theorems on real multiplication, by $\mathbb{Q}(\sqrt{6})$ on K3 surfaces and by the cyclic cubic fields $\mathbb{Q}(\zeta_7)^+$ and $\mathbb{Q}(\zeta_9)^+$ on the surfaces of Mumford type, the second of which closes the first two cases of the frontier of [19], and three constructions carried to the point where the next step is a single stated question. Figure 1 draws this plan.

1.1. The two statements

Write **HC** for the Hodge conjecture for all smooth complex projective varieties and **AB** for its restriction to abelian varieties. For a smooth projective variety X and an integer p let $\text{Hdg}^p(X)$ be the Hodge classes of degree $2p$, and let $\text{Ab}^p(X) \subseteq \text{Hdg}^p(X)$ be the span of the classes $\Gamma_*\gamma$, where B runs over the complex abelian varieties (the point included), γ over the Hodge classes on B , and Γ over the algebraic correspondences from B to X . The two statements are

(F2) every Hodge class on every complex abelian variety is algebraic, in the form in which it first appears in the closure computation of [19]: the Hodge classes of the smallest known instance, the

Its general case (Theorem 12) applies the Kuga–Satake construction outside Hodge structures of K3 type and assumes the Kuga–Satake correspondence algebraic, which is open. Section 7 lists every point at which that preprint fails.

exceptional classes on the square of a Mumford fourfold, are algebraic together with everything that can be built from them;

(F3') $\text{Hdg}^p(X) = \text{Ab}^p(X)$ for every X and every p : every Hodge class is algebraic modulo the images, under algebraic correspondences, of Hodge classes of abelian varieties.

Theorem 1.1 (Reduction). *HC holds if and only if (F2) and (F3') both hold; (F2) is equivalent to AB, and (F3') holds on every abelian variety, on every variety dominated by a product of curves or by an abelian variety, and on every variety in degrees $2p$ with $p \in \{0, 1, \dim X - 1, \dim X\}$.*

The content of the theorem is in the second sentence: (F2) is not a special case of the conjecture for abelian varieties but the whole of it, carried by its hardest classes, and (F3') is exactly what the conjecture asks of the part of the cohomology of X that no abelian variety reaches. Section 3 proves it and places the two statements in the closure graph of the known reductions, with the Lefschetz standard conjecture, André's motivated classes and the variational Hodge conjecture as the other nodes (Fig. 2).

1.2. The shape of the known locus

For a smooth projective family $\pi: \mathcal{X} \rightarrow S$ and a flat Hodge class ξ let $\Sigma(\xi) \subseteq S$ be the set of points at which ξ is algebraic.

Theorem 1.2 (Dichotomy and propagation).

- (i) $\Sigma(\xi)$ is a countable union of Zariski closed subsets of S , and either $\Sigma(\xi) = S$ or $\Sigma(\xi)$ is meagre.
- (ii) (F2) holds if and only if, for every polarised abelian scheme and every flat Hodge class ξ , algebraicity of ξ at one point of complex multiplication forces $\Sigma(\xi) = S$.
- (iii) For the Weil classes of every CM field, every $n \geq 2$ and every discriminant, $\Sigma(\xi)$ contains an explicit point of complex multiplication and is dense.
- (iv) (F3') holds for X in degree $2p$ if and only if a Hodge complement of the abelian part of $H^{2p}(X, \mathbb{Q})$ carries no nonzero Hodge class.

So both statements have the same form. There is a dense set of points, the points of complex multiplication, at which the class is known to be algebraic by explicit cycles; the locus of algebraicity is a countable union of closed strata; and the conjecture is the statement that this union is not meagre. Figure 3 draws it. Section 4 proves the theorem, after [19], whose argument we rewrite in the generality needed here.

1.3. Constructions at the first class

The rest of the paper is constructive. At each statement we go to the first class that is not known and build the object that would make it algebraic.

At (F3') the first class lives on powers of a K3 surface S with real multiplication: the transcendental lattice $T(S)_{\mathbb{Q}}$ is a vector space over a totally real field $E = \text{End}_{\text{Hdg}} T(S)_{\mathbb{Q}}$, and the elements of E are Hodge classes on $S \times S$. Section 5 proves:

Theorem 1.3 (Chains of symplectic quotients). *Let S be a projective K3 surface and $e \in E$ with $e^2 = n \in \mathbb{Q}_{>0}$. If there is a chain of projective K3 surfaces $S = S_0, S_1, \dots, S_r$ joined by dominant rational maps or rational Hodge isometries whose degrees multiply to n modulo squares, then e is algebraic on $S \times S$. In particular, for every projective K3 surface S with $\mathbb{Q}(\sqrt{6}) \subseteq E$ whose transcendental space embeds in the invariant lattice of a symplectic automorphism of order three, the real multiplication by $\sqrt{6}$ is algebraic, the Hodge conjecture holds for $S \times S$ and $S^{[2]}$, and such surfaces form two-dimensional families whose very general member has $E = \mathbb{Q}(\sqrt{6})$ exactly.*

Varesco [74] proved the case of a single quotient by a symplectic automorphism of prime order, which gives $\sqrt{2}$ and $\sqrt{3}$; van Geemen and Schütt [80] reach $\sqrt{2}, \sqrt{3}, \sqrt{7}$ through rational self-maps. The chain adds $\sqrt{6}$, the first composite value, with the lattice condition made explicit. A second construction in Section 5 starts from the universal elliptic curve over the modular curves $X_0(5)$ and $X_0(7)$: the Fricke involution lifts to an isogeny of degree 5 or 7 between the elliptic K3 surface over the base and the one over its image, and on the quadratic twists of the two surfaces by the same

pair of parameters it becomes a rational map of that degree between two K3 surfaces of the same family. Lemma 5.8 determines the sign of the square of this map: the isogeny composed with itself is $[-N]$, and on the twists at a pair exchanged by the involution the map acts on the transcendental lattice with square $-N$, a complex multiplication by $\mathbb{Q}(\sqrt{-5})$ or $\mathbb{Q}(\sqrt{-7})$, while on the two twists at the fixed points of the involution, the surfaces of [80], its square is N and it is a real multiplication. Theorem 5.9 shows that every member of the two families has that complex multiplication, algebraic because it is the pullback of the isogeny, that its endomorphism field is a CM field, and that the Hodge conjecture and (F3') hold for every power and every Hilbert scheme of points of every member; the archive counts points to show that the very general member has Picard number 16 and endomorphism field exactly $\mathbb{Q}(\sqrt{-N})$. The method also shows exactly why it stops. Proposition 5.11 describes the action of an arbitrary correspondence on the transcendental lattice: every component whose resolution has geometric genus at most one acts as a similarity, with multiplier the product of the degrees of its two projections, a composite returning to S is an element of E with rational square, and these span only the multiquadratic subfield of E ; a correspondence reaching anything else has a component of geometric genus at least two and transcendental rank at least twice that of S . Proposition 5.12 shows that the correspondences cut out of finite groups acting on covers of S , which include the graphs of the dihedral families of [80], produce only abelian fields, and Corollary 5.13 draws the consequence that the Hodge conjecture holds on $X \times X$ for every member of those families. On the higher powers of a K3 surface the Hodge classes are the elements of E and Varesco's exceptional classes [75]; Proposition 5.16 reduces the Hodge conjecture for all powers of the surfaces of Section 5 with totally real endomorphism field to the latter, and Proposition 5.17 proves (F3') and the Hodge conjecture on every power and every Hilbert scheme of points of a K3 surface of Kummer type over $\mathbb{Q}$, the surfaces whose transcendental space embeds in $U_{\mathbb{Q}}^3$, with the condition made explicit on the Hasse invariants at Picard numbers 17 and 18. Proposition 5.19 computes the transcendental spaces of the Fricke surfaces, $U_{\mathbb{Q}}^2 \oplus \langle -1 \rangle \oplus \langle -N \rangle$, so that by Lombardo their Kuga-Satake varieties are fourth powers of abelian fourfolds of Weil type for $\mathbb{Q}(\sqrt{-N})$ with discriminant one, and shows that at a member with endomorphism field $\mathbb{Q}(\sqrt{-N})$ the fourfold splits into an elliptic curve with complex multiplication by that field and a simple threefold. For the cubic field of the frontier case of [19] the multiquadratic subfield is $\mathbb{Q}$ and the field is not cyclic in general, so both constructions are blind to it. A third is not. For the two cyclic cubic fields $F = \mathbb{Q}(\zeta_7)^+$ and $\mathbb{Q}(\zeta_9)^+$ the quaternion algebra over F ramified at exactly two real places is the even Clifford algebra of a two-dimensional dihedral family of K3 surfaces of Picard number ten, and weak approximation in the norm-one group of that algebra moves the period of the surface of Mumford type into the image of the family; Theorem 5.14 makes the real multiplication by F algebraic there, through Buskin's theorem, and proves the Hodge conjecture for $S_{\lambda} \times S_{\lambda}$ and for the Hilbert square of S_{λ} . Proposition 5.15 then says exactly what the mechanism cannot reach: the algebras over these fields ramified at finite places, which no rank-twelve family carries, the cyclic cubic fields of larger conductor, which no finite group acting on $\mathbb{P}^1$ produces, and the fields with Galois group S_3 . The frontier is every pair of a cubic field and a quaternion algebra other than these two.

At (F2) the first class is a Weil class of a very general abelian $2n$ -fold of Weil type with $n \geq 3$, or the exceptional class of a Mumford fourfold. For the Weil classes the tool is the polarised semiregularity criterion of [19]: a perfect complex $\mathcal{E}$ whose Chern character is a nonzero multiple of a Weil class plus a polynomial in the polarisation, with $\dim \text{Ext}^2(\mathcal{E}, \mathcal{E})$ equal to the lower bound $r(\gamma) = (4 + \rho)n^2 - 2n$, makes the Weil classes algebraic on every member with full Hodge group. Section 6 proves:

Theorem 1.4 (The product object). *Let A, A' be very general polarised abelian varieties of Weil type for the same imaginary quadratic field, of dimensions $2m, 2m'$, with algebraic Weil classes, and let $n = m + m'$. There is a perfect complex $\mathcal{E}$ on $A \times A'$, a direct sum with shifts of exterior products of secant sheaves and powers of the polarisations, whose Chern character has the polarised shape with Hankel rank $\rho = 3$, so that $r(\gamma) = 7n^2 - 2n$, and whose first-order Hodge locus is the whole tangent space of the polarised Weil family. Every such direct sum has $\dim \text{Ext}^2(\mathcal{E}, \mathcal{E}) > r(\gamma)$, and its semiregularity map is not injective. The criterion can therefore be met at a product point only by a convolution of these pieces with nonzero differentials.*

For the exceptional classes of a Mumford fourfold X we replace the K3 surface that carries their transcendental part in [19] by an abelian sixfold Z_K over the same Shimura curve, so that the whole problem stays inside abelian varieties.

Theorem 1.5 (The sixfold link). *Let X be a Mumford fourfold with cubic field F , let $K = F(\sqrt{-\delta})$ be a CM extension inside the quaternion algebra, and let Z_K be the abelian sixfold of Shimura type attached to K over the Shimura curve of X . The Hodge classes in $H^1(X) \otimes H^1(X) \otimes H^2(Z_K)$ that identify the two copies of the nine-dimensional representation T form a line over F , spanned by a class ξ . If ξ is algebraic and K contains no imaginary quadratic field, the exceptional classes of $X \times X$ are algebraic, and so is every Hodge class on every power of X . Conversely the algebraicity of ξ is equivalent to the algebraicity of the exceptional classes together with one algebraic link from Z_K to a power of X .*

Neither theorem closes (F2). Section 6 says what each needs next: for Theorem 1.4 the Ext algebra of Markman’s secant sheaves, which no published result computes at a very general point, and which Lemma 6.8 computes at the point E_0^6 of complex multiplication, where Proposition 6.11 shows that no convolution with three powers of L meets the criterion, the thirty-two threefolds of Appendix A included, Proposition 6.13 excludes by a count of isolated Ext blocks the pieces meeting along surfaces, and Proposition 6.14 excludes every convolution with a threefold piece outside every closed chain of Ext blocks, among them those whose further pieces stay within one shift of the thirty-two, Theorem 6.23 excludes, for every polarisation of E_0^6 , every convolution of distinct threefolds and at most five line bundles, the enumeration of Remark 6.25 those with six or seven and all but thirty-one arrangements of eight, and Section 6.8 reads the support of an object meeting the criterion off the obstruction class localised to that support: a threefold of nonzero multiplicity is cut by the rest of the support in an ample divisor (Theorem 6.27), two threefolds meeting along a surface cannot carry the object between them (Theorem 6.29), no object supported on the thirty-two threefolds of one family has the polarised shape (Corollary 6.31), nor on all sixty-four of both families, by the relations along each line of four threefolds (Theorem 6.33), and a smooth complete intersection meeting the threefolds in curves cannot carry the rest (Proposition 6.35); what is left open there are the convolutions with eight or more line bundles, with or without threefolds, decided by the products along their chains; for Theorem 1.5 the algebraicity of one class in a mixed Künneth component of an abelian fourteenfold, which no known theorem produces even at the points of complex multiplication.

Three further results of Section 6 sharpen the frontier of (F2). At the point E_0^8 of [19, §18], Proposition 6.37 excludes every convolution whose shifts form two adjacent pairs at any distance, and Corollary 6.38 excludes every convolution whose shifts lie in seven consecutive integers in eighteen of the twenty-eight arrangements; the computation of Remark 6.39 decides nine others and leaves one open. Proposition 6.40 writes down a curve of genus four with a line bundle of order three at which the period map of the cyclic triple covers is a local isomorphism onto the Weil domain of $\mathbb{Q}(\sqrt{-3})$ at $n = 3$, so that the Weil classes of that domain are reached by the Prym varieties of an open set of covers. Remark 6.41 records that over twelve cubic fields the count of [19, Prop. 19.47] excludes no integral character at a sextic CM field. Section 8.3 turns to analytic constructions: the residues of Jacobian determinants of complete intersections (Proposition 8.3) and mass-minimising integral currents (Proposition 8.4), for which the conjecture on a class is exactly the vanishing of a mass gap after a multiplication and the addition of a complete intersection, and Kollár’s hypersurfaces show that the multiplication cannot be omitted.

1.4. What is proved and what is not

Table 4 in Section 8 lists every statement of the paper with its status. The reduction, the dichotomy, Theorem 1.3 and the two constructions are proved. (F2) and (F3’) are open, and with them the conjecture. We state this at the outset because the earlier preprint named in the footnote did not, and because the value of the constructions lies in the precision of the question each one leaves.

Appendix B describes the machine checks: a Lean 4 file that verifies the closure graph, the character computation behind the footnote and the dimension counts of Section 6, with no axiom beyond propositional extensionality; the Lean certificate of [19] on which the reduction rests; and programs in

Python, Julia, C, C++ and shell that recompute every finite quantity used in the proofs. No statement of the paper rests on a computation.

2. CONVENTIONS

Varieties are smooth, projective and connected over $\mathbb{C}$ unless said otherwise. Cohomology is singular cohomology with rational coefficients, $H^k(X) = H^k(X, \mathbb{Q})$, with its Hodge structure. A Hodge class of degree $2p$ is an element of $H^{2p}(X) \cap H^{p,p}$; their space is $\text{Hdg}^p(X)$. A class is algebraic if it is a rational combination of fundamental classes of subvarieties of codimension p ; by Fulton [30], intersection products, pull-backs and push-forwards along morphisms preserve algebraic classes. An algebraic correspondence from Y to X is an algebraic class Γ on $Y \times X$, acting by $\Gamma_*\gamma = \text{pr}_{X*}([\Gamma] \cup \text{pr}_Y^*\gamma)$. The Hodge group $\text{Hg}(V)$ of a rational Hodge structure V is the smallest algebraic subgroup of $\text{GL}(V)$ defined over $\mathbb{Q}$ whose real points contain the circle $h(U(1))$; for a polarisable V it is reductive, the Hodge classes in every tensor construction on V are its invariants, and the Hodge group of a variety is that of its total cohomology [27]. The Hodge conjecture for X is the statement that every Hodge class on X is algebraic; **HC** is the conjecture for all varieties and **AB** for all abelian varieties.

For an abelian variety A of dimension g with $H^1(A) = V$, the cohomology ring is $\bigwedge^\bullet V$. An abelian $2n$ -fold A is of Weil type for the imaginary quadratic field K if K acts on A up to isogeny with V a K -vector space of dimension $2n$ on which the two complex embeddings have equal multiplicity n in $H^{1,0}$. The Weil classes are the elements of $\bigwedge_K^{2n} V \subseteq \bigwedge^{2n} V$, a two-dimensional rational space of Hodge classes of degree $2n$ [83]; $K^\times$ acts on it by $\tau \mapsto \tau^{2n}$, and η^k , for η the class of a K -hermitian polarisation, carries $\tau \mapsto N_{K/\mathbb{Q}}(\tau)^k$. For the general CM field we follow [19]: a family of Weil type for a CM field F of degree $2m$ with $n \geq 2$ and discriminant δ has a connected polarised period domain $\mathcal{D}_F$ of dimension n^2 , and its very general member has Hodge group $\text{SU}(V, H)$ for the F -hermitian form H of the polarisation.

A Mumford fourfold is a simple abelian fourfold X whose Hodge group is the $\mathbb{Q}$ -form of SL_2^3 given by the norm-one group D^1 of a quaternion algebra D over a totally real cubic field F , split at one real place and definite at the other two [54]. Its Hodge ring is generated by the polarisation and one class of degree four that is not a polynomial in divisors, and $X \times X$ carries exceptional Hodge classes of degree four that are not products of divisor classes and are not Weil classes of any quotient [56, 19]. We write $T = D^0 \subseteq D$ for the trace-zero part, a nine-dimensional F -line on which $\text{Hg}(X)$ acts by conjugation; $T(-1)$ is a Hodge structure of K3 type with Hodge numbers $(1, 7, 1)$, and the exceptional classes of $X \times X$ are the real multiplication by F on the copy of $T(-1)$ inside $H^1(X) \otimes H^1(X)$.

For a K3 surface S the transcendental lattice $T(S)$ is the orthogonal complement of $\text{NS}(S)$ in $H^2(S, \mathbb{Z})$; $T(S)_\mathbb{Q}$ is an irreducible Hodge structure, and $E = \text{End}_{\text{Hdg}} T(S)_\mathbb{Q}$ is a totally real field or a CM field, acting by self-adjoint endomorphisms in the first case [84]. The K3 lattice is $\Lambda = U^3 \oplus E_8(-1)^2$, and $\Lambda_\mathbb{Q} = \Lambda \otimes \mathbb{Q}$. For a rational quadratic space W we use the discriminant $d(W) \in \mathbb{Q}^\times / \mathbb{Q}^{\times 2}$ and the Hasse invariants $\varepsilon_p(W) = \prod_{i < j} (a_i, a_j)_p$ of a diagonalisation $\langle a_1, \dots, a_k \rangle$, with $(\ , \)_p$ the Hilbert symbol; the classification of rational quadratic forms by rank, signature, discriminant and Hasse invariants, and the existence theorem for prescribed local invariants, are Serre's [69, Chap. IV].

Perfect complexes on A are objects of the bounded derived category of coherent sheaves; $\text{ch}(\mathcal{E}) \in H^{\text{ev}}(A)$ is the Chern character, additive on triangles, multiplicative on exterior products, with $\text{ch}(\mathcal{E}[1]) = -\text{ch}(\mathcal{E})$. The semiregularity map $\sigma_\mathcal{E}: \text{Ext}^2(\mathcal{E}, \mathcal{E}) \rightarrow \bigoplus_q H^{q+2}(A, \Omega^q)$ is that of Buchweitz and Flenner [17], after Bloch [18]; its injectivity implies that the Hodge locus of $\text{ch}(\mathcal{E})$ coincides, near the given point, with the locus where $\mathcal{E}$ deforms [17, 64].

Throughout, an assertion attributed to a reference is cited and not reproved; everything else is proved in the text. The phrase “very general” means outside a countable union of proper closed analytic subsets of the parameter space.

3. THE REDUCTION TO ABELIAN VARIETIES

3.1. The statement (F3') and the class $\mathcal{A}$

Definition 3.1. For a variety X and $p \geq 0$ let $\text{Ab}^p(X) \subseteq H^{2p}(X)$ be the span of the classes $\Gamma_*\gamma$ of degree $2p$, where B runs over the complex abelian varieties, the point included, γ over $\text{Hdg}^*(B)$, and Γ over

the algebraic cycles on $B \times X$ with rational coefficients. The statement (F3') is: $\text{Hdg}^p(X) = \text{Ab}^p(X)$ for every X and every p .

Each $\Gamma_*\gamma$ is a Hodge class, since cup product with an algebraic class and push-forward along a projection are morphisms of Hodge structures of the right bidegree; so $\text{Ab}^p(X) \subseteq \text{Hdg}^p(X)$. Taking B a point shows that $\text{Ab}^p(X)$ contains every algebraic class, and taking X abelian, $B = X$ and Γ the diagonal shows that (F3') holds on every abelian variety.

Proposition 3.2. *HC holds if and only if AB and (F3') both hold.*

Proof. If HC holds then AB holds, and every Hodge class is algebraic, hence in $\text{Ab}^p(X)$. Conversely let $\gamma \in \text{Hdg}^p(X)$. By (F3'), $\gamma = \sum_j \Gamma_{j*} \gamma_j$ with γ_j a Hodge class on an abelian variety B_j , algebraic by AB; then $\text{pr}_{B_j}^* \gamma_j \cup [\Gamma_j]$ is algebraic on $B_j \times X$ and so is its push-forward to X . $\square$

Proposition 3.3. *Let $\mathcal{A}$ be the class of varieties X such that $H^*(X)$ is spanned by classes Γ_*y with B abelian, Γ an algebraic cycle on $B \times X$ and $y \in H^*(B)$.*

- (i) *Every $X \in \mathcal{A}$ satisfies (F3').*
- (ii) *$\mathcal{A}$ contains the abelian varieties and the curves, and is closed under products and under images of surjective morphisms. It contains every product of curves, every symmetric product of a curve, and every variety dominated by a product of curves or by an abelian variety.*
- (iii) *Every X of dimension d satisfies (F3') in the degrees $2p$ with $p \in \{0, 1, d-1, d\}$. In particular every variety of dimension at most three satisfies it in every degree.*

Proof. (i) Let $\gamma \in \text{Hdg}^p(X)$ and write $\gamma = \sum_j \Gamma_{j*} y_j$ with $y_j \in H^*(B_j)$. Each Γ_{j*} is a morphism of Hodge structures, so after replacing y_j by its component in the degree that Γ_{j*} sends to $2p$ we may assume y_j homogeneous. Let π_j be the projector of $H^*(B_j)$ onto its Hodge classes along a complement stable under the Hodge group, which exists because that group is reductive. Then $\gamma = \sum_j \Gamma_{j*} \pi_j y_j + \sum_j \Gamma_{j*} (1 - \pi_j) y_j$. The second sum is a Hodge class whose summands lie in images of Hodge complements of the invariants; the Hodge group of $\prod_j B_j \times X$ acts on each term through a representation without invariants, so each term, being a Hodge class, is zero. Hence $\gamma = \sum_j \Gamma_{j*} (\pi_j y_j) \in \text{Ab}^p(X)$.

(ii) The point, abelian varieties and curves are in $\mathcal{A}$: for a curve C , H^0 and H^2 come from the point and $H^1(C) = H^1(J(C))$ by the Abel–Jacobi map. If $X, Y \in \mathcal{A}$ then $H^*(X \times Y) = H^*(X) \otimes H^*(Y)$ is spanned by exterior products of classes $\Gamma_* y \boxtimes \Gamma'_* y' = (\Gamma \times \Gamma')_*(y \boxtimes y')$ with $B \times B'$ abelian. If $f: X \rightarrow X'$ is surjective, then $f_* f^* = \deg(f)$ on $H^*(X')$ after cutting X by hyperplanes to the dimension of X' if needed, so every class of X' is f_* of a class of X , and $f_* \Gamma_* y = (\Gamma_f \circ \Gamma)_* y$ with Γ_f the graph of f . The listed varieties follow: a symmetric product is the image of a product of curves, and a variety dominated by a product of curves or an abelian variety is such an image.

(iii) Degrees 0 and $2d$ are spanned by the class of X and the class of a point. In degree 2 every Hodge class is a divisor class by Lefschetz [43]. In degree $2d-2$, let θ be a hyperplane class; the hard Lefschetz theorem makes $\theta^{d-2} \cup: H^2(X) \rightarrow H^{2d-2}(X)$ an isomorphism of Hodge structures, so every Hodge class of degree $2d-2$ is $\theta^{d-2} \cup \alpha$ with α a Hodge class of degree 2, algebraic by Lefschetz; a product of algebraic classes is algebraic. $\square$

Part (iii) covers every variety of dimension at most three whatever its Hodge numbers, the Calabi–Yau threefolds included, for which the Hodge numbers of the toric hypersurface families are tabulated in [11]; the first open degree is the middle degree of a fourfold.

Proposition 3.4 (Fourfolds with few zero-cycles). *Let X be a smooth projective fourfold whose group of zero-cycles is supported on a threefold: $\text{CH}_0(Z)_{\mathbb{Q}} \rightarrow \text{CH}_0(X)_{\mathbb{Q}}$ is surjective for some closed $Z \subset X$ of dimension at most three. Every uniruled fourfold is such: a general point lies on a rational curve, which meets an ample divisor D , so the point is rationally equivalent to a point of D , and the points of a dense open set generate $\text{CH}_0(X)$. Then the Hodge conjecture holds for X in degree four, and X satisfies (F3') in every degree.*

Proof. Bloch and Srinivas prove, by the decomposition of the diagonal that the hypothesis provides, that a fourfold whose zero-cycles are supported on a threefold satisfies the Hodge conjecture for cycles

of codimension two [21, Thm. 1]. So $\text{Hdg}^2(X) = \text{Alg}^2(X) \subseteq \text{Ab}^2(X)$, the class of a cycle Z being $\Gamma_* 1$ for the point and $\Gamma = Z$; the other degrees are Proposition 3.3(iii). $\square$

The uniruled fourfolds and the rationally connected fivefolds are treated in [19]; the proposition names the hypothesis on zero-cycles behind the first of them.

3.2. The statement (F2)

For an abelian variety Y let $R(Y) \subseteq H^*(Y)$ be the $\mathbb{Q}$ -subalgebra generated by the divisor classes of Y and by the pull-backs f^*w of Weil classes w of abelian varieties B of Weil type along homomorphisms $f: Y \rightarrow B$. Every element of $R(Y)$ is a Hodge class, and every Hodge class on an abelian variety of dimension at most five lies in $R(Y)$ [57].

Definition 3.5. The statement (F2) is: for every complex abelian variety Y , every Hodge class on Y that does not lie in $R(Y)$ is algebraic.

The classes in $R(Y)$ are the ones the known theorems reach when they reach anything: divisor classes by Lefschetz, Weil classes by Markman [49, 50, 51] on fourfolds of Weil type and by the propagation results of [19] wherever they propagate. So (F2) is the conjecture for abelian varieties with the known classes struck out. The next proposition says that striking them out removes nothing.

Setting 3.6. Let X be a Mumford fourfold, $V = H^1(X)$ with its polarisation ψ , $G = \text{Hg}(X) \subseteq \text{Sp}(V, \psi)$, and $e_0 \in \text{Hdg}^2(X \times X)$ an exceptional class, a Hodge class of degree four that is not a polynomial in divisor classes. Such classes exist [56];² X is simple with $\text{End}^0(X) = \mathbb{Q}$ [54].

Lemma 3.7. *Let Y be an abelian variety. Let $Y_X \subseteq Y$ be the largest abelian subvariety isogenous to a power of X and Y' the identity component of the intersection of the kernels of the homomorphisms $Y \rightarrow X$. Then $Y_X \times Y' \rightarrow Y$ is an isogeny, $\text{Hom}(X, Y') = 0$, and*

$$H^1(Y) = (V \otimes Q) \oplus H^1(Y'), \quad Q = \text{Hom}^0(Y_X, X),$$

the first summand embedded by $v \otimes f \mapsto f^*v$. Let $\text{Sp}(V, \psi)$ act on $H^*(Y) = \bigwedge^* H^1(Y)$ through the factor V . Then every class of $R(Y)$ is invariant under $\text{Sp}(V, \psi)$, and for $Y = X^m$ the invariants of $\text{Sp}(V, \psi)$ in $H^*(X^m)$ are polynomials in divisor classes.

Proof. Poincaré reducibility gives $Y \sim Y_X \times Y'$ with $Y_X \sim X^k$ and no factor X in Y' ; since X is simple, $\text{Hom}(X, Y') = \text{Hom}(Y', X) = 0$, and since $\text{End}^0(X) = \mathbb{Q}$, Q has dimension k and the displayed map is an isomorphism. A homomorphism $h: Y \rightarrow Z$ of abelian varieties decomposes up to isogeny as $h_X \times h'$, and h^* acts on H^1 as $(1 \otimes (f \mapsto f \circ h_X)) \oplus h'^*$, which commutes with the action of $\text{GL}(V)$ on the factor V . So pull-backs along homomorphisms, cup products and exterior products are $\text{GL}(V)$ -equivariant, and the classes invariant under $\text{Sp}(V, \psi)$ form a subalgebra stable under pull-back. It remains to check the generators of $R(Y)$.

A divisor class on Y has Künneth components in $\bigwedge^2(V \otimes Q)$, $(V \otimes Q) \otimes H^1(Y')$ and $\bigwedge^2 H^1(Y')$, each a Hodge class. The middle one is a homomorphism from Y' to the dual of Y_X , which is zero. The first is a Hodge class on Y_X , hence a G -invariant of $\bigwedge^2(V \otimes Q) = S^2V \otimes \bigwedge^2 Q \oplus \bigwedge^2 V \otimes S^2Q$. Since G contains the Hodge group of a simple abelian fourfold whose only Hodge classes of degree two are the divisor classes, $(S^2V)^G = 0$ and $(\bigwedge^2 V)^G = \mathbb{Q}\psi$; so the component lies in $\psi \otimes S^2Q$, which $\text{Sp}(V, \psi)$ fixes. The third component is fixed because $\text{GL}(V)$ acts trivially on $H^1(Y')$.

For a Weil class, let B be of Weil type for K and $f: Y \rightarrow B$. The field K acts on $B \sim B_X \times B'$ through $\text{End}^0(B_X) \times \text{End}^0(B')$, because $\text{Hom}(B_X, B') = 0$; so it acts on $Q_B = \text{Hom}^0(B_X, X)$ and on $H^1(B')$,

²Mumford constructed [54] abelian fourfolds X whose Hodge group is a $\mathbb{Q}$ -form of SL_2^3 acting on $H^1(X)$, of dimension eight, through the tensor product of the three two-dimensional representations; the form comes from a quaternion algebra over a totally real cubic field F that is split at one real place and ramified at the other two, and the corestriction to $\mathbb{Q}$ of that algebra is the matrix algebra, which is what makes the representation rational. Such an X is simple, has $\text{End}^0(X) = \mathbb{Q}$, is not of CM type, and its Hodge group is smaller than the full symplectic group although no endomorphism accounts for the loss; its Shimura curve is compact. The exceptional classes of [56] live in degree four on $X \times X$ (equivalently in End of H^1 as a Hodge structure of weight zero) and span the invariants of the Hodge group that are not generated by the polarisation, and no one has proved them algebraic. The fourfold enters this paper because it is the smallest variety on which (F2) has an unsettled instance that no CM point reaches directly.

and the eigenspace of $H^1(B, \bar{\mathbb{Q}})$ for an embedding σ of K is $V \otimes Q_{B,\sigma} \oplus H^1(B', \bar{\mathbb{Q}})_\sigma$. A Weil class lies in the sum over σ of the top exterior powers of these eigenspaces, and $\bigwedge^{\text{top}}(V \otimes Q_{B,\sigma} \oplus W_\sigma) = (\det V)^{\otimes \dim Q_{B,\sigma}} \otimes (\det Q_{B,\sigma})^{\otimes 8} \otimes \det W_\sigma$, on which $\text{SL}(V) \supseteq \text{Sp}(V, \psi)$ acts trivially. Pulling back along f preserves invariance.

For $Y = X^m$, $H^*(X^m) = \bigwedge^*(V \otimes \mathbb{Q}^m)$, and the invariants of $\text{Sp}(V, \psi)$ in the tensor algebra of V are spanned by complete contractions with ψ by the first fundamental theorem for the symplectic group [34, Chap. 5]; a contraction of two factors of X^m is the pull-back of the class $\psi \in H^2(X \times X)$ along the projection to those factors, a divisor class, and a contraction within one factor is $\psi \in H^2(X)$. So the invariants are polynomials in divisor classes. $\square$

Theorem 3.8. (F2) implies **AB**, and is therefore equivalent to it.

Proof. **AB** contains (F2). Assume (F2). By Lemma 3.7 the exceptional class e_0 is not invariant under $\text{Sp}(V, \psi)$, since it is not a polynomial in divisor classes; as $\text{Sp}(V, \psi)(\mathbb{Q})$ is Zariski dense in $\text{Sp}(V, \psi)$, there is $g \in \text{Sp}(V, \psi)(\mathbb{Q})$ with $ge_0 \neq e_0$. Let θ be an ample class on $X \times X$, fixed by g by the lemma.

Let A be an abelian variety and $\beta \neq 0$ a Hodge class of degree $2p$ on A , and put $Y = A \times X \times X$. The group $\text{Sp}(V, \psi)$ acts on $H^*(Y) = H^*(A) \otimes H^*(X \times X)$ diagonally, through the copies of V in both factors; on $H^*(A)$ it acts through A_X and fixes β when $\beta \in R(A)$, but we need only that $g(\beta \otimes e_0) = g\beta \otimes ge_0$. If $\beta \otimes e_0$ lay in $R(Y)$ it would be fixed by g , so $g\beta \otimes ge_0 = \beta \otimes e_0$, which forces $ge_0 \in \mathbb{Q}e_0$; and g preserves the polarisation of $X \times X$, hence the intersection form on $H^4(X \times X)$ in which e_0 has nonzero square (the Hodge–Riemann relations on the primitive Hodge classes, after subtracting the part of e_0 in $\theta \cup H^2$, which is a polynomial in divisor classes and may be dropped from e_0 at the start), so $ge_0 = \pm e_0$, and replacing g by g^2 if needed, $ge_0 = e_0$, a contradiction. So $\beta \otimes e_0 \notin R(Y)$ and (F2) makes it algebraic. In the same way $e_0 \cup \theta^4 \notin R(X \times X)$: otherwise $(ge_0 - e_0) \cup \theta^4 = 0$, and $\theta^4 \cup : H^4(X \times X) \rightarrow H^{12}(X \times X)$ is injective by hard Lefschetz. So (F2) makes every Hodge class of degree twelve on $X \times X$ algebraic.

The Hodge group of $X \times X$ is reductive, the Hodge classes of degrees four and twelve are its invariants, and Poincaré duality is an equivariant perfect pairing between H^4 and H^{12} ; so it restricts to a perfect pairing between Hdg^2 and Hdg^6 , and there is a Hodge class e' of degree twelve with $\int_{X \times X} e_0 \cup e' = 1$. Then

$$\beta = \text{pr}_{A*}((\beta \otimes e_0) \cup \text{pr}_{X \times X}^* e'),$$

a push-forward of a product of algebraic classes, is algebraic. $\square$

Theorem 1.1 is Propositions 3.2, 3.3 and Theorem 3.8 together.

3.3. The closure graph

The paper [19] records the implications between the conjecture and the statements that have been used to approach it: the Lefschetz standard conjecture (L) for all varieties, the conjecture (M) that every Hodge class is motivated in André’s sense [5], the variational Hodge conjecture (V), the statement (F3) that the conjecture holds for all varieties that are not abelian, and the two statements of this paper.³ The rules are listed in Table 1, each with the place where it is proved; the first two rows are Propositions 3.2 and Theorem 3.8.

Proposition 3.9. Under the rules of Table 1 the closure of a set of statements contains **HC** if and only if the set contains one of

³For a smooth projective variety X of dimension d with hyperplane class h , (L) asks that the inverse of the hard Lefschetz isomorphism $h^{d-2p} : H^{2p}(X, \mathbb{Q}) \rightarrow H^{2d-2p}(X, \mathbb{Q})$ be induced by an algebraic cycle on $X \times X$ for every p ; it is known for abelian varieties, surfaces and the varieties whose cohomology is spanned by algebraic classes, and open for a general threefold. A Hodge class is motivated in André’s sense when it is obtained from classes of algebraic cycles by cup products, pull-backs, push-forwards and the inverse Lefschetz operators, on X and on the products $X \times Y$; (M) asks that every Hodge class be of this kind, which is weaker than the conjecture by exactly the use of the inverse Lefschetz operator. (V) asks that a flat Hodge class in a smooth projective family be algebraic on every fibre once it is algebraic on one; Grothendieck observed that (V) follows from the conjecture, and the converse is open. (F3) concerns the varieties that are not abelian and makes no statement about abelian varieties at all; it is the complement, inside the conjecture, of (F2), and the pair (F2), (F3) is the coarsest division of the conjecture that this paper refines.

TABLE 1. The rule set. A rule with two premises needs both. The closure of a set of statements under these rules, and the list of minimal sets whose closure contains **HC**, are computed by hand in the text and checked in Lean (Appendix B).

premises	conclusion	proof
HC	(F3), (F2), (F3'), AB , (M)	definitions; [5]
(F3)	HC	[19, Prop. 20.33]
(F2)	AB	Theorem 3.8
AB	(F2)	definition
AB , (F3')	HC	Proposition 3.2
(L), (M)	HC	[5]; [19, Thm. 20.70]
(L)	AB	[19, Thm. 20.70]
(V)	AB	[19, Thm. 20.70]

$$\{F3\}, \{L, M\}, \{L, F3'\}, \{V, F3'\}, \{F2, F3'\}, \{AB, F3'\}, \quad (3.1)$$

and these six sets are the minimal ones with this property.

Proof. Each listed set derives **HC**: $\{F3\}$ by the second row; $\{L, M\}$ by the sixth; the other four through **AB**, which (F2), (L) and (V) each give, and the fifth row. For the converse, call a set closed if it is stable under the rules. A closed set without **HC** contains no (F3), does not contain (L) and (M) together, and, if it contains (F3'), contains none of **AB**, (F2), (L), (V), since each of these yields **AB** and then **HC**. The maximal closed sets without **HC** are therefore

$$\begin{aligned} M_1 &= \{F2, AB, L, V\}, \\ M_2 &= \{F2, AB, V, M\}, \\ M_3 &= \{F3', M\}, \end{aligned}$$

each of which is closed by inspection of the table. The closure of a set Σ contains **HC** exactly when Σ lies in none of the M_i . If Σ contains (F3) this holds. Otherwise Σ needs an element outside M_1 , that is (F3') or (M); one outside M_2 , that is (F3') or (L); and one outside M_3 , that is one of (F2), **AB**, (L), (V). The minimal choices are (F3') with one of (F2), **AB**, (L), (V), and (M) with (L). This gives the six sets, and each is minimal because removing any element lands in some M_i . The same computation is carried out mechanically from the rule list in the Lean file of Appendix B, which returns these six sets and no others. $\square$

Figure 2 draws the graph. The two statements of this paper are the nodes marked in colour; every route to **HC** other than (F3), which is **HC** itself, and (L)+(M) passes through (F3').

4. THE ALGEBRAIC LOCUS IN A FAMILY

Definition 4.1. Let $\pi: \mathcal{X} \rightarrow S$ be a smooth projective morphism to a smooth connected complex quasi-projective variety, and let ξ be a global section of $R^{2p}\pi_*\mathbb{Q}$ that is a Hodge class at one point. The algebraic locus of ξ is the set $\Sigma(\xi)$ of $s \in S$ at which $\xi_s \in H^{2p}(\mathcal{X}_s)$ is algebraic.

By the theorem of the fixed part [25, 4], the flat sections of a polarisable variation form a sub-Hodge structure, constant along S ; so ξ_s is a Hodge class at every point once it is one somewhere.⁴

Lemma 4.2 (Dichotomy).

⁴The theorem says that for a polarisable variation of Hodge structure $\mathcal{V}$ over a smooth connected quasi-projective base S , the space $H^0(S, \mathcal{V})$ of global flat sections, which is the subspace of a fibre $\mathcal{V}_s$ fixed by the monodromy group, is a sub-Hodge structure of $\mathcal{V}_s$, and the Hodge structure it carries does not depend on s . Deligne proved it for the variations $R^k\pi_*\mathbb{Q}$ of a smooth projective morphism, where it follows from the degeneration of the Leray spectral sequence and the weight filtration of a smooth compactification; André proved it for every polarisable variation by showing that the connected monodromy group is a normal subgroup of the derived group of the generic Mumford–Tate group. In the situation of the text the section ξ is flat, so its value at one point is fixed by the monodromy; if that value has type (p, p) at one point, the sub-Hodge structure of flat sections contains it as a Hodge class, and since that Hodge structure is the same at every point, ξ_s has type (p, p) everywhere.

The closure graph: $HC \Leftrightarrow (F2) \wedge (F3')$

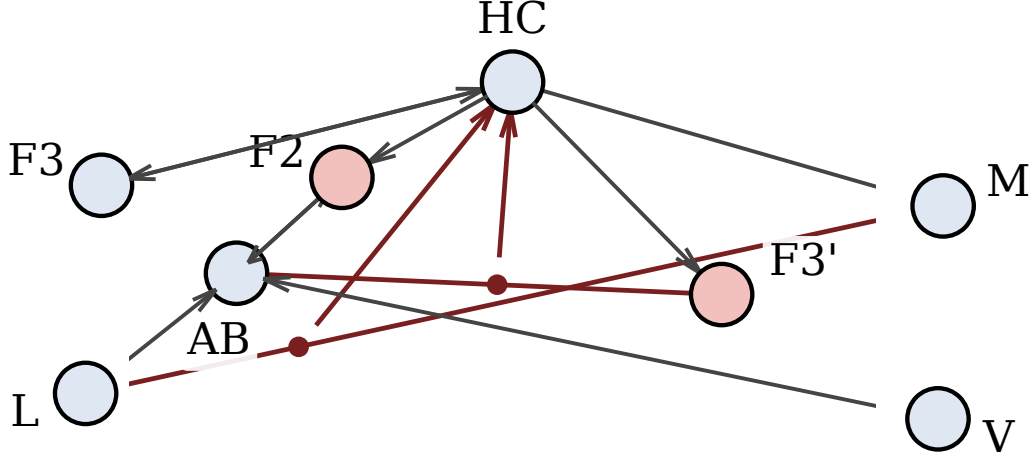


FIGURE 2. The closure graph of Table 1. Grey arrows are rules with one premise; a red fork joins the two premises of a rule that needs both. The coloured nodes are (F2) and (F3').

- (i) $\Sigma(\xi)$ is a countable union of Zariski closed subsets of S .
- (ii) Either $\Sigma(\xi) = S$ or $\Sigma(\xi)$ is meagre in S^{an} .
- (iii) The following are equivalent: $\Sigma(\xi) = S$; $\Sigma(\xi)$ has an interior point; $\Sigma(\xi)$ is not meagre; ξ_s is algebraic for the very general s . If $\Sigma(\xi)$ is dense they are equivalent to $\Sigma(\xi)$ being closed.

Proof. (i) Fix a relatively ample line bundle on $\mathcal{X}$. For a Hilbert polynomial P the relative Hilbert scheme $\text{Hilb}^P(\mathcal{X}/S)$ is projective over S [31], and the class of the fibre of the universal family in $H^{2p}(\mathcal{X}_s)$ is locally constant on each connected component of it, since a flat family of subschemes over a connected base has locally constant cycle class. A rational combination $\sum a_i [Z_i]$ equal to ξ_s is a point of a fibre product $\prod_S \text{Hilb}^{P_i}(\mathcal{X}/S)$, projective over S and with countably many connected components; on each component the condition $\sum a_i [Z_i] = \xi$ is either empty or holds throughout, because both sides are flat sections. So $\Sigma(\xi)$ is the union, over the countably many tuples (P_i, a_i) and components, of images of proper morphisms to S , each Zariski closed.

(ii) A countable union of proper Zariski closed subsets of an irreducible S has empty interior in S^{an} , and each is closed, so the union is meagre; if one of the closed sets is S itself, $\Sigma(\xi) = S$.

(iii) By the Baire category theorem⁵ S^{an} is not meagre in itself, so $\Sigma(\xi) = S$ excludes meagreness, and a set with an interior point is not meagre; by (ii) not meagre forces $\Sigma(\xi) = S$. The complement

⁵A subset of a topological space is meagre when it is a countable union of nowhere dense sets, that is, of sets whose closure has empty interior. The Baire category theorem states that a complete metric space, or a locally compact Hausdorff space, is not meagre in itself; a connected complex manifold such as S^{an} is locally compact, so the theorem applies to it. A proper Zariski closed subset of an irreducible variety has empty interior in the analytic topology, because a holomorphic function vanishing on an open set vanishes identically on a connected manifold; so a countable union of proper Zariski closed subsets is meagre, and the alternative in (ii) is exactly whether one of the countably many closed strata of (i) is the whole of S .

of a countable union of proper closed subvarieties is the very general locus, so algebraicity at the very general point means that one of the closed sets is all of S . A dense set that is also closed is S . $\square$

Lemma 4.3 (Points of CM type are dense). *Let G be a connected reductive group over $\mathbb{Q}$, $u_0: U(1) \rightarrow G_{\mathbb{R}}$ a homomorphism and $\mathcal{D}$ the orbit of u_0 under conjugation by $G(\mathbb{R})^+$. The points $u \in \mathcal{D}$ for which $u(U(1))$ lies in the real points of a maximal torus of G defined over $\mathbb{Q}$ are dense in $\mathcal{D}$.*

Proof. Let $u_1 \in \mathcal{D}$ and W a neighbourhood of u_1 . A maximal torus T_1 of $G_{\mathbb{R}}$ contains $u_1(U(1))$, since the centraliser of that compact torus is reductive and connected. Pick a regular element Y_1 in the Lie algebra $\mathfrak{t}_1$ of T_1 . The map $(g, Y) \mapsto \text{Ad}(g)Y$ from $G(\mathbb{R})^+ \times \mathfrak{t}_1$ to $\mathfrak{g}_{\mathbb{R}}$ is submersive at $(1, Y_1)$ because $\mathfrak{g}_{\mathbb{R}} = \mathfrak{t}_1 \oplus [\mathfrak{g}_{\mathbb{R}}, Y_1]$, so the image of a small neighbourhood $N_1 \times N_2$, with $N_1 u_1 N_1^{-1} \subseteq W$ and N_2 made of regular elements, contains an open set of $\mathfrak{g}_{\mathbb{R}}$, hence a rational point $Y = \text{Ad}(g)Y'$. The identity component T of the centraliser of the rational regular semisimple Y is a maximal torus over $\mathbb{Q}$ with $T_{\mathbb{R}} = gT_1g^{-1}$, and $u = gu_1g^{-1} \in W$ has $u(U(1)) \subseteq T_{\mathbb{R}}$. $\square$

When $G = \text{Hg}(A)$ and u is the Hodge cocharacter of an abelian variety A , the condition of the lemma says that $\text{Hg}(A)$ lies in a $\mathbb{Q}$ -torus, and then A is of CM type: a maximal $\mathbb{Q}$ -torus of $\text{GL}(H^1(A))$ containing it is the unit group of a commutative semisimple algebra L of dimension $2 \dim A$ commuting with $\text{Hg}(A)$, and the commutant of $\text{Hg}(A)$ is $\text{End}^0(A)$.

Proposition 4.4 (A family with a CM member through every Hodge class). *Let A_0 be an abelian variety and γ a Hodge class on it. There are a polarised abelian scheme $\pi: \mathcal{A} \rightarrow S$ over a smooth connected quasi-projective base, a flat section ξ of $R^{2p}\pi_*\mathbb{Q}$, a point s_0 with $\mathcal{A}_{s_0} \cong A_0$ and $\xi_{s_0} = \gamma$, and a point c with $\mathcal{A}_c$ of CM type; the points of CM type are dense in S .*

Proof. Let $G = \text{Hg}(A_0)$, let u_0 be the Hodge cocharacter of A_0 and $\mathcal{D}$ its $G(\mathbb{R})^+$ -orbit. Every $u \in \mathcal{D}$ defines a polarisable Hodge structure of weight one on $H^1(A_0)$ with Hodge group contained in G , so an abelian variety A_u up to isogeny; fixing the lattice $H^1(A_0, \mathbb{Z})$, the polarisation of A_0 and a level structure of level $N \geq 3$, the quotient $S = \Gamma \backslash \mathcal{D}$ by a neat congruence subgroup $\Gamma \subseteq G(\mathbb{Q})$ is a smooth quasi-projective variety carrying a polarised abelian scheme with these fibres [54, 25]. The class γ is invariant under G , so it is a flat section ξ of $R^{2p}\pi_*\mathbb{Q}$, and $\xi_{s_0} = \gamma$ at the image s_0 of u_0 . By Lemma 4.3 the points of CM type are dense in $\mathcal{D}$, hence in S . $\square$

Theorem 4.5 ((F2) is propagation from CM points). *The following are equivalent.*

- (i) (F2).
- (ii) **AB**.
- (iii) *For every polarised abelian scheme $\pi: \mathcal{A} \rightarrow S$ and every flat Hodge class ξ , algebraicity of ξ_s at one point s forces $\Sigma(\xi) = S$.*
- (iv) *The same for the points s of CM type.*
- (v) *For every π, ξ with ξ_c algebraic at a point c of CM type, $\Sigma(\xi)$ has an interior point.*

Proof. (i) and (ii) are Theorem 3.8. If (ii) holds, ξ_t is a Hodge class at every t by the theorem of the fixed part and algebraic by (ii), so (iii) holds; (iv) is a case of (iii), and (iv) and (v) are equivalent by Lemma 4.2(iii).

Assume (iv). Let F be a CM field and $n \geq 2$. The family of abelian varieties of split Weil type for F over its Shimura variety is a polarised abelian scheme on which the Weil classes are flat sections, and by [19, Thm. 1.1(i)] they are algebraic at a member isogenous to a product of abelian varieties with complex multiplication by F , a member of CM type; so by (iv) they are algebraic on every member, and by pull-back along an F -linear isogeny on every abelian variety of split Weil type for F . By André's theorem, in the form Milne gives it [52, Thm. 1 and §5], every Hodge class on an abelian variety of CM type is a sum of pull-backs, along homomorphisms, of Weil classes of abelian varieties of split Weil type; so **AB** holds for every abelian variety of CM type. Now let A_0 be any abelian variety and γ a Hodge class on it, and take π, ξ, s_0, c from Proposition 4.4. Then ξ_c is algebraic, (iv) gives $\Sigma(\xi) = S$, and $\gamma = \xi_{s_0}$ is algebraic. This is (ii). $\square$

Proposition 4.6 (The Weil classes). *For every CM field F , every $n \geq 2$ and every discriminant, the algebraic locus of the Weil classes in the polarised period domain $\mathcal{D}_F$ contains an explicit point of CM type and is dense, and it is either $\mathcal{D}_F$ or meagre.*

Proof. This is [19, Thm. 1.1], items (i) to (iii). The density is the Hecke orbit of the base point: the group $G_F(\mathbb{Q})$ of the unitary group of the hermitian form acts on $\mathcal{D}_F$ through F -linear isogenies that preserve the polarisation up to a scalar, hence preserve the algebraicity of the Weil classes, and its closure contains $G_F(\mathbb{R})^+$, which acts transitively. The dichotomy is Lemma 4.2. $\square$

Proposition 4.7 ((F3') and the abelian part). *Let $H_{\text{ab}}^{2p}(X) \subseteq H^{2p}(X)$ be the sum of the images of all maps $\Gamma_*: H^k(B) \rightarrow H^{2p}(X)$ with B abelian and Γ an algebraic cycle on $B \times X$. Then $H_{\text{ab}}^{2p}(X)$ is a sub-Hodge structure, $\text{Ab}^p(X) = \text{Hdg}^p(X) \cap H_{\text{ab}}^{2p}(X)$, and (F3') holds for X in degree $2p$ if and only if a Hodge complement $T_{\text{ab}}^{2p}(X)$ of $H_{\text{ab}}^{2p}(X)$ carries no nonzero Hodge class.*

Proof. Each Γ_* is a morphism of Hodge structures, so its image and the sum of finitely many images, which is all of $H_{\text{ab}}^{2p}(X)$ by dimension, are sub-Hodge structures. A Hodge class in $H_{\text{ab}}^{2p}(X)$ is $\sum \Gamma_{j*} y_j$, and the argument of Proposition 3.3(i) replaces each y_j by a Hodge class; so $\text{Ab}^p(X) = \text{Hdg}^p(X) \cap H_{\text{ab}}^{2p}(X)$. A Hodge complement exists because the Hodge group is reductive, and the Hodge classes of $H^{2p}(X)$ are the direct sum of those of the two summands, which gives the last assertion. $\square$

Theorem 1.2 is Lemma 4.2, Theorem 4.5, Proposition 4.6 and Proposition 4.7. Figure 3 draws the locus of Proposition 4.6.

Both statements are now of one shape. There is a dense set of points at which the class is algebraic by an explicit cycle; the locus is a countable union of closed strata; the statement is that this union is not meagre, that is, that algebraicity propagates from the dense set. The rest of the paper goes to the first point where the propagation is not known and builds the object that would carry it.

5. CONSTRUCTIONS AT (F3'): REAL MULTIPLICATION ON K3 SURFACES

5.1. Where the first class is

Let S be a projective K3 surface, $T = T(S)_{\mathbb{Q}}$ and $E = \text{End}_{\text{Hdg}} T$. The Hodge classes of $S \times S$ are $H^0 \otimes H^4$, $H^4 \otimes H^0$ and $\text{End}_{\text{Hdg}} H^2(S) = \text{End}(\text{NS}_{\mathbb{Q}}) \oplus E$, since T is irreducible with no class of type $(1,1)$ and $\text{Hom}_{\text{Hdg}}(T, \text{NS}_{\mathbb{Q}}) = 0$. The first two pieces and $\text{End}(\text{NS}_{\mathbb{Q}})$ are spanned by products of divisor classes. So the Hodge conjecture for $S \times S$ is the algebraicity of E , and by Proposition 4.7 the same classes decide (F3') for $S \times S$ once T is not reached from abelian varieties. When E is a CM field it is spanned by its elements of norm one, which are Hodge isometries of T and algebraic by Buskin's theorem [22]; see [74, Rem. 2.16]. When E is totally real, the Rosati involution is the identity on it [84], every $e \in E$ is self-adjoint for the intersection form q , and $q(ex, ey) = q(e^2x, y)$. The first class is a real multiplication.

5.2. Chains of correspondences

Lemma 5.1. *Let $f: S \dashrightarrow S'$ be a dominant rational map of K3 surfaces of degree d and Γ_f the closure of its graph. Then $f_* = \Gamma_{f*}$ and $f^* = \Gamma_{f*}^t$ are morphisms of Hodge structures, f^* maps $T(S')_{\mathbb{Q}}$ isomorphically onto $T(S)_{\mathbb{Q}}$ with $q_S(f^*a, f^*b) = dq_{S'}(a, b)$, and $f_* = d(f^*)^{-1}$ on $T(S)_{\mathbb{Q}}$.*

Proof. The graph is an algebraic cycle, so both maps are induced by algebraic classes and are morphisms of Hodge structures. Pull-back of a nowhere vanishing 2-form is a nonzero 2-form, so $f^*(T(S')_{\mathbb{Q}})$ is a sub-Hodge structure of $H^2(S)$ that meets $H^{2,0}(S)$; the smallest sub-Hodge structure of $H^2(S)$ containing $H^{2,0}$ is $T(S)_{\mathbb{Q}}$, so $f^*T(S')_{\mathbb{Q}} \supseteq T(S)_{\mathbb{Q}}$, and since $T(S')_{\mathbb{Q}}$ is irreducible and f^* is injective on it ($f_*f^* = d$), equality of dimensions gives $f^*T(S')_{\mathbb{Q}} = T(S)_{\mathbb{Q}}$. The projection formula gives $q_S(f^*a, f^*b) = q_{S'}(f_*f^*a, b) = dq_{S'}(a, b)$ and $f_*f^* = d$. $\square$

Lemma 5.2. *A Hodge isometry $\phi: T(S)_{\mathbb{Q}} \rightarrow T(S')_{\mathbb{Q}}$ is induced by an algebraic class on $S \times S'$.*

Proof. The spaces $\text{NS}(S)_{\mathbb{Q}}$ and $\text{NS}(S')_{\mathbb{Q}}$ are the orthogonal complements of $T(S)_{\mathbb{Q}}$ and $T(S')_{\mathbb{Q}}$ in the two copies of $\Lambda_{\mathbb{Q}}$, isometric to each other through ϕ ; by Witt's cancellation theorem the complements

Σ_F : dense, a countable union of closed strata,
either all of $\mathcal{D}_{n,n}$ or meagre

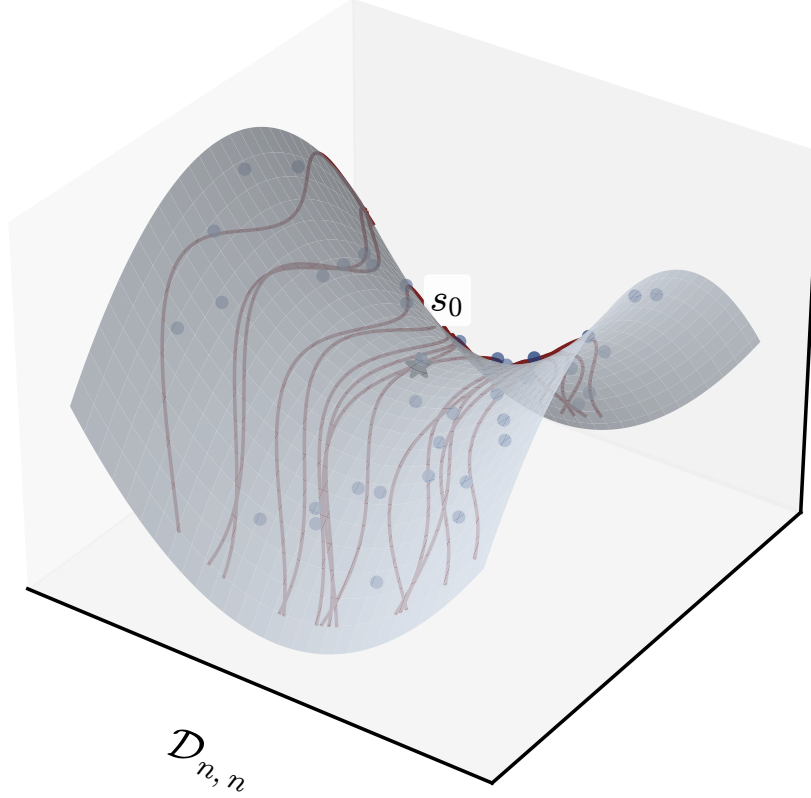


FIGURE 3. The period domain $\mathcal{D}_{n,n}$ of a Weil family as a surface. The star is the CM base point, the dots and curves are strata of the algebraic locus, dense by the Hecke orbit. The locus is a countable union of closed strata; it is either the whole surface or meagre.

are isometric, so φ extends to a Hodge isometry $H^2(S) \rightarrow H^2(S')$, which is algebraic by Buskin's theorem [22] (see [38, Thm. 0.1]). The projector of $H^2(S)$ onto $T(S)_{\mathbb{Q}}$ along $\text{NS}(S)_{\mathbb{Q}}$ is algebraic, being 1 minus the projector onto NS , which is a sum $\sum_{i,j} c_{ij} D_i \otimes D_j$ over a basis of divisor classes with (c_{ij}) the inverse Gram matrix; composing gives φ on T and zero on NS . $\square$

Theorem 5.3 (Chains). *Let S be a projective K3 surface with E totally real and $e \in E$ with $e^2 = n \in \mathbb{Q}_{>0}$. Suppose there are projective K3 surfaces $S = S_0, S_1, \dots, S_r$ and, for each i , either a dominant rational map $S_{i-1} \dashrightarrow S_i$ of degree n_i , a dominant rational map $S_i \dashrightarrow S_{i-1}$ of degree n_i , or a Hodge isometry $T(S_{i-1})_{\mathbb{Q}} \cong T(S_i)_{\mathbb{Q}}$ (and then $n_i = 1$), such that*

$$\prod_i n_i^{\pm 1} \equiv n \pmod{\mathbb{Q}^{\times 2}},$$

the sign being $+$ for maps leaving S_{i-1} and $-$ for maps arriving at it. Then e is algebraic on $S \times S$.

Proof. Composing along the chain the maps f_* for maps leaving S_{i-1} , f^* for maps arriving, and the classes of Lemma 5.2 for the isometries, gives an algebraic isomorphism $\phi: T(S_r)_{\mathbb{Q}} \rightarrow T(S)_{\mathbb{Q}}$ of Hodge structures with $q_S(\phi a, \phi b) = \mu q_{S_r}(a, b)$, where $\mu = \prod n_i^{\pm 1}$ by Lemma 5.1 (a map leaving contributes $f_* = n_i(f^*)^{-1}$, of multiplier n_i ; a map arriving contributes f^* , of multiplier n_i in the other

direction, that is n_i^{-1} for the composite in the direction of ϕ). Write $\mu = nc^2$ with $c \in \mathbb{Q}^\times$. Since e is self-adjoint with $e^2 = n$, it is a Hodge similarity of $T(S)_\mathbb{Q}$ of multiplier n . So $\psi = (c^{-1}\phi)^{-1} \circ e: T(S)_\mathbb{Q} \rightarrow T(S_r)_\mathbb{Q}$ is a Hodge isometry, algebraic by Lemma 5.2, and $e = (c^{-1}\phi) \circ \psi$ is a composite of algebraic correspondences, hence algebraic on $S \times S$. $\square$

The case $r = 1$ with $S \dashrightarrow S_1$ the quotient by a symplectic automorphism of prime order p is Varesco's theorem [74, Thm. 2.1], which gives $e = \sqrt{p}$ and is used for $p = 2, 3$. What a chain adds is a composite multiplier.

5.3. Real multiplication by $\mathbb{Q}(\sqrt{6})$

A symplectic automorphism of finite order on a K3 surface acts trivially on $T(S)$, its quotient has a K3 resolution Y [59] (for involutions and their quotients see also [16]), and $T(Y)_\mathbb{Q} \cong T(S)_\mathbb{Q}(1/\text{ord})$ by Lemma 5.1. The invariant lattice of a symplectic involution is $U^3 \oplus E_8(-2)$, and a K3 surface has a symplectic involution exactly when $E_8(-2)$ embeds primitively in its Néron–Severi lattice [53, 79]. The invariant lattice of a symplectic automorphism of order three is $U \oplus U(3)^2 \oplus A_2^2$, and such an automorphism exists exactly when the Coxeter–Todd lattice $K_{12}(-2)$ embeds primitively in the Néron–Severi lattice [32, Thm. 4.1]. Put

$$V_2 = (U^3 \oplus E_8(-2))_\mathbb{Q}, \quad V_3 = (U \oplus U(3)^2 \oplus A_2^2)_\mathbb{Q},$$

of ranks 14 and 10 and signatures (3, 11) and (3, 7); over $\mathbb{Q}$, $U(3) \cong U$ and $E_8(-2) \cong \langle -1 \rangle^8 \cong \langle -2 \rangle^8$. Varesco [74, Props. 2.5 and 2.11] shows that a projective K3 surface whose transcendental space embeds isometrically in V_2 (resp. V_3) is Hodge isometric to one with a symplectic involution (resp. a symplectic automorphism of order three).

Lemma 5.4 (Embedding criterion). *Let W and V be rational quadratic spaces with $\text{rk } V - \text{rk } W \geq 3$. Then W embeds isometrically in V if and only if the signature of V dominates that of W in both signs. In that case a complement W' exists with $V \cong W \oplus W'$.*

Proof. Necessity is clear. For sufficiency put $k = \text{rk } V - \text{rk } W \geq 3$ and let (s_+, s_-) be the difference of signatures, $s_+ + s_- = k$. A quadratic space W' of rank k and signature (s_+, s_-) with prescribed discriminant $d' = d(V)d(W)$ and prescribed Hasse invariants ε'_p exists provided the ε'_p equal 1 for almost all p , $\prod_{p \leq \infty} \varepsilon'_p = 1$, and ε'_∞, d' match the signature [69, Chap. IV, Prop. 7], with no further condition when $k \geq 3$. Take $\varepsilon'_p = \varepsilon_p(V)\varepsilon_p(W)(d(W), d')_p$, the value forced by the formula $\varepsilon_p(W \oplus W') = \varepsilon_p(W)\varepsilon_p(W')(d(W), d(W'))_p$. The product formula for these holds because it holds for V , for W and for the Hilbert symbol. Then $W \oplus W'$ and V have the same rank, signature, discriminant and Hasse invariants at every place, so they are isometric by the Hasse–Minkowski theorem [69, Chap. IV, Thm. 9], and W embeds in V . $\square$

Theorem 5.5 (Two-dimensional families with algebraic real multiplication by $\mathbb{Q}(\sqrt{6})$). *Let S be a projective K3 surface with $\mathbb{Q}(\sqrt{6}) \subseteq E$ whose transcendental space $T = T(S)_\mathbb{Q}$ embeds isometrically in V_3 .*

- (i) $\sqrt{6} \in E$ is algebraic on $S \times S$.
- (ii) The Hodge conjecture, and (F3'), hold for $S \times S$ and for the Hilbert square $S^{[2]}$.
- (iii) Such surfaces exist in two-dimensional families, whose very general member has $\text{rk } T = 8$ and $E = \mathbb{Q}(\sqrt{6})$ exactly. One such family is the one whose transcendental space is E^4 with the trace form of $q_E = \langle -2, -2, -2 + \sqrt{6}, -1 + \sqrt{6} \rangle$ over $E = \mathbb{Q}(\sqrt{6})$.

Proof. (i) By [74, Prop. 2.11] S is Hodge isometric to a K3 surface S' with a symplectic automorphism σ of order three, and by Lemma 5.2 the isometry is algebraic, so we may replace S by S' . Let Y be the minimal resolution of S'/σ and $f_1: S' \dashrightarrow Y$ the quotient map, of degree three, so that $T(Y)_\mathbb{Q} \cong T(1/3)$, of rank $\text{rk } T \leq 8$ and signature $(2, \text{rk } T - 2)$: the rank is at most 8 because T embeds in V_3 of rank 10 and signature (3, 7) with a complement that must contain the remaining positive direction, so the complement has rank at least two. Now $T(Y)_\mathbb{Q}$ has rank at most 8 and signature dominated by (3, 11), and $14 - 8 \geq 3$, so by Lemma 5.4 it embeds in V_2 . By [74, Prop. 2.5] Y is Hodge isometric to a K3 surface Y' with a symplectic involution ι ; let Z be the resolution of Y'/ι and $f_2: Y' \dashrightarrow Z$ the

TABLE 2. Rational invariants used in Theorem 5.5(iii). Hasse invariants are +1 at every prime not shown.

space	rank	signature	d	ε_2	ε_3	ε_5
$T = \text{Tr}_{E/\mathbb{Q}} q_E$	8	(2, 6)	10	+1	-1	+1
$\langle 1, -10 \rangle$	2	(1, 1)	-10	+1	+1	+1
$T \oplus \langle 1, -10 \rangle$	10	(3, 7)	-1	+1	-1	+1
V_3	10	(3, 7)	-1	+1	-1	+1
V_2	14	(3, 11)	-1	-1	+1	+1

quotient, of degree two. The chain $S' \dashrightarrow Y \cong Y' \dashrightarrow Z$ has multiplier $3 \cdot 1 \cdot 2 = 6$, and Theorem 5.3 applies to $e = \sqrt{6}$ (Fig. 4).

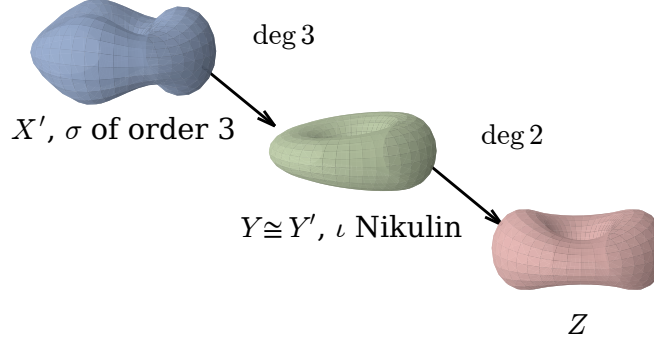
(ii) By the first paragraph of this section it remains to make E algebraic. E contains $\mathbb{Q}(\sqrt{6})$, and either $E = \mathbb{Q}(\sqrt{6})$ or E is a CM field: a totally real E has $\dim_E T \geq 3$, because the complex line $H^{2,0}$ lies in one eigenspace $T_\sigma \otimes \mathbb{C}$ and $\overline{H}^{2,0}$ in the same one, so that $T_\sigma \otimes \mathbb{R}$ has signature $(2, *)$ and $\dim_E T \geq 3$ follows from q being negative definite on the other eigenspaces together with $T \otimes \mathbb{R}$ having exactly two positive directions; and $[E : \mathbb{Q}] \cdot \dim_E T = \text{rk } T \leq 8$ then forces $[E : \mathbb{Q}] \leq 2$. If $E = \mathbb{Q}(\sqrt{6})$, (i) suffices. If E is CM, it is spanned by its norm-one elements, Hodge isometries, algebraic by Buskin's theorem. So every Hodge class on $S \times S$ is algebraic, and (F3') follows. For $S^{[2]}$ the Chow motive is a direct summand of twists of $h(S \times S)$, $h(S)$ and $\mathbf{1}$ [24], so every Hodge class on $S^{[2]}$ is the image under an algebraic correspondence of a Hodge class on $S \times S$ or S , and both statements follow.

(iii) Let $E = \mathbb{Q}(\sqrt{6})$ with real embeddings σ_1, σ_2 ($\sqrt{6} \mapsto \pm\sqrt{6}$), and let $q_E = \langle a_1, a_2, a_3, a_4 \rangle$ with a_1, a_2 negative at both places and a_3, a_4 positive at σ_1 , negative at σ_2 . The $\mathbb{Q}$ -space $T = E^4$ with the form $\text{Tr}_{E/\mathbb{Q}} \circ q_E$ has rank 8 and signature $(2, 2) + (0, 4) = (2, 6)$, and multiplication by $\sqrt{6}$ is self-adjoint of square 6. For $q_E = \langle -2, -2, -2 + \sqrt{6}, -1 + \sqrt{6} \rangle$ one computes $d(T) = 10$, $\varepsilon_2 = 1$, $\varepsilon_3 = -1$, $\varepsilon_5 = 1$ and $\varepsilon_p = 1$ otherwise, while V_3 has $d = -1$, $\varepsilon_2 = 1$, $\varepsilon_3 = -1$, $\varepsilon_5 = 1$; and $T \oplus \langle 1, -10 \rangle$ has the invariants of V_3 at every place, so $V_3 \cong T \oplus \langle 1, -10 \rangle$ by Hasse–Minkowski, with a complement containing a positive vector (Table 2; the Hilbert symbols are recomputed in the programs of Appendix B). Embed V_3 in $\Lambda_{\mathbb{Q}}$ as the invariant lattice of an order-three symplectic action, with orthogonal complement $K_{12}(-2)$, and let $N \subseteq \Lambda$ be the primitive sublattice with $N_{\mathbb{Q}} = T^{\perp}$; it contains $K_{12}(-2)$ primitively and a positive class. The Hodge structures of K3 type on T for which $\sqrt{6}$ is a Hodge endomorphism are the periods $\omega \in \mathbb{P}(T_{\sigma_1} \otimes \mathbb{C})$ with $q(\omega) = 0$, $q(\omega, \bar{\omega}) > 0$, where T_{σ_1} is the σ_1 -eigenspace, of signature $(2, 2)$; this is a domain of dimension two [74, Prop. 2.3]. For each such ω the surjectivity of the period map gives a K3 surface S_ω with $T(S_\omega)_{\mathbb{Q}} \subseteq T$, projective because N contains a positive class, carrying an order-three symplectic automorphism because $K_{12}(-2) \subseteq N$ primitively. For ω outside a countable union of proper closed analytic subsets, $T(S_\omega)_{\mathbb{Q}} = T$ and $E = \mathbb{Q}(\sqrt{6})$, by the argument of [19, Thm. 22.9], which needs only $\dim_E T \geq 3$. $\square$

In Varesco's families E is $\mathbb{Q}(\sqrt{2})$ or $\mathbb{Q}(\sqrt{3})$ [74, Thms. 2.9, 2.15]; van Geemen and Schütt obtain $\sqrt{2}, \sqrt{3}, \sqrt{7}$ from rational self-maps of elliptic K3 surfaces [80, §§6–7]; Bayer–Fluckiger, van Geemen and Schütt prove that families with real multiplication by any totally real E with $[E : \mathbb{Q}] \dim_E T \leq 21$ exist, without algebraicity [23]. The lattice condition of Theorem 5.5 is a genuine restriction: among the 165 sign-admissible diagonal forms q_E with small coefficients, 98 embed in V_3 and 67 do not (Appendix B).

5.4. Isogenies of elliptic surfaces

Varesco's quotients and the chains of Theorem 5.3 produce the square roots of the orders of symplectic automorphisms, 2 and 3, and their products. A different source of similarities is a rational self-map of a K3 surface, and the self-maps that exist in families come from isogenies of elliptic fibrations: van Geemen and Schütt obtain $\sqrt{2}$ and $\sqrt{3}$ from isogenies of degree 2 and 3 in this way, and for degrees 5 and 7 they give one surface each [80, §§6–7]. We show that the degrees 5 and 7 come in one-dimensional families, that in the families the isogeny is a complex multiplication by $\mathbb{Q}(\sqrt{-5})$ or

Theorem A for $e = \sqrt{6}$: a chain of two symplectic quotients

$$\text{multiplier } 3 \cdot 1 \cdot 2 = 6 = e^2$$

FIGURE 4. The chain of Theorem 5.5: the quotient of S' by a symplectic automorphism of order three, a Hodge isometry (algebraic by Buskin), and the quotient of Y' by a Nikulin involution. The multipliers compose to 6.

$\mathbb{Q}(\sqrt{-7})$ and not a real one, the sign of its square being decided by how the twisting function behaves under the Fricke involution, and that the families satisfy the Hodge conjecture on all their powers.

Let $N \in \{5, 7\}$ and write N^k for 125 when $N = 5$ and 49 when $N = 7$. The modular curve $X_0(N)$ is rational, with Fricke's coordinate t in which the involution $(E, C) \mapsto (E/C, E[N]/C)$ is $w(t) = N^k/t$ [80, §7.6].⁶ The universal curve with a cyclic subgroup C_t of order N over $\mathbb{P}_t^1$ is

$$\mathcal{E}_t: y^2 = x^3 - 27c_4(t)x - 54c_6(t), \quad (5.1)$$

with, for $N = 7$,

$$\begin{aligned} c_4 &= (t^2 + 13t + 49)(t^2 + 5t + 1), \\ c_6 &= (t^2 + 13t + 49)(t^4 + 14t^3 + 63t^2 + 70t - 7), \end{aligned}$$

and for $N = 5$,

$$\begin{aligned} c_4 &= (t^2 + 10t + 5)(t^2 + 22t + 125), \\ c_6 &= (t^2 + 4t - 1)(t^2 + 22t + 125)^2. \end{aligned}$$

The discriminant is $1728t(t^2 + 13t + 49)^2$ for $N = 7$ and $1728t(t^2 + 22t + 125)^3$ for $N = 5$, of degree 5 and 7. So the relatively minimal model R_N of (5.1) over $\mathbb{P}_t^1$ is a rational elliptic surface with fibres of type I_1 at $t = 0$, I_N at $t = \infty$, and at the two roots e_1, e_2 of the quadratic factor, where c_4 vanishes, of

⁶The curve $X_0(N)$ parametrises elliptic curves with a cyclic subgroup of order N ; it has genus zero exactly for $N \leq 10$ and $N \in \{12, 13, 16, 18, 25\}$, and for $N = 5$ and $N = 7$ a coordinate is the Hauptmodul $t = (\eta(\tau)/\eta(N\tau))^k$ with $k = 12/(N-1)$, so $k = 3$ for $N = 5$ and $k = 2$ for $N = 7$; this is why N^k is 125 in one case and 49 in the other. The Fricke involution sends τ to $-1/N\tau$, which exchanges the two cusps 0 and ∞ of $X_0(N)$ and, on the moduli interpretation, replaces (E, C) by the quotient E/C with the image of $E[N]$ as its distinguished subgroup. Applying it twice gives $E/E[N] \cong E$, so it is an involution, and in the coordinate t it reads $t \mapsto N^k/t$ because $\eta(-1/\tau) = \sqrt{-i\tau}\eta(\tau)$ turns the quotient of eta functions into its reciprocal up to the constant $N^{k/2}$, squared to N^k by the exponent.

type II for $N = 7$ and III for $N = 5$; the Euler numbers add up to $1 + N + 2 \cdot 2 = 12$ and $1 + N + 2 \cdot 3 = 12$. The roots satisfy $e_1 e_2 = N^k$, so w exchanges them and exchanges 0 with ∞ .

Lemma 5.6 (The Fricke isogeny over the function field). $\mathcal{E}_t/C_t \cong \mathcal{E}_{w(t)}$ over $\mathbb{C}(t)$.

Proof. The two curves have the same j -invariant, because w is the Fricke involution of $X_0(N)$. Two elliptic curves over $\mathbb{C}(t)$ with the same non-constant j -invariant differ by a quadratic twist $\chi \in \mathbb{C}(t)^\times / \mathbb{C}(t)^{\times 2}$, and a twist ramified at a point of $\mathbb{P}^1$ changes the Kodaira type of the fibre there, exchanging I_n with I_n^* , II with IV^* and III with III^* . So it suffices to show that $\mathcal{E}_t/C_t$ has the same fibre types as $\mathcal{E}_{w(t)}$, which are I_N at 0, I_1 at ∞ and II or III at e_1, e_2 : then χ is unramified on $\mathbb{P}^1$, hence constant, hence trivial over $\mathbb{C}$.

At $t = 0$ the fibre of $\mathcal{E}$ is I_1 , the Tate curve $\mathbb{G}_m/q^{\mathbb{Z}}$ over $\mathbb{C}((t))$, whose only cyclic subgroup of order N rational over $\mathbb{C}((t))$ is μ_N ; the quotient $\mathbb{G}_m/q^{N\mathbb{Z}}$ is I_N . At e_i the local monodromy $M \in \mathrm{SL}_2(\mathbb{Z})$ of $\mathcal{E}$ has order 6 or 4 and irreducible characteristic polynomial; the isogeny is given on the period lattices by a matrix $A \in M_2(\mathbb{Z})$ with $\det A = N$, and the monodromy of the quotient is $M' = AMA^{-1}$. The $\mathrm{SL}_2(\mathbb{Z})$ -classes with this characteristic polynomial are those of M and M^{-1} , which are the types II and II^* , or III and III^* . If $M' = gM^{-1}g^{-1}$ with $g \in \mathrm{SL}_2(\mathbb{Z})$, then $B = g^{-1}A$ conjugates M to M^{-1} . All rational matrices doing so form the coset $B_0 \mathbb{Q}(M)^\times$, where $B_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ for $M = \begin{pmatrix} 0 & -1 \\ 1 & \mathrm{tr} M \end{pmatrix}$ has determinant -1 and the elements of the imaginary quadratic field $\mathbb{Q}(M)$ have positive norm; so $\det B < 0$, against $\det B = N$. Hence the quotient has the type of M at e_1, e_2 . At ∞ the j -invariant of the quotient has a simple pole, so the fibre is I_1 or I_1^* , and the Euler numbers of the other fibres add up to 11; since the total is 12 the fibre is I_1 . $\square$

For $p \neq q$ in $\mathbb{P}_t^1$ let $u_{p,q} = (t-p)(t-q)$, and let $X_{p,q}$ be the elliptic K3 surface whose generic fibre is the quadratic twist $\mathcal{E}_t \otimes u_{p,q}$: the twist multiplies c_4 by u^2 and c_6 by u^3 , giving Weierstrass coefficients of degrees 8 and 12, and adds two fibres of type I_0^* at p and q when these are not among $0, \infty, e_1, e_2$, by Tate's algorithm; the same bookkeeping of Weierstrass coefficients, fibre types and their resolutions in higher dimension is carried out in [12]. For a double cover $\psi: \mathbb{P}_s^1 \rightarrow \mathbb{P}_t^1$ let X_ψ be the relatively minimal model of $\psi^* \mathcal{E}$.

Lemma 5.7.

(i) $X_{p,q}$ has Picard number at least 16: its Néron–Severi lattice contains $U \oplus A_6 \oplus D_4^2$ for $N = 7$ and $U \oplus A_4 \oplus A_1^2 \oplus D_4^2$ for $N = 5$. The same bound holds for X_ψ , whose fibres $2I_1 + 2I_N$, together with four fibres of type III when $N = 5$, and the pulled-back Mordell–Weil group of R_N , of rank 2, give rank at least 16 by the Shioda–Tate formula.

(ii) The fibrewise isogeny by C_t is a dominant rational map of degree N

$$f: X_{p,q} \dashrightarrow X_{w(p),w(q)}, \quad f: X_\psi \dashrightarrow X_{w \circ \psi}.$$

(iii) The deck involution of ψ times $[-1]$ on the fibres is a symplectic involution ι of X_ψ , and the minimal resolution of X_ψ/ι is $X_{p,q}$ with $\{p, q\}$ the branch locus of ψ . Hence $T(X_\psi)_\mathbb{Q} \cong T(X_{p,q})_\mathbb{Q}(1/2)$ as in Lemma 5.1, and the two surfaces have the same Picard number.

Proof. (i) Shioda–Tate with the fibres listed, and for X_ψ the Mordell–Weil rank $8 - 6 = 2$ of R_N . (ii) By Lemma 5.6 the quotient of the generic fibre is $\mathcal{E}_{w(t)} \otimes u_{p,q}(t)$, and

$$u_{p,q}(w(t)) = \left(\frac{N^k}{t} - p\right) \left(\frac{N^k}{t} - q\right) = \frac{pq}{t^2} u_{w(p),w(q)}(t), \quad (5.2)$$

a constant times a square times $u_{w(p),w(q)}$. So $\mathcal{E}_{w(t)} \otimes u_{p,q}(t) \cong \mathcal{E}_{w(t)} \otimes u_{w(p),w(q)}(w(t))$, which is the pull-back along the automorphism w of the base of the generic fibre of $X_{w(p),w(q)}$; pulling back along an automorphism of the base gives an isomorphic surface. For X_ψ , $(\psi^* \mathcal{E})/C = \psi^*(\mathcal{E}/C) \cong \psi^* w^* \mathcal{E}$.

(iii) The quadratic twist by the function defining the double cover is the quotient of the base change by the deck involution composed with $[-1]$ on the fibres; the involution is symplectic with eight fixed points, the 2-torsion of the two fibres over the ramification points, and the quotient's resolution is the twist. $\square$

Lemma 5.8 (The sign of the Fricke isogeny). *Let $\varphi_t: \mathcal{E}_t \rightarrow \mathcal{E}_{w(t)}$ be an isogeny over $\mathbb{C}(t)$ with kernel C_t , which exists by Lemma 5.6 and is unique up to sign, and let $\varphi_t^*(dx/y) = \lambda(t)dx/y$ in the models (5.1).*

- (i) $\varphi_{w(t)} \circ \varphi_t = [-N]$ on $\mathcal{E}_t$, and $\lambda(t)\lambda(w(t)) = -N$.
- (ii) *Let $u \in \mathbb{C}(t)$ satisfy $u(w(t)) = c(t)^2u(t)$ with $c \in \mathbb{C}(t)$, let X be the elliptic K3 surface with generic fibre $\mathcal{E}_t \otimes u$, and let $f: X \dashrightarrow X$ be the fibrewise isogeny by C_t followed by the isomorphism $(x, y) \mapsto (c^2x, c^3y)$ from $\mathcal{E}_{w(t)} \otimes u(t)$ to $\mathcal{E}_{w(t)} \otimes u(w(t))$, the fibre of X over $w(t)$. Then f^* acts on $H^{2,0}(X)$ by a constant Λ with*

$$\Lambda^2 = -\frac{N}{c(t)c(w(t))}.$$

For $u = (t - p)(t - N^k/p)$ one has $c(t)c(w(t)) = 1$ and $\Lambda^2 = -N$; for $u = t^2 - N^k$ one has $c(t)c(w(t)) = -1$ and $\Lambda^2 = N$.

Proof. (i) The composite is an endomorphism of $\mathcal{E}_t$ over $\mathbb{C}(t)$ of degree N^2 . Since j is not constant, $\text{End}_{\mathbb{C}(t)}(\mathcal{E}_t) = \mathbb{Z}$, so the composite is $[N]$ or $[-N]$, and changing φ to $-\varphi$ does not change it. We read off the sign on the upper half plane $\mathfrak{H}$. Let $\tau \mapsto t(\tau)$ be the quotient map $\mathfrak{H} \rightarrow X_0(N)$. By the moduli interpretation of $X_0(N)$ the pair $(\mathcal{E}_{t(\tau)}, C_{t(\tau)})$ is $(\mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z}), \langle 1/N \rangle)$, analytically in τ , and the Fricke involution lifts to $\tau \mapsto w(\tau) = -1/N\tau$. The map $f_\tau(z) = z/\tau$ carries $\mathbb{Z}\tau + \mathbb{Z}$ into $\mathbb{Z}w(\tau) + \mathbb{Z} = \mathbb{Z}/N\tau + \mathbb{Z}$, so it is an isogeny $\mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z}) \rightarrow \mathbb{C}/(\mathbb{Z}w(\tau) + \mathbb{Z})$, and its kernel $\tau(\mathbb{Z}w(\tau) + \mathbb{Z})/(\mathbb{Z}\tau + \mathbb{Z}) = (\mathbb{Z}/N + \mathbb{Z}\tau)/(\mathbb{Z}\tau + \mathbb{Z})$ is $\langle 1/N \rangle$. Its composite with the map of the same form at $w(\tau)$ is $f_{w(\tau)}(f_\tau(z)) = z/\tau w(\tau) = -Nz$. On the set $\mathfrak{H}'$ of τ with $j(\tau) \notin \{0, 1728\}$ the automorphisms of $\mathcal{E}_\tau$ are ± 1 , so $\varphi_{t(\tau)} = \varepsilon(\tau)f_\tau$ with $\varepsilon(\tau) \in \{\pm 1\}$; ε is continuous because φ is, hence constant on $\mathfrak{H}'$, which is $\mathfrak{H}$ minus a discrete set and connected. So $\varphi_{w(t)} \circ \varphi_t = \varepsilon^2 f_{w(\tau)} \circ f_\tau = [-N]$ on $\mathfrak{H}'$, hence identically. Pulling back dx/y along the composite gives $\lambda(t)\lambda(w(t))dx/y$ on one side and $-Ndx/y$ on the other.

(ii) Over $\mathbb{C}(t)(\sqrt{u})$ the twist by u is the same change of coordinates $(x, y) \mapsto (ux, u^{3/2}y)$ on $\mathcal{E}_t$ and on $\mathcal{E}_{w(t)}$, which multiplies dx/y by $u^{-1/2}$ on both, so the twisted isogeny pulls dx/y back to $\lambda(t)dx/y$. The isomorphism $(x, y) \mapsto (c^2x, c^3y)$ pulls dx/y back to $c(t)^{-1}dx/y$, and the holomorphic two-form of X , which is $dx/y \wedge dt$ on the Weierstrass model, restricts to the fibre over $w(t)$ as $dx/y \wedge d(w(t)) = w'(t)dx/y \wedge dt$. Hence $f^*(dx/y \wedge dt) = \Lambda(t)dx/y \wedge dt$ with $\Lambda(t) = w'(t)\lambda(t)/c(t)$. The pull-back of a holomorphic two-form along a dominant rational map of K3 surfaces is holomorphic off the finitely many points of indeterminacy, hence everywhere, so f^* preserves the line $H^{2,0}(X)$ and Λ is a constant. Then $\Lambda^2 = \Lambda(t)\Lambda(w(t)) = w'(t)w'(w(t))\lambda(t)\lambda(w(t))/c(t)c(w(t))$, where $w'(t)w'(w(t)) = (w \circ w)'(t) = 1$ and $\lambda(t)\lambda(w(t)) = -N$ by (i). For $u = (t - p)(t - N^k/p)$ equation (5.2) with $q = N^k/p$ gives $u(w(t)) = (N^k/t^2)u(t)$, so $c(t) = N^{k/2}/t$ for a fixed square root $N^{k/2}$, and $c(t)c(w(t)) = N^k/tw(t) = 1$. For $u = t^2 - N^k$, $u(w(t)) = N^{2k}/t^2 - N^k = -(N^k/t^2)u(t)$, so $c(t) = iN^{k/2}/t$ and $c(t)c(w(t)) = -1$. $\square$

The archive [13] computes λ itself: the kernel polynomial of φ_t is the factor of degree $(N - 1)/2$ of the N -th division polynomial over $\mathbb{Z}[t, x]$, Vélú's formulas give the quotient curve, and the isomorphism to (5.1) at $w(t)$ has $\lambda(t)^2 = -t^2/25$ for $N = 5$ and $-t^2/7$ for $N = 7$, so that $\lambda(t)^2\lambda(w(t))^2 = N^2$ and $\Lambda^2 = -N$ on the pair twists directly; the program code/`fricke_sign.py` carries this out, and the Lean file checks the polynomial identity it reduces to.

Theorem 5.9 (Fricke families). *Let $N \in \{5, 7\}$ and $K = \mathbb{Q}(\sqrt{-N})$. Let $\mathfrak{A}_N$ be the family of surfaces $X_p = X_{p, N^k/p}$ for $p \notin \{0, \infty, e_1, e_2, \pm\sqrt{N^k}\}$, and $\mathfrak{B}_N$ the family of surfaces X_ψ with ψ branched over a pair $\{b, N^k/b\}$.*

- (i) *Every member X has a dominant rational self-map f of degree N with $f^* \in E = \text{End}_{\text{Hdg}} T(X)_{\mathbb{Q}}$, $f^*f_* = N$, $(f^*)^2 = -N$ and $f_* = -f^*$. So $K \subseteq E$, with $\sqrt{-N} = \pm f^*$ algebraic on $X \times X$, and E is a CM field.*
- (ii) *The families are not isotrivial, and every member has $\text{rk } T(X) \in \{2, 4, 6\}$. At all but countably many members $E = K$ and the Hodge group of $T(X)$ is the unitary group of the K -hermitian form h on $T(X)$ with $\text{Tr}_{K/\mathbb{Q}} h = q_X$; at the others $T(X)$ is of CM type.*

- (iii) For every member X and every $n \geq 1$ the Hodge conjecture holds for X^n and for the Hilbert scheme $X^{[n]}$, and (F3') holds there.

Van Geemen and Schütt state that the Fricke parametrisations give one-dimensional families with complex multiplication by $\mathbb{Q}(\sqrt{-5})$ and $\mathbb{Q}(\sqrt{-7})$, and give the details for the two surfaces at the fixed points of w only [80, §7.6]; (i) and (ii) make their statement explicit, and (iii) is what the complex multiplication gives.

Proof. (i) w exchanges p and N^k/p , so Lemma 5.7(ii) gives $f: X_p \dashrightarrow X_{p'}$, the map of Lemma 5.8(ii) for $u = (t-p)(t-N^k/p)$; for $\mathfrak{B}_N$, $w \circ \psi$ has the same branch locus as ψ , so $w \circ \psi = \psi \circ \sigma$ for a Möbius transformation σ and $X_{w \circ \psi} \cong X_\psi$. The correspondence of a dominant rational map preserves NS and its transpose does too, so it preserves $T = \text{NS}^\perp$ and acts there as a morphism of Hodge structures, nonzero since $f^* \omega = \Lambda \omega$ with $\Lambda \neq 0$; on a resolution S of the graph with $p_1: S \rightarrow X$ birational and p_2 of degree N , both $p_1^* T$ and $p_2^* T$ are irreducible sub-Hodge structures of $H^2(S)$ containing its unique $(2,0)$ -line, so they coincide, and $f_* f^* = p_{2*} p_1^* p_{1*} p_2^* = N$ on T . The field E acts faithfully on the line $H^{2,0}(X)$, and f^* acts there by Λ with $\Lambda^2 = -N$ by Lemma 5.8(ii); so $(f^*)^2 = -N$ in E , and $f_* = N(f^*)^{-1} = -f^*$. For X_ψ the generic fibre is $\mathcal{E}_{\psi(s)}$ over $\mathbb{C}(s)$, f is the fibrewise isogeny over σ , and σ is an involution: in the coordinate s with $s^2 = (t-b)/(t-N^k/b)$ one computes $(w(t)-b)/(w(t)-N^k/b) = b^2/N^k s^2$, so $\sigma(s) = \kappa/s$ with $\kappa^2 = b^2/N^k$; the argument of Lemma 5.8(ii) with $c = 1$ and σ in place of w gives $\Lambda^2 = \lambda(t)\lambda(w(t))\sigma'(s)\sigma'(\sigma(s)) = -N$. The adjoint of f^* for q_X is $f_* = -f^*$, so the involution of E induced by q_X is not the identity; by Zarhin's theorem E is totally real or a CM field [84, Thm. 1.5.1], so it is a CM field, and $K = \mathbb{Q} + \mathbb{Q}f^* \subseteq E$. The class of the graph of f is algebraic, its action on $H^2(X)$ is f^* on T and an endomorphism of $\text{NS}(X)_\mathbb{Q}$, which is a sum of algebraic classes of $\text{NS} \otimes \text{NS}$; so f^* , extended by zero on NS , is an algebraic class on $X \times X$.

(ii) The elliptic fibration of X_p has labelled singular fibres I_1 at 0, I_N at ∞ , II or III at e_1, e_2 and I_0^* at p and N^k/p . An isomorphism of elliptic fibrations induces a Möbius transformation preserving this configuration; the transformations preserving $\{0, \infty\}$ and $\{e_1, e_2\}$ are $t \mapsto \lambda t$ and $t \mapsto \lambda/t$ with $\lambda e_1 \in \{e_1, e_2\}$, and since $e_1 + e_2 \neq 0$ and $e_1^2 \neq e_2^2$ they are the identity and w , which preserve $\{p, N^k/p\}$. So X_p and $X_{p'}$ have isomorphic fibrations only when $p' \in \{p, N^k/p\}$. A K3 surface has finitely many elliptic fibrations up to automorphisms [72], so the map $p \mapsto [X_p]$ to the moduli space has finite fibres and the family is not isotrivial; by local Torelli its period map has uncountable image. By (i) $T(X)_\mathbb{Q}$ is a K -vector space, so its rank is even, and it is at most 6 by Lemma 5.7(i). It is also an E -vector space, so $[E : \mathbb{Q}]$ divides $\text{rk } T(X)$; if $E \neq K$ then $[E : \mathbb{Q}] \in \{4, 6\}$, $\dim_E T(X) = 1$ and $T(X)$ is of CM type. A Hodge structure of CM type on a sublattice of $M^\perp$ is determined by its endomorphism field, a subfield of the endomorphism algebra of the sublattice, and by an eigenline of that field, so the periods of such members form a countable subset of the period domain of $M^\perp$; the period map of the family is a non-constant holomorphic map from a connected curve, its fibres over a countable set are countable, so all but countably many members have $E = K$. There the Hodge group of $T(X)$ is the unitary group of the K -hermitian form h with $\text{Tr}_{K/\mathbb{Q}} h = q_X$ by Zarhin's theorem for CM fields [84, Thm. 2.3.1]. The same holds for $\mathfrak{B}_N$ by Lemma 5.7(iii).

(iii) By (i) E is a CM field containing K . If $E = K$ then $E = \mathbb{Q} + \mathbb{Q}f^*$ is algebraic. Otherwise every element of E of norm one to its maximal real subfield is a Hodge isometry of T , extended by the identity on NS to a Hodge isometry of $H^2(X, \mathbb{Q})$, algebraic by Buskin's theorem [22]. The algebraic classes form a $\mathbb{Q}$ -subalgebra $E' \subseteq E$ containing these elements; for $x \in E$ with $\bar{x} \neq x$ and $t_1 \neq t_2$ in $\mathbb{Q}$ it contains $(t_i + x)/(t_i + \bar{x})$, hence their quotient $1 + (t_2 - t_1)/(t_1 + \bar{x})$, hence $\bar{x}$; so $E' = E$. In both cases $\text{End}_{\text{Hdg}} H^2(X) = \text{End}(\text{NS}) \oplus E$ is algebraic, and $\text{Hdg}^2(X \times X) = \text{End}_{\text{Hdg}} H^2(X) \oplus H^0 \otimes H^4 \oplus H^4 \otimes H^0$ since $H^1 = H^3 = 0$, so the Hodge conjecture holds for $X \times X$. For a K3 surface whose endomorphism field is a CM field the Hodge classes in the tensor algebra of T are generated by those of $T \otimes T$, and the Hodge conjecture holds for every power [75, Thm. 2.2, Cor. 2.3], [66, Prop. 5.2, Thm. 5.4]. It passes to the Hilbert schemes $X^{[n]}$, whose motives are direct summands of motives of powers of X [24], and (F3') on a variety follows from the Hodge conjecture on it. $\square$

The surface of [80, §7.6] with real multiplication by $\sqrt{7}$ is the twist at $\{7, -7\}$, the two fixed points of w , and the surface of [80, Prop. 7.7] is the twist at the fixed points $\pm 5\sqrt{5}$ for $N = 5$, which has Picard

The family $\mathfrak{A}_N$, $N \in \{5, 7\}$: twists at a pair exchanged by w

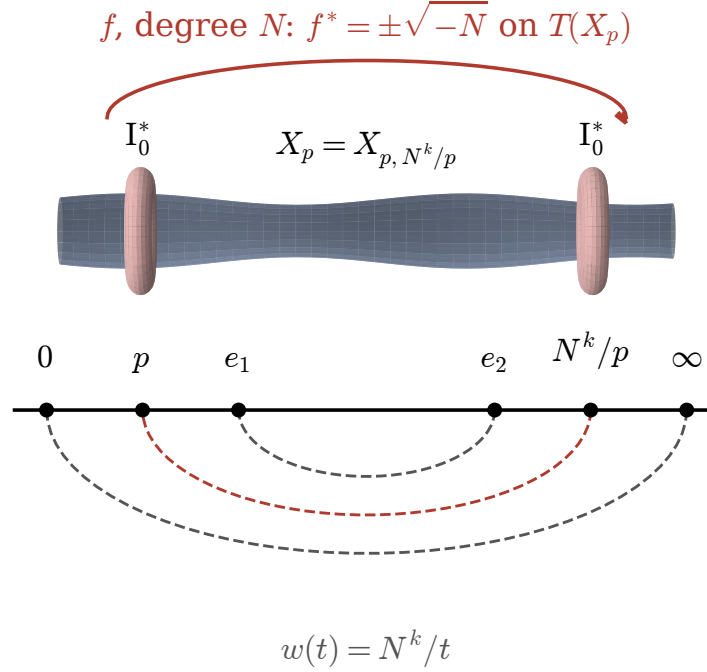


FIGURE 5. The family $\mathfrak{A}_N$ of Theorem 5.9. Over the Fricke coordinate t of $X_0(N)$ the involution $w(t) = N^k/t$ exchanges the cusps $0, \infty$ and the elliptic points e_1, e_2 ; the twist X_p at a pair $\{p, N^k/p\}$ exchanged by w is carried to itself by the fibrewise isogeny f of degree N , and f^* is the complex multiplication by $\sqrt{-N}$ on the transcendental lattice.

number 18 and complex multiplication as well: there $u = t^2 - N^k$, $c(t)c(w(t)) = -1$ and $\Lambda^2 = N$ by Lemma 5.8(ii), so f^* is the real multiplication by $\sqrt{N}$. The families $\mathfrak{A}_N$ are the twists at the pairs exchanged by w , where $c(t)c(w(t)) = 1$ and $\Lambda^2 = -N$; the two signs are the two values of the constant by which the twisting function changes under w once the square factor N^k/t^2 is removed. The chains of Theorem 5.3 apply to every member, and a member whose field E contained $\sqrt{2}$, $\sqrt{3}$ or $\sqrt{6}$ would have that element algebraic as well; by Theorem 5.9(ii) such a member is of CM type, and we know none.

Remark 5.10 (The Picard number). Theorem 5.9 does not say which of the three ranks the very general member has. The archive decides it by counting points. The surface X_1 , the twist at $\{1, N^k\}$, has a Weierstrass model over $\mathbb{Z}$ with good reduction at $\ell = 11$ for $N = 7$ and at $\ell = 7$ for $N = 5$, primes split in K ; the program code `fricke_picard.py` counts its points over $\mathbb{F}_{\ell^i}$, $i \leq 3$, fibre by fibre, subtracts the trace of Frobenius on the trivial lattice, and completes the characteristic polynomial of Frobenius on the rank-six complement of the trivial lattice by the functional equation, whose sign is fixed by the nonzero middle coefficient. The polynomials are $x^6 - 726x^3 + 11^6$ for $N = 7$ and $x^6 + 6x^5 + 56x^4 + 392x^3 + 2744x^2 + 14406x + 7^6$ for $N = 5$; in the variable x/ℓ neither has a root of unity among its roots, each is irreducible over $\mathbb{Q}$ and splits over K into a cubic and its conjugate, as the action of f^* requires. By the Tate conjecture for K3 surfaces over finite fields, used in the same way in [80, §7], $\rho(X_1) = 16$, so the very general member of $\mathfrak{A}_N$ and of $\mathfrak{B}_N$ has $\rho = 16$, $\text{rk } T = 6$, $E = K$ with $\dim_K T = 3$, and its Hodge group is the unitary group of a hermitian form of signature $(1, 2)$ on K^3 . At the primes inert in K , $\ell = 5$ for $N = 7$ and $\ell = 11$ for $N = 5$, the odd traces vanish, as they must when Frobenius exchanges the two eigenspaces of f^* . No statement of this section rests on this computation.

5.5. The reach of similarities

Every correspondence used so far acts on transcendental spaces as a Hodge similarity. The next proposition says that this is not an accident of the examples but a property of every correspondence whose components are small, and it measures what a correspondence must look like to act otherwise.

Proposition 5.11 (Structure of the transcendental action). *Let X, X' be projective K3 surfaces, $\Gamma \in \text{CH}^2(X \times X')_{\mathbb{Q}}$, and $\gamma_T: T(X)_{\mathbb{Q}} \rightarrow T(X')_{\mathbb{Q}}$ the map induced on transcendental spaces by composing the action of Γ with the orthogonal projections.*

- (i) γ_T is zero or an isomorphism of Hodge structures.
- (ii) If $\gamma_T \neq 0$ then $a = \gamma_T^\dagger \gamma_T$ lies in the maximal totally real subfield of $E = \text{End}_{\text{Hdg}} T(X)_{\mathbb{Q}}$ and is totally positive, and γ_T is a similarity exactly when $a \in \mathbb{Q}$.
- (iii) Let Γ be irreducible, S a resolution of its normalisation with projections p_1, p_2 of degrees d_1, d_2 to X, X' . If $p_1^* T(X)_{\mathbb{Q}} = p_2^* T(X')_{\mathbb{Q}}$ inside $H^2(S, \mathbb{Q})$, then γ_T is a similarity of multiplier $d_1 d_2$. This holds whenever $p_g(S) \leq 1$.
- (iv) If Γ is irreducible and γ_T is not a similarity, then $p_1^* T(X)_{\mathbb{Q}} \cap p_2^* T(X')_{\mathbb{Q}} = 0$, so $p_g(S) \geq 2$ and the transcendental rank of S is at least twice that of X .
- (v) The $\mathbb{Q}$ -algebra generated by the actions on $T(X)_{\mathbb{Q}}$ of all correspondences from X to itself whose irreducible components either fail to dominate a factor or have $p_g \leq 1$ is the multiquadratic subfield $E_2 \subseteq E$ generated by the square roots of the products $d_1 d_2$ that occur. When $[E : \mathbb{Q}]$ is odd it is $\mathbb{Q}$.

Proof. (i) $T(X)_{\mathbb{Q}}$ is an irreducible Hodge structure [84], and the image of γ_T is a sub-Hodge structure of $H^2(X')_{\mathbb{Q}}$; if nonzero it contains the $(2, 0)$ -part, hence $T(X')_{\mathbb{Q}}$, and γ_T is injective by irreducibility. (ii) a is a Hodge endomorphism of $T(X)$, so $a \in E$, and it is self-adjoint, so it lies in the fixed field of the adjoint involution, the maximal totally real subfield. The form $q'(\gamma x, \gamma y) = q(x, ay)$ is a polarisation of the Hodge structure $T(X)$ up to sign on each real eigenspace of E , and comparison with q gives $\sigma(a) > 0$ at every real place. A similarity of multiplier μ has $a = \mu$. (iii) Both $p_i^* T$ are irreducible sub-Hodge structures of $H^2(S)$ with a $(2, 0)$ -part; if $p_g(S) \leq 1$ they share the unique $(2, 0)$ -line, so they meet and coincide. When they coincide, $\gamma_T = p_{2*} p_1^* = d_2 (p_2^*)^{-1} p_1^*$ on T , so $p_2^* \gamma x = d_2 p_1^* x$; since $q_S(p_1^* x, p_1^* y) = d_1 q(x, y)$ and $q_S(p_2^* x', p_2^* y') = d_2 q'(x', y')$, this gives $d_2 q'(\gamma x, \gamma y) = q_S(p_2^* \gamma x, p_2^* \gamma y) = d_2^2 q_S(p_1^* x, p_1^* y) = d_1 d_2^2 q(x, y)$, that is $q'(\gamma x, \gamma y) = d_1 d_2 q(x, y)$. (iv) If the two subspaces met they would coincide by irreducibility and (iii) would apply; so they are independent, their $(2, 0)$ -lines are distinct, and their sum has rank $2 \text{rk } T(X)$ inside the transcendental part of $H^2(S)$. (v) A component that does not dominate a factor acts through NS or through H^0 and H^4 , hence by zero on T . The others act by similarities by (iii), a composite of similarities returning to X is an element $\Phi \in E$ with $\Phi^\dagger \Phi \in \mathbb{Q}$, and for totally real E this is $\Phi^2 \in \mathbb{Q}$; such elements generate E_2 . For a CM field E the elements with $\Phi^\dagger \Phi \in \mathbb{Q}$ are the $\mathbb{Q}^\times$ -multiples of the norm-one elements, which generate E , so the statement is about the totally real case, which is the one that matters for $(F3')$. An odd-degree field has $E_2 = \mathbb{Q}$. $\square$

The correspondences of Lemmas 5.1 and 5.2, the Fourier–Mukai kernels between a K3 surface and its moduli spaces of sheaves, the rational Hodge isometries of hyper-Kähler varieties of $\text{K3}^{[n]}$ type [48], the Fricke isogenies of Theorem 5.9 and all their composites fall under (iii) or (v). A correspondence that realises an element of a cubic field must therefore have a component whose resolution has $p_g \geq 2$ and transcendental rank at least twice that of the surface. Such correspondences exist, and the only ones we know come from finite groups.

Proposition 5.12 (Hecke correspondences). *Let Z be a smooth projective surface with an action of a finite group G , $H \leq G$ a subgroup, and X a smooth projective surface with $p_g(X) = 1$ birational to Z/H ; write $\pi: Z \dashrightarrow X$.*

- (i) For $g \in G$ the correspondence $\Gamma_g = (\pi \times \pi)_* [\text{graph of } g]$ acts on $H^2(X, \mathbb{Q})$ as the Hecke operator of the double coset HgH , and these operators span a $\mathbb{Q}$ -algebra isomorphic to a quotient of $\text{End}_G \mathbb{Q}[G/H]$ acting by Hodge endomorphisms.

- (ii) Exactly one simple factor of this algebra acts nontrivially on $H^{2,0}(X)$; it is a field, it preserves $T(X)_{\mathbb{Q}}$, and it embeds into E by algebraic correspondences. If $H^{2,0}(Z)^G = 0$ this factor is not the one on which G acts trivially.
- (iii) Every field obtained in this way is abelian over $\mathbb{Q}$. In particular a totally real cubic field F whose Galois closure has group S_3 is realised by no correspondence built from a finite group acting on a cover of X .

Proof. (i) Let $\tilde{Z} \rightarrow Z$ be an equivariant resolution and $\pi: \tilde{Z} \rightarrow X$ the quotient by H followed by the resolution; then $\pi_*\pi^* = |H|$ and $\pi_*h^* = \pi_*$ for $h \in H$, so Γ_g acts as $\pi_*g^*\pi^* = (\pi^*)^{-1} \circ \sum_{h \in HgH/H} h^* \circ \pi^*$ on $H^2(X, \mathbb{Q}) = H^2(\tilde{Z}, \mathbb{Q})^H$, the Hecke operator of HgH . The Hecke operators of (G, H) span $\text{End}_G \mathbb{Q}[G/H]$, a semisimple algebra, and the action on $H^2(\tilde{Z}, \mathbb{Q})^H$ factors through it. (ii) The idempotents of the simple factors are rational combinations of Hecke operators, so they are Hodge endomorphisms of $H^2(X)$; they sum to 1 and act on the line $H^{2,0}(X)$ by 0 or 1, so exactly one of them, e_V , acts as 1. A simple factor with nonzero image in the commutative ring $\text{End}_{\text{Hdg}} H^{2,0}(X) = \mathbb{C}$ through a ring homomorphism to a field is commutative, hence a field $\mathcal{H}_V$. For $f \in \mathcal{H}_V$, $f(T(X)_{\mathbb{Q}})$ is a sub-Hodge structure of the transcendental part of $H^2(X)$; if $f\omega \neq 0$ it contains ω , hence $T(X)_{\mathbb{Q}}$, and if $f\omega = 0$ it has no $(2,0)$ -part and lies in $\text{NS} \cap T = 0$. So $\mathcal{H}_V \rightarrow \text{End}_{\text{Hdg}} T(X)_{\mathbb{Q}} = E$ is a homomorphism from a field, injective. The factor on which G acts trivially fixes $H^{2,0}(\tilde{Z})^G = H^{2,0}(Z/G)$, which is zero by hypothesis, so it is not $\mathcal{H}_V$. (iii) The centres of the simple factors of $\text{End}_G \mathbb{Q}[G/H]$ are the character fields $\mathbb{Q}(\chi_i)$ of the irreducible constituents of $\mathbb{Q}[G/H]$, generated by sums of roots of unity, so each lies in $\mathbb{Q}(\zeta_{|G|})$ and is abelian; a commutative simple factor is its centre. A non-abelian cubic field is not a subfield of an abelian field. $\square$

For the dihedral group D_n with n odd, $H = \langle \tau \rangle$ of order two and Z the elliptic surfaces $\tilde{\mathcal{E}}_a$ of [80, §4.7], the algebra $\text{End}_{D_n} \mathbb{Q}[D_n/H]$ is $\mathbb{Q} \times \prod_{d|n, d>1} \mathbb{Q}(\zeta_d)^+$, and the factor acting on ω is $\mathbb{Q}(\zeta_n)^+$; this is the content of [80, §4.8], where the cycle $\Gamma_1 + \Gamma_{-1}$ induces $\zeta_n + \zeta_n^{-1}$. The components of these correspondences are birational to $\tilde{\mathcal{E}}_a$, which has $p_g \geq 2$, as Proposition 5.11(iv) requires.

Corollary 5.13 (Squares of the dihedral families). *Let X be an elliptic K3 surface of one of the families of [80, Thm. 1.1], with real multiplication by $F = \mathbb{Q}(\zeta_n)^+$ for $n \in \{5, 7, 9, 11\}$, and suppose $E = F$ or E is a CM field; the very general member has $E = F$. Then the Hodge conjecture holds for $X \times X$, and (F3') holds for $X \times X$ and $X^{[2]}$. This covers the cubic fields $\mathbb{Q}(\zeta_7)^+$ and $\mathbb{Q}(\zeta_9)^+$ on families of dimensions 3 and 2.*

Proof. If $E = F$, E is algebraic by Proposition 5.12(ii) and [80, §4.8]; if E is CM, by Buskin's theorem as in Theorem 5.9(iii). The rest is as there. $\square$

5.6. The cyclic cubic fields $\mathbb{Q}(\zeta_7)^+$ and $\mathbb{Q}(\zeta_9)^+$

The dihedral families reach the cyclic cubic fields only on their own members. For each of the two cubic fields they contain, $F = \mathbb{Q}(\zeta_n)^+$ with $n \in \{7, 9\}$, there is a two-dimensional family on a transcendental lattice of rank twelve: for $n = 9$ the family of [80, §5.6], and for $n = 7$ the Dickson deformation of the one-parameter family of Artebani, Sarti and Taki with Picard lattice $U \oplus E_8$ [8, Ex. 6.1], which [80] do not use, their family for $n = 7$ being the three-dimensional one of Picard number four. Two is the largest dimension a family with this real multiplication on a transcendental space of rank twelve can have, and this maximality lets the cycle travel: every K3 surface of Mumford type for F and for the quaternion algebra ramified only at infinity is a Hodge isometric image of a member of the family, and Buskin's theorem carries the cycle along the isometry. The result settles the frontier case of Section 5.8 for these two fields and this algebra.

Write $\theta = \zeta_n + \zeta_n^{-1}$, so that $F = \mathbb{Q}(\theta)$, $K = \mathbb{Q}(\zeta_n) = F(\sqrt{-\delta})$ with $-\delta = (\zeta_n - \zeta_n^{-1})^2 = \theta^2 - 4$, and $\delta = 4 - \theta^2$ is totally positive. For $n = 7$ one has $\theta^3 + \theta^2 - 2\theta - 1 = 0$ and the three real embeddings send θ to -1.802 , -0.445 and 1.247 ; for $n = 9$ one has $\theta^3 - 3\theta + 1 = 0$ and the embeddings send θ to -1.879 , 0.347 and 1.532 . Let p_n be the prime dividing n ; it is totally ramified in K , so it lies under a single place of F , and $K/\mathbb{Q}$ is unramified outside p_n . The prime 2 has order three in $(\mathbb{Z}/n)^\times / \{\pm 1\} = \text{Gal}(F/\mathbb{Q})$, so it is inert in F . Let D_∞ be the quaternion algebra over F ramified

Real multiplication by $\mathbb{Q}(\zeta_7)^+$ or $\mathbb{Q}(\zeta_9)^+$: moving the surface of Mumford type into the dihedral family

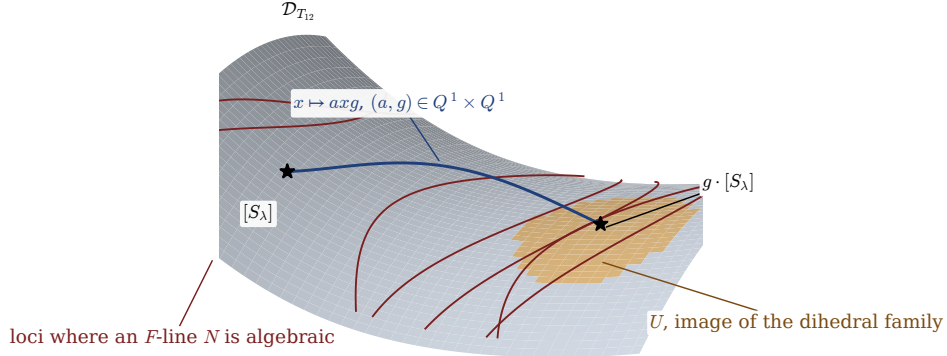


FIGURE 6. The proof of Theorem 5.14. The surface is the two-dimensional period domain of the twelve-dimensional space T_{12} with real multiplication by $F = \mathbb{Q}(\zeta_7)^+$ or $\mathbb{Q}(\zeta_9)^+$; the shaded patch is the image of the two-dimensional dihedral family, on which the multiplication is algebraic. The curves are the loci on which an F -line N becomes algebraic, and the surface of Mumford type is a point on one of them. The rational isometries $x \mapsto axg$ of the quaternion algebra are dense in the real group, so one of them moves the point into the patch; Buskin's theorem turns that isometry into a cycle.

exactly at two real places and at no finite place. Put $\varepsilon = \theta$ for $n = 7$ and $\varepsilon = -\theta$ for $n = 9$; then ε is a unit positive at exactly one real place, and $D_\infty = (-1, \varepsilon)_F$: the symbol is ramified at the two real places where $\varepsilon < 0$, split at every odd finite place because both entries are units there, and split at the one place above 2 by the parity of the number of ramified places. Its corestriction to $\mathbb{Q}$ is unramified everywhere and split at infinity, hence is $M_8(\mathbb{Q})$, so D_∞ is one of Mumford's algebras and the fourfolds of [54] exist for it. For $\mu \in F$ totally positive let q_μ be the quadratic form $\text{Tr}_{F/\mathbb{Q}}(\mu \text{tr}(y^2))$ on the nine-dimensional $\mathbb{Q}$ -space D_∞^0 of elements of reduced trace zero, of signature $(2, 7)$; the surfaces S_λ of [19, Thm. 21.4] for the fourfolds of (F, D_∞) have $T(S_\lambda)_\mathbb{Q} \cong (D_\infty^0, q_\lambda)$.

Theorem 5.14 (Real multiplication by $\mathbb{Q}(\zeta_7)^+$ and $\mathbb{Q}(\zeta_9)^+$ on the surfaces of Mumford type). *Let $n \in \{7, 9\}$, $F = \mathbb{Q}(\zeta_n)^+$ and D_∞ as above. Let S be a projective K3 surface such that $T(S)_\mathbb{Q}$ with its intersection form is isometric to (D_∞^0, q_μ) for some totally positive $\mu \in F$, by an isometry under which the multiplication by F on D_∞^0 acts on $T(S)_\mathbb{Q}$ by Hodge endomorphisms. Then $E = F$, the real multiplication by F is induced by algebraic cycles on $S \times S$, and the Hodge conjecture and (F3') hold for $S \times S$ and $S^{[2]}$. This applies to every surface S_λ of a Mumford fourfold of (F, D_∞) , at every point of its Shimura curve.*

Proof. Step 1, the family for $n = 9$. Let $\mathfrak{X} \rightarrow B$ be the two-dimensional family of [80, §5.6]: elliptic K3 surfaces with very general Picard lattice $U \oplus A_2^4$,⁷ deformations of the surfaces $y^2 = x^3 + bx + c_1 t^9 + c_0$ with a non-symplectic automorphism σ of order nine, obtained by replacing t^9 by the Dickson polynomial $p_{9,a}(t)$. The transcendental lattice T_0 of a very general member has rank twelve and discriminant group $(\mathbb{Z}/3)^4$, is a $\mathbb{Z}[\zeta_9]$ -module of rank two through σ^* , and carries the flat action of F through $\sigma^* + \sigma^{*-1}$, which on every member of the family is induced by the cycle $\Gamma_1 + \Gamma_{-1}$ of [80, §4.8] on a dihedral cover, as in Proposition 5.12. The period map of B has two-dimensional image [80, §5.6].

⁷Families of K3 surfaces polarised by a fixed lattice also occur as the fibres of Calabi–Yau threefolds, the quartic and quintic ones being studied in [10]; the chains of this section act on the fibres only, and the Hodge classes of such a total space are not considered here.

Step 1, the family for $n = 7$. The surfaces $X_b: y^2 = x^3 + bx + c_1 t^7 + c_0$ are the K3 surfaces of [8, Ex. 6.1] with $a = 0$ there: $\sigma(x, y, t) = (x, y, \zeta_7 t)$ is a non-symplectic automorphism of order seven, the fibration has a fibre of type II^* at $t = \infty$ and fourteen fibres of type I_1 , and for very general b the Picard lattice is the invariant lattice $U \oplus E_8$ of σ , with $T_0 = U^2 \oplus E_8(-1)$ [8, Table 3]. Replacing t^7 by the Dickson polynomial $p_{7,a}(t) = t^7 - 7at^5 + 14a^2t^3 - 7a^3t$ gives the surfaces⁸

$$\mathfrak{X}_a: y^2 = x^3 + bx + c_1 p_{7,a}(t) + c_0, \quad (5.3)$$

where $p_{7,a}$ is the Dickson polynomial with $p_{7,a}(v + av^{-1}) = v^7 + a^7 v^{-7}$;⁹ these are the deformations (4.4) of [80, Prop. 4.6] with $n = 7$, $\alpha = b$ and $\beta(u) = c_1 u + c_0$; by that proposition each $\mathfrak{X}_a$ is a K3 surface on whose transcendental space F acts by Hodge endomorphisms, induced by the cycle $\Gamma_1 + \Gamma_{-1}$ on the D_7 -cover as before. The polynomial $c_1 p_{7,a} + c_0$ has degree seven, so at $t = \infty$ the coefficients of the Weierstrass equation vanish to orders 8 and 5 and the fibre of type II^* survives; hence $\text{Pic}(\mathfrak{X}_a)$ contains the unimodular lattice $U \oplus E_8$ of rank ten, and since the Picard number of a very general member of a family is at most that of a very general member of any subfamily, here the locus $a = 0$, it equals $U \oplus E_8(-1)$, with $T_0 = U^2 \oplus E_8(-1)$, unimodular of rank twelve. The parameters (b, c_1, c_0, a) modulo the scalings $(x, y, t) \mapsto (\lambda^2 x, \lambda^3 y, \mu t)$, which send a to a/μ^2 , leave two, and the image of the period map is two-dimensional: otherwise it is the closure of the one-dimensional image of the locus $a = 0$, and a very general $\mathfrak{X}_a$ is isomorphic by the Torelli theorem to some $X_{b'}$, so carries a non-symplectic automorphism σ' of order seven. Such a σ' has no invariant vector in $T_0 \otimes \mathbb{Q}$, so its invariant lattice is $\text{Pic}(\mathfrak{X}_a)$; it fixes the class of a fibre and the class of the zero section O , hence preserves the fibration and O , the only effective curve in its class, and it acts on the base with order seven, since an automorphism over the identity of the base that fixes O is a group automorphism of the fibres, of order dividing six. It fixes $t = \infty$, the only reducible fibre, so it acts by $t \mapsto \zeta t + \beta$ with ζ a primitive seventh root of unity, and since the Weierstrass equation of a fibration with a chosen section is unique up to $(x, y) \mapsto (\lambda^2 x, \lambda^3 y)$, the polynomial $c = c_1 p_{7,a} + c_0$ satisfies $c(\zeta t + \beta) = \lambda^6 c(t)$. The coefficients of t^7 , t^6 and t^5 give $\lambda^6 = 1$, $\beta = 0$ and $7ac_1(\zeta^5 - 1) = 0$, so $a = 0$. Thus the direction a is not isotrivial and the image is two-dimensional. If $E \supsetneq F$ then E has degree six or twelve and $\dim_E T_0 \otimes \mathbb{Q} \leq 2$, which allows at most one modulus by the count of [80, §5.4]; so $E = F$ on the very general member.

In both cases the family is maximal: $\dim_F T_{12} = 4$ for $T_{12} = T_0 \otimes \mathbb{Q}$, the period domain of Hodge structures of K3 type on T_{12} with F acting by Hodge endomorphisms has dimension $4 - 2 = 2$, and the period map has two-dimensional image in its quotient by the arithmetic group, so by the open mapping theorem the image contains a nonempty open set U of one of the two components of that domain.

Step 2, the quadratic structure. Since the intersection form q of T_{12} is σ -invariant there is a unique hermitian form h on the two-dimensional K -space T_{12} with $q = \text{Tr}_{K/\mathbb{Q}} h$, and a unique F -bilinear form $\Phi = \text{Tr}_{K/F} h$ with $q = \text{Tr}_{F/\mathbb{Q}} \Phi$. Diagonalise $h = \langle h_1, h_2 \rangle$ with $h_i \in F^\times$. For $x = u + v\sqrt{-\delta}$ one has $\text{Tr}_{K/F}(h_1 x \bar{x}) = 2h_1(u^2 + \delta v^2)$, so

$$\Phi \cong 2h_1 \langle 1, \delta, h, h\delta \rangle, \quad h = h_2/h_1, \quad (5.4)$$

and $\langle 1, \delta, h, h\delta \rangle$ is the reduced norm form of the quaternion algebra $Q = (-\delta, -h_1 h_2)_F$; thus $(T_{12}, \Phi) \cong (Q, 2h_1 \text{nrd})$ as F -quadratic spaces. The signature of q is $(2, 10)$ and splits over the three real places of F as $(2, 2)$ at the place σ_1 whose eigenspace carries $H^{2,0}$ and $(0, 4)$ at the two others. From (5.4), signature $(2, 2)$ at σ_1 forces $h_1 h_2 < 0$ there, and $(0, 4)$ at σ_2, σ_3 forces $h_1, h_2 < 0$ there; so Q is split at σ_1 and ramified at σ_2 and σ_3 , and after exchanging h_1 and h_2 we may take $h_1 > 0$ at σ_1 and $h_1 < 0$ at the other two places.

⁸The Dickson polynomial $p_{n,a}$ is the unique polynomial of degree n with $p_{n,a}(v + a/v) = v^n + (a/v)^n$; for $a \neq 0$ the map $t \mapsto p_{n,a}(t)$ from $\mathbb{P}^1$ to itself is a cover of degree n with monodromy group the dihedral group D_n , totally ramified over ∞ [80, Lem. 4.3], since it is the quotient of $v \mapsto v^n + (a/v)^n$, on which D_n acts by $v \mapsto \zeta_n v$ and $v \mapsto a/v$, by the involution. For $a = 0$ it is the cyclic cover $t \mapsto t^n$. The identity $p_{n,a}(\mu t) = \mu^n p_{n,a/\mu^2}(t)$ follows from the defining equation with v replaced by μv .

⁹Explicitly $p_{7,a}(t) = t^7 - 7at^5 + 14a^2t^3 - 7a^3t$, the case $n = 7$ of $p_{n,a}(t) = \sum_{k \leq n/2} \frac{n}{n-k} \binom{n-k}{k} (-a)^k t^{n-2k}$; for $a = 0$ it is t^7 , so the family $\mathfrak{X}_a$ contains the surfaces X_b .

Step 3, $Q = D_\infty$. The lattice T_0 is unimodular at every prime $p \neq p_n$, and $K/\mathbb{Q}$ is ramified only at p_n , so for a place v of F not above p_n the ring $\mathcal{O}_K \otimes \mathbb{Z}_p$ is a product of unramified extensions whose trace forms are perfect, and the hermitian $\mathcal{O}_{K,v}$ -lattice $T_0 \otimes \mathbb{Z}_p$ is unimodular for h ; its determinant $h_1 h_2$ is therefore a unit of F_v times a norm from K_v . Units are norms from an unramified extension, so $(-\delta, -h_1 h_2)_v = 1$ at every finite $v \nmid p_n$, including the place above 2. The ramification set of Q has even cardinality, contains σ_2, σ_3 and is otherwise contained in the single place above p_n ; so that place is unramified as well, and $Q \cong D_\infty$. As a check independent of this argument, the rational invariants of $T_0 \otimes \mathbb{Q}$, computed from the Picard lattice by Witt cancellation in the K3 lattice, and those of $\text{Tr}_{F/\mathbb{Q}}(\varepsilon \text{nrd}_{D_\infty})$ agree for both n : signature $(2, 10)$, discriminant 1, Hasse invariant -1 at 2 and $+1$ at every odd prime, while the trace forms of the algebras over F with the same real ramification and ramification at finite places differ from them at 7 or 43 for $n = 7$ and at 3 or 19 for $n = 9$.

Step 4, the surfaces of Mumford type inside T_{12} . For $n_0 \in Q^\times$ the F -line $N = Fn_0$ has $\Phi(n_0, n_0) = 2h_1 \text{nrd}(n_0)$, and its orthogonal complement is $n_0 Q^0$, with $\Phi(n_0 y, n_0 y') = -h_1 \text{nrd}(n_0) \text{trd}(yy')$ for $y, y' \in Q^0$; so $(N^\perp, \Phi) \cong (D_\infty^0, \mu \text{trd})$ with $\mu = -h_1 \text{nrd}(n_0)$, and N is negative definite exactly when μ is totally positive. By the Hasse–Schilling norm theorem [65, Thm. 33.15] the reduced norms of $Q^\times$ are the elements of $F^\times$ positive at σ_2 and σ_3 , and $-h_1$ is negative at σ_1 and positive at the other two places, so every totally positive μ occurs. Given S as in the statement, choose n_0 with $\mu = -h_1 \text{nrd}(n_0)$ and an F -linear isometry $T(S)_\mathbb{Q} \cong n_0^\perp$; the period ω_0 of S becomes a point of the period domain of T_{12} at which Fn_0 is algebraic and F acts by Hodge endomorphisms, because ω_0 lies in the σ_1 -eigenspace. Replacing S by its complex conjugate, whose period is $\bar{\omega}_0$ and for which the conclusion is equivalent, we may take ω_0 on the component containing U .

Step 5, moving the point. The group $\text{SO}(\Phi)(\mathbb{R})$ at the place σ_1 is $(\text{SL}_2(\mathbb{R}) \times \text{SL}_2(\mathbb{R}))/\{\pm 1\}$ acting on $Q \otimes_{F, \sigma_1} \mathbb{R} \cong M_2(\mathbb{R})$ by $x \mapsto axg$, and it acts transitively on each component of the two-dimensional period domain. The group $Q^1 \times Q^1$ of pairs of elements of reduced norm one is simply connected, so by weak approximation [63, Thm. 7.8] its rational points are dense in its real points; hence there are $a, g \in Q^1(\mathbb{Q})$ with $x \mapsto axg$ carrying ω_0 into U . This map is an F -linear isometry of (T_{12}, Φ) , it carries $n_0^\perp$ onto $(an_0g)^\perp$ with $\text{nrd}(an_0g) = \text{nrd}(n_0)$, and it carries the Hodge structure of $T(S)$ to the Hodge structure on $(an_0g)^\perp$ determined by the image point, which is the period of a member X_b of the family. The rational classes of type $(1, 1)$ in T_{12} at that period are exactly $F \cdot an_0g$, since a rational class orthogonal to the period decomposes into a component in the line and a component in $(an_0g)^\perp$ that is a rational class of type $(1, 1)$ in a copy of $T(S)$, which has none. So $T(X_b)_\mathbb{Q} = (an_0g)^\perp$ with the restricted action of F , and $x \mapsto axg$ is a Hodge isometry $\varphi: T(S)_\mathbb{Q} \rightarrow T(X_b)_\mathbb{Q}$ that commutes with F .

Step 6, the cycle. $H^2(S, \mathbb{Q})$ and $H^2(X_b, \mathbb{Q})$ are both isometric to $\Lambda_\mathbb{Q}$, so by Witt cancellation $\text{NS}(S)_\mathbb{Q} \cong \text{NS}(X_b)_\mathbb{Q}$, and φ extends to a rational Hodge isometry $H^2(S, \mathbb{Q}) \rightarrow H^2(X_b, \mathbb{Q})$, which is algebraic by Buskin's theorem [22]. The action of $e \in F$ on $T(X_b)$ is induced by the cycle of Step 1, so $m_e = \varphi^{-1} \circ m_e \circ \varphi$ on $T(S)$ is induced by a cycle on $S \times S$. Since $T(S)$ has rank nine and $F \subseteq E$, E is F or a totally real field of degree nine with $\dim_E T = 1$, which [77] excludes; so $E = F$ is algebraic, and the Hodge conjecture for $S \times S$ and the rest follow as in Theorem 5.9(iii). $\square$

The next proposition says exactly which pairs (F, D) the mechanism reaches and where it stops.

Proposition 5.15 (What the dihedral mechanism reaches). *Let $n \in \{7, 9\}$ and $F = \mathbb{Q}(\zeta_n)^+$.*

- (i) *Let X be a K3 surface with a purely non-symplectic automorphism of order $n \in \{7, 9\}$ whose transcendental lattice has rank twelve and is unimodular at every prime other than p_n . Then $(T(X)_\mathbb{Q}, \Phi)$ is isometric to $(D_\infty, c \text{nrd})$ for some $c \in F^\times$, and every surface of Mumford type obtained from X or from a rational isometric image of X as in Step 4 has the algebra D_∞ . The Mumford algebras over F ramified at two finite places are reached by no family of rank twelve with such an automorphism.*
- (ii) *Let $F = \mathbb{Q}(\zeta_7)^+$ and let $T_{18} = U^2 \oplus A_6 \oplus E_8(-1)$ be the transcendental lattice of the three-dimensional family of [80, §5.4], with its F -bilinear form Φ_6 of dimension six. For every Mumford algebra D over F and every totally positive λ the space $(D^0, \lambda \text{trd})$ embeds F -isometrically into $(T_{18} \otimes \mathbb{Q}, \Phi_6)$, so the four-dimensional period domain of this family contains the Shimura curve of the surfaces S_λ of every pair (F, D) . The image of the family is a threefold in it, and Step 5 would need a rational isometry carrying the period of S_λ into that threefold.*

- (iii) A finite group acting on the base $\mathbb{P}^1$ of an elliptic surface is cyclic, dihedral, tetrahedral, octahedral or icosahedral, and the centres of the simple factors of its rational group algebra are the fields $\mathbb{Q}(\zeta_m)$, $\mathbb{Q}(\zeta_m)^+$, $\mathbb{Q}(\sqrt{-3})$, $\mathbb{Q}$ and $\mathbb{Q}(\sqrt{5})$. The only cubic fields among them are $\mathbb{Q}(\zeta_7)^+$ and $\mathbb{Q}(\zeta_9)^+$, so the base change of elliptic surfaces along covers of $\mathbb{P}^1$ reaches no cyclic cubic field of conductor at least 13.

Proof. (i) Steps 2 and 3 of the theorem use only the order of the automorphism, the rank, the signature and the unimodularity away from p_n , so they apply to X and give $\Phi \cong 2h_1 \text{nr}_{D_\infty}$. A rational isometry preserves (T, Φ) up to the F -similarity class, which determines the algebra, and Step 4 exhibits the orthogonal complement of every negative definite F -line as $(D_\infty^0, \mu \text{trd})$; an algebra $D \neq D_\infty$ has $(D^0, \lambda \text{trd}) \not\cong (D_\infty^0, \mu \text{trd})$ for all λ, μ , since the even Clifford algebra of the ternary form λtrd is D .

(ii) $T_{18} \otimes \mathbb{Q}$ is a three-dimensional hermitian K -space and $\Phi_6 = 2\langle h_1, h_1\delta, h_2, h_2\delta, h_3, h_3\delta \rangle$ has signature $(2, 4)$ at σ_1 and $(0, 6)$ at σ_2, σ_3 , by the same count as in Step 2. The form $q_3 = \lambda \text{trd}$ on D^0 has signature $(2, 1)$ at σ_1 , where D is split, and $(0, 3)$ at the two ramified places. The Witt class of $\Phi_6 - q_3$ is represented at every real place by a negative definite ternary form and at every finite place by a ternary form, because a class of odd dimension has an anisotropic representative of dimension one or three; these local ternary forms have the global discriminant $d(\Phi_6)d(q_3)$, Hasse invariants $+1$ at almost every place and satisfying the product formula, since they are products of the invariants of Φ_6 and q_3 and of Hilbert symbols of their discriminants; so by [60, Thm. 72:1] there is a global ternary form q'_3 with these localisations, and $\Phi_6 \cong q_3 \perp q'_3$ by the Hasse–Minkowski theorem. The image N of q'_3 is a negative definite F -subspace of dimension three with $N^\perp \cong (D^0, \lambda \text{trd})$, and the Hodge structures of K3 type on $N^\perp$ with F acting form the Shimura curve of the S_λ of (F, D) . The family has three moduli [80, §5.4] in a domain of dimension $6 - 2 = 4$.

(iii) The list of finite subgroups of $\text{PGL}_2(\mathbb{C})$ is Klein's. The simple factors of $\mathbb{Q}[G]$ have centres the fields $\mathbb{Q}(\chi)$ generated by the values of the irreducible characters: $\mathbb{Q}(\zeta_m)$ for the cyclic group of order m , $\mathbb{Q}(\zeta_m)^+$ for the dihedral group of order $2m$ (its two-dimensional characters take the values $\zeta_m^k + \zeta_m^{-k}$), $\mathbb{Q}$ and $\mathbb{Q}(\sqrt{-3})$ for A_4 , $\mathbb{Q}$ for S_4 , and $\mathbb{Q}$ and $\mathbb{Q}(\sqrt{5})$ for A_5 . By Proposition 5.12(iii) the correspondences cut out of the action act through these fields; $\mathbb{Q}(\zeta_m)$ is never cubic, and $\mathbb{Q}(\zeta_m)^+$ is cubic exactly for $m \in \{7, 9, 14, 18\}$, where it is $\mathbb{Q}(\zeta_7)^+$ or $\mathbb{Q}(\zeta_9)^+$. $\square$

For the fourfolds X of (F, D_∞) the theorem leaves (F2) where it was: by Theorem 6.45(v) their exceptional classes are algebraic as soon as one algebraic cycle on $S_\lambda \times X^r$ acts nontrivially on $T(S_\lambda)$, and the real multiplication, now algebraic, supplies no such cycle.

5.7. Powers of K3 surfaces and Kummer type

The theorems of this section decide $S \times S$ and $S^{[2]}$. On the higher powers the Hodge classes are described by Varesco [75, Thm. 2.16]. Let E be totally real, $m = \dim_E T$, and for each real embedding σ of E let $V_\sigma = T \otimes_{E, \sigma} \mathbb{C}$, so that $T_\mathbb{C} = \bigoplus_\sigma V_\sigma$. The line $\bigwedge^m V_\sigma \subset T_\mathbb{C}^{\otimes m}$ is spanned by a class $\det V_\sigma$ of type (m, m) , and the span of the $\det V_\sigma$ over all σ is defined over $\mathbb{Q}$: it is $\bigwedge_E^m T \subset \bigwedge^m T \subset T^{\otimes m}$, a space of Hodge classes of dimension $[E : \mathbb{Q}]$ in $H^{2m}(S^m)$, the exceptional classes. Varesco's theorem is that every Hodge class in the tensor algebra of T is obtained from the Hodge classes of $T \otimes T$, which are the elements of E , and from the exceptional classes by tensor products, permutations of the factors, linear combinations and contractions with q_S . Each of these operations carries algebraic classes on powers of S to algebraic classes: a tensor product is an exterior product, a permutation is the pull-back along an automorphism of S^k , and the contraction of two factors is the push-forward along the projection $S^k \rightarrow S^{k-2}$ after cupping with the pull-back of the class of the diagonal of $S \times S$, since $q_S(a, b) = \int_{S \times S} (a \otimes b) \cup [\Delta_S]$.

Proposition 5.16 (Hodge classes on the powers). *Let S be a projective K3 surface with E totally real and $m = \dim_E T$, so that $m \geq 3$ as in the proof of Theorem 5.5(ii).*

- (i) *The Hodge conjecture holds for every power S^n if and only if the elements of E are algebraic on $S \times S$ and the exceptional classes $\bigwedge_E^m T \subset H^{2m}(S^m)$ are algebraic. When $E = \mathbb{Q}$, $m = r = \text{rk } T$ and the exceptional class is the determinant $\det T$, which spans $\bigwedge^r T \subset H^{2r}(S^r)$.*

Where the dihedral mechanism stops: finitely ramified algebras, the threefold in $\mathcal{D}_{T_{18}}$, and the fields of finite groups on $\mathbb{P}^1$

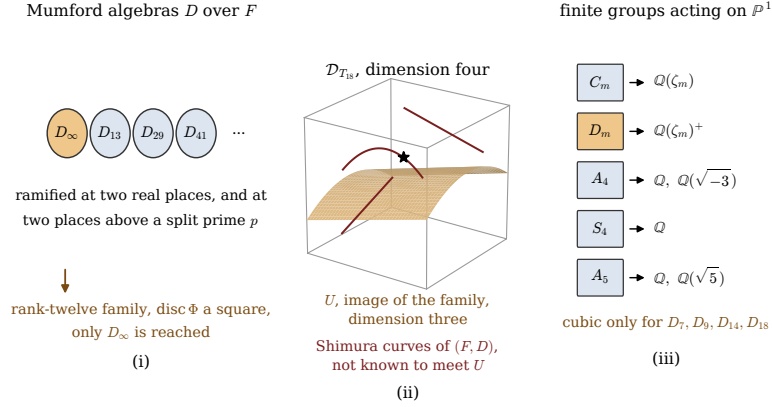


FIGURE 7. Where the dihedral mechanism stops, Proposition 5.15. (i) Over $F = \mathbb{Q}(\zeta_7)^+$ or $\mathbb{Q}(\zeta_9)^+$ the Mumford algebras are D_∞ and the algebras D_p ramified at two real places and at two places above a split prime p ; a rank-twelve family unimodular away from p_n has square discriminant and carries only D_∞ . (ii) The period domain of the rank-eighteen family is four-dimensional, the image of the family is a threefold, and the Shimura curves of the surfaces of Mumford type for the other algebras lie in the domain; whether the threefold meets them is the open transport condition. (iii) The finite subgroups of PGL_2 acting on the base produce only $\mathbb{Q}(\zeta_m)$, $\mathbb{Q}(\zeta_m)^+$, $\mathbb{Q}$, $\mathbb{Q}(\sqrt{-3})$ and $\mathbb{Q}(\sqrt{5})$; a cubic field arises only from the dihedral groups D_7 , D_9 , D_{14} and D_{18} .

- (ii) For a surface of Theorems 5.5, 5.14 or Corollary 5.13 with E totally real, E is algebraic, and the Hodge conjecture for all powers of S is equivalent to the algebraicity of $\bigwedge_E^m T$, $m = \mathrm{rk} T / [E : \mathbb{Q}]$: $m = 3$ for the surfaces S_λ of Theorem 5.14, with $\mathrm{rk} T = 9$; $m = 4$ for the very general members of the two-dimensional family of Theorem 5.14, with $\mathrm{rk} T = 12$.

Proof. (i) If the Hodge conjecture holds on all powers, both kinds of classes are algebraic, being Hodge classes on $S \times S$ and on S^m . Conversely, $H^2(S) = \mathrm{NS}(S)_\mathbb{Q} \oplus T$, and $H^0(S)$, $\mathrm{NS}(S)_\mathbb{Q}$ and $H^4(S)$ are spanned by algebraic classes and are Tate twists of trivial Hodge structures. So a Hodge class of $H^*(S^n) = \bigotimes^n (H^0 \oplus \mathrm{NS}_\mathbb{Q} \oplus T \oplus H^4)$ is a sum of exterior products of algebraic classes with Hodge classes of the tensor powers $T^{\otimes k}$, $k \leq n$, pulled back along the projections of S^n onto k of its factors. By Varesco's theorem and the remark preceding the proposition, the Hodge classes of $T^{\otimes k}$ are algebraic as soon as E and the exceptional classes are. When $E = \mathbb{Q}$ there is one embedding, $V_\sigma = T_\mathbb{C}$ and $\bigwedge_E^m T = \bigwedge^r T$.

(ii) The theorems cited prove that E is algebraic; $m = \mathrm{rk} T / [E : \mathbb{Q}]$ since T is an E -vector space. The ranks are those of the nine-dimensional space D_∞^0 and of the lattice T_0 in Step 1 of the proof of Theorem 5.14. $\square$

Proposition 5.17 (K3 surfaces of Kummer type over $\mathbb{Q}$). *Let S be a projective K3 surface, $r = \mathrm{rk} T$, let $\langle a_1, \dots, a_r \rangle$ be a diagonalisation of q_S on T , $d(T) = \prod_i a_i \in \mathbb{Q}^\times / \mathbb{Q}^{\times 2}$ its discriminant and $\varepsilon_p(T) = \prod_{i < j} (a_i, a_j)_p$ its Hasse invariant at the prime p . The following are equivalent.*

- (a) T embeds isometrically in $U_\mathbb{Q}^3$, the quadratic space $H^2(A, \mathbb{Q})$ of an abelian surface.
- (b) There is a dominant rational map from S to a Kummer surface.
- (c) $r \leq 3$; or $r = 4$ and $\varepsilon_p(T) = (-1, -1)_p$ at every prime p at which $d(T)$ is a square in $\mathbb{Q}_p$; or $r = 5$ and $\varepsilon_p(T) = (-1, -1)_p$ at every prime p .

Here $(-1, -1)_p = 1$ for p odd and $(-1, -1)_2 = -1$, so (c) asks for $\varepsilon_p(T) = 1$ at the odd primes concerned and $\varepsilon_2(T) = -1$. If (a)–(c) hold, then S lies in the class $\mathcal{A}$ of Proposition 3.3, (F3') holds for every power S^n , and the Hodge conjecture holds for every S^n and every Hilbert scheme $S^{[n]}$.

The equivalence of (a) and (b) is Ma's theorem [46, Prop. 3.2], where (a) is also shown equivalent to an embedding of lattices $T(S) \hookrightarrow U^3$, and for a Kummer surface itself the Hodge conjecture on all powers follows from that on the powers of the abelian surface [75, §0.2]. What is added here is the form (c) of the condition, which reads it off the invariants of T , and the passage from Ma's rational map to the class $\mathcal{A}$.

Proof. (a) $\Leftrightarrow$ (c). The space $V = U_{\mathbb{Q}}^3 \cong \langle 1, -1 \rangle^3$ has signature $(3, 3)$, discriminant -1 and Hasse invariants $\varepsilon_p(V) = (-1, -1)_p^3 = (-1, -1)_p$. By Witt's theorem T embeds in V if and only if there is a quadratic space W over $\mathbb{Q}$ with $T \oplus W \cong V$, and by the Hasse–Minkowski theorem [69, Ch. IV, Thm. 9] this is a condition on invariants: W must have dimension $6 - r$, signature $(1, 5 - r)$, discriminant $d(W) = d(V)d(T)^{-1} = -d(T)$ and, from $\varepsilon_p(T \oplus W) = \varepsilon_p(T)\varepsilon_p(W)(d(T), d(W))_p$ with $(d(T), -d(T))_p = 1$, Hasse invariants $\varepsilon_p(W) = (-1, -1)_p \varepsilon_p(T)$. The signature forces $r - 2 \leq 3$. A quadratic space over $\mathbb{Q}$ with prescribed dimension k , signature, discriminant and Hasse invariants exists if and only if the invariants satisfy the product formula, the real invariants are those of the signature, and at every prime the local existence condition holds [69, Ch. IV, §2.3, Prop. 6, and §3.3, Prop. 7]; the first two conditions hold here because they hold for T and for V and $T \otimes \mathbb{R} \oplus W \otimes \mathbb{R} \cong V \otimes \mathbb{R}$, and the local condition is empty for $k \geq 3$, is $\varepsilon_p(W) = 1$ at every p at which $d(W) = -1$ in $\mathbb{Q}_p^\times / \mathbb{Q}_p^{\times 2}$ for $k = 2$, and is $\varepsilon_p(W) = 1$ at every p for $k = 1$. With $k = 6 - r$, $d(W) = -d(T)$ and $\varepsilon_p(W) = (-1, -1)_p \varepsilon_p(T)$ these are the three cases of (c).

(b) and the class $\mathcal{A}$. Let $f: S \dashrightarrow \text{Km}(A)$ be dominant. By Lemma 5.1, $f^* = \Gamma_{f*}^t$ maps $T(\text{Km}(A))_{\mathbb{Q}}$ onto T . Let $\beta: \tilde{A} \rightarrow A$ be the blow-up of the sixteen points of order two and $\pi: \tilde{A} \rightarrow \text{Km}(A)$ the quotient by the involution; $\Gamma_A = (\beta \times \pi)_*[\tilde{A}]$ is an algebraic cycle on $A \times \text{Km}(A)$ with $\Gamma_{A*} = \pi_*\beta^*$, and $H^2(\text{Km}(A), \mathbb{Q}) = \pi_*\beta^*H^2(A, \mathbb{Q}) \oplus \bigoplus_{i=1}^{16} \mathbb{Q}[E_i]$ with E_i the exceptional curves. Hence $T \subseteq (\Gamma_f^t \circ \Gamma_A)_*H^2(A, \mathbb{Q})$, and $H^2(S) = \text{NS}(S)_{\mathbb{Q}} \oplus T$ is spanned by this image and by divisor classes, while $H^0(S)$ and $H^4(S)$ are spanned by the class of S and the class of a point. So $S \in \mathcal{A}$, the point being the abelian variety of dimension zero, $S^n \in \mathcal{A}$ and (F3') holds on S^n by Proposition 3.3. For the Hodge conjecture, the proof of Proposition 3.3(i) writes a Hodge class of S^n as $\sum_j \Gamma_{j*} y_j$ with y_j a Hodge class on one of the abelian varieties spanning $H^*(S^n)$, here the powers A^k with $k \leq n$. The Hodge conjecture holds for every product of surfaces with $p_g = 1$ and $q = 2$ [66], in particular for A^k , so every y_j , and with it every Hodge class of S^n , is algebraic. The Chow motive of $S^{[n]}$ is a direct summand of a sum of Tate twists of the motives $h(S^k)$, $k \leq n$ [24], so every Hodge class of $S^{[n]}$ is the image under an algebraic correspondence of a Hodge class of some S^k . $\square$

Remark 5.18 (The powers left open). (i) At $\rho(S) \geq 19$ the condition (c) is empty and S has a Shioda–Inose structure [53]. At $\rho(S) = 17$ or 18 it is a condition on the Hasse invariants, met by $U_{\mathbb{Q}}^2 \oplus \langle -2 \rangle$, by $U_{\mathbb{Q}} \oplus \langle 2, -2, -2 \rangle$ and by $\langle 2, 2, -2, -2 \rangle$, and failed at $p = 2$ and $p = 3$ by $\langle 2, 2, -6, -6 \rangle$, by $\langle 2, 2, -2, -6, -6 \rangle$ and by $\langle 2, 6, -2, -2, -2 \rangle$, the primes of failure coming in pairs by the product formula; all six are transcendental lattices of K3 surfaces of Picard number 18 or 17 by [58] and the surjectivity of the period map. For $r \leq 5$ a totally real E is $\mathbb{Q}$, since $m \geq 3$; so E is $\mathbb{Q}$ or a CM field, and in the CM case the Hodge conjecture holds on all powers of S with or without (c) [66]. At $\rho \geq 17$ the surfaces for which the Hodge conjecture on all powers is not decided by Propositions 5.16 and 5.17 are therefore those with $E = \mathbb{Q}$ and T failing (c), and for them the one class left is $\det T \in H^{2r}(S^r)$, $r \in \{4, 5\}$. For $\rho \leq 16$ no surface is of Kummer type.

(ii) The Fricke surfaces have complex multiplication by $K = \mathbb{Q}(\sqrt{-N})$ at every member, and Theorem 5.9(iii) settles the Hodge conjecture on all their powers. Their transcendental spaces $U_{\mathbb{Q}}^2 \oplus \langle -1 \rangle \oplus \langle -N \rangle$ (Proposition 5.19) have discriminant N , so they are neither of Kummer type nor in the two families of Picard number 16 for which Varesco proves the Hodge conjecture on all powers from the algebraicity of the Kuga–Satake correspondence [75, Thm. 4.3, Exs. 4.4, 4.5], whose transcendental spaces $U_{\mathbb{Q}}^2 \oplus \langle -2 \rangle^2$ and $U_{\mathbb{Q}}^2 \oplus \langle -6 \rangle \oplus \langle -2 \rangle$ have discriminants 1 and 3 modulo squares; they are settled by the complex multiplication instead. For each surface of Proposition 5.16(ii) the exceptional classes are Hodge classes of degree $2m$ on S^m reached by no correspondence of this section,

all of which act on $T \otimes T$; whether they are algebraic, and with it whether (F3') and the Hodge conjecture hold on S^m , is open. The one construction known to produce such classes is the Kuga–Satake correspondence, through which Varesco proves [75, Thm. 4.3].

Proposition 5.19 (The transcendental lattice and the Kuga–Satake variety of the Fricke surfaces). *Let $N \in \{5, 7\}$, $K = \mathbb{Q}(\sqrt{-N})$, let M be the trivial lattice $U \oplus A_4 \oplus A_1^2 \oplus D_4^2$ or $U \oplus A_6 \oplus D_4^2$ of the elliptic fibration of the members of $\mathfrak{A}_N$, and let $T = M_{\mathbb{Q}}^{\perp} \subset H^2(X, \mathbb{Q})$, which is $T(X)_{\mathbb{Q}}$ when $\rho(X) = 16$.*

- (i) $T \cong U_{\mathbb{Q}}^2 \oplus \langle -1 \rangle \oplus \langle -N \rangle$ as a quadratic space over $\mathbb{Q}$, so $d(T) = N$ and the centre of the even Clifford algebra $C^+(T)$ is K .
- (ii) The Kuga–Satake variety of T is isogenous to B^4 for an abelian fourfold B of K -Weil type with discriminant one. If $\rho(X) = 16$ and $\text{End}_{\text{Hdg}} T = K$, then B is isogenous to $E_K \times B_3$, where E_K is an elliptic curve with complex multiplication by K and B_3 is a simple abelian threefold with $\text{End}(B_3)_{\mathbb{Q}} = K$; in particular B is not simple.

Proof. (i) The root lattices are rationally diagonal: $D_n = \{x \in \mathbb{Z}^n : \sum x_i \in 2\mathbb{Z}\}$ has index two in $\mathbb{Z}^n$, so $D_n \otimes \mathbb{Q} \cong \langle 1 \rangle^n$, and $E_8 \supset D_8$ with index two gives $E_8 \otimes \mathbb{Q} \cong \langle 1 \rangle^8$; $A_n = \{x \in \mathbb{Z}^{n+1} : \sum x_i = 0\}$ is the orthogonal complement of the vector $(1, \dots, 1)$ of norm $n+1$, so $(A_n \oplus \langle n+1 \rangle) \otimes \mathbb{Q} \cong \langle 1 \rangle^{n+1}$; and $A_1^2 = \langle 2, 2 \rangle \cong \langle 1, 1 \rangle$ by the vectors $(1, 1)$ and $(1, -1)$ of $\mathbb{Z}^2$. Hence, with all forms taken positive definite, $A_4 \oplus A_1^2 \oplus D_4^2 \oplus \langle 1, 5 \rangle \cong \langle 1 \rangle^5 \oplus \langle 1 \rangle^2 \oplus \langle 1 \rangle^8 \oplus \langle 1 \rangle = \langle 1 \rangle^{16}$ and $A_6 \oplus D_4^2 \oplus \langle 1, 7 \rangle \cong \langle 1 \rangle^7 \oplus \langle 1 \rangle^8 \oplus \langle 1 \rangle = \langle 1 \rangle^{16}$ over $\mathbb{Q}$. The K3 lattice is $\Lambda = U^3 \oplus E_8(-1)^2$, so $\Lambda_{\mathbb{Q}} \cong U_{\mathbb{Q}}^3 \oplus \langle -1 \rangle^{16}$, while $M_{\mathbb{Q}} \oplus U_{\mathbb{Q}}^2 \oplus \langle -1, -N \rangle \cong U_{\mathbb{Q}}^3 \oplus \langle -1 \rangle^{16}$ by the two isometries just written. Thus $M_{\mathbb{Q}} \oplus T \cong \Lambda_{\mathbb{Q}} \cong M_{\mathbb{Q}} \oplus (U_{\mathbb{Q}}^2 \oplus \langle -1, -N \rangle)$, and Witt's cancellation theorem gives $T \cong U_{\mathbb{Q}}^2 \oplus \langle -1 \rangle \oplus \langle -N \rangle$. Its discriminant is $(-1)^2(-1)(-N) = N$, and the centre of $C^+(T)$ is $\mathbb{Q}(\sqrt{(-1)^3 d(T)}) = \mathbb{Q}(\sqrt{-N})$. The program code/fricke_lattice.py recomputes these isometries by the invariants of Hasse–Minkowski.

(ii) By Lombardo's theorem [45, Thms. 6.2, 6.4], in the form of [75, Thm. 4.1], the Kuga–Satake variety of a polarised Hodge structure of K3 type on $U_{\mathbb{Q}}^2 \oplus \langle a \rangle \oplus \langle b \rangle$ with $a, b < 0$ is isogenous to B^4 for an abelian fourfold B of $\mathbb{Q}(\sqrt{-ab})$ -Weil type with discriminant one; here $(a, b) = (-1, -N)$, $\mathbb{Q}(\sqrt{-ab}) = K$ and $C^+(T) \cong M_4(K)$. Suppose $\rho(X) = 16$ and $\text{End}_{\text{Hdg}} T = K$. By Zarhin's theorem [84, Thm. 2.3.1] the Hodge group of T is $G = \text{U}(W, h)$, the unitary group of the hermitian form h on the three-dimensional K -space $W = T$ with $\text{Tr}_{K/\mathbb{Q}} h = q$. Over $\mathbb{C}$ the two eigenspaces W_{σ} , $W_{\bar{\sigma}}$ of K on $T_{\mathbb{C}}$ have dimension three and are isotropic for q , since $q(ax, y) = q(x, \bar{a}y)$, so $T_{\mathbb{C}} = W_{\sigma} \oplus W_{\sigma}^*$ and $G_{\mathbb{C}} = \text{GL}(W_{\sigma})$ acting on W_{σ}^* by the dual representation. The Kuga–Satake Hodge structure on $H^1(B, \mathbb{Q})$ is the lift of that of T along $\text{Spin}(T) \rightarrow \text{SO}(T)$ [77, §5], so the Hodge group of B is the preimage $\tilde{G}$ of G in $\text{Spin}(T)$, and $H^1(B, \mathbb{C}) = S^+ \oplus S^-$ is the sum of the two half-spin representations of $\text{Spin}(T_{\mathbb{C}}) \cong \text{SL}_4$, on which the two eigenspaces of the centre K of $C^+(T)$ are S^+ and S^- . For the isotropic decomposition $T_{\mathbb{C}} = W_{\sigma} \oplus W_{\sigma}^*$ the spin representation is $\bigwedge^{\bullet} W_{\sigma} \otimes (\det W_{\sigma})^{-1/2}$, with S^+ the even and S^- the odd part. Restricted to $\tilde{G}_{\mathbb{C}}$, $S^+ = (\bigwedge^0 W_{\sigma} \oplus \bigwedge^2 W_{\sigma}) \otimes \det^{-1/2}$ is the sum of the character $\chi = \det^{-1/2}$ and of the irreducible representation $W_{\sigma}^* \otimes \det^{1/2}$ of dimension three, and $S^- = (W_{\sigma} \oplus \bigwedge^3 W_{\sigma}) \otimes \det^{-1/2}$ is the sum of $W_{\sigma} \otimes \det^{-1/2}$ and of χ^{-1} . The four summands are pairwise non-isomorphic, the two characters being distinct and the two three-dimensional representations having different weights, so they are the isotypic components of $H^1(B, \mathbb{C})$ under the connected group $\tilde{G}_{\mathbb{C}}$. An automorphism of $\mathbb{C}$ over $\mathbb{Q}$ carries an isotypic component to one of the same dimension, since the representation is defined over $\mathbb{Q}$; so $U_{\mathbb{C}} = \chi \oplus \chi^{-1}$ and $V_{\mathbb{C}} = S^+ \oplus S^-$ minus it are stable under all of them and are the complexifications of $\mathbb{Q}$ -subspaces U, V with $H^1(B, \mathbb{Q}) = U \oplus V$. Both are stable under $\tilde{G}$, hence sub-Hodge structures, and under K , which acts by scalars on S^+ and on S^- . By Poincaré reducibility B is isogenous to $E_K \times B_3$ with $H^1(E_K, \mathbb{Q}) = U$ and $H^1(B_3, \mathbb{Q}) = V$. The algebra $\text{End}_{\text{Hdg}}(U) \otimes \mathbb{C} = \text{End}(\chi) \oplus \text{End}(\chi^{-1})$ has dimension two and contains K , so $\text{End}(E_K)_{\mathbb{Q}} = K$; likewise $\text{End}_{\text{Hdg}}(V) \otimes \mathbb{C}$ has dimension two, so $\text{End}(B_3)_{\mathbb{Q}} = K$, a field, and B_3 is simple; and $\text{Hom}(E_K, B_3) = 0$ since U and V share no isotypic component. $\square$

Remark 5.20 (Real multiplication at the fixed points). Theorem 5.9(iii) decides every power of every Fricke surface, so the Kuga–Satake variety of Proposition 5.19(ii) is not needed for them; the proposition records what the splitting of T into the two eigenspaces of K does to the Kuga–Satake

construction. For a very general Hodge structure of K3 type on $U_{\mathbb{Q}}^2 \oplus \langle -1 \rangle \oplus \langle -N \rangle$, with endomorphism field $\mathbb{Q}$, the fourfold B is simple with a definite quaternion algebra as endomorphism algebra [77, Prop. 5.7]; the Fricke families lie in the locus of the four-dimensional period domain where the endomorphism field is K , and there B splits. Real multiplication by $\sqrt{N}$ appears on the two twists at the fixed points of w , the surfaces of [80, §7.6, Prop. 7.7], where $\Lambda^2 = N$ by Lemma 5.8(ii). A family of K3 surfaces of transcendental rank six with real multiplication by $\sqrt{5}$ or $\sqrt{7}$ on which the multiplication is algebraic is not known to us: the fixed points of w are two, so the fixed-point twists are two surfaces, and Remark 5.22 says why the chains of Theorem 5.3 do not supply one.

5.8. Where this stops

The frontier case of (F3') in [19, Thm. 21.4 and Rem. 21.5] is a K3 surface S_{λ} of Picard number 13 with $T(S_{\lambda})_{\mathbb{Q}} \cong T$, the nine-dimensional representation of a Mumford fourfold, and $E = F$ a totally real cubic field, which may have Galois group S_3 . Theorem 5.14 settles it for the two cyclic fields $F = \mathbb{Q}(\zeta_7)^+$ and $\mathbb{Q}(\zeta_9)^+$ with the algebra D_{∞} ; the frontier is now S_{λ} for every other pair (F, D) : for these two fields with the algebras ramified at two finite places, for the cyclic cubic fields of conductor at least 13, and for every field that is not cyclic. There, by Proposition 5.11(v), no composite of similarities reaches F ,¹⁰ so Theorem 5.3, the Fricke isogenies and every correspondence of the kinds listed are blind to it; by Proposition 5.11(iv) a correspondence that does reach it has an irreducible component whose resolution has $p_g \geq 2$, transcendental rank at least 18 and $b_2 \geq 31$; and by Proposition 5.12(iii) that component is not the graph of an element of a finite group acting on a cover of S_{λ} when F is not cyclic. When F is cyclic the dihedral families are the natural source, as van Geemen observed for $\mathbb{Q}(\zeta_9)^+$ [77, Rem. 7.9]; Theorem 5.14 is the statement that for the two fields of conductor 7 and 9 and the algebra D_{∞} the source actually reaches S_{λ} , and Proposition 5.15 says that it reaches nothing else. The only correspondences of this size that we know to act on a transcendental space of K3 type are the dihedral ones, which give abelian fields, and the Kuga–Satake correspondence into the cohomology of an abelian variety of dimension 2^7 , whose algebraicity is open [26, 76]; in Section 6 we replace its abelian variety by a sixfold and show what the replacement does and does not give. Within quadratic fields the method reaches $\sqrt{d}$ for $d \in \{2, 3, 6\}$ in families, and the Fricke isogenies give complex multiplication by $\mathbb{Q}(\sqrt{-5})$ and $\mathbb{Q}(\sqrt{-7})$ on one-dimensional families, where Theorem 5.9 settles the Hodge conjecture on every power; real multiplication by $\sqrt{5}$ and $\sqrt{7}$ is algebraic on the two surfaces of [80] and on no family known to us (Remark 5.20); for $\mathbb{Q}(\sqrt{10})$, $\mathbb{Q}(\sqrt{14})$ and the other composite discriminants no family is known on which the chain of Theorem 5.3 can be started. On the powers beyond the square, Proposition 5.16 reduces the Hodge conjecture for every surface of this section with totally real endomorphism field to its exceptional classes, Proposition 5.17 settles it with (F3') for the surfaces of Kummer type over $\mathbb{Q}$, Theorem 5.9 settles it for every member of the Fricke families, and Remark 5.18 records what is left.

Proposition 5.21 (Chains of symplectic quotients reach exactly $\sqrt{2}$, $\sqrt{3}$, $\sqrt{6}$). *Let S be a projective K3 surface whose endomorphism field E is totally real and different from $\mathbb{Q}$, and let $S = S_0, \dots, S_r$ be a chain as in Theorem 5.3 in which every rational map is the quotient by a finite group of symplectic automorphisms. Then no degree n_i is divisible by 5 or by 7, every n_i divides a power of 24, and the multiplier $\prod n_i^{\pm 1}$ lies in $\{1, 2, 3, 6\}$ modulo squares. In particular no such chain makes $\sqrt{10}$, $\sqrt{14}$, $\sqrt{15}$ or $\sqrt{21}$ algebraic, and the surfaces with real multiplication by these fields are not reached by Theorem 5.3 in this form.*

¹⁰Reaching F means producing, by algebraic correspondences from S_{λ} to itself, Hodge endomorphisms of $T(S_{\lambda})_{\mathbb{Q}}$ that generate the cubic field $E = F$, which is what (F3') asks of this surface, since the class of a generator of F in $H^4(S_{\lambda} \times S_{\lambda})$ is a Hodge class reached from no abelian variety as long as it is not algebraic. The correspondences built in this section, and every correspondence whose components have geometric genus at most one, act on $T(S_{\lambda})$ as similarities, and a composite of similarities is a similarity; by Proposition 5.11(ii) a similarity lies in $\mathbb{Q}$ when $[E : \mathbb{Q}]$ is odd, so it generates nothing. The correspondences coming from finite group actions act through abelian fields by Proposition 5.12(iii), and a cubic field whose Galois closure has group S_3 is not abelian. The two propositions together say what a correspondence realising F must look like: an irreducible component with $p_g \geq 2$ and transcendental rank at least eighteen, not arising from a finite group; no construction in the literature produces one.

Proof. Along the chain the transcendental spaces are isomorphic up to scaling of the form, by Lemma 5.1 and Lemma 5.2, so every S_i has the same transcendental rank t and the same endomorphism field E as S . By Zarhin [84, Thms. 2.2.1 and 2.3.1] the Hodge group of $T(S)_{\mathbb{Q}}$ is the group of E -linear special isometries of the E -quadratic space W with $T(S)_{\mathbb{Q}} = \text{Tr}_{E/\mathbb{Q}} W$, of dimension $m = t/[E : \mathbb{Q}]$ over E . If $m \leq 2$ this group is a torus, the Hodge structure $T(S)_{\mathbb{Q}}$ is of CM type, and its endomorphism field is a CM field of degree t containing E properly [23, Lemma 11.1 and Rem. 12.3]; so $m \geq 3$ and $t \geq 3[E : \mathbb{Q}] \geq 6$.

Let G be a finite group of symplectic automorphisms of a K3 surface X and $\Omega_G \subset \text{NS}(X)$ its coinvariant lattice, negative definite because G fixes the transcendental lattice and an ample class. If 5 divides $|G|$, then G contains an element of order five, whose coinvariant lattice has rank 16 [59, § 5] and lies in Ω_G , so $\text{rk } \Omega_G \geq 16$; if 7 divides $|G|$ the same argument with the rank 18 of an element of order seven gives $\text{rk } \Omega_G \geq 18$. The transcendental lattice $T(X)$ lies in $\Omega_G^{\perp} \subset H^2(X, \mathbb{Z})$, which has rank $22 - \text{rk } \Omega_G \leq 6$ and signature $(3, 19 - \text{rk } \Omega_G)$, and $T(X)$ has signature $(2, \text{rk } T(X) - 2)$; so $\text{rk } T(X) \leq 5$ when 5 divides $|G|$ and $\text{rk } T(X) \leq 3$ when 7 does. Every symplectic automorphism of finite order has order at most eight [59], so if some $n_i = |G_i|$ is not of the form $2^a 3^b$ then 5 or 7 divides it, the source of that quotient has transcendental rank at most five, and so has S ; this contradicts $t \geq 6$. Hence every n_i is of the form $2^a 3^b$, and the multiplier is $2^a 3^b$ modulo squares, that is 1, 2, 3 or 6. A multiplier n makes only e with $e^2 \equiv n$ algebraic, which gives the list. $\square$

Remark 5.22. The proposition settles the lattice question of whether a chain $S \rightarrow Y \leftarrow S'$ through a symplectic involution and a symplectic automorphism of order five, of multiplier $2/5 \equiv 10$, can start on a surface with real multiplication by $\mathbb{Q}(\sqrt{10})$: it cannot, since S' would have transcendental rank at most five and real multiplication by a real quadratic field needs rank at least six. The same bound shows that the surfaces of transcendental rank at most five with a symplectic automorphism of order five or seven have endomorphism field $\mathbb{Q}$ or a CM field; in the CM case the Hodge group is a torus and the chain closes only through the complex multiplication itself. Real multiplication by $\sqrt{10}$, $\sqrt{14}$ or any field outside $\{\mathbb{Q}(\sqrt{2}), \mathbb{Q}(\sqrt{3}), \mathbb{Q}(\sqrt{5}), \mathbb{Q}(\sqrt{6}), \mathbb{Q}(\sqrt{7})\}$, and by $\sqrt{5}$ or $\sqrt{7}$ beyond the two surfaces of [80], therefore needs a correspondence that is not a composite of symplectic quotients, Hodge isometries and fibrewise isogenies, and none is known to us.

6. CONSTRUCTIONS AT (F2)

The first classes (F2) has to reach are the Weil classes of a very general abelian $2n$ -fold of Weil type with $n \geq 3$, and the exceptional classes of a Mumford fourfold. We build an object at each.

6.1. The polarised semiregularity criterion

Let A be a very general member of the polarised family $\mathcal{D}_{n,n}$ of abelian $2n$ -folds of Weil type for an imaginary quadratic field K , with polarisation class θ and Weil plane $W(A)$. Then $\text{Hdg}^*(A) = \mathbb{Q}[\theta] \oplus W(A)$ and $W(A) \cdot \theta = 0$ [19, Thm. 4.14]. A class $\gamma \in H^{\text{ev}}(A)$ has the polarised shape if

$$\gamma = N\omega + \sum_{k=0}^{2n} c_k \theta^k, \quad N \neq 0, 0 \neq \omega \in W(A), c_k \in \mathbb{Q}.$$

Its Hankel rank $\rho \in \{0, 1, 2, 3\}$ is the rank of the Hankel matrix of the sequence $(k! c_k)_k$ capped at three; it is the number of distinct exponentials $e^{a\theta}$ needed to write the polynomial part when that number is at most two, and 3 otherwise.¹¹

¹¹The Hankel matrix of a sequence $(a_k)_{k \geq 0}$ has entries a_{i+j} . If a_k is a sum $\sum_j \lambda_j b_j^k$ of m terms with distinct b_j and nonzero λ_j , the matrix factors as $V^T \text{diag}(\lambda_j) V$ with V the Vandermonde matrix of the b_j , so its rank is m ; conversely a sequence whose Hankel matrix has finite rank m satisfies a linear recurrence of order m and is a sum of m exponentials with polynomial coefficients. For the polynomial part $\sum_k c_k \theta^k$ of a Chern character, writing it as $\sum_j \lambda_j e^{b_j \theta}$ means $k! c_k = \sum_j \lambda_j b_j^k$, so the number of exponentials needed is the Hankel rank of $(k! c_k)$, provided the degree $2n$ allows the recurrence to be read off, which it does for $m \leq 2$; the cap at three records that the criterion of Theorem 6.1 does not distinguish further.

$$r(\gamma) = (4 + \rho)n^2 - 2n: \text{ the rank the criterion demands}$$

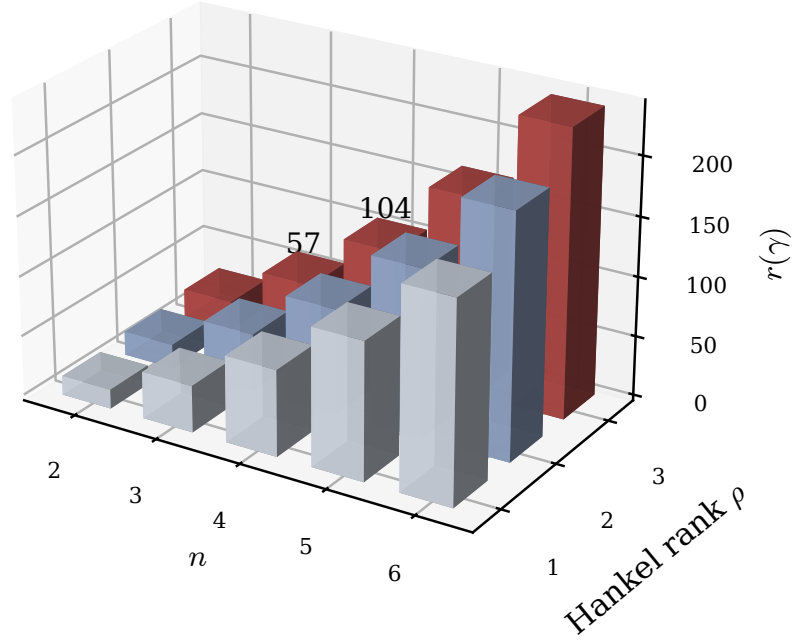


FIGURE 8. The lower bound $r(\gamma) = (4 + \rho)n^2 - 2n$ of Theorem 6.1 for $2 \leq n \leq 6$ and Hankel rank $\rho = 1, 2, 3$. The two labelled bars are the product cases of Theorem 6.3.

Theorem 6.1 (The polarised criterion [19]). *Let $\mathcal{E}$ be a perfect complex on A with $\text{ch}(\mathcal{E}) = \gamma$ of polarised shape and Hankel rank ρ . Then*

$$\dim \text{Ext}^2(\mathcal{E}, \mathcal{E}) \geq r(\gamma) := (4 + \rho)n^2 - 2n,$$

and if equality holds the semiregularity map¹² of $\mathcal{E}$ is injective, $\mathcal{E}$ deforms with A along the whole of $\mathcal{D}_{n,n}$, and the Weil classes are algebraic on every member of the family with full Hodge group. If some $c_k \neq 0$ with $1 \leq k \leq 2n - 1$, the first-order Hodge locus of γ in $H^1(A, T_A)$ is the tangent space of $\mathcal{D}_{n,n}$, of dimension n^2 . A perfect complex whose Chern character is exactly a multiple of a Weil class does not exist on a very general member, so the polynomial part is forced.

The number $r(\gamma)$ is the rank of the contraction $HT^2(A) \rightarrow H^*(A)$, $\xi \mapsto \xi \lrcorner \gamma$, on the Hochschild cohomology $HT^2 = \bigoplus_{p+q=2} H^q(A, \bigwedge^p T_A)$, of dimension $\binom{4n}{2}$; its kernel is the annihilator $\text{Ann}(\gamma)$, which meets $H^1(A, T_A)$ in the first-order Hodge locus. The semiregularity map $\sigma_{\mathcal{E}}$ of a perfect complex composes with the characteristic map $\text{ev}_{\mathcal{E}}: HT^2 \rightarrow \text{Ext}^2(\mathcal{E}, \mathcal{E})$, $\xi \mapsto \xi \lrcorner \text{At}(\mathcal{E})$, to the contraction into $\text{ch}(\mathcal{E})$; this is how the lower bound arises, and it is the form in which the criterion is used below. Figure 8 shows $r(\gamma)$; the two values that matter below are $r = 57$ at $n = 3$, $\rho = 3$ and $r = 104$ at $n = 4$, $\rho = 3$. On the product E_0^6 of six copies of the elliptic curve $\mathbb{C}/\mathbb{Z}[i]$ every twisted complex of line bundles has $\dim \text{Ext}^2 \geq 69 > 57$ [19, Thm. 18.35]; the criterion is not met there by any such object.

¹²For a perfect complex $\mathcal{E}$ on a smooth projective variety the semiregularity map of Buchweitz and Flenner [17] is the linear map $\sigma_{\mathcal{E}}: \text{Ext}^2(\mathcal{E}, \mathcal{E}) \rightarrow \prod_q H^{q+1}(X, \Omega_X^{q-1})$ sending ξ to the trace of $\xi \circ \exp(-\text{At}(\mathcal{E}))$, where $\text{At}(\mathcal{E}) \in \text{Ext}^1(\mathcal{E}, \mathcal{E} \otimes \Omega_X^1)$ is the Atiyah class; it generalises Bloch's map for a subvariety [18], which is the case of the structure sheaf of the subvariety. Its value is that the obstruction to deforming $\mathcal{E}$ along a first-order deformation of X lies in $\text{Ext}^2(\mathcal{E}, \mathcal{E})$ and is sent by $\sigma_{\mathcal{E}}$ to the obstruction to keeping $\text{ch}(\mathcal{E})$ of Hodge type; so when $\sigma_{\mathcal{E}}$ is injective, $\mathcal{E}$ deforms wherever its Chern character stays a Hodge class, and the Chern character of the deformed complex is algebraic there. The dimension count of the theorem is the arithmetic of this statement: the rank of $\sigma_{\mathcal{E}}$ is at least the rank of the contraction into $\text{ch}(\mathcal{E})$, so $\dim \text{Ext}^2$ is at least $r(\gamma)$, and equality forces injectivity.

6.2. The Weil plane of a product

Lemma 6.2. *Let A, A' be of Weil type for the same field K , of dimensions $2m, 2m'$, with the diagonal K -action on $A \times A'$. Then $W(A \times A')$ is a two-dimensional rational subspace of the four-dimensional space $W(A) \otimes W(A') \subseteq H^{2n}(A \times A')$, $n = m + m'$, and every rational $w \in W(A \times A')$ is $\sum_{i,j} a_{ij} \omega_i \otimes \omega'_j$ with $a_{ij} \in \mathbb{Q}$ for rational bases $(\omega_i), (\omega'_j)$ of $W(A), W(A')$.*

Proof. Over $\mathbb{C}$, $H^1(A \times A') = V \oplus V'$ splits into eigenspaces $V_{\pm} \oplus V'_{\pm}$ of K , and $\bigwedge_K^{2n}$ of the sum is spanned by $\det V_+ \otimes \det V'_+$ and $\det V_- \otimes \det V'_-$; these lie in the span of the four products $\omega_{\pm} \otimes \omega'_{\pm}$ of the eigenvectors of $W(A) \otimes \mathbb{C}$ and $W(A') \otimes \mathbb{C}$. Both $W(A \times A')$ and $W(A) \otimes W(A')$ are defined over $\mathbb{Q}$, so the inclusion holds over $\mathbb{Q}$. The explicit formula is [19, Eq. (36)]: $\omega_1(A \times A') = \omega_1 \otimes \omega'_1 - d \omega_2 \otimes \omega'_2$ for $K = \mathbb{Q}(\sqrt{-d})$. $\square$

6.3. An object with the polarised Chern character

Theorem 6.3 (The product object). *Let A, A' be very general in their polarised Weil families, with ample classes $\theta_A, \theta_{A'}$, $\theta = \theta_A + \theta_{A'}$, and suppose that $W(A)$ and $W(A')$ are algebraic. Then there is a perfect complex $\mathcal{E}$ on $A \times A'$, a finite direct sum with shifts of exterior products*

$$F_i \boxtimes G'_j, \quad F_i \boxtimes L'^k, \quad L^l \boxtimes G'_j, \quad L^l \boxtimes L'^k,$$

with F_i, G'_j coherent sheaves on A, A' and L, L' the ample bundles, such that

$$\text{ch}(\mathcal{E}) = Nw + \sum_{k=0}^{2n} c_k \theta^k, \quad N \neq 0, 0 \neq w \in W(A \times A'),$$

has the polarised shape with Hankel rank $\rho = 3$, so that $r(\text{ch} \mathcal{E}) = 7n^2 - 2n$, and the first-order Hodge locus of $\text{ch}(\mathcal{E})$ in $H^1(A \times A', T)$ is the tangent space of $\mathcal{D}_{n,n}$.

The hypotheses hold in the two open cases nearest to the known ones: A, A' very general Weil fourfolds of trivial discriminant, where Markman proves the Weil classes algebraic [49, 50], so that $A \times A'$ is an eightfold with $n = 4$; and A such a fourfold and $A' = B$ a Weil surface of any discriminant d , whose Weil plane is of type $(1, 1)$ and algebraic by Lefschetz, so that $A \times B$ is a sixfold of discriminant d with $n = 3$.

Proof. Since $\text{Hdg}^*(A) = \mathbb{Q}[\theta_A] \oplus W(A)$, algebraic classes on A are rational combinations of Chern characters of coherent sheaves, and the Chern character of a coherent sheaf is a Hodge class, the algebraicity of $W(A)$ gives coherent sheaves F_1, F_2 with $\text{ch}(F_i) = N_i \omega_i + p_i(\theta_A)$, $N_i \neq 0$, for a basis (ω_1, ω_2) of $W(A)$; likewise G'_1, G'_2 on A' with $\text{ch}(G'_j) = N'_j \omega'_j + p'_j(\theta_{A'})$. Fix $0 \neq w \in W(A \times A')$ rational, $w = \sum a_{ij} \omega_i \otimes \omega'_j$ by Lemma 6.2, and a common denominator M of the numbers $Ma_{ij}/(N_i N'_j)$, so that these are integers. Negative multiplicities are realised by the shift $[1]$, which changes the sign of ch .

Step 1. $\mathcal{E}_0 = \bigoplus_{i,j} (F_i \boxtimes G'_j)^{\oplus Ma_{ij}/(N_i N'_j)}$ has, by multiplicativity,

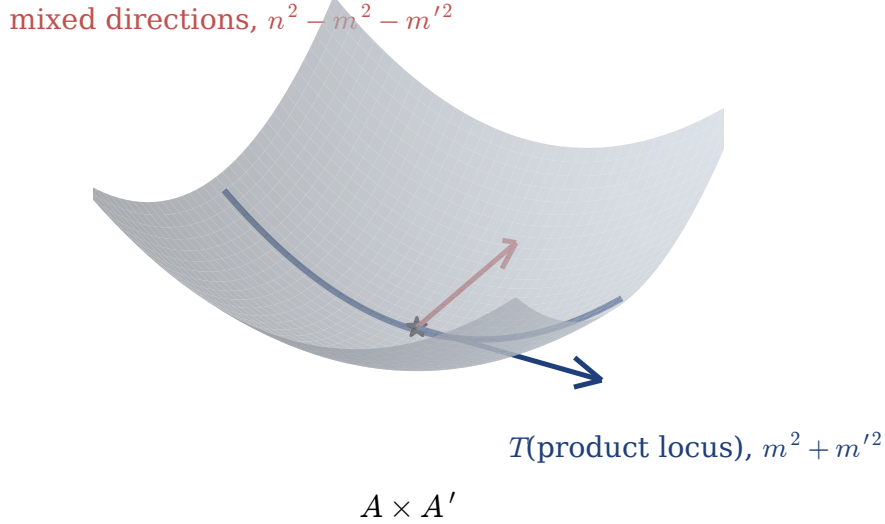
$$\text{ch}(\mathcal{E}_0) = Mw + \sum_i \omega_i \otimes q'_i(\theta_{A'}) + \sum_j q_j(\theta_A) \otimes \omega'_j + s, \quad (6.1)$$

with q'_i, q_j polynomials and $s \in \mathbb{Q}[\theta_A] \otimes \mathbb{Q}[\theta_{A'}]$.

Step 2. The classes $e^{k\theta_{A'}} = \text{ch}(L'^k)$, $k = 0, \dots, 2m'$, form a basis of $\mathbb{Q}[\theta_{A'}]/(\theta_{A'}^{2m'+1})$: the matrix $(k^j/j!)_{j,k}$ has determinant $\prod_j (1/j!)$ times the Vandermonde determinant of $0, 1, \dots, 2m'$, which is nonzero. Write $q'_i = \sum_k b_{ik} e^{k\theta_{A'}}$ and add $(F_i \boxtimes L'^k)^{\oplus -Mb_{ik}/N_i}$, after enlarging M to clear denominators; this removes $\omega_i \otimes q'_i$ and adds only terms $p_i(\theta_A) \otimes e^{k\theta_{A'}}$, which lie in $\mathbb{Q}[\theta_A] \otimes \mathbb{Q}[\theta_{A'}]$. Symmetrically remove $q_j \otimes \omega'_j$ with summands $L^l \boxtimes G'_j$.

Step 3. The classes $e^{l\theta_A + k\theta_{A'}} = \text{ch}(L^l \boxtimes L'^k)$ form a basis of $\mathbb{Q}[\theta_A]/(\theta_A^{2m+1}) \otimes \mathbb{Q}[\theta_{A'}]/(\theta_{A'}^{2m'+1})$ by the same determinant, so the remaining part in $\mathbb{Q}[\theta_A] \otimes \mathbb{Q}[\theta_{A'}]$ is cancelled by line bundles with integer multiplicities and shifts. Finally add $(L \boxtimes L')^{\otimes k}$, $k = 0, 1, 2$, with generic nonzero multiplicities μ_k . The

$\mathcal{D}_{n,n}$ near a product point: 4 (resp. 8) mixed directions obstruct every $F \boxtimes G'$



result is

$$\text{ch}(\mathcal{E}) = Mw + \sum_{k=0}^2 \mu_k e^{k\theta},$$

a polarised shape whose polynomial part is a sum of three distinct exponentials with nonzero coefficients. Three exponentials are not one of the shapes of Hankel rank at most two, so $\rho = 3$; and some c_k with $1 \leq k \leq 2n - 1$ is nonzero, so the first-order Hodge locus is the tangent space of $\mathcal{D}_{n,n}$ by Theorem 6.1. $\square$

The construction was carried out in the exterior-algebra model of $H^*(A \times A')$ for $(m, m') = (2, 1)$ and $(2, 2)$: the cross terms of $\text{ch}(\mathcal{E}_0)$ vanish, nothing is left outside $W(A \times A') \oplus \mathbb{Q}[\theta]$ after Steps 2 and 3, the Hankel rank is 3, and the first-order Hodge locus has dimension 9, resp. 16, equal to n^2 (Appendix B).

6.4. What a direct sum cannot do

The tangent space of $\mathcal{D}_{n,n}$ at a product point contains the tangent space of the product locus $\mathcal{D}_{m,m} \times \mathcal{D}_{m',m'}$, of dimension $m^2 + m'^2$, and $n^2 - m^2 - m'^2 = 2mm'$ mixed directions. In the same model, each exterior product $F \boxtimes G'$ of sheaves of polarised shape has first-order Hodge locus inside $T\mathcal{D}_{n,n}$ equal to the tangent space of the product locus: all $2mm'$ mixed directions obstruct it, four for $A \times B$ and eight for $A \times A'$ (Fig. 9). The object $\mathcal{E}$ of Theorem 6.3 has the whole tangent space as its first-order locus because its Chern character is a single polarised class and not a product. Semiregularity is a different matter.

Proposition 6.4 (Direct sums are never semiregular).

- (i) Let $\mathcal{E} = \bigoplus_k \mathcal{E}_k$ be a direct sum of at least two nonzero complexes. Then the semiregularity map $\sigma_{\mathcal{E}}$ vanishes on $\bigoplus_{k \neq l} \text{Ext}^2(\mathcal{E}_k, \mathcal{E}_l)$. So $\sigma_{\mathcal{E}}$ is injective only if $\text{Ext}^2(\mathcal{E}_k, \mathcal{E}_l) = 0$ for all $k \neq l$.

- (ii) For every direct-sum realisation $\mathcal{E}$ of Theorem 6.3, $\sigma_{\mathcal{E}}$ is not injective and $\dim \text{Ext}^2(\mathcal{E}, \mathcal{E}) > r(\text{ch} \mathcal{E})$.

Proof. (i) The semiregularity map of Buchweitz and Flenner is built from the traces $\xi \mapsto \text{tr}(\xi \circ \text{At}(\mathcal{E})^p)$ of compositions with powers of the Atiyah class [17, Sect. 4]. The Atiyah class of a direct sum is block diagonal, so for ξ in an off-diagonal block $\text{Ext}^2(\mathcal{E}_k, \mathcal{E}_l)$ the composition $\xi \circ \text{At}(\mathcal{E})^p$ is off-diagonal, and the trace of an off-diagonal block is zero.

(ii) The summands of $\mathcal{E}$ include $F_1 \boxtimes L'^k$ and $F_1 \boxtimes L'^l$ for two exponents $k < l$, because Step 2 of the proof uses every power L'^k with $0 \leq k \leq 2m'$ (the exponentials form a basis and q'_1 is a nonzero polynomial of positive degree for a sheaf of polarised shape, which has p_1 of positive degree by Theorem 6.1). By the Künneth formula

$$\text{Ext}^2(F_1 \boxtimes L'^k, F_1 \boxtimes L'^l) \supseteq \text{Ext}^2(F_1, F_1) \otimes H^0(A', L'^{l-k}), \quad (6.2)$$

and $\dim \text{Ext}^2(F_1, F_1) \geq r(\text{ch} F_1) \geq 12$ by Theorem 6.1 on the fourfold A ($n = 2$, $\rho \geq 0$), while H^0 of an ample bundle is nonzero. So an off-diagonal block is nonzero, $\sigma_{\mathcal{E}}$ has a kernel by (i), and $\dim \text{Ext}^2(\mathcal{E}, \mathcal{E}) > r$ because equality would force injectivity. $\square$

Proposition 6.5 (Convolutions). *Let $\mathcal{E}_{\delta}$ be a one-sided twisted complex on pieces $P_1, \dots, P_N$ with shifts s_a , ordered so that the differential $\delta = (\delta_{ab})_{a < b}$, $\delta_{ab} \in \text{Ext}^{1+s_a-s_b}(P_a, P_b)$, is strictly upper triangular; let $\gamma = \text{ch}(\mathcal{E}_{\delta}) = \sum_a (-1)^{s_a} \text{ch}(P_a)$.*

- (i) *The semiregularity map $\sigma_{\mathcal{E}_{\delta}}$ vanishes on every class of $\text{Ext}^2(\mathcal{E}_{\delta}, \mathcal{E}_{\delta})$ represented by a strictly upper triangular cocycle. So the criterion holds for $\mathcal{E}_{\delta}$ only if every strictly upper triangular cocycle of degree two is a coboundary.*
- (ii) *Suppose no term of the twisted differential carries a class of degree one into a diagonal block $\text{Ext}^*(P_c, P_c)$. Let $v \in T\mathcal{D}_{n,n}$ have a nonzero mixed component. If some piece $P = X \boxtimes Y$ has $\text{rk} Y c_1(X)/\theta_A \neq \text{rk} X c_1(Y)/\theta_{A'}$, then $\text{ev}_{\mathcal{E}_{\delta}}(v) \neq 0$ and $\sigma_{\mathcal{E}_{\delta}}(\text{ev}_{\mathcal{E}_{\delta}}(v)) = 0$: the criterion fails.*
- (iii) *Suppose the pieces split into two nonempty groups with $\text{Ext}^*(P_a, P_b) = 0$ in both directions for a, b in different groups. Then $\mathcal{E}_{\delta} = \mathcal{E}' \oplus \mathcal{E}''$ is the direct sum of the convolutions of the groups, and the criterion holds for $\mathcal{E}_{\delta}$ only if every $v \in T\mathcal{D}_{n,n}$ annihilates $\text{ch}(\mathcal{E}')$ and $\text{ch}(\mathcal{E}'')$ separately.*

Proof. (i) The components of $\sigma_{\mathcal{E}_{\delta}}(\xi)$ are the traces $\text{tr}(\xi \circ \text{At}(\mathcal{E}_{\delta})^p)/p!$ [17, Sect. 4]. In the harmonic model of a twisted complex the Atiyah class is represented by $[\nabla, \bar{\partial} + \delta]$ with ∇ the direct sum of the Chern connections of the pieces; it is block upper triangular with diagonal blocks $\text{At}(P_a)$ and off-diagonal blocks $\nabla \delta_{ab}$. The homotopy transfer $\xi \mapsto \sum_k (-h[\delta, \cdot])^k \xi$ that produces the chain-level representative of ξ raises or preserves the block position $b - a$, so a strictly upper triangular cocycle has a strictly upper triangular representative. The product of an upper triangular block matrix with a strictly upper triangular one is strictly upper triangular, and its trace, the sum of the traces of its diagonal blocks, is zero. The trace is a chain map for $\bar{\partial} + [\delta, \cdot]$ by cyclicity, so the vanishing holds in cohomology. Since $\sigma_{\mathcal{E}_{\delta}} \circ \text{ev}_{\mathcal{E}_{\delta}}$ is contraction into γ , of rank $r(\gamma)$, the kernel of $\sigma_{\mathcal{E}_{\delta}}$ contains the image of the strictly upper triangular classes and $\dim \text{Ext}^2 \geq r(\gamma) + \dim(\text{that image})$.

(ii) The Atiyah class of $X \boxtimes Y$ is $\text{At}(X) \otimes 1 + 1 \otimes \text{At}(Y)$, so the diagonal component of $\text{ev}_{\mathcal{E}_{\delta}}(v)$ at P is $v \lrcorner \text{At}(P)$, with trace $v \lrcorner \text{ch}_1(P)$. Write $v = v_A + v_{A'} + z \otimes d' + z' \otimes d$ with $v_A \in T\mathcal{D}_{m,m}$, $v_{A'} \in T\mathcal{D}_{m',m'}$, $z \in H^{0,1}(A)$, $d' \in H^0(T_{A'})$, $z' \in H^{0,1}(A')$, $d \in H^0(T_A)$. Since v kills $\theta = \theta_A + \theta_{A'}$ and $v_A \lrcorner \theta_A = v_{A'} \lrcorner \theta_{A'} = 0$, we have $v \lrcorner \theta_{A'} = -v \lrcorner \theta_A = -z' \wedge (d \lrcorner \theta_A)$, nonzero when $d \neq 0$ because θ_A is nondegenerate and $z' \wedge$ is injective on $H^{0,1}(A)$; the case $d = 0$, $d' \neq 0$ is symmetric. On a very general member $c_1(X)$ and $c_1(Y)$ are multiples of the polarisations, so $v \lrcorner \text{ch}_1(P) = (\text{rk} Y c_1(X)/\theta_A - \text{rk} X c_1(Y)/\theta_{A'}) v \lrcorner \theta_A \neq 0$ under the hypothesis on P . Under the assumption on the differential, the diagonal components of a class in $\text{Ext}^2(\mathcal{E}_{\delta}, \mathcal{E}_{\delta})$ are well defined, so $\text{ev}_{\mathcal{E}_{\delta}}(v) \neq 0$, while $\sigma_{\mathcal{E}_{\delta}}(\text{ev}_{\mathcal{E}_{\delta}}(v)) = v \lrcorner \gamma = 0$ because v is tangent to the Hodge locus of γ .

(iii) With no Ext groups between the groups there are no components of δ between them, so the twisted complex is block diagonal,

$$\begin{aligned} \text{Ext}^2(\mathcal{E}_{\delta}, \mathcal{E}_{\delta}) &= \text{Ext}^2(\mathcal{E}', \mathcal{E}') \oplus \text{Ext}^2(\mathcal{E}'', \mathcal{E}''), \\ \dim \text{Ext}^2(\mathcal{E}_{\delta}, \mathcal{E}_{\delta}) &\geq \dim HT^2 - \dim(\text{Ann}(\text{ch} \mathcal{E}') \cap \text{Ann}(\text{ch} \mathcal{E}')), \end{aligned} \quad (6.3)$$

with $\sigma_{\mathcal{E}_\delta} = \sigma_{\mathcal{E}' } \oplus \sigma_{\mathcal{E}''}$; the inequality holds because for each summand the contraction into its Chern character factors through ev and σ , so that $\dim \text{Ext}^2(\mathcal{E}', \mathcal{E}')$ is at least the rank of $\xi \mapsto \xi \lrcorner \text{ch} \mathcal{E}'$, and likewise for $\mathcal{E}''$. Here $r(\gamma) = \dim HT^2 - \dim \text{Ann}(\gamma)$ with $\text{Ann}(\gamma) = T\mathcal{D}_{n,n}$. Equality $\dim \text{Ext}^2 = r(\gamma)$ forces the common annihilator to be all of $T\mathcal{D}_{n,n}$. $\square$

For the pieces of Theorem 6.3 the splitting in (iii) that one would like to use separates the pieces $F_i \boxtimes Y$ carrying a secant sheaf on A from the pieces $L^l \boxtimes Y'$. Two statements settle what that splitting gives. The first computes the annihilator of the secant pieces by hand; the second shows that the splitting is never available.

Theorem 6.6 (The annihilator of the secant pieces). *Let $\mathcal{E}'$ be the sum, with the multiplicities and shifts of Theorem 6.3, of the pieces $F_i \boxtimes G'_j$ and $F_i \boxtimes L'^k$. Then*

$$\text{ch}(\mathcal{E}') = Mw + \sum_i \frac{M}{N_i} p_i(\theta_A) \otimes R_i, \quad R_i = \sum_j a_{ij} \omega'_j, \quad (6.4)$$

where (a_{ij}) is the matrix of w in the bases (ω_i) , (ω'_j) and is invertible, and the annihilator of $\text{ch}(\mathcal{E}')$ in $T\mathcal{D}_{n,n}$ is the tangent space $T\mathcal{D}_{m,m} \oplus T\mathcal{D}_{m',m'}$ of the product locus, of dimension $m^2 + m'^2$. If $\text{Ext}^*(F_i, L^l) = 0$ for every power L^l among the pieces, every convolution $\mathcal{E}_\delta$ of the pieces has $\dim \text{Ext}^2(\mathcal{E}_\delta, \mathcal{E}_\delta) \geq r(\gamma) + 2mm'$, which is 61 at $n = 3$ and 120 at $n = 4$.

Proof. Steps 1 and 2 of Theorem 6.3 add and subtract, for each i , the classes

$$\sum_j \frac{Ma_{ij}}{N_i N'_j} \text{ch}(F_i) \text{ch}(G'_j) \quad \text{and} \quad \frac{M}{N_i} \text{ch}(F_i) \otimes q'_i,$$

where $q'_i = \sum_j (a_{ij}/N'_j) p'_j$; the difference is $\text{ch}(F_i) \otimes R_i$, and summing over i with $\text{ch}(F_i) = N_i \omega_i + p_i(\theta_A)$ gives (6.4), the terms $\omega_i \otimes R_i$ adding up to Mw by the definition of a . By Lemma 6.2 $w = x \omega_+ \otimes \omega'_+ + \bar{x} \omega_- \otimes \omega'_-$ with $x \neq 0$, a tensor of rank two, so a is invertible and R_1, R_2 form a basis of the Weil plane of A' .

Write $V = H^1(A, \mathbb{C}) = V_+ \oplus V_-$ for the eigenspaces of K , $\omega_\pm = \det V_\pm$, and $V'_\pm, \omega'_\pm$ for their analogues on A' . A tangent vector $v \in H^1(T_{A \times A'})$ acts on cohomology as a derivation and decomposes as $v = v_A + v_{A'} + \mu + \mu'$ with $v_A \in H^{0,1}(A) \otimes H^0(T_A)$, $v_{A'}$ its analogue on A' , and the mixed parts $\mu \in H^{0,1}(A) \otimes H^0(T_{A'})$, $\mu' \in H^{0,1}(A') \otimes H^0(T_A)$. Let $v \in T\mathcal{D}_{n,n}$. The three Künneth components of $v \lrcorner \theta$ vanish separately: $v_A \lrcorner \theta_A = 0$, $v_{A'} \lrcorner \theta' = 0$ and $\mu \lrcorner \theta' + \mu' \lrcorner \theta_A = 0$. Contraction into a nondegenerate form is an isomorphism from $H^0(T)$ onto $H^{0,1}$, so $\mu \mapsto \mu \lrcorner \theta'$ and $\mu' \mapsto \mu' \lrcorner \theta_A$ are injective into $H^{0,1}(A) \otimes H^{0,1}(A')$, and $\mu = 0$ exactly when $\mu' = 0$. Next, $v \lrcorner (\omega_+ \otimes \omega'_+) = 0$. The term $(v_A \lrcorner \omega_+) \otimes \omega'_+$ has no component in $\Lambda^{2m} V_+ \otimes \Lambda^{2m'} V'_+$, because that component is the trace of v_A on V_+ and v_A sends $V_+^{1,0}$ into $V_+^{0,1}$ and kills $V_+^{0,1}$; it lies in $\Lambda^{2m-1} V_+ \otimes V_- \otimes \Lambda^{2m'} V'_+$, while the other three terms lie in $\Lambda^{2m} V_+ \otimes \Lambda^{2m'-1} V'_+ \otimes V'_-$, in $\Lambda^{2m-1} V_+ \otimes \Lambda^{2m'} V'_+ \otimes V'_-$ and in $\Lambda^{2m} V_+ \otimes V_- \otimes \Lambda^{2m'-1} V'_+$, four distinct weight spaces for $GL(V_+) \times GL(V_-) \times GL(V'_+) \times GL(V'_-)$. Hence $v_A \lrcorner \omega_+ = 0$, and in the same way $v_A \lrcorner \omega_- = 0$ and $v_{A'} \lrcorner \omega'_\pm = 0$. A vector of $T\mathcal{D}_{n,n}$ with vanishing mixed part therefore lies in $T\mathcal{D}_{m,m} \oplus T\mathcal{D}_{m',m'}$, and conversely every such sum kills θ and the Weil plane; the dimension $m^2 + m'^2$ is that of the two Weil families [83].

Now let $v \in T\mathcal{D}_{n,n}$ have $\mu \neq 0$. It kills Mw ; v_A kills $p_i(\theta_A)$ and $v_{A'}$ kills R_i ; so $v \lrcorner \text{ch}(\mathcal{E}') = (\mu + \mu') \lrcorner \sum_k \theta_A^k \otimes R^{(k)}$, where $p_i = \sum_k c_{ik} \theta_A^k$ and $R^{(k)} = \sum_i (M c_{ik} / N_i) R_i$. Write $\mu = \sum_s z_s \otimes d'_s$ and $\mu' = \sum_s z'_s \otimes d_s$. The contraction of $\theta_A^k \otimes R^{(k)}$ against μ' is $k \sum_s \theta_A^{k-1} (d_s \lrcorner \theta_A) \otimes (z'_s \wedge R^{(k)})$, of Künneth bidegree $(2k-1, 2m'+1)$, and against μ it is $\sum_s (z_s \wedge \theta_A^k) \otimes (d'_s \lrcorner R^{(k)})$, of bidegree $(2k+1, 2m'-1)$. These bidegrees are pairwise distinct over all k and both types, so every term vanishes on its own. Some $R^{(k)}$ is nonzero, since p_1, p_2 are not both zero and R_1, R_2 are a basis. For $R = \sigma \omega'_+ + \bar{\sigma} \omega'_-$ with $\sigma \neq 0$ the map $z' \mapsto z' \wedge R$ is injective on $H^{0,1}(A')$, because $z' \wedge \omega'_+ = z'_- \wedge \omega'_+$ and $z' \wedge \omega'_- = z'_+ \wedge \omega'_-$ lie in independent weight spaces and vanish only with $z'_\pm$; dually $d' \mapsto d' \lrcorner R$ is injective on $H^0(T_{A'})$. If the nonzero $R^{(k)}$ has $k \geq 1$, hard Lefschetz makes θ_A^{k-1} injective on $H^1(A)$, since $k-1 \leq 2m-1$, so the μ' -term vanishes only if $\sum_s (d_s \lrcorner \theta_A) \otimes z'_s = 0$, that is $\mu' = 0$, and then $\mu = 0$. If $k = 0$ the μ -term

is $(\text{id} \otimes (\cdot \lrcorner R^{(0)}))(\mu)$, which vanishes only if $\mu = 0$. Either way the mixed part is zero, against the assumption.

The bound is Proposition 6.5(iii): the common annihilator of $\text{ch}\mathcal{E}'$ and $\text{ch}\mathcal{E}''$ in $T\mathcal{D}_{n,n}$ lies in that of $\text{ch}\mathcal{E}'$, so $\dim \text{Ext}^2 \geq \dim HT^2 - m^2 - m'^2 = r(\gamma) + n^2 - m^2 - m'^2$, and $n = m + m'$. $\square$

Proposition 6.7 (Secant and line-bundle pieces are never orthogonal). *For every realisation of Theorem 6.3 there are an index i and a power L^l among the pieces with $\chi(F_i, L^l) \neq 0$. Hence $\text{Ext}^*(F_i, L^l) \neq 0$ and $\text{Ext}^*(L^l, F_i) \neq 0$, and the hypothesis of the last sentence of Theorem 6.6 fails.*

Proof. The Todd class of A is 1, $\omega_i \theta_A = 0$ and $\int \theta_A^4 = 24$, so Riemann–Roch gives $\chi(F_i, L^l) = \int p_i(-\theta_A) e^{l\theta_A} = P_i(l)$ with

$$P_i(l) = c_{i0}l^4 - 4c_{i1}l^3 + 12c_{i2}l^2 - 24c_{i3}l + 24c_{i4}, \quad (6.5)$$

a nonzero polynomial because $p_i \neq 0$. Expand $p_i = \sum_l b_{il} e^{l\theta_A}$, $0 \leq l \leq 4$, modulo θ_A^5 , which is possible since the five exponentials span $\mathbb{Q}[\theta_A]/(\theta_A^5)$, and call the set S_i of l with $b_{il} \neq 0$ the support of p_i . Then $P_i(s) = \sum_{l \in S_i} b_{il}(s-l)^4$. If P_i vanished on all of S_i with $|S_i| \geq 2$, the vector $(b_{il})_{l \in S_i}$ would lie in the kernel of the symmetric matrix $((s-l)^4)_{s,l \in S_i}$. That matrix is nonsingular: for two elements its determinant is $-(s-l)^8$, for three it is $2\prod(s-l)^4$, for the five four-element subsets of $\{0, \dots, 4\}$ it is 29952, 1032192, 6054912, 1032192, 29952, and for the whole set it is 7962624. So a support with at least two elements contains some l with $P_i(l) \neq 0$.

Step 2 of Theorem 6.3 uses the pieces $L^l \boxtimes G'_j$ for every l in the support of $q_j = \sum_i (Ma_{ij}/N_i)p_i$, and the last step uses $L^k \boxtimes L'^k$ for three values of k whose net multiplicities are nonzero. If p_1 and p_2 are independent, q_1 and q_2 span the same plane because a is invertible, so the union of their supports is $S_1 \cup S_2$; if p_2 is a multiple of p_1 , both q_j are multiples of p_1 , not both zero, with the same support. In either case every $l \in S_1 \cup S_2$ gives a piece $L^l \boxtimes G'_j$. If $|S_1| \geq 2$ or $|S_2| \geq 2$ we are done. Otherwise $p_i = b_i e^{l_i \theta_A}$ with $b_i \neq 0$. If $l_1 \neq l_2$ then $L^{l_2} \boxtimes G'_j$ is a piece and $P_1(l_2) = b_1(l_2 - l_1)^4 \neq 0$. If $l_1 = l_2 = l$ the residue of Steps 1 to 3 lies in $e^{l\theta_A} \otimes \mathbb{Q}[\theta']$, so at least two of the three final exponents k differ from l , and $P_i(k) = b_i(k-l)^4 \neq 0$. The vanishing of $\text{Ext}^*(F_i, L^l)$ would force $\chi(F_i, L^l) = 0$; Serre duality gives $\text{Ext}^*(L^l, F_i) \neq 0$ as well. $\square$

Theorem 6.6 is therefore a conditional statement whose condition never holds: Proposition 6.5(iii) cannot be applied to the splitting between secant pieces and line-bundle pieces, and no bound on $\dim \text{Ext}^2$ of a convolution follows from the Chern characters alone. The hypothesis of part (ii) can fail with line bundles only. The cone of a section of $L^l \boxtimes L'^{k'-k}$, viewed as a map $L^l \boxtimes L'^k \rightarrow L^l \boxtimes L'^{k'}$ with $k' > k$, is a twisted complex of two pieces in which the Serre dual of the section, a class of degree one, enters the diagonal block of the piece $L^l \boxtimes L'^k$; the only terms that can enter a diagonal block of a line-bundle piece are of this form, with the same power on A . Part (ii) applies to every piece $L^l \boxtimes L'^k$ with $l \neq k$ whose block receives no such term, which is how Step 3 of Theorem 6.3 uses it.

6.5. Where the product object stops

Every test that depends on $\text{ch}(\mathcal{E})$ alone is passed: polarised shape, $N \neq 0$, Hankel rank 3, first-order Hodge locus equal to $T\mathcal{D}_{n,n}$, lower bound 57 or 104. The criterion then asks for $\dim \text{Ext}^2(\mathcal{E}, \mathcal{E}) = r$, Proposition 6.4 shows that no direct sum achieves it, and Proposition 6.5 says what a convolution of the same pieces must do: kill every strictly upper triangular class of degree two, carry classes of degree one from the lower triangle into the diagonal blocks of every piece with mismatched slopes, and, since by Proposition 6.7 the secant pieces and the line-bundle pieces are never orthogonal, carry classes between the two groups in a way that cancels the mixed directions, which by Theorem 6.6 the secant pieces alone leave uncanceled. All three demands are statements about the groups $\text{Ext}^*(F_1, F_2)$, $\text{Ext}^*(F_i, L^l)$, $\text{Ext}^*(L^l, F_i)$ and their analogues on A' . Those groups are not computed in [49, 50, 51], where the only object with known Ext groups on a very general Weil fourfold of trivial discriminant is the twisted transform $\Phi(F \boxtimes F^\vee)$ with $\dim \text{Ext}^k = 1, 8, 18, 8, 1$, whose Brauer class is half the Poincaré class and which therefore cannot be mixed with untwisted line bundles; and we have not computed them. Riemann–Roch fixes $\chi(F_i, L^l) = P_i(l)$ and the Künneth formula fixes everything on the line-bundle side, so the unknown is exactly the thirteen graded groups $\text{Ext}^*(F_i, F_j)$ and $\text{Ext}^*(F_i, L^l)$. This

is the exact stopping point: the polarised criterion at a product point is a finite question about the Ext algebra of the secant sheaves, and Proposition 6.5 lists what the answer must contain for the route to stay open. The analogous question on E_0^6 , for line bundles, closed negatively with $\dim \text{Ext}^2 \geq 69$ [19, Thm. 18.35]; that is an analogy and not a proof of the answer here.

At a point of complex multiplication the thirteen groups are known. We record them, since they turn the stopping point into a statement about one explicit variety.

Lemma 6.8 (The Ext groups at a point of complex multiplication). *Let $K = \mathbb{Q}(i)$, $E_0 = \mathbb{C}/\mathbb{Z}[i]$, and let $A_0 = E_0^6$ carry the K -action by i on three factors and by $-i$ on the other three; it is a member of $\mathcal{D}_{3,3}$ for every discriminant, since the K -hermitian forms $\sum_{j \leq 3} a_j z_j \bar{w}_j - \sum_{j > 3} a_j z_j \bar{w}_j$ with $a_j > 0$ are polarisations of every discriminant class. Let L be ample on A_0 with class θ , let $Z, Z' \subseteq A_0$ be abelian subvarieties of dimension three with $Z \cap Z'$ finite, and put $\chi_Z = (\theta|_Z)^3/6$, a positive integer.*

- (i) *Every Hodge class on A_0 is a polynomial in divisor classes, and every product of three divisor classes is a rational combination of classes $[Z]$ of abelian threefolds. The Weil classes are such combinations, and no Weil class is a multiple of a single $[Z]$. For every Z , $\text{ch}(\mathcal{O}_Z) = [Z]$ and $\text{ch}(\mathcal{O}_Z \otimes L^a) = [Z]e^{a\theta}$.*
- (ii) *$\text{Ext}^k(\mathcal{O}_Z \otimes L^a, \mathcal{O}_{Z'} \otimes L^b) = 0$ for $k \neq 3$, and $\dim \text{Ext}^3 = |Z \cap Z'| = (Z \cdot Z')$, for all a, b .*
- (iii) *The groups $\text{Ext}^k(\mathcal{O}_Z \otimes L^a, \mathcal{O}_Z \otimes L^b)$ are given by (6.6) below, with $N_Z \cong \mathcal{O}_Z^3$. For $a = b$ it has dimension $\binom{6}{k}$ and the characteristic map $HT^2(A_0) \rightarrow \text{Ext}^2$ is onto; for $b > a$ it lives in degrees 0 to 3, of dimension $\binom{3}{k}(b-a)^3\chi_Z$; for $b < a$ in degrees 3 to 6, of dimension $\binom{3}{k-3}(a-b)^3\chi_Z$.*
- (iv) *For $b > a$, $\text{Ext}^k(\mathcal{O}_Z \otimes L^a, L^b)$ is concentrated in degree 3 and $\text{Ext}^k(L^a, \mathcal{O}_Z \otimes L^b)$ in degree 0, both of dimension $(b-a)^3\chi_Z$; for $b < a$ the first is concentrated in degree 6 and the second in degree 3, both of dimension $(a-b)^3\chi_Z$; for $a = b$ the dimensions are $\binom{3}{k-3}$ and $\binom{3}{k}$.*
- (v) *For $b > a$, $\text{Ext}^k(L^a, L^b) = H^k(A_0, L^{b-a})$ is concentrated in degree 0, of dimension $(b-a)^6\theta^6/720$; for $b < a$ in degree 6; for $a = b$ it is $H^k(\mathcal{O}_{A_0})$, of dimension $\binom{6}{k}$.*

Proof. (i) The Hodge group of A_0 is the norm-one torus K^1 acting by z on V_+ and $\bar{z}$ on V_- , so over $\mathbb{C}$ the Hodge classes of degree $2k$ are $\wedge^k V_+ \otimes \wedge^k V_-$, which is spanned by the products of k elements of $V_+ \otimes V_- = \text{Hdg}^2(A_0)_{\mathbb{C}}$; since $\text{Hdg}^2 = \text{NS}_{\mathbb{Q}}$ by Lefschetz, the $\mathbb{Q}$ -span of the products of k divisor classes has full dimension $\binom{6}{k}^2$. The classes $f^*[0]$, $f \in \text{Hom}(E_0^6, E_0) = \mathbb{Z}[i]^6$, are the rank-one hermitian forms $v\bar{v}^T$ and span $\text{NS}_{\mathbb{Q}}$. For f_1, f_2, f_3 independent over K the map $f = (f_1, f_2, f_3): A_0 \rightarrow E_0^3$ is a surjective homomorphism, hence flat, and $f_1^*[0]f_2^*[0]f_3^*[0] = f^*[0] = [f^{-1}(0)]$ is a positive multiple of the class of the abelian threefold $\ker(f)^\circ$; for dependent f_i the product is zero. A Weil class w satisfies $w\theta = 0$, while $[Z]\theta = i_*(\theta|_Z) \neq 0$ because $(\theta|_Z)^3 > 0$. Since Z is a translate of a subgroup, N_Z is trivial and $\text{td}(N_Z) = 1$, so Grothendieck–Riemann–Roch gives $\text{ch}(i_*\mathcal{O}_Z) = i_*(1) = [Z]$ and $\text{ch}(\mathcal{O}_Z \otimes L^a) = [Z]e^{a\theta}$.

(ii) Since $\dim Z \cap Z' = 0$ and the tangent spaces are translation invariant, $T_Z \oplus T_{Z'} = T_{A_0}$ at every point of $Z \cap Z'$, so Z and Z' are locally transverse complete intersections and $Z \cap Z'$ is reduced. In local coordinates with $Z = \{x_1 = x_2 = x_3 = 0\}$ and $Z' = \{y_1 = y_2 = y_3 = 0\}$, the Koszul resolution of $\mathcal{O}_Z$ gives $\mathcal{E}xt^k(\mathcal{O}_Z, \mathcal{O}_{Z'})$ as the cohomology of the Koszul complex of x_1, x_2, x_3 on $\mathcal{O}_{Z'}$, a regular sequence there; so $\mathcal{E}xt^k = 0$ for $k \neq 3$ and $\mathcal{E}xt^3 = \mathcal{O}_{Z \cap Z'}$. The local-to-global spectral sequence has one row, $H^{k-3}(Z \cap Z', \mathcal{O})$, and twisting by line bundles does not change a sheaf on a finite reduced scheme; the intersection number equals $|Z \cap Z'|$ by transversality.

(iii) Let $\pi: A_0 \rightarrow A_0/Z$ be the quotient. The normal bundle N_Z is the restriction of the trivial bundle $\pi^*T_{A_0/Z}$, so it extends to all of A_0 , and by the theorem of Arinkin and Căldăraru [6] the derived restriction $i^*i_*\mathcal{O}_Z$ is formal, $i^*i_*\mathcal{O}_Z \cong \bigoplus_p \wedge^p N_Z^\vee[p]$. Hence

$$\text{Ext}^k(\mathcal{O}_Z \otimes L^a, \mathcal{O}_Z \otimes L^b) = \text{Ext}_Z^k(i^*i_*\mathcal{O}_Z, L^{b-a}|_Z) = \bigoplus_{p+q=k} H^q(Z, L^{b-a}|_Z) \otimes \wedge^p N_Z. \quad (6.6)$$

For $a = b$ the dimension is $\sum_{p+q=k} \binom{3}{q} \binom{3}{p} = \binom{6}{k}$; for $b \neq a$ the cohomology of $L^{b-a}|_Z$ on the abelian threefold Z sits in degree 0 or 3 according to the sign of $b - a$, with dimension $|b - a|^3\chi_Z$ by Riemann–Roch, which gives the two ranges of degrees and their dimensions. The characteristic map sends

$H^2(\mathcal{O}_{A_0})$ onto $H^2(\mathcal{O}_Z)$ by restriction, $H^1(T_{A_0}) = H^1(\mathcal{O}) \otimes T_0$ onto $H^1(\mathcal{O}_Z) \otimes N_{Z,0}$ through the quotient $T_0 \rightarrow N_{Z,0}$, and $H^0(\wedge^2 T_{A_0}) = \wedge^2 T_0$ onto $\wedge^2 N_{Z,0}$; all three are surjective, the first because $H^*(A_0) \rightarrow H^*(Z)$ is onto for an abelian subvariety.

(iv) By Grothendieck duality for the regular embedding, whose relative dualising sheaf $\det N_Z$ is trivial, and by adjunction,

$$\begin{aligned} \mathrm{Ext}^k(\mathcal{O}_Z \otimes L^a, L^b) &= H^{k-3}(Z, L^{b-a}|_Z), \\ \mathrm{Ext}^k(L^a, \mathcal{O}_Z \otimes L^b) &= H^k(Z, L^{b-a}|_Z). \end{aligned}$$

On Z the cohomology of an ample bundle is concentrated in degree 0 and that of an anti-ample one in degree 3, of dimension $\chi = c_1^3/6$ by Riemann–Roch, and $c_1(L^m|_Z)^3 = m^3(\theta|_Z)^3$; for $m = 0$ one gets $H^*(\mathcal{O}_Z)$.

(v) is the same statement on A_0 , with $\chi(L^m) = m^6 \theta^6 / 720$. $\square$

Lemma 6.9 (Threefolds that meet along a positive-dimensional set). *Let $Z, Z' \subseteq A_0$ be distinct abelian threefolds, let $W = Z \cap Z'$, of dimension $d \in \{0, 1, 2\}$, and put $c = 3 - d$ and $m = b - a$.*

- (i) $\mathcal{H}om(\mathcal{O}_Z, \mathcal{O}_{Z'}) = 0$, and $\mathcal{E}xt^q(\mathcal{O}_Z, \mathcal{O}_{Z'}) \cong \mathcal{O}_W^{\binom{d}{q-c}}$ for $c \leq q \leq 3$, zero otherwise.
- (ii) $\mathrm{Ext}^k(\mathcal{O}_Z \otimes L^a, \mathcal{O}_{Z'} \otimes L^b) = 0$ unless $c \leq k \leq 3$ when $m > 0$, $3 \leq k \leq 3 + d$ when $m < 0$, and $c \leq k \leq 3 + d$ when $m = 0$. In particular the group vanishes in degrees $k \leq 2$ when $m < 0$, and $\mathrm{Ext}^1 \neq 0$ only when $d = 2$ and $m \geq 0$, where $\mathrm{Ext}^1 = H^0(W, L^m|_W)$.
- (iii) For the threefolds $Z_\zeta, Z_{\zeta'}$ of Appendix A, d is the number of indices j with $\zeta_j = \zeta'_j$, W is the product of the shared curves with $\prod_{\zeta_j \neq \zeta'_j} |\zeta_j - \zeta'_j|^2$ points, and for $d = 2$ and $m = 0$, $\dim \mathrm{Ext}^1 = |\zeta_l - \zeta'_l|^2 \in \{2, 4\}$, l being the index at which they differ.

Proof. (i) W is a closed subgroup scheme of A_0 , reduced in characteristic zero, so a disjoint union of translates of an abelian variety of dimension d , and it is nonempty since Z and Z' contain the origin. At a point of W choose translation-invariant linear coordinates in which $Z = \{x_1 = x_2 = x_3 = 0\}$. The restrictions of x_1, x_2, x_3 to Z' are linear forms cutting out W , of codimension c in Z' ; after a linear change $x_1, \dots, x_c$ restrict to a regular sequence on Z' with zero locus W and $x_{c+1}, \dots, x_3$ restrict to zero. The Koszul resolution of $\mathcal{O}_Z$ computes $\mathcal{E}xt^q(\mathcal{O}_Z, \mathcal{O}_{Z'})$ as the cohomology of the Koszul complex of these restrictions on $\mathcal{O}_{Z'}$, the tensor product of the Koszul complex of the regular sequence, with cohomology $\mathcal{O}_W$ in degree c , and of d complexes $\mathcal{O}_{Z'} \xrightarrow{0} \mathcal{O}_{Z'}$; so $\mathcal{E}xt^q = \wedge^{q-c}(\mathcal{O}_W^d)$. Since $c \geq 1$, $\mathcal{H}om = 0$.

(ii) The local-to-global spectral sequence has $E_2^{p,q} = H^p(W, \mathcal{E}xt^q \otimes L^m|_W)$ with $0 \leq p \leq d$ and $c \leq q \leq 3$. On each component of W the bundle $L^m|_W$ is ample for $m > 0$, with cohomology in degree 0 only, anti-ample for $m < 0$, with cohomology in degree d only, and trivial for $m = 0$. This gives the three ranges. For $k = 1$ the only possible term is $E_2^{0,1}$, nonzero only when $c = 1$ and $m \geq 0$, and it survives to E_∞ because $E_2^{1,0} = E_2^{2,0} = 0$.

(iii) In the coordinates of Appendix A, $Z_\zeta = \{u_j + \zeta_j v_j = 0, j = 1, 2, 3\}$; a shared index gives a form vanishing on $Z_{\zeta'}$, and an index with $\zeta_j \neq \zeta'_j$ gives the form $(\zeta_j - \zeta'_j)v_j$ on $Z_{\zeta'}$, whose zero locus in $C_{\zeta'_j}$ has $|\zeta_j - \zeta'_j|^2$ points by Lemma A.1(ii). So W is as stated, and $\mathrm{Ext}^1 = H^0(W, \mathcal{O}_W)$ has dimension the number of components of W . $\square$

Theorem 6.1 applies at A_0 with the same number $r = 57$: the rank of the contraction of HT^2 into $\mathrm{ch}(\mathcal{E})$ is invariant under $\mathrm{SU}(V, H)(\mathbb{R})$, which fixes the Weil classes and θ and acts transitively on $\mathcal{D}_{3,3}$, so it is the rank computed at a very general point, and the argument of Buchweitz and Flenner is local on the base. An object $\mathcal{E}$ with $\dim \mathrm{Ext}^2(\mathcal{E}, \mathcal{E}) = 57$ at A_0 would therefore deform along all of $\mathcal{D}_{3,3}$ and prove the Weil classes algebraic on every member with full Hodge group. By (i) every Chern character of polarised shape is realised at A_0 by a convolution of pieces $\mathcal{O}_{Z_a} \otimes L^{k_a}[s_a]$ and $L^k[t]$, with at least two subvariety pieces, since w is not a multiple of one $[Z]$, and at least three powers of L , since the Hankel rank is three; by (ii) to (v) every group between two pieces is known. Appendix A writes the classes, the intersection numbers and every dimension of the lemma in coordinates for the threefolds that are

products of elliptic curves of the three Weil surfaces, and exhibits the signed sum of thirty-two of them equal to $32(\alpha + \bar{\alpha})$. What the lemma decides is how the diagonal classes can disappear.

Proposition 6.10 (No cancellation in two steps). *Let $\mathcal{E}$ be a convolution of pieces $F_a[s_a]$ of the types of Lemma 6.8, whose threefolds meet pairwise in finite sets, and no two of which are isomorphic sheaves placed at shifts differing by at most one. A class of $\text{Ext}^2(F_a, F_a)$ is cancelled in the spectral sequence of the convolution only through a closed chain $a = a_0, a_1, \dots, a_m = a$ of pieces with $\text{Ext}^{k_i}(F_{a_{i-1}}, F_{a_i}) \neq 0$ and $\sum_i k_i = m$, and no such chain has $m = 2$. The chains with $m = 3$ are exactly those through three twists $k < k' < k''$ of one threefold, $\mathcal{O}_Z \otimes L^k[s] \rightarrow \mathcal{O}_Z \otimes L^{k'}[s+1] \rightarrow \mathcal{O}_Z \otimes L^{k''}[s+2] \rightarrow \mathcal{O}_Z \otimes L^k[s]$, with degrees 0, 0, 3. Hence every cancellation among the diagonal classes of degree two is a Massey product of length at least three.*

Proof. The Maurer–Cartan components of the twisted complex and the class η whose differential is computed all have total degree one, that is, a component in $\text{Ext}^k(F_a, F_b)$ has $k + s_b - s_a = 1$. A product of m such components along a closed chain lands in $\text{Ext}^{\sum k_i - m + 2}(F_a, F_a)$, since the shifts cancel around the chain and the m -fold product lowers the degree by $m - 2$; so it reaches degree two exactly when $\sum k_i = m$. For $m = 2$ both components have degree at most two. Between two pieces that are not isomorphic sheaves, Lemma 6.8 gives the degrees: 3 and 3 between $\mathcal{O}_Z$ and $\mathcal{O}_{Z'}$; 0 to 3 one way and 3 to 6 the other between two twists of one $\mathcal{O}_Z$; 3 and 3 between $\mathcal{O}_Z \otimes L^a$ and L^b with $a < b$, 6 and 0 with $a > b$, and 3 to 6 against 0 to 3 for $a = b$; 0 and 6 between two powers of L . In every case the two degrees add up to at least three. Between two copies of one sheaf at shifts s, s' the two degrees are $1 - s' + s$ and $1 - s + s'$, which are both at most two only when $|s - s'| \leq 1$, the case excluded. For $m = 3$ the edges of degree at most two are, by the same list, the edges that raise the twist and the edges $L^a \rightarrow \mathcal{O}_Z \otimes L^a$; a closed chain cannot consist of them alone, so one edge has degree at least three and the other two have degree zero. Two edges of degree zero from a line bundle lead to a power of L or to a subvariety piece of strictly larger twist, from which every edge back to the line bundle has degree six; from $\mathcal{O}_Z \otimes L^k$ they lead only to $\mathcal{O}_Z \otimes L^{k''}$ with $k'' > k$ through an intermediate twist k' , and the edge back has degree three, through $H^3(Z, L^{k-k''}|_Z) \otimes \wedge^0 N_Z$. These are the chains displayed, and their three groups are nonzero. $\square$

For line-bundle pieces alone the same bookkeeping holds on E_0^6 , and there the Massey products of length four are the ones that act, with the result $\dim \text{Ext}^2 \geq 69$ [19, Thm. 18.35]. With subvariety pieces and three powers of L no closed chain of any length exists, and the criterion fails for a reason that involves no product at all.

Proposition 6.11 (Convolutions with three line bundles fail). *Let $\mathcal{E}$ be a convolution at A_0 of pieces $\mathcal{O}_{Z_a} \otimes L^{k_a}[s_a]$ and $L^{l_b}[t_b]$ in which no sheaf occurs twice, the threefolds Z_a are pairwise distinct, there are at most three line bundles, and $\text{Ext}^1(\mathcal{O}_{Z_a} \otimes L^{k_a}, \mathcal{O}_{Z_b} \otimes L^{k_b}) = 0$ whenever $s_a = s_b$. Suppose $\text{ch}(\mathcal{E})$ has the polarised shape. Then no term of the twisted differential carries a class of total degree one into a diagonal block, and*

$$\dim \text{Ext}^2(\mathcal{E}, \mathcal{E}) > r(\text{ch} \mathcal{E}) :$$

the criterion fails for $\mathcal{E}$. The hypothesis on Ext^1 holds when the threefolds meet pairwise in finite sets, and for the thirty-two threefolds Z_ζ of Appendix A placed at one twist with shifts of the parity of $\zeta_1 \zeta_2 \zeta_3$.

Proof. Step 1, no closed chain. As in the proof of Proposition 6.10, a term of the differential landing in a diagonal block runs along a closed chain $a_0, a_1, \dots, a_m = a_0$ of pieces with a nonzero component in $\text{Ext}^{k_i}(F_{a_{i-1}}, F_{a_i})$, $k_i = 1 + s_{a_{i-1}} - s_{a_i}$, so that $\sum_i k_i = m$; the pieces of the chain are distinct, because the differential is strictly upper triangular in the one-sided order. We show that no such chain exists. Between two distinct threefold pieces $\text{Hom} = 0$ by Lemma 6.9(i), and a component of degree one needs $s_a = s_b$, which the hypothesis excludes; so every edge between two threefold pieces has degree at least two. By Lemma 6.8(iv) and (v), an edge $L^l \rightarrow \mathcal{O}_Z \otimes L^k$ has degree 0 when $k > l$, 3 when $k < l$ and 0 to 3 when $k = l$, so a degree at most one forces $k \geq l$; an edge $\mathcal{O}_Z \otimes L^k \rightarrow L^l$ has degree 3 when $l > k$, 6 when $l < k$ and 3 to 6 when $l = k$, so a degree at most four forces $l \geq k$; and an

edge between two line bundles has degree 0 when it raises the twist and 6 when it lowers it. A chain of threefold pieces alone has $\sum_i k_i \geq 2m > m$. Otherwise the chain visits $\ell \leq 3$ line-bundle pieces; between consecutive visits it either passes directly, by an edge that raises the twist (u such edges in all) or lowers it (w in all), or runs through $r \geq 1$ threefold pieces, entered with degree e and left with degree x ; let R be the number of runs. Then $R + u + w = \ell$, $m = \ell + \sum_{\text{runs}} r$, a run contributes at least $e + x + 2(r - 1)$ to $\sum_i k_i$, and so

$$\sum_i k_i - m \geq \sum_{\text{runs}} (e + x + r - 3) + 5w - u,$$

where each bracket is at least $0 + 3 + 1 - 3 = 1$. If $w \geq 1$ then $u \leq \ell - R - w \leq 2 - R$ and the right side is at least $R + 5 - (2 - R) > 0$. If $w = 0$ then $u = \ell - R \leq 3 - R$ and the right side is at least $2R - 3$, so $R \leq 1$; $R = 0$ is a cycle of edges that all raise the twist, which cannot close; and for $R = 1$ the single bracket must equal u , so $r \leq 2$ and $e + x \leq 5 - r$, that is $(e, x) \in \{(0, 3), (0, 4), (1, 3)\}$ for $r = 1$ and $(e, x) = (0, 3)$ for $r = 2$. In both cases the run enters the first threefold, of twist k , from L^l with $k \geq l$, leaves the last one, of twist k' , for $L^{l'}$ with $l' \geq k'$, and returns from $L^{l'}$ to L^l by $u \geq 1$ edges that raise the twist, so $l' < l$ and $k' < k$. For $r = 1$ this reads $k < k$. For $r = 2$ the internal edge goes from the twist k to the strictly smaller twist k' and has degree two, which Lemma 6.9(ii) forbids. So there is no closed chain.

Step 2, the class $\text{ev}_{\mathcal{E}}(v)$. By Step 1 the diagonal components of a class in $\text{Ext}^2(\mathcal{E}, \mathcal{E})$ are well defined, as in Proposition 6.5(ii), and for $v \in H^1(A_0, T_{A_0})$ the diagonal components of $\text{ev}_{\mathcal{E}}(v)$ are the classes $\text{ev}_{F_a}(v) = v_{\perp} \text{At}(F_a)$ of the pieces, the Atiyah class of the twisted complex being block upper triangular with diagonal blocks $\text{At}(F_a)$. For a piece $F = \mathcal{O}_Z \otimes L^k$ the semiregularity map sends $\text{ev}_F(v)$ to $v_{\perp} \text{ch}(F) = v_{\perp}[Z] + k v_{\perp}([Z]\theta) + \dots$, whose component of lowest degree is $v_{\perp}[Z]$. There is at least one threefold piece, since the Weil component of $\text{ch}(\mathcal{E})$ is nonzero and powers of L contribute none. Step 3 gives $v \in T\mathcal{D}_{3,3}$ with $v_{\perp}[Z_a] \neq 0$ for such a piece, so $\text{ev}_{\mathcal{E}}(v) \neq 0$; and $\sigma_{\mathcal{E}}(\text{ev}_{\mathcal{E}}(v)) = v_{\perp} \text{ch}(\mathcal{E}) = 0$, because $T\mathcal{D}_{3,3}$ annihilates θ and the Weil plane. So $\sigma_{\mathcal{E}}$ has a nonzero kernel, while its rank is at least the rank $r(\text{ch}\mathcal{E})$ of $\sigma_{\mathcal{E}} \circ \text{ev}_{\mathcal{E}}$; hence $\dim \text{Ext}^2(\mathcal{E}, \mathcal{E}) \geq r(\text{ch}\mathcal{E}) + 1$.

Step 3, no $[Z]$ is annihilated by $T\mathcal{D}_{3,3}$. The tangent space $T\mathcal{D}_{3,3} \subseteq H^1(A_0, T_{A_0}) = \text{Hom}(V^{1,0}, V^{0,1})$ consists of the K -linear maps that are skew for the polarisation; it is the part $\mathfrak{g}^{-1,1}$ of the Lie algebra $\mathfrak{g}$ of $\text{SU}(V, H)$ in the Hodge structure that A_0 induces on it, and the contraction $v_{\perp}$ on $H^*(A_0) = \bigwedge^* V_{\mathbb{C}}$ is the derivation by which $\mathfrak{g}_{\mathbb{C}}$ acts. Suppose $v_{\perp}[Z] = 0$ for all $v \in \mathfrak{g}^{-1,1}$. Since $[Z]$ is real it is annihilated by the conjugate space $\mathfrak{g}^{1,-1}$ as well, and $\mathfrak{g}^{-1,1} \oplus \mathfrak{g}^{1,-1}$, the off-diagonal blocks of $\mathfrak{sl}(V_+ \oplus V_-)$, generates $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}_6$; so $[Z]$ is invariant under the connected group $\text{SU}(V, H)$, in particular under its rational points. Every $g \in \text{SU}(V, H)(\mathbb{Q}) \subseteq \text{GL}_K(V) = \text{End}^0(A_0)^{\times}$ is a quasi-isogeny of A_0 , because E_0 has complex multiplication by K ; for an integer n with $\varphi = ng$ an isogeny, $\varphi^*[Z] = [Z]$ gives $\varphi_*[Z] = \varphi_*\varphi^*[Z] = \deg(\varphi)[Z]$, so the class of the abelian threefold $\varphi(Z)$, whose tangent space is gT_Z , is a positive multiple of $[Z]$. The group $\text{SU}(V, H)(\mathbb{C}) \cong \text{SL}_6(\mathbb{C})$ acts transitively on the three-dimensional subspaces of V_+ , which determine the K -subspaces of V , the condition $gT_Z \cap T_Z = 0$ is Zariski open and nonempty in $\mathfrak{g}$, and $\text{SU}(V, H)(\mathbb{Q})$ is Zariski dense in the connected group $\text{SU}(V, H)$ [20, Cor. 18.3]. So some g makes $Z \cap \varphi(Z)$ finite; it contains the origin, and $(Z \cdot \varphi(Z)) = |Z \cap \varphi(Z)| \geq 1$ by Lemma 6.8(ii). But $(Z \cdot \varphi(Z))$ is a positive multiple of $[Z]^2$, and $[Z]^2$ is a multiple of $f^*([0]^2) = 0$ for the map $f: A_0 \rightarrow E_0^3$ of Lemma 6.8(i), since $[0]^2 = 0$ on E_0^3 . This contradiction proves the claim, and the proposition.

For the last sentence of the statement: when the threefolds meet in finite sets every group between two of them sits in degree three by Lemma 6.8(ii); and two of the Z_{ζ} at the same shift have the same value of $\zeta_1 \zeta_2 \zeta_3$, hence differ in at least two coordinates, so $d \leq 1$ and $\text{Ext}^1 = 0$ by Lemma 6.9(ii). $\square$

Proposition 6.11 settles the configuration that Appendix A writes down, the thirty-two threefolds Z_{ζ} at one twist with three powers of L : no convolution of these thirty-five pieces meets the criterion. By Lemma 6.8(i) every object meeting the criterion at A_0 is a convolution of pieces $\mathcal{O}_Z \otimes L^k[s]$ and $L^l[t]$, so what remains are the convolutions with four or more line bundles, with a sheaf at two shifts, with a threefold at several twists, or with two threefold pieces at the same shift meeting along an abelian surface, as two of the Z_{ζ} with two equal coordinates do. Closed chains exist among them. With the threefolds at twist 0 and the line bundles of positive twist, every edge from a line bundle into a threefold piece and every edge out of one has degree three, so the shortest closed chain through a threefold piece

runs through five line bundles at five consecutive shifts, $L^{l_1}[\sigma - 2] \rightarrow L^{l_2}[\sigma - 1] \rightarrow L^{l_3}[\sigma] \rightarrow L^{l_4}[\sigma + 1] \rightarrow L^{l_5}[\sigma + 2] \rightarrow \mathcal{O}_Z[\sigma] \rightarrow L^{l_1}[\sigma - 2]$ with $0 < l_1 \leq \dots \leq l_5$ and degrees $0, 0, 0, 0, 3, 3$; one threefold at three twists carries the chain of Proposition 6.10; and two pieces $\mathcal{O}_{Z_\zeta} \otimes L^k[s]$, $\mathcal{O}_{Z_{\zeta'}} \otimes L^k[s]$ with two equal coordinates carry the chain of length two with degrees $1, 1$, through the groups $H^0(W, \mathcal{O}_W)$ of Lemma 6.9(iii). The question at the stopping point is whether the diagonal components $(\text{ev}_{F_a}(v))_a$, $v \in T\mathcal{D}_{3,3}$, lie in the image of the products along such chains; the Hodge locus of the obstruction, the only other way the criterion could fail, is settled by Theorem 6.1 itself. The next subsection answers the question twice without computing a product: by counting the blocks that no chain reaches, and by observing that one threefold piece outside every closed chain decides it.

6.6. Counting the remaining configurations at E_0^6

The four kinds of convolution left open by Proposition 6.11 can be tested by a bound that uses no product at all. Fix pieces F_a with shifts s_a and a minimal A_∞ -model of their full subcategory, so that $\text{Hom}^\bullet(\mathcal{E}, \mathcal{E})$ for a convolution $\mathcal{E}$ of the F_a is the direct sum of the blocks $\text{Ext}^k(F_a, F_b)$ placed in total degree $k + s_b - s_a$, with the differential $x \mapsto \sum_{p,q} m_{p+q+1}(\delta, \dots, \delta, x, \delta, \dots, \delta)$ built from the twisting cochain δ , whose components lie in the blocks of total degree one. Call a block $\text{Ext}^k(F_a, F_b)$ of total degree two *isolated* when no block of total degree one maps to it and it maps to no block of total degree three along such a term: that is, there is no chain $a \rightarrow \dots \rightarrow a'$ and $b' \rightarrow \dots \rightarrow b$ of pieces with every step a nonzero block of total degree one and $(a', b') \neq (a, b)$ carrying a block of total degree one, and no such chain $a' \rightarrow \dots \rightarrow a$, $b \rightarrow \dots \rightarrow b'$ with (a', b') carrying a block of total degree three.

Lemma 6.12 (Isolated blocks). *For every twisting cochain δ on the pieces F_a , the isolated blocks of total degree two inject into $\text{Ext}^2(\mathcal{E}, \mathcal{E})$; so $\dim \text{Ext}^2(\mathcal{E}, \mathcal{E})$ is at least the sum of their dimensions.*

Proof. An element of an isolated block is a cycle, since every term of its differential lands in a block of total degree three reached from it along a chain of components of δ , and there is none; and it is not a boundary, since every term of the differential of a class of total degree one lands in a block reached from that class along such a chain, and none reaches an isolated block. The blocks are direct summands, so the cycles of the isolated blocks meet the boundaries only in zero. $\square$

The bound depends only on the Ext groups of Lemmas 6.8 and 6.9 and on the shifts, not on δ , so it can be computed over all configurations of a family at once. We did this for the configurations of the four kinds within a window of twists 1 to 4 and shifts -3 to 3 around the thirty-five pieces, keeping those whose Chern character has the polarised shape, and recorded for each the sums C^1, C^2, C^3 of the block dimensions in total degrees one to three, the Euler characteristic $C^2 - C^1 - C^3$ and the isolated blocks (Table 3). The programs are in the archive of the Data availability section.¹³

Proposition 6.13 (What counting decides). *At $A_0 = E_0^6$, with the pieces of Appendix A:*

- (i) *For the thirty-five pieces of Proposition 6.11 with the three line bundles at any shifts in the window, $\dim \text{Ext}^2(\mathcal{E}, \mathcal{E}) \geq 2605$ for every twisting cochain, by isolated blocks alone.*
- (ii) *Adding to them a threefold $Z_{\zeta'}$ meeting Z_ζ along an abelian surface, at the shift of Z_ζ and at a neighbouring shift so that the Chern character keeps the polarised shape, gives $\dim \text{Ext}^2 \geq 5325$: the configuration of two threefold pieces at the same shift meeting along a surface fails for every twisting cochain.*
- (iii) *Adding one further copy of a threefold at a second shift within the window never gives the polarised shape.*

¹³The configurations (a), (c) and (d) of Table 3 and the quotient chains of Proposition 5.21 are the candidates C1, C2, C3 and C7 in the record of Anaxia, the construction engine of the BLLM project [14], run on the stopping points of this paper. The engine encodes those stopping points as typed targets, builds candidate objects from the pieces of Appendix A and the Ext dimensions of Lemmas 6.8 and 6.9, computes for each the Chern character exactly and the two bounds of this subsection, discards the candidates that a finite check refutes, keeping a record of the number that stopped each one, and proposes the survivors; the test of Proposition 6.14 and the enumeration of Remark 6.25 were first run on its graph of the blocks of total degree one, the latter as its record C8. The engine proposes and does not decide: every statement of this paper is decided by the proofs in the text and by the exact programs of Appendix B, which recompute its tables independently. Release 3.0.5 of the engine is the GitHub release given in [14]; the library for the pieces of E_0^6 is in the development branch of its repository (pull request 20, 19 October 2026).

TABLE 3. The isolated-block bound of Lemma 6.12 over the configurations at $A_0 = E_0^6$. “Polarised” counts the configurations of the family in the window whose Chern character has the polarised shape; “least” is the smallest bound among them, with the configuration attaining it. The criterion needs $\dim \text{Ext}^2 = 57$.

family (least bound attained at)	polarised	least
thirty-five pieces, L^1, L^2, L^3 at shifts 0, 0, 0	1	7213
the same at shifts 2, 3, 4 or $-3, -2, -1$	2	2605
(a) four line bundles ($L^1[-2], L^2[-1], L^3[0], L^4[1]$)	2401	0
(b) one threefold at a second shift	0	
(c) one threefold at three twists ($Z_{(1,1,1)} \otimes L^1$ at $[-1], [0]$ and $\otimes L^2$ at $[1], [2]$)	432	0
(d) a second threefold meeting Z_ζ along a surface, at the shift of Z_ζ and a neighbouring one ($Z_{(1,1,-1)}$ at $[0], [1]$)	6	5325

(iv) For four line bundles and for one threefold at three twists the isolated blocks are 0 at the configurations of Table 3 and the Euler characteristic is negative throughout the window: the bound decides nothing there.

Proof. (i), (ii) and (iv) are Lemma 6.12 applied to the block dimensions of Lemmas 6.8 and 6.9, enumerated over the window; (iii) is the enumeration of Chern characters, since a second copy of $\mathcal{O}_{Z_\zeta}$ at a shift of the other parity changes the Weil component by $\pm 2[Z_\zeta]$, which no choice of line bundles compensates. $\square$

Proposition 6.13 closes the configuration (d) of two threefolds at one shift and every placement of the thirty-five pieces. For the configurations (a) and (c) the bound is silent, and a different argument decides them: it suffices that one threefold piece lies outside every closed chain. The configuration (a) with more line bundles is decided in Section 6.7, by hand up to five and by the enumeration of the archive up to seven.

Let Γ be the directed graph whose vertices are the pieces F_a and which has an edge $a \rightarrow b$, for $a \neq b$, when the block of total degree one, $\text{Ext}^{1+s_a-s_b}(F_a, F_b)$, is nonzero. The chains of Lemma 6.12 are its paths, a closed chain in the sense of Proposition 6.10 is one of its directed cycles, and the components of every twisting cochain δ sit on its edges. Call a piece *chain-free* when it lies on no directed cycle of Γ ; this is a property of the pieces and their shifts alone.

Proposition 6.14 (A piece outside every closed chain). *Let $\mathcal{E}$ be a convolution at A_0 of pieces $\mathcal{O}_{Z_a} \otimes L^{k_a}[s_a]$ and $L^{l_b}[t_b]$ whose Chern character has the polarised shape, and suppose that some threefold piece $F_a = \mathcal{O}_{Z_a} \otimes L^{k_a}[s_a]$ is chain-free. Then, for every twisting cochain δ ,*

$$\dim \text{Ext}^2(\mathcal{E}, \mathcal{E}) \geq r(\text{ch } \mathcal{E}) + 1,$$

and the criterion fails for $\mathcal{E}$.

Proof. Let $\eta \in \text{Hom}^1(\mathcal{E}, \mathcal{E})$ have components η_{bc} in the blocks of total degree one, and consider the component of its differential in the block $\text{Ext}^2(F_a, F_a)$. It is a sum of terms $m_{p+q+1}(\delta, \dots, \delta, \eta_{bc}, \delta, \dots, \delta)$ with $p+q \geq 1$, and such a term is nonzero only if its arguments are composable and run from F_a back to F_a : the p components of δ on one side of η_{bc} along edges of Γ from a to b , the q components on the other side along edges from c to a , and $\eta_{bc} \neq 0$. If $b = c$, the $p+q \geq 1$ components of δ form a closed walk of positive length, which is impossible because δ is strictly upper triangular in the one-sided order. If $b \neq c$, then $b \rightarrow c$ is an edge of Γ , and $a \rightarrow \dots \rightarrow b \rightarrow c \rightarrow \dots \rightarrow a$ is a closed walk through a with at least two edges, whose consecutive vertices are distinct; its first return to a , with repeated vertices shortcut, is a directed cycle of Γ through a . Since F_a is chain-free, no term survives: the differential of every element of total degree one has zero component in $\text{Ext}^2(F_a, F_a)$. Hence the map $\pi_a: \text{Ext}^2(\mathcal{E}, \mathcal{E}) \rightarrow \text{Ext}^2(F_a, F_a)$ that reads off the component of a cocycle in that block is well defined on cohomology.

As in the proof of Proposition 6.5(ii), the Atiyah class of the twisted complex is block upper triangular with diagonal blocks $\text{At}(F_b)$, and the homotopy transfer to the minimal model only raises the block position, so it preserves the diagonal blocks of an upper triangular element; hence for

$v \in H^1(A_0, T_{A_0})$ the class $\text{ev}_{\mathcal{E}}(v)$ satisfies $\pi_a(\text{ev}_{\mathcal{E}}(v)) = \text{ev}_{F_a}(v) = v \lrcorner \text{At}(F_a)$. By Step 3 of the proof of Proposition 6.11 there is $v \in T\mathcal{D}_{3,3}$ with $v \lrcorner [Z_a] \neq 0$, and then $\text{ev}_{F_a}(v) \neq 0$, because the semiregularity map of F_a sends it to $v \lrcorner \text{ch}(F_a)$, whose component of lowest degree is $v \lrcorner [Z_a]$. So $\text{ev}_{\mathcal{E}}(v) \neq 0$, while $\sigma_{\mathcal{E}}(\text{ev}_{\mathcal{E}}(v)) = v \lrcorner \text{ch}(\mathcal{E}) = 0$, since $T\mathcal{D}_{3,3}$ annihilates θ and the Weil plane. The kernel of $\sigma_{\mathcal{E}}$ is nonzero and its rank is at least $r(\text{ch}(\mathcal{E}))$, the rank of $\sigma_{\mathcal{E}} \circ \text{ev}_{\mathcal{E}}$; the bound follows, and equality in Theorem 6.1 is excluded. $\square$

Corollary 6.15 (Configurations with a chain-free piece). *Let $\mathcal{E}$ be a convolution at A_0 whose Chern character has the polarised shape and whose pieces are the thirty-two threefolds $\mathcal{O}_{Z_{\zeta}}$ of Appendix A at twist 0, at the shifts s and $s+1$ of the parity of $\zeta_1 \zeta_2 \zeta_3$, together with further pieces, line bundles $L^l[t]$ and threefolds $\mathcal{O}_Z \otimes L^k[t]$, all of positive twist. The criterion fails for $\mathcal{E}$, for every twisting cochain, in each of the following cases.*

- (i) *The further pieces are at most four line bundles, of pairwise distinct twists: then Γ has no directed cycle.*
- (ii) *Every further piece sits at a shift at most $s+2$: then no edge of Γ ends at a piece of shift $s+1$, and the sixteen pieces there are chain-free.*
- (iii) *The further pieces are at most five line bundles, at any shifts: then a cycle through a piece at shift σ needs five line bundles at the shifts $\sigma-2, \dots, \sigma+2$, and the sixteen pieces at one of the two shifts are chain-free.*

Both bounds are sharp: $L^l[s+3]$ enters every piece at shift $s+1$, and six line bundles $L^{l_1}[s-2], \dots, L^{l_6}[s+3]$ with $0 < l_1 \leq \dots \leq l_6$ put each of the thirty-two pieces on a cycle of length six.

Proof. Along a directed cycle $a_0, a_1, \dots, a_m = a_0$ of Γ write $k_i = 1 + s_{a_{i-1}} - s_{a_i}$ for the degree of the block on the edge $a_{i-1} \rightarrow a_i$; the shifts cancel around the cycle, so $\sum_i k_i = m$. The degrees available on the edges are read off from Lemmas 6.8 and 6.9. Two of the thirty-two pieces at one shift have the same value of $\zeta_1 \zeta_2 \zeta_3$, hence differ in at least two coordinates, so $d \leq 1$ and the degrees between them are 2 and 3; as $\text{Hom} = 0$ between distinct threefolds and a degree one would need equal shifts, every edge between two of the thirty-two has degree at least 2. An edge from a line bundle of positive twist into a piece of twist 0 has degree 3, and so has an edge from a piece of twist 0 into a line bundle of positive twist; an edge into a piece of twist 0 from a threefold piece of positive twist has degree at least 3; an edge between two line bundles has degree 0 when it raises the twist, 6 when it lowers it, and $1+t-t' \geq 0$ between two copies of one line bundle at shifts t and t' . Parts (i) and (iii) use of the thirty-two pieces only that they have one twist, pairwise distinct threefolds and no Ext^1 at equal shifts, as in Proposition 6.11.

(i) A cycle among pieces of twist 0 alone has $\sum_i k_i \geq 2m$, so a cycle visits $\ell' \geq 1$ line-bundle pieces, $\ell' \leq 4$. Between two consecutive visits it either runs through $r \geq 1$ pieces of twist 0, entered and left with degree 3 and with $r-1$ internal edges of degree at least 2, which contributes at least $2r+4$ to $\sum_i k_i$ and r to m , or it passes directly along an edge that raises the twist, u such edges in all, or lowers it, w in all. With R runs, $R+u+w = \ell'$ and

$$0 = \sum_i k_i - m \geq \sum_{\text{runs}} (r+4) + 6w - \ell' \geq 4R + 5w - u.$$

If $R \geq 1$ this gives $u \geq 4$ and $\ell' \geq 5$. If $R = 0$ it gives $w = 0$, and a cycle whose edges all raise the twist does not close. So Γ has no cycle; every threefold piece is chain-free, there is one, and Proposition 6.14 applies.

(ii) An edge $u \rightarrow v$ into a piece v at shift $s+1$ has degree $1 + s_u - (s+1) = s_u - s$, at most 2 when $s_u \leq s+2$. No line bundle and no threefold piece of positive twist has an edge of degree at most 2 into a piece of twist 0, and an edge into v from one of the thirty-two would have degree 0 or 1, which do not occur. So no edge ends at v .

(iii) Let a cycle pass through a piece v of twist 0 at shift σ , and let q count its direct edges between two copies of one line bundle, so that $R+u+w+q = \ell' \leq 5$ and, the q edges contributing at least 0, $0 \geq 4R+5w-u-q$ as in (i). Since $R \geq 1$, this forces $R = 1$, $w = 0$, $u+q = 4$, $\ell' = 5$ and equality throughout: the only run is v itself, and the four direct edges have degree 0, so each raises the shift by one, while the two edges of degree 3 at v lower it by two. The five line-bundle pieces therefore sit

at the shifts $\sigma - 2, \dots, \sigma + 2$, one at each, with twists that do not decrease along the shifts. Five line bundles occupy at most five shifts, and the five shifts around s differ from the five around $s + 1$; so the pieces at one of the two shifts are chain-free.

For the sharpness, $\text{Ext}^3(L^l, \mathcal{O}_{Z_\zeta}) \neq 0$ is the edge $L^l[s + 3] \rightarrow \mathcal{O}_{Z_\zeta}[s + 1]$; and the six line bundles carry the cycle $L^1[s - 2] \rightarrow \dots \rightarrow L^5[s + 2] \rightarrow \mathcal{O}_{Z_\zeta}[s] \rightarrow L^1[s - 2]$ through every piece at shift s and the cycle $L^2[s - 1] \rightarrow \dots \rightarrow L^6[s + 3] \rightarrow \mathcal{O}_{Z_\zeta}[s + 1] \rightarrow L^2[s - 1]$ through every piece at shift $s + 1$, each edge being one of the nonzero groups listed. $\square$

Remark 6.16 (The configurations of Table 3). Corollary 6.15(i) covers the configuration (a) of the table, (ii) covers (c), whose further twists sit at shifts at most $2 = s + 2$, and (iii) covers every configuration of the thirty-two pieces with at most five line bundles. The programs of the archive rebuild Γ from the dimensions of Lemmas 6.8 and 6.9 and find, in the window of the table, no cycle in any of the 2401 configurations of (a); at least fifteen chain-free threefold pieces in each of the 3024 configurations of (c) with further twists $1 \leq k < k' \leq 3$, twenty-two at the configuration of the table; no edge into the sixteen pieces at shift 1 from the thirty-two or from the 1560 pieces of positive twist at shifts -3 to 2 , while $L^1[3]$ and $\mathcal{O}_{Z_{(1,1,1)}} \otimes L^1[3]$ do enter; in each of the 28812 configurations of four or five line bundles of twists 1 to 5 at shifts -3 to 3 one class of sixteen pieces chain-free, the other class, when it lies on cycles, having line bundles at its five shifts; and with $L^1[-2], \dots, L^6[3]$ every piece on a cycle. The Lean file checks the counts 32 and 22 of chain-free pieces at the configurations (a) and (c) of the table, the absence of entering edges from the line bundles L^1 to L^4 and the twists 1 to 3 of the sixty-four threefolds Z_ζ at shifts -3 to 2 , the entering edges at shift 3 and the six-cycles. No statement above rests on these computations.

6.7. Chains through line bundles

Corollary 6.15 needs the threefolds at one twist and the further pieces of positive twist, and it stops at five line bundles because six of them put every threefold piece on a cycle. This subsection removes both restrictions. It allows the threefolds any twists and any shifts, it works for every polarisation of A_0 and hence at every discriminant, and in place of the absence of cycles it uses a property of the cycles that the Maurer–Cartan equation of the twisted complex forces. By hand this excludes every convolution of distinct threefolds and at most five line bundles; the enumeration of Remark 6.25, a computation of the archive, extends the exclusion to six and seven line bundles and to all but thirty-one arrangements of eight. The next subsection turns from counting pieces to the support of any object meeting the criterion.

Setting 6.17 (Threefolds and line bundles at A_0). Throughout this subsection $\mathcal{E} = \mathcal{E}_\delta$ is a convolution at $A_0 = E_0^6$, in the sense of Proposition 6.5, of threefold pieces $\mathcal{O}_{Z_\zeta} \otimes L^{k_\zeta}[s_\zeta]$, $\zeta \in \mu_4^3$, and line-bundle pieces $L^{l_i}[t_i]$ such that (S1) each threefold Z_ζ occurs in at most one piece, (S2) the exponents l_i are pairwise distinct, and (S3) two threefold pieces at one shift differ in at least two coordinates of ζ , so that $\text{Ext}^1 = 0$ between them by Lemma 6.9(ii) and (iii). The twists k_ζ , l_i and the shifts are arbitrary integers. For $h \in \mathbb{Z}_{>0}^6$ let $\theta_h = \sum_j (h_j a_j + h_{3+j} b_j)$ be the class of the hermitian form of Lemma 6.8 with coefficients h , so that $\theta_{(1,\dots,1)} = \theta$ is the class of L , let $\mathcal{D}_h$ be the family $\mathcal{D}_{3,3}$ of the discriminant of θ_h , which contains (A_0, θ_h) , and suppose that $\text{ch}(\mathcal{E})$ has the polarised shape with respect to θ_h : $\text{ch}(\mathcal{E}) = N\omega + \sum_k c_k \theta_h^k$ with $N \neq 0$ and $0 \neq \omega \in W$. The Weil plane W of (A.4) does not depend on h .

Theorem 6.1 applies at (A_0, θ_h) with $r = 57$ for every h , by the argument given before Proposition 6.10 with $\text{SU}(V, H_h)$ in place of $\text{SU}(V, H)$: the group fixes W and θ_h and acts transitively on $\mathcal{D}_h$. The tangent space $T\mathcal{D}_h \subseteq H^1(A_0, T_{A_0})$ consists of the K -linear maps skew for θ_h ; it annihilates θ_h and W , and Step 3 of the proof of Proposition 6.11 holds for it word for word, $\text{SU}(V, H_h)(\mathbb{Q})$ being again a group of quasi-isogenies of A_0 Zariski dense in $\text{SU}(V, H_h)$: for every abelian threefold Z there is $v \in T\mathcal{D}_h$ with $v \lrcorner [Z] \neq 0$. Suppose now that the criterion holds for $\mathcal{E}$, that is $\dim \text{Ext}^2(\mathcal{E}, \mathcal{E}) = 57$. Then $\sigma_{\mathcal{E}}$ is injective, and for every $v \in T\mathcal{D}_h$ the class $\text{ev}_{\mathcal{E}}(v) \in \text{Ext}^2(\mathcal{E}, \mathcal{E})$, which $\sigma_{\mathcal{E}}$ sends to $v \lrcorner \text{ch}(\mathcal{E}) = 0$, vanishes. Everything below draws consequences from this vanishing.

Let Γ be the graph of Section 6.6 on the pieces F_a of $\mathcal{E}$, and let $\Gamma_\delta \subseteq \Gamma$ be the subgraph of the edges $a \rightarrow b$ on which the component δ_{ab} is nonzero. Since δ is strictly upper triangular in the one-sided

order, Γ_δ has no directed cycle, and its walks are paths. The twisted complex lives over the minimal A_∞ -model of the pieces, whose underlying graded space is $\bigoplus_{a,b} \text{Ext}^*(F_a, F_b)$ with the block $\text{Ext}^k(F_a, F_b)$ in total degree $k + s_b - s_a$. The cochain δ satisfies the Maurer–Cartan equation $\sum_{k \geq 2} m_k(\delta, \dots, \delta) = 0$, and the differential of the twisted complex is $D(\eta) = \sum_{p,q \geq 0} m_{p+q+1}(\delta, \dots, \delta, \eta, \delta, \dots, \delta)$ with p components of δ before η and q after it. A term of either sum is nonzero only if its arguments are composable: along a path of Γ_δ for the Maurer–Cartan equation, and, for a term of $D(\eta)_{ad}$ with the argument η_{bc} , along a path of Γ_δ from a to b , the edge $b \rightarrow c$ of Γ , and a path of Γ_δ from c to d .

Lemma 6.18 (The cocycle of a tangent vector). *For $v \in H^1(A_0, T_{A_0})$ the class $\text{ev}_{\mathcal{E}}(v)$ is represented in the minimal model by a cocycle $c(v)$ whose component in the diagonal block $\text{Ext}^2(F_a, F_a)$ is $\text{ev}_{F_a}(v) = v \lrcorner \text{At}(F_a)$ and whose component in a block (a, c) with $a \neq c$ is a sum of terms each of which is a composite along a path of Γ_δ from a to c . In particular $c(v)_{ac} = 0$ when Γ_δ has no path from a to c .*

Proof. As in the proof of Proposition 6.5(i), the Atiyah class of the twisted complex is represented in the harmonic model by $[\nabla, \bar{\partial} + \delta]$, block upper triangular with diagonal blocks $\text{At}(F_a)$ and off-diagonal blocks $\nabla \delta_{ab}$, which vanish off the edges of Γ_δ ; so $v \lrcorner [\nabla, \bar{\partial} + \delta]$ has diagonal blocks $v \lrcorner \text{At}(F_a)$ and off-diagonal blocks on the edges of Γ_δ only. The homotopy transfer $\xi \mapsto \sum_k (-h[\delta, \cdot])^k \xi$ to the minimal model composes each block of ξ with components of δ on either side, the homotopy h acting block by block; so every term of the component (a, c) of the transferred cocycle is a composite of one block of $v \lrcorner [\nabla, \bar{\partial} + \delta]$, sitting on a vertex or an edge of Γ_δ , with components of δ along edges of Γ_δ leading from a to it and from it to c . For $a = c$ only the term with $k = 0$ occurs, since a composite of $k \geq 1$ components of δ from a back to a would be a cycle of Γ_δ . $\square$

Call a directed cycle of Γ *effective* for δ when exactly one of its edges lies outside Γ_δ .

Proposition 6.19 (Every threefold piece lies on an effective cycle). *Suppose $\mathcal{E} = \mathcal{E}_\delta$ meets the criterion. Then for every $v \in T\mathcal{D}_h$ there is η of total degree one with $D(\eta) = c(v)$, and for every piece F_a*

$$\text{ev}_{F_a}(v) = \sum m_{p+q+1}(\delta, \dots, \delta, \eta_{bc}, \delta, \dots, \delta),$$

the sum running over the effective cycles $a \rightarrow \dots \rightarrow b \rightarrow c \rightarrow \dots \rightarrow a$ of Γ through a , with $b \rightarrow c$ the edge outside Γ_δ and p, q the numbers of edges before and after it. In particular every threefold piece lies on an effective cycle whose edge outside Γ_δ carries a nonzero component of η .

Proof. The class $\text{ev}_{\mathcal{E}}(v)$ vanishes, so its representative $c(v)$ of Lemma 6.18 is a coboundary $D(\eta)$. The diagonal component $D(\eta)_{aa}$ is the sum of the terms $m_{p+q+1}(\delta, \dots, \eta_{bc}, \dots, \delta)$ whose arguments are composable from a to a : a path of Γ_δ from a to b , the edge $b \rightarrow c$ with $\eta_{bc} \neq 0$, and a path of Γ_δ from c to a . If $b = c$ the term is $m_1(\eta_{aa}) = 0$ when $p = q = 0$, and otherwise its components of δ form a cycle of Γ_δ ; so $b \neq c$. The two paths are increasing in the one-sided order, so they meet only in a ; the edge $b \rightarrow c$ is not in Γ_δ , as otherwise Γ_δ would contain a cycle; and the walk is a directed cycle of Γ with exactly one edge outside Γ_δ . By Lemma 6.18 the left side is $c(v)_{aa} = \text{ev}_{F_a}(v)$. For a threefold piece choose $v \in T\mathcal{D}_h$ with $v \lrcorner [Z_a] \neq 0$; then $\text{ev}_{F_a}(v) \neq 0$, as in the proof of Proposition 6.14, so some term of the sum is nonzero. $\square$

Lemma 6.20 (Two rules). *Let $a = L^{l_a}[t]$, $b = L^{l_b}[t+1]$ and $c = L^{l_c}[t+2]$ be line-bundle pieces of $\mathcal{E}$ with $l_a < l_b < l_c$, so that $a \rightarrow b \rightarrow c$ is a path of Γ whose edges carry sections, $\delta_{ab} \in H^0(L^{l_b-l_a})$ and $\delta_{bc} \in H^0(L^{l_c-l_b})$.*

- (i) *The path $a \rightarrow b \rightarrow c$ is not the only path of Γ_δ from a to c : either δ vanishes on one of its two edges, or Γ_δ has another path from a to c .*
- (ii) *Suppose $a \rightarrow b \rightarrow c$ is the only path of Γ from a to c , let $v \in T\mathcal{D}_h$ and let η satisfy $D(\eta) = c(v)$. If $\delta_{bc} = 0 \neq \delta_{ab}$ then $\eta_{bc} = 0$, and if $\delta_{ab} = 0 \neq \delta_{bc}$ then $\eta_{ab} = 0$.*

Proof. (i) The component (a, c) of the Maurer–Cartan equation is the sum, over the paths of Γ_δ from a to c , of the products $m_k(\delta, \dots, \delta)$ along them. If $a \rightarrow b \rightarrow c$ were the only such path it would read $m_2(\delta_{ab}, \delta_{bc}) = 0$, and m_2 on sections is their product, which is nonzero for two nonzero sections of line bundles on the integral variety A_0 .

(ii) A term of $D(\eta)_{ac}$ runs along a walk of Γ from a to c made of a path of Γ_δ , one edge carrying η and a path of Γ_δ . If the walk repeats a vertex, the repetition is a vertex common to the two paths of Γ_δ , and shortcutting it leaves a path of Γ_δ from a to c ; under the hypothesis this path is $a \rightarrow b \rightarrow c$ with both components nonzero, which is excluded in either case. So every term runs along a path of Γ from a to c , that is along $a \rightarrow b \rightarrow c$, and $D(\eta)_{ac} = \pm m_2(\delta_{ab}, \eta_{bc}) \pm m_2(\eta_{ab}, \delta_{bc})$. The right side $c(v)_{ac}$ vanishes by Lemma 6.18, since Γ_δ has no path from a to c . When $\delta_{bc} = 0 \neq \delta_{ab}$ this gives $\delta_{ab}\eta_{bc} = 0$, the product of the nonzero section δ_{ab} with $\eta_{bc} \in \text{Ext}^0(L^b, L^c) = H^0(L^{c-b})$, so $\eta_{bc} = 0$; the other case is the same with the roles exchanged. $\square$

Lemma 6.21 (Both parities occur). *Grade $H^*(A_0, \mathbb{C})$ by $\mathbb{Z}^3$, giving dz_j and $d\bar{z}_{3+j}$ the weight e_j and $d\bar{z}_j$, dz_{3+j} the weight $-e_j$. Then a_j, b_j, θ and every θ_h have weight 0, c_j has weight $2e_j$, $\bar{c}_j$ has weight $-2e_j$, W lies in the weights $\pm 2(e_1 + e_2 + e_3)$, and the weight-zero part of $[Z_\zeta]$ is $X_0 = \prod_j (a_j + b_j)$ for every ζ . Consequently $\sum_\zeta (-1)^{s_\zeta} = 0$ over the threefold pieces of $\mathcal{E}$: there are threefold pieces at even shifts and at odd shifts.*

Proof. The weights are read off from (A.1), (A.2) and (A.4), and in $[Z_\zeta] = \prod_j (a_j + b_j + \bar{\zeta}_j c_j + \zeta_j \bar{c}_j)$ a product of one term from each factor has weight zero only if it takes a_j or b_j from every factor, the e_j being independent. The part of $\text{ch}(\mathcal{E})$ in H^6 is $\sum_\zeta (-1)^{s_\zeta} [Z_\zeta] + P\theta^3$ with $P = \sum_i (-1)^{i_1} l_i^3 / 6$, by Lemma 6.8(i), and it equals $N\omega + c_3\theta_h^3$. Taking the parts of weight zero, with $Q = \sum_\zeta (-1)^{s_\zeta}$,

$$QX_0 + P\theta^3 = c_3\theta_h^3.$$

The squarefree cubic monomials in $a_1, b_1, \dots, a_3, b_3$ are linearly independent. The coefficient of $a_j b_j a_k$, $j \neq k$, is 0 in X_0 , 6 in θ^3 and $6h_j h_{3+j} h_k$ in θ_h^3 , so $P = c_3 h_j h_{3+j} h_k$, and likewise $P = c_3 h_j h_{3+j} h_{3+k}$. If $c_3 \neq 0$ these equations, over all $j \neq k$, give $h_k = h_{3+k}$ and $h_1 h_4 = h_2 h_5 = h_3 h_6$, so all six h_k equal one value h and $P = c_3 h^3$; if $c_3 = 0$ then $P = 0$. The coefficient of $a_1 a_2 a_3$ is 1 in X_0 , 6 in θ^3 and $6h_1 h_2 h_3$ in θ_h^3 , so $Q + 6P = 6c_3 h_1 h_2 h_3$, which equals $6P$ in both cases. Hence $Q = 0$. There is a threefold piece, since $N\omega \neq 0$ and the line bundles contribute nothing to the Weil component; so there are pieces of both parities. $\square$

The edges of Γ between line-bundle pieces are of two kinds, by Lemma 6.8(v): an *up* edge $L^l[s] \rightarrow L^{l'}[s+1]$ with $l < l'$, of degree 0, and a *down* edge $L^l[s] \rightarrow L^{l'}[s-5]$ with $l > l'$, of degree 6; in general an edge of degree k joins the shifts s and $s - (k-1)$. By Lemma 6.8(iv) an edge $L^l \rightarrow \mathcal{O}_Z \otimes L^k$ has degree 0 when $k > l$, 3 when $k < l$ and 0 to 3 when $k = l$, an edge $\mathcal{O}_Z \otimes L^k \rightarrow L^l$ has degree 3 when $l > k$, 6 when $l < k$ and 3 to 6 when $l = k$, and by Lemma 6.9(ii) and (S3) an edge between two threefold pieces has degree at least 2, and at least 3 when it lowers the twist. A *run* is a maximal path of Γ through threefold pieces from a line-bundle piece $L^l[s]$ to a line-bundle piece $L^{l'}[s-\beta]$; its *excess* is β , and it is *decreasing* when $l' < l$.

Lemma 6.22 (The shape of a cycle). *Let C be a directed cycle of Γ through a threefold piece, with ℓ line-bundle pieces, $R \geq 1$ runs, u up edges and w down edges, so that $\ell = R + u + w$.*

- (i) *A run through r threefold pieces has excess at least r , with equality only when the twists do not decrease along it. A decreasing run has excess at least 3, with equality only for $L^l \rightarrow \mathcal{O}_Z \otimes L^k \rightarrow \mathcal{O}_{Z'} \otimes L^{k'} \rightarrow L^{l'}$ with $k' \leq l' < l \leq k$ and degrees 0, 3, 3, whose two threefold pieces have the parity of shift of $L^{l'}$.*
- (ii) *$u = \sum_{\text{runs}} \beta + 5w$. If $\ell \leq 7$ then $w = 0$, exactly one run is decreasing, and $\ell \geq 2R + 2$.*
- (iii) *If $\ell \leq 5$ then $R = 1$: the line-bundle pieces of C sit at ℓ consecutive shifts, joined by up edges in the order of their exponents, and the run leads from the highest back to the lowest. If $\ell = 4$ the run is the one of (i) with excess 3, and both its threefold pieces have the parity of the lowest shift.*

Proof. (i) An edge of degree k lowers the shift by $k-1$. Along a run the entering edge has degree $e \geq 0$, the $r-1$ internal edges have degree at least 2 and the leaving edge has degree $x \geq 3$, so the excess is at least $-1 + (r-1) + 2 = r$, with equality only for $e = 0$, internal degrees 2 and $x = 3$, that is $l \leq k_1 \leq \dots \leq k_r \leq l'$ for the twists. Let $l' < l$. With $r = 1$ the excess $(e-1) + (x-1)$ is at least 4: $e = 0$ forces $k \geq l > l'$, hence $x = 6$; $e = 3$ gives $2 + (x-1) \geq 4$; and $e \in \{1, 2\}$ forces $k = l > l'$, hence

$x = 6$. With $r = 2$ and degrees e, d, x the excess $e + d + x - 3$ equals 3 only for $(e, d, x) = (0, 3, 3)$: $e = 0$ forces $k_1 \geq l > l'$, and then either $k_2 \geq k_1 > l'$, so $x = 6$, or $k_2 < k_1$, so $d \geq 3$ and $x \geq 3$, with $x = 3$ exactly when $k_2 \leq l'$; $e = 3$ gives at least $3 + 2 + 3$; $e = 1$ forces $k_1 = l$, and then $k_2 \geq l > l'$ gives $x = 6$ while $k_2 < k_1$ gives $d \geq 3$, each at least 7; $e = 2$ gives at least $2 + 2 + 3$. With $r \geq 3$ the excess is at least $r + 1 \geq 4$. Along the run with $(e, d, x) = (0, 3, 3)$ from L^l at shift s the shifts are $s + 1$, $s - 1$ and $s - 3$.

(ii) The shifts return to their value around C : each up edge adds 1, each down edge subtracts 5 and each run subtracts its excess. Each run has excess at least 1, so $\ell = R + u + w = R + \sum \beta + 6w \geq 2R + 6w$, and $w \geq 1$ forces $\ell \geq 8$. Let $w = 0$. The exponents also return to their value, and the up edges raise them, so some run is decreasing. With $D \geq 1$ decreasing runs, of excess at least 3, $\ell = R + \sum \beta \geq R + 3D + (R - D) = 2R + 2D$; so $\ell \leq 7$ forces $D = 1$, and $\ell \geq 2R + 2$.

(iii) $\ell \leq 5$ gives $R = 1$ by (ii), and $u = \ell - 1$: the line-bundle pieces form a chain of $\ell - 1$ up edges, each raising the shift by one and the exponent, closed by one run from the last to the first, which is decreasing. For $\ell = 4$ the run has excess $u = 3$, so it is the run of (i), and the parity of its threefold pieces is that of its exit, the lowest line bundle. $\square$

Theorem 6.23 (Convolutions with at most five line bundles fail). *In the Setting 6.17, if $\mathcal{E}$ has at most five line-bundle pieces then $\dim \text{Ext}^2(\mathcal{E}, \mathcal{E}) \geq 58$: the criterion fails for $\mathcal{E}$, for every twisting cochain, for every twist of the threefolds and for every polarisation θ_h .*

Proof. Suppose the criterion holds. By Lemma 6.21 there are threefold pieces of both parities of shift, and by Proposition 6.19 each lies on an effective cycle carrying a nonzero component of η , in particular on a directed cycle of Γ . A cycle through a threefold piece has at least four line-bundle pieces by Lemma 6.22(ii), so with at most three there is none. Let there be four or five, $L_1, \dots, L_n$ in the order of their exponents.

With four, every cycle through a threefold piece uses all four, at consecutive shifts in the order of the exponents, and its threefold pieces have the parity of the lowest shift, by Lemma 6.22(iii); so the threefold pieces on cycles have one parity, and a threefold piece of the other parity lies on no cycle.

With five, a cycle through a threefold piece has four or five line bundles, at consecutive shifts in the order of the exponents. If the five are not at five consecutive shifts in that order, no cycle uses all five, and two cycles of four, omitting L_m and $L_{m'}$ with $m \neq m'$, have the same lowest shift: the three line bundles common to both occupy three of four consecutive shifts in each; if these are consecutive, $x, x + 1, x + 2$, the omitted bundles sit at $x - 1$ or at $x + 3$, and at the same one of the two, since L_m at $x - 1$ and $L_{m'}$ at $x + 3$ would put the five at five consecutive shifts in the order of the exponents; and if they leave a gap, both omitted bundles fill it. So again the threefold pieces on cycles have one parity, and a threefold piece of the other parity lies on no cycle.

It remains to treat L_i at the shift $\sigma - 3 + i$, $i = 1, \dots, 5$. The cycles through threefold pieces are C_5 through all five, C_{1234} and C_{2345} , each a chain of up edges followed by one run. The only path of Γ from L_i to L_{i+2} is $L_i \rightarrow L_{i+1} \rightarrow L_{i+2}$: a path leaving the chain of up edges enters a run, which ends at least three shifts below its start, at L_1 or L_2 , and from there the up edges lead only through line bundles already visited. By Lemma 6.20(i) applied to the three triples, δ vanishes on one of the edges $L_1 \rightarrow L_2$, $L_2 \rightarrow L_3$, on one of $L_2 \rightarrow L_3$, $L_3 \rightarrow L_4$, and on one of $L_3 \rightarrow L_4$, $L_4 \rightarrow L_5$. An effective cycle has exactly one edge outside Γ_δ . So C_5 , with two such edges, is not effective; C_{1234} is effective only when its edge outside Γ_δ is $L_2 \rightarrow L_3$, and then $\delta_{12} \neq 0$, so that $\eta_{23} = 0$ by Lemma 6.20(ii); and C_{2345} is effective only when that edge is $L_3 \rightarrow L_4$, with $\delta_{23} \neq 0$, so that $\eta_{34} = 0$. No effective cycle carries a nonzero component of η on its edge outside Γ_δ .

Each case contradicts Proposition 6.19. So the criterion fails, and since Theorem 6.1 gives $\dim \text{Ext}^2 \geq 57$ with equality excluded, $\dim \text{Ext}^2 \geq 58$. $\square$

Lemma 6.24 (The shifts along a cycle form an interval). *The shifts of the line-bundle pieces along a directed cycle of Γ , or along a path of Γ between two line-bundle pieces the second of which is at the higher shift, take every value between their least and their greatest. Hence, if the shifts of the line-bundle pieces of $\mathcal{E}$ do not form an interval of integers, the line bundles split along a gap into two groups such that every cycle of Γ through a threefold piece, and every path $a \rightarrow b \rightarrow c$ of Lemma 6.20, has its line bundles in one group.*

Proof. Between consecutive line-bundle pieces of a cycle or a path the shift rises by exactly 1, along an up edge, or falls, along a down edge or a run; from the least shift to the greatest it therefore passes through every intermediate value. The three line bundles of Lemma 6.20 are at three consecutive shifts. $\square$

Remark 6.25 (Six, seven and eight line bundles). The rules used in the proof of Theorem 6.23 depend on the shifts of the line bundles and the order of their exponents alone. The program of the archive described in Appendix B runs through every arrangement of $n \leq 8$ line bundles whose shifts form an interval, builds the graph of up edges, down edges and runs, admitting every excess and every parity of threefold shift that the degrees of Lemmas 6.8 and 6.9 allow for a run, and, for every set of up edges on which δ is taken to vanish that Lemma 6.20(i) admits, records the parities of the threefold pieces on the cycles with at most one edge in that set whose edge in the set is not forced to carry $\eta = 0$ by Lemma 6.20(ii). Runs and down edges are taken to carry nonzero components, and the condition that the edge outside Γ_δ point downwards in the one-sided order is dropped; both choices enlarge the set of cycles counted, so an arrangement for which no choice reaches both parities is one for which Proposition 6.19 and Lemma 6.21 exclude every twisting cochain. The counts are: 75 arrangements of four line bundles and 541 of five, none reaching a parity; 4683 of six and 47293 of seven, of which 4 and 140 reach one parity and none both; and 545835 of eight, of which 31 reach both. By Lemma 6.24 an arrangement whose shifts do not form an interval splits into arrangements of fewer line bundles, each contributing the parities it reaches, and with at most eight line bundles at most one part has six or more, so at most one parity is reached. Hence, in the Setting 6.17, every convolution with at most seven line-bundle pieces fails the criterion for every polarisation θ_h , and so does every convolution with eight line bundles outside the thirty-one arrangements; those have their eight line bundles at four to seven consecutive shifts with at most three at one shift, the simplest being $L^1[0], L^2[1], L^3[2], L^4[3], L^5[3], L^6[4], L^7[5], L^8[6]$ with $l_1 < \dots < l_8$. These two statements are the output of the enumeration and rest on it; Theorem 6.23 rests on nothing computed.

6.8. The support of an object meeting the criterion

The results of this subsection do not count pieces. They start from one consequence of the criterion and read off where an object meeting it can be supported.¹⁴ Let $\mathcal{E}$ be a perfect complex on A_0 whose Chern character has the polarised shape with respect to θ_h and which meets the criterion, $\dim \text{Ext}^2(\mathcal{E}, \mathcal{E}) = 57$, so that $\sigma_{\mathcal{E}}$ is injective. For $v \in T\mathcal{D}_h$ the obstruction $\text{ob}_{\mathcal{E}}(v) = v \lrcorner \text{At}(\mathcal{E}) \in \text{Ext}^2(\mathcal{E}, \mathcal{E})$ to deforming $\mathcal{E}$ along v is sent by $\sigma_{\mathcal{E}}$ to $v \lrcorner \text{ch}(\mathcal{E})$ [17], which vanishes because $T\mathcal{D}_h$ annihilates θ_h and W ; so

$$\text{ob}_{\mathcal{E}}(v) = 0 \quad \text{for every } v \in T\mathcal{D}_h. \quad (6.7)$$

Everything below is drawn from (6.7). The open subsets $U \subseteq A_0$ are Zariski open; inside the proofs we pass to the analytic spaces, which changes no Ext group and no Atiyah class on the projective A_0 [68], and the cohomology with supports and the Chern characters with supports are those of the analytic spaces. For a closed subset Z of an open U we write $H_Z^q(U, F)$ and $\mathcal{H}_Z^q(F)$ for the cohomology and the local cohomology sheaves with supports in Z [35]. Two facts are used throughout: if Z is a submanifold of codimension c and F is locally free then $\mathcal{H}_Z^q(F) = 0$ for $q \neq c$ (depth below c , the Čech complex of c local equations above c), so that $H_Z^{c+p}(U, F) = H^p(U, \mathcal{H}_Z^c(F))$; and if $C \subseteq Z$ is closed of codimension at least $c + 1$ then $\mathcal{H}_C^0(\mathcal{H}_Z^c(F)) = 0$, because this sheaf embeds into $\mathcal{H}_C^c(F)$, which vanishes. So a section of $\mathcal{H}_Z^c(F)$ that vanishes outside C vanishes.

For the threefolds we use the coordinates of Appendix A. Fix $\zeta \in \mu_4^3$ and $T = Z_\zeta$. The linear forms $n_m = z_m + \zeta_m z_{3+m}$ on the universal cover cut out the lift of T ; their differentials dn_m , and the dz_m for $m \leq 3$, are translation-invariant forms on A_0 and together a frame of $\Omega_{A_0}^1$. On a tubular neighbourhood N of T the n_m themselves are single-valued, and $\pi(z) = z - \sum_m \zeta_m n_m(z) \partial_{3+m}$ retracts N onto T with

¹⁴The statements of this subsection on the supports of threefold pieces, Theorems 6.29, Proposition 6.30, Corollary 6.31, Proposition 6.35 and Proposition 6.36, were first found, with their proofs and the exact checks of Appendix B, by release 3.1.0 of Anaxia, the construction engine of the BLLM project [15], run on the stopping points of this paper; the general form of the locality theorem, Theorem 6.27, followed in the repository of the engine. The engine proposes; every statement is decided by the proofs below.

$z_m \circ \pi = z_m$; the pair (n, π) identifies N with $T \times D^3$, and a holomorphic function on $V \times (D^*)^3$, for $V \subseteq T$ open and D^* the punctured disc, is a Laurent series in n_1, n_2, n_3 whose coefficients are holomorphic on V . The local cohomology $\mathcal{H}_T^3(\mathcal{O})$ is computed by the Čech complex of the cover of $N \setminus T$ by the three open sets $\{n_m \neq 0\}$, all of whose intersections are Stein; so on T the sheaf $\mathcal{H}_T^3(\mathcal{O})$ is the sheaf of Laurent tails $\sum_{a,b,c \geq 1} g_{abc} n_1^{-a} n_2^{-b} n_3^{-c}$ with $g_{abc} \in \mathcal{O}_T$, and $\mathcal{H}_T^3(F) = \mathcal{H}_T^3(\mathcal{O}) \otimes F$ for F locally free. The part of $\mathcal{H}_T^3(\mathcal{O})$ annihilated by the ideal I_T is $\mathcal{O}_T \cdot [1/n_1 n_2 n_3]$. We write $\rho_T: \mathcal{H}_T^3(\mathcal{O}) \rightarrow \mathcal{O}_T$ for the map sending the class of $\sum g_{abc} n_1^{-a} n_2^{-b} n_3^{-c}$ to g_{111} , a morphism of sheaves of vector spaces on T with $\rho_T(g[1/n_1 n_2 n_3]) = g|_T$ for g holomorphic near T , extended coefficientwise to $\mathcal{H}_T^3(\Omega^q)$ in the frame above. For a coherent sheaf G on U supported in T its multiplicity $\mu_T(G)$ is the length of G at the generic point of T , the sum over $k \geq 0$ of the ranks of the $\mathcal{O}_T$ -modules $I_T^k G / I_T^{k+1} G$; and $\mu_T(\mathcal{E}|_U) = \sum_q (-1)^q \mu_T(H^q(\mathcal{E}|_U))$ when every $H^q(\mathcal{E}|_U)$ is supported in T .

Lemma 6.26 (The Chern character with supports). *Let U be open, $Z \subseteq U$ closed, and $\mathcal{F}$ a perfect complex on U whose cohomology sheaves are supported in Z .*

- (i) *The trace $R\mathcal{H}om(\mathcal{F}, \mathcal{F}) \rightarrow \mathcal{O}_U$ factors through a unique morphism $\text{tr}^Z: R\mathcal{H}om(\mathcal{F}, \mathcal{F}) \rightarrow R\Gamma_Z \mathcal{O}_U$. Setting $\text{ch}_k^Z(\mathcal{F}) = \text{tr}^Z(\text{At}(\mathcal{F})^k)/k! \in H_Z^k(U, \Omega^k)$ defines classes that forget to $\text{ch}_k(\mathcal{F})$, are compatible with restriction to open subsets and with enlarging Z , are additive in exact triangles, and satisfy*

$$v \lrcorner \text{ch}_{k+1}^Z(\mathcal{F}) = \frac{1}{k!} \text{tr}^Z(\text{ob}_{\mathcal{F}}(v) \circ \text{At}(\mathcal{F})^k) \in H_Z^{k+2}(U, \Omega^k)$$

for $v \in H^1(U, T_U)$, where $\text{ob}_{\mathcal{F}}(v) = v \lrcorner \text{At}(\mathcal{F})$. In particular $\text{ob}_{\mathcal{F}}(v) = 0$ implies $v \lrcorner \text{ch}_{k+1}^Z(\mathcal{F}) = 0$ for every k .

- (ii) *For $T = Z_\zeta$ and every open U meeting T the class $[T] := \text{ch}_3^T(\mathcal{O}_{T \cap U})$ in $H^0(U, \mathcal{H}_T^3(\Omega^3))$ is*

$$[T] = c \left[\frac{dn_1 \wedge dn_2 \wedge dn_3}{n_1 n_2 n_3} \right],$$

with one constant $c \neq 0$, the same for all sixty-four threefolds.

- (iii) *If the cohomology sheaves of $\mathcal{F}$ are supported in $T \cap U$ then $\text{ch}_3^T(\mathcal{F}) = \mu_T(\mathcal{F})[T]$. If they are supported in a union Z of threefolds T_i of the sixty-four, any two meeting in dimension at most two, then $\text{ch}_3^Z(\mathcal{F}) = \sum_i \mu_{T_i}(\mathcal{F})[T_i]_Z$, where $[T_i]_Z$ is the image of $[T_i]$ in $H_Z^3(U, \Omega^3)$.*

Proof. (i) Let $j: U \setminus Z \rightarrow U$. In the triangle $R\Gamma_Z \mathcal{O} \rightarrow \mathcal{O} \rightarrow Rj_* \mathcal{O}_{U \setminus Z}$ the third term receives no nonzero morphism from a complex K whose cohomology sheaves are supported in Z , since $\text{Hom}(K, Rj_* \mathcal{O}_{U \setminus Z}) = \text{Hom}(K|_{U \setminus Z}, \mathcal{O}_{U \setminus Z}) = 0$; so $\text{Hom}(K, R\Gamma_Z \mathcal{O}) \rightarrow \text{Hom}(K, \mathcal{O})$ is bijective. The complex $R\mathcal{H}om(\mathcal{F}, \mathcal{F})$ has cohomology supported in Z , which gives tr^Z and its uniqueness; the same bijection applied to the sources $\mathcal{F}^\vee \otimes \mathcal{F} \otimes \Omega^k$ transports every identity of the trace to tr^Z : its compatibility with restriction to open subsets, with the morphism $R\Gamma_Z \rightarrow R\Gamma_{Z'}$ for $Z \subseteq Z'$, its additivity in triangles, and its graded cyclicity. The Atiyah class $\text{At}(\mathcal{F}) \in \text{Ext}^1(\mathcal{F}, \mathcal{F} \otimes \Omega^1)$ is compatible with restriction and with triangles, so the classes ch_k^Z have the listed properties, and cyclicity applied to the $k+1$ factors of $\text{tr}^Z(\text{At}^{k+1})$ gives $v \lrcorner \text{tr}^Z(\text{At}^{k+1}) = (k+1) \text{tr}^Z((v \lrcorner \text{At}) \circ \text{At}^k)$, which is the displayed identity.

(ii) The trace is $\mathcal{O}_U$ -linear and $\text{At}(\mathcal{O}_T)^3$ is a morphism between $\mathcal{O}_T$ -modules, so $[T]$ is annihilated by I_T on every open subset and is a section of $\mathcal{O}_{T \cap U} [1/n_1 n_2 n_3] \otimes \Omega^3$. By the compatibility with restriction it suffices to take $U = A_0$, where the coefficients are holomorphic functions on the compact connected T , hence constants: $[T] = \sum_a g_a [1/n_1 n_2 n_3] \omega_a$ with $g_a \in \mathbb{C}$ and ω_a running through the wedge products of three elements of the frame $dn_1, dn_2, dn_3, dz_1, dz_2, dz_3$. For $0 < \lambda < 1$ the map $\varphi_\lambda(z) = z - (1-\lambda) \sum_m \xi_m n_m(z) \partial_{3+m}$ is a biholomorphism of N onto a smaller tubular neighbourhood of T , fixes T pointwise, and satisfies $n_m \circ \varphi_\lambda = \lambda n_m$, $z_m \circ \varphi_\lambda = z_m$ and $\varphi_\lambda^* \mathcal{O}_T = \mathcal{O}_T$. The Atiyah class, the trace and the local cohomology are natural under biholomorphisms onto open subsets, and a section of $\mathcal{H}_T^3(\Omega^3)$ is determined by its restriction to any neighbourhood of T ; so $\varphi_\lambda^*[T] = [T]$. If ω_a contains a of the dn_m , then φ_λ^* multiplies $g_a [1/n_1 n_2 n_3] \omega_a$ by λ^{a-3} ; so $g_a = 0$ unless $a = 3$, and $[T] = c [dn_1 \wedge dn_2 \wedge dn_3 / n_1 n_2 n_3]$. The automorphisms $z_{3+m} \mapsto i^{e_m} z_{3+m}$ of A_0 permute the sixty-four threefolds transitively and carry the forms n_m, dn_m of one to those of another, so c does not depend on

ζ . Forgetting supports sends $[T]$ to $\text{ch}_3(\mathcal{O}_T) \in H^3(A_0, \Omega^3)$, which is the class of T by the Riemann–Roch theorem for the embedding of T [30, Ch. 15] (the Chern character defined by the Atiyah class is the usual one on the projective A_0 [17]); and $[Z_\zeta]\theta^3 = 48 \neq 0$ by Lemma A.1, so $c \neq 0$.

(iii) Both identities are between sections of a sheaf, so they may be checked locally on U and, by the second fact recalled above, off a closed subset of codimension at least four. For the first, ch_3^T is additive and $\mathcal{F}$ is an iterated extension of its shifted cohomology sheaves, so let G be a coherent sheaf supported in T ; near a point of T some $I_T^k G$ vanishes, and G is an iterated extension of the $\mathcal{O}_T$ -modules $G_k = I_T^k G / I_T^{k+1} G$, of ranks r_k with $\sum_k r_k = \mu_T(G)$. Each G_k is locally free of rank r_k off a proper closed analytic subset of T , where it is locally $\mathcal{O}_T^{r_k}$ and $\text{ch}_3^T(G_k) = r_k[T]$ by additivity and (ii). For the second, let C be the union of the pairwise intersections of the T_i , of codimension at least four. On $U \setminus C$ the T_i are disjoint, $\text{R}\Gamma_Z = \bigoplus_i \text{R}\Gamma_{T_i}$ by the Mayer–Vietoris triangle for supports, and $\mathcal{F} \cong \text{R}\Gamma_Z \mathcal{F} = \bigoplus_i \text{R}\Gamma_{T_i} \mathcal{F}$, a sum of perfect complexes with cohomology supported in the T_i , whose multiplicities along T_i are those of $\mathcal{F}$; so there $\text{ch}_3^Z(\mathcal{F}) = \sum_i \mu_{T_i}(\mathcal{F})[T_i]_Z$ by additivity and the first identity. $\square$

Theorem 6.27 (Locality). *Let $\mathcal{E}$ be as above, let $T = Z_\zeta$, and let $U \subseteq A_0$ be open with $T \cap U \neq \emptyset$, every cohomology sheaf of $\mathcal{E}|_U$ supported in T , and $\mu_T(\mathcal{E}|_U) \neq 0$. Then the union S of the components of $T \setminus U$ of codimension one in T is a nonempty ample divisor on T . In particular, if T is an irreducible component of the support of $\mathcal{E}$ along which $\mathcal{E}$ has nonzero multiplicity, then the other components of the support cut T in a set whose divisorial part is a nonempty ample divisor.*

The case of a shifted line bundle on $T \cap U$ is the case $\mu_T = \pm 1$ with one cohomology sheaf.

Proof. Let $v \in T\mathcal{D}_h$. Since T is a subgroup, its normal bundle is $\mathcal{O}_T \otimes N_0$ with $N_0 = T_0 A_0 / T_0 T$, and we write $\kappa_T(v) \in H^1(T, \mathcal{O}_T) \otimes N_0$ for the image of v under $H^1(A_0, T_{A_0}) \rightarrow H^1(T, T_{A_0}|_T) \rightarrow H^1(T, N_T)$, and $\kappa_T(v)_m \in H^1(T, \mathcal{O}_T)$ for its image under $dn_m: N_0 \rightarrow \mathbb{C}$; on a cover of A_0 by open sets U_a with v represented by vector fields v_{ab} , the class $\kappa_T(v)_m$ is represented by the functions $dn_m(v_{ab})|_T$. By (6.7) and Lemma 6.26(i), (iii),

$$0 = v \lrcorner \text{ch}_3^T(\mathcal{E}|_U) = \mu_T(\mathcal{E}|_U) v \lrcorner [T]$$

in $H_T^4(U, \Omega^2) = H^1(T \cap U, \mathcal{H}_T^3(\Omega^2))$. The right-hand side is represented by the cocycle $v_{ab} \lrcorner [T] = c[(v_1 dn_2 \wedge dn_3 - v_2 dn_1 \wedge dn_3 + v_3 dn_1 \wedge dn_2) / n_1 n_2 n_3]$ with $v_m = dn_m(v_{ab})$, and ρ_T sends it to $c(\kappa_T(v)_1 dn_2 \wedge dn_3 - \kappa_T(v)_2 dn_1 \wedge dn_3 + \kappa_T(v)_3 dn_1 \wedge dn_2)$ restricted to $T \cap U$. As $\mu_T(\mathcal{E}|_U)c \neq 0$, every $\kappa_T(v)_m$ restricts to zero on $T \cap U$; the dn_m are a basis of $N_0^\vee$, so $\kappa_T(v)$ lies in the kernel of the restriction $H^1(T, \mathcal{O}_T) \otimes N_0 \rightarrow H^1(T \cap U, \mathcal{O}_T) \otimes N_0$, for every $v \in T\mathcal{D}_h$.

We compute that kernel. Let C be the union of the components of $T \setminus U$ of codimension at least two in T , so that $T \cap U = (T \setminus S) \setminus C$. Since T is smooth, $\mathcal{O}_T$ has depth at least two along C , the local cohomology $H_C^1(T \setminus S, \mathcal{O})$ vanishes, and $H^1(T \setminus S, \mathcal{O}) \rightarrow H^1(T \cap U, \mathcal{O})$ is injective; so the kernel is that of $H^1(T, \mathcal{O}) \rightarrow H^1(T \setminus S, \mathcal{O})$, which is zero when $S = \emptyset$. Let $S \neq \emptyset$, let $H \subseteq T$ be the connected component of the stabiliser $\{x \in T : S + x = S\}$, an abelian subvariety, and let $q: T \rightarrow T' = T/H$. Each component of S is H -invariant, H being connected, so $S = q^{-1}(S')$ for the divisor $S' = q(S)$ of T' , whose stabiliser is the finite group of components of the stabiliser of S ; hence S' is ample [55, §6, Application 1] and $T' \setminus S'$ is affine. By Poincaré reducibility there is an abelian subvariety $H' \subseteq T$ with $H \times H' \rightarrow T$ an isogeny, and the composite $H \times H' \rightarrow T \rightarrow T'$ is the projection to H' followed by the isogeny $H' \rightarrow T'$. So the preimage of $T \setminus S$ in $H \times H'$ is $H \times U'$ with $U' \subseteq H'$ the preimage of the affine $T' \setminus S'$ under a finite map, which is affine. Pullback along the finite étale map $H \times U' \rightarrow T \setminus S$ is injective on $H^1(\mathcal{O})$, since composed with the trace it is multiplication by the degree, and by the Künneth formula and $H^1(U', \mathcal{O}) = 0$,

$$H^1(H \times U', \mathcal{O}) = H^1(H, \mathcal{O}) \otimes H^0(U', \mathcal{O}).$$

The composite $H^1(T, \mathcal{O}) \rightarrow H^1(H \times U', \mathcal{O})$ factors through $H^1(H \times H', \mathcal{O}) = H^1(H, \mathcal{O}) \oplus H^1(H', \mathcal{O})$, an isomorphism for the isogeny, followed by the restriction to $H \times U'$, which is injective on the first summand and zero on the second. So the kernel is the second summand, which is $q^* H^1(T', \mathcal{O})$: the classes of the invariant $(0, 1)$ -forms on T vanishing on the tangent space of H . Hence $\kappa_T(v)(\bar{u}) = 0$ for every $v \in T\mathcal{D}_h$ and every $u \in T_0 H$, where $H = T$ when $S = \emptyset$.

Finally we compute $\kappa_T(v)$ in the coordinates of Appendix A. Write $v = \sum_{k,l} v_{lk} d\bar{z}_k \otimes \partial_l$, so that v replaces $d\bar{z}_l$ by $\sum_k v_{lk} d\bar{z}_k$. The K -linear maps have $v_{lk} \neq 0$ only when one of k, l is at most 3 and the other at least 4, and

$$v \lrcorner \theta_h = \frac{i}{2} \sum_{k < l} (h_l v_{lk} - h_k v_{kl}) d\bar{z}_k \wedge d\bar{z}_l$$

vanishes exactly when $h_l v_{lk} = h_k v_{kl}$ for all $k < l$; so $T\mathcal{D}_h$ is the nine-dimensional space of the v with $v_{3+m,j} = Y_{mj}$ and $v_{j,3+m} = (h_{3+m}/h_j)Y_{mj}$ for $j, m \leq 3$, Y any complex 3×3 matrix, and the other entries zero. The threefold $T = Z_\zeta$ is $\{z_j + \zeta_j z_{3+j} = 0, j = 1, 2, 3\}$, with tangent space spanned by $w_j = \zeta_j \partial_j - \partial_{3+j}$ and normal space N_0 with basis $f_m = [\partial_{3+m}]$, so that $[\partial_m] = \bar{\zeta}_m f_m$. The class $\kappa_T(v)$ is that of the invariant form $\sum_{k,l} v_{lk} d\bar{z}_k|_T \otimes [\partial_l]$, and on $\bar{w}_j = \bar{\zeta}_j \bar{\partial}_j - \bar{\partial}_{3+j}$, where $d\bar{z}_j(\bar{w}_j) = \bar{\zeta}_j$ and $d\bar{z}_{3+j}(\bar{w}_j) = -1$,

$$\kappa_T(v)(\bar{w}_j) = \sum_{m=1}^3 \left(\bar{\zeta}_j Y_{mj} - \frac{h_{3+j}}{h_m} \bar{\zeta}_m Y_{jm} \right) f_m.$$

Let $0 \neq u = \sum_j u_j w_j \in T_0 T$ with $u_a \neq 0$, choose $b \neq a$, and let Y have the entry $Y_{ab} = 1$ and all others 0. Then

$$\kappa_T(v)(\bar{u}) = \bar{\zeta}_b \left(\bar{u}_b f_a - \frac{h_{3+a}}{h_b} \bar{u}_a f_b \right) \neq 0.$$

So $T_0 H = 0$: S is nonempty, H is trivial, $S = S'$ has finite stabiliser, and S is ample. $\square$

Theorem 6.27 says where the rest of the support must cut a threefold of nonzero multiplicity; the next results say where it can lie. Fix ζ with $\zeta_1 \zeta_2 \zeta_3 = \pm 1$, let $T = Z_\zeta$, fix $j \in \{1, 2, 3\}$ with $\{j, k, l\} = \{1, 2, 3\}$, and let $\zeta^{(j)}$ be ζ with the sign of the j -th coordinate changed, $T' = Z_{\zeta^{(j)}}$. On the j -th factor B_j the curves C_{ζ_j} and $C_{-\zeta_j}$ meet in the set P_j of the four points of order two of C_{ζ_j} ($|2\zeta_j|^2 = 4$ by Lemma A.1), so

$$S := T \cap T' = P_j \times C_{\zeta_k} \times C_{\zeta_l}$$

is a disjoint union of four surfaces, and $Z = T \cup T'$ is cut out near S by $n_j n'_j = n_k = n_l = 0$, where $n'_j = z_j - \zeta_j z_{3+j}$ is the j -th linear form of T' . For $y \in C_{\zeta_k} \times C_{\zeta_l}$ let $\Gamma_y = C_{\zeta_j} \times \{y\} \subseteq T$ and $\Gamma'_y = C_{-\zeta_j} \times \{y\} \subseteq T'$; each meets S in the four points $P_j \times \{y\}$. On C_{ζ_j} one has $z_{3+j} = -\bar{\zeta}_j z_j$, so $n'_j|_{\Gamma_y} = 2z_j$ is a coordinate vanishing at the four points and $dn'_j|_{\Gamma_y} = 2dz_j$ is a nonzero invariant form; on $C_{-\zeta_j}$ likewise $n_j|_{\Gamma'_y} = 2z_j$ and $dn_j|_{\Gamma'_y} = 2dz_j$.

Lemma 6.28 (The curves through the four points). *Let $y = (y_k, y_l) \in C_{\zeta_k} \times C_{\zeta_l}$ with $y_k \notin C_\xi$ for $\xi \neq \zeta_k$ and $y_l \notin C_\xi$ for $\xi \neq \zeta_l$, which excludes finitely many values of each coordinate. The threefolds Z_ξ , $\xi \in \mu_4^3$, meeting Γ_y or Γ'_y are those with $\xi_k = \zeta_k$ and $\xi_l = \zeta_l$; among the thirty-two with $\xi_1 \xi_2 \xi_3 = \pm 1$ they are T and T' alone.*

Proof. $Z_\xi \cap \Gamma_y = (C_{\zeta_j} \cap C_{\zeta_j}) \times (\{y_k\} \cap C_{\xi_k}) \times (\{y_l\} \cap C_{\xi_l})$; the first factor contains the origin, and $y_k \in C_{\xi_k}$ for $\xi_k \neq \zeta_k$ only for the finitely many $y_k \in C_{\xi_k} \cap C_{\zeta_k}$. The same holds for Γ'_y . The condition $\xi_k = \zeta_k$, $\xi_l = \zeta_l$, $\xi_1 \xi_2 \xi_3 = \pm 1$ leaves $\xi_j = \pm \zeta_j$. $\square$

Theorem 6.29 (Two threefolds meeting along a surface). *Let $\mathcal{E}$ be as above and let $U \subseteq A_0$ be open such that every cohomology sheaf of $\mathcal{E}|_U$ is supported in $Z = T \cup T'$ and $\Gamma_y \cup \Gamma'_y \subseteq U$ for general $y \in C_{\zeta_k} \times C_{\zeta_l}$. Then $\mu_T(\mathcal{E}|_U) = \mu_{T'}(\mathcal{E}|_U) = 0$.*

Proof. Let $\mu = \mu_T(\mathcal{E}|_U)$ and $\mu' = \mu_{T'}(\mathcal{E}|_U)$, so that by Lemma 6.26(iii)

$$\text{ch}_3^Z(\mathcal{E}|_U) = \mu[T]_Z + \mu'[T']_Z.$$

Let $v \in T\mathcal{D}_h$ be the vector whose matrix Y , in the notation of the proof of Theorem 6.27, is E_{jk} , the matrix with the single entry $Y_{jk} = 1$. By (6.7) and Lemma 6.26(i) the class $v \lrcorner \text{ch}_3^Z(\mathcal{E}|_U)$ vanishes in $H_Z^4(U, \Omega^2)$. The Mayer–Vietoris sequence for supports

$$H_S^4(U, \Omega^2) \rightarrow H_T^4(U, \Omega^2) \oplus H_{T'}^4(U, \Omega^2) \rightarrow H_Z^4(U, \Omega^2)$$

is exact, so some $s \in H_S^4(U, \Omega^2)$ maps to the pair

$$(\mu v \lrcorner [T], \pm \mu' v \lrcorner [T']).$$

Near S the forms $dn_j, dn'_j, dn_k, dn_l, dz_k, dz_l$ are a frame of Ω^1 , the functions n_j, n'_j, n_k, n_l are transverse coordinates to S , and the coordinates z_k, z_l of T restrict to coordinates on each component of S ; as S is smooth of codimension four, $H_S^4(U, \Omega^2) = H^0(S \cap U, \mathcal{H}_S^4(\Omega^2))$, and s is a Laurent tail

$$s = \sum_{\alpha, \beta, \gamma, \delta \geq 1} c_{\alpha\beta\gamma\delta} n_j^{-\alpha} n'_j{}^{-\beta} n_k^{-\gamma} n_l^{-\delta}$$

in each coefficient of the frame of Ω^2 , the $c_{\alpha\beta\gamma\delta}$ holomorphic functions on $S \cap U$. The map $H_S^4(U, \Omega^2) \rightarrow H_T^4(U, \Omega^2) = H^1(T \cap U, \mathcal{H}_T^3(\Omega^2))$ forgets the support: since $R\Gamma_S = R\Gamma_S R\Gamma_T$ it is the connecting map of $0 \rightarrow \mathcal{H}_T^3(\Omega^2) \rightarrow j_* \mathcal{H}_T^3(\Omega^2)|_{T \setminus S} \rightarrow \mathcal{H}_S^1(\mathcal{H}_T^3(\Omega^2)) \rightarrow 0$, which sends the tail s , read as a section over $(T \cap U) \setminus S$ of $\mathcal{H}_T^3(\Omega^2)$ whose coefficients have poles along S , to its class. Applying ρ_T to that class gives the class in $H^1(T \cap U, \mathcal{O}_T)$ of the principal parts $\sum_{\beta} c_{1\beta 11} n'_j{}^{-\beta}$ along S , and symmetrically for T' with $\sum_{\alpha} c_{\alpha 111} n_j^{-\alpha}$.

We compare the coefficients of $dn_j \wedge dn_l$. In $v_{\perp}[T']$ this coefficient is zero, because $[T'] = c[dn'_j \wedge dn_k \wedge dn_l / n'_j n_k n_l]$ contains no dn_j ; in $v_{\perp}[T]$ it is $\pm c[dn_k(v) / n_j n_k n_l]$, and ρ_T sends $\mu v_{\perp}[T]$ to $\pm \mu c \kappa_T(v)_k$. Let $c_{\alpha\beta\gamma\delta}$ be the coefficients of $dn_j \wedge dn_l$ in s . Restricting to the curve $\Gamma'_y \subseteq U$, y general, the T' side gives that the class of the principal parts $\sum_{\alpha} c_{\alpha 111}(p) n_j^{-\alpha}$ at the four points $p \in P_j \times \{y\}$ vanishes in $H^1(\Gamma'_y, \mathcal{O})$; by Serre duality its pairing with the holomorphic form $dn_j|_{\Gamma'_y}$, the sum of the residues $\sum_p c_{1111}(p)$, is zero. Restricting to $\Gamma_y \subseteq U$, the T side gives that the class of the principal parts $\sum_{\beta} c_{1\beta 11}(p) n'_j{}^{-\beta}$ equals $\pm \mu c \kappa_T(v)_k|_{\Gamma_y}$ in $H^1(\Gamma_y, \mathcal{O})$. Pair both with $dn'_j|_{\Gamma_y}$. With the normalisation in which a class represented by a Dolbeault form ϕ pairs with ω to $(2\pi i)^{-1} \int \phi \wedge \omega$, a class given by principal parts g_p pairs to $\sum_p \text{Res}_p(g_p \omega)$ up to a sign independent of the class; so the left side pairs to $\pm \sum_p c_{1111}(p) = 0$. The class $\kappa_T(v)_k$ is represented by the invariant form $\sum_b (v_{kb} + \zeta_k v_{3+k,b}) d\bar{z}_b|_T$, since $dn_k(\partial_k) = 1$, $dn_k(\partial_{3+k}) = \zeta_k$ and dn_k kills the other ∂_m ; on Γ_y only $d\bar{z}_j$ and $d\bar{z}_{3+j} = -\zeta_j d\bar{z}_j$ survive, and the entries of v for $Y = E_{jk}$ are $v_{k,3+j} = h_{3+j}/h_k$ and $v_{3+k,j} = 0$ among those that occur, so

$$\begin{aligned} \kappa_T(v)_k|_{\Gamma_y} &= -\zeta_j \frac{h_{3+j}}{h_k} d\bar{z}_j|_{\Gamma_y}, \\ \frac{1}{2\pi i} \int_{\Gamma_y} \kappa_T(v)_k \wedge dn'_j &= -\frac{\zeta_j h_{3+j}}{\pi i h_k} \int_{C_{\zeta_j}} d\bar{z}_j \wedge dz_j \neq 0. \end{aligned}$$

Hence $\mu = 0$. The coefficients of $dn'_j \wedge dn_l$ give $\mu' = 0$ in the same way, with the roles of T and T' exchanged: there $v_{\perp}[T]$ has coefficient zero, the restriction to Γ_y paired with dn'_j gives $\sum_p c'_{1111}(p) = 0$ for the coefficients c' of that component, and on Γ'_y , where $d\bar{z}_{3+j} = \zeta_j d\bar{z}_j$, the class $\kappa_{T'}(v)_k|_{\Gamma'_y} = \zeta_j (h_{3+j}/h_k) d\bar{z}_j$ pairs with dn_j to a nonzero number. $\square$

Proposition 6.30 (Where the rest of the support lies). *Let $\mathcal{E}$ be as above, let $\zeta_1 \zeta_2 \zeta_3 = \pm 1$, and suppose that $T = Z_{\xi}$ lies in the support of $\mathcal{E}$ with $\mu_T(\mathcal{E}|_{U_T}) \neq 0$, where U_T is the complement of the components of the support other than T . Then for each $j \in \{1, 2, 3\}$ there is an irreducible component R_j of the support of $\mathcal{E}$ of dimension at least three, not one of the thirty-two threefolds Z_{ξ} with $\xi_1 \xi_2 \xi_3 = \pm 1$, which meets $\Gamma_y \cup \Gamma'_y$ for every $y \in C_{\zeta_k} \times C_{\zeta_l}$; and $R_j \cap (T \cup T')$ has an irreducible component that maps onto $C_{\zeta_k} \times C_{\zeta_l}$.*

Proof. Let R be the union of the components of the support not contained in $T \cup T'$ and $U = A_0 \setminus R$. Then every cohomology sheaf of $\mathcal{E}|_U$ is supported in $T \cup T'$ and $\mu_T(\mathcal{E}|_U) = \mu_T(\mathcal{E}|_{U_T}) \neq 0$, so by Theorem 6.29 the curves $\Gamma_y \cup \Gamma'_y$ are not contained in U for general y : the closed set $R \cap (T \cup T')$ projects onto a dense, hence the whole, subset of $C_{\zeta_k} \times C_{\zeta_l}$ under the proper projection (z_k, z_l) . Some irreducible component R_j of R then has $R_j \cap (T \cup T')$ projecting onto $C_{\zeta_k} \times C_{\zeta_l}$, so one of its components does, and R_j meets every $\Gamma_y \cup \Gamma'_y$. If $\dim R_j \leq 2$ then $R_j = R_j \cap (T \cup T') \subseteq T \cup T'$, which is excluded; so $\dim R_j \geq 3$. By Lemma 6.28 no Z_{ξ} with $\xi_1 \xi_2 \xi_3 = \pm 1$ other than T and T' meets Γ_y or Γ'_y for general y . $\square$

Corollary 6.31 (Objects supported on the thirty-two threefolds). *No perfect complex $\mathcal{E}$ on A_0 whose support lies in the union of the thirty-two threefolds Z_{ξ} , $\xi_1 \xi_2 \xi_3 = \pm 1$, has a Chern character of*

polarised shape with respect to some θ_h and meets the criterion. In particular no convolution of pieces $\mathcal{O}_{Z_\xi} \otimes L^k[s]$ over these threefolds, at any twists and shifts, does so; the same holds for the thirty-two threefolds with $\xi_1 \xi_2 \xi_3 = \pm i$.

Proof. Suppose $\mathcal{E}$ has polarised shape and meets the criterion. For each ξ with $\xi_1 \xi_2 \xi_3 = \pm 1$ and each j , let U be the complement of the thirty threefolds other than Z_ξ and $Z_{\xi(j)}$. Every cohomology sheaf of $\mathcal{E}|_U$ is supported in $Z_\xi \cup Z_{\xi(j)}$, and $\Gamma_y \cup \Gamma'_y \subseteq U$ for general y by Lemma 6.28; so $\mu_{Z_\xi}(\mathcal{E}) = 0$ by Theorem 6.29. Two distinct threefolds meet in dimension at most two, so Lemma 6.26(iii) with $U = A_0$ and Z the union of the thirty-two gives $\text{ch}_3^Z(\mathcal{E}) = \sum_\xi \mu_{Z_\xi}(\mathcal{E})[Z_\xi]_Z = 0$, and forgetting supports, $\text{ch}_3(\mathcal{E}) = 0$. But $\text{ch}_3(\mathcal{E}) = N\omega + c_3\theta_h^3$ with $N\omega \neq 0$, and $\omega \in W$ and θ_h^3 are linearly independent, W having the weights $\pm 2(e_1 + e_2 + e_3)$ and θ_h^3 weight zero (Lemma 6.21). A convolution of pieces over the threefolds is supported in their union. The automorphism $z_4 \mapsto iz_4$ of A_0 preserves θ , every θ_h and W , and exchanges the two families of thirty-two threefolds. $\square$

Corollary 6.31 treats one family at a time; the threefolds of both families together need one more relation, read off from a line of four of them. Fix $j \in \{1, 2, 3\}$ with $\{j, k, l\} = \{1, 2, 3\}$ and $\xi_k, \xi_l \in \mu_4$. For $a \in \mu_4$ let $T_a = Z_x$ with $x_j = a$, $x_k = \xi_k$, $x_l = \xi_l$; two of the four lie in each family. Let $C_a = \{z_j + az_{3+j} = 0\} \subseteq B_j$, let $Q = \bigcup_a C_a$, a divisor on B_j , and let $\Pi = C_{\xi_k} \times C_{\xi_l} \subseteq B' = B_k \times B_l$, a surface, so that $A_0 = B_j \times B'$ and

$$Z := \bigcup_{a \in \mu_4} T_a = Q \times \Pi.$$

Two of the T_a meet in $(C_a \cap C_b) \times \Pi$, a union of surfaces, so Lemma 6.26(iii) applies to Z . The divisor Q is ample: the connected component H of its stabiliser $\{x \in B_j : Q + x = Q\}$ is an abelian subvariety which, being connected, leaves each C_a invariant, so $H \subseteq C_1 \cap C_i$, a finite set, and Q has finite stabiliser [55, §6, Application 1]. Hence $B_j \setminus Q$ is affine.

Lemma 6.32 (The four threefolds through a surface). *Let $\mathcal{E}$ be as above and let $U \subseteq A_0$ be open such that every cohomology sheaf of $\mathcal{E}|_U$ is supported in Z and $Q \times \{y\} \subseteq U$ for general $y \in \Pi$. Then the multiplicities $\mu_a = \mu_{T_a}(\mathcal{E}|_U)$ satisfy*

$$\sum_{a \in \mu_4} \mu_a = 0, \quad \sum_{a \in \mu_4} a \mu_a = 0,$$

that is, $\mu_a = ma^2$ for one integer m : $\mu_1 = \mu_{-1} = -\mu_i = -\mu_{-i}$. In particular, if one of the four threefolds carries no cohomology sheaf of $\mathcal{E}|_U$, none of them has nonzero multiplicity.

Proof. Let $n_a = z_j + az_{3+j}$, $n_k = z_k + \xi_k z_{3+k}$, $n_l = z_l + \xi_l z_{3+l}$ and $f = \prod_a n_a$, so that near Z the set Z is cut out by $f = n_k = n_l = 0$, a regular sequence, $Q = \{f = 0\}$ in B_j and $\Pi = \{n_k = n_l = 0\}$ in B' . As for one threefold, the pair (n_k, n_l) and the retraction along $\partial_{3+k}, \partial_{3+l}$ identify a tubular neighbourhood of Π in B' with $\Pi \times D^2$, and the local cohomology $\mathcal{H}_Z^3(F)$ of a locally free sheaf F , computed by the Čech complex of the cover of the complement of Z by $\{f \neq 0\}, \{n_k \neq 0\}, \{n_l \neq 0\}$, is the sheaf on Z of Laurent tails $\sum_{\beta, \gamma \geq 1} g_{\beta\gamma} n_k^{-\beta} n_l^{-\gamma}$ whose coefficients $g_{\beta\gamma}$ are sections of $\mathcal{H}_Q^1(F)$: over $W \times V$, with $W \subseteq B_j$ and $V \subseteq \Pi$ open and Stein, a coefficient is a section of F on $(W \setminus Q) \times V$ modulo the sections on $W \times V$. In particular $\mathcal{H}_{C_a}^1(F) \subseteq \mathcal{H}_Q^1(F)$, and $[T_a] = c[dn_a \wedge dn_k \wedge dn_l / n_a n_k n_l]_Z$ has the single tail $n_k^{-1} n_l^{-1}$ with coefficient $c[dn_a / n_a] \wedge dn_k \wedge dn_l$, where $[dn_a / n_a] \in H^0(C_a, \mathcal{H}_{C_a}^1(\Omega_{B_j}^1))$ is the class of the form dn_a / n_a with a pole along C_a .

Let $v \in T\mathcal{D}_h$ have the matrix $Y = E_{jk}$, as in the proof of Theorem 6.29: then v replaces dz_{3+j} by $d\bar{z}_k$ and dz_k by $(h_{3+j}/h_k)d\bar{z}_{3+j}$ and kills the other dz_m , so $dn_a(v) = a d\bar{z}_k$, $dn_k(v) = (h_{3+j}/h_k)d\bar{z}_{3+j}$ and $dn_l(v) = 0$. By (6.7) and Lemma 6.26(i), (iii),

$$\begin{aligned} 0 &= v \lrcorner \text{ch}_3^Z(\mathcal{E}|_U) = \sum_a \mu_a v \lrcorner [T_a]_Z \\ &= c \sum_a \mu_a \left[\frac{a d\bar{z}_k dn_k \wedge dn_l - (h_{3+j}/h_k) d\bar{z}_{3+j} dn_a \wedge dn_l}{n_a n_k n_l} \right]_Z \end{aligned}$$

in $H_Z^4(U, \Omega^2) = H^1(Z \cap U, \mathcal{H}_Z^3(\Omega^2))$, the classes of the invariant $(0, 1)$ -forms $d\bar{z}_k, d\bar{z}_{3+j}$ in $H^1(U, \mathcal{O})$ acting by cup product. The decomposition $\Omega_{A_0}^2 = \Omega_{B_j}^2 \oplus (\Omega_{B_j}^1 \boxtimes \Omega_{B'}^1) \oplus \Omega_{B'}^2$ into locally free summands splits the local cohomology; $dn_k \wedge dn_l$ lies in the third summand and $dn_a \wedge dn_l$ in the second, so the second summand of the identity reads

$$[d\bar{z}_{3+j}] \cup \sum_a \mu_a \left[\frac{dn_a \wedge dn_l}{n_a n_k n_l} \right]_Z = 0$$

in $H^1(Z \cap U, \mathcal{H}_Z^3(\Omega_{B_j}^1 \boxtimes \Omega_{B'}^1))$. Reading off the coefficient of $dn_l n_k^{-1} n_l^{-1}$ in the frame dn_k, dn_l, dz_k, dz_l of $\Omega_{B'}^1$ is a morphism of sheaves of vector spaces $\mathcal{H}_Z^3(\Omega_{B_j}^1 \boxtimes \Omega_{B'}^1) \rightarrow \mathcal{H}_Q^1(\Omega_{B_j}^1) \boxtimes \mathcal{O}_\Pi$ on Z , linear over the functions pulled back from B_j ; restricting to $Q \times \{y\} \subseteq Z \cap U$ for general y , where the $(0, 1)$ -form $d\bar{z}_{3+j}$ restricts to the same form on B_j , gives

$$[d\bar{z}_{3+j}] \cup \sum_a \mu_a [dn_a/n_a] = 0$$

in $H^1(Q, \mathcal{H}_Q^1(\Omega_{B_j}^1)) = H_Q^2(B_j, \Omega_{B_j}^1)$. Since $B_j \setminus Q$ is affine, $H^1(B_j \setminus Q, \Omega^1) = 0$, and the exact sequence $H^1(B_j \setminus Q, \Omega^1) \rightarrow H_Q^2(B_j, \Omega^1) \rightarrow H^2(B_j, \Omega^1)$ shows that forgetting the support is injective on $H_Q^2(B_j, \Omega^1)$; it is compatible with the cup product by $H^1(B_j, \mathcal{O})$. The argument of Lemma 6.26(ii) in codimension one, with the scaling $z \mapsto z - (1 - \lambda)\bar{a}n_a(z)\partial_{3+j}$ of B_j , which fixes C_a , multiplies n_a by λ and preserves z_j , shows that $\text{ch}_1^{C_a}(\mathcal{O}_{C_a}) = c_1[dn_a/n_a]$ in $H^0(C_a, \mathcal{H}_{C_a}^1(\Omega^1))$ with a constant c_1 , the same for the four curves since $z_{3+j} \mapsto iz_{3+j}$ permutes them and carries n_a to n_{ia} , and $c_1 \neq 0$ since forgetting the support gives $\text{ch}_1(\mathcal{O}_{C_a}) = [C_a] \neq 0$. So the image of $[dn_a/n_a]$ in $H^1(B_j, \Omega^1)$ is $c_1^{-1}[C_a]$, and

$$[d\bar{z}_{3+j}] \cup \sum_a \mu_a [C_a] = 0 \quad \text{in } H^2(B_j, \Omega^1) = H^{1,2}(B_j).$$

By Lemma A.1(i) with $(u, v) = (z_j, z_{3+j})$, $[C_a] = \frac{i}{2}(dz_j + adz_{3+j}) \wedge (d\bar{z}_j + \bar{a}d\bar{z}_{3+j})$, so

$$[C_a] \wedge d\bar{z}_{3+j} = \frac{i}{2}(dz_j + adz_{3+j}) \wedge d\bar{z}_j \wedge d\bar{z}_{3+j},$$

and the invariant forms $dz_j \wedge d\bar{z}_j \wedge d\bar{z}_{3+j}$ and $dz_{3+j} \wedge d\bar{z}_j \wedge d\bar{z}_{3+j}$ are linearly independent in $H^{1,2}(B_j)$. Their coefficients $\sum_a \mu_a$ and $\sum_a a\mu_a$ vanish. Separating real and imaginary parts of the second relation gives $\mu_1 = \mu_{-1}$ and $\mu_i = \mu_{-i}$, and the first then gives $\mu_i = -\mu_1$; so $\mu_a = \mu_1 a^2$, and if some μ_a vanishes all do. $\square$

Theorem 6.33 (Objects supported on the sixty-four threefolds). *No perfect complex $\mathcal{E}$ on A_0 whose support lies in the union of the sixty-four threefolds Z_x , $x \in \mu_4^3$, has a Chern character of polarised shape with respect to some θ_h and meets the criterion. In particular no convolution of pieces $\mathcal{O}_{Z_x} \otimes L^k[s]$ over threefolds of both families, at any twists and shifts, does so.*

Proof. Suppose $\mathcal{E}$ has polarised shape and meets the criterion, and write $\mu_x = \mu_{Z_x}(\mathcal{E})$. Fix j and (ξ_k, ξ_l) as above and let U be the complement of the sixty threefolds $Z_{x'}$ with $(x'_k, x'_l) \neq (\xi_k, \xi_l)$. Every cohomology sheaf of $\mathcal{E}|_U$ is supported in $Z = \bigcup_a T_a$, and $\mu_{T_a}(\mathcal{E}|_U) = \mu_x$ for $x_j = a$, the generic point of T_a lying in U . For $y = (y_k, y_l) \in \Pi$ with $y_k \notin C_\xi$ for $\xi \neq \xi_k$ and $y_l \notin C_\xi$ for $\xi \neq \xi_l$, which excludes finitely many values of each coordinate, $Z_{x'} \cap (Q \times \{y\}) = (C_{x'_j} \cap Q) \times (\{y_k\} \cap C_{x'_k}) \times (\{y_l\} \cap C_{x'_l})$ is empty unless $(x'_k, x'_l) = (\xi_k, \xi_l)$; so $Q \times \{y\} \subseteq U$ for general y , and Lemma 6.32 gives that μ_x/x_j^2 does not depend on x_j when x_k, x_l are fixed. This holds for $j = 1, 2, 3$, so $\mu_x/(x_1^2 x_2^2 x_3^2) = \mu_x/\sigma(x)^2$ is independent of each coordinate of x separately, hence constant: $\mu_x = m\sigma(x)^2$ for one integer m . Two distinct threefolds meet in dimension at most two, so Lemma 6.26(iii) with $U = A_0$ and Z the union of the sixty-four, followed by forgetting supports and (A.5) with $m = 2$ there, gives

$$\text{ch}_3(\mathcal{E}) = \sum_x \mu_x [Z_x] = m \sum_{\omega \in \mu_4} \omega^2 S(\omega) = m \prod_{j=1}^3 T_{2,j} = 0.$$

But $\text{ch}_3(\mathcal{E}) = N\omega + c_3\theta_h^3 \neq 0$, as in the proof of Corollary 6.31. A convolution of pieces over the threefolds is supported in their union. $\square$

The theorem contains Corollary 6.31, since a complex supported on one family has $\mu_x = 0$ on the other, hence $m = 0$; the corollary is kept for its proof, which localises to two threefolds and is what Proposition 6.30 uses. The relations of Lemma 6.32 hold along every line of four threefolds in the support, whatever else the support contains, as long as the rest of the support misses $Q \times \{y\}$ for general y .

Theorem 6.33 removes every convolution built from threefold pieces alone, of one family or of both, at any twists and shifts; it leaves the convolutions whose support reaches outside the sixty-four threefolds. For these the next two results say what the rest of the support can be. By (A.4) the Weil class is $\omega = (p + qi)\alpha + (p - qi)\bar{\alpha}$ with $p, q \in \mathbb{Q}$; for $x \in \mu_4^3$ let $\sigma(x) = x_1x_2x_3$ and $c(x) = (x_1^2, x_2^2, x_3^2) \in \{\pm 1\}^3$. The fibres of c are the eight orbits of the group $\{\pm 1\}^3$ acting on μ_4^3 by sign changes; $\sigma(x)^2 = \prod_j c(x)_j$, so the four orbits with $\prod_j c_j = 1$ make up the family $\sigma = \pm 1$ and the other four the family $\sigma = \pm i$.

Lemma 6.34 (The four orbits). *Let $v: \mu_4^3 \rightarrow \mathbb{Q}$ satisfy $\sum_x v_x [Z_x] \equiv N\omega$ modulo $\theta_h H^4(A_0, \mathbb{C})$. Then for every $c \in \{\pm 1\}^3$*

$$\sum_{c(x)=c} v_x \sigma(x) = \begin{cases} Np/4, & \prod_j c_j = 1, \\ -iNq/4, & \prod_j c_j = -1. \end{cases}$$

So if $Np \neq 0$ each orbit of the family $\sigma = \pm 1$ carries an x with $v_x \neq 0$, and if $Nq \neq 0$ each orbit of the family $\sigma = \pm i$ does.

Proof. For $\varepsilon \in \{\pm 1\}^3$ let $X_\varepsilon = \prod_j (c_j \text{ if } \varepsilon_j = 1, \bar{c}_j \text{ if } \varepsilon_j = -1)$, of weight 2ε ; X_ε spans the part of $H^6(A_0, \mathbb{C})$ of that weight, $H^4(A_0, \mathbb{C})$ has no class of weight $\pm 2\varepsilon$, and θ_h has weight zero, so the component of weight 2ε of $\theta_h H^4$ is zero and $\alpha = X_{(1,1,1)}$, $\bar{\alpha} = X_{(-1,-1,-1)}$. In $[Z_x] = \prod_j (a_j + b_j + \bar{x}_j c_j + x_j \bar{c}_j)$ the coefficient of X_ε is $\prod_{\varepsilon_j=1} \bar{x}_j \prod_{\varepsilon_j=-1} x_j = \sigma(x) \prod_{\varepsilon_j=1} \bar{x}_j^2 = \sigma(x) \prod_{\varepsilon_j=1} c(x)_j$, as $x_j^4 = 1$. Writing $W_c = \sum_{c(x)=c} v_x \sigma(x)$, the coefficient of X_ε in $\sum_x v_x [Z_x]$ is $\sum_c W_c \chi_\varepsilon(c)$ with $\chi_\varepsilon(c) = \prod_{\varepsilon_j=1} c_j$, and the χ_ε are the eight characters of $\{\pm 1\}^3$. The hypothesis says that this coefficient is $N(p + qi)$ for $\varepsilon = (1, 1, 1)$, $N(p - qi)$ for $\varepsilon = (-1, -1, -1)$ and 0 otherwise. Fourier inversion gives $8W_c = N(p + qi)c_1c_2c_3 + N(p - qi)$, which is the claim. $\square$

Proposition 6.35 (A smooth complete intersection cannot carry the rest). *Let $\mathcal{E}$ be as above and suppose that its support lies in $Y \cup \bigcup_{\sigma(x)=\pm 1} Z_x$, where $Y \subseteq A_0$ is a smooth complete intersection of dimension $d \leq 5$ of divisors in the linear systems $|L_h^{m_i}|$ of powers of the polarisation L_h . Then:*

- (i) *$Np \neq 0$, and every orbit of the family $\sigma = \pm 1$ contains a threefold $Z_x \not\subseteq Y$ along which $\mathcal{E}$ has nonzero multiplicity.*
- (ii) *If Y meets every Z_x in dimension at most one, then $\mathcal{E}$ does not exist: no such complex has polarised shape and meets the criterion.*

Proof. (i) Let $Y_1, \dots, Y_c$ be the divisors, $c = 6 - d \geq 1$. Each $A_0 \setminus Y_i$ is affine and smooth, as is every intersection of them, so has vanishing homology above degree six [7]; the Mayer–Vietoris spectral sequence of the cover of $A_0 \setminus Y$ by these c open sets gives $H_q(A_0 \setminus Y) = 0$ for $q > 5 + c$, and Lefschetz duality $H^q(A_0, Y) \cong H_{12-q}(A_0 \setminus Y)$ gives $H^q(A_0, Y) = 0$ for $q \leq d$. So $H^{2d-6}(A_0, \mathbb{Q}) \rightarrow H^{2d-6}(Y, \mathbb{Q})$ is onto, as $2d - 6 < d$. For $\eta \in H^{2d-6}(A_0)$ the projection formula gives $i_* i^* \eta = \eta \cup [Y] = (\prod_i m_i) \theta_h^c \eta$, so $i_* H^{2d-6}(Y) \subseteq \theta_h H^4(A_0)$. Now let $\Sigma = Y \cup \bigcup_{x \in I} Z_x$, I the set of x with $\sigma(x) = \pm 1$ and $Z_x \subseteq \text{Supp } \mathcal{E}$, $Z_x \not\subseteq Y$. By the Riemann–Roch theorem for singular varieties [30, Ch. 18], applied to the cohomology sheaves of $\mathcal{E}$, which are coherent sheaves on infinitesimal neighbourhoods of Σ , the class $\text{ch}_3(\mathcal{E}) \in H^6(A_0, \mathbb{Q})$ is the image of a class $\tau \in A_3(\Sigma)_{\mathbb{Q}}$ (the Todd class of A_0 being 1). The localisation sequence $A_3(Y) \rightarrow A_3(\Sigma) \rightarrow A_3(\Sigma \setminus Y) \rightarrow 0$ is exact, $A_3(\Sigma \setminus Y)$ is free on the classes of the three-dimensional components $Z_x \setminus Y$, $x \in I$, and the component of τ there is $\sum_{x \in I} \mu_{Z_x}(\mathcal{E}) [Z_x \setminus Y]$, the top-dimensional part of the Riemann–Roch class of a sheaf being its cycle. Hence

$$\text{ch}_3(\mathcal{E}) - \sum_{x \in I} \mu_{Z_x}(\mathcal{E}) [Z_x] \in i_* H^{2d-6}(Y) \subseteq \theta_h H^4(A_0),$$

and $\text{ch}_3(\mathcal{E}) = N\omega + c_3\theta_h^3$ with $\theta_h^3 \in \theta_h H^4$. Lemma 6.34 applies to $v_x = \mu_{Z_x}(\mathcal{E})$ for $x \in I$ and $v_x = 0$ otherwise: since v vanishes on the family $\sigma = \pm i$, $Nq = 0$, so $Np \neq 0$ as $N\omega \neq 0$, and each orbit of the family $\sigma = \pm 1$ contains $x \in I$ with $\mu_{Z_x}(\mathcal{E}) \neq 0$.

(ii) Take x from (i), $T = Z_x$. The multiplicity of $\mathcal{E}$ along T on the complement of the other components equals $\mu_{Z_x}(\mathcal{E}) \neq 0$, so Proposition 6.30 gives a component R_1 of the support, of dimension at least three and not among the thirty-two threefolds, hence contained in Y , such that $R_1 \cap (Z_x \cup Z_{x(1)})$ contains a surface. But $Y \cap (Z_x \cup Z_{x(1)})$ has dimension at most one. $\square$

The hypotheses of Proposition 6.35(ii) hold, for instance, when the rest of the support is a smooth curve or a smooth surface cut out by divisors of the polarisation and meeting the threefolds in finitely many points; the smooth divisors Y meet every Z_x in a surface and are only constrained by (i) and by Theorem 6.27. When the support is large the localised obstruction says nothing at all:

Proposition 6.36 (Where the localised obstruction stops). *Let $\mathcal{E}$ be as above and suppose that its support contains an ample effective divisor Y . Then $v \lrcorner \text{ch}_{k+1}^Z(\mathcal{E}|_U) = 0$ for every $v \in T\mathcal{D}_h$, every $k \geq 0$, every open $U \subseteq A_0$ and every closed $Z \subseteq U$ containing the support of $\mathcal{E}|_U$. In particular a convolution of line-bundle pieces $L^i[t_i]$ and threefold pieces with $\sum_i (-1)^{t_i} \neq 0$ has support A_0 , and the criterion for it is decided by its products alone.*

Proof. Let Σ be the support of $\mathcal{E}$ and $X = A_0 \setminus Y$, an affine variety, so that $H^1(X, T_X) = 0$ and $H^{k+1}(X, \Omega^k) = 0$ for $k \geq 0$. The restriction of v to X vanishes, so the image of $v \lrcorner \text{ch}_{k+1}^\Sigma(\mathcal{E})$ under $H_\Sigma^{k+2}(A_0, \Omega^k) \rightarrow H_{\Sigma \setminus Y}^{k+2}(X, \Omega^k)$, which is $v|_X \lrcorner \text{ch}_{k+1}^{\Sigma \setminus Y}(\mathcal{E}|_X)$, vanishes; by the exact sequence

$$H_Y^{k+2}(A_0, \Omega^k) \rightarrow H_\Sigma^{k+2}(A_0, \Omega^k) \rightarrow H_{\Sigma \setminus Y}^{k+2}(X, \Omega^k)$$

the class $v \lrcorner \text{ch}_{k+1}^\Sigma(\mathcal{E})$ comes from a class $b \in H_Y^{k+2}(A_0, \Omega^k)$. The sequence

$$H^{k+1}(X, \Omega^k) \rightarrow H_Y^{k+2}(A_0, \Omega^k) \rightarrow H^{k+2}(A_0, \Omega^k)$$

shows that b is determined by its image in $H^{k+2}(A_0, \Omega^k)$, which is $v \lrcorner \text{ch}_{k+1}(\mathcal{E}) = 0$. So $v \lrcorner \text{ch}_{k+1}^\Sigma(\mathcal{E}) = 0$, and its restriction to U , followed by the enlargement of supports from $\Sigma \cap U$ to Z , is $v \lrcorner \text{ch}_{k+1}^Z(\mathcal{E}|_U)$. A complex of rank $\sum_i (-1)^{t_i} \neq 0$ has a cohomology sheaf of positive rank, whose support is A_0 . $\square$

What remains at E_0^6 . By Theorem 6.33 every convolution of threefold pieces fails, of one family or of both, at any twists and shifts, which covers the threefolds at several twists of Setting 6.17; Theorem 6.23 and Remark 6.25 cover the convolutions with at most seven line bundles, and eight line bundles outside thirty-one arrangements. Left are the convolutions with eight line bundles in those thirty-one arrangements or with nine or more line bundles, with or without threefold pieces at any twists. For a convolution of line bundles the support is A_0 when $\sum_i (-1)^{t_i} \neq 0$, which happens for eight of the thirty-one arrangements, where the rank is ± 2 ; there Proposition 6.36 says that only the products decide. For the other twenty-three the rank is zero and the support is a proper closed subset containing every threefold piece; Theorem 6.27 and Propositions 6.30 and 6.35 constrain it, and the products decide the rest. The products along the chains of Proposition 6.10 and along the cycles through the line bundles are computable in the theta-function model of [19, §18], where the fourfold products of line-bundle pieces at distance at least two were found to give $\dim \text{Ext}^2 \geq 69$; we have not computed them, and (F2) at E_0^6 stands or falls with them.

6.9. Seven consecutive shifts at E_0^8

At $n = 4$ the same route was taken in [19, §18] on E_0^8 , $E_0 = \mathbb{C}/\mathbb{Z}[i]$, with the 128 line bundles L_ζ , $\zeta \in \mu_4^4$, $\prod \zeta = \pm 1$, whose signed Chern characters add up to a multiple of the Weil class, and three multiples M_1, M_2, M_3 of the polarisation; the criterion asks for $\dim \text{Ext}^2(E, E) = 104$. In the setting of [19, Lemma 18.36] the pieces of one parity, $\prod \zeta = \varepsilon$, have shifts of one parity and the pieces of the other parity have shifts of the other, and [19, Cor. 18.48] excludes every convolution whose 128 shifts lie in six consecutive integers. We extend this to seven. The groups used below are the groups $\mathcal{H}^k(W, V)$ between two pieces in the harmonic model of the convolution [19, Lemma 18.24], as in [19, Thm. 18.37 and Lemma 18.39]: for pieces W, V that differ in exactly three coordinates and have

$d_W = d_V + 1$, the groups $\mathcal{H}^3(W, V)$, of total dimension 640 over the 28 ratios W/V of this kind, are reached by no coboundary, and a term of d_E on them passes through a piece at the shift $d_W + 1$, $d_W + 2$, $d_V - 1$ or $d_V - 2$.

Proposition 6.37 (Two separated adjacent pairs). *Let E be a convolution as in [19, Lemma 18.36], let $g \geq 4$, and suppose that the shifts of the pieces L_ζ take exactly the four values s , $s - 1$, $s - g$ and $s - g - 1$. Then for every order, every choice of the shifts of the M_i and every placement of the t_i , $\dim \text{Ext}^2(E, E) \geq 640$, and E does not meet the polarised criterion.*

Proof. Let P, Q, R, T be the sets of pieces at the shifts $s, s - 1, s - g, s - g - 1$; all four are nonempty. Two shifts that differ by one belong to pieces of opposite parity, so P and Q have opposite parities, and so do R and T ; P and R have the same parity when g is even and opposite parities when g is odd. Let W and V differ in exactly three coordinates with $d_W = d_V + 1$. Then $W \in P$ and $V \in Q$, or $W \in R$ and $V \in T$, since these are the only pairs of occupied shifts at distance one. A term of d_E on $\mathcal{H}^3(W, V)$ passes through a piece at one of the shifts $s + 1, s + 2, s - 2, s - 3$ when $W \in P$, and $s - g + 1, s - g + 2, s - g - 2, s - g - 3$ when $W \in R$. Since $g \geq 4$ none of these is among $s, s - 1, s - g, s - g - 1$, so no term acts and the groups $\mathcal{H}^3(W, V)$ inject into $\text{Ext}^2(E, E)$.

Let Γ be the group of ratios of two pieces, of order 128, acting simply transitively on the pieces, and let $S_3 \subset \Gamma$ be the 28 ratios that move exactly three coordinates, with the weight given by the dimensions of the groups $\mathcal{H}^3$; every ratio in S_3 changes the parity. Put $X = P \cup T$ if g is even and $X = P \cup R$ if g is odd. In both cases X is a union of one set of each parity, and its complement is the union of the other two. The edges of the Cayley graph of (Γ, S_3) join pieces of opposite parity, so the edges between X and its complement join P to Q and R to T , and these are exactly the pairs (W, V) above. Hence the dimensions of the surviving groups add up to the weight of the cut of X . The set X is neither empty nor everything, and the proof of [19, Prop. 18.47] shows, through [19, Lemma 18.40], that every such cut weighs at least 640. $\square$

For $g = 4$ this is [19, Prop. 18.47]. Together with the results of [19, §18] it settles most arrangements in seven consecutive integers.

Corollary 6.38 (Seven consecutive shifts). *Let E be a convolution as in [19, Lemma 18.36] whose 128 pieces L_ζ have their shifts in $\{s, s - 1, \dots, s - 6\}$. Let $U \subseteq \{s, s - 2, s - 4, s - 6\}$ and $U' \subseteq \{s - 1, s - 3, s - 5\}$ be the values taken by the two parities. Then $\dim \text{Ext}^2(E, E) \geq 640$ for every order, every choice of the shifts of the M_i and every placement of the t_i , except possibly when U is one of*

$$\{s, s - 2, s - 6\}, \quad \{s, s - 4, s - 6\}, \quad \{s, s - 2, s - 4, s - 6\},$$

U' is one of

$$\{s - 1\}, \quad \{s - 3\}, \quad \{s - 5\}, \quad \{s - 1, s - 5\},$$

and (U, U') is neither $(\{s, s - 2, s - 6\}, \{s - 5\})$ nor $(\{s, s - 4, s - 6\}, \{s - 1\})$.

Proof. If the shifts do not take both values s and $s - 6$ they lie in six consecutive integers and [19, Cor. 18.48] applies; so $s, s - 6 \in U$. If U' contains two values two apart it is $\{s - 1, s - 3\}$, $\{s - 3, s - 5\}$ or $\{s - 1, s - 3, s - 5\}$, and [19, Thm. 18.41] applies to the parity at U' for every U . Otherwise U' is $\{s - 1\}$, $\{s - 3\}$, $\{s - 5\}$ or $\{s - 1, s - 5\}$. Let $U = \{s, s - 6\}$. For $U' = \{s - 3\}$ the two parities sit at $s - 3$ and at $s - 3 \pm 3$, and [19, Thm. 18.42] gives $\dim \text{Ext}^2 \geq 67584$. For $U' = \{s - 1\}$, [19, Prop. 18.43] with $y = s - 1$ applies, since the pieces at $y + 1 = s$ exist and no piece has the shift $s - 2$ or $s + 2$; for $U' = \{s - 5\}$ its second form with $y = s - 5$ applies, since the pieces at $s - 6$ exist and no piece has the shift $s - 4$ or $s - 8$. For $U' = \{s - 1, s - 5\}$ the shifts take exactly the values $s, s - 1, s - 5, s - 6$, and Proposition 6.37 applies with $g = 5$. The same two applications of [19, Prop. 18.43] settle $U = \{s, s - 4, s - 6\}$ with $U' = \{s - 1\}$, where no piece has the shift $s - 2$ or $s + 2$, and $U = \{s, s - 2, s - 6\}$ with $U' = \{s - 5\}$, where no piece has the shift $s - 4$ or $s - 8$. $\square$

Remark 6.39 (Nine of the ten remaining arrangements). Nine of the ten arrangements left by Corollary 6.38 were decided by computation, which no statement of this paper uses. For each arrangement and each distribution of the 64 pieces of each parity over its values, a lower bound for $\dim \text{Ext}^2(E, E)$ was obtained as the sum of three families of surviving groups: the groups $\mathcal{H}^4$ at spread two between

pieces of one parity on the 13 ratios of [19, Lemma 18.39(ii)], bounded below by the cut bound of [19, Lemma 18.40]; the groups $\mathcal{H}^3$ between adjacent shifts, less what the terms into $\mathcal{H}^5$ and $\mathcal{H}^6$ can remove; and the groups $\mathcal{H}^5$ at spread three, on which d_E vanishes [19, Lemma 18.39(iii)], less what the coboundaries can reach. The dimension carried by each set of edges between the sets of pieces at two shifts was bounded by the per-piece degrees, by the cut bounds and by the expander mixing lemma for the weighted Cayley graphs of Γ , with the largest nontrivial eigenvalue of each weighting, and the sum was minimised by linear programming, solved in double precision, for every distribution. Over eleven of the twelve arrangements of the statement, with the 1953 or 39711 distributions of the parity with three or four values and the 1 or 63 of the other, the least bound obtained is 125.8, against 104 required. For $U = \{s, s-2, s-4, s-6\}$ and $U' = \{s-1, s-5\}$ the bound is below 104 for 585 753 of the 2 501 793 distributions and vanishes for some of them, so the method decides nothing there and that arrangement remains open. The program is in the archive [13]. So every convolution whose 128 shifts lie in seven consecutive integers fails, by hand in eighteen of the twenty-eight arrangements and by the computation in nine others, except possibly in the arrangement just named; Proposition 6.37 settles the separated pairs at every distance.

6.10. Weil sixfolds from cyclic triple covers of curves of genus four

Abelian covers of curves give explicit Weil families, and for most of them the image of the moduli of the covers in $\mathcal{D}_F$ is meagre by a dimension count [19, Prop. 23.13(iii)]. The one boundary case is that of étale cyclic covers of degree $\ell \in \{3, 4, 6\}$ of curves of genus four, where the moduli of the covers and $\mathcal{D}_F$, $F = \mathbb{Q}(\zeta_\ell)$, $n = 3$, both have dimension nine, and the image is open if the multiplication map

$$\mu_\eta : H^0(K_B \otimes \eta) \otimes H^0(K_B \otimes \eta^{-1}) \longrightarrow H^0(K_B^{\otimes 2}) \quad (6.8)$$

is surjective at one point [19, Prop. 23.13(iv)]. For $\ell = 3$ the Prym map of these covers is generically finite, by Faber [28], and surjectivity of μ_η at a general point follows. We exhibit an explicit point at which μ_η is an isomorphism.

Proposition 6.40 (An explicit cover). *For $\lambda \in \mathbb{C} \setminus \{0, 1\}$ let $B_\lambda \subset \mathbb{P}^1 \times \mathbb{P}^1$ be the curve $x^3 + y^3 + 1 + \lambda x^3 y^3 = 0$, in affine coordinates x, y , and $\eta = \mathcal{O}(1, -1)|_{B_\lambda}$.*

- (i) *B_λ is a smooth curve of genus four with $K_{B_\lambda} = \mathcal{O}(1, 1)|_{B_\lambda}$.*
- (ii) *η has order exactly three.*
- (iii) *$H^0(K \otimes \eta)$ and $H^0(K \otimes \eta^{-1})$ are spanned by $1, x, x^2$ and by $1, y, y^2$, and μ_η is an isomorphism.*
- (iv) *The period map of the étale cyclic triple covers of curves of genus four, with values in the domain $\mathcal{D}_F$ of $(\mathbb{Q}(\sqrt{-3}), 3)$ -Weil type that contains the χ -part of H^1 of the cover $C_\lambda \rightarrow B_\lambda$ defined by η , is a local isomorphism at (B_λ, η) . Its image contains an open subset of $\mathcal{D}_F$, and the Weil classes are algebraic on every member of $\mathcal{D}_F$ as soon as they are algebraic on the abelian sixfolds $H_\chi(C_i)$ for the covers C_i of a non-meagre subset of a neighbourhood of (B_λ, η) .*

Proof. (i) In bihomogeneous coordinates the curve is $F = x_1^3 y_0^3 + x_0^3 y_1^3 + x_0^3 y_0^3 + \lambda x_1^3 y_1^3$, of bidegree $(3, 3)$. In the chart $x_0 = y_0 = 1$ the partial derivatives are $3x^2(1 + \lambda y^3)$ and $3y^2(1 + \lambda x^3)$. At a singular point with $x = 0$ the equation gives $y^3 = -1$ and then $\partial_y F = 3y^2 \neq 0$; similarly $y \neq 0$. So $x^3 = y^3 = -1/\lambda$, and the equation becomes $1 - 1/\lambda = 0$, excluded. The equation is symmetric in x and y ; the substitution $(x, y) = (1/x', \lambda^{-1/3} y')$, after multiplication by x'^3 , carries it to the same form in x', y' with λ replaced by $1/\lambda$; and the substitution $(x, y) = (\lambda^{-1/3}/x', \lambda^{-1/3}/y')$, after multiplication by $\lambda x'^3 y'^3$, carries it to itself. So the other three charts reduce to the first. By adjunction $K_B = (K_{\mathbb{P}^1 \times \mathbb{P}^1} + B)|_B = \mathcal{O}(1, 1)|_B$, of degree $(1, 1) \cdot (3, 3) = 6 = 2g - 2$, so $g = 4$.

(ii) Let D_x and D_y be the intersections of B with the lines $x = \infty$ and $y = \infty$. At $x = \infty$ the equation reads $1 + \lambda y^3 = 0$, so D_x consists of three points (∞, c) with $c^3 = -1/\lambda$, and at each of them $u = 1/x$ is a local parameter, since the derivative of $1 + u^3 y^3 + u^3 + \lambda y^3$ in y is $3\lambda c^2 \neq 0$; along B , $1 + \lambda y^3 = -u^3(1 + y^3)$, so $\varphi = y^3 + 1/\lambda$ vanishes to order three there. Its other zeros would have $y^3 = -1/\lambda$ and $x \neq \infty$, hence $x^3(1 + \lambda y^3) = 0 = 1 + y^3$, which forces $\lambda = 1$. Its poles are at D_y , three points at each of which $1/y$ is a local parameter by the same computation, of order three. So $\text{div } \varphi = 3D_x - 3D_y$, and since $D_x \in |\mathcal{O}(1, 0)|_B|$ and $D_y \in |\mathcal{O}(0, 1)|_B|$, $\eta^{\otimes 3} \cong \mathcal{O}_B$. If η were trivial, D_y would lie in $|\mathcal{O}_B(1, 0)|$.

From $0 \rightarrow \mathcal{O}(-2, -3) \rightarrow \mathcal{O}(1, 0) \rightarrow \mathcal{O}_B(1, 0) \rightarrow 0$ and $H^0(\mathcal{O}(-2, -3)) = H^1(\mathcal{O}(-2, -3)) = 0$ by the Künneth formula, every section of $\mathcal{O}_B(1, 0)$ is $a + bx$, so every divisor in that system is a fibre $x = c$ of B . The three points of D_y have $x^3 = -1/\lambda$, three distinct values of x , so D_y is not such a fibre.

(iii) $K \otimes \eta = \mathcal{O}(2, 0)|_B$ and $K \otimes \eta^{-1} = \mathcal{O}(0, 2)|_B$. The kernels $\mathcal{O}(-1, -3)$ and $\mathcal{O}(-3, -1)$ of the restrictions have no cohomology in degrees 0 and 1, again by the Künneth formula, so restriction gives $H^0(K \otimes \eta) = H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(2, 0))$, spanned by $1, x, x^2$, and $H^0(K \otimes \eta^{-1}) = H^0(\mathcal{O}(0, 2))$, spanned by $1, y, y^2$. The product of the two is $H^0(\mathcal{O}(2, 2))$, of dimension nine, and μ_η is its restriction to B , with values in $H^0(\mathcal{O}_B(2, 2)) = H^0(K^{\otimes 2})$, of dimension $3g - 3 = 9$. The kernel of the restriction is $H^0(\mathcal{O}(-1, -1)) = 0$, so μ_η is injective between spaces of dimension nine, hence an isomorphism.

(iv) By [19, Prop. 23.13(i), (ii)] the χ -part H_χ of $H^1(C_\lambda, \mathbb{Q})$, for a character χ of order three, is a polarised Hodge structure with an action of $\mathbb{Q}(\sqrt{-3})$ whose two eigenspaces have dimension $2g - 2 = 6$, with $h^{1,0}$ of the χ -eigenspace equal to $h^0(K \otimes \eta) = 3$ for the appropriate sign of η , so it is of $(\mathbb{Q}(\sqrt{-3}), 3)$ -Weil type. By Griffiths' description of the differential of the period map [82, Ch. 10], as in the proof of [19, Prop. 23.13(iv)], the transpose of the differential at (B_λ, η) is μ_η after Serre duality. Both sides have dimension nine, so by (iii) the differential is an isomorphism and the period map is a local isomorphism there; its image contains an open set $\Omega \subseteq \mathcal{D}_F$. A local biholomorphism sends a non-meagre subset of a neighbourhood to a non-meagre subset of Ω , so under the hypothesis of the last assertion the algebraic locus of the Weil classes in $\mathcal{D}_F$ is not meagre, and by Lemma 4.2 it is all of $\mathcal{D}_F$. $\square$

The curve B_λ is a cyclic cover of $\mathbb{P}^1 \times \mathbb{P}^1$ in disguise: it is invariant under μ_3^2 acting on x and y , and its quotient is the line $X + Y + 1 + \lambda XY = 0$ in the coordinates $X = x^3, Y = y^3$. For $\ell = 4$ and $\ell = 6$ the same argument needs a curve of genus four with a line bundle of order four or six for which μ_η is surjective, and we have not found one.

6.11. Integral characters at sextic CM fields

For a sextic CM field $F = F_0(\sqrt{-q})$, with F_0 totally real cubic and $q \in F_0$ totally positive, [19, Prop. 19.47] gives a count that would exclude a minimal object for the polarised criterion whose character has a component v in the flat secant space $S(0, q)$ with $\chi(v, v)$ above an explicit threshold, and [19, Prop. 19.49] shows that the condition of the count holds whenever $\chi(v, v) \leq -26$, computes $\chi(v, v) = -8(\text{Nm}(q)c_0^2 + \text{Tr}(\text{Nm}(q)q^{-1}f^2) + \text{Tr}(qg^2) + c_3^2)$ on the rational points $v(c_0, f, g, c_3)$, and exhibits integral points with $\chi = -32$ for $q = 1$.

Remark 6.41 (No exclusion over twelve cubic fields). For the twelve totally real cubic fields of discriminant 49, 81, 148, 169, 229, 257, 316, 321, 361, 404, 469, 473, whose rings of integers are generated by one element, and for $q = 1$ and $q = k + \alpha$ with α the generator and k the least integer making q totally positive, the lattice of integral points of $S(0, q)$ was computed exactly, in the model of [19, Prop. 19.49], with the Gram matrix of $-\chi/8$, which is integral. No nonzero integral point has $-\chi \leq 24$, and since $\chi \in 8\mathbb{Z}$ on the lattice, every nonzero integral point has $\chi \leq -32$ and meets the condition of [19, Prop. 19.47(iii)]. For $q = 1$ the least value $-\chi = 32$ is attained by four points of shape $N_w = 2, r^2 = 30, r^3 = 40$; for $q = k + \alpha$ it is 32 for the discriminants 49, 81 and 288, 384, 912, 352, 480, 600, 960, 544, 2400, 1208 for the other ten. So over these fields the count of [19, Prop. 19.47] excludes no integral character of a minimal object, and the route through it is closed there. This is a computation of the archive [13], and nothing in the paper depends on it.

6.12. The exceptional classes of a Mumford fourfold

Keep Setting 3.6 and write ψ for the polarisation form on V , $T = D^0$, $q_\lambda(x, y) = \text{Tr}_{F/\mathbb{Q}}(\lambda \text{trd}(xy))$ for $\lambda \in F$, and m_a for multiplication by $a \in F$ on T . The copy $T_X(-1) \subseteq V \otimes V$, $x \mapsto ((v, w) \mapsto \psi(xv, w))$, is symmetric and carries the form induced by $\psi \otimes \psi$, equal to $-4\text{trd}(xy)$ summed over the three real places, that is q_1 up to the scalar -4 . The exceptional classes of $X \times X$ are the elements m_a , $a \in F \setminus \mathbb{Q}$, of $\text{End}_{\text{Hdg}} T_X(-1) = F$ [19, Prop. 21.2], and in [19, Thm. 21.4] their algebraicity is derived from the Kuga–Satake class of a K3 surface S_λ with $T(S_\lambda)_\mathbb{Q} \cong (T, q_\lambda)$. We replace S_λ by an abelian variety.

Setting 6.42 (The sixfold Z_K). Let $K = F(\sqrt{-\delta}) \subseteq D$ be a CM quadratic extension of F inside D , $\delta \in F$ totally positive; every maximal commutative CM subfield of D is of this form since D is definite at two real places. Let D carry the left action of $G = D^1$ and the right action of the torus $K^1 = \ker(N_{K/F})$. Over $\bar{\mathbb{Q}}$, $D = D_1 \oplus D_2 \oplus D_3$ with $D_i = V_i \otimes W_i$, V_i the standard representation of the i -th factor of G and $W_i = \chi_i \oplus \chi_i^{-1}$ that of the i -th factor of K^1 . Fix a point t of the Shimura curve C of X , with Hodge cocharacter μ_X nontrivial only at the split place σ_3 , and let $\mu_Z = \mu' \mu''$, with μ' the action of μ_X by left multiplication and μ'' the cocharacter of K^1 of weight $\pm \frac{1}{2}$ on $\chi_1^{\pm 1}$, $\chi_2^{\pm 1}$ and 0 on $\chi_3^{\pm 1}$. Then $D_{\mathbb{C}}$ has type $(1, 0)$ on $V_1 \chi_1$, $V_2 \chi_2$, $V_3^{1,0} W_3$ and $(0, 1)$ on the complements, so D carries a rational Hodge structure of weight one with $h^{1,0} = 6$ and Hodge group inside $G \cdot K^1$. For $c \in K$ with $\bar{c} = -c$ put $\psi_c(x, y) = \text{Tr}_{F/\mathbb{Q}} \text{trd}(\bar{x} y c)$. The abelian sixfold $Z_K = Z_{K,t}$ is the one with $H^1(Z_K) = D$ and polarisation ψ_c for c in the cone where ψ_c is positive; over a finite cover of C these form an abelian scheme [70], and we use only the fibre at t .

Lemma 6.43. ψ_c is alternating, invariant under left multiplication by D^1 and right multiplication by K^1 , and the ψ_c , $c \in K$ with $\bar{c} = -c$, fill the three-dimensional space of all $G \times K^1$ -invariant alternating forms on D . Hence $\text{NS}(Z_K)_{\mathbb{Q}}$ has dimension three.

Proof. The reduced trace is invariant under conjugation and cyclic, so $\psi_c(y, x) = \text{Tr} \text{trd}(\bar{y} x c) = \text{Tr} \text{trd}(\bar{c} \bar{x} y) = \text{Tr} \text{trd}(\bar{x} y \bar{c})$, and ψ_c is alternating exactly when $\bar{c} = -c$. For g with $\text{nrd}(g) = 1$, $\text{trd}(\bar{g} x g y c) = \text{trd}(\bar{x} \bar{g} g y c) = \text{trd}(\bar{x} y c)$. For $k \in K^1$, $\text{trd}(\bar{k} x k y c) = \text{trd}(\bar{k} \bar{x} y k c) = \text{trd}(\bar{x} y k c \bar{k}) = \text{trd}(\bar{x} y c)$ because K is commutative and $k \bar{k} = 1$. Over $\bar{\mathbb{Q}}$ the invariant alternating forms live in the three summands $\wedge^2(V_i \otimes W_i)^{G \times K^1} = \wedge^2 V_i \otimes (\chi_i \chi_i^{-1})$, one dimension each, and the cross terms $D_i \otimes D_j$ carry the nontrivial characters $\chi_i^{\pm 1} \chi_j^{\pm 1}$; so the space has dimension $3 = \dim_{\mathbb{Q}} K^-$, which the ψ_c fill since $c \mapsto \psi_c$ is injective. Hodge classes of degree two are the invariant alternating forms, as the Hodge group lies in $G \cdot K^1$ and the invariants of $G \cdot K^1$ are the three forms, which are of type $(1, 1)$. $\square$

Lemma 6.44 (Where T sits in Z_K).

- (i) The K^1 -invariants of $\wedge^2 D_i$ form a four-dimensional space: $S^2 V_i \otimes \wedge^2 W_i$, the adjoint representation of $\mathfrak{sl}(V_i)$, and $\wedge^2 V_i \otimes \mathbf{1}$. The sum T_Z of the three adjoint pieces is defined over $\mathbb{Q}$, is a sub-Hodge structure of $H^2(Z_K)$, and is isomorphic, as a Hodge structure with G -action, to $T_X(-1)$.
- (ii) The form induced on $\wedge^2 D$ by ψ_c restricts on T_Z to q_{c^2} , up to a rational scalar independent of the place; here $c^2 = -N_{K/F}(c) = -f^2 \delta \in F$ for $c = f\sqrt{-\delta}$.

Proof. (i) $\wedge^2(V_i \otimes W_i) = S^2 V_i \otimes \wedge^2 W_i \oplus \wedge^2 V_i \otimes S^2 W_i$; the torus acts trivially on $\wedge^2 W_i = \chi_i \chi_i^{-1}$ and on one summand of $S^2 W_i = \chi_i^2 \oplus \mathbf{1} \oplus \chi_i^{-2}$. The cross terms carry nontrivial characters. The Galois group permutes the places, so T_Z is defined over $\mathbb{Q}$; as a G -representation it is the adjoint representation T . The cocharacter μ'' acts trivially on K^1 -invariants and the centre acts on $\wedge^2 D$ with weight two, so the Hodge structure on T_Z is the one induced by the adjoint action of μ' , twisted to weight two; this is the recipe that gives $T_X(-1)$ at the same point t .

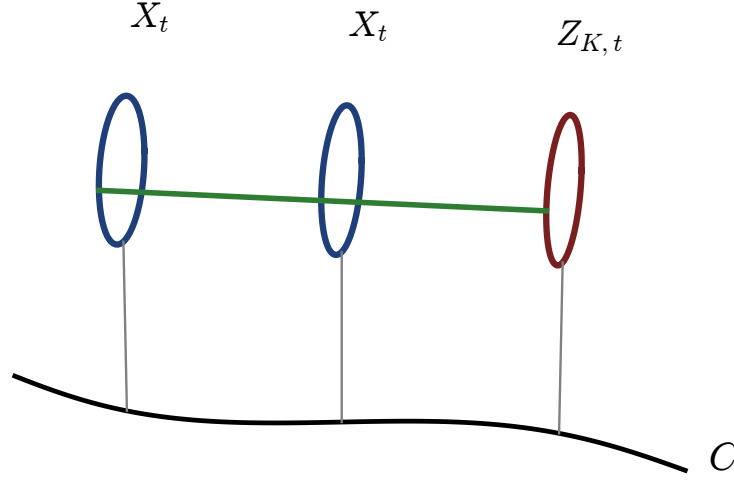
(ii) By Schur's lemma the induced form is, on each irreducible T_i , a multiple of $\text{trd}(xy)$. In the split model at the place i the element c acts on D_i by $\sigma_i(c) \text{diag}(1, -1)$, whose square is $\sigma_i(c^2)$; the multiple is $\sigma_i(c)^2$ times a rational constant that is the same at every place, since the computation in the weight basis is identical at the three places. The sum over the places is q_{c^2} up to that constant. $\square$

Theorem 6.45 (The sixfold link). Let ξ be a nonzero element of $\text{Hom}_{\text{Hdg}}(T_X(-1), T_Z) \subseteq H^4(X \times X \times Z_K)$.

- (i) $\text{Hom}_{\text{Hdg}}(T_X(-1), T_Z) = F \cdot \iota$ for the identification ι of Lemma 6.44, so $\xi = m_e \circ \iota$ for a unique $e \in F^\times$. The $G \times K^1$ -invariant classes of degree four on $X \times X \times Z_K$ have dimension 26, and ξ and $\psi \otimes \text{NS}(Z_K)$ are the only ones in the Künneth component $H^1(X) \otimes H^1(X) \otimes H^2(Z_K)$.
- (ii) If ξ is algebraic, then $\xi^\dagger \xi$, the composite with the adjoint for the Lefschetz pairing on $X \times X$ and the intersection pairing on Z_K , is an algebraic endomorphism of $T_X(-1)$ equal to $m_{a(e)}$ with $a(e) = \kappa e^2 c^2 \in F$, $\kappa \in \mathbb{Q}^\times$.

The sixfold link over the Shimura curve

$$\xi = m_e \circ \iota: T_X(-1) \rightarrow T_Z$$



$\xi_t \in H^1(X) \otimes H^1(X) \otimes H^2(Z_K)$ (green), flat along C

FIGURE 10. The sixfold link. Two copies of the Mumford fourfold X_t and the sixfold $Z_{K,t}$ sit over the same point t of the Shimura curve C ; the class ξ_t in the mixed Künneth component identifies the two copies of T and is flat along C .

- (iii) If $a(e) \notin \mathbb{Q}$, the exceptional classes of $X \times X$ are algebraic at t , hence at every point of C , and so is every Hodge class on every power of X .
- (iv) $a(e) \notin \mathbb{Q}$ for every $e \in F^\times$ if and only if K contains no imaginary quadratic field. If $K = F \cdot k$ with $k = \mathbb{Q}(\sqrt{-d})$ and $c = f\sqrt{-d}$, then $a(e) \in \mathbb{Q}$ exactly for $e \in \mathbb{Q}f^{-1}$.
- (v) Conversely, if the exceptional classes are algebraic and some algebraic cycle on $X^r \times Z_K$ acts nonzero on T_Z , then every ξ is algebraic.

Proof. (i) Both sides are the irreducible G -representation T with $\text{End}_{\text{Hdg}} = F$ [19, Prop. 21.2], so $\text{Hom}_{\text{Hdg}} = F\iota$. The count is the dimension of the kernel of the six raising and lowering operators of $\mathfrak{sl}_2^3$ on the torus-weight-zero part of $\wedge^4(V \oplus V \oplus D)$, computed by Künneth type: $(1, 1, 2)$ gives $3 + 3$, the line $F\iota$ and $\psi \otimes \text{NS}(Z_K)$; $(2, 2, 0)$ gives $4 = \mathbb{Q} \oplus F$, the Hodge classes of $X \times X$ in $H^2 \otimes H^2$; $(0, 0, 4)$ gives $6 = S^2\text{NS}(Z_K)$; $(2, 0, 2)$ and $(0, 2, 2)$ give 3 each; and $(4, 0, 0)$, $(0, 4, 0)$, $(3, 1, 0)$, $(1, 3, 0)$ one each, products of divisor classes. The sum is 26. The Hodge group of the product lies in $G \cdot K^1$, so 26 bounds the Hodge classes from above; equality is not needed.

(ii) The adjoint of a correspondence for the Lefschetz pairing on $H^2(X \times X)$ and the polarised intersection pairing on Z_K is the transposed cycle composed with cup products with powers of the polarisations and the inverse Lefschetz isomorphism, which is algebraic on an abelian variety [40, §2]; so $\xi^\dagger \xi$ is algebraic. By Lemma 6.44, $T_X(-1)$ carries $-4q_1$ and T_Z carries a rational multiple of q_{c^2} , and for $\xi = m_e$, $q_{c^2}(m_e x, m_e y) = \text{Tr}(c^2 e^2 \text{trd}(xy)) = q_1(c^2 e^2 x, y)$, so $\xi^\dagger \xi = \kappa m_{e^2 c^2}$.

(iii) An irrational $a \in F$ generates the cubic field, so every m_b , $b \in F$, is a polynomial in $\xi^\dagger \xi$, algebraic on $T_X(-1)$. Compose with the product map $\mu: H^2(X \times X) \supseteq T_X(-1) \rightarrow H^2(X)$ and its adjoint, both built from ψ , the diagonal and θ and hence algebraic [19, Lem. 21.3]: the composite

$\mu \circ (m_b \otimes 1) \circ \mu^\dagger$ acts on the primitive part U of $H^2(X)$ by the element $v(\text{Tr}b - b)/2$ of F , irrational when b is. With the identity and the projector onto $\mathbb{Q}\theta$ this generates $\mathbb{Q} \times F = \text{End}_{\text{Hdg}} H^2(X)$, so by [19, Prop. 21.2] the exceptional classes are algebraic. The class ξ is flat along C and so are the exceptional classes, so by Lemma 4.2 algebraicity at one point of the curve C gives it on a countable set or everywhere; and [19, Cor. 20.30] shows that algebraicity at one point of C gives it at every point, and then every Hodge class on every power of X is algebraic [19, Thm. 21.7].

(iv) $a(e) = \kappa e^2 f^2 \delta$ is rational iff $e^2 f^2 \delta \in \mathbb{Q}$. If $-\delta \notin \mathbb{Q}^\times (F^\times)^2$ then $\delta \notin \mathbb{Q}^\times (F^\times)^2$ and $e^2 f^2 \delta \in \mathbb{Q}$ is impossible. If $-\delta = -dg^2$ with $d \in \mathbb{Q}^\times$, $g \in F^\times$, then with $c = f\sqrt{-d}$ one gets $a(e) = \kappa e^2 f^2 d$, rational iff $(ef)^2 \in \mathbb{Q}$ iff $ef \in \mathbb{Q}$, a cubic field having no irrational element with rational square. And $-\delta \in \mathbb{Q}^\times (F^\times)^2$ exactly when $K \supseteq \mathbb{Q}(\sqrt{-d})$ for some d .

(v) By [19, Thm. 21.7] the Hodge conjecture holds for all powers of X once the exceptional classes are algebraic. A cycle on $X^r \times Z_K$ acting nonzero on the irreducible T_Z carries it isomorphically onto a summand of $H^{2k}(X^r)(k-1)$; projecting onto that summand and composing with an isomorphism to $T_X(-1) \subseteq H^2(X \times X)$, a Hodge class on X^{r+2} and algebraic, gives an algebraic nonzero element of $\text{Hom}_{\text{Hdg}}(T_Z, T_X(-1)) = F$; composing with the algebraic m_a gives every element of that line, and transposing gives every ξ . $\square$

Proposition 6.46 (Hyper-Kähler carriers give ± 1). *Let $Y_1, \dots, Y_m$ be projective hyper-Kähler varieties of $K3^{[n_j]}$ type whose transcendental spaces are Hodge isometric to $(T, r_j q_{\lambda_j})$, $r_j \in \mathbb{Q}^\times$, $\lambda_j \in F$ totally positive. Every composite of rational Hodge isometries between them that returns to Y_j acts on T by ± 1 , and every composite of Hodge similarities by an element of $\mathbb{Q}$.*

Proof. A Hodge isometry $(T, r_j q_{\lambda_j}) \rightarrow (T, r_k q_{\lambda_k})$ is a Hodge endomorphism of T , hence m_a for some $a \in F$, and it is an isometry iff $r_j \lambda_j = r_k \lambda_k a^2$. A composite returning to j is m_a with $a^2 = 1$, and a composite of similarities is m_a with $a^2 \in \mathbb{Q}$, which in a cubic field forces $a \in \mathbb{Q}$. The isometries are algebraic by [22, 48], which is why the statement matters: they realise m_b between S_λ and $S_{\lambda b^{-2}}$ and between their moduli spaces of sheaves, but every return is trivial. $\square$

6.13. Where the link stops

Theorem 6.45 replaces the Kuga–Satake variety of [19, Thm. 21.4], of dimension 2^7 , by an abelian sixfold over the same curve, and it replaces the irrationality that had to be extracted from anticommuting Clifford blocks by the arithmetic of K : for every K that is not a compositum $F \cdot k$ every link closes the target. Its logical status is the same as the Kuga–Satake link: by (v) it is equivalent to the target together with one algebraic link, not a reduction to something smaller.

The missing step is the algebraicity of one class $\xi \in H^4(X \times X \times Z_K)$. It is exceptional: its component in $V \otimes V$ is the nine-dimensional T , not a multiple of ψ , so it is not invariant under $\text{Sp}(V)$ and not in the Lefschetz algebra of $X \times X \times Z_K$ (Lemma 3.7); the theorems that prove the conjecture when the Hodge ring is generated by divisors do not apply, and no Weil class of Z_K or of $X \times X$ is involved. At the points of complex multiplication of C the class ξ_c is a Hodge class on a CM abelian variety in a mixed Künneth component between $X_c \times X_c$ and $Z_{K,c}$ with $\text{Hom}(X_c, Z_{K,c}) = 0$, so it is not a pull-back from one factor nor a product of lower-degree Hodge classes, and the generation theorems for CM abelian varieties do not reach it either. Unlike the Kuga–Satake class, the link is therefore not known even at the countably many special points. This is the price of staying inside abelian varieties, and it is the exact statement that (F2) needs at its smallest instance.

7. WHERE THE EARLIER ATTEMPT FAILS

The preprint [9] of the two authors claimed the conjecture in full. We record here each point at which it fails, so that the present paper can be read as its replacement and not as a continuation. Numbers refer to that preprint.

The secant cycle (Definition 3, Lemma 4). For an abelian $2n$ -fold A of Weil type the preprint takes $\Sigma = \text{Sec}^n(A^\vee)$, the closure of the union of the $(n-1)$ -planes spanned by n points of $A^\vee$ in a projective embedding, and the cycle $Z_\Sigma = \Sigma \cap A^\vee$, asserted to have codimension n . Every point of $A^\vee$ lies on such a plane, so $A^\vee \subseteq \Sigma$ and $Z_\Sigma = A^\vee$: the intersection is not proper. The sheaf $\mathcal{O}_\Sigma|_{A^\vee}$ is $\mathcal{O}_{A^\vee}$, whose Fourier–Mukai transform is the skyscraper sheaf at the origin shifted by $-2n$, with Chern character

Hodge classes of a very general Weil $2n$ -fold, $n = 3$: η^n and the Weil line carry different characters

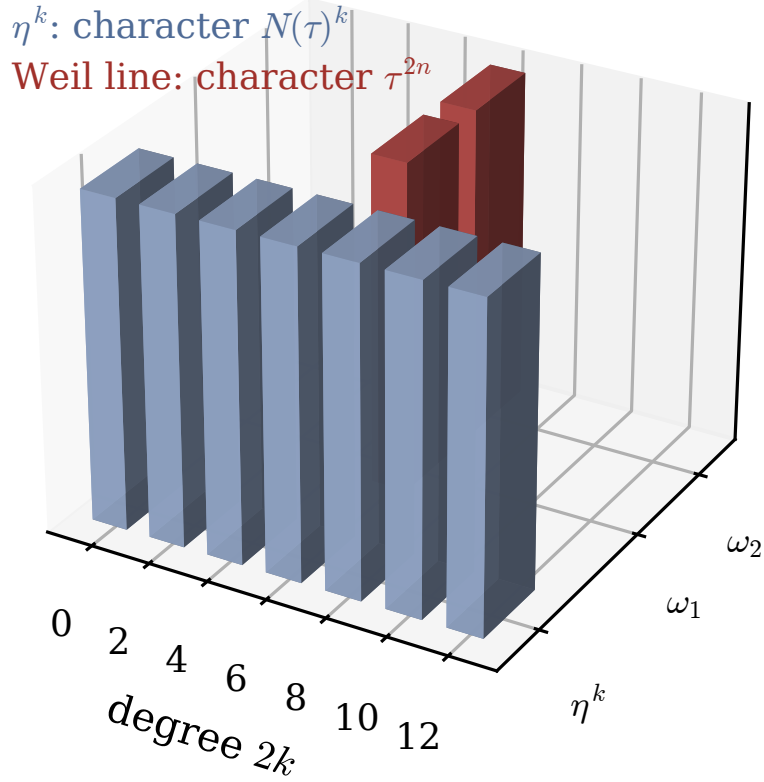


FIGURE 11. The Hodge classes of a very general abelian sixfold of Weil type and the characters of $K^\times$ on them: η^k carries $N(\tau)^k$, the Weil plane carries τ^{2n} . An eigenspace for N^n in degree $2n$ is a multiple of η^n .

the class of a point. So $\text{ch}_k = 0$ for $1 \leq k < 2n$, contrary to Lemma 4, and Proposition 2 has nothing to identify with a Weil class.

The polarisation (Definition 3). The polarisation is taken principal, so $\chi(L) = 1$ and the embedding is into $\mathbb{P}^0$; the dimension count needs $\chi(L) = 2n^2 + 2n$. A principal polarisation is never very ample.

The dimension of the Weil space (Proposition 1). $\det_K H^1(A)$ is a two-dimensional $\mathbb{Q}$ -space of Hodge classes [83], so at a very general point $\text{Hdg}^n(A) = \mathbb{Q}\eta^n \oplus W(A)$ has dimension three, not two.

The eigencharacter (Lemmas 1, 2, Proposition 2, Theorem 4). Equation (1) of the preprint gives $\tau^* \eta = N(\tau) \eta$ for $\tau \in K^\times$, so η^n has character $N(\tau)^n$, while Proposition 2 asserts character 1. The group $K^\times$ acts on $W(A) = \bigwedge_K^{2n} H^1(A)$ by $\tau \mapsto \tau^{2n}$, which differs from $N(\tau)^n = (\tau \bar{\tau})^n$ unless $(\tau/\bar{\tau})^n = 1$; so the Weil classes are not in the N^n -eigenspace, and an N^n -eigenspace of degree $2n$ is a multiple of η^n (Fig. 11). The scaling of the secant sheaf by d^n under an isogeny of degree d (Lemma 1) is asserted without proof.

Step 4 of Theorem 4. The points of a Shimura variety with smaller Mumford–Tate group are the points of proper special subvarieties, most of which are not of CM type; and Pohlmann’s theorem [62] covers the CM types whose Hodge ring is generated by divisors, not all CM abelian varieties.

The reduction to Weil classes (Theorem 3, Section 5). Moonen and Zarhin describe every Hodge class as a polynomial in divisor classes and Weil classes of isogeny factors in dimension at most five [57],

and the description is not known beyond; simple CM abelian varieties can carry Weil classes [83, 71]; and not every factor of type IV is of Weil type.

Theorem 12. The Kuga–Satake construction exists for Hodge structures of K3 type, with $h^{2,0} = 1$ [42, 26]; Theorem 12 applies it to arbitrary primitive classes. Its algebraicity, credited there to Deligne, is open: Deligne proved that the correspondence is absolutely Hodge [27], and its algebraicity is known only in special families [76, 74]. Since the case split of Section 1.4 rests on this step for the classes of coniveau zero, it is incomplete.

None of these points is repaired here. The present paper does not use secant cycles, the eigencharacter argument or the Kuga–Satake construction outside K3 type, and it claims no case of the conjecture that is not proved in Sections 3 to 6.

8. WHAT IS PROVED AND WHAT IS OPEN

Table 4 lists the statements. Three remarks on what the table says.

The two open statements are not independent problems that happen to add up to the conjecture; they are two halves of one phenomenon. Both say that the algebraic locus of a flat Hodge class, a countable union of closed strata through a dense set of points of complex multiplication, is not meagre. For (F2) the dense set is supplied by explicit cycles at CM points and the first instance is a single class on an abelian fourteenfold or a single Ext group on an eightfold; for (F3') the dense set is supplied by the K3 surfaces with complex multiplication and by the chains of Section 5, and the first instance is a cubic real multiplication on one K3 surface of Picard number 13, which Theorem 5.14 supplies for the fields $\mathbb{Q}(\zeta_7)^+$ and $\mathbb{Q}(\zeta_9)^+$ with the algebra ramified at two real places, and which stays open for every other pair of a cubic field and an algebra.

Each construction of this paper ends at a question with a definite answer. Whether some convolution of the pieces of Theorem 6.3 has $\dim \text{Ext}^2 = 57$ is a finite computation in the Ext algebra of two sheaves on a fourfold; at the point E_0^6 of complex multiplication every group entering it is written down in Lemma 6.8, Proposition 6.11 answers it in the negative for every convolution with three powers of L , Proposition 6.13 answers it in the negative for two threefolds at one shift and for every placement of the thirty-five pieces, Proposition 6.14 with Corollary 6.15 for every further piece of positive twist within one shift of the thirty-two, Theorem 6.23 for at most five line bundles at any twists and every polarisation, the enumeration of Remark 6.25 for six and seven and for eight outside thirty-one arrangements, Corollary 6.31 and Theorem 6.33 for every convolution of threefold pieces, of one family or of both, at any twists and shifts, and for the configurations left, eight or more line bundles with or without threefolds, it is the rank of the products along the chains of Proposition 6.10 and along the cycles through the line bundles, the support of the object being constrained by Theorem 6.27 and Propositions 6.30 and 6.35. Whether the link ξ is algebraic is a question about one Hodge class on one abelian variety, equivalent by Theorem 6.45(v) to the exceptional classes together with one more cycle. Whether a correspondence acting on $T(S_\lambda)$ otherwise than by a similarity exists is a question about one K3 surface; for $F = \mathbb{Q}(\zeta_7)^+$ or $\mathbb{Q}(\zeta_9)^+$ and $D = D_\infty$ Theorem 5.14 answers it, and for every other pair (F, D) it is open, with Proposition 5.15 marking the three places where the present mechanism fails. A negative answer to the first closes that route, as the analogous question on E_0^6 closed; a positive answer to any of them proves the corresponding statement at its first instance.

8.1. Specialisation, degeneration and blow-ups

Three devices are often proposed for passing from the members where a class is known to be algebraic to the others, and we record exactly what each one proves.

Proposition 8.1. *Let $\pi : \mathcal{X} \rightarrow B$ be a smooth projective family over an irreducible base and ξ a flat Hodge class on its fibres.*

- (i) *If ξ_t is algebraic at the generic point of a curve $C \subseteq B$ then it is algebraic at every point of C . The converse implication, from one point of C to its generic point, holds when ξ_{t_0} is the Chern character of a perfect complex whose semiregularity map is injective, and in no greater generality is it known.*

TABLE 4. Every statement of the paper with its status. “Proved” means proved in the text from the cited results; “open” means not proved here or elsewhere to our knowledge.

statement	where	status	what it needs
HC $\Leftrightarrow$ (F2) $\wedge$ (F3')	Thm. 1.1	proved	—
(F3') for fourfolds with zero-cycles on a threefold	Prop. 3.4	proved from [21]	—
(F2) $\Leftrightarrow$ AB	Thm. 3.8	proved	—
six minimal sufficient sets	Prop. 3.9	proved, checked in Lean	—
dichotomy of the algebraic locus	Lem. 4.2	proved	—
(F2) as propagation from CM points	Thm. 4.5	proved	—
Weil loci dense with CM base point	Prop. 4.6	proved in [19]	—
chains make e with $e^2 \in \mathbb{Q}$ algebraic	Thm. 5.3	proved	a chain of the right multiplier
symplectic chains reach exactly $\sqrt{2}, \sqrt{3}, \sqrt{6}$	Prop. 5.21	proved	—
$\sqrt{6}$ algebraic on two-dimensional families	Thm. 5.5	proved	—
$\sqrt{-5}, \sqrt{-7}$ algebraic on one-dimensional families	Thm. 5.9	proved	—
HC on all powers and Hilbert schemes of the Fricke surfaces	Thm. 5.9	proved	—
similarities act by multipliers $d_1 d_2$	Prop. 5.11	proved	—
finite-group correspondences give abelian E	Prop. 5.12	proved	—
HC for $X \times X$, X in the dihedral families	Cor. 5.13	proved	—
(F3') for S_λ , $F = \mathbb{Q}(\zeta_7)^+$ or $\mathbb{Q}(\zeta_9)^+$, $D = D_\infty$	Thm. 5.14	proved	—
where the dihedral mechanism stops	Prop. 5.15	proved	—
(F3') for S_λ , other cubic F or other D	Sect. 5.8	open	a non-similarity correspondence
Hodge classes on all powers of S : E and $\bigwedge_E^m T$	Prop. 5.16	proved from [75]	—
K3 surfaces of Kummer type over $\mathbb{Q}$: (F3') and HC on all powers	Prop. 5.17	proved	—
Fricke Kuga–Satake fourfold: $B \sim E_K \times B_3$	Prop. 5.19	proved	—
$\bigwedge_E^m T$ for the other surfaces of Sect. 5	Rem. 5.18	open	a cycle on S^m
(F3') in general	—	open	—
product object of polarised shape	Thm. 6.3	proved	—
direct sums never semiregular	Prop. 6.4	proved	—
annihilator of the secant pieces is the product locus	Thm. 6.6	proved	—
secant and line-bundle pieces orthogonal	Prop. 6.7	false for every realisation	—
(P2') at a product point	Sect. 6.5	open	Ext* of secant sheaves
the Ext groups at the CM point E_0^6	Lem. 6.8	proved	—
no two-step cancellation of diagonal classes	Prop. 6.10	proved	—
three line bundles at E_0^6	Prop. 6.11	proved, fail	—
isolated Ext blocks at E_0^6	Prop. 6.13	proved, fail	—
a threefold piece outside every closed chain	Prop. 6.14	proved, fail	—
further pieces within one shift of the thirty-two	Cor. 6.15	proved, fail	—
at most five line bundles, any twists, every polarisation	Thm. 6.23	proved, fail	—
six or seven line bundles; eight outside thirty-one cases	Rem. 6.25	enumeration, fail	—
the rest of the support cuts a threefold in an ample divisor	Thm. 6.27	proved	—
two threefolds meeting along a surface carry nothing	Thm. 6.29	proved	—
objects on the thirty-two threefolds of one family	Cor. 6.31	proved, fail	—
objects on the sixty-four threefolds of both families	Thm. 6.33	proved, fail	—
a smooth complete intersection cannot carry the rest	Prop. 6.35	proved, fail	—
an ample divisor in the support empties the localised obstruction	Prop. 6.36	proved	—
(P2') at E_0^6 : eight or more line bundles, with or without threefolds	Sect. 6.8	open	the chain products
E_0^8 : shifts in seven consecutive integers	Cor. 6.38	18 of 28 proved	—
an étale Prym map of triple covers at genus four	Prop. 6.40	proved	—
sextic CM fields: the count excludes nothing	Rem. 6.41	enumeration	—
sixfold link $\Rightarrow$ exceptional classes	Thm. 6.45	proved	—
algebraicity of the link ξ	Sect. 6.13	open	a cycle on $X \times X \times Z_K$
dominant maps, blow-ups, derived equivalences preserve the defect	Prop. 8.2	proved	—
residues of complete intersections	Prop. 8.3	proved	rational residues
HC for α as a vanishing mass gap	Prop. 8.4	proved	—
exceptional classes of $X \times X$	—	open	—
(F2) in general	—	open	—
the Hodge conjecture	—	open	(F2) and (F3')

The defect $\delta = \dim \text{Hdg}/\text{Alg}$ never drops: blow-up, domination, derived equivalence, degeneration

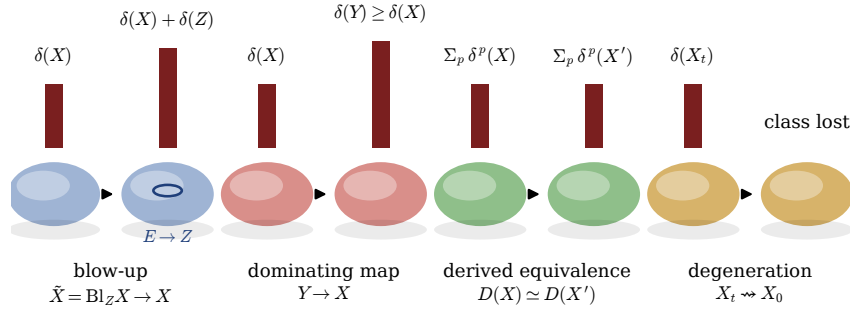


FIGURE 12. Where the defect goes under surgery. A blow-up along Z carries the Hodge classes of X to $\tilde{X}$ and adds those of Z , so the quotient Hdg/Alg grows by the defect of the centre; a dominating map $Y \rightarrow X$ carries the defect of X into Y ; a Fourier–Mukai equivalence permutes degrees and preserves the total defect; a degeneration to X_0 loses the class only in the special fibre, from which no cycle lifts.

- (ii) Let $\tilde{X}$ be the blow-up of a smooth projective variety X along a smooth closed subvariety Z of codimension c . Then $\text{Hdg}^k(\tilde{X})/\text{Alg}^k(\tilde{X}) \cong \text{Hdg}^k(X)/\text{Alg}^k(X) \oplus \bigoplus_{i=1}^{c-1} \text{Hdg}^{k-i}(Z)/\text{Alg}^{k-i}(Z)$. In particular the Hodge conjecture for $\tilde{X}$ is equivalent to the Hodge conjecture for X and for Z , and no sequence of blow-ups and blow-downs reduces the defect of a variety.
- (iii) For the Weil classes of $\mathcal{D}_{n,n}$ and the exceptional class of a Mumford fourfold, every degeneration to a fibre with a semistable model on which the limit class is algebraic has its lift to the generic fibre as its missing step, and for the Mumford fourfold no degeneration keeping the class of Hodge type exists, since the Hodge locus of the class is a compact Shimura curve.

Proof. (i) Specialisation of cycles along a curve [30, Ch. 20] sends an algebraic cycle on the generic fibre to one on each special fibre with the same flat cohomology class; this is the direction used by Schoen’s degeneration and by Proposition 4.6. In the other direction the algebraic locus is a countable union of closed sets by Lemma 4.2, and a point in one stratum says nothing about the stratum being the whole base; the one proved lift is the semiregularity theorem of Bloch, Buchweitz and Flenner [18, 17], which is the content of Theorem 6.1. (ii) Manin’s decomposition of the motive of a blow-up [44], $h(\tilde{X}) = h(X) \oplus \bigoplus_{i=1}^{c-1} h(Z)(-i)$, is realised by the algebraic correspondences π^* and $j_* \circ p^* \circ$ (powers of the relative hyperplane class), which are compatible with Hodge classes and with algebraic classes; so both the Hodge classes and the algebraic classes of $\tilde{X}$ decompose accordingly, and the quotients add. A birational map between smooth projective varieties factors through blow-ups and blow-downs along smooth centres only in special cases, but every birational map is dominated by a sequence of blow-ups, and (ii) shows that the defect of the top variety is at least that of the bottom one, so the defect cannot be removed on the way. (iii) A flat class has a limit in the limit mixed Hodge structure of a semistable fibre, and on a totally degenerate fibre the graded pieces are combinatorial and every class is algebraic on the components; the return to the generic fibre is the lift of (i), with the same unproved step. For a Mumford fourfold the Hodge locus of the exceptional class is the compact Shimura curve of [19, Thm. 21.7], which has no boundary, so no fibre at which the class is still of Hodge type degenerates. $\square$

8.2. Dominating maps, birational models and derived equivalences

Write $\delta^p(X) = \dim \text{Hdg}^p(X) - \dim \text{Alg}^p(X)$ for the defect of the conjecture in degree $2p$, $\delta_{\text{ab}}^p(X) = \dim \text{Hdg}^p(X) - \dim \text{Ab}^p(X)$ for the defect of (F3’), and $\delta(X) = \sum_p \delta^p(X)$. The conjecture for X is $\delta(X) = 0$. The following proposition records how the surgeries of birational geometry move these numbers; Figure 12 draws it.

Proposition 8.2 (Defects under surgery). *Let X, Y be smooth complex projective varieties.*

- (i) If $f: Y \rightarrow X$ is surjective and generically finite, then $\delta^p(Y) \geq \delta^p(X)$ and $\delta_{\text{ab}}^p(Y) \geq \delta_{\text{ab}}^p(X)$ for every p . In particular no blow-up along a singular centre, weighted blow-up, resolution of a modification or finite cover of X has a smaller defect than X .
- (ii) If X and Y are birational, then $\delta^2(X) = \delta^2(Y)$ and $\delta^{d-1}(X) = \delta^{d-1}(Y)$, $d = \dim X$; for $3 \leq p \leq d-2$ the difference $\delta^p(X) - \delta^p(Y)$ is a signed sum of defects $\delta^{p-j}(Z)$, $j \geq 1$, of smooth projective varieties Z of dimension less than d .
- (iii) If $\Gamma \in \text{CH}^*(X \times Y)_{\mathbb{Q}}$ induces an isomorphism from $H^*(X, \mathbb{Q})$ onto $H^*(Y, \mathbb{Q})$ whose inverse is induced by a cycle, for instance the kernel of an equivalence between the derived categories of X and Y , then $\delta(X) = \delta(Y)$ and $\delta_{\text{ab}}(X) = \delta_{\text{ab}}(Y)$.
- (iv) Let $\mathcal{X} \rightarrow \Delta$ be a projective family, smooth over the punctured disc, with a flat Hodge class ξ on the smooth fibres that is algebraic on a crepant resolution $\hat{X}$ of the special fibre X_0 , or on the components of a semistable model of X_0 . Then ξ is algebraic on X_t for $t \neq 0$ if and only if the specialisation $H^{2p}(X_0) \rightarrow H^{2p}(X_t)$ sends an algebraic class to an algebraic class, which is the statement (V) for the family; the monodromy plays no role, since it fixes ξ .

Proof. (i) f_*f^* is multiplication by the degree, so f^* is an injective morphism of Hodge structures and $f^*\text{Hdg}^p(X) \subseteq \text{Hdg}^p(Y)$. If $f^*\alpha$ is algebraic then so is $\deg(f)\alpha = f_*f^*\alpha$; hence $\text{Alg}^p(Y) \cap f^*\text{Hdg}^p(X) = f^*\text{Alg}^p(X)$ and f^* embeds $\text{Hdg}^p(X)/\text{Alg}^p(X)$ into $\text{Hdg}^p(Y)/\text{Alg}^p(Y)$. For Ab the same argument applies because f_* of a class $\Gamma_*\gamma$ reached from an abelian variety is $(\Gamma_f \circ \Gamma)_*\gamma$, reached from the same one. The coarse space of a weighted blow-up has quotient singularities and $f_*f^* = \text{id}$ with rational coefficients, so the argument is unchanged there.

(ii) By weak factorisation [3] a birational map between smooth projective varieties is a composite of blow-ups and blow-downs along smooth centres with smooth projective intermediate varieties, and Proposition 8.1(ii) gives the change of δ^p at each step as $\pm \sum_{j=1}^{c-1} \delta^{p-j}(Z)$. For $p = 2$ the terms are $\delta^1(Z)$ and $\delta^0(Z)$, which vanish by the Lefschetz (1,1) theorem; for $p = d-1$ they are $\delta^{d-1-j}(Z)$ with $\dim Z \leq d-2$, hence $\delta^q(Z)$ with $q \geq \dim Z - 1$ after $j \geq 1$, and these vanish by hard Lefschetz and Lefschetz (1,1) on Z .

(iii) By the projection formula Γ_* sends $\bigoplus_p \text{Hdg}^p(X)$ into $\bigoplus_p \text{Hdg}^p(Y)$, since $[\Gamma]$ is a sum of classes of type (i, i) and cup product and push-forward are morphisms of Hodge structures up to Tate twists; it sends algebraic classes to algebraic classes because $[\Gamma] \cup \text{pr}_X^*[Z]$ is the class of a cycle and the push-forward of a cycle is a cycle; and it sends Ab to Ab by composing correspondences. The inverse does the same in the other direction, so the two total quotients are isomorphic. The kernel of a Fourier–Mukai equivalence and the kernel of its inverse are cycles by Mukai’s and Orlov’s description of the inverse functor [61], which is the equivalence with kernel the dual of the original one shifted by the dimension.

(iv) ξ is monodromy invariant, so by the local invariant cycle theorem it is the specialisation of a class in $H^{2p}(X_0)$, which pulls back to $\hat{X}$ or to the components of the semistable model; the hypothesis says that the pulled back class is algebraic there, and push-forward keeps it algebraic on X_0 . Cycles specialise from the generic fibre to the special fibre [30, Ch. 20] and not the other way, so the remaining step is exactly the one in the statement. If the limit cycle on X_0 lifts to the family, its restriction to X_t has class ξ_t ; conversely if ξ_t is algebraic the specialisation of a representing cycle is a cycle on X_0 with class the limit of ξ , and the specialisation map sends that class to ξ_t . The vanishing cycles, on which the monodromy acts, are orthogonal to the invariant part, and ξ is invariant, so no condition on them enters. $\square$

Items (i) to (iii) say that every surgery available in birational and derived geometry carries the defect of X into its output as long as the open class survives there, and loses the class only in outputs of lower dimension, the centres, where it has become a defect of those varieties. Both open statements of this paper sit in degree four, (F2) for the Weil classes and the exceptional classes, (F3') for the classes of $H^4(S \times S)$ on a K3 surface, and in degree four the defect is a strict birational invariant by (ii) and a derived invariant by (iii); so the open classes are present, and not algebraic, on every birational model and every Fourier–Mukai partner of the varieties that carry them. Item (iv) is the reason that degenerations to products of elliptic curves, where every Hodge class is algebraic [57], do not close

(F2): the class returns to the smooth fibre only through (V). Flows of Kähler metrics change no complex structure between their surgeries, and the surgeries themselves are the contractions and flips of the minimal model programme, covered by (i) and (ii); the surgery of Perelman has no counterpart for Hodge structures, since it cuts a manifold along spheres and a Hodge structure has no local pieces to cut.

The devices therefore relocate the two open statements without weakening them. The one true reduction in shape is Theorem 4.5: the whole of (F2) is equivalent to a lift from CM points along polarised abelian schemes, and the lift along each scheme is a question about one semiregular object at one point.

8.3. Residues and mass-minimising currents

Two analytic descriptions of algebraic classes are exact, and we record what each reduces the conjecture to. The first is Griffiths' residue calculus on hypersurfaces [33]. For a smooth hypersurface $X = \{F = 0\} \subset \mathbb{P}^{n+1}$ of even dimension $n = 2m$ and degree d , with Jacobian ring $R(F)$ and $\Omega = \sum_i (-1)^i x_i dx_0 \wedge \dots \wedge \widehat{dx_i} \dots \wedge dx_{n+1}$, the residues of the forms $P\Omega/F^{q+1}$ identify $H_{\text{prim}}^{n-q,q}(X)$ with $R(F)_{(q+1)d-n-2}$.

Proposition 8.3 (Residues of complete intersections). *Let $X = \{F = 0\}$ be as above, and suppose that $F = \sum_{i=1}^{m+1} f_i g_i$ with $Z = \{f_1 = \dots = f_{m+1} = 0\}$ a complete intersection of codimension $m+1$ in $\mathbb{P}^{n+1}$ whose ideal is generated by the f_i .*

- (i) *Let θ be the hyperplane class and J_H the Jacobian determinant of $(f_1, g_1, \dots, f_{m+1}, g_{m+1})$, a polynomial of degree $(m+1)(d-2)$. Then*

$$[Z] = \frac{\deg Z}{d} \theta^m - \frac{m!}{d} \text{res} \left(\frac{J_H \Omega}{F^{m+1}} \right)^{m,m}$$

[81, Thm. 1.1].

- (ii) *Let $J(F) \subseteq R(F)_{(m+1)(d-2)}$ be the span of the classes of the determinants J_H over all such decompositions of F . If the rational span of the residues of $J(F)$ and θ^m contains $\text{Hdg}^m(X)$, the Hodge conjecture holds for X .*

Proof. (i) is the cited theorem; the degree of the determinant is $\sum_i (\deg f_i + \deg g_i) - (n+2) = (m+1)d - 2m - 2$. (ii) By (i) every residue of an element of $J(F)$ is a rational combination of θ^m and of classes of complete intersections in X , so it is algebraic. The Hodge conjecture for a hypersurface holds outside the middle degree by the Lefschetz hyperplane theorem, and in the middle degree it is the statement that $\text{Hdg}^m(X)$ consists of algebraic classes. $\square$

The proposition replaces a cycle by a polynomial, and the open step becomes the statement that the polynomials of $J(F)$ have enough rational residues. Deciding whether a given element of $R(F)$ has a rational residue requires its periods, so the residue calculus moves the conjecture from cycles to transcendental numbers without closing it. On a Fermat hypersurface $\sum_i x_i^d = 0$ the linear subspaces cut out by $x_{2i} = \zeta_i x_{2i+1}$, $\zeta_i^d = -1$, are of this form, since $x_{2i}^d + x_{2i+1}^d$ factors into linear forms; whether their residues and those of the other decompositions exhaust $\text{Hdg}^m(X)$ is again a question about periods.

The second description uses currents and is the analogue for cycles of the flows of geometric analysis. Let X be a smooth complex projective variety of dimension N , h an ample class with Kähler form ω , and $0 < p < N$. For $\beta \in H_{2(N-p)}(X, \mathbb{Z})$ let $M(\beta)$ be the infimum of the masses of the integral currents in β , put $\text{vol}(\beta) = \int_\beta \omega^{N-p}/(N-p)!$, and call $g(\beta) = M(\beta) - \text{vol}(\beta)$ the gap of β . We identify $H^{2p}(X, \mathbb{Z})$ with $H_{2(N-p)}(X, \mathbb{Z})$ by Poincaré duality.

Proposition 8.4 (Mass-minimising currents).

- (i) *Every class β contains an integral current of mass $M(\beta)$, and $g(\beta) \geq 0$, with equality if and only if β is the class of an effective algebraic cycle with integer coefficients.*
- (ii) *Let $\alpha \in H^{2p}(X, \mathbb{Z})$. The image of α in $H^{2p}(X, \mathbb{Q})$ is algebraic if and only if $g(e\alpha + kh^p) = 0$ for some integers $e \geq 1$ and $k \geq 0$.*

- (iii) *There are X and α with $g(\alpha + kh^p) > 0$ for every $k \geq 0$ although $g(e\alpha) = 0$ for some $e \geq 1$: for a very general hypersurface $X \subset \mathbb{P}^4$ of degree D divisible by ℓ^3 for an integer $\ell > 1$ prime to 6, $p = 2$, and the generator α of $H^4(X, \mathbb{Z})$.*

Proof. (i) By the compactness theorem of Federer and Fleming [29] the integral currents in β of mass at most $M(\beta) + 1$ form a compact set for the flat topology, and mass is lower semicontinuous, so the infimum is attained. For an integral current T in β , Wirtinger's inequality, that $\omega^{N-p}/(N-p)!$ has comass one with equality exactly on the complex $(N-p)$ -planes with their canonical orientation, gives $M(T) \geq T(\omega^{N-p}/(N-p)!) = \text{vol}(\beta)$, the last value depending only on β because ω is closed [36]. If equality holds, T is calibrated by $\omega^{N-p}/(N-p)!$, hence a closed positive current of bidimension $(N-p, N-p)$ [36], and by King's theorem [39] a closed positive integral current of that bidimension is $\sum_i n_i [V_i]$ with n_i positive integers and V_i irreducible analytic subvarieties, which are algebraic by Chow's theorem. Conversely, for an effective cycle $Z = \sum n_i V_i$ the tangent planes of the smooth locus of each V_i are complex, so the current $[Z]$ has mass $\text{vol}([Z])$ and $g([Z]) = 0$.

(ii) If $g(e\alpha + kh^p) = 0$, then $e\alpha + kh^p$ is the class of an effective cycle by (i), and $\alpha = ([Z] - kh^p)/e$ in $H^{2p}(X, \mathbb{Q})$. Conversely let α be rationally algebraic, so that $e_0\alpha - [Z^+] + [Z^-]$ is a torsion class for an integer $e_0 \geq 1$ and effective cycles $Z^\pm$; multiplying by its order, $e\alpha = [Z^+] - [Z^-]$ in $H^{2p}(X, \mathbb{Z})$. Write $Z^- = \sum_i c_i V_i$. For r large the ideal sheaf of each V_i twisted by $\mathcal{O}(rh)$ is globally generated, so p general members of $|rh|$ containing V_i meet in a complete intersection Y_i of codimension p containing V_i , and $[Y_i] = r^p h^p$; the cycle Z'_i of Y_i minus V_i is effective. Then $e\alpha + \sum_i c_i r^p h^p = [Z^+ + \sum_i c_i Z'_i]$, effective, and (i) gives $g = 0$ with $k = \sum_i c_i r^p$.

(iii) By the Lefschetz hyperplane theorem $H^4(X, \mathbb{Z}) \cong \mathbb{Z}\alpha$ with $\int_X \alpha \cdot h = 1$, and $h^2 = D\alpha$, the class of a complete intersection, so $g(D\alpha) = 0$. By Kollár's theorem [41], in the form of [73, Thm. 2], every curve on X has degree divisible by ℓ . The class $\alpha + kh^2$ has degree $1 + kD$, prime to ℓ , so it is not the class of an effective cycle, and $g(\alpha + kh^2) > 0$ by (i). $\square$

The analytic part of the variational approach is therefore complete. A minimiser exists in every class without any flow, so a mass-decreasing flow of a cycle, with whatever surgeries along its singularities, can at best reach it; what such a flow cannot provide is the equality case, and the equality case is the whole conjecture: by (ii) the Hodge conjecture for a class α is the vanishing of the gap of $e\alpha + kh^p$ for one pair (e, k) . Part (iii) shows that the multiplier e cannot be omitted, since integral minimisers of a class that is rationally algebraic need not be calibrated for any k , and so no argument by integral currents alone, including one modelled on Ricci flow with surgery, proves the conjecture without passing to rational coefficients. For the Weil classes of $\mathcal{D}_{n,n}$ and the exceptional classes of a Mumford fourfold the gap of $e\alpha + kh^p$ is not known to vanish for any (e, k) .

Nothing in the paper assumes an open statement, and no section depends on a computation that is not also proved by hand in the text. The machine checks of Appendix B recompute the finite quantities; they do not carry any proof.

APPENDIX A. THE POINT OF COMPLEX MULTIPLICATION IN COORDINATES

This appendix works out, in coordinates, the objects of Lemma 6.8 and Proposition 6.10 on $A_0 = E_0^6$: the Hodge ring, the Weil plane, an explicit family of abelian threefolds whose classes combine to the Weil class, their intersection numbers, and the dimensions of every Ext group between the pieces built from them.

A.1. The forms and the Hodge ring

Write $z_1, \dots, z_6$ for the coordinates of $A_0 = E_0^6$, $E_0 = \mathbb{C}/\mathbb{Z}[i]$, group the factors into the three surfaces $B_j = E_0^2$ with coordinates (z_j, z_{3+j}) , $j = 1, 2, 3$, and let $K = \mathbb{Q}(i)$ act by i on z_1, z_2, z_3 and by $-i$ on z_4, z_5, z_6 . Put

$$\begin{aligned} a_j &= \frac{i}{2} dz_j \wedge d\bar{z}_j, & b_j &= \frac{i}{2} dz_{3+j} \wedge d\bar{z}_{3+j}, \\ c_j &= \frac{i}{2} dz_j \wedge d\bar{z}_{3+j}, & \bar{c}_j &= \frac{i}{2} dz_{3+j} \wedge d\bar{z}_j. \end{aligned} \tag{A.1}$$

TABLE 5. The Ext groups between two pieces at $A_0 = E_0^6$, from Lemma 6.8; $m = b - a$, $\chi_Z = 8$ for the threefolds Z_ζ .

from	to	degrees k	$\dim \text{Ext}^k$
$\mathcal{O}_Z L^a$	$\mathcal{O}_Z L^a$	$0, \dots, 6$	$\binom{6}{k}: 1, 6, 15, 20, 15, 6, 1$
$\mathcal{O}_Z L^a$	$\mathcal{O}_Z L^b, m > 0$	$0, \dots, 3$	$\binom{3}{k} 8m^3$
$\mathcal{O}_Z L^a$	$\mathcal{O}_Z L^b, m < 0$	$3, \dots, 6$	$\binom{3}{k-3} 8 m ^3$
$\mathcal{O}_Z L^a$	$\mathcal{O}_{Z'} L^b$	3	$(Z \cdot Z') \in \{8, 16, 32, 64\}$
$\mathcal{O}_Z L^a$	$L^b, m > 0$	3	$8m^3$
$\mathcal{O}_Z L^a$	$L^b, m < 0$	6	$8 m ^3$
$\mathcal{O}_Z L^a$	L^a	$3, \dots, 6$	$\binom{3}{k-3}$
L^a	$\mathcal{O}_Z L^b, m > 0$	0	$8m^3$
L^a	$\mathcal{O}_Z L^b, m < 0$	3	$8 m ^3$
L^a	$\mathcal{O}_Z L^a$	$0, \dots, 3$	$\binom{3}{k}$
L^a	$L^b, m > 0$	0	m^6
L^a	$L^b, m < 0$	6	$ m ^6$
L^a	L^a	$0, \dots, 6$	$\binom{6}{k}$

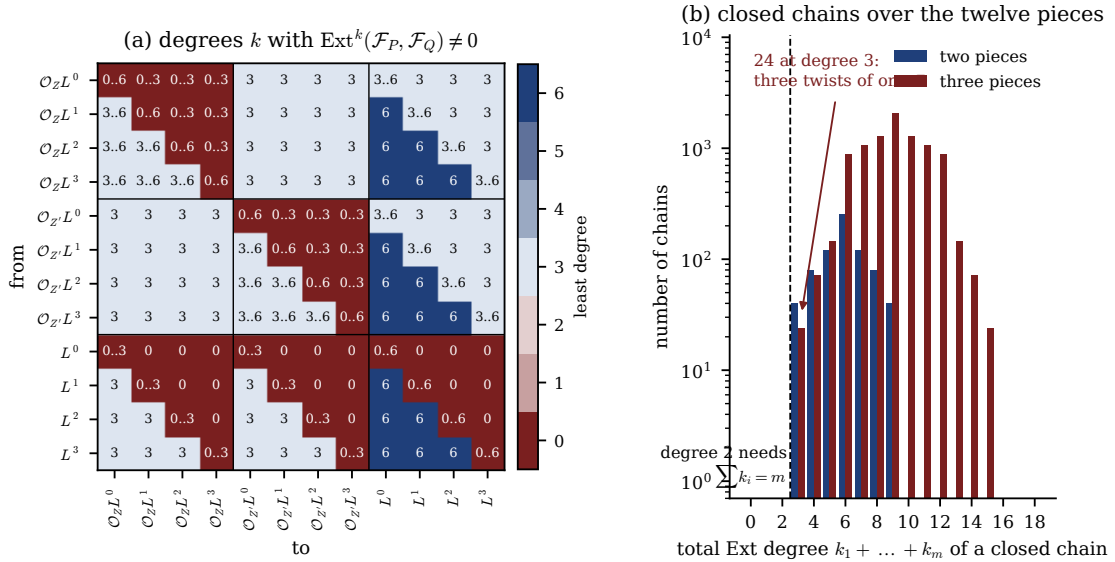


FIGURE 13. The degrees of the Ext groups between the twelve pieces $\mathcal{O}_Z L^k$, $\mathcal{O}_{Z'} L^k$, L^k , $k = 0, \dots, 3$, of Table 5. Left: the degrees in which $\text{Ext}^k(\mathcal{F}_P, \mathcal{F}_Q)$ is nonzero, coloured by the least one; degree zero occurs only from a lower twist to a higher one and from L^a to $\mathcal{O}_Z L^a$. Right: the number of closed chains of two and of three distinct pieces by total degree, on a logarithmic scale; no two-piece chain has total degree below three, and the three-piece chains of total degree three are those of Proposition 6.10.

In the real coordinates $z_j = x_j + iy_j$ one has $a_j = dx_j \wedge dy_j$, the volume form of the j -th factor, of integral one over $\mathbb{C}/\mathbb{Z}[i]$,¹⁵ and $c_j = \frac{1}{2}(dx_j \wedge dy_{3+j} - dy_j \wedge dx_{3+j}) + \frac{i}{2}(dx_j \wedge dx_{3+j} + dy_j \wedge dy_{3+j})$, so that the real and imaginary parts of c_j are rational classes and $\bar{c}_j$ is the complex conjugate of c_j . All four forms have type $(1, 1)$. The polarisation is

$$\theta = \sum_{j=1}^3 (a_j + b_j), \quad \theta^6 = 6! a_1 b_1 a_2 b_2 a_3 b_3, \quad \int_{A_0} \frac{\theta^6}{6!} = 1, \quad (\text{A.2})$$

¹⁵The fundamental domain of $\mathbb{Z}[i]$ is the unit square, so $\int_{E_0} dx \wedge dy = 1$; this normalisation makes every form in (A.1) integral on the lattice $\mathbb{Z}[i]^6$, and all intersection numbers below are integers computed as integrals of top forms.

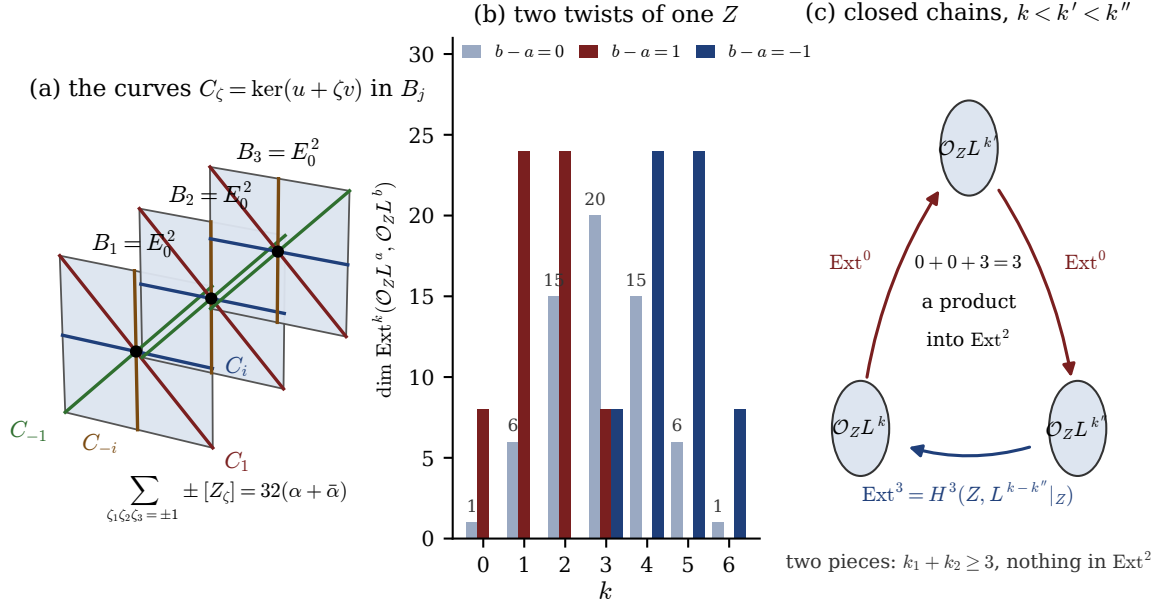


FIGURE 14. The point E_0^6 of complex multiplication. Left: the three Weil surfaces $B_j = E_0^2$ and the four elliptic curves C_ζ in each, kernels of $u + \zeta v$; a threefold Z_ζ picks one curve in each surface, and the 32 threefolds with $\zeta_1 \zeta_2 \zeta_3 = \pm 1$, signed by that product, add up to $32(\alpha + \bar{\alpha})$. Middle: the dimensions of $\text{Ext}^k(\mathcal{O}_Z \otimes L^a, \mathcal{O}_Z \otimes L^b)$ by degree for $b - a = 0$ (the $\binom{6}{k}$), $b - a = 1$ (degrees 0 to 3) and $b - a = -1$ (degrees 3 to 6). Right: the closed chains of Proposition 6.10; every two-step chain has total degree at least three, and the shortest chains reaching degree two run through three twists of one threefold with degrees 0, 0, 3.

the principal polarisation of the product; the hermitian forms $\sum_{j \leq 3} a'_j z_j \bar{w}_j - \sum_{j > 3} a'_j z_j \bar{w}_j$ of Lemma 6.8 with all $a'_j = 1$ give θ itself.

The Hodge group of A_0 is the circle acting by z on $H^{1,0}$ and by $\bar{z}$ on $H^{0,1}$, so the Hodge classes of degree $2k$ are exactly the rational classes of type (k, k) , of dimension $\binom{6}{k}^2$:

$$\dim \text{Hdg}^{2k}(A_0) = 1, 36, 225, 400, 225, 36, 1 \quad (k = 0, \dots, 6). \quad (\text{A.3})$$

In degree two these are the 36 hermitian forms on $\mathbb{Z}[i]^6$, the Picard number of E_0^6 . The K -eigenspaces of $H^1(A_0, \mathbb{C})$ are spanned by $dz_1, dz_2, dz_3, d\bar{z}_4, d\bar{z}_5, d\bar{z}_6$ (eigenvalue i) and by their conjugates, so the Weil plane $W = \bigwedge_K^6 H^1(A_0, K) \subseteq H^6(A_0, \mathbb{Q})$ is spanned over $\mathbb{C}$ by

$$\alpha = c_1 c_2 c_3 \propto dz_1 dz_2 dz_3 d\bar{z}_4 d\bar{z}_5 d\bar{z}_6 \quad \text{and} \quad \bar{\alpha} = \bar{c}_1 \bar{c}_2 \bar{c}_3, \quad (\text{A.4})$$

and over $\mathbb{Q}$ by $\alpha + \bar{\alpha}$ and $i(\alpha - \bar{\alpha})$. The relation $W\theta = 0$ is visible term by term: $c_j a_j$ contains dz_j twice and $c_j b_j$ contains $d\bar{z}_{3+j}$ twice, so $\alpha(a_k + b_k) = 0$ for each k , since α already contains dz_k and $d\bar{z}_{3+k}$.

A.2. The elliptic curves C_ζ and the threefolds Z_ζ

For $\zeta \in \mu_4 = \{1, i, -1, -i\}$ let $C_\zeta \subseteq B_j$ be the kernel of $(u, v) \mapsto u + \zeta v$, an elliptic curve isomorphic to E_0 , and for $\zeta = (\zeta_1, \zeta_2, \zeta_3) \in \mu_4^3$ let $Z_\zeta = C_{\zeta_1} \times C_{\zeta_2} \times C_{\zeta_3} \subseteq A_0$, an abelian threefold.

Lemma A.1. (i) $[C_\zeta] = a_j + b_j + \bar{\zeta} c_j + \zeta \bar{c}_j$ in $H^2(B_j)$, and $[Z_\zeta] = \prod_j [C_{\zeta_j}]$.

(ii) $(C_\zeta \cdot C_{\zeta'}) = |\zeta - \zeta'|^2 \in \{0, 2, 4\}$ on B_j ; $Z_\zeta \cap Z_{\zeta'}$ is finite exactly when $\zeta_j \neq \zeta'_j$ for all j , and then $|Z_\zeta \cap Z_{\zeta'}| = \prod_j |\zeta_j - \zeta'_j|^2 \in \{8, 16, 32, 64\}$.

(iii) $\theta|_{Z_\zeta}$ has degree two on each factor, $(\theta|_{Z_\zeta})^3 = 48$ and $\chi_{Z_\zeta} = 8$; in particular $[Z_\zeta]\theta \neq 0$.

(iv) Writing $S(\omega) = \sum_{\zeta_1 \zeta_2 \zeta_3 = \omega} [Z_\zeta]$ for $\omega \in \mu_4$,

$$\sum_{\omega \in \mu_4} \omega^m S(\omega) = \prod_{j=1}^3 T_{m,j}, \quad T_{m,j} = \sum_{\zeta \in \mu_4} \zeta^m [C_\zeta], \quad (A.5)$$

$$T_{m,j} = 4(a_j + b_j), 4c_j, 0, 4\bar{c}_j \quad \text{for } m \equiv 0, 1, 2, 3 \pmod{4},$$

and therefore

$$S(1) - S(-1) = \sum_{\zeta_1 \zeta_2 \zeta_3 = 1} [Z_\zeta] - \sum_{\zeta_1 \zeta_2 \zeta_3 = -1} [Z_\zeta] = 32(\alpha + \bar{\alpha}). \quad (A.6)$$

Proof. (i) C_ζ is the fibre over 0 of the surjective homomorphism $f_\zeta: B_j \rightarrow E_0$, $w = u + \zeta v$, so $[C_\zeta] = f_\zeta^*[0]$ with $[0] = \frac{i}{2}dw \wedge d\bar{w}$, and

$$\begin{aligned} dw \wedge d\bar{w} &= (du + \zeta dv) \wedge (d\bar{u} + \bar{\zeta} d\bar{v}) \\ &= du \wedge d\bar{u} + dv \wedge d\bar{v} + \bar{\zeta} du \wedge d\bar{v} + \zeta dv \wedge d\bar{u}, \end{aligned}$$

which is the formula. A product of subvarieties of the factors has the product class.

(ii) On B_j the only nonzero products of two of the four forms are $a_j b_j$ and $c_j \bar{c}_j = \bar{c}_j c_j = -a_j b_j$ (reorder $dud\bar{v}dv d\bar{u}$ to $-dud\bar{u}dv d\bar{v}$), and $\int_{B_j} a_j b_j = 1$. Hence $[C_\zeta][C_{\zeta'}] = 2 - \bar{\zeta}\zeta' - \zeta\bar{\zeta}' = |\zeta - \zeta'|^2$. Directly: a point of $C_\zeta \cap C_{\zeta'}$ has $(\zeta - \zeta')v = 0$, so v runs over the kernel of $\zeta - \zeta'$ on E_0 , of order $|\zeta - \zeta'|^2$, and $u = -\zeta v$ is then determined. The statement for Z_ζ follows factor by factor.

(iii) $\int_{C_\zeta} (a_j + b_j) = \int_{B_j} (a_j + b_j)[C_\zeta] = 2$, since $a_j c_j, a_j \bar{c}_j, b_j c_j, b_j \bar{c}_j$ all repeat a differential. So $\theta|_{Z_\zeta} = 2(p_1 + p_2 + p_3)$ with p_j the point class of the j -th factor, $(\theta|_{Z_\zeta})^3 = 3! \cdot 8 = 48$, and $\chi_{Z_\zeta} = 48/6 = 8$.

(iv) The left side of (A.5) is $\sum_{\zeta} (\zeta_1 \zeta_2 \zeta_3)^m \prod_j [C_{\zeta_j}]$, which factors as the product of the three sums $T_{m,j}$; these are read off from (i), using $\sum_{\zeta \in \mu_4} \zeta^m = 4$ or 0 according as $4 \mid m$ or not. Then $S(1) - S(-1) = \frac{1}{2} \sum_{\omega} (\omega + \omega^3) S(\omega) = \frac{1}{2} (\prod_j T_{1,j} + \prod_j T_{3,j}) = \frac{1}{2} (64\alpha + 64\bar{\alpha})$. $\square$

Equation (A.6) is the analogue, with abelian threefolds in place of line bundles, of the signed sum of the 32 line bundles on E_0^6 in [19, Prop. 18.21]: the 32 threefolds Z_ζ with $\zeta_1 \zeta_2 \zeta_3 = \pm 1$, taken with the sign $\zeta_1 \zeta_2 \zeta_3$, have classes adding up to 32 times the real Weil class. Together with three powers of L this is a Chern character of polarised shape and Hankel rank three at A_0 , and by Lemma 6.8(i) it is one of the shapes a convolution must realise. Two of these threefolds with $\zeta_j = \zeta'_j$ for one or two indices j meet along a curve or a surface, so Lemma 6.8(ii) does not apply to them and Lemma 6.9 gives their groups; Proposition 6.11 shows that no convolution of these thirty-five pieces meets the criterion. Figure 14 draws the curves, the dimensions of the Ext groups and the chains.

A.3. The dimensions in Lemma 6.8

For the threefolds $Z = Z_\zeta$ the numbers of Lemma 6.8 are explicit. Table 5 lists, for every pair of piece types, the degrees k in which Ext^k is nonzero and its dimension; here $m = b - a$ is the difference of the twists, $\chi_Z = 8$, $Z' = Z_{\zeta'}$ with $\zeta_j \neq \zeta'_j$ for all j , and $(Z \cdot Z') = \prod_j |\zeta_j - \zeta'_j|^2$.

Each row of the table restates the corresponding item of Lemma 6.8, with $\chi(L^m|_Z) = m^3 \chi_Z = 8m^3$ from (iii) above and $\chi(L^m) = m^6 \theta^6 / 720 = m^6$ from (A.2). Two features of the table drive Proposition 6.10. Every group between two pieces of different twist is concentrated in degree 0, in degree 3 or 6, or in the range 0 to 3 for two twists of one threefold; and a group in degree at most two between distinct pieces occurs only from a lower twist to a higher one, or from L^a to $\mathcal{O}_Z L^a$. The bookkeeping of the proposition is then the following count. A closed chain of m pieces whose edges have Ext degrees $k_1, \dots, k_m$ carries an m -fold product into $\text{Ext}^{\sum k_i - m + 2}$ of the first piece. For $m = 2$ the table gives, for

each unordered pair of distinct piece types, the two degrees and their sum:

$$\begin{array}{lll}
 (\mathcal{O}_Z L^a, \mathcal{O}_Z L^b) & 3+3 & = 6, \\
 (\mathcal{O}_Z L^a, \mathcal{O}_Z L^b), b > a & (0 \text{ to } 3) + (3 \text{ to } 6) & \geq 3, \\
 (\mathcal{O}_Z L^a, L^b), b > a & 3+3 & = 6, \\
 (\mathcal{O}_Z L^a, L^b), b < a & 6+0 & = 6, \\
 (\mathcal{O}_Z L^a, L^a) & (3 \text{ to } 6) + (0 \text{ to } 3) & \geq 3, \\
 (L^a, L^b) & 0+6 & = 6,
 \end{array} \tag{A.7}$$

so no two-step product reaches degree two. For $m = 3$ the sum must be 3 with two edges of degree zero, and the table leaves exactly the ascending chains $\mathcal{O}_Z L^k \rightarrow \mathcal{O}_Z L^{k'} \rightarrow \mathcal{O}_Z L^{k''}$, $k < k' < k''$, closed by the degree-three edge $H^3(Z, L^{k-k''}|_Z)$, of dimension $8(k'' - k)^3$. The program code/cm_point_ext.py enumerates the chains of two and three pieces over four twists and two threefolds and confirms both statements, together with the identities of Lemma A.1. Figure 13 shows the table as the matrix of the twelve pieces, and the distribution of the total degree over every closed chain of two and of three distinct pieces: the two-piece chains begin at total degree three, and the twenty-four ordered three-piece chains of total degree three are the eight ascending triples of twists of one threefold, each read from its three starting pieces.

APPENDIX B. MACHINE CHECKS

Every finite quantity used in the proofs is recomputed by programs in the archive [13], and the combinatorial and arithmetic statements among them are also proved in Lean 4. No statement proved in the text rests on these checks; the enumerations of Proposition 6.13 and Remarks 6.25, 6.39 and 6.41 are their output and are stated as such, and the rest guard against slips in the hand computations. Lean. The file lean/HodgeClosure.lean (Lean 4.24.0, no Mathlib) contains sixty-two theorems. Section 1 encodes the eight statements and the rule set of Table 1 as data, computes closures by iteration, and proves by evaluation that the closures of $\{F2, F3'\}$, $\{F3\}$, $\{L, M\}$, $\{L, F3'\}$ and $\{V, F3'\}$ contain **HC**, that the closures of $\{F2\}$, $\{F3'\}$, $\{L\}$, $\{M\}$, $\{V\}$, $\{AB\}$, $\{L, V\}$ and $\{M, F3'\}$ do not, and that the list of minimal sufficient sets, computed from the rule set alone, has exactly six members, those of Proposition 3.9. Section 2 works in $\mathbb{Z}[\sqrt{-d}]$ and proves that the hermitian form η transforms under τ by $N(\tau)$, that the norm is multiplicative, that η is not $K^\times$ -invariant, and that the characters $N(\tau)^n$ and τ^{2n} differ at explicit τ (Section 7). Section 3 proves the dimension counts of Section 6: $3n < n^2$ for $n \geq 4$, $m^2 + m'^2 < (m + m')^2$ for $m, m' \geq 1$ with $2mm'$ mixed directions, the values $r = 57$ and 104 of the lower bound, and the Hodge class count 26 of Theorem 6.45(i) as a sum of Künneth contributions. Section 4 proves the polynomial identities behind the Fricke families of Section 5.4: $c_4^3 - c_6^2 = 1728t(t^2 + 13t + 49)^2$ at level 7 and $1728t(t^2 + 22t + 125)^3$ at level 5, by exact arithmetic on coefficient lists, together with the root products 7^2 and 5^3 , the Euler numbers 12 and 24, the Shioda–Tate counts giving Picard number at least 16, the transcendental rank 6, the dimensions $(n + 1)/2$ of the dihedral Hecke algebras, and the polynomial identity behind the sign of Lemma 5.8: with A' , B' the Vêlu coefficients of the quotient of (5.1) by C_t , computed from the kernel polynomial by Newton’s identities on coefficient lists, and $\tilde{A}(t) = t^4 A(N^k/t)$, $\tilde{B}(t) = t^6 B(N^k/t)$, the identity $N^{k-1} B' \tilde{A} + A' \tilde{B} = 0$ holds at both levels, which is $\lambda(t)^2 = -t^2/N^{k-1}$. Section 5 proves the Dickson identities $p_7(t + t^{-1}) = t^7 + t^{-7}$ and $p_9(t + t^{-1}) = t^9 + t^{-9}$ behind the families of Theorem 5.14, that 2 has order three in $(\mathbb{Z}/7)^\times / \pm 1$ and in $(\mathbb{Z}/9)^\times / \pm 1$, so that 2 is inert in both cubic fields, and the sign pattern of the conjugates of θ at the three real places. Section 6 proves the finite statements of Appendix A: the binomial identity $\sum_q \binom{3}{q} \binom{3}{k-q} = \binom{6}{k}$, the intersection numbers $(C_\zeta \cdot C_{\zeta'}) = |\zeta - \zeta'|^2$ and $\theta \cdot C_\zeta = 2$ in the basis of the $(1, 1)$ -forms of B_j with the pairing $a_j b_j = 1$, $c_j \bar{c}_j = -1$, the twisted sums $T_{m,j}$, the identity $S(1) - S(-1) = 32(\alpha + \bar{\alpha})$ as an equality of the 64 coefficients of the triple products over $\mathbb{Z}[i]$, the counts 48, 8, 75 and 18 with the Euler characteristics, and the two chain statements of Proposition 6.10 over the twelve pieces of Appendix A.3, each by decide. Section 7 encodes the pieces of Table 3 and the Ext dimensions of Lemmas 6.8 and 6.9, and proves that the isolated blocks of Lemma 6.12 sum to 7213, 2605, 2605, 0, 0 and 5325 at the six configurations named in the table; the reachability relation along blocks of total degree one is supplied as a certificate, which the kernel checks to be reflexive, to contain every such block and to be closed under one more step, before the

sum is evaluated by `decide +kernel`. From the same certified relation it proves that the configurations (a) and (c) of the table have 32 and 22 threefold pieces outside every directed cycle of the graph Γ of Section 6.6; and, by evaluating the Ext dimensions on every pair, that no edge of Γ enters a piece at shift 1 from the thirty-two pieces or from the pieces $\mathcal{O}_{Z_\zeta} \otimes L^k[t]$, $\zeta \in \mu_4^3$, $1 \leq k \leq 3$, and $L^l[t]$, $1 \leq l \leq 4$, with $-3 \leq t \leq 2$, that $L^1[3]$, $L^4[3]$ and $\mathcal{O}_{Z_{(1,1,1)}} \otimes L^1[3]$ do enter, and that $L^1[-2], \dots, L^6[3]$ put each of the thirty-two pieces on a cycle of length six. Every theorem is checked by `#print axioms`: fifty-two use no axiom, the two inequalities proved with the arithmetic tactic use `propext` and `Quot.sound`, and the six isolated-block counts and the two chain-free counts of Section 7 use `propext`; `native_decide` is not used.

The Lean certificate of [19], `HodgeObstruction.lean` (Lean 4.34.0, four hundred and nine theorems, each axiom-free or depending on `propext` only), carries the finite computations of that paper; it is cited and not repeated here. The reduction of Section 3 uses only the hand-proved parts of [19]. `Python.code/sqrt6_chain.py` (sympy) implements Hilbert symbols and Hasse invariants exactly, verifies the product formula on random pairs, classifies $U_{\mathbb{Q}}$, $E_8(-1)_{\mathbb{Q}}$ and the invariant lattices, reproduces Varesco's embedding results for $\sqrt{2}$ and $\sqrt{3}$ as controls, searches the 165 sign-admissible small forms q_E over $\mathbb{Q}(\sqrt{6})$ (98 embed in V_3 , 67 do not), exhibits the complements $\langle 1, -10 \rangle$ and $\langle 1, -1, -1, -1, -3, -30 \rangle$, and checks both orders of the chain: twenty-five checks. `code/fricke_families.py` computes the fibre types, Euler numbers and Picard bounds of the Fricke curves and their twists at levels 5 and 7, checks numerically that the roots of $j(t) = j(\tau)$ are carried by $t \mapsto N^k/t$ to the j -invariants of the N -isogenous curves, verifies the twist identity $u_{p,q}(w(t)) = pq u_{w(p),w(q)}(t)/t^2$, the Möbius transformations preserving the labelled points, the embedding of every rank-six form of signature (2,4) in V_2 and V_3 after any scaling, and the Hecke algebras of $(D_n, \langle \tau \rangle)$ for $n = 5, 7, 9, 11$: forty-one checks. `code/product_object.py` builds the exterior-algebra model of $H^*(A \times A')$ for $(m, m') = (2, 1), (2, 2)$, the tangent space of $\mathcal{D}_{n,n}$ and of the product locus, the first-order Hodge loci of the exterior products and of the object of Theorem 6.3, the Vandermonde determinant, the vanishing of the cross terms, and the Hankel rank and $r(\gamma)$ of the final class; `code/convolution_checks.py` computes in the same model the rank of $v \mapsto v \lrcorner \theta_A$ on $T\mathcal{D}_{n,n}$, the contraction ranks of $\text{ch}(\mathcal{E}')$ and $\text{ch}(\mathcal{E}'')$ in Proposition 6.5(iii), and the trace of an upper triangular times a strictly upper triangular block matrix. In these two programs ranks are computed in floating point with a relative tolerance of 10^{-7} and agree with the closed formulas. `code/sixfold_link.py` checks the alternation and invariances of ψ_c , the decomposition of the K^1 -invariants, the ratio of the induced forms at the three places, the symmetry of $T_X(-1)$, the count 26 and the weight support of ξ ; `code/link_weights.py` computes the Hodge classes of degree four on $X \times X \times Z_K$ under the three groups of Section 6.13 and the vanishing of $\text{Hom}(X_c, Z_{K,c})$ at the compositum CM point. `code/zeta9_invariants.py` computes the rank, signature, discriminant and Hasse invariants of the trace form $\text{Tr}_{F/\mathbb{Q}}(-\theta \text{ nrd})$ on the quaternion algebra $(-1, -\theta)_F$, $F = \mathbb{Q}(\zeta_9)^+$, and of the transcendental lattice of the dihedral family of Theorem 5.14 for $n = 9$, and checks that they agree while three contrasting algebras fail; `code/zeta7_invariants.py` does the same for $n = 7$, with $(-1, \theta)_F$, $F = \mathbb{Q}(\zeta_7)^+$, the lattice $U^2 \oplus E_8(-1)$ and the competing algebras ramified at 7 or 43, and checks the Dickson identity of (5.3); `code/annihilator_exact.py` verifies in exact arithmetic over $\mathbb{Q}(\sqrt{-3})$, at $(m, m') = (2, 1)$ and $(2, 2)$, the formula (6.4) and the dimension $m^2 + m'^2$ of the annihilator in Theorem 6.6, component by component; `code/riemann_roch_pieces.py` computes the polynomial (6.5) and the thirty-one supports of Proposition 6.7; `code/cm_point_ext.py` checks Lemma 6.8 and Proposition 6.10: the identity $\sum \binom{3}{q} \binom{3}{p} = \binom{6}{k}$, the Koszul cohomology of a transverse intersection in exact arithmetic, the Euler characteristics of the twisted groups, the enumeration of the closed chains of two and three pieces with their degrees, and the identities of Lemma A.1 in the exterior algebra of the twelve real one-forms of E_0^6 ; `code/cm_point_counts.py` enumerates the configurations of Table 3, computes their Chern characters exactly in the Hodge ring of E_0^6 , and recomputes for each the sums C^1, C^2, C^3 , the Euler characteristic and the isolated blocks of Lemma 6.12; `code/cm_point_chains.py` builds the graph Γ of Section 6.6 from the same dimensions, finds its directed cycles through the strongly connected components, and checks the statements

of Remark 6.16 together with the absence of closed chains among the thirty-five pieces of Proposition 6.11. `code/line_bundle_chains.py` builds, for every arrangement of the shifts of $n \leq 8$ line bundles forming an interval, the graph of up, down and run edges, with a run table derived from the degrees of Lemmas 6.8 and 6.9, applies the rules of Proposition 6.19 and Lemma 6.20 over every set of vanishing up edges, and reports the counts of Remark 6.25, with the ranks $\sum_i (-1)^{t_i}$ of the thirty-one arrangements of eight: five checks. `code/threefold_supports.py` checks the exact arithmetic of Section 6.8: the entries of the vectors of $T\mathcal{D}_h$ and the restrictions $\kappa_T(v)_k|_{\Gamma_y}$, $\kappa_{T'}(v)_k|_{\Gamma_y}$ of Theorem 6.29 for every ζ and every pair $j \neq k$, the four points of $C_{\zeta_j} \cap C_{-\zeta_j}$ and the coordinates $n'_j|_{\Gamma_y} = n_j|_{\Gamma_y} = 2z_j$, the threefolds meeting a general Γ_y (Lemma 6.28), the coefficients $\sigma(x) \prod_{\epsilon=1} c(x)_j$ of X_ϵ in $[Z_x]$ and the Fourier inversion of Lemma 6.34, the vanishing of $\theta_h X_\epsilon$ and the absence of the weights $\pm 2\epsilon$ from $H^4(A_0)$, and for Lemma 6.32 and Theorem 6.33 the values $dn_a(v)$, $dn_k(v)$, $dn_l(v)$ at $Y = E_{jk}$, the classes $[C_a] \wedge d\bar{z}_{3+j}$ and the two relations they impose, the solutions $\mu_a = ma^2$ of the relations, the intersection numbers $Q \cdot C_b = 8$ and $Q^2 = 32$ and the finiteness of $C_1 \cap C_i$, the threefolds meeting $Q \times \{y\}$ for general y , the passage from the relations along the lines to $\mu_x = m\sigma(x)^2$, and the vanishing $\sum_x \sigma(x)^2 [Z_x] = 0$: sixteen checks. `code/kummer_type.py` implements the Hasse invariants and the criterion (c) of Proposition 5.17, compares it on fourteen quadratic spaces of ranks four and five with a direct search for a complement W with $T \oplus W \cong U_{\mathbb{Q}}^3$ by the invariants of Hasse–Minkowski, computes the discriminants of $U_{\mathbb{Q}}^3$, of the transcendental spaces of the Fricke surfaces and of the two families of [75], names the primes at which the failing examples of Remark 5.18 fail, and counts the spaces of ranks four and five with coefficients in $\pm\{1, 2, 3, 5, 6, 10, 15, 30\}$ that meet the condition (1084 of 1296 and 1304 of 4320). `code/fricke_sign.py` (sympy) computes the N -th division polynomial of (5.1) over $\mathbb{Z}[t, x]$, isolates the factor of degree $(N-1)/2$ that is the kernel polynomial of C_t , applies Vélú’s formulas, checks that the quotient has the j -invariant of $\mathcal{E}_{w(t)}$, computes $\lambda(t)^2 = -t^2/25$ and $-t^2/7$, the products $\lambda(t)\lambda(w(t)) = \pm N$ with the sign fixed by the analytic model of Lemma 5.8, the twist constants $c(t)c(w(t)) = 1$ and -1 and the resulting $\Lambda^2 = -N$ and N , the identity checked in Lean, and the composite $f_{w(\tau)} \circ f_\tau = [-N]$ with kernel $\langle 1/N \rangle$ in the analytic model: thirty-two checks. `code/fricke_picard.py` counts the points of X_1 over $\mathbb{F}_{\ell^i}$, $i \leq 3$, for $\ell = 11$ at level 7 and $\ell = 7$ at level 5, in exact arithmetic with Zech logarithms, forms the traces of Frobenius on the rank-six complement of the trivial lattice, completes the characteristic polynomial by Newton’s identities and the functional equation, checks that the sign of the functional equation is determined, that the polynomial has integer coefficients, is irreducible over $\mathbb{Q}$, splits over $\mathbb{Q}(\sqrt{-N})$ into a cubic and its conjugate and has no cyclotomic factor after the normalisation $x \mapsto \ell x$, that the odd traces vanish at the inert primes 5 and 11, and that the same counts on the rational surfaces R_N give the traces of a rational elliptic surface: twelve checks. `code/fricke_lattice.py` recomputes, by the same invariants, the rational isometries $A_n \oplus \langle n+1 \rangle \cong \langle 1 \rangle^{n+1}$, $D_4 \cong \langle 1 \rangle^4$, $A_1^2 \cong \langle 1 \rangle^2$, $E_8 \cong \langle 1 \rangle^8$ of the proof of Proposition 5.19(i), the isometries $T \cong U_{\mathbb{Q}}^2 \oplus \langle -1 \rangle \oplus \langle -N \rangle$ for $N = 5, 7$ both by cancellation and from the invariants of the K3 lattice and of M , the pairs (a, b) with $U_{\mathbb{Q}}^2 \oplus \langle a, b \rangle \cong T$ in a box, the signed discriminant $-N$, and the dimension count of the Hodge groups: twenty-one checks. `code/closure_graph.py` recomputes the closures and the minimal sets, and `code/bypass_checks.py` the Horn closure with the facts of the literature on abelian varieties adjoined and the blow-up bookkeeping of Proposition 8.1(ii). `code/cover_boundary.py` (sympy) checks the smoothness of B_λ , the divisor of $y^3 + 1/\lambda$, the nontriviality of η and the rank nine of the multiplication map of Proposition 6.40. `code/sextic_cubic_fields.py`, with `code/sextic_lattice.py` from the archive of [19], computes in exact arithmetic the lattices of integral points of $S(0, q)$ of Remark 6.41 over the twelve cubic fields, their Gram matrices and their short vectors. `code/efour_seven.py` (numpy, scipy) recomputes exactly the weights and the largest nontrivial character sums of the weighted Cayley graphs of Section 6.9 and runs the linear programs of Remark 6.39 over every distribution of the pieces in the twelve arrangements, in double precision.

Julia, C, C++ and shell. `code/closure_graph.jl`, `code/closure_graph.c` and `code/closure_graph.cpp` recompute the closures of all 256 subsets of the eight statements and the six minimal sets by three independent implementations of the fixed-point iteration; `code/eigencharacter.jl` recomputes the characters of Section 7 in exact integer arithmetic, and `code/hilbert.c` the Hilbert

symbols and the invariants of Table 2; `code/run_all.sh` runs every program and the Lean file and prints a line per check. The expected outputs are recorded in `code/EXPECTED.md`.

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DATA AND CODE AVAILABILITY

The $\text{\LaTeX}$ source of the paper, the Python, Julia, C, C++ and shell programs of Appendix B and the Lean 4 file are published as GitHub release v2.7.0, <https://github.com/creelie/fermat-hodge-/releases/tag/v2.7.0> [13]; the compiled paper is the only file of the folder preprints of that release. The candidate configurations of Section 6.6 and the first enumerations of their chains, among them that of Remark 6.25, were produced with Anaxia, the construction engine of the BLLM project [14], release v3.0.5, <https://github.com/creelie/eBLLM/releases/tag/v3.0.5>, and the support results of Section 6.8 with release v3.1.0 [15], <https://github.com/creelie/eBLLM/releases/tag/v3.1.0>; the point counts of Remark 5.10 follow the Frobenius test of its module `k3frob` (commit 8782bd3), which first showed that a Fricke self-map is a complex and not a real multiplication, at level 13; every statement of the paper is decided by the proofs in the text and by the programs of the release, not by the engine. Deep Bhattacharjee carried out the constructions on (F2) and (F3’) and the Lean, Julia, C, C++, Python and shell verifications with the help of AI assistance (Claude, Anthropic). Both authors approved the final version.

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