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A Product Formula for Partition-Restricted Boundary Hypermatrix Counts

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Abstract

We prove the Koprowski–Lewis product conjecture for nondegenerate $2 \times (k+1) \times k$ hypermatrices over finite fields with two partition-shaped layers of prescribed zeros. The result holds for every positive k, every finite field, and every admissible pair of partitions. After fixing the first slice, nondegeneracy is equivalent to simplicity of a partial linear map. Applying the simple-extension theorem of Arora, Ram, and Venkateswarlu inside the successive allowed row spaces gives the product directly. We include an elementary proof of that extension theorem and justify the equivalence with the defining projective critical locus of the hyperdeterminant. The complete counting statement and its semantic equivalences have been checked in Lean.

1 The counting formula

Let $\mathbb{F}$ be a finite field with q elements and let $k \geq 1$. For $r \geq 1$, put $[r]_q = 1 + q + \cdots + q^{r-1}$. Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ and $\mu = (\mu_1, \ldots, \mu_k)$ be weakly decreasing sequences of nonnegative integers satisfying

$$\mu_j \leq \lambda_j \leq k + 1 - j, \qquad \mu_j \leq k - j \qquad (1 \leq j \leq k). \tag{1}$$

Consider ordered pairs of $(k+1) \times k$ matrices (A, B) over $\mathbb{F}$ such that the last λ_j entries of column j of A and the last μ_j entries of column j of B vanish.

Write K for an algebraic closure of $\mathbb{F}$. Call (A, B) *pencil-nondegenerate* if

$$\operatorname{rank}_K(sA + tB) = k \quad \text{for every } (s,t) \in K^2 \setminus \{(0,0)\}. \tag{2}$$

The field K is essential: checking only $\mathbb{F}$-rational directions is not the condition in this paper. Section 5 proves that (2) is exactly hyperdeterminantal nondegeneracy in this boundary format.

Theorem 1.1. *The number of nondegenerate pairs (A, B) with these zero restrictions is*

$$q^{k^2}(q-1)^{2k} \prod_{j=1}^{k} [k + 2 - j - \lambda_j]_q \prod_{j=1}^{k} [k + 1 - j - \mu_j]_q. \tag{3}$$

This is Conjecture 3.1 of Koprowski and Lewis [1]. Their work proves several special cases and expresses the general count, after normalization, as a polynomial with nonnegative coefficients whose value at $q = 1$ counts hyperrook placements. A public manuscript of Zeng [3] treats all partition restrictions for $k = 3$ by a finite polynomial census. Theorem 1.1 covers arbitrary k without a census of shapes.

The linear-algebraic ingredient is already known: Lemma 2.10 of Arora, Ram, and Venkateswarlu [2] counts simple extensions of a partial linear map by one domain dimension. Our contribution is its application with both nested zero restrictions retained, together with the resulting full hypermatrix formula. For completeness, Section 3 supplies an elementary induction proof of the extension count; the count itself is not claimed as a new theorem.

2 Simple partial maps and pencils

Definition 2.1. For a subspace $D \subseteq V$, a linear map $f : D \to V$ is *simple* if the only subspace $L \subseteq D$ for which $f(L) \subseteq L$ is $L = 0$.

This definition excludes every nonzero invariant subspace, not just invariant lines over the base field. It is unchanged if one replaces V by any subspace containing both D and $f(D)$. Restricting a simple map to a smaller domain preserves simplicity.

Lemma 2.2. *Simplicity of a finite-dimensional partial map is preserved and reflected by extension of the base field.*

Proof. Let $\dim D = k$, and define

$$D_0 = D, \qquad D_{i+1} = \{u \in D : f(u) \in D_i\}.$$

This is a descending sequence. If $D_{i+1} = D_i$, the common subspace is invariant and all subsequent terms are the same. Each invariant subspace is contained in every D_i. Consequently f is simple if and only if $D_k = 0$: otherwise some step among the first k fails to decrease dimension and gives a nonzero invariant subspace.

Each D_{i+1} is obtained from D_i by a preimage and an intersection of subspaces defined by linear equations. Row reduction of those equations over the base field remains valid over any extension field. Thus scalar extension gives precisely the corresponding sequence and preserves every dimension. In particular it preserves and reflects $D_k = 0$. □

Proposition 2.3. *Regard A, B as maps $U \to V$, where $\dim U = k$ and $\dim V = k+1$. Condition (2) holds if and only if A is injective and the map $f : \operatorname{im} A \to V$ determined by $f(Ax) = Bx$ is simple.*

Proof. Condition (2) includes $(s, t) = (1, 0)$, hence forces A to be injective. Assume this and extend scalars to K. A nonzero vector x in the kernel of $sA + tB$ must have $t \neq 0$. Its image $u = Ax \neq 0$ satisfies $f(u) = -(s/t)u$, giving an invariant line. Conversely, on any nonzero invariant subspace the restriction of f is an endomorphism over an algebraically closed field. It has an eigenvector $u = Ax \neq 0$, which supplies a singular pencil direction. Lemma 2.2 identifies simplicity over K with simplicity over $\mathbb{F}$. □

3 Counting one-dimensional extensions

Lemma 3.1 (Simple-extension count, cf. [2, Lemma 2.10])**.** *Let $U \subset U' \subseteq V$ have dimensions $r, r+1, m$, respectively, over $\mathbb{F}$, and let $f : U \to V$ be simple. The number of simple maps $g : U' \to V$ extending f is $q^m - q^{r+1}$.*

Proof. Fix $v \in U' \setminus U$. An extension g_w is uniquely specified by $g_w(v) = w \in V$. We prove by strong induction on r that exactly q^{r+1} values of w give a nonzero invariant subspace. We allow $U' = V$ in the induction; then the asserted number of simple extensions is zero.

Let $\pi : V \to V/U'$ be the quotient map, and put

$$H = \{u \in U : f(u) \in U'\}, \qquad h = \dim H, \qquad C = \pi \circ f.$$

Rank–nullity gives $\dim \operatorname{im} C = r - h$. If $\pi(w) \notin \operatorname{im} C$, the equation $C(u) + \alpha\pi(w) = 0$ forces $\alpha = 0$. Any invariant subspace of g_w lies in $\ker(\pi \circ g_w)$, so is then contained in U. Simplicity of f forces that subspace to be zero. These values of w are all good.

For each $c \in \operatorname{im} C$, choose $u_c \in U$ with $C(u_c) = c$. The values of w with $\pi(w) = c$ are uniquely $w = f(u_c) + y$, where $y \in U'$. Set $v_c = v - u_c$. Directly,

$$K_c := \ker(\pi \circ g_w) = H \oplus \mathbb{F}v_c, \qquad g_w(v_c) = y. \tag{4}$$

Indeed, $C(u) + \alpha c = 0$ is equivalent to $u + \alpha u_c \in H$, and $u + \alpha v = (u + \alpha u_c) + \alpha(v - u_c)$. The subspace K_c has dimension $h + 1$ and is independent of y.

Every invariant subspace of g_w is contained in K_c; conversely, an invariant subspace for $g_w|_{K_c} : K_c \to U'$ is invariant for g_w. The map $f|_H : H \to U'$ is simple. If $h < r$, induction applied to $H \subset K_c \subseteq U'$ therefore shows that exactly q^{h+1} of the choices y are bad. There are q^{r-h} values of c, giving $q^{r-h}q^{h+1} = q^{r+1}$ bad extensions in total.

If $h = r$, then $H = U$ and $\operatorname{im} C = 0$. In the only relevant fibre $c = 0$, choose $u_0 = 0$. Now $K_0 = U'$, and all q^{r+1} choices $y \in U'$ give endomorphisms of the nonzero space U'. They are all bad. This also proves the base case $r = 0$ without an induction call. Subtracting the bad count from the q^m extensions proves the lemma. □

4 Keeping the two nested restrictions

Write $n = k + 1$, $a_j = n - \lambda_j$, and $b_j = n - \mu_j$. In $V = \mathbb{F}^n$, let F_d be the span of the first d standard basis vectors. The hypotheses give

$$a_1 \le \cdots \le a_k, \quad b_1 \le \cdots \le b_k, \quad j \le a_j \le b_j, \quad j + 1 \le b_j. \tag{5}$$

The column restrictions are exactly $A_j \in F_{a_j}$ and $B_j \in F_{b_j}$.

First count the injective matrices A. If columns $1, \ldots, j - 1$ are independent, their span lies in F_{a_j} and has q^{j-1} elements. Hence

$$\#\{\text{allowed injective } A\} = \prod_{j=1}^{k} (q^{a_j} - q^{j-1}). \tag{6}$$

Fix one such A, and put $D_j = \langle A_1, \ldots, A_j \rangle$, with $D_0 = 0$.

We choose the columns of B in order, retaining exactly the prefixes for which $f_{j-1} : D_{j-1} \to V$, $f_{j-1}(A_i) = B_i$, is simple. At step j, take the ambient space to be F_{b_j}. This is legitimate: (5) gives $D_j \subseteq F_{b_j}$, and monotonicity of the b_i gives $f_{j-1}(D_{j-1}) \subseteq F_{b_j}$. Simplicity does not change when this smaller ambient space is used.

Lemma 3.1, applied to $D_{j-1} \subset D_j \subseteq F_{b_j}$, gives $q^{b_j} - q^j$ simple extensions. Each one is specified uniquely by its value $B_j = f_j(A_j)$, so it is precisely an allowed choice of the new column. Every retained prefix has the same number of continuations. All prefixes of a simple final map are simple, so none of the desired pairs is discarded. Proposition 2.3 identifies the final retained maps with the desired nondegenerate pencils. Consequently, for every fixed allowed injective A,

$$\#\{\text{allowed nondegenerate } B \mid A\} = \prod_{j=1}^{k} (q^{b_j} - q^j). \tag{7}$$

Multiplying (6) and (7) counts actual ordered matrices; no change-of-basis quotient or stabilizer division occurs. Finally, $q^d - q^e = q^e(q-1)[d-e]_q$ for $d > e$, and $\sum_{j=1}^{k}((j-1)+j) = k^2$. This proves Theorem 1.1 for pencil-nondegeneracy.

5 Hyperdeterminantal nondegeneracy

The boundary-format hyperdeterminant used in [1] vanishes on the projective critical locus of the multilinear form

$$F(s,t,x,y) = y^{\mathsf{T}}(sA+tB)x = \sum_{i=1}^{k+1}\sum_{j=1}^{k}(A_{ij}s + B_{ij}t)x_j y_i. \tag{8}$$

Here (s,t), x, and y must each be nonzero. The vanishing of all first partial derivatives is equivalent to

$$Cx = 0, \qquad y^{\mathsf{T}}C = 0, \qquad y^{\mathsf{T}}Ax = y^{\mathsf{T}}Bx = 0, \quad C = sA + tB. \tag{9}$$

The value of F then vanishes as well.

Proposition 5.1. *Over an algebraic closure, a nonzero critical triple for (8) exists if and only if some nonzero pencil direction has rank less than k.*

Proof. A critical triple gives a nonzero kernel vector x of C, so one implication is immediate. Conversely suppose $Cx = 0$ with $x \neq 0$. If $t \neq 0$, set $v = Ax$; if $t = 0$, set $v = Bx$ (then $s \neq 0$ and $Ax = 0$). Consider

$$d : K^k \times K \longrightarrow K^{k+1}, \qquad d(u,c) = Cu + cv.$$

It is a map between spaces of equal dimension and has the nonzero kernel vector $(x, 0)$. Its image is therefore proper, and some nonzero linear functional y^{T} annihilates that image. Thus $y^{\mathsf{T}}C = 0$ and $y^{\mathsf{T}}v = 0$. If $t \neq 0$, $sAx + tBx = 0$ also implies $y^{\mathsf{T}}Bx = 0$; if $t = 0$, the remaining equation follows from $Ax = 0$. All of (9) hold. □

This argument is valid in every characteristic. It proves equality of the relevant zero loci, which is exactly what the nonzero count needs; no normalization of a defining hyperdeterminant polynomial is required. It completes the source interpretation of Theorem 1.1.

6 Specializations and formal verification

For $\lambda = \mu = 0$, formula (3) becomes the unrestricted boundary-format count

$$q^{k^2}(q-1)^{2k}[k+1]_q!\,[k]_q!, \qquad [r]_q! = \prod_{i=1}^{r}[i]_q.$$

At the other extreme, $\lambda_j = k+1-j$ and $\mu_j = k-j$ make every q-integer factor equal to 1. The proof treats these cases and all intermediate shapes uniformly.

In the accompanying project `boundary-hypermatrix-count`, the declarations `Boundary.full_pencil_target` and `Boundary.full_source_target` establish the full product for, respectively, the algebraic-closure pencil condition and the literal multilinear-polynomial critical-point condition. The extension theorem is proved in the project, rather than assumed as an axiom. The formalization implements scalar-extension invariance by an equivalent square trajectory matrix: the map $(x_0, \ldots, x_k) \mapsto (Bx_i - Ax_{i+1})_{i=0}^{k-1}$ is injective exactly when A is injective and its partial map is simple, and nonvanishing

of its determinant is unchanged by field embeddings. Formal partial differentiation supplies every equation in (9).

The 22-module build uses Lean 4.27.0 and mathlib revision `a3a10db0e9d66acbebf76c5e6a135066525ac900`; all dependency revisions are in `lake-manifest.json`. Run `python3 scripts/verify.py -packages /path/to/packages` with those dependencies provisioned. Saved successful logs and the statement audit are identified in `verification/result.json`. The final transitive axiom lists contain only `propext`, `Classical.choice`, and `Quot.sound`. No proof placeholder, new mathematical axiom, or native computation is used. The pinned compiler and library proof objects are trusted. This verification concerns the mathematical statements; it does not certify the prose or establish bibliographic priority.

References

[1] B. Koprowski and J. B. Lewis, *Enumeration of nondegenerate* $2 \times (k+1) \times k$ *hypermatrices*, arXiv:2602.22129v1 (2026). https://arxiv.org/abs/2602.22129v1.

[2] A. Arora, S. Ram, and A. Venkateswarlu, *Unimodular polynomial matrices over finite fields*, J. Algebraic Combin. **53** (2021), 1299–1312. https://doi.org/10.1007/s10801-020-00963-2.

[3] X. Zeng, *All partition-restricted counts of nondegenerate* $2\times4\times3$ *hypermatrices*, public manuscript, AI-Has-Taste repository, accessed 14 October 2026.