

THREE CONJECTURES ON THE FIRST TWO GRAPH EIGENVALUES

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ABSTRACT. Let $\lambda_1(G) \geq \lambda_2(G)$ be the two largest adjacency eigenvalues of a finite simple graph G on at least two vertices. We resolve the three conjectures appearing as 1.15–1.17 in Kannan, Kumar and Pragada. First, for every noncomplete graph, we prove $\lambda_1(G)^2 + \lambda_2(G)^2 \leq 2 \sum_{e \in E(G)} (1 - 1/c(e))$, where $c(e)$ is the largest order of a clique containing e . Second, we construct connected $K_5 - e$ -free graphs of arbitrarily large size that violate the exact color-critical bound proposed by Li, Liu and Zhang. Third, for every fixed F with $\chi(F) = r + 1 \geq 3$, we prove, uniformly over F -free graphs with $m = e(G) \rightarrow \infty$,

$$\lambda_1(G)^2 + \lambda_2(G)^2 \leq 2 \left(1 - \frac{1}{r}\right) m - \frac{r-2}{r} \Delta(G) + o_F(m),$$

where $\Delta(G)$ is the maximum degree. The coefficient of the degree term is best possible. The arguments use the half-plane complete-positivity theorem of Coutinho, Liu, Spier, Tang and Zhang, together with an explicit connected construction, weighted clique removal and spectral separation of vertices of high degree. Theorems 1.1–1.3 are formalized in Lean 4 and registered as PALOMAR-2026-10-15-000015 v1, PALOMAR-2026-10-16-000012 v1 and PALOMAR-2026-10-17-000005 v1, respectively.

1. INTRODUCTION

Throughout, graphs are finite and simple, and a forbidden copy is a subgraph that need not be induced. Write $m = e(G)$, let $\Delta(G)$ be the maximum degree, and order the adjacency eigenvalues as $\lambda_1(G) \geq \dots \geq \lambda_n(G)$. For an edge e , define $c_G(e)$ to be the largest number of vertices in a clique containing e . The subscript is omitted when the graph is clear. All edge sums are unordered unless both vertex indices are displayed. Our graphs may be disconnected; [6] assumes connectedness unless otherwise specified.

Bollobás and Nikiforov [1] conjectured that every noncomplete graph satisfies

$$\lambda_1(G)^2 + \lambda_2(G)^2 \leq 2m \left(1 - \frac{1}{\omega(G)}\right).$$

The planar Gram results of Coutinho, Liu, Spier, Tang and Zhang [4] give this inequality and its extension to nonnegative edge weights. Earlier cases

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include triangle-free graphs, proved by Lin, Ning and Wu [8], regular graphs, proved by Zhang [9], and graphs with few triangles, treated by Kumar and Pragada [7]. Coutinho, Spier and Zhang [10] obtained general bounds with weaker constants by conic programming. Kannan, Kumar and Pragada [6] formulated a local refinement and an asymptotic forbidden-subgraph extension. They also recorded an exact color-critical conjecture of Li, Liu and Zhang [15]. We prove the local refinement and a stronger asymptotic estimate, and disprove the exact color-critical statement.

Conjecture 1.15 of [6] reads:

For any graph $G \neq K_n$, we have

$$\lambda_1^2(G) + \lambda_2^2(G) \leq \sum_{e \in E(G)} 2 \left(1 - \frac{1}{c(e)}\right).$$

Our first result proves this inequality, including for disconnected graphs.

Theorem 1.1. *If G is a noncomplete graph on at least two vertices, then*

$$(1) \quad \lambda_1(G)^2 + \lambda_2(G)^2 \leq 2 \sum_{e \in E(G)} \left(1 - \frac{1}{c(e)}\right).$$

This proves Conjecture 1.15 of [6]. The corresponding one-eigenvalue bound was proved by Liu and Ning [16]; its weighted extension is given in [17]. In Section 3 we prove the nonnegative weighted form of Theorem 1.1. The scalar input is Bradač's local Motzkin–Straus inequality [2]. Its unweighted Turán consequence was obtained independently by Malec and Tompkins [11]. The passage to two eigenvalues uses the complete-positivity theorem from [4].

A graph is *color-critical* if deleting some edge lowers its chromatic number. Li, Liu and Zhang [15, Conjecture 5.4], also recorded as Conjecture 1.16 of [6], asked whether every fixed color-critical F with $\chi(F) = r + 1 \geq 4$ satisfies $\lambda_1(G)^2 + \lambda_2(G)^2 \leq 2(1 - 1/r)m$ for all F -free graphs of sufficiently large size. Their formulation has no connectivity hypothesis. Proposition 4.1 gives a disconnected counterexample for each $r \geq 3$. The next theorem supplies connected counterexamples, so it also applies under the default convention of [6].

Theorem 1.2. *Let $F = K_5 - e = K_3 \vee I_2$. For every integer $s \geq 3$, form G_s by joining $K_{s,s,s}$ and K_4 with one edge. Then F is color-critical with $\chi(F) = 4$, and G_s is connected and F -free. Moreover, $e(G_s) = 3s^2 + 7$ and*

$$(2) \quad \lambda_1(G_s)^2 + \lambda_2(G_s)^2 - \frac{4}{3}e(G_s) \geq \frac{656s^2 - 1366s + 62}{3(37s - 1)^2} > 0.$$

The obstruction persists for arbitrarily large m . Its positive excess has a lower bound tending to 656/4107. Thus an asymptotic error of order $o(m)$ cannot in general be replaced by zero.

Conjecture 1.17 of [6] reads:

For any F -free graph G where $\chi(F) \geq 3$, we have

$$\lambda_1^2(G) + \lambda_2^2(G) \leq 2m \left(1 - \frac{1}{\chi(F) - 1} + o(1) \right).$$

The limiting parameter in $o(1)$ is left implicit there. We fix F and let $m \rightarrow \infty$. On connected graphs this is equivalent to letting the order tend to infinity, since $n - 1 \leq m \leq \binom{n}{2}$. Our uniform edge formulation also allows arbitrary isolated vertices. The case $\chi(F) = 3$ was proved in [6, Theorem 4.5].

Our main general theorem adds an optimal maximum-degree term.

Theorem 1.3. *Fix a finite graph F with $\chi(F) = r + 1 \geq 3$, and put $\kappa = 1 - 1/r$. For every $\varepsilon > 0$ there is a positive integer $M = M(F, \varepsilon)$ such that every F -free graph G with $m = e(G) \geq M$ has at least two vertices and satisfies*

$$(3) \quad \lambda_1(G)^2 + \lambda_2(G)^2 \leq 2 \left(1 - \frac{1}{r} \right) m - \frac{r-2}{r} \Delta(G) + \varepsilon m.$$

The threshold is independent of the order of G , and G may be disconnected.

The coefficient $2\kappa - 1 = (r-2)/r$ is optimal even among connected graphs. Indeed, the star $K_{1,t}$, for $t \geq 2$, is F -free whenever $\chi(F) \geq 3$. Its eigenvalues are $\sqrt{t}$, 0 with multiplicity $t-1$, and $-\sqrt{t}$, so

$$m = \Delta(G) = t, \quad \lambda_1(G)^2 + \lambda_2(G)^2 = t.$$

An estimate with degree coefficient c and a uniform $o_F(m)$ error would therefore give $1 \leq 2\kappa - c + o(1)$ as $t \rightarrow \infty$. Hence $c \leq 2\kappa - 1$.

Dropping the nonpositive degree term in (3) proves Conjecture 1.17 of [6]. For $r \geq 3$, the correction gives a strict asymptotic improvement when $\Delta(G)$ is a positive proportion of m . When $\Delta(G) = o(m)$, the unspecified $o_F(m)$ error may dominate the degree term; the estimate supplies no general second-order expansion in that regime.

Theorems 1.1, 1.2 and 1.3 are mechanically verified in Lean 4 with Mathlib in the Palomar records [18, 19, 20], respectively. The records permit only the standard axioms of propositional extensionality, quotient soundness and classical choice. Section 8 gives the correspondence with the checked Lean declarations.

For the proofs, define for every real symmetric matrix A

$$F_2(A) = (\lambda_1(A)_+)^2 + (\lambda_2(A)_+)^2, \quad x_+ = \max\{x, 0\},$$

with missing eigenvalues replaced by zero. This is the sum of the squares of the two largest positive eigenvalues, padded with zeros. Lemma 2.1 identifies it with the ordered top-two square sum for noncomplete graphs and for all sufficiently large F -free graphs.

The comparison with known bounds involves both the spectral statistic and the forbidden graph. With $\kappa = 1 - 1/r$, the relevant edge bounds are:

Source and forbidden graph	Quantity	Upper bound
[4]: K_{r+1} , $r \geq 2$	$F_2(A_G)$	$2\kappa m$
[12]: arbitrary F , $r \geq 2$, large m	λ_1^2	$2\kappa m + o_F(m)$
[15]: color-critical F , $r \geq 3$, large m	λ_1^2	$2\kappa m$
Theorem 1.3: arbitrary F , $r \geq 2$, large m	$F_2(A_G)$	$2\kappa m - (2\kappa - 1)\Delta(G) + o_F(m)$

Here F is fixed with $\chi(F) = r + 1$, and the last remainder is uniform over all F -free graphs. The Perron-eigenvalue form of the asymptotic extension was already proved by Li, Liu and Zhang [12, Theorem 1.2], and is recalled in [15, Theorem 1.3]. Their argument reduces a minimal counterexample to a dense graph using Perron-vector estimates. Our argument instead uses a rank-two spectral lift, weighted deletion and separation of high-degree vertices to control both selected eigenvalues. Liu, Sun, Wang and Wu [13] prove the separate local walk conjecture from [6]. Li [14] proves flat-probability-vector Turán estimates and uses an entropy argument to transfer weighted inequalities to the Perron eigenvalue. Jayarajan, Kannan, Pragada and Roy [5] obtain vertex-local bounds for the full positive square energy $\sum_{\lambda_i(G) > 0} \lambda_i(G)^2$; their budget is a sum over vertices.

The main difficulty in the asymptotic theorem is that n may be much larger than $\sqrt{m}$, and vertices of degree comparable to m can concentrate the relevant eigenvectors. We first prove weighted clique removal for a flat vertex measure. This gives the required Gram estimate when $\Delta(G) = o(m)$. We then separate a small set of vertices of high degree by truncating the singular values of their incidence matrix. The separated bipartite part costs at most one unit of spectral energy per edge, while the residual graph costs 2κ . This difference produces the maximum-degree term.

Section 2 fixes the spectral conventions and states the Gram inputs. Section 3 proves Theorem 1.1, and Section 4 proves Theorem 1.2. The asymptotic argument occupies Sections 5 and 6. Section 7 gives equality examples, delocalization and a lower bound on the remainder. Section 8 identifies the companion Lean statements.

2. SPECTRAL PRELIMINARIES AND GRAM INPUTS

The positive-part convention allows the same matrix estimates to be used before any noncompleteness assumption. The following comparison connects these estimates with the ordered-eigenvalue statements.

Lemma 2.1. *If G is noncomplete and has at least two vertices, then $\lambda_2(G) \geq 0$ and $F_2(A_G) = \lambda_1(G)^2 + \lambda_2(G)^2$. If F has $t \geq 2$ vertices, then every F -free graph with $m > \binom{t-1}{2}$ is noncomplete.*

Proof. Two nonadjacent vertices give a zero 2×2 principal adjacency submatrix. Interlacing yields $\lambda_2(G) \geq 0$, proving the first assertion. A complete

graph on at least t vertices contains F as a non-induced subgraph. A complete F -free graph thus has at most $t-1$ vertices and at most $\binom{t-1}{2}$ edges. $\square$

For real matrices of the same size, write $\langle B, C \rangle = \text{tr}(B^\top C)$, $\|B\|_F^2 = \langle B, B \rangle$, and $\|B\|_{\text{op}}$ for the operator norm. Let J be the all-ones matrix. A symmetric matrix C is *completely positive* if $C = \sum_{a=1}^q p_a p_a^\top$ for finitely many entrywise nonnegative vectors p_a . For a positive semidefinite matrix X , define

$$(4) \quad \mathcal{M}(X) = X^{\circ 2} + \sum_{i < j: X_{ij} < 0} X_{ij}^2 (e_i - e_j)(e_i - e_j)^\top,$$

where $X^{\circ 2}$ is the entrywise square. Directly from this definition,

$$(5) \quad \mathcal{M}(X)_{ij} = (X_{ij})_+^2 \quad (i \neq j), \quad \langle J, \mathcal{M}(X) \rangle = \|X\|_F^2.$$

Indeed, when $X_{ij} < 0$, the corresponding off-diagonal entry of $X^{\circ 2}$ is cancelled by its correction term. Each correction term has zero sum of entries.

Our only imported Gram result is the following half-plane theorem. We cite the unrefereed repository manuscript [4]; its pinned Lean development contains a machine-checked proof under the declaration `BN.matrix_theorem`. The rank-two estimate used after removal is derived from the localized weighted bound in Theorem 3.3 below.

Theorem 2.2 (Coutinho–Liu–Spier–Tang–Zhang). *Let $z_1, \dots, z_n \in \mathbb{R}^2$ lie in a closed half-plane whose boundary contains the origin, and set $X = (z_i^\top z_j)_{i,j}$. Then $\mathcal{M}(X)$ is completely positive.*

We will also use three standard consequences of the min–max principle. Compression cannot increase the first or second positive eigenvalue, with zero padding. For an orthogonal direct sum, $F_2(C \oplus D) \leq F_2(C) + F_2(D)$. Finally, if C, D have the same size and both operator norms are at most L , Weyl’s inequality gives

$$(6) \quad |F_2(C) - F_2(D)| \leq 4L\|C - D\|_{\text{op}}.$$

Here $x \mapsto x_+^2$ is $2L$ -Lipschitz on $[-L, L]$. Zero padding also covers matrices of order less than two.

3. THE EDGE-LOCALIZED BOUND

The scalar certificate depends only on the clique number of each edge. Its extension to completely positive matrices is immediate, so the main spectral step is to choose a lift to which Theorem 2.2 applies.

Lemma 3.1 (Bradač’s scalar inequality). *Define $W_{ij} = c_G(ij)/(c_G(ij) - 1)$ when $ij \in E(G)$, and $W_{ij} = 0$ otherwise. For every entrywise nonnegative vector y ,*

$$(7) \quad y^\top W y \leq \left(\sum_i y_i \right)^2.$$

Consequently, every completely positive matrix C satisfies

$$(8) \quad \langle W, C \rangle \leq \langle J, C \rangle.$$

Proof. By homogeneity, it suffices to consider $\sum_i y_i = 1$. Choose a maximizer of $y^\top W y$ on this compact simplex with support of smallest possible size. If two vertices i, j in its support are nonadjacent, transfer their combined mass between them, keeping every other coordinate fixed. Since $W_{ii} = W_{jj} = W_{ij} = 0$, the objective is affine in the transferred mass. An endpoint of this interval is also a maximizer, contradicting minimal support. Thus the support is a clique.

If its size is $k \geq 2$, every edge in the support has $c_G(ij) \geq k$. Since $t/(t-1)$ decreases for $t > 1$,

$$y^\top W y \leq \frac{k}{k-1} \sum_{i \neq j} y_i y_j = \frac{k}{k-1} \left(1 - \sum_i y_i^2 \right) \leq 1.$$

A support of size one gives zero. This proves (7). Apply it to each nonnegative vector in a completely positive decomposition $C = \sum_a p_a p_a^\top$ and sum, obtaining $\langle W, C \rangle \leq \sum_a (\sum_i p_a(i))^2 = \langle J, C \rangle$. $\square$

Proposition 3.2. *Let B be symmetric, entrywise nonnegative, and zero on the diagonal, with $B_{ij} = 0$ whenever $ij \notin E(G)$. Then*

$$(9) \quad F_2(B) \leq 2 \sum_{ij \in E(G)} \left(1 - \frac{1}{c_G(ij)} \right) B_{ij}^2.$$

Proof. Let $S = F_2(B)$. If $S = 0$, the assertion is immediate. Otherwise choose a nonnegative unit Perron eigenvector u , and form

$$X = \sum_{\ell} \lambda_{\ell} u_{\ell} u_{\ell}^\top,$$

where the sum consists of the two largest positive eigenvalues of B , or all positive eigenvalues if there are fewer than two; take $u_1 = u$. At a multiple eigenvalue, the orthonormal choice is arbitrary. The spectral theorem gives $\langle B, X \rangle = \|X\|_F^2 = S$. The Gram vectors of X have the form $(\sqrt{\lambda_1} u(i), \sqrt{\lambda_2} u_2(i))$, with a zero second coordinate when needed. They lie in a common closed half-plane. Theorem 2.2, (5), and Lemma 3.1 therefore give

$$T := \sum_{i,j} W_{ij} (X_{ij})_+^2 = \langle W, \mathcal{M}(X) \rangle \leq \langle J, \mathcal{M}(X) \rangle = S.$$

Since B is entrywise nonnegative, weighted Cauchy-Schwarz on the ordered edges yields

$$S \leq \sum_{i,j} B_{ij} (X_{ij})_+ \leq \left(\sum_{i,j: ij \in E(G)} \frac{B_{ij}^2}{W_{ij}} \right)^{1/2} T^{1/2}.$$

Squaring and dividing by $S > 0$ proves (9). $\square$

Proof of Theorem 1.1. Apply Proposition 3.2 to $B = A_G$. Lemma 2.1 identifies $F_2(A_G)$ with the two ordered eigenvalue squares. $\square$

The rank-two Gram inequality of [4] follows from Proposition 3.2 by the same variational argument.

Theorem 3.3 (Rank-two Gram estimate). *Let $r \geq 2$. If $\omega(H) \leq r$ and X is positive semidefinite of rank at most two, then*

$$(10) \quad \sum_{i,j} (A_H)_{ij} (X_{ij})_+^2 \leq \left(1 - \frac{1}{r}\right) \|X\|_F^2.$$

Proof. Set $Y_{ij} = (A_H)_{ij} (X_{ij})_+$ and

$$T = \|Y\|_F^2 = \langle Y, X \rangle = \sum_{i,j} (A_H)_{ij} (X_{ij})_+^2.$$

If $T = 0$, the result is immediate. Let C and X_R be the compressions of Y and X to $R = \text{ran } X$. Then $X_R \succeq 0$ and $\|X_R\|_F = \|X\|_F$. Writing C_+ for the positive spectral part of C , positive semidefiniteness and Frobenius Cauchy–Schwarz give

$$\langle Y, X \rangle = \langle C, X_R \rangle \leq \langle C_+, X_R \rangle \leq \|C_+\|_F \|X\|_F.$$

The compression has order at most two, so interlacing yields $\|C_+\|_F^2 \leq F_2(Y)$. Every edge of H has $c_H(ij) \leq r$. Proposition 3.2, applied to Y , therefore gives

$$T^2 \leq F_2(Y) \|X\|_F^2 \leq \left(1 - \frac{1}{r}\right) T \|X\|_F^2.$$

Divide by $T > 0$. The positive part in (10) is entrywise before squaring; the proof imposes no half-plane condition on X . $\square$

4. COLOR-CRITICAL COUNTEREXAMPLES

A bounded component with an additional positive eigenvalue already destroys the exact color-critical bound.

Proposition 4.1. *For integers $r \geq 3$ and $s \geq 1$, put*

$$F_r = K_{r+2} - e, \quad D_{r,s} = \underbrace{K_{s,\dots,s}}_{r \text{ parts}} \dot{\cup} K_{r+1}.$$

Then F_r is color-critical with $\chi(F_r) = r + 1$, and $D_{r,s}$ is F_r -free. Its edge count tends to infinity with s , and

$$(11) \quad \lambda_1(D_{r,s})^2 + \lambda_2(D_{r,s})^2 - 2 \left(1 - \frac{1}{r}\right) e(D_{r,s}) = 1.$$

Proof. The endpoints of the missing edge in F_r can share a color, and the remaining r vertices form a clique. Thus $\chi(F_r) = r + 1$. Delete an edge between two vertices of that clique. Its endpoints can share a second color, leaving an r -coloring, so F_r is color-critical.

The graph F_r is connected, so a copy in $D_{r,s}$ must lie in one component. The multipartite component is r -colorable, whereas K_{r+1} has fewer than $r+2$ vertices. Hence $D_{r,s}$ is F_r -free. Its two positive eigenvalues are $(r-1)s$ and r , and

$$e(D_{r,s}) = \frac{r(r-1)}{2}s^2 + \frac{r(r+1)}{2}.$$

Substitution gives

$$(r-1)^2s^2 + r^2 - ((r-1)^2s^2 + r^2 - 1) = 1.$$

□

This refutes the unrestricted formulation in [15, Conjecture 5.4]. To obtain connected examples for the convention of [6], we join the two components for $r=3$ by one bridge. A two-dimensional compression certifies the remaining excess without computing the full spectrum.

Proof of Theorem 1.2. The graph $F = K_3 \vee I_2$ has chromatic number four. Delete an edge of the core triangle. Its endpoints can then share one color, the remaining core vertex takes a second color, and the two independent vertices take a third. Hence F is color-critical.

Let $T = K_{s,s,s}$, let a be a vertex in its first part, and join a to a vertex b of a disjoint K_4 . The resulting graph G_s is connected and has $3s^2 + 7$ edges. Every edge of F lies in a triangle, whereas the bridge ab lies in no triangle of G_s . An F -copy cannot use this bridge. Since F is connected, a copy avoiding the bridge would lie entirely in T or entirely in K_4 . The first possibility contradicts $\chi(F) = 4$, and the second contradicts $v(F) = 5$. Thus G_s is F -free.

Write $A = A_{G_s}$. Let U equal one on T and zero on K_4 . Define $V(a) = s-1$, let $V = -1$ on the other $s-1$ vertices in the first part of T , let $V = 0$ on the other two parts, and let $V = 3s$ on every vertex of K_4 . Direct summation gives

$$(12) \quad U^\top V = 0, \quad \|U\|^2 = 3s, \quad \|V\|^2 = s(37s-1),$$

$$(13) \quad U^\top AU = 6s^2, \quad U^\top AV = 3s, \quad V^\top AV = s(114s-6).$$

For the last identity, the internal contribution of T is zero, the K_4 contribution is $108s^2$, and the bridge contributes $6s(s-1)$.

Compression to the orthonormal vectors $U/\sqrt{3s}$ and $V/\sqrt{s(37s-1)}$ gives

$$(14) \quad C_s = \begin{pmatrix} \frac{2s}{\sqrt{3/(37s-1)}} & \sqrt{3/(37s-1)} \\ \sqrt{3/(37s-1)} & (114s-6)/(37s-1) \end{pmatrix}.$$

Its trace is positive, and

$$\det C_s = \frac{228s^2 - 12s - 3}{37s-1} > 0.$$

Both compression eigenvalues are therefore positive. Interlacing and (14) imply

$$(15) \quad \begin{aligned} \lambda_1(G_s)^2 + \lambda_2(G_s)^2 &\geq \text{tr}(C_s^2) \\ &= 4s^2 + \frac{(114s - 6)^2}{(37s - 1)^2} + \frac{6}{37s - 1}. \end{aligned}$$

Subtracting $4(3s^2 + 7)/3$ gives the rational expression in (2). Its numerator $P(s) = 656s^2 - 1366s + 62$ satisfies $P(3) = 1868$ and $P(s + 1) - P(s) = 1312s - 710 > 0$ for $s \geq 3$. This proves the strict inequality. Finally, $3s^2 + 7$ is unbounded, so no edge threshold can make the proposed exact bound hold for this F . $\square$

5. WEIGHTED REMOVAL AND FLAT GRAM MATRICES

Fix F with $\chi(F) = r + 1 \geq 3$, and write $\kappa = 1 - 1/r$. We first replace clique-freeness by F -freeness when the diagonal mass of a Gram matrix is spread over many vertices. The deletion statement below is stronger than the scalar estimate needed for one probability vector: it retains a clique-free graph on which the rank-two Gram theorem can be applied.

We use the ordinary graph removal lemma in its labelled form: for a fixed graph K and every $\xi > 0$, there is $\delta > 0$ such that a graph on N vertices with at most $\delta N^{v(K)}$ labelled copies of K can be made K -free by deleting at most ξN^2 edges; see [3, Theorem 1.1]. We apply this only to $K = K_{r+1}$. The labelled and unlabelled versions differ by a constant depending only on K . For $\rho > 0$, choose a valid labelled K_k -removal threshold $\delta_k(\rho) \in (0, 1]$ with unordered deletion budget ρN^2 .

Lemma 5.1 (Weighted clique removal). *For every $\varepsilon > 0$ there is $\eta = \eta(F, \varepsilon) > 0$ with the following property. If G is F -free and p is a probability vector on $V(G)$ with $\max_i p_i \leq \eta$, then G has a spanning K_{r+1} -free subgraph G_0 such that*

$$(16) \quad \sum_{ij \in E(G) \setminus E(G_0)} p_i p_j \leq \varepsilon.$$

Writing $k = r + 1$ and $t = v(F)$, one may take

$$\eta(F, \varepsilon) = \frac{\delta_k(\varepsilon/2)^{t^k}}{2^{\binom{t}{2}}}.$$

Proof. Put $k = r + 1$ and $t = v(F)$. For a graph J , let $t_p(J, G)$ be the probability that a map obtained by independently sampling every vertex image from p is a homomorphism. Every homomorphism $F \rightarrow G$ identifies two vertices, since G is F -free. A union bound gives

$$(17) \quad t_p(F, G) \leq \binom{t}{2} \sum_i p_i^2 \leq \binom{t}{2} \max_i p_i.$$

Cloning a vertex into t pairwise nonadjacent twins replaces a conditional factor h by h^t . More precisely, condition on all other vertex images, let $g \in \{0, 1\}$ indicate their edge requirements, and let $h \in [0, 1]$ be the conditional probability of meeting the adjacency requirements of the vertex being cloned. The original and cloned densities are $\mathbb{E}[gh]$ and $\mathbb{E}[gh^t]$. Jensen's inequality gives $\mathbb{E}[gh^t] = \mathbb{E}[(gh)^t] \geq (\mathbb{E}[gh])^t$. Clone the k original vertices of K_k , one after another. Writing $K_k(t)$ for the complete k -partite graph with t vertices in each part, we obtain

$$(18) \quad t_p(K_k(t), G) \geq t_p(K_k, G)^{t^k}.$$

A proper k -coloring embeds F into $K_k(t)$, so $t_p(K_k(t), G) \leq t_p(F, G)$. Equations (17) and (18) therefore make $t_p(K_k, G)$ arbitrarily small by making $\max p_i$ sufficiently small.

First fix a deletion tolerance $\rho > 0$ and suppose p is rational. Ignore its zero coordinates, and replace vertex i by Np_i independent clones, for a sufficiently large common multiple N . Place all edges between clone classes corresponding to edges of G . The resulting graph B has K_k -homomorphism density $t_p(K_k, G)$. A clique homomorphism into a loopless graph is injective, so the ordinary removal lemma gives a K_k -free subgraph $B' \subseteq B$ after at most ρN^2 edge deletions whenever $\max_i p_i \leq \delta_k(\rho)^{t^k} / \binom{t}{2}$.

Independently choose one uniformly random clone of every original positive-weight vertex. Use the edges of B' between these representatives to define a subgraph of G , and isolate the original zero-weight vertices. This graph is K_k -free. If d_{ij} edges were deleted between two clone classes, the probability that the representative pair is missing is $d_{ij}/(N^2 p_i p_j)$. The expected unordered deletion cost is thus exactly

$$\sum_{ij \in E(G): p_i p_j > 0} p_i p_j \frac{d_{ij}}{N^2 p_i p_j} = \frac{\sum d_{ij}}{N^2} \leq \rho.$$

Some choice of representatives has cost at most this expectation.

For a real probability vector, take η as displayed in the statement. Approximate p by a rational probability vector p' with the same support, $\max_i p'_i \leq 2\eta$, and $\|p - p'\|_1 \leq \varepsilon/4$. Apply the rational assertion with $\rho = \varepsilon/2$; the bound $2\eta = \delta_k(\varepsilon/2)^{t^k} / \binom{t}{2}$ permits this. For every set E_0 of unordered pairs,

$$(19) \quad \left| \sum_{ij \in E_0} p_i p_j - \sum_{ij \in E_0} p'_i p'_j \right| \leq \sum_{i,j} |p_i p_j - p'_i p'_j| \leq 2\|p - p'\|_1.$$

The resulting p -cost is at most $\varepsilon/2 + 2\|p - p'\|_1 \leq \varepsilon$. $\square$

Lemma 5.2 (Flat Gram estimate). *For every $\varepsilon > 0$ there is $\eta = \eta(F, \varepsilon) > 0$ such that, whenever G is F -free and $X \neq 0$ is positive semidefinite of rank*

at most two with $\max_i X_{ii}/\text{tr } X \leq \eta$, one has

$$(20) \quad \sum_{i,j} (A_G)_{ij} (X_{ij})_+^2 \leq (\kappa + \varepsilon) \|X\|_F^2.$$

Proof. Set $w_i = X_{ii}$, $\ell = \text{tr } X > 0$, and $p_i = w_i/\ell$. Positive semidefiniteness gives $X_{ij}^2 \leq w_i w_j$. Apply Lemma 5.1 with deletion cost $\varepsilon/4$. The loss in the ordered sum in (20) is at most $2\ell^2(\varepsilon/4)$. Since $\text{rank } X \leq 2$, Cauchy–Schwarz on its nonnegative eigenvalues gives $\ell^2 \leq 2\|X\|_F^2$. The loss is therefore at most $\varepsilon\|X\|_F^2$. On the retained K_{r+1} -free graph, apply Theorem 3.3. $\square$

The diagonal of the positive spectral lift is controlled directly by the vertex degrees. This avoids a cutoff between large and small eigenvalues and gives a uniform low-degree statement.

Lemma 5.3 (Low maximum degree). *For every $\varepsilon > 0$ there is $\delta = \delta(F, \varepsilon) > 0$ such that every F -free graph G with $m > 0$ and $\Delta(G) \leq \delta m$ satisfies*

$$(21) \quad F_2(A_G) \leq 2(\kappa + \varepsilon)m.$$

Proof. Let $A = A_G$, and form its full two-positive-eigenvalue lift

$$Z = \sum_{\ell} \lambda_{\ell} u_{\ell} u_{\ell}^{\text{T}}, \quad S = \|Z\|_F^2 = F_2(A).$$

If $S \leq m$, then $S \leq 2\kappa m$, since $\kappa \geq 1/2$. Assume $S > m$. The matrix Z^2 is a spectral truncation of A^2 , so $Z^2 \preceq A^2$. For every vertex v ,

$$Z_{vv}^2 \leq (Z^2)_{vv} \leq (A^2)_{vv} = d(v), \quad \text{tr } Z \geq \sqrt{S}.$$

Consequently,

$$(22) \quad \frac{\max_v Z_{vv}}{\text{tr } Z} \leq \sqrt{\frac{\Delta(G)}{S}} \leq \sqrt{\frac{\Delta(G)}{m}}.$$

Choose $\delta = \eta^2$, where η is the threshold in Lemma 5.2 for this ε . Since $\langle A, Z \rangle = S$ and A is entrywise nonnegative, Cauchy–Schwarz on the $2m$ ordered edges gives

$$S^2 \leq \left(\sum_{i,j} A_{ij} (Z_{ij})_+ \right)^2 \leq 2m \sum_{i,j} A_{ij} (Z_{ij})_+^2 \leq 2m(\kappa + \varepsilon)S.$$

Divide by $S > 0$. $\square$

6. SEPARATING VERTICES OF HIGH DEGREE

Lemma 5.3 applies to the part of the graph left after a suitable degree cutoff. The next estimate isolates the spectral contribution of the removed vertices. Its error depends on the interaction of the residual adjacency matrix with the incidence matrix, which can be small even when the incidence matrix itself has large norm.

Lemma 6.1 (Spectral separation). *Let G have $m > 0$ edges, let $H \subseteq V(G)$ have h vertices, and let $D = A_{G-H}$. Put $\Delta_0 = \Delta(G - H)$ and $q = e(H, V(G) \setminus H)$. Then*

$$(23) \quad F_2(A_G) \leq F_2(D) + q + 4\sqrt{2}m \left[3 \left(\frac{2h\Delta_0}{m} \right)^{1/4} + \frac{h}{\sqrt{m}} \right].$$

When $h\Delta_0 = 0$, the fourth-root term is zero.

Proof. Write

$$A_G = \begin{pmatrix} A_H & B \\ B^\top & D \end{pmatrix}, \quad \hat{B} = B/\sqrt{m}, \quad \hat{D} = D/\sqrt{m}.$$

For $i \in H$ and $j \notin H$, the entry $(DB^\top)_{ji}$ is at most $d_{G-H}(j)$. Thus

$$(24) \quad \|\hat{D}\hat{B}^\top\|_{\text{op}}^2 \leq \|\hat{D}\hat{B}^\top\|_F^2 \leq \frac{h \sum_j d_{G-H}(j)^2}{m^2} \leq \frac{2h\Delta_0}{m}.$$

First suppose $a = (2h\Delta_0/m)^{1/4} > 0$. Truncate the singular value decomposition of $\hat{B}$, retaining exactly the singular values greater than a , and denote the result by B_1 . Let P be the orthogonal projection onto the retained right singular space and $Q = I - P$. Then $\|\hat{B} - B_1\|_{\text{op}} \leq a$. If $B_1 = U\Sigma V^\top$ on the retained spaces, then $P = VV^\top$ and

$$\hat{D}P = \hat{D}\hat{B}^\top U \Sigma^{-1} V^\top.$$

The identity is vacuous if the retained space is zero. By (24),

$$(25) \quad \|\hat{D}P\|_{\text{op}} \leq \frac{\|\hat{D}\hat{B}^\top\|_{\text{op}}}{a} \leq a.$$

Define

$$A_0 = \begin{pmatrix} 0 & B_1 \\ B_1^\top & Q\hat{D}Q \end{pmatrix}.$$

Since $\hat{D} - Q\hat{D}Q = P\hat{D} + Q\hat{D}P$, its operator norm is at most $2a$. Also $\|A_H\|_{\text{op}} \leq h$, and hence

$$(26) \quad \|A_G/\sqrt{m} - A_0\|_{\text{op}} \leq h/\sqrt{m} + 3a.$$

The bipartite block with incidence matrix B_1 acts on the coordinates indexed by H together with $\text{ran } P$. The block $Q\hat{D}Q$ acts on the orthogonal space $\text{ran } Q$. The positive eigenvalues of the bipartite block are the singular values of B_1 , so its F_2 -value is at most $\|B_1\|_F^2 \leq \|\hat{B}\|_F^2 = q/m$. Compression and the direct-sum inequality give

$$(27) \quad F_2(A_0) \leq q/m + F_2(D)/m.$$

Both $A_G/\sqrt{m}$ and A_0 have operator norm at most $\sqrt{2}$. For the first, use $\|A_G\|_F^2 = 2m$. For the second, use the orthogonal splitting above, $\|B_1\|_{\text{op}} \leq \|\hat{B}\|_F = \sqrt{q/m} \leq 1$, and $\|\hat{D}\|_{\text{op}} \leq \sqrt{2}$. Apply (6) with $L = \sqrt{2}$, and then (26) and (27). Multiplication by m proves (23).

If $h = 0$, the assertion is immediate. If $h > 0$ and $\Delta_0 = 0$, then $D = 0$. Take $B_1 = \widehat{B}$ without truncation and take A_0 to be its bipartite adjacency block. Only the error $h/\sqrt{m}$ remains. The same norm comparison proves the boundary case. $\square$

A dyadic cutoff makes both terms in the separation error small, uniformly over all graphs of a given size.

Lemma 6.2 (Choice of the separated set). *For $m \geq 16$, put $j_* = \lfloor (\log_2 m)/4 \rfloor$. Every graph with m edges has a set H containing a maximum-degree vertex such that*

$$(28) \quad 1 \leq h = |H| \leq m^{1/4}, \quad \frac{h\Delta(G-H)}{m} \leq \frac{4}{j_* + 1}.$$

Proof. Order the original degrees as $d_1 \geq d_2 \geq \dots$, padding by zeros. For $h = 2^j$ with $j \geq 1$,

$$h d_{h+1} \leq 2 \sum_{i=h/2+1}^h d_i.$$

These blocks are disjoint as j varies, and the $j = 0$ term d_2 is at most $2d_1$. Therefore

$$\sum_{j=0}^{j_*} 2^j d_{2^j+1} \leq 2 \sum_i d_i = 4m.$$

Choose $h = 2^j$ minimizing $h d_{h+1}$, and let H consist of the h highest-degree vertices. There are enough vertices, since $n \geq \sqrt{2m} \geq m^{1/4}$. Every residual degree is at most d_{h+1} , proving (28). $\square$

Proof of the bound in Theorem 1.3. Choose H by Lemma 6.2, and put

$$m' = e(G-H), \quad q = e(H, V(G) \setminus H), \quad e_H = e(G[H]).$$

Thus $m = m' + q + e_H$. Write

$$\beta_m = \frac{4}{\lfloor (\log_2 m)/4 \rfloor + 1}, \quad \zeta_m = 4\sqrt{2} \left[3(2\beta_m)^{1/4} + m^{-1/4} \right].$$

The separation estimate gives

$$(29) \quad F_2(A_G) \leq F_2(A_{G-H}) + q + \zeta_m m, \quad \zeta_m \rightarrow 0.$$

We claim that, for every $\sigma > 0$, all sufficiently large m satisfy

$$(30) \quad F_2(A_{G-H}) \leq 2\kappa m' + \sigma m,$$

uniformly over F -free G . Choose $\alpha > 0$ with $2(1 - \kappa)\alpha \leq \sigma$. If $m' \leq \alpha m$, then

$$F_2(A_{G-H}) \leq 2m' \leq 2\kappa m' + \sigma m.$$

If $m' > \alpha m$, then, since $h \geq 1$,

$$\frac{\Delta(G-H)}{m'} \leq \frac{\beta_m}{\alpha} \rightarrow 0.$$

Apply Lemma 5.3 with parameter $\sigma/2$. For all sufficiently large m , its conclusion is $F_2(A_{G-H}) \leq 2\kappa m' + \sigma m' \leq 2\kappa m' + \sigma m$. This proves (30), including the case $m' = 0$.

Set $c = 2\kappa - 1 \geq 0$. Since H contains a maximum-degree vertex, every edge incident to this vertex is counted among the $q + e_H$ edges incident to H . Hence $q + e_H \geq \Delta(G)$. Combining (29) and (30), we obtain

$$\begin{aligned} F_2(A_G) &\leq 2\kappa m' + q + (\sigma + \zeta_m)m \\ &= 2\kappa m - c(q + e_H) - e_H + (\sigma + \zeta_m)m \\ &\leq 2\kappa m - c\Delta(G) + (\sigma + \zeta_m)m. \end{aligned}$$

Given $\varepsilon > 0$, take $\sigma = \varepsilon/2$ and then choose $M(F, \varepsilon)$ large enough for (30) and $\zeta_m \leq \varepsilon/2$ to hold whenever $m \geq M$. This proves the uniform estimate for $F_2(A_G)$. Increase M , if necessary, so that $M \geq 16$ and $M > \binom{v(F)-1}{2}$. Lemma 2.1 then identifies $F_2(A_G)$ with the ordered top-two square sum for every graph under consideration, proving (3). $\square$

7. EQUALITY EXAMPLES AND NEAR EXTREMALITY

Stars establish the optimal coefficient in the introduction. Mixtures of a regular multipartite component and a star also attain the degree-penalized expression, with both components carrying a positive proportion of the edges.

Let $T = K_{s, \dots, s}$ have r equal parts, and put $G = T \dot{\cup} K_{1, M}$ and $a_r = r(r-1)/2$. This graph is r -colorable and has $m = a_r s^2 + M$ edges. Its only positive eigenvalues are $(r-1)s$ and $\sqrt{M}$. If $M \geq (r-1)s$, then $\Delta(G) = M$, and

$$(31) \quad F_2(A_G) = (r-1)^2 s^2 + M = 2\kappa m - (2\kappa - 1)\Delta(G).$$

Taking $M = \lfloor \theta s^2 \rfloor$, for fixed $\theta > 0$, gives $\Delta(G)/m \rightarrow \theta/(a_r + \theta)$. Thus equality in this expression occurs along sequences with any prescribed limiting degree-to-edge ratio in $(0, 1)$, as well as the star endpoint 1.

For $r \geq 3$, the positive degree coefficient also constrains asymptotically extremal graphs.

Corollary 7.1. *Fix F with $\chi(F) = r + 1 \geq 4$. If G_j are F -free, $m_j = e(G_j) \rightarrow \infty$, and $F_2(A_{G_j})/(2m_j) \rightarrow \kappa$, then $\Delta(G_j) = o(m_j)$. For the full two-positive-eigenvalue lift Z_j ,*

$$\frac{\max_v (Z_j)_{vv}}{\text{tr } Z_j} \rightarrow 0.$$

Every selected unit eigenvector whose eigenvalue is at least a fixed positive multiple of $\sqrt{m_j}$ satisfies $\max_v |u_j(v)|^2 \rightarrow 0$.

Proof. For all sufficiently large j , Lemma 2.1 identifies $F_2(A_{G_j})$ with the ordered top-two square sum. Theorem 1.3, in uniform little-oh form, gives

$$(2\kappa - 1)\Delta(G_j) \leq 2\kappa m_j - F_2(A_{G_j}) + o_F(m_j) = o(m_j).$$

Now use (22) with $F_2(A_{G_j}) \sim 2\kappa m_j$. For a unit adjacency eigenvector, the eigenvector equation and Cauchy–Schwarz give

$$\lambda^2 u(v)^2 = \left(\sum_{w \sim v} u(w) \right)^2 \leq d(v) \sum_{w \sim v} u(w)^2 \leq d(v).$$

The asserted eigenvector bound follows. $\square$

For $r = 2$, stars satisfy $F_2(A_G) = m$ and $\Delta(G) = m$. The central coordinate of their unit Perron vector has squared magnitude $1/2$. Thus neither the degree conclusion nor diagonal flatness extends to this boundary.

Disconnected equality examples are also essential. For positive integers s, t , the graph

$$\underbrace{K_{s, \dots, s}}_{r \text{ parts}} \dot{\cup} \underbrace{K_{t, \dots, t}}_{r \text{ parts}}$$

has exactly two positive eigenvalues and satisfies both $F_2(A_G) = 2\kappa e(G)$ and equality in (1). These examples, already noted in [6], allow two components of different regularities. The degree and lift conclusions above therefore leave room for more than one multipartite component.

Remark 7.2. Balanced complete multipartite graphs already force a square-root lower bound on the remainder in the degree-penalized estimate. Fix F with $\chi(F) = r + 1 \geq 4$, and suppose $E : \mathbb{N} \rightarrow \mathbb{R}$ satisfies

$$F_2(A_G) \leq 2 \left(1 - \frac{1}{r} \right) m - \frac{r-2}{r} \Delta(G) + E(m)$$

for every F -free graph with sufficiently many edges. For $s \geq 2$, the balanced complete r -partite graph $T_r(rs)$ is F -free and has

$$m_s = \frac{r(r-1)}{2} s^2, \quad \Delta(T_r(rs)) = (r-1)s,$$

$$F_2(A_{T_r(rs)}) = (r-1)^2 s^2 = 2 \left(1 - \frac{1}{r} \right) m_s.$$

Consequently, for all sufficiently large s ,

$$(32) \quad \begin{aligned} E(m_s) &\geq \frac{(r-2)(r-1)}{r} s, \\ \liminf_{s \rightarrow \infty} \frac{E(m_s)}{\sqrt{m_s}} &\geq \frac{r-2}{r} \sqrt{\frac{2(r-1)}{r}}. \end{aligned}$$

For $r = 3$, this constant is $2/(3\sqrt{3})$. Thus an $o(\sqrt{m})$ remainder is impossible for every fixed forbidden graph of chromatic number at least four. The connected $K_5 - e$ -free construction gives the same leading constant along $m = 3s^2 + 7$, with an additional bounded contribution.

8. FORMALIZATION

The Palomar records [18, 19, 20] verify Theorems 1.1, 1.2 and 1.3, respectively, together with the companion statements listed below. Each record preserves the formal statements, proofs and immutable source version.

Mathematical statement	Lean declaration
Theorem 1.1	<code>LocalBN.edgeLocalizedNoncompleteUnordered</code>
Theorem 1.2	<code>KKP.connected_counterexample_family</code>
Arbitrarily large counterexamples	<code>KKP.unbounded_connected_counterexamples</code>
Theorem 1.3	<code>KKP117.strengthened_ordered_asymptotic</code>
Positive-eigenvalue form	<code>KKP117.strengthened_asymptotic</code>

The packages use Lean 4.35.0-rc3 with Mathlib. The selected proofs are checked by Lean’s kernel and the independent kernels NanoDa and ConRon, under the standard axiom policy stated above. The formal statements retain the uniform ε – M quantifiers. The asymptotic formalization proves the clique-removal and spectral inputs and uses the proved BN Gram theorem. The written proof derives that estimate in Theorem 3.3 from Proposition 3.2. The coefficient’s optimality, Proposition 4.1 and the additional examples and consequences in Section 7 are established by the written arguments.

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OpenAI ChatGPT and Codex assisted with mathematical exploration, Lean formalization, manuscript preparation and review. The author reviewed the AI-assisted material and takes full responsibility for the manuscript.

REFERENCES

- [1] B. Bollobás and V. Nikiforov, Cliques and the spectral radius, *J. Combin. Theory Ser. B* **97** (2007), 859–865. doi:10.1016/j.jctb.2006.12.002.
- [2] D. Bradač, A generalization of Turán’s theorem, arXiv:2205.08923 (2022). <https://arxiv.org/abs/2205.08923>.
- [3] D. Conlon and J. Fox, Graph removal lemmas, arXiv:1211.3487 (2012). <https://arxiv.org/abs/1211.3487>.
- [4] G. Coutinho, Y. Liu, T. J. Spier, Q. Tang and S. Zhang, The Bollobás–Nikiforov inequality for nonnegative edge weights, repository manuscript, fixed source version. <https://github.com/ShengtongZhang-alt/BN/blob/5f4741254398424b730c9a893220a51e1cd829c5/docs/sol.tex>.
- [5] A. Jayarajan, M. R. Kannan, S. Pragada and R. Roy, A vertex-localized positive square-energy strengthening of Turán’s theorem, arXiv:2608.24861v2 (2026). <https://arxiv.org/abs/2608.24861v2>.
- [6] M. R. Kannan, H. Kumar and S. Pragada, Localization of spectral Turán-type theorems, arXiv:2512.01409v2. <https://arxiv.org/abs/2512.01409v2>.
- [7] H. Kumar and S. Pragada, Bollobás–Nikiforov conjecture for graphs with not so many triangles, *Linear Algebra Appl.* **727** (2025), 1–9. doi:10.1016/j.laa.2025.07.035.

- [8] H. Lin, B. Ning and B. Wu, Eigenvalues and triangles in graphs, *Combin. Probab. Comput.* **30** (2021), 258–270. doi:10.1017/S0963548320000462.
- [9] S. Zhang, On the first two eigenvalues of regular graphs, *Linear Algebra Appl.* **686** (2024), 102–110. doi:10.1016/j.laa.2024.01.002.
- [10] G. Coutinho, T. J. Spier and S. Zhang, Conic programming to understand sums of squares of eigenvalues of graphs, arXiv:2411.08184v1 (2024). <https://arxiv.org/abs/2411.08184v1>.
- [11] D. Malec and C. Tompkins, Localized versions of extremal problems, *European J. Combin.* **112** (2023), Article 103715. doi:10.1016/j.ejc.2023.103715.
- [12] Y. Li, H. Liu and S. Zhang, An edge-spectral Erdős–Stone–Simonovits theorem and its stability, arXiv:2508.15271v2 (2026). <https://arxiv.org/abs/2508.15271v2>.
- [13] F. Liu, S. Sun, Y. Wang and Q. Wu, Local Turán inequalities for walks and the spectral radius, arXiv:2605.02191v2 (2026). <https://arxiv.org/abs/2605.02191v2>.
- [14] Y. Li, An entropy bridge from weighted to spectral Turán theorems, arXiv:2608.24388v2 (2026). <https://arxiv.org/abs/2608.24388v2>.
- [15] Y. Li, H. Liu and S. Zhang, Edge-spectral Turán theorems for color-critical graphs with applications, arXiv:2511.15431v3 (2026), accepted for publication in *SIAM J. Discrete Math.* <https://arxiv.org/abs/2511.15431v3>.
- [16] L. Liu and B. Ning, Local properties of the spectral radius and Perron vector in graphs, *J. Combin. Theory Ser. B* **176** (2026), 241–253. doi:10.1016/j.jctb.2025.09.001.
- [17] L. Liu and B. Ning, A new spectral Turán theorem for weighted graphs and consequences, arXiv:2510.26410v1 (2025). <https://arxiv.org/abs/2510.26410v1>.
- [18] I. A. Shaik, Localized two-eigenvalue bound for finite noncomplete graphs, Palomar, PALOMAR-2026-10-15-000015 v1 (2026). <https://palomar-registry.org/entry?id=PALOMAR-2026-10-15-000015&version=1>.
- [19] I. A. Shaik, Connected counterexamples to the exact color-critical two-eigenvalue bound, Palomar, PALOMAR-2026-10-16-000012 v1 (2026). <https://palomar-registry.org/entry?id=PALOMAR-2026-10-16-000012&version=1>.
- [20] I. A. Shaik, A sharp asymptotic spectral bound for forbidden subgraphs, Palomar, PALOMAR-2026-10-17-000005 v1 (2026). <https://palomar-registry.org/entry?id=PALOMAR-2026-10-17-000005&version=1>.

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